Null controllability of a linearized Korteweg-de Vries equation by backstepping approach

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Abstract

This paper deals with the controllability problem of a linearized Korteweg-de Vries equation on bounded interval. The system has a homogeneous Dirichlet boundary condition and a homogeneous Neumann boundary condition at the right end-points of the interval, a non homogeneous Dirichlet boundary condition at the left end-point which is the control. We prove the null controllability by using a backstepping approach, a method usually used to handle stabilization problems.

Keywords. Korteweg-de Vries, backstepping, spectral theory, stabilization, null controllability.
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1 Introduction

In this paper we consider the null controllability of the following linearized KdV control system:

$$\begin{cases} u_t + u_{xxx} + u_x = 0 & \text{in } (0,+\infty)\times(0,L), \\ u(t,L) = u_x(t,L) = 0 & \text{on } (0,+\infty), \\ u(t,0) = \kappa(t) & \text{on } (0,+\infty), \end{cases} \tag{1.1}$$

where $\kappa(t) \in \mathbb{R}$ is the control term.

This control system is a little different from the pioneer one raised by Lionel Rosier [33], the only difference is the place where control acts: $u(t,0)$ instead of $u_x(t,L)$. For the model given by Lionel Rosier, one surprisingly finds that the controllability of system depends on the length of the interval, more precisely the system is controllable if and only if

$$L \notin \mathcal{N} := \Big\{ 2\pi\sqrt{\frac{l^2+lk+k^2}{3}};\ l,k \in \mathbb{N}^* \Big\}. \tag{1.2}$$

This system has a great interest in mathematical and engineering points, the controllability [4, 5, 7, 9, 11, 33, 35] and stabilization [8, 14, 17, 32, 38] has been studied for years.

In the case of system (1.1), there is no critical length (but some regularity difficulties occur). The controllability was first discovered by Lionel Rosier in [34], which was further discussed in [6, 21]. In [6], Jean-Michel Coron and Eduardo Cerpa have given the rapid stabilization for this

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system by using the backstepping method. Recently, in [16], using the backstepping approach Jean-Michel Coron and Hoai-Minh Nguyen proved the null controllability and the semi-global finite time stabilization for a class of heat equations. This turned out to be a method to reach possibly the semi-global finite time stabilization for those systems which can be rapidly stabilized by means of backstepping methods. At the same time, this provides a visible way to get the null controllability instead of using Hilbert Uniqueness Method by which one doesn't know explicitly what the control is.

In this paper, we use this new method developped by Jean-Michel Coron and Hoai-Minh Nguyen to prove the null controllability of system (1.1).

Theorem 1. *For any given $T > 0$, the control system (1.1) is null controllable in time T.*

Remark 1. *Instead of using Carleman estimate in [34], we proved the null controllability of linearized KdV control system (1.1). Let us recall that the exact controllability of system (1.1) fails, which is proved in [34].*

Introduced by Jean-Michel Coron and Brigitte d'Andréa-Novel in [12] for PDE stabilization problems, the backstepping method has been well used and developped in recent 20 years (see [10, 13, 14, 15]). One can see [25], due to Miroslav Krstic and his collaborators, for an excellent starting point to get inside this method. Generally speaking, when we deal with a disspative equation, by simply considering its eigenvalues (without feedback law) we can get asymptotic stability. Backstepping method aims at changing the spectre (let the eigenvalues be away from imaginary-axis) by adding a feedback on the control term (normally a boudary control), thus making the transformed equation decay exponentially (even rapid decay). Normally we need to solve a kernel equation. In the case of the heat equation the kernel equation is just a wave equation, but in this paper the kernel equation turned out to be a third-order equation which generates some difficulties in both estimation and well-posedness.

This paper is organized as follows. Section 2 is a preliminary part in which we present the rapid stabilization obtained in [6]. In Section 3, we give explicitly the control and give some estimates which will lead to the null controllability. In Section 4, we give the null controllability. The article ends with an appendix where the proof of the uniqueness of solution to the kernel equation, Proposition 1, which is essential in this paper, is given.

2 Preliminary

2.1 Well-posedness of the control system

We start with the non-homogeneous linear Cauchy problem

$$\begin{cases} u_t + u_{xxx} + u_x = \tilde{h} & \text{in } (T_1, T_2) \times (0, L), \\ u_x(t, L) = u(t, L) = 0 & \text{on } (T_1, T_2), \\ u(t, 0) = \kappa(t) & \text{on } (T_1, T_2), \\ u(0, x) = u_0(x) & \text{on } (0, L), \end{cases} \tag{2.1}$$

for

$$-\infty < T_1 < T_2 < +\infty, \tag{2.2}$$

$$u_0 \in L^2(0, L), \tag{2.3}$$

$$\tilde{h} \in L^1(T_1, T_2; L^2(0, L)), \tag{2.4}$$

$$\kappa \in L^2(T_1, T_2). \tag{2.5}$$

Definition 1. *Let T_1, T_2 be given. A solution to Cauchy problem (2.1)–(2.5) is a function $u \in C^0([T_1, T_2]; L^2(0, L))$ such that, for every $\tau \in [T_1, T_2]$ and for every $\phi \in C^3([T_1, \tau] \times [0, L])$ such that*

$$\phi(t, 0) = \phi(t, L) = \phi_x(t, 0) = 0, \forall t \in [T_1, \tau], \tag{2.6}$$

one has

$$\begin{aligned} -\int_{T_1}^{\tau}\int_0^L (\phi_t + \phi_x + \phi_{xxx})u dx dt - \int_{T_1}^{\tau} \kappa(t)\phi_{xx}(t, 0)dt - \int_{T_1}^{\tau}\int_0^L \phi\tilde{h} dx dt \\ + \int_0^L u(\tau, x)\phi(\tau, x)dx - \int_0^L u_0\phi(T_1, x)dx = 0. \end{aligned} \tag{2.7}$$

It is easy to see the uniqueness of the solution to Cauchy problem (2.1)–(2.5), one can get the details in the book by Jean-Michel Coron [9]. For the existence of the solution, in [2], Jerry L. Bona, Shu Ming Sun and Bing-Yu Zhang proved the following result.

Lemma 1. *If $h \in H^{1/3}(T_1, T_2)$, then Cauchy problem (2.1) has one and only one solution. This solution is in $C^0([T_1, T_2]; L^2(0, L)) \cap L^2(T_1, T_2; H^1(0, L))$, and there exist a constant $c_1 > 0$ depending only on $T_2 - T_1$ such that*

$$\begin{aligned} \|u\|_{C^0([T_1,T_2];L^2(0,L))} + \|u\|_{L^2(T_1,T_2;H^1(0,L))} + Sup_{x\in[0,L]}\|u_x(\cdot, x)\|_{L^2(T_1,T_2)} \\ \leqslant c_1\left(\|u_0\|_{L^2(0,L)} + \|\kappa\|_{H^{1/3}(T_1,T_2)} + \|\tilde{h}\|_{L^1(T_1,T_2;L^2(0,L))}\right). \end{aligned} \tag{2.8}$$

2.2 Rapid stabilization of (1.1)

We recall the results given in [6]. Given a positive parameter λ, we consider the following equations in the triangle $\mathcal{T} := \{(x, y) : x \in [0, L], y \in [x, L]\}$:

$$\begin{cases} k_{xxx} + k_{yyy} + k_x + k_y + \lambda k = 0 & \text{in } \mathcal{T}, \\ k(x, L) = 0 & \text{on } [0, L], \\ k(x, x) = 0 & \text{on } [0, L], \\ k_x(x, x) = \frac{\lambda}{3}(L - x) & \text{on } [0, L], \end{cases} \tag{2.9}$$

and

$$\begin{cases} l_{xxx} + l_{yyy} + l_x + l_y - \lambda l = 0 & \text{in } \mathcal{T}, \\ l(x, L) = 0 & \text{on } [0, L], \\ l(x, x) = 0 & \text{on } [0, L], \\ l_x(x, x) = \frac{\lambda}{3}(L - x) & \text{on } [0, L]. \end{cases} \tag{2.10}$$

In [6], it is pointed that both (2.9) and (2.10) have solution in $C^3(\mathcal{T})$, in Section 3 we give some estimations on $\|k\|_{C^0(\mathcal{T})}$ with respect to λ. Actually, the solutions of the equation (2.9) and of (2.10) satisfies the following condition

$$k_{xy}(x, x) = -\frac{\lambda}{3}, \tag{2.11}$$

$$l_{xy}(x, x) = -\frac{\lambda}{3}, \tag{2.12}$$

respectively. Properties (2.11) and (2.12) can be checked as follows, we perform the change of variables:

$$t = y - x, \quad s = x + y \tag{2.13}$$

and define

$$G(s,t) := k(x,y). \tag{2.14}$$

Then equation (2.9) of k becomes the following equation of G:

$$\begin{cases} 6G_{tts} + 2G_{sss} + 2G_s + \lambda G = 0 & \text{in } \mathcal{T}_0, \\ G(s, 2L - s) = 0 & \text{on } [L, 2L], \\ G(s,0) = 0 & \text{on } [0, 2L], \\ G_t(s,0) = \frac{\lambda}{6}(s - 2L) & \text{on } [0, 2L], \end{cases} \tag{2.15}$$

where $\mathcal{T}_0 := \{(s,t); t \in [0,L], s \in [t, 2L - t]\}$. From (2.15) one can easily get that

$$G_{tts}(s,0) = 0 \text{ in } [0, 2L]. \tag{2.16}$$

Hence $G_{tt}(s,0) = G_{tt}(2L,0)$. In order to calculate $G_{tt}(2L,0)$, one observes from (2.15) that

$$G_{tt}(2L,0) = 2G_{ts}(2L,0) = \frac{\lambda}{3}. \tag{2.17}$$

Direct calculations show that

$$k_{xy}(x,x) = -G_{tt}(s,0) = -\frac{\lambda}{3}, \tag{2.18}$$

which concludes (2.11), the proof of (2.12) is similar.

Now, let us define a continuous transformation $\Pi_\lambda : L^2(0,L) \to L^2(0,L)$ by

$$\omega(x) = \Pi_\lambda(u(x)) := u(x) - \int_x^L k(x,y)u(y)dy, \tag{2.19}$$

and its inverse $\Pi_\lambda^{-1} : L^2(0,L) \to L^2(0,L)$ by

$$u(x) = \Pi_\lambda^{-1}(\omega(x)) := \omega(x) + \int_x^L l(x,y)\omega(y)dy. \tag{2.20}$$

Let us point that the reason why the inverse of transformation Π_λ is given by (2.20) is that $k(x,y)$ and $l(x,y)$ are related by the formula

$$l(x,y) - k(x,y) = \int_x^y k(x,\eta)l(\eta,y)d\eta, \tag{2.21}$$

which is due to the uniqueness of solution k of (2.9). In fact, one can define

$$\tilde{l}(x,y) := k(x,y) + \int_x^y k(x,\eta)l(\eta,y)d\eta. \tag{2.22}$$

Hence, one only need to prove that $l = \tilde{l}$. Direct calculations show that $\tilde{l}$ satisfies

$$\begin{cases} \tilde{l}_{xxx} + \tilde{l}_{yyy} + \tilde{l}_x + \tilde{l}_y - \lambda l = 0 & \text{in } \mathcal{T}, \\ \tilde{l}(x,L) = 0 & \text{on } [0,L], \\ \tilde{l}(x,x) = 0 & \text{on } [0,L], \\ \tilde{l}_x(x,x) = \frac{\lambda}{3}(L - x) & \text{on } [0,L], \\ \tilde{l}_{xy}(x,x) = -\frac{\lambda}{3} & \text{on } [0,L]. \end{cases} \tag{2.23}$$

Comparing (2.23) with (2.10) and (2.12), one knows that $l_0 := l - \tilde{l}$ satisfies

$$\begin{cases} l_{0xxx} + l_{0yyy} + l_{0x} + l_{0y} = 0 & \text{in } \mathcal{T}, \\ l_0(x, L) = 0 & \text{on } [0, L], \\ l_0(x, x) = 0 & \text{on } [0, L], \\ l_{0x}(x, x) = 0 & \text{on } [0, L], \\ l_{0xy}(x, x) = 0 & \text{on } [0, L]. \end{cases} \tag{2.24}$$

From the uniqueness of the solution of (2.24), the following proposition, which will be proved in Appendix A, one gets that $l_0 = 0$.

Proposition 1. *Equation (2.24) has a unique solution in $C^3(\mathcal{T})$.*

Remark 2. *Proposition 1 is necessary in this paper. In the following section we will construct precisely a solution of equation (2.9)(and (2.10) respectively), the proof of Theorem 1 relies on the estimations of this solution. Proposition 1 ensures the solution we will construct satisfies (2.21) hence (2.20).*

We can find that after the transformation Π_λ, the solution of (1.1) with

$$\kappa(t) = \int_0^L k(0, y)u(t, y)dy, \tag{2.25}$$

is mapped into the trajectory $\omega(t, x)$, which is the solution of the linear system

$$\begin{cases} \omega_t + \omega_{xxx} + \omega_x + \lambda\omega = 0 & \text{in } (0, +\infty) \times (0, L), \\ \omega(t, L) = \omega_x(t, L) = 0 & \text{on } t(0, +\infty), \\ \omega(t, 0) = 0 & \text{on } (0, +\infty). \end{cases} \tag{2.26}$$

For system (2.26), one can easily obtain exponential decay of the solution

$$\|\omega(t, \cdot)\|_{L^2(0,L)} \leqslant e^{-\lambda t}\|\omega(0, \cdot)\|_{L^2(0,L)}. \tag{2.27}$$

Hence the solution of (1.1) with feedback law (2.25) satisfies:

$$\begin{aligned} \|u(t, \cdot)\|_{L^2(0,L)} &\leqslant \|\Pi_\lambda^{-1}\|_{L^2(0,L)\to L^2(0,L)}\|\omega(t, \cdot)\|_{L^2(0,L)} \\ &\leqslant e^{-\lambda t}\|\Pi_\lambda^{-1}\|_{L^2(0,L)\to L^2(0,L)}\|\omega(0, \cdot)\|_{L^2(0,L)} \\ &\leqslant e^{-\lambda t}\|\Pi_\lambda^{-1}\|_{L^2(0,L)\to L^2(0,L)}\|\Pi_\lambda\|_{L^2(0,L)\to L^2(0,L)}\|u(0, \cdot)\|_{L^2(0,L)}. \end{aligned} \tag{2.28}$$

2.3 Well-posedness of system (2.26)

For a parameter $\lambda > 0$, we consider the following linear operator $\mathcal{A}_\lambda : \mathcal{D}(\mathcal{A}_\lambda) \subset L^2(0, L) \to L^2(0, L)$ with

$$\mathcal{D}(\mathcal{A}_\lambda) := \{f \in H^3(0, L); f(0) = f(L) = f_x(L) = 0\}, \tag{2.29}$$

$$\mathcal{A}_\lambda := -f_x - f_{xxx} - \lambda f, \forall f \in \mathcal{D}(\mathcal{A}_\lambda). \tag{2.30}$$

As already seen in the special case when $\lambda = 0$ in [9, page 38–43], here we also have the following properties:

$$\mathcal{D}(\mathcal{A}_\lambda) \text{ is dense in } L^2(0, L), \tag{2.31}$$

$$\mathcal{A}_\lambda \text{ is closed}, \tag{2.32}$$

$$\mathcal{A}_\lambda \text{ and } A^* \text{ are dissipative}. \tag{2.33}$$

Hence, $\mathcal{A}_\lambda$ generates a strongly continuous semigroup of linear operator $\{S_\lambda(t)\}_{t\geqslant 0}$ on $L^2(0,L)$. Furthermore, for every initial data $\omega_0 \in \mathcal{D}(\mathcal{A}_\lambda)$, system (2.26) has one and only one solution $\omega(t,x) \in C^0([0,+\infty);L^2(0,L))$, which also satisfies:

$$\omega \in C^1([0,+\infty);L^2(0,L)) \cap C^0([0,+\infty);\mathcal{D}(\mathcal{A}_\lambda)), \tag{2.34}$$

$$\|\omega_x(\cdot,0)\|_{L^2(0,T)} \leqslant \|\omega_0\|_{L^2(0,L)}, \forall T>0, \tag{2.35}$$

$$\|\omega\|_{L^2(0,T;H^1(0,L))} \leqslant C_T\|\omega_0\|_{L^2(0,L)}, \text{ where } C_T \text{ only depends on T} > 0. \tag{2.36}$$

By standard methods, it follows that when $\omega_0 \in L^2(0,L)$, (2.26) has one and only one solution $\omega(t,x) \in C^0([0,+\infty);L^2(0,L))$, which also satisfies (2.35) and (2.36).

One can see more details on the results and proofs of this subsection in Jean-Michel Coron's book [9, page 38–43 and page 374-377]. Although only the special case where $\lambda = 0$ is considered, the general case $\lambda > 0$ follows by considering $e^{\lambda t}\omega$.

Remark 3. *Inequality (2.35) is a hidden inequality, which was first found by Lionel Rosier in [33]. Inequality (2.36) is the Kato smoothing effect.*

3 Control design

Inspired by the work of Jean-Michel Coron and Hoai-Minh Nguyen in [16], we want to give a piecewise control such that, on each piece the solution of (1.1) can be transformed to a solution of (2.26). More precisely, we select:

$$\{\lambda_n\}, \text{ increasing positive numbers that tends to infinity,} \tag{3.1}$$

$$\{t_n\}, \text{ increasing numbers with } t_0 = 0 \text{ that tends to } T \text{ as } n \text{ tends to infinity.} \tag{3.2}$$

We first define

$$u(0) := u_0 \text{ and } u(T) := 0, \tag{3.3}$$

and then, for $t_n < t \leqslant t_{n+1}$, we successively define

$$u(t) := \Pi_{\lambda_n}^{-1} S_{\lambda_n}(t-t_n)\Pi_{\lambda_n}u(t_n), \tag{3.4}$$

$$\kappa(t) := \int_0^L k_{\lambda_n}(0,y)u(t,y)dy, \tag{3.5}$$

where S_{λ_n} is the semigroup given in Section 2.3.

One can verify that (the proof is given at the end of this section):

Lemma 2. *Thus defined $u(t)|_{t_n\leqslant t\leqslant t_{n+1}}$ is a solution of (2.1) with $T_1 = t_n, T_2 = t_{n+1}, \tilde{h} = 0$ and $\kappa(t)$ given by (3.5).*

Notice that if we define

$$s_0 := 0 \text{ and } s_n := \sum_{k=0}^{n-1} \lambda_k(t_{k+1}-t_k) \text{ for } n \geqslant 1, \tag{3.6}$$

then, from (3.4) and (2.28) we get that, for $t_n < t \leqslant t_{n+1}$,

$$\|u(t,\cdot)\|_{L^2} \leqslant e^{-s_n}\|u_0\|_{L^2}\prod_{k=0}^{n}\left(\|\Pi_{\lambda_k}^{-1}\|\|\Pi_{\lambda_k}\|\right), \tag{3.7}$$

$$|\kappa(t)| \leqslant e^{-s_n}\|u_0\|_{L^2}\|k_{\lambda_n}(0,\cdot)\|_{L^2}\prod_{k=0}^{n}\left(\|\Pi_{\lambda_k}^{-1}\|\|\Pi_{\lambda_k}\|\right). \tag{3.8}$$

Hence, if we have a good estimation on k_λ, it will be possible to get $u(t) \to 0$ when t tends to T. Actually, we have the following estimation:

Lemma 3. *Let $\lambda > 2$, equation (2.9) has a unique solution $k_\lambda \in C^3(\mathcal{T})$ (respectively (2.10) has a unique solution $l_\lambda \in C^3(\mathcal{T})$) which satisfies*

$$\|k_\lambda\|_{C^0(\mathcal{T})} \leqslant e^{(1+L)^2\sqrt{\lambda}} \text{ and } \|l_\lambda\|_{C^0(\mathcal{T})} \leqslant e^{(1+L)^2\sqrt{\lambda}}. \tag{3.9}$$

The existence of solution to (2.9) is given in [6], here we proceed the same method introduced there in order to get (3.9). We give the proof of uniqueness of solution in Appendix A.

Let us make the following change of variable:

$$t = y - x, \quad s = x + y \tag{3.10}$$

and define

$$G(s,t) := k(x,y). \tag{3.11}$$

Then we can transform equation (2.9) into an integral equation of $G(s,t)$ (see (21) in [6]):

$$\begin{aligned} G(s,t) &= -\frac{\lambda t}{6}(2L - t - s) \\ &+ \frac{1}{6}\int_s^{2L-t}\int_0^t\int_0^\tau \left(2G_{sss} + 2G_s + \lambda G\right)(\eta,\xi)d\xi d\tau d\eta, \end{aligned} \tag{3.12}$$

in $\mathcal{T}_0 := \{(s,t); t \in [0,L], s \in [t, 2L-t]\}$.

We use the method of successive approximations to give a solution of function (3.12). We take

$$G^1(s,t) := -\frac{\lambda t}{6}(2L - t - s) \tag{3.13}$$

and define

$$G^{n+1}(s,t) = \frac{1}{6}\int_s^{2L-t}\int_0^t\int_0^\tau \left(2G^n_{sss} + 2G^n_s + \lambda G^n\right)(\eta,\xi)d\xi d\tau d\eta. \tag{3.14}$$

For instance, we can see that

$$G^2(s,t) = \frac{1}{108}\Big\{t^3(\lambda - \lambda^2 L + \frac{\lambda^2 t}{4})(2L - t - s) + \frac{t^3\lambda^2}{4}[(2L-t)^2 - s^2]\Big\}. \tag{3.15}$$

But unfortunately, we can't perform such explicit calculation each time. We try to estimate $G^n(s,t)$ from another way. At first we give following observations:

Suppose that $f(s,t) := g(s)h(t)$, then

$$\begin{aligned} &\int_s^{2L-t}\int_0^t\int_0^\tau \frac{\partial^m}{\partial s^m}f(\eta,\xi)d\xi d\tau d\eta \\ =&\int_s^{2L-t}\int_0^t\int_0^\tau \frac{\partial^m}{\partial s^m}g(\eta)h(\xi)d\xi d\tau d\eta \\ =&\int_s^{2L-t}\frac{\partial^m}{\partial s^m}g(\eta)d\eta\int_0^t\int_0^\tau h(\xi)d\xi d\tau. \end{aligned} \tag{3.16}$$

We define by $\mathbb{P}$ the space of polynomials of one variable on $\mathbb{R}$. We define operator $\mathbf{T}$ by

$$\begin{aligned}&\mathbf{T}:\mathbb{P}\otimes\mathbb{P}\to\mathbb{P}\otimes\mathbb{P}\\&g(s)h(t)\mapsto\frac{1}{6}\int_s^{2L-t}\int_0^t\int_0^\tau\left(2\frac{\partial^3}{\partial s^3}+2\frac{\partial}{\partial s}+\lambda Id\right)(g\cdot h)(\eta,\xi)d\xi d\tau d\eta.\end{aligned}\tag{3.17}$$

Equality (3.16) tells us that (3.17) is well defined. In fact,

$$\mathbf{T}\big(g(s)h(t)\big)=g_{\mathbf{T}}(s,t)h_{\mathbf{T}}(t),\tag{3.18}$$

where $g_{\mathbf{T}}(s,t)$ and $h_{\mathbf{T}}(t)$ are given by

$$g_{\mathbf{T}}(s,t):=\frac{1}{6}\int_s^{2L-t}\left(2\frac{\partial^3}{\partial s^3}+2\frac{\partial}{\partial s}+\lambda Id\right)(g)(\eta)d\eta,\tag{3.19}$$

$$h_{\mathbf{T}}(t):=\int_0^t\int_0^\tau h(\xi)d\xi d\tau.\tag{3.20}$$

We observe that, if

$$|h(t)|\leqslant t^n,\forall\,0\leqslant t\leqslant L,\tag{3.21}$$

then, when $t\in[0,L]$ we have

$$|h_{\mathbf{T}}(t)|=|\int_0^t\int_0^\tau h(\xi)d\xi d\tau|\leqslant\frac{1}{(n+1)(n+2)}t^{n+2}.\tag{3.22}$$

As for $g_{\mathbf{T}}(s,t)$, we notice that, if $g(s)=s^m$ with $m\geqslant 3$, then

$$\begin{aligned}g_{\mathbf{T}}(s,t)=&\frac{1}{6}\Big(2m(m-1)(2L-t)^{m-2}-2m(m-1)s^{m-2}\\&+2(2L-t)^m-2s^m+\frac{\lambda(2L-t)^{m+1}}{m+1}-\frac{\lambda s^{m+1}}{m+1}\Big).\end{aligned}\tag{3.23}$$

Hence, this inspires us to separate $\mathbf{T}$ into following 6 linear operators $\{\mathbf{T}_i\}_{1\leqslant i\leqslant 6}$ from $\mathbb{P}\otimes\mathbb{P}$ to $\mathbb{P}\otimes\mathbb{P}$:

$$\mathbf{T}_1:s^mh(t)\mapsto\begin{cases}\frac{1}{6}\big(2m(m-1)(2L-t)^{m-2}\big)h_{\mathbf{T}}(t), & \text{when } m\geqslant 3,\\0, & \text{when } 0\leqslant m\leqslant 2,\end{cases}\tag{3.24}$$

$$\mathbf{T}_2:s^mh(t)\mapsto\begin{cases}-\frac{1}{6}\big(2m(m-1)s^{m-2}\big)h_{\mathbf{T}}(t), & \text{when } m\geqslant 3,\\0, & \text{when } 0\leqslant m\leqslant 2,\end{cases}\tag{3.25}$$

$$\mathbf{T}_3:s^mh(t)\mapsto\begin{cases}\frac{1}{6}\big(2(2L-t)^m\big)h_{\mathbf{T}}(t), & \text{when } m\geqslant 1,\\0, & \text{when } m=0,\end{cases}\tag{3.26}$$

$$\mathbf{T}_4:s^mh(t)\mapsto\begin{cases}-\frac{1}{6}\big(2s^m\big)h_{\mathbf{T}}(t), & \text{when } m\geqslant 1,\\0, & \text{when } m=0,\end{cases}\tag{3.27}$$

$$\mathbf{T}_5 : s^m h(t) \mapsto \frac{\lambda}{6}\frac{(2L-t)^{m+1}}{m+1} h_{\mathbf{T}}(t), \tag{3.28}$$

$$\mathbf{T}_6 : s^m h(t) \mapsto -\frac{\lambda}{6}\frac{s^{m+1}}{m+1} h_{\mathbf{T}}(t). \tag{3.29}$$

Since that $\mathbf{T}_i$ is linear, we have

$$\mathbf{T}_i(0) = 0. \tag{3.30}$$

From (3.17)–(3.29), we know that

$$\mathbf{T} = \sum_{i=1}^{6} \mathbf{T}_i, \text{ on } \mathbb{P} \otimes \mathbb{P}. \tag{3.31}$$

Hence,

$$\begin{aligned} G^{n+1}(s,t) &= \mathbf{T} G^n(s,t) \\ &= \Big(\sum_{i=1}^{6} \mathbf{T}_i\Big) G^n(s,t) \\ &= \Big(\sum_{i=1}^{6} \mathbf{T}_i\Big)^n G^1(s,t). \end{aligned}$$

By (3.13)

$$G^1(s,t) = -\frac{\lambda}{6} t(2L-t) + \frac{\lambda}{6}(st) = I(s,t) + J(s,t), \tag{3.32}$$

where

$$I(s,t) := -\frac{\lambda}{6} t(2L-t) \text{ and } J(s,t) := \frac{\lambda}{6}(st). \tag{3.33}$$

Let us define the sets

$$\mathbb{A}_n := \Big\{(x_1, x_2, ..., x_n); x_i \in \{1,2,3,4,5,6\}, \forall\, 1 \leqslant i \leqslant n\Big\}, \forall\, n \geqslant 1. \tag{3.34}$$

For any $n \in \mathbb{N}^*$, for any $a = (a(1), a(2), ..., a(n)) \in \mathbb{A}_n$, let us define the operator

$$\mathbf{T}_a := \mathbf{T}_{a(n)} \mathbf{T}_{a(n-1)} ... \mathbf{T}_{a(1)}. \tag{3.35}$$

We define additionally $\mathbb{A} := \{a_0\}$ and $\mathbf{T}_{a_0} := Id$ (identity operator on $\mathbb{P}$).

Hence for any $n \in \mathbb{N}$, we have

$$\begin{aligned} G^{n+1}(s,t) &= \mathbf{T}^n \Big(I + J\Big) \\ &= \sum_{a \in \mathbb{A}_n} (\mathbf{T}_a I) + \sum_{a \in \mathbb{A}_n} (\mathbf{T}_a J). \end{aligned} \tag{3.36}$$

Now we are going to use mathematical induction to conclude following lemma:

Lemma 4. *For every $\lambda \geqslant 2$, for every $n \in \mathbb{N}$, and for every $a \in \mathbb{A}_n$, $\mathbf{T}_a I$ and $\mathbf{T}_a J$ are of the form $s^l h(t)$ and satisfy*

$$|h(t)| \leqslant (\frac{\lambda}{6})^{n+1} \frac{t^{2n+1}}{(2n+1)!} (2L+1)^{n+1-l} \frac{1}{l!}, \quad t \in [0, L]. \tag{3.37}$$

Proof of Lemma 4. When $n = 0$, one can check that Lemma 4 holds. We suppose that when $n = k$ Lemma 4 holds, where $k \geqslant 0$, then we consider the case where $n = k + 1$.

For any $n \geqslant 1$, and any $a := (a(1), a(2), ..., a(n+1)) \in \mathbb{A}_{n+1}$, let us define

$$\varrho(a) := (a(1), a(2), ..., a(n)). \tag{3.38}$$

For any $a \in \mathbb{A}_1$, let us define

$$\varrho(a) := a_0. \tag{3.39}$$

Hence for any $a := (a(1), a(2), ..., a(k+1)) \in \mathbb{A}_{k+1}$, we have

$$\mathbf{T}_a = \mathbf{T}_{a(k+1)}\mathbf{T}_{\varrho(a)}. \tag{3.40}$$

From the assumption, we know that

$$\mathbf{T}_{\varrho(a)} I = s^l h(t). \tag{3.41}$$

If $\mathbf{T}_{\varrho(a)} I = 0$, then we conclude the proof.
If $\mathbf{T}_{\varrho(a)} I = s^l h(t)$, then we know from (3.37) that

$$|h(t)| \leqslant \Big(\frac{\lambda}{6}\Big)^{k+1} \frac{t^{2k+1}}{(2k+1)!}(2L+1)^{1+k-l}\frac{1}{l!}, \quad t \in [0, L]. \tag{3.42}$$

For $\mathbf{T}_1$, we know that $\mathbf{T}_1(s^l h(t)) = 0$ if $l \leqslant 2$. Therefore, we only need to consider the case where $l \geqslant 3$. From (3.20)–(3.22), (3.24) and (3.42), we know that

$$\begin{aligned} |\mathbf{T}_1(s^l h(t))| &= \Big|\frac{2}{6}\big(l(l-1)(2L-t)^{l-2}\big)h_{\mathbf{T}}(t)\Big| \\ &\leqslant \frac{2}{6}\big(l(l-1)(2L-t)^{l-2}\big)\Big(\frac{\lambda}{6}\Big)^{1+k}\frac{t^{2k+3}}{(2k+3)!}(2L+1)^{1+k-l}\frac{1}{l!} \\ &\leqslant \Big(\frac{\lambda}{6}\Big)^{k+2}\frac{t^{2k+3}}{(2k+3)!}(2L+1)^{k-1}\frac{1}{(l-2)!}. \end{aligned} \tag{3.43}$$

Notice that $\mathbf{T}_1(s^l h(t))$ is of the form $s^0 g(t)$, so one can see from (3.43) that (3.37) is satisfied in this case.
With the same procedure, we can check that:

$$|\mathbf{T}_2(s^l h(t))| \leqslant \Big(\frac{\lambda}{6}\Big)^{k+2}\frac{t^{2k+3}}{(2k+3)!}(2L+1)^{1+k-l}\frac{s^{l-2}}{(l-2)!}, \tag{3.44}$$

$$|\mathbf{T}_3(s^l h(t))| \leqslant \Big(\frac{\lambda}{6}\Big)^{k+2}\frac{t^{2k+3}}{(2k+3)!}(2L+1)^{1+k}\frac{1}{l!}, \tag{3.45}$$

$$|\mathbf{T}_4(s^l h(t))| \leqslant \Big(\frac{\lambda}{6}\Big)^{k+2}\frac{t^{2k+3}}{(2k+3)!}(2L+1)^{1+k-l}\frac{s^l}{l!}, \tag{3.46}$$

$$|\mathbf{T}_5(s^l h(t))| \leqslant \Big(\frac{\lambda}{6}\Big)^{k+2}\frac{t^{2k+3}}{(2k+3)!}(2L+1)^{k+2}\frac{1}{(l+1)!}, \tag{3.47}$$

$$|\mathbf{T}_6(s^l h(t))| \leqslant \Big(\frac{\lambda}{6}\Big)^{k+2}\frac{t^{2k+3}}{(2k+3)!}(2L+1)^{1+k-l}\frac{s^{l+1}}{(l+1)!}. \tag{3.48}$$

Hence, we complete the proof. □

By the same idea of partition and Lemma 4, we can conclude that

Lemma 5. *For every $\lambda \geqslant 2$, for every $n \in \mathbb{N}$, and for every $a \in \mathbb{A}_n$, $\mathbf{T}_a I$ and $\mathbf{T}_a J$ are of the form $s^l h(t)$ and satisfy*

$$|\partial_t(h(t))| \leqslant 2\Big(\frac{\lambda}{6}\Big)^{n+1} \frac{t^{2n}}{(2n)!}(2L+1)^{n+3-l}\frac{1}{l!}, \quad t \in [0, L]. \tag{3.49}$$

Remark 4. *One can get similar estimates for C^2-norm even C^n-norm. However, since in this paper we don't need such precise estimates, we omit them. For C^2-norm one can directly obtain these estimations by using Lemma 4 (just like the way to get Lemma 5), but for C^n-norm (with $n \geq 3$) which would be rather complicated. Furthermore, one can see from [25] that for the heat equation the kernel is analytic in the triangle, to know if the kernel we obtained in this article is also analytic, it is an interesting open problem.*

We come back to the estimation (3.36). From Lemma 4, we know that for every $n \in \mathbb{N}$, for every $a \in \mathbb{A}_n$, for every $m \in \mathbb{N}$ and $t \in [0, L]$,

$$\Big|\frac{\partial^m}{\partial s^m}\Big(\mathbf{T}_a I\Big)(s,t)\Big| \leqslant \Big(\frac{\lambda}{6}\Big)^{1+n} \frac{t^{2n+1}}{(2n+1)!}(2L+1)^{n+1}, \tag{3.50}$$

$$\Big|\frac{\partial^m}{\partial s^m}\Big(\mathbf{T}_a J\Big)(s,t)\Big| \leqslant \Big(\frac{\lambda}{6}\Big)^{1+n} \frac{t^{2n+1}}{(2n+1)!}(2L+1)^{n+1}, \tag{3.51}$$

which together with (3.36) imply that

$$|\frac{\partial^m}{\partial s^m} G^{n+1}(s,t)| \leqslant \frac{\lambda^{1+n}}{3} \frac{t^{2n+1}}{(2n+1)!}(2L+1)^{n+1}, \tag{3.52}$$

which implies that the series

$$\sum_{n=1}^{+\infty} \frac{\partial^m}{\partial s^m} G^n(s,t) \text{ is uniformly convergent in } \mathcal{T}_0. \tag{3.53}$$

The same approach shows that the series

$$\sum_{n=1}^{+\infty} \frac{\partial^{m+1}}{\partial t \partial s^m} G^n(s,t) \text{ is uniformly convergent in } \mathcal{T}_0. \tag{3.54}$$

We define

$$G(s,t) := \sum_{n=1}^{+\infty} G^n(s,t), \tag{3.55}$$

which is the solution of (3.12) (see [6, page 1691]). First we can estimate $|G|$ from (3.52), (3.53) and (3.55):

$$|G(s,t)| \leqslant \frac{\sqrt{(2L+1)\lambda}}{3} e^{\sqrt{(2L+1)\lambda}t}, \text{ in } \mathcal{T}_0. \tag{3.56}$$

Hence,

$$|G(s,t)| \leqslant \frac{e^{(1+L)\sqrt{(2L+1)\lambda}}}{3} \leqslant e^{(1+L)^2\sqrt{\lambda}}, \text{ in } \mathcal{T}_0. \tag{3.57}$$

It only remains to prove that $G(s,t) \in C^3(\mathcal{T}_0)$. Actually, from (3.53) and (3.54) we know that it suffice to prove $G_{tt}, G_{tts}, G_{ttt} \in C^0(\mathcal{T}_0)$, we know from (2.15) that $G_{tts} \in C^0(\mathcal{T}_0) \in C^0(\mathcal{T}_0)$. For the other two, one notice that from (3.12):

$$
\begin{aligned}
6G_t(s,t) = &-\lambda(2L-s-2t) - \int_0^t \int_0^\tau \left(2G_{sss} + 2G_s + \lambda G\right)(2L-t,\xi) d\xi d\tau \\
&+ \int_s^{2L-t} \int_0^t \left(2G_{sss} + 2G_s + \lambda G\right)(\eta,\xi) d\xi d\eta, \\
6G_{tt}(s,t) = &\, 2\lambda + \int_0^t \int_0^\tau \left(2G_{ssss} + 2G_{ss} + \lambda G_s\right)(2L-t,\xi) d\xi d\tau \\
&- \int_0^t \left(2G_{sss} + 2G_s + \lambda G\right)(2L-t,\xi) d\xi - \int_0^t \left(2G_{sss} + 2G_s + \lambda G\right)(2L-t,\xi) d\xi \\
&+ \int_s^{2L-t} \left(2G_{sss} + 2G_s + \lambda G\right)(\eta,t) d\eta, \\
6G_{ttt}(s,t) = &- \int_0^t \int_0^\tau \left(2G_{sssss} + 2G_{sss} + \lambda G_{ss}\right)(2L-t,\xi) d\xi d\tau + \int_0^t \left(2G_{ssss} + 2G_{ss} + \lambda G_s\right)(2L-t,\xi) d\xi \\
&- 2\left(2G_{sss} + 2G_s + \lambda G\right)(2L-t,t) d\xi + 2\int_0^t \left(2G_{ssss} + 2G_{ss} + \lambda G_s\right)(2L-t,\xi) d\xi \\
&+ \int_s^{2L-t} \left(2G_{ssst} + 2G_{st} + \lambda G_t\right)(\eta,t) d\eta - \left(2G_{sss} + 2G_s + \lambda G\right)(2L-t,t),
\end{aligned}
$$

with (3.53)–(3.54), we get the continuity of G_{tt} and G_{ttt}, which completes the proof of Lemma 3.

Remark 5. *As we can see from [16, 25] for the heat equation, the L^∞-norm of the kernel k_λ is of the form $exp(\lambda^{1/2})$. One may naturally wonder the sharp estimate on L^∞-norm of kernel k_λ is of the form $exp(\lambda^{1/3})$ for the KdV case, since KdV is of order 3. We do not know how to get such estimate.*

At last, it remains to give the proof of Lemma 2.

Proof of Lemma 2. It is equivalent to prove the following statement:

(S) Given $u_0 \in L^2(0,L), \lambda > 0, s > 0$, one has

$\kappa(t) := \int_0^L k_\lambda(0,\cdot)\left(\Pi_\lambda^{-1} S_\lambda(t)\Pi_\lambda u_0\right)(\cdot) dy \in L^2(0,s)$, and

$u(t) := \Pi_\lambda^{-1} S_\lambda(t)\Pi_\lambda u_0$ is the solution of (2.1) on $[0,s]$ with $\tilde{h} = 0$.

We only prove the case where $u_0 \in \Pi_\lambda^{-1}\mathcal{D}(\mathcal{A}_\lambda)$, since standard method then leads to the general case of (S). From Subsection 2.3, we know that

$$
\omega(t) := S_\lambda(t)\Pi_\lambda u_0 \in C^0([0,s];\mathcal{D}(\mathcal{A}_\lambda)) \cap C^1([0,s];L^2(0,L)), \tag{3.58}
$$

which shows that $\kappa(t) \in C^0([0,s])$. Direct calculation, thanks to (2.10), shows that (similar to

page 1690 in [6]):

$$u_t = \omega_t + \int_x^L (l_y + l_{yyy} - \lambda l)(x,z)\omega(z)dz$$
$$- l_y(x,x)\omega_x(x) + l_{yy}(x,x)\omega(x), \tag{3.59}$$
$$u_x = \omega_x(x) + \int_x^L l_x(x,z)\omega(z)dz, \tag{3.60}$$
$$u_{xx} = \omega_{xx}(x) - l_x(x,x)\omega(x) + \int_x^L l_{xx}(x,z)\omega(z)dz, \tag{3.61}$$
$$u_{xxx} = \omega_{xxx} - \Big(l_{xx}(x,x) + l_{xy}(x,x)\Big)\omega(x)$$
$$- l_x(x,x)\omega_x(x) - l_{xx}(x,x)\omega(x) + \int_x^L l_{xxx}(x,z)\omega(x)dz, \tag{3.62}$$

all these calculations are on $C^0([0,T];L^2(0,L))$. Hence, with (2.10) and (3.59)–(3.62) we get that

$$u(t,x) \in C^1([0,s];L^2(0,L)) \cap C^0([0,s];H^3(0,L)), \tag{3.63}$$
$$u_t + u_x + u_{xxx} = 0, \text{ in } L^2(0,L), \tag{3.64}$$
$$u(t,L) = u_x(t,L) = 0, \tag{3.65}$$
$$u(t,0) = \kappa(t), \tag{3.66}$$

which show that u verifies Definition 1. □

Remark 6. *In fact, by using (2.9) and the hidden inequality (2.35), we can also prove that $\kappa(t) \in H^1(0,s)$ with its norm controlled by $\|u_0\|_{L^2(0,L)}$.*

4 Null controllability

Finally, we are able to prove the null controllability (Theorem 1) by constructing a piecewise continuous bounded control. The way of constructing this control is explained in Section 3, thanks to Lemma 2, (3.1)–(3.8), we only need to find good sequences $\{\lambda_n\}$ and $\{t_n\}$ such that:

$$\lim_{n\to+\infty} e^{-s_n} \prod_{k=0}^{n} \Big(\|\Pi_{\lambda_k}^{-1}\|\|\Pi_{\lambda_k}\|\Big) = 0, \tag{4.1}$$

$$\lim_{n\to+\infty} e^{-s_n}\|k_{\lambda_n}(0,\cdot)\|_{L^2} \prod_{k=0}^{n} \Big(\|\Pi_{\lambda_k}^{-1}\|\|\Pi_{\lambda_k}\|\Big) = 0, \tag{4.2}$$

and that

$$u(t)|_{0\leqslant t\leqslant T} \text{ is a solution of (2.1) with } \tilde{h} = 0, \kappa(t) \text{ given by (3.5).} \tag{4.3}$$

Thanks to Lemma 2, from Definition 1, (3.7), (3.8), (4.1) and (4.2), one can easily deduce that $u(t)|_{0\leqslant t\leqslant T}$ is the solution of (1.1). It remains to prove that (4.1)–(4.2) hold.

From the definition of Π_λ and Π_λ^{-1}, (2.19)–(2.20), we know that

$$\|\Pi_\lambda\|_{L^2\to L^2} \leqslant (1 + L\|k_\lambda\|_{C^0(\mathcal{T})}) \leqslant e^{2(1+L)^2\sqrt{\lambda}}, \tag{4.4}$$
$$\|\Pi_\lambda^{-1}\|_{L^2\to L^2} \leqslant (1 + L\|l_\lambda\|_{C^0(\mathcal{T})}) \leqslant e^{2(1+L)^2\sqrt{\lambda}}, \tag{4.5}$$
$$\|k_\lambda(0,\cdot)\|_{L^2} \leqslant \sqrt{L}\|l_\lambda\|_{C^0(\mathcal{T})} \leqslant e^{2(1+L)^2\sqrt{\lambda}}, \tag{4.6}$$

where Lemma 3 is used. Hence it suffice to select $\{\lambda_n\}$ and $\{t_n\}$ such that

$$e^{-s_n}\prod_{k=0}^{n} e^{6(1+L)^2\sqrt{\lambda_k}} \longrightarrow 0. \tag{4.7}$$

Inspired by the construction of Jean-Michel Coron and Hoai-Minh Nguyen in [16, Proposition 1], we choose $t_n := T - 1/n^2$ and $\lambda_n := 2n^8$. One can easily verify that (4.7) holds, which completes the proof.

Remark 7. *To deal with heat equations by using backstepping approach, one need to study wave equation instead, which is already well investigated. In this article, we study KdV system whose order is 3, where the kernel system (see (A.1)) becomes a third order "wave-like" equation. For this reason, we encountered some difficulties, Lemma 3 for estimation and Proposition 1 for uniqueness. We believe this method as well as those techniques introduced in this paper could be used to other systems whose order is greater than 3. As we know, backstepping method can be well used for first-order hyperbolic systems for rapid stablization, see [24], but since it is of order one we are not sure if some good estimation could be obtained for null controllability or even finite time stabilization. However, looking for [18, 23, 26], this might be possible for quasilinear hyperbolic systems.*

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A Proof of Proposition 1

In this part, we give the proof of uniqueness of solution to equation (2.24). As the function is defined in a triangle $\mathcal{T}$, we extend l_0 by 0 in lower triangle $[0,L]\times[0,L]\setminus\mathcal{T}$, we will denote by h the extended function. Since on the diagonal $x=y$, $C^3(\mathcal{T})$ function l_0 satisfies

$$l_{0x} = l_{0y} = l_{0xx} = l_{0xy} = l_{0yy} = 0,$$

which ensures the extended function, h, to be a $H^3([0,L]\times[0,L])$ function. Moreover, h satisfies

$$\begin{cases} h_{xxx} + h_{yyy} + h_x + h_y = 0 & \text{in } [0,L]\times[0,L], \\ h(x,L) = 0 & \text{on } [0,L], \\ h(x,0) = h_y(x,0) = h_{yy}(x,0) = 0 & \text{on } [0,L], \\ h(L,y) = h_x(L,y) = h_{xx}(L,y) = 0 & \text{on } [0,L]. \end{cases} \tag{A.1}$$

By simply change of variables $\tilde{x} = L - x$ and $\tilde{y} = L - y$, it suffice to prove that solution $h \in H^3([0,L]\times[0,L])$ of equation

$$\begin{cases} h_{xxx} + h_{yyy} + h_x + h_y = 0 & \text{in } [0,L]\times[0,L], \\ h(x,0) = 0 & \text{on } [0,L], \\ h(x,L) = h_y(x,L) = h_{yy}(x,L) = 0 & \text{on } [0,L], \\ h(0,y) = h_x(0,y) = h_{xx}(0,y) = 0 & \text{on } [0,L]. \end{cases} \tag{A.2}$$

is 0.

We can notice that (A.2) is similar to a wave equation, it is quite natural to consider the eigenfunctions of operator (with respect to y variable):

$$\mathcal{A}_y : \mathcal{D}(\mathcal{A}_y) \subset L^2(0,L) \to L^2(0,L), \tag{A.3}$$

$$\mathcal{D}(\mathcal{A}_y) := \{f \in H^3(0,L); f(0) = f(L) = f_y(L) = 0\}, \tag{A.4}$$

$$\mathcal{A}_y f := -f_y - f_{yyy}, \forall f \in \mathcal{D}(\mathcal{A}_y). \tag{A.5}$$

If the eigenfunctions, $\{\varphi_n(y)\}_n$ form a Riesz basis of $L^2(0,L)$, then the fourier series decomposition

$$h(x,y) = \sum_n \phi_n(x) \cdot \varphi_n(y) \tag{A.6}$$

could easily linfer the uniqueness required. Unfortunately, this operator is a non-self-adjoint operator and eigenfunctions do not form a Riesz basis, see [31].

Another idea is to investigate the completeness of eigenfunctions, $\{\psi(y)_n\}_n$, of the adjoint operator $\mathcal{A}_y^*$:

$$\mathcal{A}_y^* : \mathcal{D}(\mathcal{A}_y^*) \subset L^2(0,L) \to L^2(0,L), \tag{A.7}$$

$$\mathcal{D}(\mathcal{A}_y^*) := \{f \in H^3(0,L); f(0) = f(L) = f_y(0) = 0\}, \tag{A.8}$$

$$\mathcal{A}_y^* f := -f_y - f_{yyy}, \forall f \in \mathcal{D}(\mathcal{A}_y^*). \tag{A.9}$$

In fact, suppose that $\{\psi_n(y)\}_n$ is an eigenfunction of the adjoint operator $\mathcal{A}_y^*$, then from (A.2) as well as the boundary conditions of h and ψ one can deduce that

$$(\partial_{xxx} + \partial_x - \lambda_n)\langle \psi_n(\cdot), h(x,\cdot)\rangle_{L^2(0,L)} = 0. \tag{A.10}$$

Combine (A.10) with the fact that

$$\langle \psi_n(\cdot), h(0,\cdot)\rangle_{L^2(0,L)} = \partial_x \langle \psi_n(\cdot), h(0,\cdot)\rangle_{L^2(0,L)} = \partial_{xx} \langle \psi_n(\cdot), h(0,\cdot)\rangle_{L^2(0,L)} = 0, \tag{A.11}$$

we obtain

$$\langle \psi_n(\cdot), h(x,\cdot)\rangle_{L^2(0,L)} = 0, \quad \forall x \in [0,L]. \tag{A.12}$$

If $\{\psi_n(y)\}_n$ is complete in $L^2(0,L)$, then $h(x,0)$ is 0. However, we don't know the completeness of the eigenfunctions $\{\psi_n(y)\}_n$.

More generally, one could consider eigenfunctionals or generalized eigenfunctions, following Gel'fand and coauthors [19, 20]. More precisely, in the non-self-adjoint cases it is not always possible expand a function as the sum of eigenfunctions. In order to avoid this problem, one uses different generalizations of eigenfunctions. For example, the generalization introduced by John Locker and the coauthors and called augmented eigenfunctions, which is itself a generalization of Gel'fand's eigenfunctions (allow the appearance of remainder functionals). This generalization turned out to be a powerful tool to investigate the initial-boundary value problem (IBVP), one can find an almost complete investigation from the papers [27, 28, 37]. In general, suppose that Φ a function space defined on the closure of an real interval I with sufficient smoothness and decay conditions, that $\mathcal{L}$ a linear operator defined on Φ. Let γ be an oriented contour in $\mathbb{C}$ and let $\mathbf{E} = \{\mathbf{E}_\lambda : \lambda \in \gamma\}$ be a family of functionals (imagine as a family of eigenfunction when γ is only defined on a discrete set). Then the corresponding remainder functionals $\mathbf{R}_\lambda \in \Phi'$ with respect to eigenvalues λ is

$$\mathbf{R}_\lambda(\phi) := \lambda^n \mathbf{E}_\lambda(\phi) - \mathbf{E}_\lambda(\mathcal{L}\phi), \quad \forall \phi \in \Phi, \forall \lambda \in \gamma. \tag{A.13}$$

One is interested in the cases in which one of the following two conditions is satisfied:

$$\int_\lambda e^{i\lambda x}\mathbf{R}_\lambda(\phi)d\lambda = 0, \quad \forall\phi\in\Phi, \forall x\in I, \tag{A.14}$$

and

$$\int_\lambda \frac{e^{i\lambda x}}{\lambda^n}\mathbf{R}_\lambda(\phi)d\lambda = 0, \quad \forall\phi\in\Phi, \forall x\in I, \tag{A.15}$$

where (A.14) (*resp.* (A.15)) is called the type I (*resp.* type II) condition of augmented eigenfunctions of $\mathcal{L}$ up to the integration along γ.

As we can see above, study of augmented eigenfunctions involved very complicated asymptotic calculations. In John Locker's work this method is only used to study the evolution equations based on a good tranform pair, which doesn't seem to be a good (easy) option to our problem (A.2). Instead of augmented eigenfunctions, John Locker [28] also considered generalized eigenspace $\mathcal{E}_\mathcal{G}$ given by

$$\bigcup \mathcal{N}((\lambda_i I-\mathcal{L})^{m_i}),\ \textit{union for all } m_i\in\mathbb{N},\ \textit{and } \lambda_i \textit{ eigenvalues},$$

where L denote the operator, $\mathcal{N}$ denote the kernel. More precisely, to linearized KdV operator he proved:

Theorem 2. *Let $L>0$, let a be a constant. For differential operator $Lf := f_{xxx}+af_x$ with boundary conditions*

$$f(0)=f(L)=0, \tag{A.16}$$

$$f_x(0)+\beta f_x(L)=0, \tag{A.17}$$

the generalized eigenfunction space $\mathcal{E}_\mathcal{G}$ is complete in $L^2(0,L)$ space when $\beta\neq 0$.

Remark 8. *One can see that when $\beta=0$, it does not seem to be known whether generalized eigenfunction space $\mathcal{E}_\mathcal{G}$ is complete in $L^2(0,L)$. This is one of the reasons much more complicated augmented eigenfunctions are introduced (the other reasons are about the regularities or more general boundary conditions). Actually, this case can be regarded as the limit of the cases when the coupling constant β approaches 0.*

In fact, in [28] John Locker only considered the operator $Lf := f_{xxx}$, one can easily verify with the same prove that the same result holds when we add f_x.

In order to solve our problem, we need to use another kind of generalized eigenfunctions which is more general than $\mathcal{E}_\mathcal{G}$ but less than augmented eigenfunctions, namely eigenfunctions and associated functions (*e.a.f.*). The definition of *e.a.f.*, which is defined on equations with λ as a parameter, is rather complicated, one can see [29, chapter 1] and [30] for precise description on this subject.

With eigenfunctions and associated functions, Andrei A. Shkalikov in [36] proved the following theorem:

Theorem 3. *The eigenfunctions and associated functions of the boundary-value problem generated by an ordinary differential equation with separated boundary conditions*

$$l(y)-\lambda^n y = y^{(n)}+p_{n-2}(x)y^{(n-2)}+...+p_0(x)y-\lambda^n y=0, \tag{A.18}$$

$$U_j(y)=\sum_{k=0}^{n-1}\alpha_{jk}y^{(k)}(0)=0,\ \textit{with } j=1,2,...,l, \tag{A.19}$$

$$U_j(y)=\sum_{k=0}^{n-1}\beta_{jk}y^{(k)}(L)=0,\ \textit{with } j=1,2,...,n-l, \tag{A.20}$$

form a complete system in the space $L^2[0,L]$*, where* $p_i(x)$ *are arbitrary summable functions, and* $l > n - l > 0$*.*

Apply Theorem 3 to our case (linearised KdV), we get:

Corollary 1. *For the ordinary differential equation with seperated boundary conditions*

$$\mu(f) = l(f) - \lambda^3 f = f_{yyy} + f_y - \lambda^3 f = 0, \tag{A.21}$$
$$U_1(f) = f(0) = 0, \tag{A.22}$$
$$U_2(f) = f_y(0) = 0, \tag{A.23}$$
$$U_3(f) = f(L) = 0, \tag{A.24}$$

the eigenfunctions and associated functions form a complete system in the space $L^2[0,L]$*.*

Finally we are able to proof Proposition 1:

Proof of Proposition 1. Let us consider the boundary-value problem (A.21)–(A.24). For this boundary problem, let λ_0 be an eigenvalue, and let $\varphi_0(y) = \varphi(y)$ an eigenfunction for the eigenvalue λ_0. Then the associated functions associated with the eigenfunction $\varphi(y)$ are given by the functions

$$\varphi_1(y), \varphi_2(y), ..., \varphi_k(y)$$

who satisfy the boundary conditions

$$U_1(\varphi_i) = U_2(\varphi_i) = U_3(\varphi_i) = 0, \quad \forall i = 0, 1, ..., k, \tag{A.25}$$

and, for $\lambda = \lambda_0$, the following relations

$$\mu(\varphi_i) + \frac{1}{1!}\frac{\partial}{\partial\lambda}\mu(\varphi_{i-1}) + ... + \frac{1}{i!}\frac{\partial^i}{\partial\lambda^i}\mu(\varphi_0) = 0, \quad \forall i = 0, 1, ..., k. \tag{A.26}$$

Now we want to prove that for all those functions (*e.a.f.*), we have

$$\langle h(x,\cdot), \varphi_i(\cdot)\rangle_{L^2(0,L)} = 0, \quad \forall x \in [0,L], \quad \forall i = 0, 1, ..., k. \tag{A.27}$$

At first, for φ_0, as what we have done in (A.12), clearly

$$\langle h(x,\cdot), \varphi_0(\cdot)\rangle_{L^2(0,L)} = 0, \quad \forall x \in [0,L]. \tag{A.28}$$

For φ_1, (A.26) shows that

$$(\varphi_1)_{yyy} + (\varphi_1)_y - \lambda_0^3\varphi_1 - 3\lambda_0^2\varphi_0 = 0. \tag{A.29}$$

Hence from (A.2), (A.25), (A.28) and (A.29) we get that

$$\begin{aligned} 0 &= \langle(\partial_x^3 + \partial_x + \partial_y^3 + \partial_y)h(x,\cdot), \varphi_1(\cdot)\rangle_{L^2(0,L)} \\ &= (\partial_x^3 + \partial_x)\langle h(x,\cdot), \varphi_1(\cdot)\rangle_{L^2(0,L)} - \langle h(x,\cdot), (\partial_y^3 + \partial_y)\varphi_1(\cdot)\rangle_{L^2(0,L)} \\ &= (\partial_x^3 + \partial_x)\langle h(x,\cdot), \varphi_1(\cdot)\rangle_{L^2(0,L)} - \langle h(x,\cdot), -\lambda_0^3\varphi_1(\cdot) - 3\lambda_0^2\varphi_0(\cdot)\rangle_{L^2(0,L)} \\ &= (\partial_x^3 + \partial_x + \lambda_0^3)\langle h(x,\cdot), \varphi_1(\cdot)\rangle_{L^2(0,L)}. \end{aligned} \tag{A.30}$$

Then using the the fact that

$$\langle h(0,\cdot), \varphi_1(\cdot)\rangle_{L^2(0,L)} = \partial_x\langle h(0,\cdot), \varphi_1(\cdot)\rangle_{L^2(0,L)} = \partial_{xx}\langle h(0,\cdot), \varphi_1(\cdot)\rangle_{L^2(0,L)} = 0, \tag{A.31}$$

we get

$$\langle h(x,\cdot),\varphi_1(\cdot)\rangle_{L^2(0,L)}=0,\quad \forall x\in[0,L]. \tag{A.32}$$

Repeating this argument we get (A.27), which combined with Corollary 1 shows that

$$h(x,\cdot)=0,\quad \forall x\in[0,L]. \tag{A.33}$$

Hence the proof of Proposition 1 is completed. □

Remark 9. *One can find that, for y-variable, we only used* 3 *boundary conditions during the proof,* $h(x,L)=h_y(x,L)=h(x,0)=0$*, to deduce the uniqueness of the solution* h*. It is quite natural since when one consider* 4 *boundary conditions for a third order differential operator, the eigenfunctions could never be a basis.*

One may also wander if we can get the uniqueness of h *by using the other* 3 *boundary conditions in y-variable:* $h(x,L)=h_y(x,L)=h_{yy}(x,L)=0$*? Unfortunately, for these* 3 *boundary conditions Theorem 3 cannot be applied, as we can observe from* (A.19) *and* (A.20) *that there should be boundary conditions on both side. That is the reason why it is difficult to get the uniqueness of* h *by using Carleman estimate, see [1, Chapter 4], [3] and [22], which is the standard way to solve the unique continuation problem: we have to use the fact that* $h(x,0)=0$ *and it is not clear how it should be used.*

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