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PHASE FIELD MODELS FOR THIN ELASTIC STRUCTURES WITH TOPOLOGICAL CONSTRAINT

PATRICK W. DONDL, ANTOINE LEMENANT, AND STEPHAN WOJTOWYTSCH

ABSTRACT. This article is concerned with the problem of minimising the Willmore energy in the class of *connected* surfaces with prescribed area which are confined to a container. We propose a phase field approximation based on De Giorgi's diffuse Willmore functional to this variational problem. Our main contribution is a penalisation term which ensures connectedness in the sharp interface limit.

For sequences of phase fields with bounded diffuse Willmore energy and bounded area term, we prove uniform convergence in two ambient space dimensions and a certain weak mode of convergence on curves in three dimensions. This enables us to show Γ -convergence to a sharp interface problem that only allows for connected structures. The topological contribution is based on a geodesic distance chosen to be small between two points that lie on the same connected component of the transition layer of the phase field.

We furthermore present numerical evidence of the effectiveness of our model. The implementation relies on a coupling of Dijkstra's algorithm in order to compute the topological penalty to a finite element approach for the Willmore term.

CONTENTS

1. Introduction	1
2. Phase Fields	5
2.1. General Background	5
2.2. Equipartition of Energy	6
2.3. Connectedness	6
2.4. Main Results	7
3. Proofs	8
3.1. Proof of Γ -Convergence	8
3.2. Auxiliary Results I	10
3.3. Auxiliary Results II	17
3.4. Proof of the Main Results	24
4. Computer Implementation	27
5. Conclusions	29
References	31

1. INTRODUCTION

In this article, we consider a diffuse interface approximation of a variational problem arising in the study of thin elastic structures. Our particular question is motivated by the problem of

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predicting the shape of certain biological objects, such as mitochondria, which consist of an elastic lipid bilayer and are confined by an additional outer mitochondrial membrane of significantly smaller surface area. When modelling such structures, in addition to the confinement and surface area constraints, a challenging topological side condition naturally arises as the biological membranes are typically connected and do not self-intersect. Incorporating this property into a diffuse interface model will be the main concern of this article.

In order to describe the the locally optimal shape of thin elastic structures, we will rely on suitable bending energies. A well-known example of such a variational characterisation of biomembranes is given by the Helfrich functional [Hel73, Can70]

$$(1.1) \quad \mathcal{E}^H(\Sigma) = \chi_0 \int_{\Sigma} (H - H_0)^2 d\mathcal{H}^2 + \chi_1 \int_{\Sigma} K d\mathcal{H}^2$$

where Σ denotes the two-dimensional membrane surface in \mathbb{R}^3 and H and K denote its mean and Gaussian curvatures. The parameters χ_0 , χ_1 and H_0 are the bending moduli and the spontaneous curvature of the membrane. Both integrals are performed with respect to the two-dimensional Hausdorff measure \mathcal{H}^2 . The special case when $H_0, \chi_1 = 0$ is known as Willmore's energy

$$\mathcal{W}(\Sigma) = \int_{\Sigma} H^2 d\mathcal{H}^2.$$

Extrema (in particular, local minimisers) of the Willmore functional are therefore of interest in models for biological membranes, but they also arise naturally in pure differential geometry as the stereographic projections of compact minimal surfaces in S^3 . An introduction from that point of view is given in [PS87], along with a number of examples. From the point of view of the calculus of variations, a natural approach to energies as the ones above is via varifolds [All72].

In the class of closed surfaces, the second term in the Helfrich functional is of topological nature. If the minimisation problem is considered only among surfaces of prescribed topological type it can be neglected due to the Gauss-Bonnet theorem. The spontaneous curvature is realistically expected to be non-zero and can have tremendous influence. It should be noted that the full Helfrich energy depends also on the orientation of a surfaces for $H_0 \neq 0$ and not only on its induced (unoriented) varifold. Große-Brauckmann [GB93] gives an example of surfaces M_k of constant mean curvature $H \equiv 1$ converging to a doubly covered plane. This demonstrates that, unlike the Willmore energy, the Helfrich energy need not be lower semi-continuous under varifold convergence for some parameters.

Even for Willmore's energy rigorous results are hard to obtain. The existence of smooth minimising Tori was proved by Simon [Sim93], and later generalised to surfaces of arbitrary genus in [BK03]. The long-standing Willmore conjecture that $\mathcal{W}(T) \geq 4\pi^2$ for all Tori embedded in \mathbb{R}^3 was recently established in [MN14], and the large limit genus of the minimal Willmore energy for closed orientable surfaces in \mathbb{R}^3 has been investigated in [KLS10]. The existence of smooth minimising surfaces under isoperimetric constraints has been established in [Sch12]. A good account of the Willmore functional in this context can be found in [KS12].

The case of surfaces constrained to the unit ball was studied in [MR14] and a scaling law for the Willmore energy was found in the regimes of surface area just exceeding 4π and the large area limit. While the above papers adopt an external approach in the language of varifold geometry, a parametrised approach has been developed in [Riv14] and related papers.

Recently, existence of minimisers for certain Helfrich-type energies among axially symmetric surfaces under an isoperimetric constraint was proved by Choksi [CV13]. Other avenues of research consider Willmore surfaces in more general ambient spaces.

Results for the gradient flow of the Willmore functional are still few. Short time existence for sufficiently smooth initial data and long time existence for small initial energy have been

demonstrated in [KS01, KS12] and it has been shown that Willmore flow can drive smooth initial surfaces to self-intersections in finite time in [MS03]. A level set approach to Willmore flow is discussed in [DR04].

Studies of numerical implementations of Willmore flow are, for example, due to Garcke, Dziuk, Elliott *et al.* in [BGN08, Dzi08, DE07]. Particularly interesting here is also an implementation of a two-step time-discretisation algorithm due to Rumpf and Balzani [BR12].

Phase-field approximations of the functional in (1.1), on the other hand, often provide a more convenient approach to gradient flows or minimisation of the Willmore or Helfrich functionals. Such phase-field models are generally based on a competition between a multi-well functional penalising deviation of the phase field function from the minima (usually ± 1) and a gradient-penalising term preventing an overly sharp transition. The prototype is the Modica-Mortola [MM77, Mod87] functional

$$S_\varepsilon(u) = \frac{1}{c_0} \int_\Omega \frac{\varepsilon}{2} |\nabla u|^2 + \frac{1}{\varepsilon} W(u) dx,$$

which is also known as the Cahn-Hilliard energy. Here, W is a function with minimizers at ± 1 , usually taken to be $W(s) = \frac{1}{4}(s^2 - 1)^2$. The normalising constant c_0 is taken such that S_ε converges in the sense of Γ -convergence to the perimeter functional of the jump-set of a function that is ± 1 almost everywhere in the domain Ω . The reason is that a transition in u from $+1$ to -1 always requires a certain minimal energy (per unit area of the transition), but this minimal energy can be attained in the limit by an optimal transition (of tanh-shape orthogonal to the interface, in this case). In particular if a coupling of the surface to a bulk term is desired, phase field models can provide an excellent alternative to a parametric discretisation [Du10].

The idea for applying this approach to Willmore's energy goes back to De Giorgi [DG91]. For a slight modification of De Giorgi's functional, reading

$$\mathcal{E}_\varepsilon^{\text{dG}}(u) = \frac{1}{c_0} \int_\Omega \frac{1}{\varepsilon} \left(\varepsilon \Delta u - \frac{1}{\varepsilon} W'(u) \right)^2 dx + \lambda S_\varepsilon(u) =: \mathcal{W}_\varepsilon(u) + \lambda S_\varepsilon(u)$$

Γ -convergence to the sum of Willmore's energy and the λ -fold perimeter functional was finally proved by Röger and Schätzle [RS06] in $n = 2, 3$ dimensions, providing the lower energy bound, after Bellettini and Paolini [BP93] had provided the recovery sequence over a decade earlier. First analytic evidence had previously been presented in the form of asymptotic expansions in [DLRW05]. This and other phase field approximations of Willmore's energy and their L^2 -gradient flows are reviewed in [BMO13]. A convergence result for a diffuse approximation of certain more general Helfrich-type functionals has also been derived by Bellettini and Mugnai [BM10].

Regarding implementation of phase-field models for Willmore's and Helfrich's energy, we refer to the work by Misbah *et al.* in [BKM05] and Du *et al.* [DLRW05, DLW05, DLW06, DW07, DLRW07, DLRW09, Du10, WD07]. In [FRW13], the two-step algorithm for surface evolution has been extended to phase fields. A numerical implementation of the Helfrich functional can be found, for example, in the work by Campelo and Hernandez-Machado [CHM06].

Starting with the above given phase-field approximation of Willmore's energy, the confinement and the surface area condition are both easily incorporated. The confinement is simply given by the domain of the phase field function with suitable boundary condition; the surface area of the transition layer can for example be fixed by an energy term which penalises the deviation of S_ε from a given target value. This leaves the topological side condition as our main challenge.

An often cited advantage of phase fields is that they are capable of changing their topology; in that sense our endeavour is non-standard. It should be noted that our phase fields may still change their topology (at least in three dimensions), only connectedness is enforced. Additionally, non-interpenetration and confinement constraints are equally difficult to enforce in sharp interface evolutions. So for the moment we neglect the topological term in the Helfrich functional and focus

on the more imminent topological property of connectedness. Still, we remark that our results can be extended to the diffuse Helfrich functionals from [BM10].

Approaches of regularising limit interfaces have been developed by Bellettini in [Bel97] and investigated analytically and numerically in [ERR14]. The approaches work by introducing non-linear terms of the phase-field in order to control the Willmore energies of the level sets individually and exclude transversal crossings (which phase fields for De Giorgi's functional can develop). These regularisations may prevent loss of connectedness along a gradient flow in practice, but do not lead to a variational statement via Γ -convergence.

Previous work in [DMR11] provides a first attempt at an implementation of a topological constraint in a phase-field model for elastic strings modelled by the one-dimensional version of the Willmore energy, namely Euler's elastica. This approach was complemented by a method put forth in [DMR14], which relies on a second phase field subject to an auxiliary minimisation problem used to identify connected components of the transition layer. For this model, a Γ -convergence result was obtained, showing that limits of bounded-energy sequences must describe a connected structure. Unfortunately, the complicated nested minimisation problem makes it unsuitable for computation.

In this article, we propose a different topological penalty term in the energy functional, inspired by [BLS15, LS14], which uses a carefully constructed distance function to detect connected components. The basic idea is that this geodesic distance between two points in the domain should be small if and only if the points can be connected by a curve lying in the phase field transition region. Connected components of the transition layer will now have a finite distance, and thus the occurrence of more than one such component can be sensed by the functional.

Along our proof, we obtain a few useful technical results about the convergence of phase field approximations to the limit problem which appear to be new in this context. Namely, for sequences along which $\mathcal{E}_\varepsilon^{\text{dG}}$ remains bounded, we prove convergence of the transition layers to the limit curve in the Hausdorff distance and uniform convergence of the phase field to ± 1 away from the limit curve in two dimensions. This resembles a similar result for minimisers of the Modica-Mortola functional among functions with prescribed integral from [CC95] and more generally stationary states from [HT00].

In three dimensions, we show that neither result is true, but prove weak analogues. We also show that phase fields are uniformly L^∞ -bounded in terms of their Willmore energy, also in three dimensions.

The paper is organised as follows. In section 2 we give a brief introduction to phase fields for Willmore's problem in general (sections 2.1 and 2.2) and to our approach to connectedness for the transition layers (section 2.3). Our main results are subsequently listed in section 2.4.

Section 3 is entirely dedicated to the proofs of our main results. We directly proceed to show Γ -convergence of our functionals (section 3.1). In sections 3.2 and 3.3, we produce all the auxiliary estimates that will be needed in the later proofs. These are then used to show convergence of the phase-fields, Hausdorff-convergence of the transition layers, and connectedness of the support of the limit measure (section 3.4).

In section 4 we briefly show numerical evidence of the effectiveness of our approach. To that end, we compare it to diffuse Willmore flow without a topological term and to the penalisation proposed in [DMR11]. We have been unable to implement the functional developed in [DMR14] in practice, so a comparison with that could not be drawn. In section 5 we discuss a few easy extensions of our main results and related open problems.

The numerical implementation of our functional will be discussed further in a forth-coming article [DW15]. We use Dijkstra's algorithm to compute the geodesic distance function used in the topological term of our energy functional. The weight in the geodesic distance is chosen to be exactly zero along connected components of the transition layer, which implies that it only needs

to be computed once per detected connected component. This makes the functional extremely efficient from an implementation point of view.

2. PHASE FIELDS

2.1. General Background. A phase-field approach is a method which lifts problems of $(n - 1)$ -dimensional manifolds which are the boundaries of sets to problems of scalar fields on n -dimensional space. Namely, instead of looking at ∂E , we study smooth approximations of $\chi_E - \chi_{E^c}$. Fix $\Omega \Subset \mathbb{R}^n$, open. Then our model space is

$$X = \{u \in W_{loc}^{2,2}(\mathbb{R}^n) \mid u \equiv -1 \text{ outside } \Omega\}.$$

If $\partial\Omega \in C^2$, this can be identified with $X_a = \{u \in W^{2,2}(\Omega) \mid u = \partial_\nu u = 0 \text{ on } \partial\Omega\}$, otherwise the first formulation is technically easier to work with. The boundary conditions are chosen to model sets which are totally contained in Ω and whose boundaries may only touch $\partial\Omega$ tangentially. Then (as well as in more general settings) it is known [Mod87] that the functionals

$$S_\varepsilon : L^1(\Omega) \rightarrow \mathbb{R}, \quad S_\varepsilon(u) = \begin{cases} \frac{1}{c_0} \int_\Omega \frac{\varepsilon}{2} |\nabla u|^2 + \frac{1}{\varepsilon} W(u) \, dx & u \in X \\ +\infty & \text{else,} \end{cases}$$

with $W(u) = (u^2 - 1)^2/4$ and $c_0 = \int_{-1}^1 \sqrt{2W(u)} \, du = 2\sqrt{2}/3$, Γ -converge to

$$S_0(u) = \frac{1}{2} |Du|(\bar{\Omega})$$

with respect to strong L^1 -convergence for functions $u \in BV(\mathbb{R}^n, \{-1, 1\})$ (and $+\infty$ else). The limit $S_0(u)$ agrees with the perimeter functional $\text{Per}(\{u = 1\})$. The double-well potential W forces sequences of bounded energy $S_\varepsilon(u_\varepsilon)$ to converge to a function of the above form as $\varepsilon \rightarrow 0$, and the functional has originally been studied in the context of minimal surfaces, see e.g. [Mod87, CC95, CC06]. We will view them in a slightly different light. If u_ε is a finite energy sequence, we denote the Radon measure given by the approximations of the area functional as

$$\mu_\varepsilon(U) = \frac{1}{c_0} \int_U \frac{\varepsilon}{2} |\nabla u_\varepsilon|^2 + \frac{1}{\varepsilon} W(u_\varepsilon) \, dx.$$

Taking the first variation of this measure, and integrating by parts, we get

$$\delta\mu_\varepsilon(\phi) = \frac{1}{c_0} \int_\Omega \left(-\varepsilon \Delta u_\varepsilon + \frac{1}{\varepsilon} W'(u_\varepsilon) \right) \phi \, dx.$$

If we take the integrand

$$v_\varepsilon := -\varepsilon \Delta u_\varepsilon + \frac{1}{\varepsilon} W'(u_\varepsilon),$$

we obtain a diffuse analogue of the mean curvature as the variation of surface area. A modified version of a conjecture by De Giorgi then proposes that squaring v_ε , integrating over Ω and dividing by ε should be a Γ -convergent approximation of $\mathcal{W}(\partial E)$ for C^2 -regular sets E , again with respect to strong L^1 -convergence. Dividing by ε is needed to convert the volume integral into a diffuse surface integral over the transition layer with width of order ε . First analytic evidence for this was provided via formal asymptotic expansions in [DLRW05] and the conjecture was proved for $n = 2, 3$ in [RS06]. From now on, we will restrict ourselves to these dimensions. We thus introduce the diffuse Willmore functional

$$\mathcal{W}_\varepsilon(u) = \begin{cases} \frac{1}{c_0} \int_\Omega \frac{1}{\varepsilon} \left(\varepsilon \Delta u - \frac{1}{\varepsilon} W'(u) \right)^2 \, dx & u \in X \\ +\infty & \text{else} \end{cases}$$

and the Radon measures associated with a sequence u_ε with bounded energy $S_\varepsilon(u_\varepsilon) + \mathcal{W}_\varepsilon(u_\varepsilon)$

$$\mu_\varepsilon(U) = \frac{1}{c_0} \int_U \frac{\varepsilon}{2} |\nabla u_\varepsilon|^2 + \frac{1}{\varepsilon} W(u_\varepsilon) dx, \quad \alpha_\varepsilon(U) = \frac{1}{c_0} \int_U \frac{1}{\varepsilon} v_\varepsilon^2 dx.$$

By Young's inequality $\frac{\varepsilon}{4} a^2 + \varepsilon b^2 \geq ab$ we can control the BV -norms of $G(u_\varepsilon) = \int_{-1}^{u_\varepsilon} \sqrt{2W(s)} ds$ and invoking the compactness theorems for BV -functions and Radon measures to see that there are $u \in BV_{loc}(\mathbb{R}^n)$ and finite Radon measures μ, α with support in $\bar{\Omega}$ such that, up to a subsequence,

$$u_\varepsilon \rightarrow u \text{ strongly in } L^1(\Omega), \quad \mu_\varepsilon \xrightarrow{*} \mu, \quad \alpha_\varepsilon \xrightarrow{*} \alpha, \quad |Du| \leq 2\mu.$$

Clearly by construction $u = \chi_E - \chi_{E^c}$ for some $E \subset \Omega$, and from the above argument is not difficult to see that E is a Caccioppoli set. In three dimensions, however, E may well be empty using the construction of [MR14], where two converging spheres are connected by a catenoid with bounded Willmore energy. A diagonal sequence shows that the same is possible with phase fields.

In [RS06] it has been shown that μ is in fact the mass measure of an integral $n-1$ -varifold with $|\partial^* E| \leq \mu$ and $H_\mu^2 \cdot \mu \leq \alpha$ for generalised mean curvature H_μ of μ . Unlike μ, α can still behave wildly.

We aim to minimise the Willmore energy among connected surfaces with given area S . To prescribe area $\mu(\bar{\Omega}) = S$, we include a penalisation term of

$$\frac{1}{\varepsilon^\sigma} (S_\varepsilon - S)^2$$

in the energy functional.

2.2. Equipartition of Energy. The Cahn-Hilliard energy forces phase fields to be close to ± 1 in phase and make transitions between the phases along layers of width proportional to ε . So while it make sense that the terms $\frac{\varepsilon}{2} |\nabla u_\varepsilon|^2$ and $\frac{1}{\varepsilon} W(u_\varepsilon)$ should scale the same way to give us the surface measure, it is by no means obvious that, for sequences u_ε of bounded \mathcal{W}_ε energy, they make an equal contribution to the measure μ_ε . The difference of their contributions is controlled by the discrepancy measures

$$\xi_\varepsilon(U) = \frac{1}{c_0} \int_U \frac{\varepsilon}{2} |\nabla u_\varepsilon|^2 - \frac{1}{\varepsilon} W(u_\varepsilon) dx,$$

their positive parts $\xi_{\varepsilon,+}(U) = \frac{1}{c_0} \int_U \left(\frac{\varepsilon}{2} |\nabla u_\varepsilon|^2 - \frac{1}{\varepsilon} W(u_\varepsilon) \right)_+ dx$ and their total variation measures $|\xi_\varepsilon|(U) = \frac{1}{c_0} \int_U \left| \frac{\varepsilon}{2} |\nabla u_\varepsilon|^2 - \frac{1}{\varepsilon} W(u_\varepsilon) \right| dx$. We will see that for a suitable recovery sequence, $|\xi_\varepsilon|$ vanishes exponentially in ε ; more generally the discrepancy goes to zero in general sequences u_ε along which $\alpha_\varepsilon + \mu_\varepsilon$ stays bounded as shown in [RS06, Propositions 4.4, 4.9]. The result has been improved along subsequences in [BM10, Theorem 4.6] to L^p -convergence for $p < 3/2$ and convergence of Radon measures for the gradients of the discrepancy densities.

The control of these measures is extremely important since they appear in the derivative of the diffuse $(n-1)$ -densities $r^{1-n} \mu_\varepsilon(B_r(x))$.

2.3. Connectedness. In order to ensure that the support of the limiting measure μ is connected we include an auxiliary term C_ε in the energy functional. To define it, we use an adapted geodesic distance

$$d^{F(u)}(x, y) = \inf \left\{ \int_K F(u) d\mathcal{H}^1 \mid K \text{ connected, } x, y \in K, \mathcal{H}^1(K) \leq \omega(\varepsilon) \right\}, \quad \omega(\varepsilon) \rightarrow \infty \text{ as } \varepsilon \rightarrow 0,$$

which is integrated against a suitable weight function. We thus take

$$C_\varepsilon(u) = \frac{1}{\varepsilon^2} \int_\Omega \int_\Omega \phi(u(x)) \phi(u(y)) d^{F(u)}(x, y) dx dy,$$

where $\phi \in C_c(-1, 1)$ resembles a bump, i.e.,

$$\phi \geq 0, \quad \{\phi > 0\} = (\rho_1, \rho_2) \Subset (-1, 1), \quad \int_{-1}^1 \phi(u) du > 0$$

and $F \in C^{0,1}(\mathbb{R})$ is chosen to be zero where ϕ does not vanish:

$$F \geq 0, \quad F \equiv 0 \text{ on } [\rho_1, \rho_2], \quad F(-1), F(1) > 0.$$

The heuristic idea of this is that if the support of the limiting measure $\text{spt}(\mu)$ is connected, then so should the set $\{\rho_1 < u_\varepsilon < \rho_2\}$. These level sets away from ± 1 can be heuristically viewed as approximations of $\text{spt}(\mu)$, and in other situations, they can be seen to Hausdorff-converge against it, see [CC95]. This is not quite true in our situation (at least in three dimensions where we can get additional points in the Hausdorff limit), but the intuition still holds.

If $\{\phi(u_\varepsilon) > 0\} = \{\rho_1 < u_\varepsilon < \rho_2\}$ is connected, we can connect any two points $x, y \in \Omega$ such that $\phi(u_\varepsilon(x))\phi(u_\varepsilon(y)) > 0$ with a curve of length zero, hence $d^{F(u_\varepsilon)}(x, y) = 0$ and both the integrand and the double integral vanish.

If on the other hand $\text{spt}(\mu)$ is disconnected, then we expect that $d^{F(u_\varepsilon)}$ should be able to separate different connected components such that $\liminf_{\varepsilon \rightarrow 0} C_\varepsilon(u_\varepsilon) > 0$. The core part of our proof is concerned with precisely that. We need to show that ϕ detects components of the interface and that $d^{F(u_\varepsilon)}$ separates them. For the first result, we need to understand the structure of the interfaces converging to μ , for the second part, we need to exclude the possibility of phase fields building tunnels between different connected components of $\text{spt}(\mu)$ which collapse away in the mass measure but keep the distance close to zero along a sequence. This is easy in two dimensions where $1 = n - 1$ but hard in three dimensions where curves have codimension two.

2.4. Main Results. We have the following new results for phase-fields with bounded Willmore energy and perimeter.

Theorem 2.1. *Let $n = 2$ and $u_\varepsilon \in X$ a sequence such that*

$$\limsup_{\varepsilon \rightarrow 0} (\mathcal{W}_\varepsilon + S_\varepsilon)(u_\varepsilon) < \infty.$$

Denote $\mu = \lim_{\varepsilon \rightarrow 0} \mu_\varepsilon$ for a subsequence along which the limit exists and take any $\Omega' \Subset \mathbb{R}^n \setminus \text{spt}(\mu)$. Then $|u_\varepsilon| \rightarrow 1$ uniformly on Ω' .

For this and the following theorems, we take the continuous representative of u_ε which exists since $W^{2,2} \hookrightarrow C^0$ in $n = 2, 3$ dimensions.

Theorem 2.2. *Let $n = 2$ and $D \Subset (-1, 1)$. Then $u_\varepsilon^{-1}(D) \rightarrow \text{spt}(\mu)$ in the Hausdorff distance. If $n = 3$, up to a subsequence, $u_\varepsilon^{-1}(D)$ Hausdorff-converges to a set K which contains $\text{spt}(\mu)$.*

For our application to connectedness, we define the total energy of an ε -phase field as

$$(2.1) \quad \mathcal{E}_\varepsilon(u) = \begin{cases} \mathcal{W}_\varepsilon(u) + \varepsilon^{-\sigma} (S_\varepsilon(u) - S)^2 + \varepsilon^{-\kappa} C_\varepsilon(u) & u \in X \\ +\infty & \text{else} \end{cases}$$

for $\sigma \in (0, 4), \kappa > 1$. Using our previous results, we can show the following.

Remark 2.3. Existence of minimisers for the functional \mathcal{E}_ε is a simple exercise in the direct method of the calculus of variations, since uniform convergence of a minimising sequence for fixed ε guarantees convergence of the distance term.

Theorem 2.4. *Let $n = 2, 3$ and $u_\varepsilon \in X$ a sequence such that $\liminf_{\varepsilon \rightarrow 0} \mathcal{E}_\varepsilon(u_\varepsilon) < \infty$. Then the diffuse mass measures μ_ε converge weakly* to a measure μ with connected support $\text{spt}(\mu) \subset \overline{\Omega}$ and area $\mu(\overline{\Omega}) = S$.*

Using [RS06], μ is also the mass measure of an integral varifold. The main result of [RS06] can be applied to deduce Γ -convergence of our functionals in the following sense:

Corollary 2.5. *Let $n = 2, 3$, $S > 0$, $\Omega \Subset \mathbb{R}^n$ and $E \Subset \Omega$, with smooth boundary $\partial E \in C^2$ with area $\mathcal{H}^{n-1}(\partial E) = S$. Then*

$$\Gamma(L^1(\Omega)) - \lim_{\varepsilon \rightarrow 0} \mathcal{E}_\varepsilon(\chi_E - \chi_{E^c}) = \begin{cases} \mathcal{W}(\partial E) & \partial E \text{ is connected} \\ +\infty & \text{otherwise} \end{cases}$$

Here we have adopted the notation of [RS06] where Γ -convergence is said to hold at a point if the lim inf- and lim sup-inequalities hold at that point. This distinction is necessary since it is not clear what Γ -convergence properties hold at other points $u \in BV(\Omega, \{-1, 1\})$ since \mathcal{W}_ε does not converge to the L^1 -lower semi-continuous envelope of \mathcal{W} at non-embedded sets even in two dimensions.

This issue stems from the fact that the stationary Allen-Cahn equation $-\Delta u + W'(u) = 0$ admits global saddle-solutions which have zeros along the coordinate axes and are positive in the first and third quadrants and negative in the second and fourth ones or even more degenerate ones, see [dPKPW10]. These solutions can be used to approximate transversal crossings with zero Willmore energy, while the lower semi-continuous envelope of Willmore's energy/Euler's elastica energy becomes infinite at those points. Interestingly enough, the crossing has zero Willmore-energy as a varifold, so that the energy can still be justified, despite the fact that it does not give the sensible result in our situation.

While heuristic considerations suggest – and numerical simulations appear to confirm – that at least in $n = 2$ dimensions and when $0 \notin \text{spt}(\phi)$, our additional term in the energy might prevent saddles, we do not investigate convergence at non-embedded points further.

Remark 2.6. Uniform convergence does generally fail in three dimensions. This can be seen by taking a sequence u^ε as the optimal interface approximation which will be given in the proof of Corollary 2.5 and adding to it a perturbation $g((x - x_0)/\varepsilon)$ for any $g \in C_c^\infty(\mathbb{R}^n)$. It can easily be verified that the energy will remain finite (at least if $|g| \ll 1$) but clearly the functions do not converge uniformly.

Heuristic arguments show that for modifications $u_\varepsilon = u^\varepsilon + \varepsilon^\alpha g(\varepsilon^{-\beta} \cdot)$, the coefficients $\alpha = \beta = 1$ should be optimal in three dimensions and $\alpha = 1/2$, $\beta = 1$ if $n = 2$, so we expect a convergence rate of $\sqrt{\varepsilon}$ in the two-dimensional case.

Remark 2.7. Since $S_\varepsilon(u_\varepsilon)$ is uniformly bounded, standard Modica-Mortola arguments show that $u_\varepsilon \rightarrow \chi_E - \chi_{\Omega \setminus E}$ in L^p for $p < 4/3$. We are going to show in Lemma 3.1 that the sequence u_ε is uniformly bounded in $L^\infty(\mathbb{R}^n)$ when additionally $\limsup_{\varepsilon \rightarrow 0} \mathcal{W}(u_\varepsilon) < \infty$, which implies L^p -convergence for all $p < \infty$.

3. PROOFS

The proofs are organised in the following way. First, anticipating the results of section 3.4, we show Γ -convergence of \mathcal{E}_ε to \mathcal{W} . We decided to move the proof to the beginning, since it is virtually independent of all the other proofs in this article and can be isolated, but introduces the optimal interface transitions which will be needed later. After that, we proceed chronologically with technical Lemmata (sections 3.2 and 3.3) and the proof of Theorems 2.1, 2.2 and 2.4 in section 3.4.

3.1. Proof of Γ -Convergence. We now proceed to prove Corollary 2.5.

Proof of the lim inf-inequality: It will follow from Theorem 2.4 that $\mathcal{E}_\varepsilon(u_\varepsilon) \rightarrow \infty$ if ∂E is disconnected. If ∂E is connected, the main part of this inequality is to show that if $u_\varepsilon \rightarrow \chi_E - \chi_{E^c}$

in $L^1(\Omega)$ and $\mu_\varepsilon(\Omega) \leq S + 1$, then $\liminf_{\varepsilon \rightarrow 0} \mathcal{E}_\varepsilon(u_\varepsilon) \geq \mathcal{W}(\partial E)$. Since $\mathcal{E}_\varepsilon \geq \mathcal{W}_\varepsilon$ and enforces the surface area estimate, we obtain with [RS06] that

$$\liminf_{\varepsilon \rightarrow 0} \mathcal{E}_\varepsilon(u_\varepsilon) \geq \mathcal{W}(\partial E).$$

□

Proof of the limsup-inequality: We may restrict our analysis to the case of connected boundaries with area $\mathcal{H}^{n-1}(\partial E) = S$. In this proof, we will construct a sequence $u^\varepsilon \in X$ such that

$$\lim_{\varepsilon \rightarrow 0} \|u^\varepsilon - (\chi_E - \chi_{E^c})\|_{1,\Omega} = 0, \quad \lim_{\varepsilon \rightarrow 0} \mathcal{E}_\varepsilon(u^\varepsilon) = \mathcal{W}(\partial E).$$

The construction is standard and holds for arbitrary $n \in \mathbb{N}$. We take the solution of the one dimensional problem and apply it to the signed distance function of ∂E , thus recreating the shape of ∂E with an (approximate) optimal profile for the transition from -1 to 1 . For $\delta > 0$ consider

$$U_\delta := \{x \in \mathbb{R}^n \mid \text{dist}(x, \partial E) < \delta\}.$$

Since $E \Subset \Omega$, $U_\delta \subset \Omega$ for all sufficiently small δ , and since $\partial E \in C^2$ is embedded, there is $\delta > 0$ such that

$$\psi : \partial E \times (-\delta, \delta) \rightarrow U_\delta, \quad \psi(x, t) = x + t \nu_x$$

is a diffeomorphism. We denote the partial inverse to ψ which projects onto ∂E by π and by $d(x) = \text{sdist}(x, \partial E)$ the signed distance of x from ∂E which is positive inside E and negative outside E , C^2 -smooth on U_δ and satisfies (see e.g. [GT01, Section 14.6])

$$\nabla d(x) = \nu_{\pi(x)}, \quad \Delta d(x) = H_{\pi(x)} + C_{\pi(x)} \cdot d(x) + O(d(x)^2).$$

Let us consider the optimal transition between -1 and 1 in one dimension. This optimal profile is a stationary point of S_1 , i.e. a solution of $-q'' + W'(q) = 0$ (and thus a zero energy point of \mathcal{W}_1) with the side conditions that $\lim_{t \rightarrow \pm\infty} q(t) = \pm 1$. Note that the optimal profile satisfies

$$q'' = W'(q) \quad \Rightarrow \quad q'' q' = W'(q) q' \quad \Rightarrow \quad \frac{d}{dt} ((q')^2 - W(q)) = 0,$$

so we find that $(q')^2 - W(q) \equiv c$. We look for transitions where both $(q')^2$ and $W(q)$ are integrable over the whole real line, and since $\lim_{t \rightarrow \pm\infty} W(q(t)) = 0$, we see that $c = 0$ and $(q')^2 \equiv W(q)$. This gives us equipartition of energy already before integration. For simplicity, we focus on $W(q) = (q^2 - 1)^2/4$ which has the optimal interface $q(t) = \tanh(t/\sqrt{2})$. Note that the functional rescales appropriately under dilations of the parameter space so that we have equipartition of energy before integration also in the ε -problem. The disadvantage of the hyperbolic tangent is that it makes the transition between the roots of W only in infinite space, so we choose to work with approximations $q_\varepsilon \in C^\infty(\mathbb{R})$ such that

- (1) $q_\varepsilon(t) = q(t)$ for $|t| \leq \delta/(3\varepsilon)$,
- (2) $q_\varepsilon(t) = 1$ for $t \geq \delta/(2\varepsilon)$,
- (3) $q_\varepsilon(-t) = -q_\varepsilon(t)$,
- (4) $q'_\varepsilon > 0$.

Then we set

$$u^\varepsilon(x) = q_\varepsilon(d(x)/\varepsilon).$$

A direct calculation establishes that with this choice of u^ε , we have $|S_\varepsilon(u^\varepsilon) - S| \leq \varepsilon^\gamma$ for all $\gamma < 2$, $|\xi_\varepsilon| \leq \varepsilon^m$ for all $m \in \mathbb{N}$ and $\lim_{\varepsilon \rightarrow 0} \mathcal{W}_\varepsilon(u^\varepsilon) = \mathcal{W}(\partial E)$. It remains to show that $\lim_{\varepsilon \rightarrow 0} \varepsilon^{-\kappa} C_\varepsilon(u^\varepsilon) = 0$. We will show that even $C_\varepsilon(u^\varepsilon) \equiv 0$ along this sequence. Since ∂E is connected and ψ is a diffeomorphism, all the level sets

$$\{u^\varepsilon = \rho\} = \psi(\partial E, q_\varepsilon^{-1}(\rho))$$

are connected manifolds for $\rho \in (-1, 1)$. We know that

$$\{\phi(u^\varepsilon) > 0\} = \{\rho_1 < u^\varepsilon < \rho_2\}$$

and pick any $\rho \in (\rho_1, \rho_2)$. Now let $x, y \in \Omega$, $\phi(u^\varepsilon(x)), \phi(u^\varepsilon(y)) > 0$. We can construct a curve from x to y by setting piecewise

$$\begin{aligned} \gamma_1 : [d(x), \rho] &\rightarrow \Omega, & \gamma_1(t) &= \pi(x) + t\nu_{\pi(x)}, \\ \gamma_3 : [\rho, d(y)] &\rightarrow \Omega, & \gamma_3(t) &= \pi(y) + t\nu_{\pi(y)} \end{aligned}$$

and γ_2 any curve connecting $\gamma_1(\rho)$ to $\gamma_3(\rho)$ in $\{u^\varepsilon = \rho\}$. This curve exists since connected manifolds are path-connected. The curve $\gamma = \gamma_3 \oplus \gamma_2 \oplus \gamma_1$ connects x and y and satisfies by construction $\phi(\gamma(t)) > 0$, so $F(\gamma(t)) \equiv 0$. Therefore we deduce

$$d^{F(u^\varepsilon)}(x, y) = 0$$

if $\phi(u^\varepsilon(x)), \phi(u^\varepsilon(y)) \neq 0$, noting that the connecting curves have uniformly bounded length and $\omega(\varepsilon) \rightarrow \infty$. Thus in particular

$$\frac{1}{\varepsilon^2} \int_{\Omega \times \Omega} \phi(u^\varepsilon(x)) \phi(u^\varepsilon(y)) d^{F(u^\varepsilon)}(x, y) dx dy \equiv 0.$$

□

3.2. Auxiliary Results I. A lot of our proofs will be inspired by [RS06] which again draws from [HT00]. We will generally cite [RS06] because it treats the more relevant case for our study. In this section, we will prove a number of auxiliary results which concern either general properties of phase fields or properties away from the support of the limiting measure μ which will enable us to investigate their convergence later. Results concerning phase interfaces are postponed until section 3.3. We start with a partial regularity lemma which is a slight improvement upon [RS06, Proposition 3.6].

Lemma 3.1. *Let $n = 2, 3$, $\Omega \Subset \mathbb{R}^n$ and $u_\varepsilon \in X$ such that $\bar{\alpha} := \limsup_{\varepsilon \rightarrow 0} \mathcal{W}_\varepsilon(u_\varepsilon) < \infty$. Then there exists $\varepsilon_0 > 0$ such that for all $\varepsilon < \varepsilon_0$*

- (1) $\|u_\varepsilon\|_{\infty, \mathbb{R}^n} \leq C$ where C is a constant depending only on $\bar{\alpha}$ and n .
- (2) u_ε is $1/2$ -Hölder continuous on ε -balls, i.e.

$$|u_\varepsilon(x) - u_\varepsilon(y)| \leq \frac{C}{\sqrt{\varepsilon}} |x - y|^{\frac{1}{2}} \quad \forall x \in \mathbb{R}^n, y \in B(x, \varepsilon).$$

Again, the constant C depends only on $\bar{\alpha}$ and n .

Proof. We will argue using Sobolev embeddings for blow ups of u_ε onto the natural length scale. In the first step, we show the set $\{|u_\varepsilon| > 1\}$ to be small. In the second step, we estimate the L^2 -norm of the blow ups, in the third we estimate the full $W^{2,2}$ -norm and use suitable embedding theorems to conclude the proof of regularity.

Step 1. First observe that

$$\begin{aligned}
 \alpha_\varepsilon(\Omega) &\geq \alpha_\varepsilon(\{u_\varepsilon > 1\}) \\
 &= \frac{1}{c_0} \int_{\{u_\varepsilon > 1\}} \frac{1}{\varepsilon} \left(\varepsilon \Delta u_\varepsilon - \frac{1}{\varepsilon} W'(u_\varepsilon) \right)^2 dx \\
 &= \frac{-2}{c_0 \varepsilon} \int_{\partial\{u_\varepsilon > 1\}} W'(u_\varepsilon) \partial_\nu u_\varepsilon d\mathcal{H}^{n-1} \\
 &\quad + \frac{1}{c_0} \int_{\{u_\varepsilon > 1\}} \varepsilon (\Delta u_\varepsilon)^2 + \frac{2}{\varepsilon} W''(u_\varepsilon) |\nabla u_\varepsilon|^2 + \frac{1}{\varepsilon^3} (W'(u_\varepsilon))^2 dx \\
 &\geq \frac{1}{c_0} \int_{\{u_\varepsilon > 1\}} \varepsilon (\Delta u_\varepsilon)^2 + \frac{4}{\varepsilon} |\nabla u_\varepsilon|^2 + \frac{1}{\varepsilon^3} (W'(u_\varepsilon))^2 dx
 \end{aligned}$$

using that $W''(t) \geq 2$ for $t \geq 1$. The boundary integral vanishes since $u_\varepsilon \in W^{2,2} \hookrightarrow C^{0,1/2}$ is continuous and $W'(1) = 0$ if $\{u_\varepsilon > 1\}$ is of finite perimeter. If this is not the case, take $\theta \searrow 1$ converging from above such that $\{u_\varepsilon > \theta\}$ is of finite perimeter. This holds for almost all $\theta \in \mathbb{R}$. The sign of the boundary integral can be determined since $\partial_\nu u_\varepsilon < 0$ on the boundary of $\{u_\varepsilon > \theta\}$ and $W'(\theta) > 0$ for $\theta > 1$ so that the same inequality can still be established. By symmetry, the same argument works for $\{u_\varepsilon < -1\}$.

Step 2. Let $x_\varepsilon \in \Omega$ be an arbitrary sequence and define the blow up sequence $\tilde{u}_\varepsilon : \mathbb{R}^n \rightarrow \mathbb{R}$ by

$$\tilde{u}_\varepsilon(y) = u_\varepsilon(x_\varepsilon + \varepsilon y).$$

Then we observe that

$$\begin{aligned}
 \int_{B(0,2)} \tilde{u}_\varepsilon^2 dx &= \int_{B(0,2)} (|\tilde{u}_\varepsilon| - 1 + 1)^2 dy \\
 &\leq \int_{B(0,2)} ((|\tilde{u}_\varepsilon| - 1)_+ + 1)^2 dy \\
 &\leq 2 \int_{B(0,2)} (|\tilde{u}_\varepsilon| - 1)_+^2 + 1 dy \\
 &\leq 2 \varepsilon^{3-n} \int_{\{u_\varepsilon > 1\}} \frac{1}{\varepsilon^3} W'(u_\varepsilon)^2 dy + 2^{n+1} \omega_n \\
 &\leq 2 \{2^n \omega_n + c_0 \varepsilon^{3-n} \alpha_\varepsilon(\Omega)\}.
 \end{aligned}$$

As usual, ω_n denotes the volume of the n -dimensional unit ball. In exactly the same way with a slightly simpler argument we obtain

$$\int_{B(0,2)} (W'(\tilde{u}_\varepsilon))^2 dy \leq C(\bar{\alpha}, n).$$

Step 3. Now a direct calculation shows that

$$\begin{aligned}
 \int_{B(0,2)} (\Delta \tilde{u}_\varepsilon - W'(\tilde{u}_\varepsilon))^2 dy &= \int_{B(0,2)} (\varepsilon^2 \Delta u_\varepsilon - W'(u_\varepsilon))^2 (x_\varepsilon + \varepsilon y) dy \\
 &= c_0 \varepsilon^{3-n} \alpha_\varepsilon(B(x_\varepsilon, 2\varepsilon)).
 \end{aligned}$$

Thus

$$\begin{aligned}
 \|\Delta \tilde{u}_\varepsilon\|_{2,B(0,2)} &\leq \|\Delta \tilde{u}_\varepsilon - W'(\tilde{u}_\varepsilon)\|_{2,B(0,2)} + \|W'(\tilde{u}_\varepsilon)\|_{2,B(0,2)} \\
 &\leq \sqrt{c_0 \varepsilon^{3-n} \alpha_\varepsilon(\Omega)} + \sqrt{C(\bar{\alpha}, n)}.
 \end{aligned}$$

In total, we see that

$$\|\tilde{u}_\varepsilon\|_{2,B(0,2)} + \|\Delta \tilde{u}_\varepsilon\|_{2,B(0,2)} \leq C(\bar{\alpha}, n)$$

for all $0 < \varepsilon < 1$ so small that $\alpha_\varepsilon(\Omega) \leq \alpha(\Omega) + 1$. Using the elliptic estimate from [GT01, Theorem 9.11], we see that

$$\|\tilde{u}_\varepsilon\|_{2,2,B(0,1)} \leq C(\bar{\alpha}, n),$$

where we absorbed the constant depending only on n and the radii into the big constant. Using the Sobolev embeddings

$$W^{2,2}(B(0,1)) \hookrightarrow W^{1,6}(B(0,1)) \hookrightarrow C^{0,1/2}(\overline{B(0,1)})$$

we deduce that

$$\|\tilde{u}_\varepsilon\|_{0,1/2,B(0,1)} \leq C(n, \bar{\alpha}),$$

again absorbing the embedding constants into the constant. In particular, this shows that

$$\|\tilde{u}_\varepsilon\|_{\infty,B(0,1)} \leq C(n, \bar{\alpha}).$$

But since this holds for all sequences x_ε , we can deduce that

$$\|u_\varepsilon\|_{\infty,\mathbb{R}^n} \leq C(n, \bar{\alpha}).$$

Furthermore, for $x, z \in \mathbb{R}^n$ with $|x - z| < \varepsilon < \varepsilon_0$, we choose $x_\varepsilon = x$ to deduce

$$|u_\varepsilon(x) - u_\varepsilon(y)| = |\tilde{u}_\varepsilon(0) - \tilde{u}_\varepsilon((y-x)/\varepsilon)| \leq C(n, \bar{\alpha}) |(y-x)/\varepsilon|^{\frac{1}{2}} = \frac{C(n, \bar{\alpha})}{\sqrt{\varepsilon}} |x-y|^{\frac{1}{2}}.$$

□

Remark 3.2. Without prescribing boundary conditions as in our modified space X , the result could still be salvaged on compactly contained subsets. Techniques for estimating quantities over $\{|u_\varepsilon| > 1\}$ in that case can be found in [RS06, Proposition 3.5], which we include below for the readers' convenience.

Proposition 3.3. [RS06, Proposition 3.5] For $n = 2, 3$, $\Omega \subseteq \mathbb{R}^n$, $\varepsilon > 0$, $u_\varepsilon \in C^2(\Omega)$, $v_\varepsilon \in C^0(\Omega)$,

$$-\varepsilon \Delta u_\varepsilon + \frac{1}{\varepsilon} W'(u_\varepsilon) = v_\varepsilon \quad \text{in } \Omega,$$

and $\Omega' \Subset \Omega$, $0 < r < \text{dist}(\Omega', \partial\Omega)$, we have

$$\int_{\{|u_\varepsilon| \geq 1\} \cap \Omega'} W'(u_\varepsilon)^2 \leq C_k (1 + r^{-2k} \varepsilon^{2k}) \varepsilon^2 \int_{\Omega} v_\varepsilon^2 + C_k r^{-2k} \varepsilon^{2k} \int_{\{|u_\varepsilon| \geq 1\} \cap \Omega} W'(u_\varepsilon)^2$$

for all $k \in \mathbb{N}_0$.

A useful rescaling property is the following observation from the proof of [RS06, Theorem 5.1].

Lemma 3.4. Let $u_\varepsilon : B(x, r) \rightarrow \mathbb{R}$, $\lambda > 0$ and $\hat{u}_\varepsilon : B(0, r/\lambda) \rightarrow \mathbb{R}$ with

$$\hat{u}_\varepsilon(y) = u_\varepsilon(x + \lambda y).$$

Set $\hat{r} := r/\lambda$, $\hat{\varepsilon} := \varepsilon/\lambda$,

$$\hat{\mu}_\varepsilon := \frac{1}{c_0} \left(\frac{\hat{\varepsilon}}{2} |\nabla \hat{u}_\varepsilon|^2 + \frac{1}{\hat{\varepsilon}} W(\hat{u}_\varepsilon) \right) \mathcal{L}^n, \quad \hat{\alpha}_\varepsilon := \frac{1}{c_0 \hat{\varepsilon}} \left(\hat{\varepsilon} \Delta \hat{u}_\varepsilon - \frac{1}{\hat{\varepsilon}} W'(\hat{u}_\varepsilon) \right) \mathcal{L}^n.$$

Then

$$\hat{r}^{1-n} \hat{\mu}_\varepsilon(B(0, \hat{r})) = r^{1-n} \mu_\varepsilon(B(x, r)), \quad \hat{r}^{3-n} \hat{\alpha}_\varepsilon(B(0, \hat{r})) = r^{3-n} \alpha_\varepsilon(B(x, r)).$$

The discrepancy measures $\xi_{\varepsilon, \pm}$ behave like μ under rescaling.

Proof. This can be seen by a simple calculation similar to the one in the proof of 3.1. □

For the reader's convenience we include the following classical monotonicity result.

Lemma 3.5. [RS06, Lemma 4.2] *For $x \in \mathbb{R}^n$ we have*

$$\begin{aligned} \frac{d}{d\rho} (\rho^{1-n} \mu_\varepsilon(B(x, \rho))) &= -\frac{\xi_\varepsilon(B(x, \rho))}{\rho^n} + \frac{1}{c_0 \rho^{n+1}} \int_{\partial B(x, \rho)} \varepsilon \langle y - x, \nabla u_\varepsilon \rangle^2 d\mathcal{H}^{n-1}(y) \\ &\quad + \frac{1}{c_0 \rho^n} \int_{B(x, \rho)} v_\varepsilon \langle y - x, \nabla u_\varepsilon \rangle dy \end{aligned}$$

In low dimensions $n = 2, 3$, the second and third term in the monotonicity formula can easily be estimated after integration. While the result is known, we fixed minor details in the proof of [RS06, Proposition 4.5], so we include it here for completeness.

Lemma 3.6. [RS06, Proposition 4.5] *Let $0 < r < R < \infty$ if $n = 3$ and $0 < r < R \leq 1$ if $n = 2$, then*

$$\begin{aligned} (3.1) \quad r^{1-n} \mu_\varepsilon(B(x, r)) &\leq 3 R^{1-n} \mu_\varepsilon(B(x, R)) + 2 \int_r^R \frac{\xi_{\varepsilon,+}(B(x, \rho))}{\rho^n} d\rho \\ &\quad + \frac{1}{2(n-1)^2} \alpha_\varepsilon(B(x, R)) + \frac{r^{3-n}}{(n-1)^2} \alpha_\varepsilon(B(x, r)) \\ &\quad + \frac{R_0^2 R^{1-n}}{(n-1)^2} \alpha_\varepsilon(B(x, R)) \end{aligned}$$

where $R_0 := \min\{R, R_\Omega\}$ and R_Ω is a radius such that $\Omega \subset B(0, R_\Omega/2)$

Proof. Without loss of generality we may assume that $x = 0$ and write $B_\rho := B(0, \rho)$, $f(\rho) = \rho^{1-n} \mu_\varepsilon(B_\rho)$. Observe that for any function $g : B_R \rightarrow \mathbb{R}$ we have

$$\begin{aligned} \int_r^R \rho^{-n} \int_{B_\rho} g(x) dx d\rho &= \int_{B_R} g(x) \int_{\max\{|x|, r\}}^R \rho^{-n} d\rho dx \\ &= \frac{1}{n-1} \int_{B_R} g(x) \left(\frac{1}{\max\{|x|, r\}^{n-1}} - \frac{1}{R^{n-1}} \right) dx \end{aligned}$$

and

$$\int_r^R \rho^{-(n+1)} \int_{\partial B_\rho} g(x) d\mathcal{H}^{n-1} d\rho = \int_{B_R \setminus B_r} \frac{g(x)}{|x|^{n+1}} dx.$$

Using this to integrate the derivative we obtain using Young's inequality

$$\begin{aligned}
f(R) - f(r) &= \int_r^R f'(\rho) \, d\rho \\
&= \int_r^R \frac{-\xi_\varepsilon(B_\rho)}{\rho^n} \, d\rho + \frac{1}{c_0} \int_{B_R \setminus B_r} \frac{\varepsilon \langle \nabla u_\varepsilon, y \rangle^2}{|y|^{n+1}} + \frac{1}{n-1} \frac{v_\varepsilon \langle y, \nabla u_\varepsilon \rangle}{|y|^{n-1}} \, dy \\
&\quad + \frac{1}{(n-1)c_0 r^{n-1}} \int_{B_r} v_\varepsilon \langle y, \nabla u_\varepsilon \rangle \, dy - \frac{1}{(n-1)c_0 R^{n-1}} \int_{B_R} v_\varepsilon \langle y, \nabla u_\varepsilon \rangle \, dy \\
&\geq \int_r^R \frac{-\xi_{\varepsilon,+}(B_\rho)}{\rho^n} \, d\rho \\
&\quad + \frac{1}{c_0} \int_{B_R \setminus B_r} \frac{\varepsilon \langle \nabla u_\varepsilon, y \rangle^2}{|y|^{n+1}} - \frac{1}{n-1} \left((n-1) \varepsilon \frac{\langle y, \nabla u_\varepsilon \rangle^2}{|y|^{2(n-1)}} + \frac{1}{4(n-1)\varepsilon} v_\varepsilon^2 \right) \, dy \\
&\quad - \frac{1}{c_0 r^{n-1}} \int_{B_r} \lambda \frac{\varepsilon \langle y, \nabla u_\varepsilon \rangle^2}{2|y|^2} + \frac{1}{2\lambda(n-1)^2 \varepsilon} v_\varepsilon^2 \, dy \\
&\quad - \frac{1}{c_0 R^{n-1}} \int_{B_R} \lambda \frac{\varepsilon \langle y, \nabla u_\varepsilon \rangle^2}{2|y|^2} + \frac{1}{2\lambda(n-1)^2 \varepsilon} v_\varepsilon^2 \, dy \\
&\geq \int_r^R \frac{-\xi_{\varepsilon,+}(B_\rho)}{\rho^n} \, d\rho - \frac{1}{4(n-1)^2} \int_{B_R \setminus B_r} \frac{1}{\varepsilon} v_\varepsilon^2 \, dy \\
&\quad - \lambda f(r) - \frac{1}{2\lambda} \frac{r^2}{(n-1)^2 r^{n-1}} \int_{B_r} \frac{v_\varepsilon^2}{\varepsilon} \, dy \\
&\quad - \lambda f(R) - \frac{1}{2\lambda} \frac{R_0^2}{(n-1)^2 R^{n-1}} \int_{B_R} \frac{v_\varepsilon^2}{\varepsilon} \, dy.
\end{aligned}$$

where $\lambda \in (0, 1)$. Here we used that $n = 2, 3$ to obtain that $2(n-1) \leq n+1$, so that $|y|^{n+1} \leq |y|^{2(n-1)}$ for all $|y|$ if $n = 3$ and for $|y| \leq 1$ if $n = 2$. When we bring all the relevant terms to the other side, this shows that

$$\begin{aligned}
(1 + \lambda) f(R) - (1 - \lambda) f(r) &\geq - \int_r^R \frac{\xi_{\varepsilon,+}(B_\rho)}{\rho^n} \, d\rho - \frac{1}{4(n-1)^2} \alpha_\varepsilon(B_R \setminus B_r) \\
&\quad - \frac{r^{3-n}}{2\lambda(n-1)^2} \alpha_\varepsilon(B_r) - \frac{R_0^2}{2\lambda(n-1)^2 R^{n-1}} \alpha_\varepsilon(B_R).
\end{aligned}$$

Setting $\lambda = 1/2$ and multiplying by two proves the Lemma. \square

Remark 3.7. If $n = 3$, we may let $R \rightarrow \infty$ and subsequently $\varepsilon \rightarrow 0$, $r \rightarrow 0$ and finally $\lambda \rightarrow 0$ that we have

$$\limsup_{r \rightarrow 0} r^{1-n} \mu(B(x, r)) \leq \frac{1}{4(n-1)^2} \alpha(\bar{\Omega})$$

at every point $x \in \mathbb{R}^3$ such that $\alpha(\{x\}) = 0$ (i.e. when $\lim_{r \rightarrow 0} \alpha(B_r) = 0$). Using the results of [RS06], μ is an integral varifold, so this yields a Li-Yau-type [LY82] inequality

$$\theta^*(\mu, x) = \limsup_{r \rightarrow 0} \frac{\mu(B(x, r))}{\pi r^2} \leq \frac{1}{16\pi} \alpha(\bar{\Omega}).$$

This inequality is usually found with a 4 in place of the 16 which stems from a different normalisation of the mean curvature and $\mathcal{W}(\mu)$ in the place of α , see also the proof of [Top98, Lemma 1] or [KS12, Proposition 2.1.1].

In $n = 2$ dimensions, we cannot do this since we had to assume $R \leq 1$. Indeed, an inequality of this type cannot hold since circles with large enough radii have arbitrarily small elastic energy. Still, setting $R = 1$, a similar bound on the multiplicity in terms of $\bar{\alpha}$ and S can still be obtained.

The version we will use of the above inequality is the simplified expression

$$r^{1-n} \mu_\varepsilon(B(x, r)) \leq 3 R^{1-n} \mu_\varepsilon(B(x, R)) + 3 \alpha_\varepsilon(B(x, R)) + 2 \int_r^R \frac{\xi_{\varepsilon,+}(B(x, \rho))}{\rho^n} d\rho.$$

This holds generally if $n = 3$, and when $R \leq 1$ if $n = 2$. Furthermore, we have the following estimate for the positive part of the discrepancy measures.

Lemma 3.8. [RS06, Lemma 3.1] *Let $n = 2, 3$. Then there are $\delta_0 > 0, M \in \mathbb{N}$ such that for all $0 < \delta \leq \delta_0, 0 < \varepsilon \leq \rho$ and*

$$\rho_0 := \max\{2, 1 + \delta^{-M} \varepsilon\} \rho$$

we have

$$\begin{aligned} \rho^{1-n} \xi_{\varepsilon,+}(B(x, \rho)) &\leq C \delta \rho^{1-n} \mu_\varepsilon(B(x, 2\rho)) + C \delta^{-M} \varepsilon^2 \rho^{1-n} \int_{B(x, \rho_0)} \frac{1}{\varepsilon} v_\varepsilon^2 dx \\ &\quad + C \delta^{-M} \varepsilon^2 \rho^{1-n} \int_{B(x, \rho_0) \cap \{|u_\varepsilon| > 1\}} \frac{1}{\varepsilon^3} W'(u_\varepsilon)^2 dx + \frac{C \varepsilon \delta}{\rho}. \end{aligned}$$

The following lemma is the key ingredient in order to obtain our required convergence results. In a slight abuse of notation, we will denote the functionals defined by the same formulas by $\mathcal{W}_\varepsilon, S_\varepsilon$ again, although they are given on spaces over $B_1 := B(0, 1)$, not Ω .

Lemma 3.9. *Let $n = 2, 3, \theta \in (0, 1), 0 < \eta < 1/2$. Consider the subsets*

$$Y^2 := \{u \in W^{2,2}(B_1) : |u(0)| \leq \theta\}$$

for $n = 2$ dimensions and

$$Y_\varepsilon^3 := \left\{ u \in W^{2,2}(B_1) : |u(0)| \leq \theta \text{ and } \alpha_\varepsilon(B_{\sqrt{\varepsilon}}) + \int_{B_{\sqrt{\varepsilon}} \cap \{|u_\varepsilon| > 1\}} \frac{W'(u)^2}{\varepsilon^3} dx \leq \varepsilon^\eta \right\}.$$

Define $F_\varepsilon : W^{2,2}(B_1) \rightarrow [0, \infty)$ as

$$\mathcal{F}_\varepsilon(u) = \mathcal{W}_\varepsilon(u) + S_\varepsilon(u).$$

Then $\theta_0 := \liminf_{\varepsilon \rightarrow 0} \inf_{u \in Y^2} \mathcal{F}_\varepsilon(u) > 0$ if $n = 2$ and

$$\theta_0 := \liminf_{\varepsilon \rightarrow 0} \inf_{u \in Y_\varepsilon^3} \mathcal{F}_\varepsilon(u) > 0$$

if $n = 3$. The same works if instead $u(0) \geq 1/\theta$.

Proof: By Hölder continuity, the condition that $|u(0)| \leq \theta$ leads to the creation of an infinitesimal diffuse mass density $\varepsilon^{1-n} \mu_\varepsilon(B_\varepsilon) \geq c_{n, \bar{\alpha}, \theta}$. We will use the monotonicity formula to integrate this up to show that if $\alpha = 0$, macroscopic mass is created as well. In three dimensions, there is an additional technical complication which forces us to make two steps, one from the ε -scale to the length scale of $\sqrt{\varepsilon}$ and a second one to the original scale.

Step 1. In a first step we show that the diffuse mass densities are uniformly bounded on large enough length scales. For $x \in \bar{\Omega}$, set $f_\varepsilon(\rho) := \rho^{1-n} \mu_\varepsilon(B(x, \rho))$. Without loss of generality, we may assume that $f_\varepsilon(1) \leq 1$ for small enough $\varepsilon > 0$ and that $\alpha_\varepsilon(B_{3/4}) + \int_{B_{3/4} \cap \{|u_\varepsilon| > 1\}} \frac{1}{\varepsilon^3} W'(u_\varepsilon)^2 dx \leq 1$ due to [RS06, Proposition 3.5] (included here as Proposition 3.3).

Take $\delta = \log(\varepsilon)^{-2}$ in Lemma 3.8 to obtain from Lemma 3.6 that for $0 < r < R = 1$ we have

$$\begin{aligned}
f_\varepsilon(r) &\leq 3f_\varepsilon(R) + 3\alpha_\varepsilon(B_R) + 2 \int_r^R \frac{\xi_{\varepsilon,+}(B_\rho)}{\rho^n} d\rho \\
&\leq C_{\bar{\alpha},n} + C \int_r^R \frac{1}{\log(\varepsilon)^2} \frac{f_\varepsilon(2\rho)}{\rho} + \frac{\varepsilon}{\log(\varepsilon)^2 \rho^2} \\
&\quad + \varepsilon^2 \log(\varepsilon)^{2M} \rho^{-n} \left(\alpha_\varepsilon(B_\rho) + \int_{B_\rho \cap \{|u_\varepsilon| > 1\}} \frac{W'(u_\varepsilon)^2}{\varepsilon^3} dx \right) d\rho \\
&\leq C_{\bar{\alpha},n} + \frac{C}{\log(\varepsilon)^2} \int_r^R \frac{f_\varepsilon(2\rho)}{\rho} d\rho + \frac{C\varepsilon}{\log(\varepsilon)^2} \left(\frac{1}{r} - \frac{1}{R} \right) \\
&\quad + \frac{C\varepsilon^2 \log(\varepsilon)^{2M}}{n-1} (r^{1-n} - R^{1-n}) \\
&\leq C_{\bar{\alpha},n} + \frac{C}{\log(\varepsilon)^2} \int_r^{2R} \frac{f_\varepsilon(\rho)}{\rho} d\rho
\end{aligned}$$

for a uniform constant $C_{\bar{\alpha},n}$ for $r \geq \varepsilon$ if $n = 2$ and for $r \geq \sqrt{\varepsilon}$ if $n = 3$. We use Grönwall's inequality backwards in time to deduce that

$$f_\varepsilon(r) \leq C_{\bar{\alpha},n} \exp\left(\frac{C}{\log(\varepsilon)^2} \int_r^{2R} \frac{1}{\rho} d\rho\right) \leq C_{\bar{\alpha},n}$$

on $[\varepsilon, \infty)$ if $n = 2$ and on $[\sqrt{\varepsilon}, \infty)$ if $n = 3$.

Step 2. If $n = 3$ and additionally

$$\alpha_\varepsilon(B_{\sqrt{\varepsilon}}) + \int_{B_{\sqrt{\varepsilon}} \cap \{|u_\varepsilon| > 1\}} \frac{W'(u)^2}{\varepsilon^3} dx \leq \varepsilon^\eta,$$

we can estimate the terms in the second line more sharply to obtain uniform boundedness of $f_\varepsilon(r)$ also for $\varepsilon \leq r \leq R = \sqrt{\varepsilon}$.

Step 3. Now we turn to the proof of the statement. For a contradiction, assume that $(\alpha_\varepsilon + \mu_\varepsilon)(B_1) \rightarrow 0$ for a suitable sequence u_ε . The functions u_ε are $C^{0,1/2}$ -Hölder continuous with Hölder constant $C/\sqrt{\varepsilon}$ for a uniform $C \geq 0$ on $B_{1/2}$, as can be obtained like in Lemma 3.1. The only difference is that we need to use [RS06, Proposition 3.5] (included here as Proposition 3.3) to estimate $\int_{\{|u_\varepsilon| > 1\}} \frac{1}{\varepsilon^3} W'(u_\varepsilon)^2 dx$ due to the lack of boundary values. It follows that $u(x) \leq \frac{1+\theta}{2}$ for $x \in B(0, c\varepsilon)$ for some uniform $c > 0$, which implies

$$\varepsilon^{1-n} \mu_\varepsilon(B_\varepsilon) \geq \frac{1}{\varepsilon^n} \int_{B(0, c\varepsilon)} W\left(\frac{1+\theta}{2}\right) dx =: c_{n,\bar{\alpha},\theta} > 0.$$

In the following, we will assume $n = 3$. The two-dimensional case follows with an easier argument of the same type. First we deduce that

$$\begin{aligned}
f_\varepsilon(\varepsilon) &\leq 3f_\varepsilon(\sqrt{\varepsilon}) + 3\alpha_\varepsilon(B_{\sqrt{\varepsilon}}) \\
&\quad + C \int_\varepsilon^{\sqrt{\varepsilon}} \frac{1}{\log(\varepsilon)^2 \rho} + \frac{\varepsilon}{\log(\varepsilon)^2 \rho^2} + \varepsilon^2 \log(\varepsilon)^{2M} \rho^{-n} \varepsilon^\beta d\rho.
\end{aligned}$$

Since the second line goes to zero as $\varepsilon \rightarrow 0$ and $\alpha_\varepsilon \rightarrow 0$ by assumption, we deduce that $f_\varepsilon(\sqrt{\varepsilon}) \geq c_{n,\bar{\alpha},\theta}/4$ for all sufficiently small $\varepsilon > 0$. Finally, we obtain

$$\begin{aligned} f_\varepsilon(\sqrt{\varepsilon}) &\leq 3f_\varepsilon(1) + 3\alpha_\varepsilon(B_1) \\ &\quad + C \int_{\sqrt{\varepsilon}}^1 \frac{1}{\log(\varepsilon)^2 \rho} + \frac{\varepsilon}{\log(\varepsilon)^2 \rho^2} + \varepsilon^2 \log(\varepsilon)^{2M} \rho^{-n} d\rho. \end{aligned}$$

Again, the terms in the second line vanish with $\varepsilon \rightarrow 0$ and $\alpha_\varepsilon(B_1) \rightarrow 0$ by assumption, thus

$$f_\varepsilon(1) \geq \frac{c_{n,\bar{\alpha},\theta}}{16}$$

for all sufficiently small $\varepsilon > 0$. But this contradicts the assumption that $(\alpha_\varepsilon + \mu_\varepsilon)(B_1) \rightarrow 0$, so we are done. The case $u(0) \geq 1/\theta$ follows similarly in two dimensions and by an immediate contradiction to the definition of Y_ε^3 in three dimensions. \square

The estimate above is not sharp, but suffices for our purposes. After this Lemma, everything is in place to show uniform convergence of phase fields in two dimensions in section 3.4, while further results will be needed for Hausdorff convergence of the transition layers. In a very weak phrasing, Lemma 3.10 suffices, more precise versions (and the application to connectedness) need the entire section 3.3. The reader mainly interested in the convergence of phase fields may skip ahead now.

3.3. Auxiliary Results II. In this section, we will derive technical results concerning how phase field approximations interact with the function ϕ as needed for the functional C_ε to impose connectedness. While the previous section focused on estimates away from the interface, here we investigate the structure of transition layers close to $\text{spt}(\mu)$. The following Lemma is a special case of [RS06, Proposition 3.4] with a closer attention to constants and the limit $\varepsilon \rightarrow 0$ already taken.

Lemma 3.10. *Let $x \in \mathbb{R}^n$, $r, \delta > 0$, $0 < \tau < 1 - 1/\sqrt{2}$. Then*

$$\limsup_{\varepsilon \rightarrow 0} \mu_\varepsilon(\{u_\varepsilon \geq 1 - \tau\} \cap B(x, r)) \leq 4\tau \mu(B(x, r + \delta)).$$

For all $x \in \mathbb{R}^n$, there are only countably many radii $r > 0$ such that $\mu(\partial B(x, r)) > 0$. This follows from the fact that μ is finite and that there are at most finitely many radii such that $\mu(\partial B(x, r)) \geq 1/k$, so that the union of those sets is countable. Thus for any $r > 0$ there is $t \in (0, r)$ such that $\mu(\partial B(x, t)) = 0$. Letting $\delta \rightarrow 0$ at such a radius t (and using that the discrepancy measures go to zero) gives us the following result (compare also [DMR14, Lemma 9]).

Corollary 3.11. *For all $x \in \text{spt}(\mu)$, $r > 0$ and $\tau < 1/8$ we have*

$$\liminf_{\varepsilon \rightarrow 0} \frac{1}{\varepsilon} \mathcal{L}^n(\{|u_\varepsilon| < 1 - \tau\} \cap B(x, r)) > 0.$$

The following arguments rely more on the rectifiable structure of the measure μ that we are approximating. Specifically, we introduce the diffuse normal direction by

$$\nu_\varepsilon := \frac{\nabla u_\varepsilon}{|\nabla u_\varepsilon|}$$

when $\nabla u_\varepsilon \neq 0$ and 0 else. To work with varifolds, we introduce the Grassmannian $G(n, n-1)$ of $n-1$ -dimensional subspaces of \mathbb{R}^n . We refer readers unfamiliar with varifolds or countably rectifiable sets to the excellent source [Sim83]; an introduction with other focus which is easier to find and covers most results relevant for us is [KP08]. Recall the following result.

Lemma 3.12. [RS06, Propositions 4.1, 5.1] *Define the $n - 1$ -varifold $V_\varepsilon := \mu_\varepsilon \otimes \nu_\varepsilon$ by*

$$V_\varepsilon(f) = \int_{\mathbb{R}^n \times G(n, n-1)} f(x, \langle \nu_\varepsilon \rangle^\perp) d\mu_\varepsilon \quad \forall f \in C_c(\mathbb{R}^n \times G(n, n-1)).$$

Then there is an integral varifold V such that $V_\varepsilon \rightarrow V$ as Radon measures on $\mathbb{R}^n \times G(n, n-1)$ (varifold convergence). The limit satisfies

$$\mu_V = \mu, \quad H_\mu^2 \mu \leq \alpha$$

where μ_V is the mass measure of V and H_μ denotes the generalised mean curvature of μ . In particular, $\mathcal{W}(\mu) \leq \bar{\alpha}$.

The following result is a suitably adapted version of [RS06, Proposition 5.5] for our purposes. It shows that given small discrepancy measures and small oscillation of the gradient, a bounded energy sequence looks very much like an optimal interface in small balls. Using our improved bounds from Lemma 3.1, we can drop most of their technical assumptions.

Lemma 3.13. *Let $\delta, \tau > 0$ and denote $\nu_{\varepsilon, n} = \langle \nu_\varepsilon, e_n \rangle$. Then there exist $0 < L < \infty$ depending on δ and τ only and $\gamma > 0$ depending on $\bar{\alpha}, \delta$ and τ such that the following holds for all $x \in \mathbb{R}^n$. If*

- (1) $|u_\varepsilon(x)| \leq 1 - \tau$ and
- (2) $|\xi_\varepsilon|(B(x, 4L\varepsilon)) + \int_{B(x, 4L\varepsilon)} 1 - \nu_{\varepsilon, n}^2 d\mu_\varepsilon \leq \gamma(4L\varepsilon)^{n-1}$

then also

- *the blow up $\tilde{u}_\varepsilon(y) = u_\varepsilon(x + \varepsilon y)$ is $C^{0,1/4}$ -close to an optimal profile as introduced in the construction of the lim sup-inequality:*

$$|\pm \tilde{u}_\varepsilon - q(y_n - t_1)|_{0,1/4, B(0, 3L)} < \delta$$

holds, where $t_1 := q^{-1}(u_\varepsilon(x))$.

- $|u(\hat{x}, x_n + t)| \geq 1 - \tau/2$ for all $L\varepsilon \leq |t| \leq 3L\varepsilon$, where $\hat{x} = (x_1, \dots, x_{n-1})$ and u changes sign in between.

Proof. Without loss of generality, we may assume that $x = 0$ and write $B_r := B(0, r)$. Recall that $q'(t) = \sqrt{2W(q(t))}$ and $\lim_{t \rightarrow \pm\infty} q(t) = \pm 1$. Thus we can pick $L > 0$ such that $|q(t)| \geq 1 - \tau/4$ for all $t > L$.

Assume for a contradiction that there is no constant $\gamma > 0$ such that the results of the Lemma hold. Then for $\gamma^j \rightarrow 0$, there must be a sequence u_ε^j such that $|u_\varepsilon^j(0)| \leq 1 - \tau$, $\mathcal{W}_\varepsilon(u_\varepsilon^j) \leq \bar{\alpha} + 1$ and

$$|\xi_\varepsilon^j|(B_{4L\varepsilon}) + \int_{B_{4L\varepsilon}} 1 - \nu_{\varepsilon, n}^2 d\mu_\varepsilon \leq \gamma^j (4L\varepsilon)^{n-1},$$

but the conclusions of the Lemma do not hold. Considering the blow ups $\tilde{u}^j : B_{4L} \rightarrow \mathbb{R}$ with $\tilde{u}^j(y) = u_\varepsilon^j(\varepsilon y)$ we obtain

$$\|\tilde{u}^j\|_{2,2, B_{3L}} \leq C_{\bar{\alpha}, n, L}$$

like in Lemma 3.1, hence there is $\tilde{u} \in W^{2,2}(B_{3L})$ such that

$$\tilde{u}^j \rightharpoonup \tilde{u} \quad \text{in } W^{2,2}(B_{3L}).$$

Since $W^{2,2}$ embeds compactly into $W^{1,2}$ and L^4 , we see that

$$\begin{aligned} \int_{B_{3L}} \left| |\nabla \tilde{u}|^2/2 - W(\tilde{u}) \right| dx &= \lim_{j \rightarrow \infty} \int_{B_{3L}} \left| |\nabla \tilde{u}^j|^2/2 - W(\tilde{u}^j) \right| dx \\ &\leq \lim_{j \rightarrow \infty} \varepsilon^{1-n} \xi_\varepsilon^j(B_{4L\varepsilon}) \\ &\leq \liminf_{j \rightarrow \infty} (4L)^{n-1} \gamma^j \\ &= 0 \end{aligned}$$

and when we set $\hat{\nabla}u = (\partial_1 u, \dots, \partial_{n-1} u)$, we get

$$\begin{aligned}
 \int_{B_{3L}} |\hat{\nabla}u| dx &= \lim_{j \rightarrow \infty} \int_{B_{3L}} |\hat{\nabla}\tilde{u}^j| dx \\
 &= \lim_{j \rightarrow \infty} \int_{B_{3L}} \sqrt{|\nabla\tilde{u}_j|^2 - |\partial_n \tilde{u}_j|^2} dx \\
 &\leq \liminf_{j \rightarrow \infty} \int_{B_{4L}} |\nabla\tilde{u}^j| \sqrt{1 - (\tilde{\nu}_n^j)^2} dx \\
 &\leq \liminf_{j \rightarrow \infty} (\omega_n (4L)^n)^{1/2} \left(\int_{B_{4L}} |\nabla\tilde{u}^j|^2 (1 - (\tilde{\nu}_n^j)^2) dx \right)^{\frac{1}{2}} \\
 &\leq \liminf_{j \rightarrow \infty} \sqrt{8L\omega_n} \left(\varepsilon^{1-n} \int_{B_{4L\varepsilon}} 1 - (\nu_n^j)^2 d\mu_\varepsilon \right)^{\frac{1}{2}} \\
 &\leq \liminf_{j \rightarrow \infty} \sqrt{8L\omega_n} \gamma^j \\
 &= 0.
 \end{aligned}$$

Thus we can see that

$$|\nabla\tilde{u}|^2 = 2W(\tilde{u}), \quad \nabla\tilde{u} = (0, \dots, 0, \partial_n \tilde{u}).$$

Clearly, this means that $\tilde{u}(y) = p(y_n)$ for a function p with $p' = \pm\sqrt{2W(p)}$. Using that $|\tilde{u}(0)| \leq 1 - \tau$ and the Picard-Lindelöf theorem on the uniqueness of the solutions to ODEs, we see that $p(y_n) = \pm q(y_n - \bar{y})$ for some $\bar{y} \in \mathbb{R}$ which can easily be fixed by the initial condition for $p(0)$.

Since weak $W^{2,2}$ -convergence implies strong $C^{0,1/4}$ -convergence in $n = 2, 3$ dimensions, we see that there is $j \in \mathbb{N}$ such that the claim of the Lemma holds for u_ε^j contradicting our assumption. Thus the Lemma is proven. \square

To deal with the rectifiable sets in the next section more easily we prove a structure result for rectifiable sets. The result seems standard, but we have been unable to find a reference for it. As usual, we call a function on a closed set differentiable if it admits a differentiable extension to a larger open set.

Lemma 3.14. *Let M be a countably k -rectifiable set in \mathbb{R}^n . Denote by B the closed unit ball in k dimensions. Then there exist injective C^1 -functions $f_i : B \rightarrow \mathbb{R}^n$ with $\nabla f_i \neq 0$ on B such that*

$$\mathcal{H}^k \left(M \setminus \bigcup_{i=1}^{\infty} f_i(B) \right) = 0$$

and such that $f_i(B) \cap f_j(B) = \emptyset$ for all $i \neq j$.

Proof. According to [KP08, Lemma 5.4.2] or [Sim83, Lemma 11.1] there is a countable collection of C^1 -maps $g_i : \mathbb{R}^k \rightarrow \mathbb{R}^n$ such that

$$M \subset N \cup \bigcup_{i=1}^{\infty} g_i(\mathbb{R}^k)$$

where $\mathcal{H}^k(N) = 0$. Without loss of generality, N is assumed to be disjoint from the other sets. First we need to make the individual maps g_i one-to-one. To do that, we define the set where injectivity fails in a bad way:

$$A_i := \{x \in \mathbb{R}^k \mid \forall r > 0 \exists y \in B(x, r) \text{ such that } g_i(x) = g_i(y)\}.$$

Due to the failure of local injectivity, we see that the Jacobian $J_{g_i}(x)$ vanishes on A_i . Since g_i is a C^1 -function, the set $D_i := J_{g_i}^{-1}(0)$ is closed and by the Morse-Sard Lemma [Fed69, 3.4.3] then

$$\mathcal{H}^k(g_i(D_i)) = 0.$$

Set $U_i := \mathbb{R}^k \setminus D_i$. Now as in [EG92, Chapter 1.5, Corollary 2] we can use Vitali's covering theorem [EG92, Chapter 1.5, Theorem 1] to obtain a countable selection of closed balls B_i^j such that f_i is injective with non-vanishing gradient on B_i^j for all $j \in \mathbb{N}$ and

$$\mathcal{L}^k \left(U_i \setminus \bigcup_{j=1}^{\infty} B_i^j \right) = 0.$$

Since the boundary of a k -ball has Hausdorff dimension $k - 1$, we could equally well take open balls. Since C^1 -functions map sets of \mathcal{L}^k -measure zero to sets of \mathcal{H}^k -measure zero, we have shown that we can write

$$M \subset \tilde{N} \cup \bigcup_{j=1}^{\infty} \tilde{g}_i(B^\circ)$$

where $\mathcal{H}^k(\tilde{N}) = 0$, $\tilde{g}_i : B \rightarrow \mathbb{R}^n$ is one-to-one, C^1 , and has a non-vanishing gradient everywhere on the closed ball B . The functions \tilde{g}_m are obtained by rescaling suitable restrictions of g_i from B_i^j to the unit ball. Finally, we have to cut out the sets that get hit by more than one function \tilde{g}_m . Inductively, we define

$$\tilde{U}_m := B^\circ \setminus \tilde{g}_m^{-1} \left(\bigcup_{l=1}^{m-1} \tilde{g}_l(B) \right).$$

Finally, we use Vitali's Lemma again to pick collections of closed balls \tilde{B}_m^l such that

$$\mathcal{L}^k \left(\tilde{U}_m \setminus \bigcup_{l=1}^{\infty} \tilde{B}_m^l \right) = 0.$$

Rescaling the restricted functions from these balls and translating to the unit ball gives us the result. \square

The proof of the following Lemma strongly resembles that of the integrality of μ in [RS06, Lemma 4.2]. It is technically easier because we do not need the multi-layeredness of the approximating interfaces, but entails different complications because blow up to the tangent space is not possible here. Instead we use the preceding Lemma for a similar result on local C^1 -flatness.

Lemma 3.15. *Let $\phi \in C^0(\mathbb{R})$ such that $\phi \geq 0$ and $\int_{-1}^1 \phi(u) du > 0$. If $x \in \text{spt}(\mu)$, then*

$$\liminf_{\varepsilon \rightarrow 0} \frac{1}{\varepsilon} \int_{B(x, \varepsilon)} \phi(u_\varepsilon) dx > 0$$

for all $r > 0$.

Proof. Step 1. As usual, we assume that $x = 0$, $\mu(\partial B(x, r/2)) = 0$ and denote $B = B(x, r/2)$. This means that all the ε -balls of positive integral we are going to find will actually lie in $B(x, r)$ and is a purely technical condition. Let ζ be a small constant to be specified later. For further use, denote by \hat{B} the closed unit ball in \mathbb{R}^{n-1} .

As μ is an integral varifold, we know that $\text{spt}(\mu)$ is rectifiable. This means that there are countably many C^1 -functions $f_i : \hat{B} \rightarrow \mathbb{R}^n$ like in Lemma 3.14 such that

$$\text{spt}(\mu) \subset M_0 \cup \bigcup_{i=1}^{\infty} f_i(\hat{B}), \quad \mathcal{H}^{n-1}(M_0) = 0 \quad f_i(\hat{B}) \cap f_j(\hat{B}) = \emptyset$$

for $i \neq j$. Since μ has second integrable mean curvature $\mu|H_\mu^2 \leq \alpha$, we can further use the Li-Yau inequality from Remark 3.7 to bound the maximum multiplicity of μ uniformly by

$$\theta_{\max} \leq \frac{\alpha(\bar{\Omega})}{16\pi},$$

at least \mathcal{H}^{n-1} -almost everywhere. Now since $\mathcal{H}^{n-1}(\text{spt}(\mu)) < +\infty$ we can find $N \in \mathbb{N}$ such that

$$\mathcal{H}^{n-1} \left((\text{spt}(\mu) \cap B) \setminus \bigcup_{i=1}^N f_i(\hat{B}) \right) < \frac{\zeta}{\theta_{\max}}.$$

Since f_i is injective and has non-vanishing tangent maps everywhere, $M := \bigcup_{i=1}^N f_i(\hat{B}^\circ)$ is a C^1 -manifold. We observe that

$$\mathcal{H}^{n-1}(\text{spt}(\mu) \cap B \setminus M) < \frac{\zeta}{\theta_{\max}}$$

and hence

$$\mu(B \setminus M) < \zeta.$$

Since the maps in question are smooth and the unit discs are orientable, for every i we can pick a continuous unit normal field to $f_i(\hat{B})$ (e.g. using cross products). Since the discs are compact and disjoint (thus a positive distance apart), the fields defined on each disc separately induce a continuous unit vector field on the union of their closures.

Now we use the Tietze-Urysohn extension theorem to obtain a vector field X on B such that $X = \nu_M$ on M and projecting on the unit ball we ensure $|X| \leq 1$. After an easy modification, we may assume that $|X| = 1$ on a neighbourhood of M . We then define

$$G : \mathbb{R}^n \times G(n, n-1) \rightarrow \mathbb{R}, \quad G(x, S) = \langle X_x, \nu_S \rangle^2$$

where ν_S is one of the unit normals to S . Note that G is continuous since X is. Using the non-negativity of G and the fact that $T_x\mu = T_xM$ for \mathcal{H}^{n-1} -almost every $x \in M \cap \text{spt}(\mu)$ we interpret μ as dual to $C^0(\mathbb{R}^n \times G(n, n-1))$ and observe

$$\begin{aligned} \langle \mu, G \rangle &= \int_{\text{spt}(\mu)} \theta(x) G(x, T_x\mu) d\mathcal{H}^{n-1} \\ &\geq \int_{\text{spt}(\mu) \cap M} \theta(x) G(x, T_x\mu) d\mathcal{H}^{n-1} \\ &\geq \int_{\text{spt}(\mu) \cap M} \theta(x) G(x, T_xM) d\mathcal{H}^{n-1} \\ &= \int_{\text{spt}(\mu) \cap M} \theta(x) d\mathcal{H}^{n-1} \\ &= \mu(M) \\ &\geq \mu(B) - \zeta. \end{aligned}$$

Step 2. By varifold convergence, we know that $\lim_{\varepsilon \rightarrow 0} \langle \mu_\varepsilon, G \rangle = \langle \mu, G \rangle \geq \mu(B) - \zeta$, and $|X|, |\nu_\varepsilon| \leq 1$ so

$$\begin{aligned} \limsup_{\varepsilon \rightarrow 0} \int_B |1 - \langle \nu_\varepsilon, X \rangle|^2 d\mu_\varepsilon &= \limsup_{\varepsilon \rightarrow 0} \int_B 1 - \langle \nu_\varepsilon, X \rangle^2 d\mu_\varepsilon \\ &\leq \limsup_{\varepsilon \rightarrow 0} (\mu_\varepsilon(B) - \langle \mu_\varepsilon, G \rangle) \\ &\leq \zeta. \end{aligned}$$

For $\gamma, \varepsilon, L > 0$ we define the set

$$U_{\varepsilon, \gamma, L} := \left\{ x \in B \mid \frac{1}{(4L\varepsilon)^{n-1}} \int_{B(x, 4L\varepsilon)} |1 - \langle \nu_\varepsilon, X \rangle|^2 d\mu_\varepsilon > \gamma/4 \right\}.$$

Let x_1, \dots, x_K be points in $U_{\varepsilon, \gamma, L}$ being maximal for the property that the balls $B(x_i, 4L\varepsilon)$ are disjoint. Then by definition

$$\zeta \geq \int_B |1 - \langle \nu_\varepsilon, X \rangle|^2 d\mu_\varepsilon \geq \sum_{i=1}^K \int_{B(x_i, 4L\varepsilon)} |1 - \langle \nu_\varepsilon, X \rangle|^2 d\mu_\varepsilon \geq K (4L\varepsilon)^{n-1} \gamma/4.$$

At the same time, we know that the balls $B(x_i, 8L\varepsilon)$ cover $U_{\varepsilon, \gamma, L}$ because otherwise we could bring in more disjoint balls, therefore

$$\begin{aligned} \frac{\mathcal{L}^n(U_{\varepsilon, \gamma, L})}{\varepsilon} &\leq K \frac{\omega_n (8L\varepsilon)^n}{\varepsilon} \\ &\leq \frac{4\zeta}{\gamma (4L\varepsilon)^{n-1}} \frac{\omega_n (8L\varepsilon)^n}{\varepsilon} \\ &= 2^{n+4} \omega_n L \zeta / \gamma. \end{aligned}$$

For a given γ , we choose $\zeta = \zeta(\gamma)$ such that this is $\leq \mu(B)/4$.

Step 3. Knowing that $|\xi_\varepsilon|(B) \rightarrow 0$, we can use the same argument as in the second step to show for

$$V_{\varepsilon, \gamma, L} := \left\{ x \in B \mid \frac{|\xi_\varepsilon|(B(x, 4L\varepsilon))}{(4L\varepsilon)^{n-1}} > \gamma/2 \right\}$$

the estimate

$$\frac{\mathcal{L}^n(V_{\varepsilon, \gamma, L})}{\varepsilon} \leq \mu(B)/4$$

for all sufficiently small $\varepsilon > 0$.

Step 4. Now choose U such that U is a neighbourhood of M on which $|X| = 1$ and $\tau > 0$ like in Corollary 3.11 satisfying

$$\liminf_{\varepsilon \rightarrow 0} \mu_\varepsilon(U \cap \{|u_\varepsilon| \leq 1 - \tau\}) \geq \frac{3\mu(B)}{4}.$$

This is easily achieved when $\mu(M) > 3\mu(B)/4$. Furthermore we take $\delta \ll 1$ suitably small for small deviations of the optimal interface to behave similarly enough, L and γ as in Lemma 3.13 and $\zeta = \zeta(\gamma)$. Using steps one through three, we see that

$$\begin{aligned} &\liminf_{\varepsilon \rightarrow 0} \frac{\mathcal{L}^n(\{|u_\varepsilon| \leq 1 - \tau\} \cap U \setminus (U_{\varepsilon, \gamma, L} \cup V_{\varepsilon, \gamma, L}))}{\varepsilon} \\ &\geq \liminf_{\varepsilon \rightarrow 0} \frac{\mathcal{L}^n(\{|u_\varepsilon| \leq 1 - \tau\} \cap U)}{\varepsilon} - \frac{\mathcal{L}^n(U_{\varepsilon, \gamma, L})}{\varepsilon} - \frac{\mathcal{L}^n(V_{\varepsilon, \gamma, L})}{\varepsilon} \\ &\geq 3\mu(B)/4 - \mu(B)/4 - \mu(B)/4 \\ &= \mu(B)/4. \end{aligned}$$

Using the argument of step 2, but this time in reverse, we can see that there are at least K points x_1, \dots, x_K in $\{|u_\varepsilon| \leq 1 - \tau\} \cap U \setminus (U_{\varepsilon, \gamma, L} \cup V_{\varepsilon, \gamma, L})$ such that the balls $B(x_i, 4L\varepsilon)$ are disjoint with

$$K \geq \frac{\mu(B)}{8^{n+1} L^n \varepsilon^{n-1}}.$$

Step 5. To apply Lemma 3.13, we must “freeze” the coefficients of the vector field X to a single unit vector. We compute

$$\begin{aligned}
 & \frac{1}{(4L\varepsilon)^{n-1}} \left| \int_{B(x_i, 4L\varepsilon)} (1 - \langle \nu_\varepsilon, X \rangle^2) - (1 - \langle \nu_\varepsilon, X_i \rangle^2) d\mu_\varepsilon \right| \\
 &= \frac{1}{(4L\varepsilon)^{n-1}} \left| \int_{B(x_i, 4L\varepsilon)} \langle \nu_\varepsilon, X_i \rangle^2 - \langle \nu_\varepsilon, X \rangle^2 d\mu_\varepsilon \right| \\
 &= \frac{1}{(4L\varepsilon)^{n-1}} \left| \int_{B(x_i, 4L\varepsilon)} \langle \nu_\varepsilon, X_i - X \rangle \langle \nu_\varepsilon, X_i + X \rangle d\mu_\varepsilon \right| \\
 &\leq |X_i + X|_{C^0(B(x_i, 4L\varepsilon))} \cdot |X_i - X|_{C^0(B(x_i, 4L\varepsilon))} \frac{1}{(4L\varepsilon)^{n-1}} \cdot \int_{B(x_i, 4L\varepsilon)} d\mu_\varepsilon \\
 &\leq 2C_{\bar{\alpha}, L, n} |X_i - X|_{C^0(B(x_i, 4L\varepsilon))}
 \end{aligned}$$

for all X_i such that $|X_i| \leq 1$. When we set $X_i = X(x_i)$, the last term converges to zero - so eventually it is smaller than $\gamma/4$ and

$$\frac{1}{(4L\varepsilon)^{n-1}} \int_{B(x_i, 4L\varepsilon)} 1 - \langle \nu_\varepsilon, X_i \rangle^2 d\mu_\varepsilon < \gamma/2.$$

Since $x_i \in U$, we finally see that $|X_i| = 1$ and Lemma 3.13 can be applied.

Step 6. Since u_ε is $C^{0,1/4}$ -close to a one-dimensional optimal profile on $B(x_i, 3L\varepsilon)$ which transitions from -1 to 1 , we see that for each $s \in (-(1-\tau), (1-\tau))$ there must be a point $y_i \in B(x_i, 3L\varepsilon)$ such that $u_\varepsilon(y_i) = s$. By Hölder continuity, we deduce that

$$\int_{B(x_i, 3L\varepsilon)} \phi(u_\varepsilon) dx \geq \bar{\theta} \varepsilon^n$$

for a constant $\bar{\theta}$ depending on the support of ϕ and on $\bar{\alpha}, n$ for the Hölder constant. Since the balls are disjoint by construction, we can add this up to

$$\begin{aligned}
 \frac{1}{\varepsilon} \int_B \phi(u_\varepsilon) dx &\geq \frac{1}{\varepsilon} \sum_{j=1}^M \int_{B(x_j, 3L\varepsilon)} \phi(u_\varepsilon) dx \\
 &\geq \frac{1}{\varepsilon} M \bar{\theta} \varepsilon^n \\
 &\geq \frac{\mu(B) \bar{\theta}}{8^{n+1} L^n} \\
 &> 0.
 \end{aligned}$$

This concludes the proof. \square

We need one final result from geometric measure theory before we move on to our main results. The Lemma is standard knowledge for smooth manifolds with a vastly different proof.

Lemma 3.16. *There is a universal constant $C > 0$ such that for all integral varifolds V with finite Willmore energy and compact connected support we have*

$$\mathcal{W}(V) \geq C \text{ if } n = 3 \text{ and } \quad \mathcal{W}(V) \geq \frac{C}{|V|(\mathbb{R}^2)} \text{ if } n = 2.$$

Proof. For a contradiction, assume that there is a sequence of varifolds V_i in \mathbb{R}^n for $n = 2, 3$ such that $|V_i|(\mathbb{R}^n) = 1$ and $\mathcal{W}(V_i) \leq 1/i$. We observe that connectedness already implies $\text{diam}(V_i) \leq$

$|V_i|(\mathbb{R}^n) = 1$ for $n = 2$. If $n = 3$, we can use techniques from [Sim93] to bound the diameter of $\text{spt}(V)$ in terms of its Willmore energy by

$$\text{diam}(V) \leq \frac{2}{\pi} \sqrt{\mathcal{W}(V) \cdot |V|(\mathbb{R}^3)},$$

see also [Top98, Lemma 1] for a derivation of this inequality and the Li-Yau inequality. The arguments are presented in the context of immersions but can easily be applied to integral varifolds with connected support. Thus we know that up to translation we can assume $\text{spt}(V_i) \subset B(0, 1)$, at least for large enough i . Then we may invoke Allard's compactness theorem to find that there is an integral varifold V such that $V_i \rightarrow V$ and

- $\text{spt}(V) \subset B(0, 1)$,
- $|V|(\mathbb{R}^n) = 1$, and
- $\mathcal{W}(V) = 0$.

But this contradicts the result of [MR14, Theorem 1] which shows that

$$\mathcal{W}(V) \geq c_n |V|(\mathbb{R}^n) > 0.$$

The proof is presented there for $n = 3$, but can easily be adapted $n = 2$ as well. \square

For $n = 3$, another approach would be to use the Li-Yau inequality, which would give us an explicit constant $C = 16\pi$. We chose this proof instead because it works for both $n = 2$ and $n = 3$ the same way.

3.4. Proof of the Main Results. Having dealt with the necessary auxiliary results, we can proceed to prove our main results. We use the terminology of Lemma 3.9.

Proof of Theorem 2.1. Let $\Omega' \Subset \mathbb{R}^n \setminus \text{spt}(\mu)$ and $\tau > 0$. Assume there is a sequence x_ε in Ω' such that $|u_\varepsilon(x_\varepsilon)| \leq 1 - \tau$ or $|u_\varepsilon(x_\varepsilon)| \geq 1 + \tau$. By definition, $x_\varepsilon \in \Omega$ and using compactness, there is $x \in \overline{\Omega' \cap \Omega}$ such that $x_\varepsilon \rightarrow x$.

Let $r > 0$ such that $B(x, 3r) \subset \mathbb{R}^n \setminus \text{spt}(\mu)$. Due to convergence, $B(x_\varepsilon, r) \subset B(x, 2r)$ for all sufficiently small $\varepsilon > 0$. If we use the rescaling property from Lemma 3.4 and the minimisation property of Lemma 3.9 for $n = 2$, we see that

$$\begin{aligned} \alpha(\overline{B(x, 2r)}) &\geq \frac{1}{r} \liminf_{\varepsilon \rightarrow 0} (r \alpha_\varepsilon(B(x_\varepsilon, r)) + r^{-1} \mu_\varepsilon(B(x_\varepsilon, r))) \\ &\geq \frac{1}{r} \liminf_{\varepsilon \rightarrow 0} \inf_{u \in Y^2} \mathcal{F}_\varepsilon(u) \\ &\geq \theta_0/r \end{aligned}$$

with $\hat{\varepsilon} = \varepsilon/r$ as in Lemma 3.4. Letting $r \rightarrow 0$, we obtain a contradiction. \square

If $n = 3$, neither the rescaling property of Lemma 3.4 nor the minimisation property of Lemma 3.9 hold in as strong formulations. As we sketched in Remark 2.6, uniform convergence is, in fact, false. It is still an open question in which sense phase fields converge away from the interface in three dimensions.

Proof of Theorem 2.2: We assume that $u_\varepsilon \rightarrow u$ strongly in $L^1(\Omega)$ and that $\mu_\varepsilon \rightarrow \mu$ for our sequence or a suitable subsequence (not relabelled). Let $D \Subset (-1, 1)$, then $u_\varepsilon^{-1}(D) \Subset \Omega$. We can consider $\overline{u_\varepsilon^{-1}(D)}$ instead without changing the Hausdorff limit to conform with standard approaches. By the usual compactness results (see e.g. [KP08, Theorem 1.6.6]), there is a compact set K such that a further subsequence of $u_\varepsilon^{-1}(D)$ converges to K in Hausdorff distance. K can be computed as the Kuratowski lower limit

$$K = \{x \in \mathbb{R}^n \mid \exists x_\varepsilon \in u_\varepsilon^{-1}(D) \text{ such that } x_\varepsilon \rightarrow x\}.$$

Step 1. We first show that $K \subset \text{spt}(\mu)$ if $n = 2$. Assume $x \in K \setminus \text{spt}(\mu)$. Then there exists $r > 0$ such that $B(x, r) \Subset \mathbb{R}^2 \setminus \text{spt}(\mu)$. This means already that $|u_\varepsilon| \rightarrow 1$ uniformly on $B(x, r)$ showing that $u_\varepsilon^{-1}(D) \cap B(x, r) = \emptyset$ for small enough ε , contradicting our assumption.

Step 2. Now we prove that $\text{spt}(\mu) \subset K$ in $n = 2, 3$ dimensions. It suffices to show that for $x \in \text{spt}(\mu)$, $s \in D$ and $r > 0$ we have $u_\varepsilon^{-1}(s) \cap B(x, r) \neq \emptyset$ for all sufficiently small ε , which implies Hausdorff convergence.

This is an easier version of the proof of Lemma 3.15. Again, we use Corollary 3.11 to show that we have a point $y \in B(x, r/2)$ to use Lemma 3.13 on and then use Lemma 3.13 to see that we get points in the pre-image of s close to y since u_ε is C^0 -close to an optimal interface in $B(y, 3L\varepsilon)$.

If $n = 2$, the uniqueness of the limit also shows convergence for the whole sequence. \square

Finally, we show that C_ε enforces connectedness.

Proof of Theorem 2.4: Let u_ε be a sequence such that $\mathcal{E}_\varepsilon(u_\varepsilon)$ is bounded. Then in particular $|\mu_\varepsilon(\Omega) - S| \leq \varepsilon^{\sigma/2}$, so $\mu_\varepsilon(\mathbb{R}^n) = \mu_\varepsilon(\Omega)$ is bounded and $\mu_\varepsilon \rightarrow \mu$ for some Radon measure μ – for this and other properties see [EG92, Chapter 1]. Clearly

$$\mu(\overline{\Omega}) \geq \limsup_{\varepsilon \rightarrow 0} \mu_\varepsilon(\overline{\Omega}) = S$$

and on the other hand

$$\mu(\overline{\Omega}) \leq \mu(\mathbb{R}^n) \leq \liminf_{\varepsilon \rightarrow 0} \mu_\varepsilon(\mathbb{R}^n) = S$$

so $\mu(\overline{\Omega}) = S$. If $U = \mathbb{R}^n \setminus \overline{\Omega}$, we have

$$\mu(U) \leq \liminf_{\varepsilon \rightarrow 0} \mu_\varepsilon(U) = 0,$$

so $\text{spt}(\mu) = \bigcap_{U \text{ open}, \mu(U)=0} U^c \subset \overline{\Omega}$. It remains to show that $\text{spt}(\mu)$ is connected. For a contradiction, assume that $\text{spt}(\mu)$ has at least two components. Since components are relatively closed, they are also compact. As $\text{spt}(\mu)$ is closed and contained in $\overline{\Omega}$, it is also compact. Since every connected component of $\text{spt}(\mu)$ by itself induces an integral varifold, Lemma 3.16 shows that there are only finitely many connected components of $\text{spt}(\mu)$. This means that also the complement of a component is relatively closed, hence compact.

Let C_1 be a connected component of $\text{spt}(\mu)$ and set $C_2 := \text{spt}(\mu) \setminus C_1$. Since C_1, C_2 are compact and disjoint, we see that

$$\delta := \text{dist}(C_1, C_2) = \min_{x \in C_1, y \in C_2} |x - y| > 0.$$

We define the disjoint open sets

$$U_1 := \{x \in \mathbb{R}^n \mid \text{dist}(x, C_1) < \delta/3\}, \quad U_2 := \{x \in \mathbb{R}^n \mid \text{dist}(x, C_2) < \delta/3\}.$$

Then we can consider two cases.

Case 1. We assume that there is $\beta < \kappa$ such that $\liminf_{\varepsilon \rightarrow 0} \varepsilon^{-\beta} \text{dist}^{F(u_\varepsilon)}(U_1, U_2) > 0$. Then

$$\begin{aligned} \liminf_{\varepsilon \rightarrow 0} \varepsilon^{-\beta} C_\varepsilon(u_\varepsilon) &\geq \liminf_{\varepsilon \rightarrow 0} \int_{U_1} \phi(u_\varepsilon(x)) \, dx \cdot \liminf_{\varepsilon \rightarrow 0} \int_{U_2} \phi(u_\varepsilon(y)) \, dy \\ &\quad \cdot \liminf_{\varepsilon \rightarrow 0} \varepsilon^{-\beta} \text{dist}^{F(u_\varepsilon)}(U_1, U_2) \\ &> 0 \end{aligned}$$

by our assumption and Lemma 3.15. Thus

$$\liminf_{\varepsilon \rightarrow 0} \varepsilon^{-\kappa} C_\varepsilon(u_\varepsilon) = \infty,$$

which clearly contradicts our original assumption.

Case 2. We assume that $1 < \beta < \kappa$ and that $\liminf_{\varepsilon \rightarrow 0} \varepsilon^{-\beta} \text{dist}^{F(u_\varepsilon)}(U_1, U_2) = 0$. Then at least for a suitable subsequence of $\varepsilon \rightarrow 0$ (not relabeled) we have points $x_\varepsilon \in \partial U_1$, $y_\varepsilon \in \partial U_2$ and connected sets K_ε such that

$$x_\varepsilon, y_\varepsilon \in K_\varepsilon, \quad \varepsilon^{-\beta} \int_{K_\varepsilon} F(u_\varepsilon) d\mathcal{H}^1 \leq 1.$$

Clearly

$$\mathcal{H}^1(\{F(u_\varepsilon) \geq 1\} \cap K_\varepsilon) \leq \varepsilon^\beta.$$

Without loss of generality we may assume that $K_\varepsilon \cap (U_1 \cup U_2) = \emptyset$, otherwise take a connected component of $K_\varepsilon \setminus (U_1 \cup U_2)$ meeting both sets. For the purpose of this proof we will make a few simplifying assumptions, namely that

$$\{\phi > 0\} = (\rho_1, \rho_2), \quad -1 < \theta_1 < \rho_1 < \rho_2 < \theta_2 < 1, \quad F \geq 2 \text{ outside } (\theta_1, \theta_2).$$

This allows us to neglect at least some constants and obviously has no effect on the structure of the functional. Now assume that there is $z_\varepsilon \in K_\varepsilon$ such that $u_\varepsilon(z_\varepsilon)$ does not lie in $[\theta_1, \theta_2]$. Since F is continuous and u_ε is Hölder continuous on ε -balls, there is a constant $c > 0$ such that

$$F(u_\varepsilon(z)) \geq 1 \quad \forall z \in B(z_\varepsilon, c\varepsilon).$$

But since K_ε is connected and not contained in $B(z_\varepsilon, \varepsilon)$, it must contain a point $w_\varepsilon \in \partial B(z_\varepsilon, c\varepsilon)$, otherwise we directly reach a contradiction. Furthermore, the connected component L_ε of $K_\varepsilon \cap B(z_\varepsilon, \varepsilon)$ containing z_ε and w_ε projects to $f(L_\varepsilon) = [0, c\varepsilon]$ under the one-Lipschitz map $f(z) = |z - z_\varepsilon|$. Thus

$$\mathcal{H}^1(L_\varepsilon) \geq \mathcal{H}^1(f(L_\varepsilon)) \geq c\varepsilon.$$

On the other hand, by construction $L_\varepsilon \subset K_\varepsilon \cap \{F(u_\varepsilon) \geq 1\}$, so

$$c\varepsilon \leq \mathcal{H}^1(L_\varepsilon) \leq \varepsilon^\beta.$$

For all sufficiently small $\varepsilon > 0$, this is a contradiction (as $\beta > 1$), and we see that $u_\varepsilon \in [\theta_1, \theta_2]$ on K_ε . If $n = 2$, this contradicts uniform convergence and the proof is finished. If $n = 3$, we need a further argument.

Take some $\gamma, \eta > 0$ to be fixed later. For fixed $\varepsilon > 0$, without loss of generality, we may assume that $x_\varepsilon = 0$ and $y_\varepsilon = r_\varepsilon e_1$ for $r_\varepsilon \geq \delta/3$. Denote by π_1 the projection on the x^1 -coordinate. Since K_ε is connected, we observe that $[0, \delta/3] \subset \pi_1(K_\varepsilon)$. Thus by construction, there are $N_\varepsilon \geq \delta/(6\varepsilon^\gamma)$ points $x_\varepsilon^i \in K_\varepsilon \cap \pi_1^{-1}(2i \cdot \varepsilon^\gamma)$ such that the balls $B(x_\varepsilon^i, \varepsilon^\gamma)$ are disjoint. Assume that there are $M_\varepsilon > 0$ balls among these such that

$$\alpha_\varepsilon(B(x_\varepsilon^i, \varepsilon^\gamma)) + \int_{B(x_\varepsilon^i, \varepsilon^\gamma) \cap \{|u_\varepsilon| > 1\}} \frac{W'(u_\varepsilon)^2}{\varepsilon^3} dx \geq \varepsilon^\eta.$$

Since the balls are disjoint and referring back to the first step in the proof of Lemma 3.1, this implies

$$2\alpha_\varepsilon(\mathbb{R}^n) \geq M_\varepsilon \varepsilon^\eta,$$

so $M_\varepsilon \leq 2(\bar{\alpha} + 1)/\varepsilon^\eta$ for all small ε . We deduce that

$$\frac{M_\varepsilon}{N_\varepsilon} \leq \frac{2(\bar{\alpha} + 1)/\varepsilon^\eta}{\delta/(6\varepsilon^\gamma)} = \frac{12(\bar{\alpha} + 1)}{\delta} \varepsilon^{\gamma - \eta}.$$

When we fix $\gamma = 1/2$, and $0 < \eta < \gamma$ we see that $M_\varepsilon/N_\varepsilon \rightarrow 0$ so eventually for

$$\left(1 - \frac{M_\varepsilon}{N_\varepsilon}\right) N_\varepsilon \sim N_\varepsilon$$

balls we have $\alpha_\varepsilon(B(x_\varepsilon^i, \varepsilon^\gamma)) + \int_{B(x_\varepsilon^i, \varepsilon^\gamma)} \frac{W'(u_\varepsilon)^2}{\varepsilon^3} dx < \varepsilon^\eta$. Denote by \mathcal{A}_ε the collection of the centres of these balls. As $x_\varepsilon^i \in K_\varepsilon \subset \mathbb{R}^n \setminus (U_1 \cup U_2)$, for every $0 < r < \delta/6$ we know that $B(x_\varepsilon^i, 2r) \subset \mathbb{R}^n \setminus \text{spt}(\mu)$ and

$$\alpha_\varepsilon(B(x_\varepsilon^i, \varepsilon^\gamma)) + \int_{B(x_\varepsilon^i, \varepsilon^\gamma) \cap \{|u_\varepsilon| > 1\}} \frac{W'(u_\varepsilon)^2}{\varepsilon^3} dx < \varepsilon^\eta.$$

When we consider the rescaling $\hat{u}_\varepsilon : B(0, 1) \rightarrow \mathbb{R}$, $\hat{u}_\varepsilon(y) = u_\varepsilon(x_\varepsilon^i + ry)$ for some sequence $x_\varepsilon^i \in \mathcal{A}_\varepsilon$ and denote $\hat{\varepsilon} = \varepsilon/r$ we obtain (in $n = 3$ dimensions) that

$$\hat{\alpha}_\varepsilon(B(0, \hat{\varepsilon}^\gamma r^{\gamma-1})) + \int_{B(0, \hat{\varepsilon}^\gamma r^{\gamma-1}) \cap \{|\hat{u}_\varepsilon| > 1\}} \frac{W(\hat{u}_\varepsilon)}{\hat{\varepsilon}^3} dx \leq r^n \varepsilon^\eta.$$

Without loss of generality, we can take $r \leq 1$ and since we fixed $\gamma \in (0, 1)$, this implies

$$\hat{\alpha}_\varepsilon(B(0, \hat{\varepsilon}^\gamma)) + \int_{B(0, \hat{\varepsilon}^\gamma) \cap \{|\hat{u}_\varepsilon| > 1\}} \frac{W(\hat{u}_\varepsilon)}{\hat{\varepsilon}^3} dx \leq \hat{\varepsilon}^\eta.$$

Also, by construction $\hat{\mu}_\varepsilon \rightarrow 0$ weakly as a Radon measure on $B(0, 1)$. With the same argument that gave us the lower bound on N_ε , for any $s \in (0, \delta)$ we can obtain a sequence of points $\tilde{x}_\varepsilon = x_\varepsilon^{i_\varepsilon} \in \mathcal{A}_\varepsilon$ such that $\tilde{x}_\varepsilon \rightarrow x$ for some $x \in \mathbb{R}^n$ with $\pi_1(x) = s$. In particular, this gives us infinitely many such limit points. We will show that each of them is an atom of α with a certain minimal size, which obviously gives us a contradiction. By the weak convergence of Radon measures,

$$\alpha(\overline{B(x, 2r)}) \geq \limsup_{\varepsilon \rightarrow 0} \alpha_\varepsilon(\overline{B(x, 2r)}) \geq \limsup_{\varepsilon \rightarrow 0} \alpha_\varepsilon(B(\tilde{x}_\varepsilon, r))$$

since for small enough $\varepsilon > 0$ we have $B(\tilde{x}_\varepsilon, r) \subset B(x, 2r)$. But since $\hat{\mu}_\varepsilon \rightarrow 0$ we have

$$\limsup_{\varepsilon \rightarrow 0} \alpha_\varepsilon(B(\tilde{x}_\varepsilon, r)) = \limsup_{\hat{\varepsilon} \rightarrow 0} (\hat{\alpha}_\varepsilon + \hat{\mu}_\varepsilon)(B(0, 1)) \geq \theta_{\bar{\alpha}, n, \theta_1, \theta_2}$$

using Lemma 3.9. Letting $r \rightarrow 0$ shows that in fact $\alpha(\{x\}) \geq \theta_{\bar{\alpha}, n, \theta_1, \theta_2}$ for uncountably many points, leading to the contradiction we were looking for. \square

4. COMPUTER IMPLEMENTATION

In a computer simulation, we try to find local minimisers of the ε -problem in $n = 2$ dimensions by following a finite element implementation of the time-normalised L^2 -gradient flow of \mathcal{E}_ε with $\varepsilon = 1.5 \cdot 10^{-2}$, $\kappa = 1$, $\sigma = 2$. The pictures below are obtained using roughly 250.000 H^2 -conforming basis functions and a time-step of $\varepsilon \cdot 10^{-5}$. The distance function $d^{F(u_\varepsilon)}$ and its gradient are implemented via Dijkstra's algorithm in a fashion similar to the one of [BCPS10]. We note that $\omega(\varepsilon)$ can be chosen so large that it does not pose a restriction for the algorithm computing the distance on our grid. The initial condition is the same for all simulations and can be seen in figure 1 on the left; the domain is the disc of radius 1. There was no penalisation of the discrepancy measure in our simulations.

Note that the phase field in this figure is already relaxed by running the gradient flow approximately up to time $t = 7.5 \cdot 10^{-5}$, so that a smooth transition layer could form from the simple sharp "true" initial condition.

For practical purposes, we use two functions ϕ_1, ϕ_2 with support close to 1 and -1 , respectively, rather than just one ϕ . Quite obviously, our proofs easily extend to that situation. By keeping level sets close to the edges connected, we create barriers that prevent the interface from splitting apart early in the process. The implementation will be described in greater detail in a forth-coming article [DW15].

We see in figure 1 that without the inclusion of the topological term, the transition layer disintegrates into several connected components along the gradient flow of $\mathcal{W}_\varepsilon + \varepsilon^{-\sigma} (S_\varepsilon - S)^2$.

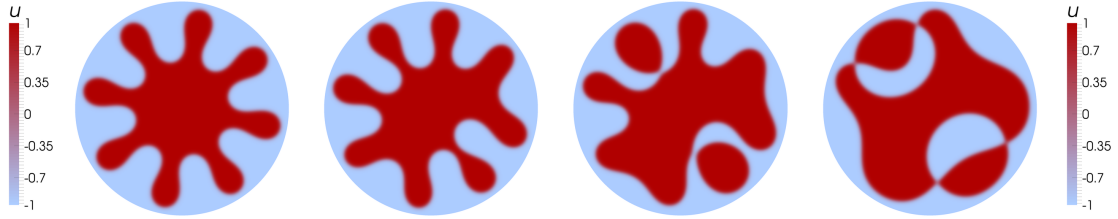


FIGURE 1. Gradient flow of $\mathcal{W}_\varepsilon + \varepsilon^{-\sigma} (S_\varepsilon - S)^2$. From left to right: phase field u for approximately $t = 7.5 \cdot 10^{-5}$, $t = 3 \cdot 10^{-4}$, $t = 7.5 \cdot 10^{-4}$ and $t = 1.8 \cdot 10^{-3}$.

To compare implementations of topological side conditions, we include the topological term suggested in [DMR11], which penalises a deviation of a diffuse signed curvature integral from 2π in the simulation. This term prevents the initial pinch-off, but at a later time, the interface will pinch off in a more complicated way which keeps the diffuse winding number close to 2π . The trick is to pinch off simultaneously at several points as seen in figure 2. The far right plot in figure 2 illustrates the diffuse curvature density as distributed along the curve at pinch off time. We can observe the formation of a circle with negative total curvature $\approx -2\pi$ (due to the phase field switching in the other direction from $+1$ to -1), and two components with total curvature $\approx 2\pi$ so that the total curvature of the whole interface stays close to 2π .

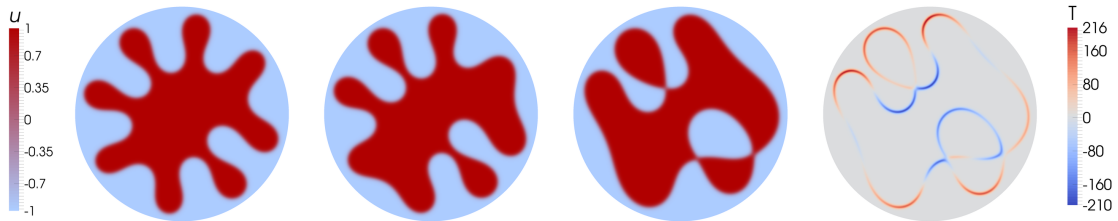


FIGURE 2. Gradient flow with penalty on a diffuse winding number as suggested in [DMR11]. From left to right: phase field u for approximately $t = 3 \cdot 10^{-4}$, $t = 7.5 \cdot 10^{-4}$ and $t = 1.8 \cdot 10^{-3}$, then a plot of the diffuse winding number density denoted T at time $t = 1.8 \cdot 10^{-3}$.

In figure 3, a flow for \mathcal{E}_ε with the additional term of C_ε on the other hand can be seen to stably flow past those singular situations. The three left plots both here and in 2 correspond approximately to the same times in the simulation.

Comparing the three scenarios above, we observe that there is virtually no difference in the plots at time $3 \cdot 10^{-4}$ and that the plots for both modified (penalised using either the old or the new method) functionals at time $7.5 \cdot 10^{-4}$ still look very similar. It can thus be argued that the topological condition does not affect the shape of the curve in a major way except when it has to in order to prevent loss of connectedness.

In figure 4, we see non-trivial geometric changes along the gradient flow for later times. This demonstrates the necessity of continuing the flow beyond the critical times.

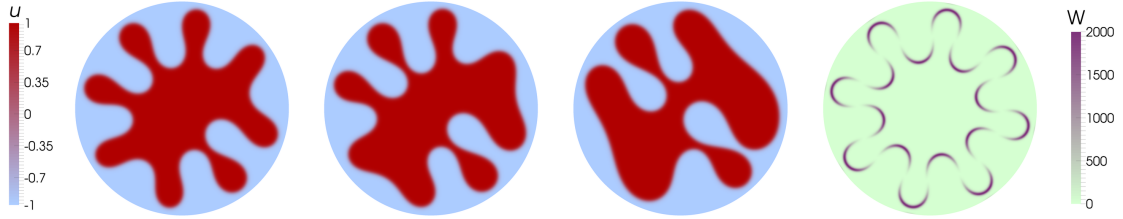


FIGURE 3. Evolution including our new topological penalty term C_ε . From left to right: phase field u for approximately $t = 3 \cdot 10^{-4}$, $t = 7.5 \cdot 10^{-4}$ and $t = 1.8 \cdot 10^{-3}$, then a plot of the diffuse Willmore energy density (denoted W here) of the initial condition.

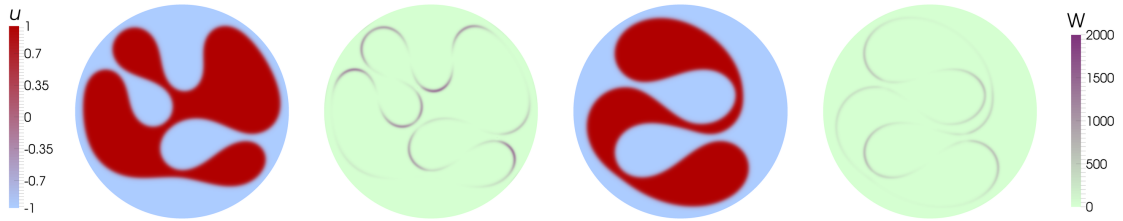


FIGURE 4. Evolution including our new topological penalty term C_ε for long times. From left to right: phase field u and diffuse Willmore energy density (denoted W here) first for approximately $t = 6.6 \cdot 10^{-3}$ and then for approximately $t = 3.6 \cdot 10^{-2}$.

It should be emphasised that our focus is not on implementing an approximate Willmore flow using phase fields but on finding minimisers of the diffuse interface problem using a gradient flow. Existence of Willmore flow for long time and topological changes along it are still an open field of research.

5. CONCLUSIONS

In this paper, we have developed a strategy to enforce connectedness of diffuse interfaces. The strategy fares well in applications and can efficiently be implemented and seems to be more generally applicable to a wider class of problems. We claim that our results can be extended to the following situations.

- We can include a hard volume constraint, for example

$$\frac{1}{\varepsilon} \left(\frac{1}{2} \int_{\Omega} u_\varepsilon + 1 \, dx - V \right)^2$$

for $0 < V < \min\{\mathcal{L}^n(\Omega), c_n S^{n/(n-1)}\}$ or a soft volume constraint like

$$F \left(\frac{1}{2} \int_{\Omega} u_\varepsilon + 1 \, dx \right)$$

for continuous functions $F \geq 0$. Here c_n is the constant from the iso-perimetric inequality in n dimensions.

- Another popular constraint compatible with our functional and results is minimising a distance from a given configuration as

$$A_\varepsilon(u) = \int_{\Omega} |u - g| \, d\lambda$$

where λ is a finite Radon measure on Ω and $g \in L^1(\Omega)$. This functional originates in problems in image segmentation, but in our context it can be understood as prescribing certain points to lie inside or outside the membrane according to experimental data; a computationally stable choice would be for example

$$g = \begin{cases} 1 & \Omega_1 \\ 0 & \Omega_2, \\ -1 & \Omega_3 \end{cases}, \quad \lambda = \mathcal{L}^n \llcorner \Omega.$$

This urges the functional to have $\Omega_1 \subset E$ and $\Omega_3 \subset E^c$ without preference on Ω_2 ; the energetic drive to form transitions on Ω_2 is compensated by the volume constraint and disappears as $\varepsilon \rightarrow 0$. As for the volume contribution, the constraint can be included in a hard or soft penalisation.

In a soft penalisation, it suffices to have $\overline{\Omega_1} \cap \overline{\Omega_3} = \emptyset$ for the existence of minimisers. In a hard penalisation, a sharp interface competitor of area S must be constructed or the hard area constraint should be dropped in favour of a soft term like S_ε or $(S_\varepsilon - S)^2$.

- Using [BM10, Theorem 4.1], we could take Bellettini's approximation of the Helfrich energy

$$\mathcal{E}_\varepsilon^{\text{Hel}}(u) = \int_{\Omega} \frac{2 + \chi}{2\varepsilon} v_{u,\varepsilon}^2 - \frac{\chi}{2\varepsilon} \left| \varepsilon \nabla^2 u - \frac{W'(u)}{\varepsilon} \nu_u \otimes \nu_u \right|^2 \, dx$$

for $\chi \in (-2, 0)$ in place of the diffuse Willmore energy \mathcal{W}_ε . Here $v_{u,\varepsilon}$ is the usual Willmore density associated with u and $\nu_u = \nabla u / |\nabla u|$ is the diffuse normal like in Lemma 3.12.

- We can use the same modelling techniques for a finite collection of membranes inside an elastic container. The outer container is modelled by a phase field U_ε and the inner membranes by phase fields $u_\varepsilon^1, \dots, u_\varepsilon^N$. The governing energy is composed by a sum of the individual elastic energies \mathcal{E}_ε modified by bending moduli $\chi_i > 0$, interaction energies I_ε preventing interpenetration and confinement energies T_ε :

$$\begin{aligned} \mathcal{F}_\varepsilon(U_\varepsilon, u_\varepsilon^1, \dots, u_\varepsilon^N) &= \mathcal{E}_\varepsilon(U_\varepsilon) + \sum_{i=1}^N \chi^i \mathcal{E}_\varepsilon(u_\varepsilon^i) + \frac{1}{\varepsilon^\beta} \sum_{i=1}^N T_\varepsilon(u_\varepsilon^i, U_\varepsilon) \\ &\quad + \frac{1}{\varepsilon^\beta} \sum_{i=1}^N \sum_{j \neq i} I_\varepsilon(u_\varepsilon^i, u_\varepsilon^j) \end{aligned}$$

for some $\beta > 0$ where for example

$$I_\varepsilon(u, v) = \int_{\Omega} (u + 1)^2 (v + 1)^2 \, dx$$

and $T_\varepsilon(u, U) = I_\varepsilon(u, 2 - U)$. These energies prevent u, v from being close to +1 at the same time preventing penetrations between different membranes and u being close to 1 where $U \approx -1$ enforcing the confinement of u to the elastic container. Surface area can be prescribed for each phase field individually and other modifications can be included. The domain Ω can be chosen as a ball large enough to have no influence on the problem. This

appears to be an interesting way to treat the mitochondria-shape problem mentioned in the introduction.

An interesting open question is whether it is possible to develop to a finer control of the topology of limit interfaces in three dimensions. While in two dimensions, the only closed connected 1-manifold is the circle, orientable closed surfaces in \mathbb{R}^3 are fully classified by their genus $g \in \mathbb{N}$. This is linked to Gaussian curvature K by the Gauss-Bonnet theorem

$$\int_{\Sigma} K d\mathcal{H}^2 = 4\pi(1 - g).$$

Although there are diffuse approximations χ_ε of this invariant [DLW05, DLRW07], investigations into their use in prescribing a topology in Willmore problems have not been conducted, neither analytically nor numerically.

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