Geometric side of a local relative trace formula

Patrick Delorme, Pascale Harinck, Sofiane Souaifi

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P. Delorme*, P. Harinck, S. Souaifi

Abstract

Following a scheme suggested by B. Feigon, we investigate a local relative trace formula in the situation of a reductive p-adic group G relative to a symmetric subgroup $H = \underline{H}(F)$ where $\underline{H}$ is split over the local field F of characteristic zero and $G = \underline{G}(F)$ is the restriction of scalars of $\underline{H}_{IE}$ relative to a quadratic unramified extension E of F. We adapt techniques of the proof of the local trace formula by J. Arthur in order to get a geometric expansion of the integral over $H \times H$ of a truncated kernel associated to the regular representation of G.

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Contents

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Introduction

In this article, we investigate a local relative trace formula in the situation of p-adic groups relative to a symmetric subgroup. This work is inspired by the recent results of B. Feigon ([F]), where she investigated what she called a local relative trace formula on $PGL(2)$ and a local Kuznetsov trace formula for $U(2)$.

Before we describe our setting and results, we would to explain on the toy model of finite groups the framework of the formulas of B. Feigon. We even start with the more general framework of the relative trace formula initiated by H. Jacquet ([J], see also [O] for an account of some applications of this relative trace formula).

Let G be a finite group and let H, H', Γ be subgroups of G. We endow any finite set with the counting measure. We denote by r the right regular representation of G on $L^2(\Gamma\backslash G)$ and we consider the H-fixed linear form ξ on $L^2(\Gamma\backslash G)$ defined by

$$\xi = \sum_{h\in H\cap\Gamma\backslash H} \delta_{\Gamma h} \tag{0.1}$$

where $\delta_{\Gamma h}$ is the Dirac measure of the coset Γh, or in other words

$$\xi(\psi) = \int_{H\cap\Gamma\backslash H} \psi(\Gamma h)dh, \quad \psi \in L^2(\Gamma\backslash G).$$

We define similarly ξ' relative to H'.
We view ξ, ξ' as elements of $L^2(\Gamma\backslash G)$ and we form the coefficient $c_{\xi,\xi'}(g) = (r(g)\xi, \xi')$. Integrating over functions on G, it defines a "distribution" Θ on G which is right invariant by H and left invariant by H'. The relative trace formula in this context gives two expressions of $\Theta(f)$ for f a function on G, the first one, called the geometric side, in terms of orbital integrals, and the second one, called the spectral side, in terms of irreducible representations of G.

First we deal with the geometric side. For this purpose we introduce suitable orbital integrals. For $\gamma \in \Gamma$, we set $[\gamma] := (H' \cap \Gamma)\gamma(H \cap \Gamma)$ and one introduces two subgroups of $H' \times H$

$$(H' \times H)_\gamma = \{(h', h) | h'\gamma h^{-1} = \gamma\}, (H' \cap \Gamma \times H \cap \Gamma)_\gamma = (H' \times H)_\gamma \cap (\Gamma \times \Gamma).$$

Then, we define the orbital integral of a function f on G by

$$I([\gamma], f) = \int_{(H'\times H)_\gamma\backslash(H'\times H)} f(h'\gamma h^{-1})dh'dh.$$

Let f be a function on G. Since $r(g)\delta_{\Gamma h} = \delta_{\Gamma hg^{-1}}$, the definition of ξ and ξ' gives

$$\Theta(f) = \sum_{g\in G} f(g)\Theta(g) = \sum_{g\in G} f(g)\frac{1}{vol(\Gamma\cap H)}\frac{1}{vol(\Gamma\cap H')}\sum_{h\in H}\sum_{h'\in H'}(\delta_{\Gamma hg^{-1}},\delta_{\Gamma h'}).$$

Changing g in $g^{-1}h$ and using the fact that $(\delta_{\Gamma g},\delta_{\Gamma h'})$ is equal to 1 for $g\in\Gamma h'$ and to zero otherwise, one gets

$$\Theta(f) = \frac{1}{vol(\Gamma\cap H)}\frac{1}{vol(\Gamma\cap H')}\sum_{h\in H}\sum_{h'\in H'}\sum_{\gamma\in\Gamma} f(h'\gamma h). \tag{0.2}$$

A simple computation of volumes leads to the geometric expression of Θ in terms of orbital integrals

$$\Theta(f) = \sum_{[\gamma]\in H'\cap\Gamma\backslash\Gamma/\Gamma\cap H} vol((H'\cap\Gamma\times H\cap\Gamma)_\gamma\backslash(H'\times H)_\gamma)I([\gamma],f). \tag{0.3}$$

Let us turn to the spectral side. We decompose $L^2(\Gamma\backslash G)$ into isotypic components $\oplus_{\pi\in\hat{G}}\mathcal{H}_\pi$ The restriction of ξ and ξ' to $\mathcal{H}_\pi$ will be denoted ξ_π and ξ'_π respectively. The spectral formula for Θ is the simple equality

$$\Theta = \sum_{\pi\in\hat{G}} c_{\xi_\pi,\xi'_\pi}. \tag{0.4}$$

Notice that it might be also interesting to decompose further the representation into irreducible representations and the restriction of ξ to each of them will be called a period.

There is a third interpretation of the distribution Θ. If f is a function on G, then the operator $r(f)$ on $L^2(\Gamma\backslash G)$ is an integral operator whose kernel K_f is the function on $\Gamma\backslash G\times\Gamma\backslash G$ given by

$$K_f(x,y) = \sum_{\gamma\in\Gamma} f(x^{-1}\gamma y).$$

By (0.2), one gets easily the following expression of $\Theta(f)$

$$\Theta(f) = \int_{(H'\cap\Gamma\backslash H')\times(H\cap\Gamma\backslash H} K_f(h',h)dh'dh. \tag{0.5}$$

This point of view is probably the best one. But it is important to have the representation theoretic meaning of Θ.

The toy model for the local relative trace formula of B. Feigon appears as a particular case of the above relative trace formula. In that case, the groups G, H and H' are products $G_1\times G_1$, $H_1\times H_1$ and $H'_1\times H'_1$ respectively and Γ is the diagonal of $G_1\times G_1$. Then $\Gamma\backslash G$ identifies with G_1 and the right representation corresponds

to the representation R of $G_1 \times G_1$ on $L^2(G_1)$ given by $[R(x,y)\phi](g) = \phi(x^{-1}gy)$. Then, we have

$$\xi(\psi) = \int_{H_1} \psi(h)dh, \quad \psi \in L^2(G_1).$$

The spectral side is more concrete. If $(\pi_1, \mathcal{H}_{\pi_1})$ is an irreducible unitary representation of G_1 then $G_1 \times G_1$ acts on $\mathrm{End}(\mathcal{H}_{\pi_1})$ by an irreducible representation denoted by π. It is unitary if we use the scalar product associated to the Hilbert-Schmidt norm. Moreover $L^2(G_1)$ is canonically isomorphic to the direct sum $\oplus_{\pi_1 \in \hat{G_1}} \mathrm{End}(\mathcal{H}_{\pi_1})$. Let P_π be the orthogonal projector onto the space of invariant vectors under H_1. Then, the period map ξ_π, which is a linear form on $\mathrm{End}(\mathcal{H}_{\pi_1})$, is given by

$$\xi_\pi(T) = \int_{H_1} Tr(\pi_1(h)T)dh = (T, P_\pi), \quad T \in \mathrm{End}(\mathcal{H}_{\pi_1}).$$

One further decomposes ξ_π by using an orthonormal basis $(\eta_{\pi_1,i})$ of the space of H_1-invariant vectors. We will use the identification of $\mathrm{End}(\mathcal{H}_{\pi_1})$ with the tensor product of $\mathcal{H}_{\pi_1}$ with its conjugate complex vector space. In this identification, one has

$$P_\pi = \sum_i \eta_{\pi_1,i} \otimes \eta_{\pi_1,i}.$$

We define similar notations for ξ' relative to H'. Then, for two functions f_1, f_2 on G_1, the spectral side (0.4) can be written

$$\Theta(f_1 \otimes f_2) = \sum_{\pi_1 \in \hat{G_1}} \sum_{i,i'} c_{\eta_{\pi_1,i},\eta'_{\pi_1,i'}}(f_1) c_{\eta_{\pi_1,i},\eta'_{\pi_1,i'}}(f_2).$$

For the geometric side, we define the integral orbital of a function f on G_1 by

$$I(g,f) = \int_{(H'_1 \times H_1)_g \backslash H'_1 \times H_1} f(h'gh^{-1})dhdh'$$

which depends only on the double coset $H'_1 g H_1$. Then one gets by (0.3) the equality

$$\Theta(f_1 \otimes f_2) = \sum_{g \in H'_1 \backslash G_1 / H_1} v(g) I(g,f_1) I(g,f_2)$$

where the $v(g)$'s are positive constants depending on volumes. Hence the final form of the local relative trace formula is:

$$\sum_{g \in H'_1 \backslash G_1 / H_1} v(g) I(g,f_1) I(g,f_2) = \sum_{\pi_1 \in \hat{G_1}} \sum_{i,i'} c_{\eta_{\pi_1,i},\eta'_{\pi_1,i'}}(f_1) c_{\eta_{\pi_1,i},\eta'_{\pi_1,i'}}(f_2).$$

This formula allows to invert the orbital integrals $I(g,f_1)$. For this purpose, one chooses $g_1 \in G_1$ and takes for f_2 the Dirac measure at g_1. Then $I(g_1,f_2) = 1$ and the other orbital integrals of f_2 are zero. Hence

$$v(g_1) I(g_1,f_1) = \sum_{\pi_1 \in \hat{G_1}} \sum_{i,i'} c_{\eta_{\pi_1,i},\eta'_{\pi_1,i'}}(f_1) c_{\eta_{\pi_1,i},\eta'_{\pi_1,i'}}(f_2).$$

In order to make the formula more precise, one needs to compute the constants $c_{\eta_{\pi_1,i},\eta'_{\pi_1,i'}}(f_2)$.

The inversion of orbital integrals is one of our motivations to investigate a local relative trace formula in the situation of p-adic groups relative to a symmetric subgroup H and we will take $H = H'$.

In this article, we consider a reductive algebraic group $\underline{H}$ defined over a non archimedean local field F of characteristic 0. We fix a quadratic unramified extension E of F and we consider the group $\underline{G} := \mathrm{Res}_{\mathrm{E/F}}\underline{H}$ obtained by restriction of scalars of $\underline{H}$, where here $\underline{H}$ is considered as a group defined over E. We denote by H and G the group of F-points of $\underline{H}$ and $\underline{G}$ respectively. Then G is isomorphic to $\underline{H}(E)$ and H appears as the fixed points of G under the involution of G induced by the nontrivial element of the Galois group of E/F. We assume that $\underline{H}$ is split over F and we fix a maximal split torus A_0 of H. The groups G and H correspond to G_1 and $H_1 = H'_1$ respectively in our example of a local relative trace formula for finite groups.

The starting point of our study is the analogue to the expression (0.5). We consider the regular representation R of $G \times G$ on $L^2(G)$ given by $(R(g_1, g_2)\psi)(x) = \psi(g_1^{-1}xg_2)$. Then for $f = f_1 \otimes f_2$ where f_1 and f_2 are two smooth compactly supported functions on G, the corresponding operator $R(f)$ is an integral operator on $L^2(G)$ with smooth kernel

$$K_f(x, y) = \int_G f_1(xg)f_2(gy)dg = \int_G f_1(g)f_2(x^{-1}gy)dg.$$

As H may be not compact, even modulo the split component A_H of the center of H, we have to truncate this kernel to integrate it. We multiply this kernel by a product of functions $u(x, T)u(y, T)$ where $u(\cdot, T)$ is the characteristic function of a large compact subset in $A_H\backslash H$ depending on a parameter $T \in a_0 = \mathrm{Rat}(A_0) \otimes_{\mathbb{Z}} \mathbb{R}$ ($\mathrm{Rat}(A_0)$ is the group of F-rational characters of A_0) as in [Ar3] (cf. (2.7)). As H is split, we have $A_H = A_G$. Hence the kernel K_f is left invariant by the diagonal $diag(A_H)$ of A_H and we can integrate the truncated kernel over $diag(A_H)\backslash H \times H$. We set

$$K^T(f) := \int_{diag(A_H)\backslash(H\times H)} K_f(x_1, x_2)u(x_1, T)u(x_2, T)d\overline{(x_1, x_2)}.$$

In [Ar3], J. Arthur studies the integral of $K_f(x, x)u(x, T)$ over $A_G\backslash G$ to obtain its local trace formula on reductive groups.

We study the geometric expression of the distribution $K^T(f)$ and its dependence on the parameter T. Our main results (Theorem 2.3 and Corollary 2.11) assert that $K^T(f)$ is asymptotic as T approaches infinity to another distribution $J^T(f)$ of the form

$$J^T(f) = \sum_{k=0}^{N} p_{\xi_k}(T, f)e^{\xi_k(T)} \tag{0.6}$$

where $\xi_0 = 0, \dots \xi_N$ are distinct points of the dual space ia_0^* and each $p_{\xi_k}(T, f)$ is a polynomial function in T. Moreover, the constant term $\tilde{J}(f) := p_0(0, f)$ of $J^T(f)$ is well-defined and uniquely determined by $K^T(f)$. We give an explicit expression of this constant term in terms of weighted orbital integrals.

These results are analogous to those of [Ar3] for the group case. Our proof follows closely the study by J. Arthur of the geometric side of his local trace formula which we were able to adapt under our assumptions to the case of double truncations.

In the first section, we introduce notation on groups and on symmetric spaces according to [RR]. The starting point of our study is the Weyl integration formula established in [RR], which takes into account the (H, H)-double classes of σ-regular elements of G (cf. (1.30) and (1.32)). These double classes are express in terms of σ-torus, that is torus whose elements are anti-invariant by σ. Under our assumptions, there is a bijective correspondence $S \to S_\sigma$ between maximal tori of H and maximal σ-tori of G which preserves H-conjugacy classes.

Then the Weyl integration formula can be written in terms of Levi subgroups $M \in \mathcal{L}(A_0)$ of H containing A_0 and M-conjugacy classes of maximal anisotropic tori of M (cf. (1.33)):

$$\int_G f(g)dg = \sum_{M \in \mathcal{L}(A_0)} c_M \sum_{S \in \mathcal{T}_M} \sum_{x_m \in \kappa_S} c_{S,x_m} \int_{S_\sigma} |\Delta_\sigma(x_m\gamma)|_{\mathrm{F}}^{1/2} \int_{diag(A_M)\backslash H \times H} f(h^{-1}x_m\gamma l) d\overline{(h,l)} d\gamma$$

where κ_S is a finite subset of G, c_M and c_{S,x_m} are positive constants, $\mathcal{T}_M$ is a suitable set of anisotropic tori of M and Δ_σ is a jacobian.

A fundamental result for our proofs concerns the orbital integral $\mathcal{M}(f)$ of a compactly smooth function f on G. It is defined on σ-regular points by

$$\mathcal{M}(f)(x_m\gamma) = |\Delta_\sigma(x_m\gamma)|_{\mathrm{F}}^{1/4} \int_{diag(A_S)\backslash H \times H} f(h^{-1}x_m\gamma l) d\overline{(h,l)},$$

where S is a maximal torus of H, $x_m \in \kappa_S$ and $\gamma \in S_\sigma$ such that $x_m\gamma$ is σ-regular. As in the group case using the exponential map and the property that each root of S_σ has multiciplity 2 in the Lie algebra of G, we prove that the orbital integral is bounded on the subset of σ-regular points of G (cf. Theorem 1.2).

In the second section, we explain the truncation process based on the notion of (H, M)-orthogonal sets and prove our main results. Using the Weyl integration formula, we can write

$$K^T(f) = \sum_{M \in \mathcal{L}(A_0)} c_M \sum_{S \in \mathcal{T}_M} \sum_{x_m \in \kappa_S} c_{S,x_m} \int_{S_\sigma} K^T(x_m, \gamma, f) d\gamma$$

where

$$K^T(x_m, \gamma, f) = |\Delta_\sigma(x_m\gamma)|_{\mathrm{F}}^{1/2} \int_{diag(A_M)\backslash H \times H} \int_{diag(A_M)\backslash H \times H} f_1(y_1^{-1}x_m\gamma y_2)$$

$$\times f_2(x_1^{-1}x_m\gamma x_2)u_M(x_1,y_1,x_2,y_2,T)d\overline{(x_1,x_2)}d\overline{(y_1,y_2)}$$

and

$$u_M(x_1,y_1,x_2,y_2,T)=\int_{A_H\backslash A_M}u(y_1^{-1}ax_1,T)u(y_2^{-1}ax_2,T)da.$$

The function $J^T(f)$ is obtained in a similar way to $K^T(f)$ where we replace the weight function $u_M(x_1,y_1,x_2,y_2,T)$ by another weight function $v_M(x_1,y_1,x_2,y_2,T)$.

The weight function v_M is given by

$$v_M(x_1,y_1,x_2,y_2,T):=\int_{A_H\backslash A_M}\sigma_M(h_M(a),\mathcal{Y}_M(x_1,y_1,x_2,y_2,T))da$$

where $\sigma_M(\cdot,\mathcal{Y})$ is the function of [Ar3] depending on a (H,M)-orthogonal set $\mathcal{Y}$ and $\mathcal{Y}_M(x_1,y_1,x_2,y_2,T)$ is a (H,M)- orthogonal set obtained as the "minimum" of two (H,M)- orthogonal sets $\mathcal{Y}_M(x_1,y_1,T)$ and $\mathcal{Y}_M(x_2,y_2,T)$ (cf. (2.4), Lemma 2.2 and (2.11)). If $\mathcal{Y}_1$ and $\mathcal{Y}_2$ are two (H,M)-orthogonal positive sets then the "minimum" $\mathcal{Z}$ of $\mathcal{Y}_1$ and $\mathcal{Y}_2$ satisfies the property that the convex hull $\mathcal{S}_M(\mathcal{Z})$ in $a_H\backslash a_M$ of the points of $\mathcal{Z}$ is the intersection of the convex hulls $\mathcal{S}_M(\mathcal{Y}_1)$ and $\mathcal{S}_M(\mathcal{Y}_2)$ in $a_H\backslash a_M$ of the points of $\mathcal{Y}_1$ and $\mathcal{Y}_2$ respectively.

If $\|T\|$ is large relative to $\|x_i\|,\|y_i\|,i=1,2$ then $\sigma_M(\cdot,\mathcal{Y}_M(x_1,y_1,x_2,y_2,T))$ is just the characteristic function of $\mathcal{S}_M(\mathcal{Y}_M(x_1,y_1,x_2,y_2,T))$. In that case, this function is equal to the product of $\sigma_M(\cdot,\mathcal{Y}_M(x_1,y_1,T))$ and $\sigma_M(\cdot,\mathcal{Y}_M(x_2,y_2,T))$.

Our proofs consist to establish good estimates of $|u_M((x_1,y_1,x_2,y_2,T)-v_M(x_1,y_1,x_2,y_2,T)|$ when $x_i,y_i,i=1,2$ satisfy $f_1(y_1^{-1}x_m\gamma y_2)f_1(x_1^{-1}x_m\gamma x_2)\neq 0$ for some $\gamma\in S_\sigma$ and $x_m\in\kappa_S$. Then, using that orbital integrals are bounded, we deduce our result on $|K^T(f)-J^T(f)|$.

This work is a first step towards a local relative trace formula. For the spectral side, we have to prove that $K^T(f)$ is asymptotic to a distribution $k^T(f)$ which is of general form (0.6) and constructed from spectral data. We hope that we can express the constant term of $k^T(f)$ in terms of regularized local period integrals introduced by B. Feigon in [F] in the same way than Jacquet-Lapid-Rogawski regularized period integrals for automorphic forms in [JLR]. We plan to explicit such a local relative trace formula for $PGL(2)$.

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1 Preliminaries

1.1 Reductive p-adic groups

Let F be a non archimedean local field of characteristic 0 and odd residual characteristic q. Let $|\cdot|_{\mathrm{F}}$ denote the normalized valuation on F.

For an algebraic variety $\underline{M}$ defined over F, we identify $\underline{M}$ with $\underline{M}(\overline{\mathrm{F}})$ where $\overline{\mathrm{F}}$ is an algebraic closure of F and we set $M := \underline{M}(\mathrm{F})$.

We will use conventions like in [W2]. One considers various algebraic groups $\underline{J}$ defined over F, and sentences like

> " let M be an algebraic group" will mean " let M be the F-points of an algebraic group $\underline{M}$ defined over F" and " let A be a split torus " will mean " let A be the group of F-points of a torus, $\underline{A}$, defined and split over F ." (1.1)

If J is an algebraic group, one denotes by $\mathrm{Rat}(J)$ the group of its rational characters defined over F. If V is a vector space, V^* will denote its dual. If V is real, $V_{\mathbb{C}}$ will denote its complexification.

Let $\underline{G}$ be an algebraic reductive group defined over F. We fix a maximal split torus A_0 of G and we denote by M_0 its centralizer in G.

We denote by A_G the maximal split torus of the center of G and we define

$$a_G := \mathrm{Hom}_{\mathbb{Z}}(\mathrm{Rat}(G), \mathbb{R}).$$

One has the canonical map $h_G : G \to a_G$ which is defined by

$$e^{\langle h_G(x), \chi\rangle} = |\chi(x)|_{\mathrm{F}}, \quad x \in G, \chi \in \mathrm{Rat}(G). \tag{1.2}$$

The restriction of rational characters from G to A_G induces an isomorphism

$$\mathrm{Rat}(G) \otimes_{\mathbb{Z}} \mathbb{R} \simeq \mathrm{Rat}(A_G) \otimes_{\mathbb{Z}} \mathbb{R}. \tag{1.3}$$

Notice that $\mathrm{Rat}(A_G)$ appears as a generating lattice in the dual space a_G^* of a_G and

$$a_G^* \simeq \mathrm{Rat}(G) \otimes_{\mathbb{Z}} \mathbb{R}. \tag{1.4}$$

The kernel of h_G, which is denoted by G^1, is the intersection of the kernels of $|\chi|_{\mathrm{F}}$ for all character $\chi \in \mathrm{Rat}(G)$ of G. The groupe G^1 is distinguished in G and contains the derived group G_{der} of G. Moreover, it is well-known that

> the group G^1 is generated by the compact subgroups of G. (1.5)

G. Henniart has communicated to us an unpublished proof of this result by N. Abe, F. Herzig, G. Henniart and M.F. Vigneras.

One denotes by $a_{G,\mathrm{F}}$ (resp., $\tilde{a}_{G,\mathrm{F}}$) the image of G (resp., A_G) by h_G. Then G/G^1 is isomorphic to the lattice $a_{G,\mathrm{F}}$. (1.6)

If P is a parabolic subgroup of G with Levi subgroup M, we keep the same notation with M instead of G.

The inclusions $A_G \subset A_M \subset M \subset G$ determine a surjective morphism $a_{M,\mathrm{F}} \to a_{G,\mathrm{F}}$ (resp., an injective morphism, $\tilde{a}_{G,\mathrm{F}} \to \tilde{a}_{M,\mathrm{F}}$) which extends uniquely to a surjective linear map h_{MG} from a_M to a_G (resp., injective map between a_G and a_M). The second map allows to identify a_G with a subspace of a_M and the kernel of the first one, a_M^G, satisfies

$$a_M = a_M^G \oplus a_G. \tag{1.7}$$

For $M = M_0$, we set $a_0 := a_{M_0}$ and $a_0^G := a_{M_0}^G$. We fix a scalar product $(\cdot,\cdot)$ on a_0 which is invariant under the Weyl group $W(G, A_0)$ of (G, A_0). Then a_G identifies with the fixed point set of a_0 by $W(G, A_0)$ and a_0^G is an invariant subspace of a_0 under $W(G, A_0)$. Hence, it is the orthogonal subspace to a_G in a_0. The space a_G^* might be viewed as a subspace of a_0^* by (1.7). Moreover, by definition of the surjective map $a_0 \to a_G$, one deduces that

if $m_0 \in M_0$ then $h_G(m_0)$ is the orthogonal projection of $h_{M_0}(m_0)$ onto a_G. (1.8)

From (1.7) applied to (M, M_0) instead of (G, M), one obtains a decomposition $a_0 = a_0^M \oplus a_M$. From the $W(G, A_0)$ invariance of the scalar product, one gets:

The decomposition $a_0 = a_0^M \oplus a_M$ is an orthogonal decomposition.
The space a_M^* appears as a subspace of a_0^* and, in the identification of a_0 with a_0^* given by the scalar product, a_M^* identifies with a_M. (1.9)

The decomposition $a_M = a_M^G \oplus a_G$ is orthogonal relative to the restriction to a_M of the $W(G, A_0)$-invariant inner product on a_0 and the natural map h_{MG} is identified with the orthogonal projection of a_M onto a_G.

In particular, $a_{G,\mathrm{F}}$ is the orthogonal projection of $a_{M,\mathrm{F}}$ onto a_G. Moreover, we have $\tilde{a}_{G,\mathrm{F}} = a_G \cap \tilde{a}_{M,\mathrm{F}}$ (cf. [Ar3] (1.4)). (1.10)

By a Levi subgroup of G, we mean a group M containing M_0 which is the Levi component of a parabolic subgroup of G. If P is a parabolic subgroup containing M_0 then it has a unique Levi subgroup denoted by M_P which contains M_0. We will denote by N_P the unipotent radical of P.

For a Levi subgroup M, we write $\mathcal{L}(M)$ for the finite set of Levi subgroups of G which contain M and we also let $\mathcal{P}(M)$ denote the finite set of parabolic subgroups P with $M_P = M$.

Let K be the fixator of a special point in the apartment of A_0 in the Bruhat-Tits building. We have the Cartan decomposition

$$G = KM_0K. \tag{1.11}$$

If $P = M_P N_P$ is a parabolic subgroup of G containing M_0, then

$$G = PK = M_P N_P K. \tag{1.12}$$

If $x \in G$, we can write

$$x = m_P(x)n_P(x)k_P(x), \quad m_P(x) \in M_P, n_P(x) \in N_P, k_P(x) \in K. \tag{1.13}$$

We set

$$h_P(x) := h_{M_P}(m_P(x)). \tag{1.14}$$

The point $m_P(x)$ is defined up an element of K but $h_p(x)$ does not depend of this choice.

We introduce a norm $\| \cdot \|$ on G as in ([W2] §I.1.) (called height function in ([Ar3])). Let $\Lambda_0 : G \to GL_n(\mathrm{F})$ be an algebraic embedding. For $g \in G$, we write

$$\Lambda_0(g) = (a_{i,j})_{i,j=1\ldots n}, \quad \Lambda_0(g^{-1}) = (b_{i,j})_{i,j=1\ldots n}.$$

We set

$$\|g\| := \sup_{i,j} \sup(|a_{i,j}|_{\mathrm{F}}, |b_{i,j}|_{\mathrm{F}}). \tag{1.15}$$

If $\Lambda : G \to GL_d(\mathrm{F})$ is another algebraic embedding then the norm $\| \cdot \|_\Lambda$ attached to Λ as above is equivalent to $\| \cdot \|$ in the following sense: there are a positive constant C_Λ and a positive integer d_Λ such that

$$\|g\|_\Lambda \leqslant C_\Lambda \|g\|^{d_\Lambda}.$$

This allows us to use results of [W2] for estimates on norms.
The following properties of $\| \cdot \|$ are immediate consequences of definition:

$$1 \leqslant \|x\| = \|x^{-1}\|, \quad x \in G, \tag{1.16}$$

$$\|xy\| \leqslant \|x\|\|y\|, \quad x, y \in G. \tag{1.17}$$

In order to have estimates, we introduce the following notation. Let r be a positive integer. Let f and g be two positive functions defined over a subset W of G^r .

We write $f(x) \preccurlyeq g(x), x \in W$ if and only if there are a positive constant c and a positive integer d such that $f(x) \leqslant cg(x)^d$ for all $x \in W$. (1.18)

We write $f(x) \approx g(x), x \in W$ if $f(x) \preccurlyeq g(x), x \in W$ and $g(x) \preccurlyeq f(x), x \in W$. (1.19)

If f_1, f_2 and f_3 are positive functions on G^r, we clearly have

if $f_1(x) \preccurlyeq f_2(x), x \in W$ and $f_2(x) \preccurlyeq f_3(x), x \in W$ then $f_1(x) \preccurlyeq f_3(x), x \in W$,
if $f_1(x) \approx f_2(x), x \in W$ and $f_2(x) \approx f_3(x), x \in W$ then $f_1(x) \approx f_3(x), x \in W$.

Moreover, if f_1, f_2, g_1 and g_2 are positive functions on G^r which take values greater or equal to 1, we obtain easily the following properties:

1. for all positive integer d, we have $f_1(x) \approx f_1(x)^d, x \in W$,
2. if $f_1(x) \preccurlyeq g_1(x), x \in W$ and $f_2(x) \preccurlyeq g_2(x), x \in W$ then $(f_1 f_2)(x) \preccurlyeq (g_1 g_2)(x), x \in W$, (1.20)
3. if $f_1(x) \approx g_1(x), x \in W$ and $f_2(x) \approx g_2(x), x \in W$ then $(f_1 f_2)(x) \approx (g_1 g_2)(x), x \in W$.

Since $\|x\| = \|xyy^{-1}\| \preccurlyeq \|xy\|\|y\|$ and $\|xy\| \preccurlyeq \|x\|\|y\|$, we obtain

If Ω is a compact subset of G, then $\|x\| \approx \|x\omega\|, \quad x \in G, \omega \in \Omega$. (1.21)

Let $P = M_P N_P$ be a parabolic subgroup of G containing M_0. Then, each $x \in G$ can be written $x = m_P(x) n_P(x) k$ where $m_P(x) \in M_P, n_P(x) \in N_P$ and $k \in K$. By ([W2] Lemma II.3.1), we have

$$\|m_P(x)\| + \|n_P(x)\| \preccurlyeq \|x\|, \quad x \in G. \tag{1.22}$$

Recall that G^1 is the kernel of $h_G : G \to a_G$. Let us prove that

$$\|xa\| \approx \|x\|\|a\|, \quad x \in G^1, a \in A_G. \tag{1.23}$$

According to the Cartan decomposition (1.11), if $g \in G$, we denote by $m_0(g)$ an element of M_0 such that there exist $k, k' \in K$ with $g = k m_0(g) k'$. Notice that $\|h_{M_0}(m_0(g))\|$ does not depend on our choice of $m_0(g)$. By (1.21), one has

$$\|g\| \approx \|m_0(g)\|, \quad g \in G, \tag{1.24}$$

and by ([W2]) 1.1.(6)) we have

$$\|m_0\| \approx e^{\|h_{M_0}(m_0)\|}, \quad m_0 \in M_0. \tag{1.25}$$

Let $x \in G^1$ and $a \in A_G$. Then $m_0(x) \in G^1 \cap M_0$ and $m_0(xa) = m_0(x)a$. Thus, one has $h_G(m_0(x)) = 0$. We deduce from (1.8) that $h_{M_0}(m_0(x))$ belongs to a_0^G. Since

$h_{M_0}(m_0(x)a) = h_{M_0}(m_0(x)) + h_{M_0}(a)$ and $h_{M_0}(a) \in a_G$, we obtain by orthogonality that

$$\frac{1}{2}(\|h_{M_0}(m_0(x))\| + \|h_{M_0}(a)\|) \leqslant \|h_{M_0}(m_0(x)a)\| \leqslant \|h_{M_0}(m_0(x))\| + \|h_{M_0}(a)\|.$$

Hence (1.23) follows from (1.24) and (1.25).

We denote by $C_c^\infty(G)$ the space of smooth functions on G with compact support. We normalize Haar measures according to [Ar3] §1. Unless otherwise stated, the Haar measure on a compact group will be normalized to have total volume 1.

Let M be a Levi subgroup of G. We fix a Haar measure on a_M so that the volume of the quotient $a_M/\tilde{a}_{M,\mathrm{F}}$ equals 1.

Let $P = MN_P \in \mathcal{P}(M)$. We denote by δ_P the modular function of P given by

$$\delta_P(mn) = e^{2\rho_P(h_M(m))}, m \in M, n \in N_P,$$

where $2\rho_P$ is the sum of roots, with multiplicity, of (P, A_M). Let $\bar{P} = MN_{\bar{P}}$ be the the parabolic subgroup which is opposite to P. If dn is a Haar measure on N_P then the number

$$\gamma(P) = \int_{N_P} e^{2\rho_{\bar{P}}(h_{\bar{P}}(n))} dn$$

is finite. Moreover, the measure $\gamma(P)^{-1}dn$ is independent of the choice of dn and thus defines a canonical Haar measure on N_P.

If dm is a Haar measure on M then there exists a unique Haar measure dg on G, independent of the choice of the parabolic subgroup P, such that

$$\int_G f(g)dg = \frac{1}{\gamma(P)\gamma(\bar{P})} \int_{N_P}\int_M\int_{N_{\bar{P}}} f(nm\bar{n})\delta_P(m)^{-1}d\bar{n}\ dm\ dn,$$

for $f \in C_c^\infty(G)$. We say that dm and dg are compatible. Compatibility has the obvious transitivity property relative to Levi subgroups of M. Using the Iwasawa decomposition (1.12), these measures satisfy

$$\int_G f(g)dg = \frac{1}{\gamma(P)} \int_K\int_M\int_{N_P} f(mnk)dn\ dm\ dk.$$

1.2 The symmetric space $H\backslash G$

Let E be an unramified quadratic extension of F. Thus $\mathrm{E} = \mathrm{F}[\tau]$ where τ^2 is not a square in F. We denote by σ the nontrivial element of the Galois group $\mathcal{G}al(\mathrm{E}/\mathrm{F})$ of E/F. The normalized valuation $|\cdot|_\mathrm{E}$ on E satisfies $|x|_\mathrm{E} = |x|_\mathrm{F}^2$ for $x \in \mathrm{F}$.

If $\underline{J}$ is an algebraic group defined over F, as usual we denote by J its group of points over F. Let $\underline{J} \times_\mathrm{F} \mathrm{E}$ be the group, defined over E, obtained from $\underline{J}$ by extension of scalars. We consider the group

$$\tilde{\underline{J}} := \mathrm{Res}_{\mathrm{E}/\mathrm{F}}(\underline{J} \times_\mathrm{F} \mathrm{E})$$

defined over F, obtained by restriction of scalars.

With our convention, one has $\tilde{J} = \underline{J}(\mathrm{F})$ and $\tilde{J}$ is isomorphic to $\underline{J}(\mathrm{E})$.

Let $\underline{H}$ be a reductive group defined over F. In all this article, we assume that $\underline{H}$ is split over F and we set $\underline{G} := \underline{\tilde{H}}$ and $G := \tilde{H}$. We fix a maximal split torus A_0 of H. Then A_0 is also a maximal split torus of G and we have $A_H = A_G$.

The nontrivial element σ of $\mathcal{G}al(\mathrm{E}/\mathrm{F})$ induces an involution of $\underline{G}$ defined over F, which we denote by the same letter. This automorphism σ extends to an E-automorphism σ_{E} on $\underline{G} \times_{\mathrm{F}} \mathrm{E}$.

We consider the canonical map φ defined over F from $\underline{G}$ to $(\underline{H} \times_{\mathrm{F}} \mathrm{E}) \times (\underline{H} \times_{\mathrm{F}} \mathrm{E})$ by $\varphi(g) = (g, \sigma(g))$.

Then, φ extends uniquely to an isomorphism Ψ defined over E from $\underline{G} \times_{\mathrm{F}} \mathrm{E}$ to $(\underline{H} \times_{\mathrm{F}} \mathrm{E}) \times (\underline{H} \times_{\mathrm{F}} \mathrm{E})$ such that $\Psi(g) = (g, \sigma(g))$ for all $g \in \underline{G}$ and if $\Psi(g) = (g_1, g_2)$ then $\Psi(\sigma_{\mathrm{E}}(g)) = (g_2, g_1)$. (1.26)

Now we turn to the description of the geometric structure of the symmetric space $\mathcal{S} = H\backslash G$ according to [RR] sections 2 and 3.

Let $\underline{\mathfrak{g}}$ be the Lie algebra of $\underline{G}$ and $\mathfrak{g}$ be the Lie algebra of its F-points. We will say that $\mathfrak{g}$ is the Lie algebra of G and the Lie algebra $\mathfrak{h}$ of H consists of the elements of $\mathfrak{g}$ invariant by σ. We denote by $\mathfrak{q}$ the space of antiinvariant elements of $\mathfrak{g}$ by σ. Thus, one has $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{q}$ and $\mathfrak{g}$ may be identified with $\mathfrak{h} \otimes_{\mathrm{F}} \mathrm{E}$.

As in ([RR] §2.), we say that a subspace $\mathfrak{c}$ of $\mathfrak{q}$ is a Cartan subspace of $\mathfrak{q}$ if $\mathfrak{c}$ is a maximal abelian subspace of $\mathfrak{q}$ made of semisimple elements. As $\mathrm{E} = \mathrm{F}[\tau]$, the multiplication by τ induces an isomorphism between the set of Cartan subspaces of $\mathfrak{q}$ and the set of Cartan subalgebras of $\mathfrak{h}$ which preserves H-conjugacy classes.

We denote by $\underline{\mathcal{P}}$ the connected component of 1 in the set of x in $\underline{G}$ such that $\sigma(x) = x^{-1}$. Then the map $\underline{p}$ from $\underline{G}$ to $\underline{\mathcal{P}}$ defined by $\underline{p}(x) = x^{-1}\sigma(x)$ induces an isomorphim of affine varieties $\underline{p} : \underline{H}\backslash\underline{G} \to \underline{\mathcal{P}}$.

A torus $\underline{A}$ of $\underline{G}$ is called a σ-torus if $\underline{A}$ is a torus defined over F contained in $\underline{\mathcal{P}}$. Notice that such torus are called σ-split torus in [RR]. We prefer change the terminology as σ-tori are not necessarily split over F. Each σ-torus is the centralizer in $\underline{\mathcal{P}}$ of a Cartan subspace of $\mathfrak{q}$, or equivalently of a Cartan subalgebra of $\mathfrak{h}$.

Let S be a maximal torus of H. We denote by $\underline{S}_\sigma$ the connected component of $\underline{\tilde{S}} \cap \underline{\mathcal{P}}$. Then $\underline{S}_\sigma$ is a σ-torus defined over F which identifies with the antidiagonal $\{(s, s^{-1}); s \in \underline{S}\}$ of $\underline{S} \times \underline{S}$ by the isomorphism (1.26). Thus, $\underline{S}_\sigma$ is a maximal σ-torus and each maximal σ-torus arises in this way . The H-conjugacy classes of maximal tori of H are in bijective correspondence with the H-conjugacy classes of maximal σ-tori of G by the map $S \mapsto S_\sigma$. The roots of $\underline{S}$ (resp.; $\underline{S}_\sigma$) in $\underline{\mathfrak{h}} = Lie(\underline{H})$ (resp.;

$\underline{\mathfrak{q}} \otimes_{\mathrm{F}} \bar{\mathrm{F}}$) are the restrictions of the roots of $\underline{\tilde{S}}$ in $\underline{\mathfrak{g}} = Lie(\underline{G})$.

Therefore, each root of $\underline{S}$ (resp.; $\underline{S}_\sigma$) in $\underline{\mathfrak{g}}$ has multiplicity two. If $\underline{\tilde{S}}$ splits over a finite extension F' of F, we denote by $\Phi(S'_\sigma, \mathfrak{g}')$ (resp.; $\Phi(S', \mathfrak{h}')$) the set of roots of $\underline{S}_\sigma(\mathrm{F}')$ in $\mathfrak{g} \otimes_{\mathrm{F}} \mathrm{F}'$ (resp.; $\underline{S}(\mathrm{F}')$ in $\mathfrak{h} \otimes_{\mathrm{F}} \mathrm{F}'$).
Let $\underline{\tilde{\mathfrak{s}}}$ be the Lie algebra of $\underline{\tilde{S}}$. Then, the differential of each root α of $\Phi(\tilde{S}', \mathfrak{g}')$ defines a linear form on $\tilde{\mathfrak{s}} \otimes_{\mathrm{F}} \mathrm{F}'$ which we denote by the same letter. (1.27)

Let $\mathcal{G}al(\overline{\mathrm{F}}/\mathrm{F})$ be the Galois group of $\overline{\mathrm{F}}/\mathrm{F}$. By ([RR] §3), the set of (H, S_σ)-double cosets in $\underline{H}\underline{S}_\sigma \cap G$ are parametrized by the finite set I of cohomology classes in $H^1(\mathcal{G}al(\overline{\mathrm{F}}/\mathrm{F}), \underline{H} \cap \underline{S}_\sigma)$ which split in both $\underline{H}$ and $\underline{S}_\sigma$. To each such classe m, we attach an element $x_m \in G$ of the form $x_m = h_m a_m^{-1}$ with $h_m \in \underline{H}$ and $a_m \in \underline{S}_\sigma$ such that $m_\gamma = h_m^{-1}\gamma(h_m) = a_m^{-1}\gamma(a_m)$ for all $\gamma \in \mathcal{G}al(\overline{\mathrm{F}}/\mathrm{F})$.

1.1 Lemma. *Let $x \in G$ such that $x = hs$ with $h \in \underline{H}$ and $s \in \underline{\tilde{S}}$. Then, xSx^{-1} is a maximal torus of H and there exists $h' \in H$ such that $x' = h'x$ centralizes the split connected component A_S of S.*

Proof :

Replace S by a H-conjugate if necessary, we may assume that $A := A_S$ is contained in the fixed maximal split torus A_0 of H. Since H is split, A_0 is also a maximal split torus of G.

Since $x = hs \in G$, the torus $\underline{S}' := x\underline{S}x^{-1}$ is equal to $h\underline{S}h^{-1} \subset \underline{H}$. Thus $\underline{S}'$ is defined over F and contained in $\underline{H}$ and we obtain the first assertion.

Let $S' := \underline{S}'(\mathrm{F})$ and let A' be the split connected component of S'. There exists $h_1 \in H$ such that $h_1 A' h_1^{-1} \subset A_0$. We set $x_1 = h_1 x$, thus we have $A_1 := x_1 A x_1^{-1} \subset A_0$.

Let $M = Z_G(A)$ and $M_1 = Z_G(A_1) = x_1 M x_1^{-1}$. Then A_0 and $x_1 A_0 x_1^{-1}$ are maximal split tori of M_1. Therefore, there exists $y_1 \in M_1$ such that $y_1 x_1 A_0 x_1^{-1} y_1^{-1} = A_0$. As H is split, the Weyl group of A_0 in G coincides with the Weyl group of A_0 in H. Thus, there exist $h_2 \in N_H(A_0)$ and $v \in Z_G(A_0)$ such that $z := y_1 x_1 = h_2 v$.

For $a \in A \subset A_0$, one has $zaz^{-1} = h_2 a h_2^{-1} = y_1 x_1 a x_1^{-1} y_1^{-1} = x_1 a x_1^{-1}$ since $x_1 a x_1^{-1} \in A_1$ and $y_1 \in M_1$. One deduces that $x' := h_2^{-1} h_1 x$ centralizes A. □

Thus, for each maximal torus S of H, we can fix a finite set of representatives $\kappa_S = \{x_m\}_{m \in I}$ of the (H, S_σ)-double cosets in $\underline{H}\underline{S}_\sigma \cap G$ such that each element x_m may be written $x_m = h_m a_m^{-1}$ where $h_m \in \underline{H}$ centralizes A_S and $a_m \in \underline{S}_\sigma$. Hence x_m centralizes A_S. (1.28)

1.3 Weyl integration formula and orbital integrals

We first recall basic notions on the symmetric space according to ([RR], §3). An element x in $\underline{G}$ is called σ-semisimple if the double coset $\underline{H}x\underline{H}$ is Zariski closed. This is equivalent to say that $\underline{p}(x)$ is a semisimple point of $\underline{G}$. We say that a semisimple

element x is σ-regular if this closed double coset $\underline{H}x\underline{H}$ is of maximal dimension. This is equivalent to say that the centralizer of $\underline{p}(x)$ in $\mathfrak{q}$ (resp.; $\underline{\mathcal{P}}$) is a Cartan subspace of $\mathfrak{q}$ (resp.; a maximal σ-torus of $\underline{G}$).

We denote by $G^{\sigma-reg}$ the set of σ-regular elements of G.

For $g \in G$, we denote by $D_G(g)$ the coefficient of the least power of t appearing nontrivially in $\det(t+1-\mathrm{Ad}(g))$. We define the H-biinvariant function Δ_σ on G by $\Delta_\sigma(x) = D_G(\underline{p}(x))$. Then by ([RR], Lemma 3.2. and Lemma 3.3), the set of $g \in G$ such that $\Delta_\sigma(g) \neq 0$ coincides with $G^{\sigma-reg}$.
Let S be a maximal torus of H with Lie algebra $\mathfrak{s}$. Then $\tilde{\mathfrak{s}} := \mathfrak{s} \otimes_{\mathrm{F}} \mathrm{E}$ identifies with the Lie algebra of $\tilde{S}$. For $g \in x_m S_\sigma$ with $x_m \in \kappa_S$, one has

$$\Delta_\sigma(g) = D_G(\underline{p}(g)) = \det(1 - \mathrm{Ad}(\underline{p}(g)))_{\mathfrak{g}/\tilde{\mathfrak{s}}}. \tag{1.29}$$

By ([RR] Theorem 3.4 (1)), the set $G^{\sigma-reg}$ is a disjoint union

$$G^{\sigma-reg} = \bigcup_{\{S\}_H} \bigcup_{x_m \in \kappa_S} H\big((x_m S_\sigma) \cap G^{\sigma-reg}\big)H, \tag{1.30}$$

where $\{S\}_H$ runs the H-conjugacy classes of maximal tori of H.

If $x_m \in \kappa_S$ then $x_m = h_m a_m$ for some $h_m \in \underline{H}$ and $a_m \in \underline{S}_\sigma$, hence $\underline{p}(x_m) = a_m^{-2}$ commutes with S and S_σ. Therefore for $\gamma \in S_\sigma$, we have

$$\underline{p}(x_m\gamma) = \underline{p}(x_m)\gamma^{-2} \quad \text{and} \quad Hx_m\gamma S = Hx_m\gamma.$$

We have the following Weyl integration formula (cf. [RR] Theorem 3.4 (2)):

Let f be a compactly supported smooth function on G, then we have

$$\int_G f(y)dy = \sum_{\{S\}_H} \sum_{x_m \in \kappa_S} c^0_{S,x_m} \int_{S_\sigma} |\Delta_\sigma(x_m\gamma)|_{\mathrm{F}}^{1/2} \int_{S\backslash H} \int_H f(hx_m\gamma l)dhd\bar{l}d\gamma, \tag{1.31}$$

where the constants c^0_{S,x_m} are explicitly given in ([RR] Theorem 3.4 (1)).

For our purposes, we need another version of this Weyl integration formula.
Let S be a maximal torus of H. We denote by A_S its split connected component. Since the quotient $A_S\backslash S$ is compact, by our choice of measure, the integration over $S\backslash H$ in the Weyl formula above can be replaced by an integration over $A_S\backslash H$. Moreover, it is convenient to change h into h^{-1}. As every $x_m \in \kappa_S$ commutes with A_S (cf. (1.28)), one can replace the integration over $(A_S\backslash H)\times H$, by an integration over $diag(A_S)\backslash(H\times H)$ where $diag(A_S)$ is the diagonal of A_S. This gives the following Weyl integration formula equivalent to (1.31):

$$\int_G f(y)dy = \sum_{\{S\}_H} \sum_{x_m \in \kappa_S} c^0_{S,x_m} \int_{S_\sigma} |\Delta_\sigma(x_m\gamma)|_{\mathrm{F}}^{1/2} \int_{diag(A_S)\backslash(H\times H)} f(h^{-1}x_m\gamma l)d\overline{(h,l)}d\gamma. \tag{1.32}$$

We will now describe the H-conjugacy classes of maximal tori of H in terms of Levi subgroups M in $\mathcal{L}(A_0)$ and M-conjugacy classes of some tori of M.

Let $M \in \mathcal{L}(A_0)$. We denote by $N_H(M)$ its normalizer in H. If S is a maximal torus of M, we denote by $W(M,S)$ (resp. $W(H,S)$) its Weyl group in M (resp. H). We choose a set $\mathcal{T}_M$ of representatives for the M-conjugacy classes of maximal tori S in M such that $A_M\backslash S$ is compact. For $M, M' \in \mathcal{L}(A_0)$, we write $M \sim M'$ if M and M' are conjugate under H.

Let S be a maximal torus of H whose split connected component A_S is contained in A_0. Then, the centralizer M of A_S belongs to $\mathcal{L}(A_0)$ and S is a maximal torus of M such that $A_M\backslash S$ is compact. If S' is a maximal torus conjugated to S by H such that $A_{S'}$ is contained in A_0, then the centralizer M' of $A_{S'}$ in H belongs to $\mathcal{L}(A_0)$ and $M' \sim M$.

Since each maximal torus of H is H-conjugate to a maximal torus S such that $A_S \subset A_0$, we obtain a surjective map $S \mapsto \{S\}_H$ from the set of S in $\mathcal{T}_M$ where M runs a system of representatives of $\mathcal{L}(A_0)_{/\sim}$ to the set of H-conjugacy classes of maximal tori of H.

Let $M \in \mathcal{L}(A_0)$. By ([Ko] (7.12.3)), the cardinal of the class of M in $\mathcal{L}(A_0)_{/\sim}$ is equal to

$$\frac{|W(H,A_0)|}{|W(M,A_0)||N_H(M)/M|}$$

where $N_H(M)$ is the normalizer of M in H.
By ([Ko] Lemma 7.1), if S is a maximal torus of M, then the number of M-conjugacy classes of maximal torus S' in M such that S' is H-conjugate to S is equal to

$$\frac{|N_H(M)/M||W(M,S)|}{|W(H,S)|}.$$

Therefore, we can rewrite (1.32) as follows:

$$\int_G f(g)dg = \sum_{M\in\mathcal{L}(A_0)} c_M \sum_{S\in\mathcal{T}_M}\sum_{x_m\in\kappa_S} c_{S,x_m}\int_{S_\sigma} |\Delta_\sigma(x_m\gamma)|_{\mathrm{F}}^{1/2}\int_{diag(A_M)\backslash H\times H} f(h^{-1}x_m\gamma l)d\overline{(h,l)}d\gamma \tag{1.33}$$

where

$$c_M = \frac{|W(M,A_0)|}{|W(H,A_0)|} \quad \text{and} \quad c_{S,x_m} = \frac{|W(H,S)|}{|W(M,S)|}c^0_{S,x_m}.$$

Let $f \in C_c^\infty(G)$. We define the orbital integral $\mathcal{M}(f)$ of f on $G^{\sigma-reg}$ as follows. Let S a maximal torus of H. For $x_m \in \kappa_S$ and $\gamma \in S_\sigma$ with $x_m\gamma \in G^{\sigma-reg}$, we set

$$\begin{aligned}\mathcal{M}(f)(x_m\gamma) &:= |\Delta_\sigma(x_m\gamma)|_{\mathrm{F}}^{1/4}\int_{diag(A_S)\backslash(H\times H)} f(h^{-1}x_m\gamma l)d\overline{(h,l)}\\ &= |\Delta_\sigma(x_m\gamma)|_{\mathrm{F}}^{1/4}\int_{S\backslash H}\int_H f(hx_m\gamma l)dhd\bar{l}.\end{aligned} \tag{1.34}$$

Our definition corresponds, up to a positive constant, to Definition 3.8 of [RR]. Indeed, by definition of Δ_σ, we have $\Delta_\sigma(x_m\gamma) = D_G(\underline{p}(x_m\gamma))$. Since we can write $x_m = h_m a_m$ with $h_m \in \underline{H}$ and $a_m \in \underline{S}_\sigma$, we have $\underline{p}(x_m\gamma) = \underline{p}(x_m)\gamma^{-2} = a_m^{-2}\gamma^{-2}$ for $\gamma \in S_\sigma$. Let F' be an extension of E such that $\tilde{\underline{S}}$ splits over F' and $a_m \in \underline{S}_\sigma(\mathrm{F}')$. Since each root α of $\underline{S}_\sigma(\mathrm{F}')$ in $\mathfrak{g} \otimes \mathrm{F}'$ have multiplicity $m(\alpha) = 2$, using notation of (1.27), we obtain:

$$\Delta_\sigma(x_m\gamma) = \prod_{\alpha \in \Phi(S'_\sigma, \mathfrak{g}')} (1 - \underline{p}(x_m)^\alpha \gamma^{-2\alpha})^2 = \prod_{\alpha \in \Phi(S'_\sigma, \mathfrak{g}')} (\gamma^\alpha - \underline{p}(x_m)^\alpha \gamma^{-\alpha})^2,$$

hence

$$\begin{aligned} |\Delta_\sigma(x_m\gamma)|_{\mathrm{F}'}^{1/4} &= \prod_{\alpha \in \Phi(S'_\sigma, \mathfrak{g}')} |(\gamma^\alpha - \underline{p}(x_m)^\alpha \gamma^{-\alpha})^{m(\alpha)-1}|_{\mathrm{F}'}^{1/2}, \\ &= \prod_{\alpha \in \Phi(S'_\sigma, \mathfrak{g}')} |(\gamma^\alpha - \underline{p}(x_m)^\alpha \gamma^{-\alpha})|_{\mathrm{F}'}^{1/2}. \end{aligned}$$

Then, the Weyl integration formula (1.31) in terms of orbital integrals is given as in ([RR] page 126) by

$$\int_G f(y)dy = \sum_{\{S\}_H} \sum_{x_m \in \kappa_S} c^0_{S,x_m} \int_{S_\sigma} |\Delta_\sigma(x_m\gamma)|_{\mathrm{F}}^{1/4} \mathcal{M}(f)(x_m\gamma)d\gamma.$$

1.2 Theorem. *Let $f \in C_c^\infty(G)$ and S be a maximal torus of H. Let $x_m \in \kappa_S$.*

1. *There exists a compact set Ω in S_σ such that, for any γ in the complementary of Ω with $x_m\gamma \in G^{\sigma-reg}$, one has $\mathcal{M}(f)(x_m\gamma) = 0$.*

2. $$\sup_{\gamma \in S_\sigma;\ x_m\gamma \in G^{\sigma-reg}} |\mathcal{M}(f)(x_m\gamma)| < +\infty.$$

Proof :

The proof follows that of the group case (cf. [HC3] proof of Theorem 14). We write it for convenience of the reader.

1. Let ω be the support of f. We consider the set ω_S of elements γ in S_σ such that $x_m\gamma$ is in the closure of $H\omega H$. For $g \in G$, we consider the polynomial function

$$\det(1 - t - \mathrm{Ad}\ \underline{p}(g)) = t^n + q_{n-1}(g)t^{n-1} + \ldots + q_l(g)t^l \tag{1.35}$$

where l is the rank of G and n its dimension. Each q_j is a $H \times H$ biinvariant regular function on G, thus it is bounded on $x_m\omega_S$. Therefore, the roots of $\det(1 - t - \mathrm{Ad}\ \underline{p}(g))$ are bounded on $x_m\omega_S$.

For $\gamma \in S_\sigma$, we have $\underline{p}(x_m\gamma) = \underline{p}(x_m)\gamma^{-2}$. We choose a finite extension F' of F such that $\tilde{\underline{S}}$ splits over F' and $\underline{p}(x_m) \in \underline{S}_\sigma(\mathrm{F}')$. Using notation of (1.27), the roots of $\det(1 - t - \mathrm{Ad}\ \underline{p}(x_m\gamma))$ are the numbers $(1 - \underline{p}(x_m)^\alpha \gamma^{-2\alpha})$ for $\alpha \in \Phi(S'_\sigma, \mathfrak{g}')$. Since

these roots are bounded on $x_m\omega_S$, we deduce that the maps $\gamma \to \gamma^\alpha$, $\alpha \in \Phi(S'_\sigma, \mathfrak{g}')$, are bounded on ω_S. This implies that ω_S is bounded. Then, the closure Ω of ω_S satisfies the first assertion.

2. By *1.*, if $\gamma \notin \Omega$ then $\mathcal{M}(f)(x_m\gamma) = 0$. Thus, it is enough to prove that for each $\gamma_0 \in S_\sigma$, there exists a neighborhood V_{γ_0} of γ_0 in S_σ such that

$$\sup_{\gamma \in V_{\gamma_0}, x_m\gamma \in G^{\sigma-reg}} |\mathcal{M}(f)(x_m\gamma)| < +\infty. \tag{1.36}$$

Let $y_0 := \underline{p}(x_m\gamma_0)$. We first assume that y_0 is central in G. Then, we have $\Delta_\sigma(x_m\gamma_0\gamma) = D_G(y_0\gamma^{-2}) = D_G(\gamma^{-2})$ for $\gamma \in S_\sigma$ and $x_m\gamma_0 h(x_m\gamma_0)^{-1} \in H$ for $h \in H$. We define the function f_0 on G by $f_0(g) := f(x_m\gamma_0 g)$. Then, we have $\mathcal{M}(f_0)(\gamma) = \mathcal{M}(f)(x_m\gamma_0\gamma)$ for $\gamma \in S_\sigma \cap G^{\sigma-reg}$. Thus, we are reduced to the case $y_0 = 1$. As in the group case, we use the exponential map " exp " which is well defined in a neighborhood of 0 in $\mathfrak{g}$ since the characteristic of F is equal to zero (cf. [HC4] §10). As in ([HC1] proof of Lemma 15), we can choose a H-invariant open neighborhood V_0 of 0 in $\mathfrak{h}$ such that the map $X \in V_0 \mapsto \exp(\tau X)$ is an isomorphism and an homeomorphism onto its image and there is a H-invariant function $\varphi \in C_c^\infty(\mathfrak{h})$ such that $\varphi(X) = 1$ for $X \in V_0$. We define $\bar{f}$ in $C_c^\infty(\mathfrak{h})$ by $\bar{f}(X) = \varphi(X)\int_H f(h\exp(\tau X))dh$.

Let $\mathfrak{s}$ be the Lie algebra of S. For $X \in \mathfrak{s}$, we set $\eta(X) = |\det(\mathrm{ad}X)_{\mathfrak{h}/\mathfrak{s}}|_{\mathrm{F}}$. We consider a finite extension F' of F such that $\tilde{\underline{S}}$ splits over F' and $\underline{p}(x_m) \in \underline{S}_\sigma(\mathrm{F}')$. We use notation of (1.27). Since each root of S'_σ in $\mathfrak{g}'$ has multiplicity 2, we have for $X \in V_0$

$$\frac{|\Delta_\sigma(\exp\tau X)|_{\mathrm{F}'}^{1/2}}{\eta(X)} = \frac{|D_{G'}(\exp(-2\tau X)|_{\mathrm{F}'}^{1/2}}{\eta(X)} = \frac{\prod_{\alpha\in\Phi(S',\mathfrak{h}')}|1-e^{2\tau\alpha(X)}|_{\mathrm{F}'}}{\prod_{\alpha\in\Phi(S',\mathfrak{h}')}|\alpha(X)|_{\mathrm{F}'}}$$

$$= |2\tau|_{\mathrm{F}'}^{|\Phi(S',\mathfrak{h}')|}\prod_{\alpha\in\Phi(S',\mathfrak{h}')}|1+\tau\alpha(X)+\frac{4\tau^2\alpha(X)^2}{3!}+\dots|_{\mathrm{F}'}.$$

We can reduce V_0 in such way that each term of this product is equal to 1. Thus, we obtain

$$\mathcal{M}(f)(\exp\tau X) = |2\tau|_{\mathrm{F}'}^{|\Phi(S',\mathfrak{h}')|}\eta(X)^{1/2}\int_{H/S}\Big(\int_H f(h\exp\tau\mathrm{Ad}(l)X)dh\Big)d\bar{l}$$

$$= |2\tau|_{\mathrm{F}'}^{|\Phi(S',\mathfrak{h}')|}\eta(X)^{1/2}\int_{H/S}\bar{f}(\mathrm{Ad}(l)X)d\bar{l},$$

for $X \in V_0$. The estimate (1.36) follows from the result on the Lie algebra (cf. [HC3] Theorem 13).

If $y_0 = \underline{p}(x_m\gamma_0)$ is not central in G, we consider the centralizer $\underline{\mathcal{Z}}$ of y_0 in $\underline{H}$. Let $\underline{\mathcal{Z}}^0$ be the connected neutral component of $\underline{\mathcal{Z}}$. By([Bo], III.9), the group $\underline{\mathcal{Z}}^0$ is

defined over F. As usual, we set $\underline{\tilde{\mathcal{Z}}}^0 := \mathrm{Res}_{\mathrm{E/F}}(\underline{\mathcal{Z}}^0 \times_{\mathrm{F}} \mathrm{E})$ and we denote by $\underline{\tilde{\mathfrak{z}}}$ its Lie algebra. By definition of $\tilde{\mathfrak{z}}$, one has

$$|\det(1 - \mathrm{Ad}(y_0))_{\mathfrak{g}/\tilde{\mathfrak{z}}}|_{\mathrm{F}} \neq 0.$$

Thus, there exists a neighborhood V of 1 in S_σ such that, for all $\gamma \in V$, then

$$|\det(1 - \mathrm{Ad}(y_0\gamma^{-2}))_{\mathfrak{g}/\tilde{\mathfrak{z}}}|_{\mathrm{F}} = |\det(1 - \mathrm{Ad}(y_0)_{\mathfrak{g}/\tilde{\mathfrak{z}}})|_{\mathrm{F}} \neq 0. \tag{1.37}$$

From ([HC3] Lemma 19), there exist a neighborhood V_1 of y_0 in $\tilde{S}$ and a compact subset $\overline{C_G}$ of $\tilde{\mathcal{Z}}^0\backslash G$ such that, if $g \in G$ satsifies $g^{-1}V_1 g \cap \underline{p}(\omega) \neq \varnothing$ then its image $\bar{g}$ in $\tilde{Z}^0\backslash G$ belongs to $\overline{C_G}$ (here ω is the support of f).

We choose a neighborhood W of 1 in S_σ such that $W \subset V$ and $\underline{p}(x_m\gamma_0\gamma) = y_0\gamma^{-2} \in V_1$ for all $\gamma \in W$. By ([Bo], III 9.1), the quotient $\mathcal{Z}^0\backslash H$ is a closed subset of $\tilde{\mathcal{Z}}^0\backslash G$, hence

$$\begin{aligned}&\text{the set } \overline{C} := \overline{C_G} \cap \mathcal{Z}^0\backslash H \text{ is a compact subset of } \mathcal{Z}^0\backslash H \text{ such that if } l \in H\\ &\text{satisfies } l^{-1}y_0\gamma^{-2}l \in \underline{p}(\omega) \text{ for some } \gamma \in W \text{ then its image } \bar{l} \text{ in } \mathcal{Z}^0\backslash H\\ &\text{belongs to } \overline{C}.\end{aligned} \tag{1.38}$$

Let $\gamma \in W$ such that $x_m\gamma_0\gamma \in G^{\sigma-reg}$. One has

$$\int_{S\backslash H}\int_H f(hx_m\gamma_0\gamma l)dhd\bar{l} = \int_{\mathcal{Z}^0\backslash H}\int_{S\backslash\mathcal{Z}^0}\int_H f(hx_m\gamma_0\gamma\xi l)dhd\bar{\xi}d\bar{l}. \tag{1.39}$$

By the choice of W, the map

$$\bar{l} \in \mathcal{Z}^0\backslash H \mapsto \int_{S\backslash\mathcal{Z}^0}\int_H f(hx_m\gamma_0\gamma\xi l)dhd\bar{\xi}$$

vanishes outside $\bar{C}$. We choose $u \in C_c^\infty(H)$ such that the map $\overline{u} \in C_c^\infty(\mathcal{Z}^0\backslash H)$ defined by $\overline{u}(\bar{l}) := \int_{\mathcal{Z}^0} u(\xi l)d\xi$ is equal to 1 if $\bar{l} \in \overline{C}$. As u and f are compactly supported, the map

$$\Phi : z \in \tilde{\mathcal{Z}}^0 \mapsto \int_H u(l)\int_H f(hx_m\gamma_0 zl)dhdl$$

is well-defined. Since $y_0 = \underline{p}(x_m\gamma_0) = (x_m\gamma_0)^{-1}\sigma(x_m\gamma_0)$, we have $\xi(x_m\gamma_0)^{-1}\sigma(x_m\gamma_0) = (x_m\gamma_0)^{-1}\sigma(x_m\gamma_0)\xi$ for $\xi \in \mathcal{Z}^0$. Hence, $x_m\gamma_0\xi(x_m\gamma_0)^{-1} \in H$. Thus Φ is left invariant by $\mathcal{Z}^0$.

We claim that $\Phi \in C_c^\infty(\mathcal{Z}^0\backslash\tilde{\mathcal{Z}}^0)$. Indeed, fix l in the support of u. If $f(hx_m\gamma_0 zl)$ is nonzero for some $h \in H$ and $z \in \tilde{\mathcal{Z}}^0$ then $\underline{p}(hx_m\gamma_0 zl) = \underline{p}(x_m\gamma_0 zl)$ belongs to $\underline{p}(\omega)$, where ω is the support of f. Since z commutes with $y_0 = \underline{p}(x_m\gamma_0)$, we have $\underline{p}(x_m\gamma_0 zl) = l^{-1}y_0\underline{p}(z)\sigma(l)$. As u is compactly supported, we deduce that $\Phi(z) = 0$ when $\underline{p}(z)$ is outside a compact set. Hence, the map Φ is a compactly supported function on $\mathcal{Z}^0\backslash\tilde{\mathcal{Z}}^0$.

By assumption, the function f is right invariant by a compact open subgroup of G. Thus f is right invariant by some compact open subgroup of H. We denote by $\tau_l f$ the right translate of f by an element $l \in G$. Since u is compactly supported, the vector space generating by $\tau_l f$, when $l \in H$ runs the support of u, is finite dimensional. Hence, one can find a compact open subgroup J_1 of $\tilde{Z}^0$ such that for each l in the support of u, the function $\tau_l f$ is right invariant by J_1. This implies that Φ is smooth and our claim follows.

Therefore, there exists $\varphi \in C_c^\infty(\tilde{\mathcal{Z}}^0)$ such that

$$\Phi(z) = \int_{\mathcal{Z}^0} \varphi(\xi z) d\xi = \int_H u(l) \int_H f(h x_m \gamma_0 z l) dh dl, \quad z \in \tilde{\mathcal{Z}}^0.$$

We obtain

$$\int_{S\backslash\mathcal{Z}^0} \int_{\mathcal{Z}^0} \varphi(\xi_1 \gamma \xi_2) d\xi_1 d\bar{\xi_2} = \int_H u(l) \Big(\int_{S\backslash\mathcal{Z}^0} \int_H f(h x_m \gamma_0 \gamma \xi_2 l) dh d\bar{\xi_2} \Big) dl$$

$$= \int_{\mathcal{Z}^0\backslash H} \int_{\mathcal{Z}^0} u(\xi_1 l) \Big(\int_{S\backslash\mathcal{Z}^0} \int_H f(h x_m \gamma_0 \gamma \xi_2 \xi_1 l) dh d\bar{\xi_2} \Big) d\xi_1 d\bar{l}$$

$$= \int_{\mathcal{Z}^0\backslash H} \overline{u}(\bar{l}) \Big(\int_{S\backslash\mathcal{Z}^0} \int_H f(h x_m \gamma_0 \gamma \xi_2 l) dh d\bar{\xi_2} \Big) d\bar{l}.$$

By definition, the map $\bar{u}$ is equal to 1 on the compact set $\overline{C}$. By definition of $\overline{C}$ (cf. (1.38) and (1.3)), we obtain

$$\int_{S\backslash\mathcal{Z}^0} \int_{\mathcal{Z}^0} \varphi(\xi_1 \gamma \xi_2) d\xi_1 d\bar{\xi_2} = \int_{S\backslash H} \int_H f(h x_m \gamma_0 \gamma l) dh d\bar{l}.$$

By (1.37) and the choice of W, one has

$$|D_G(y_0 \gamma^{-2})|_{\mathrm{F}} = |D_{\tilde{\mathcal{Z}}^0}(\gamma^{-2})|_{\mathrm{F}} |\det(1 - \mathrm{Ad}(y_0))_{\mathfrak{g}/\tilde{\mathfrak{z}}}|_{\mathrm{F}}, \quad \gamma \in W.$$

Then, we deduce that for $\gamma \in W$ satisfying $x_m \gamma_0 \gamma \in G^{\sigma - reg}$, one has

$$\mathcal{M}(f)(x_m \gamma_0 \gamma) = |\det(1 - \mathrm{Ad}(y_0))_{\mathfrak{g}/\mathfrak{z}}|_{\mathrm{F}}^{1/4} |D_{\tilde{\mathcal{Z}}^0}(\gamma^{-2})|_{\mathrm{F}}^{1/4} \int_{S\backslash\mathcal{Z}^0} \int_{\mathcal{Z}^0} \varphi(\xi_1 \gamma \xi_2) d\xi_1 d\bar{\xi_2}.$$

Since $|D_{\tilde{\mathcal{Z}}^0}(\gamma^{-2})|_{\mathrm{F}}$ coincides with the function $|\Delta_\sigma|_{\mathrm{F}}$ for the group $\tilde{Z}^0$ evaluated at γ (cf. (1.29)), one deduces the estimate (1.36) for f applying the first case to φ defined on $\tilde{\mathcal{Z}}^0$. □

2 Geometric side of the local relative trace formula

2.1 Truncation

In this section, we will recall some results of ([Ar3], §3), needed in the sequel. We keep notation of §1.1 for the group H. Since H is split, one has $M_0 = A_0$. We fix a Levi subgroup $M \in \mathcal{L}(A_0)$ of H. Let $P \in \mathcal{P}(M)$. We recall that A_M denotes the maximal split connected component of M.

We denote by Σ_P the set of roots of A_M in the Lie algebra of P, Σ_P^r the subset of reduced roots and Δ_P the subset of simple roots.

For $\beta \in \Delta_P$, the "co-root" $\check{\beta} \in a_M$ is defined as usual as follows: if $P \in \mathcal{P}(A_0)$ is a minimal parabolic subgroup, then $\check{\beta} = 2(\beta, \beta)^{-1}\beta$, where a_0^* identifies with a_0 by the scalar product on a_0. In the general case, we choose $P_0 \in \mathcal{P}(A_0)$ contained in P. Then, there exists a unique $\alpha \in \Delta_{P_0}$ such that $\beta = \alpha_{|a_M}$. The "co-root" $\check{\beta}$ is the projection of $\check{\alpha}$ onto a_M with respect to the decomposition $a_0 = a_M \oplus a_0^M$. This projection does not depend of the choice of P_0.

We denote by a_P^+ the positive Weyl chamber of elements $X \in a_M$ satisfying $\alpha(X) > 0$ for all $\alpha \in \Sigma_P$.

Let $M \in \mathcal{L}(A_0)$. A set of points in a_M indexed by $P \in \mathcal{P}(M)$

$$\mathcal{Y} = \mathcal{Y}_M := \{Y_P \in a_M; P \in \mathcal{P}(M)\}$$

is called a (H, M)-orthogonal set if for all adjacent parabolic subgroups P, P' in $\mathcal{P}(M)$ whose chambers in a_M share the wall determined by the simple root $\alpha \in \Delta_P \cap (-\Delta_{P'})$, one has $Y_P - Y_P' = r_{P,P'}\check{\alpha}$ for a real number $r_{P,P'}$. The orthogonal set is called positive if each of the numbers $r_{P,P'}$ are nonnegative. This is the case for example if the number

$$d(\mathcal{Y}) = \inf_{\{\alpha \in \Delta_P; P \in \mathcal{P}(M)\}} \alpha(Y_P) \tag{2.1}$$

is nonnegative.
One example is the set

$$\{-h_P(x); P \in \mathcal{P}(M)\},$$

defined for any point $x \in H$. This is a positive (H, M)-orthogonal set by ([Ar1] Lemma 3.6).

> If L belongs to $\mathcal{L}(M)$ and Q is a group in $\mathcal{P}(L)$, we define Y_Q to be the projection onto a_L of any point Y_P, with $P \in \mathcal{P}(M)$ and $P \subset Q$. Then Y_Q is independent of P and $\mathcal{Y}_L := \{Y_Q; Q \in \mathcal{P}(L)\}$ is a (H, L)-orthogonal set. (2.2)

We shall write $\mathcal{S}_M(\mathcal{Y})$ for the convex hull in a_M/a_H of a (H, M)-orthogonal set $\mathcal{Y}$. Notice that $\mathcal{S}_M(\mathcal{Y})$ does only depend on the projection onto a_M^H of each $Y_P \in \mathcal{Y}$, $P \in \mathcal{P}(M)$.

Let $P \in \mathcal{P}(M)$. If each Y_P is in the positive Weyl chamber a_P^+ (this condition is equivalent to say that $d(\mathcal{Y})$ is positive), we have a simple description of $\mathcal{S}_M(\mathcal{Y}) \cap a_P^+$ ([Ar3] Lemma 3.1). We denote by $(\omega_\gamma^P)_{\gamma \in \Delta_P}$ the set of weights, that is the dual basis in $(a_M^H)^*$ of the set of co-roots $\{\check{\gamma}; \gamma \in \Delta_P\}$. Then, we have

$$\mathcal{S}_M(\mathcal{Y}) = \{X \in a_P^+; \omega_\gamma^P(X - Y_P) \leqslant 0, \gamma \in \Delta_P\}. \tag{2.3}$$

We now recall a decomposition of the characteristic function of $\mathcal{S}_M(\mathcal{Y})$ valid when $\mathcal{Y}$ is positive. (cf. [Ar3] (3.8)). Suppose that Λ is a point in $a_{M,\mathbb{C}}^*$ whose real part $\Lambda_R \in a_M^*$ is in general position. If $P \in \mathcal{P}(M)$, we define Δ_P^Λ the set of simple roots $\alpha \in \Delta_P$ such that $\Lambda_R(\check{\alpha}) < 0$. Let φ_P^Λ be the characteristic function of the set of $X \in a_M$ such that $\omega_\alpha^P(X) > 0$ for each $\alpha \in \Delta_P^\Lambda$ and $\omega_\alpha^P(X) \leqslant 0$ for each α in the complement of Δ_P^Λ in Δ_P. We define

$$\sigma_M(X, \mathcal{Y}) := \sum_{P \in \mathcal{P}(M)} (-1)^{|\Delta_P^\Lambda|} \varphi_P^\Lambda(X - Y_P). \tag{2.4}$$

By ([Ar3], §3 p22), the function $\sigma_M(\cdot, \mathcal{Y})$ vanishes on the complement of $\mathcal{S}_M(\mathcal{Y})$ and is bounded. Moreover, if $\mathcal{Y}$ is positive then $\sigma_M(\cdot, \mathcal{Y})$ is exactly the characteristic function of $\mathcal{S}_M(\mathcal{Y})$. (2.5)

The following Lemma will allow us to define the minimum of two orthogonal sets.

For $P \in \mathcal{P}(M)$, we denote by $(\tilde{\omega}_\gamma^P)_{\gamma \in \Delta_P}$ the set of coweights, that is the dual basis in a_M^H of the roots $\{\gamma; \gamma \in\in \Delta_P\}$.

2.1 Lemma. *Let P and P' two adjacent parabolic subgroups in $\mathcal{P}(M)$ whose chambers in a_M share the wall determined by the simple root $\alpha \in \Delta_P \cap (-\Delta_{P'})$. Then:*

1. *For all β in $\Delta_P - \{\alpha\}$, there exists a unique β' in $\Delta_{P'} - \{\alpha\}$ such that $\beta' = \beta + k_\beta \alpha$ where k_β is a nonnegative integer. Moreover, the map $\beta \mapsto \beta'$ is a bijection between $\Delta_P - \{\alpha\}$ and $\Delta_{P'} - \{-\alpha\}$.*

2. *For all β in $\Delta_P - \{\alpha\}$, one has $\tilde{\omega}_{\beta'}^{P'} = \tilde{\omega}_\beta^P$.*

Proof :

We denote by $\mathbb{N}$ the set of nonnegative integers and by $\mathbb{N}^*$ the subset of positive integers.

1. As P and P' are adjacent, we have $\Sigma_{P'} = \big(\Sigma_P - \{\alpha\}\big) \cup \{-\alpha\}$. Let $\beta \in \Delta_P - \{\alpha\}$. If $\beta \in \Delta_{P'}$ then we set $\beta' := \beta$.

Assume that β is not in $\Delta_{P'}$. Since $\beta \in \Sigma_{P'}$, there exists $\Theta \subset \Delta_{P'} - \{-\alpha\}$ such that $\beta = \sum_{\delta \in \Theta} n_\delta \delta - k_\beta \alpha$ where the n_δ's are positive integers and k_β is a nonnegative integer. Each δ in Θ belongs to Σ_P. Therefore, there are nonnegative integers $(r_{\delta,\eta})_{\eta \in \Delta_P}$ such that $\delta = \sum_{\eta \in \Delta_P} r_{\delta,\eta} \eta$. We set $\beta_1 := \sum_{\delta \in \Theta} n_\delta \delta = \beta + k_\beta \alpha$.

Let $\gamma \in \Delta_P - \{\alpha\}$. If $\gamma \neq \beta$, one has $\beta_1(\tilde{\omega}_\gamma^P) = \beta(\tilde{\omega}_\gamma^P) = 0$. Thus, for each $\delta \in \Theta$, we have $r_{\delta,\gamma} = 0$, hence $\delta = r_{\delta,\beta}\beta + r_{\delta,\alpha}\alpha$.

On the other hand, one has $\beta_1(\tilde{\omega}_\beta^P) = \beta(\tilde{\omega}_\beta^P) = 1$. Thus, for all $\delta \in \Theta$, one has $\sum_{\delta\in\Theta} n_\delta r_{\delta,\beta} = 1$. Since $n_\delta \in \mathbb{N}^*$ and $r_{\delta,\beta} \in \mathbb{N}$, one deduces that there exists a unique $\delta_0 \in \Theta$ such that $r_{\delta_0,\beta} \neq 0$ and we have $n_{\delta_0} = r_{\delta_0,\beta} = 1$. This implies that $\Theta = \{\delta_0\}$ and $\beta = \delta_0 - k_\beta\alpha$. We can take $\beta' := \delta_0$. Hence, we obtain the existence of β' in all cases.

If $\beta'_1 \in \Delta_{P'}$ satisfies $\beta'_1 = \beta + k^1_\beta\alpha$ then $\beta' = \beta'_1 + (k_\beta - k^1_\beta)\alpha$. Since the roots β'_1, β' and $-\alpha$ belong to the set of simple roots $\Delta_{P'}$, we deduce that $\beta'_1 = \beta'$. This gives the unicity of β'.

Let γ and β be in Δ_P such that $\gamma' = \beta'$. Then we have $\beta = \gamma + (k_\gamma - k_\beta)\alpha$. Since γ, β and α belong to Δ_P, the same argument as above leads to $\beta = \gamma$. Hence, the map $\beta \mapsto \beta'$ is injective.

2. Let $\beta \in \Delta_P - \{\alpha\}$. By definition, we have $\beta' = \beta + k_\beta\alpha \in \Delta_{P'} - \{-\alpha\}$ with $k_\beta \in \mathbb{N}$. Thus we have $\alpha(\tilde{\omega}_{\beta'}^{P'}) = \alpha(\tilde{\omega}_\beta^P) = 0$ and $\beta(\tilde{\omega}_{\beta'}^{P'}) = \beta'(\tilde{\omega}_{\beta'}^{P'}) = 1$. If $\gamma \in \Delta_P - \{\beta, \alpha\}$, then $\gamma' = \gamma + k_\gamma\alpha$ is different from β' by (*1.*), thus we have $\gamma(\tilde{\omega}_{\beta'}^{P'}) = \gamma'(\tilde{\omega}_{\beta'}^{P'}) = 0$. One deduces that $\tilde{\omega}_{\beta'}^{P'} = \tilde{\omega}_\beta^P$. □

For Y^1 and Y^2 in a_M, we denote by $\inf^P\{Y^1, Y^2\}$ the unique element Z in a_M^H such that, for all $\gamma \in \Delta_P$, one has $(\tilde{\omega}_\gamma^P, Z) = \inf\{(\tilde{\omega}_\gamma^P, Y^1), (\tilde{\omega}_\gamma^P, Y^2)\}$. (2.6)

2.2 Lemma. *Let $\mathcal{Y}^1 = \{Y_P^1, P \in \mathcal{P}(M)\}$ and $\mathcal{Y}^2 = \{Y_P^2, P \in \mathcal{P}(M)\}$ be two (H, M)-orthogonal sets. Let $\mathcal{Z} := \inf(\mathcal{Y}^1, \mathcal{Y}^2)$ be the set of $Z_P := \inf^P\{Y_P^1, Y_P^2, \}$ when P runs $\mathcal{P}(M)$.*

1. *The set $\mathcal{Z}$ is a (H, M)-orthogonal set.*
2. *If $d(\mathcal{Y}^j) > 0$ for $j = 1, 2$ then $d(\mathcal{Z}) > 0$. In this case, the convex hull $\mathcal{S}_M(\mathcal{Z})$ is the intersection of $\mathcal{S}_M(\mathcal{Y}^1)$ and $\mathcal{S}_M(\mathcal{Y}^2)$.*

Proof :

1. Let P and P' two adjacent parabolic subgroups in $\mathcal{P}(M)$ whose chambers in a_M share the wall determined by the simple root $\alpha \in \Delta_P \cap (-\Delta_{P'})$. Let $\gamma \in \Delta_P - \{\alpha\}$. By definition of orthogonal sets, for $j = 1$ or 2, one has $(\tilde{\omega}_\gamma^P, Y_P^j) = (\tilde{\omega}_\gamma^P, Y_{P'}^j)$. By Lemma 2.1, we have $\tilde{\omega}_\gamma^P = \tilde{\omega}_{\gamma'}^{P'}$. Hence we obtain $(\tilde{\omega}_\gamma^P, Z_P) = (\tilde{\omega}_{\gamma'}^{P'}, Z_{P'})$ and $(\tilde{\omega}_{\gamma'}^{P'}, Z_{P'}) = (\tilde{\omega}_\gamma^P, Z_{P'})$. Since the scalar product on a_0 identifies a_M to a_M^*, one deduces that $Z_P - Z_{P'}$ is proportional to $\check{\alpha}$.

2. Let $j \in \{1, 2\}$ and $P \in \mathcal{P}(M)$. By definition, we have $d(\mathcal{Y}^j) > 0$ if and only if $\alpha(Y_P^j) > 0$ for all $\alpha \in \Delta_P$. By ([Ar1] Corollary 2.2), this implies that $(\tilde{\omega}_\alpha^P, Y_P^j) > 0$ for all $\alpha \in \Delta_P$. Let $\alpha \in \Delta_P$. Writing

$$Y_P^j = (\tilde{\omega}_\alpha^P, Y_P^j)\alpha + \sum_{\beta\in\Delta_P-\{\alpha\}} (\tilde{\omega}_\beta^P, Y_P^j)\beta + X^j,$$

with $X^j \in a_H$, the condition $\alpha(Y_P^j) > 0$ is equivalent to

$$\sum_{\beta\in\Delta_P-\{\alpha\}} (\tilde{\omega}_\gamma^P, Y_P^j)[-(\beta,\alpha)] < (\tilde{\omega}_\alpha^P, Y_P^j)(\alpha,\alpha).$$

Since the real numbers $(\tilde{\omega}_\beta^P, Y_P^j)$ for $\beta \in \Delta_P$ and $-(\beta,\alpha)$ for $\alpha \neq \beta$ in Δ_P are nonnegative, one deduces that

$$\sum_{\beta\in\Delta_P-\{\alpha\}} (\tilde{\omega}_\beta^P, Z_P)[-(\beta,\alpha)] = \sum_{\beta\in\Delta_P-\{\alpha\}} \inf\big((\tilde{\omega}_\beta^P, Y_P^1), (\tilde{\omega}_\beta^P, Y_P^2)\big)[-(\beta,\alpha)]$$

$$\leqslant \inf\Big(\sum_{\beta\in\Delta_P-\{\alpha\}} (\tilde{\omega}_\beta^P, Y_P^1)[-(\beta,\alpha)], \sum_{\beta\in\Delta_P-\{\alpha\}} (\tilde{\omega}_\beta^P, Y_P^2)[-(\beta,\alpha)]\Big)$$

$$< \inf\big((\tilde{\omega}_\alpha^P, Y_P^1), (\tilde{\omega}_\alpha^P, Y_P^2)\big)(\alpha,\alpha) = (\tilde{\omega}_\alpha^P, Z_P)(\alpha,\alpha).$$

One deduces that $\alpha(Z_P) > 0$ for $\alpha \in \Delta_P$, thus $d(\mathcal{Z}) > 0$.

For the property of the convex hulls, it is enough to prove that, for all $P \in \mathcal{P}(M)$, one has $a_P^+ \cap \mathcal{S}_M(\mathcal{Y}^1) \cap \mathcal{S}_M(\mathcal{Y}^2) = a_P^+ \cap \mathcal{S}_M(\mathcal{Z})$. By ([Ar3], Lemma 3.1), one has

$$a_P^+ \cap \mathcal{S}_M(\mathcal{Y}^j) = \{X \in a_P^+; \omega_\gamma^P(X - Y_P^j) \leqslant 0, \gamma \in \Delta_P\}.$$

Since $\tilde{\omega}_\gamma^P = c_\gamma \omega_\gamma^P$ for $\gamma \in \Delta_P$, where c_γ is a positive real number, the assertion follows easily. □

2.2 The truncated kernel

We consider the regular representation R of $G \times G$ on $L^2(G)$ defined by

$$\big(R(y_1,y_2)\phi\big)(x) = \phi(y_1^{-1}xy_2), \quad \phi \in L^2(G), y_1, y_2 \in G.$$

Consider $f \in C_c^\infty(G \times G)$ of the form $f(y_1,y_2) = f_1(y_1)f_2(y_2)$ with $f_j \in C_c^\infty(G)$. Then

$$R(f) := \int_G \int_G f_1(y_1)f_2(y_2)R(y_1,y_2)dy_1dy_2$$

is an integral operator with smooth kernel

$$K_f(x,y) = \int_G f_1(xg)f_2(gy)dg = \int_G f_1(g)f_2(x^{-1}gy)dg.$$

In our case (H is split), one has $A_H = A_G$, and the kernel K_f is invariant by the diagonal $diag(A_H)$ of A_H. Since H is not compact, we introduce truncation to integrate this kernel on $diag(A_H)\backslash(H \times H)$.

We fix a point T in $a_{0,\mathrm{F}}$. If $P_0 \in \mathcal{P}(A_0)$, let T_{P_0} be the unique translate by the Weyl group $W(H, A_0)$ of T in the closure $\bar{a}^+_{P_0}$ of the positive Weyl chamber $a^+_{P_0}$. Then

$$\mathcal{Y}_T := \{T_{P_0}; P_0 \in \mathcal{P}(A_0)\}$$

is a (H, A_0)-orthogonal set. We shall assume that the number

$$d(T) := \inf_{\alpha \in \Delta_{P_0},\, P_0 \in \mathcal{P}(A_0)} \alpha(T_{P_0})$$

is suitable large. This means that the distance from T to any of the root hyperplanes in a_0 is large.

We denote by $u(\cdot, T)$ the characteristic function in $A_H\backslash H$ of the set of points x such that

$$x = k_1 a k_2 \text{ with } a \in A_H\backslash A_0, k_1, k_2 \in K \text{ and } h_{A_0}(a) \in \mathcal{S}_{A_0}(\mathcal{Y}_T), \tag{2.7}$$

where $H = KA_0K$ is the Cartan decomposition of H.

We consider $u(\cdot, T)$ as a A_H-invariant function on H. Thus, there is a compact set Ω_T of H such that if $u(x, T) \neq 0$ then $x \in A_H\Omega_T$. Let Ω be a compact subset of G containing the support of f_1 and f_2. We consider $g \in G$ and $x_1, x_2 \in H$ such that $f_1(g)f_2(x_1^{-1}gx_2)u(x_1, T)u(x_2, T) \neq 0$. Thus, there are ω_1, ω_2 in Ω_T and a_1, a_2 in A_H such that $x_1 = \omega_1 a_1$, $x_2 = \omega_2 a_2$ and we have $g \in \Omega$ and $x_1^{-1}gx_2 = \omega_1^{-1}g\omega_2 a_1^{-1}a_2 \in \Omega$ since $A_H = A_G$. One deduces that $a_1^{-1}a_2$ lies a compact subset of A_H. Therefore the map $(g, x_1, x_2) \mapsto f_1(g)f_2(x_1^{-1}gx_2)u(x_1, T)u(x_2, T)$ is a compactly supported function on $G \times diag(A_H)\backslash(H \times H)$.
Hence, we can define

$$K^T(f) := \int_{diag(A_H)\backslash H\times H} K_f(x_1, x_2)u(x_1, T)u(x_2, T)d\overline{(x_1, x_2)}.$$

By Fubini's Theorem, we have

$$K^T(f) = \int_G \int_{diag(A_H)\backslash H\times H} f_1(g)f_2(x_1^{-1}gx_2)u(x_1, T)u(x_2, T)d\overline{(x_1, x_2)}dg.$$

We apply the Weyl integration formula (1.33). Thus, we obtain

$$K^T(f) = \sum_{M\in\mathcal{L}(A_0)} c_M \sum_{S\in\mathcal{T}_M} \sum_{x_m\in\kappa_S} c_{S,x_m} \int_{S_\sigma} K^T(x_m, \gamma, f)d\gamma, \tag{2.8}$$

where, for $S \in \mathcal{T}_M$, $x_m \in \kappa_S$ and $\gamma \in S_\sigma$, we have

$$K^T(x_m, \gamma, f) = |\Delta_\sigma(x_m\gamma)|^{1/2}_{\mathrm{F}} \int_{diag(A_M)\backslash H\times H} \int_{diag(A_H)\backslash H\times H} f_1(y_1^{-1}x_m\gamma y_2)$$

$$\times f_2(x_1^{-1}y_1^{-1}x_m\gamma y_2x_2)u(x_1,T)u(x_2,T)d\overline{(x_1,x_2)}d\overline{(y_1,y_2)}.$$

We recall that each x_m in κ_S and γ in S_σ commute with A_M for $S \in \mathcal{T}_M$.
We first replace (x_1,x_2) by (y_1x_1,y_2x_2) in the integral over $\overline{(x_1,x_2)}$. The resulting integral over $diag(A_H)\backslash H\times H$ can be expressed as a double integral over $a \in A_H\backslash A_M$ and $(x_1,x_2) \in diag(A_M)\backslash H\times H$ which depends on $\overline{(y_1,y_2)} \in diag(A_M)\backslash H\times H$. Since A_M commutes with $x_m \in \kappa_S$ and $\gamma \in S_\sigma$, we obtain

$$K^T(x_m,\gamma,f) = |\Delta_\sigma(x_m\gamma)|_{\mathrm{F}}^{1/2}\int_{diag(A_M)\backslash H\times H}\int_{diag(A_M)\backslash H\times H} f_1(y_1^{-1}x_m\gamma y_2)$$

$$\times f_2(x_1^{-1}x_m\gamma x_2)u_M(x_1,y_1,x_2,y_2,T)d\overline{(x_1,x_2)}d\overline{(y_1,y_2)} \tag{2.9}$$

where

$$u_M(x_1,y_1,x_2,y_2,T) = \int_{A_H\backslash A_M} u(y_1^{-1}ax_1,T)u(y_2^{-1}ax_2,T)da.$$

Our goal is to prove that $K^T(f)$ is asymptotic to an expression $J^T(f)$ where $J^T(f)$ is obtained in a similar way to $K^T(f)$ where we replace the weight function $u_M(x_1,y_1,x_2,y_2,T)$ by another weight function $v_M(x_1,y_1,x_2,y_2,T)$ defined as follows.

We fix $M \in \mathcal{L}(A_0)$ and $P \in \mathcal{P}(M)$. Let $P_0 \in \mathcal{P}(A_0)$ be contained in P. We denote by T_P the projection of T_{P_0} on a_M according to the decomposition $a_0 = a_M \oplus a_0^M$. By (2.2), the set $\mathcal{Y}_M(T) := \{T_P; P \in \mathcal{P}(M)\}$ is a (H,M)-orthogonal set independent of the choices of P_0. Moreover, by ([Ar3] (3.2)), we have $d(\mathcal{Y}_M(T)) \geqslant d(T) > 0$. Thus, $\mathcal{Y}_M(T)$ is positive.

For x,y in H, we set

$$Y_P(x,y,T) := T_P + h_P(y) - h_{\overline{P}}(x).$$

By ([Ar3], page 30), the set $\mathcal{Y}_M(x,y,T) := \{Y_P(x,y,T); P \in \mathcal{P}(M)\}$ is a (H,M)-orthogonal set, which is positive when $d(T)$ is sufficiently large relative to x and y.

For x_1,x_2,y_1 and y_2 in H, we set

$$Z_P(x_1,y_1,x_2,y_2,T) := \inf{}^P(Y_P(x_1,y_1,T),Y_P(x_2,y_2,T)) \tag{2.10}$$

where $\inf^P$ is defined in (2.6) and

$$\mathcal{Y}_M(x_1,y_1,x_2,y_2,T) := \{Z_P(x_1,y_1,x_2,y_2,T); P \in \mathcal{P}(M)\}. \tag{2.11}$$

By Lemma 2.6, the set $\mathcal{Y}_M(x_1,y_1,x_2,y_2,T)$ is a (H,M)-orthogonal set. Moreover, when $d(T)$ is large relative to x_i,y_i, for $i=1,2$, one has $d(\mathcal{Y}_M(x_1,y_1,x_2,y_2,T)) > 0$,

hence this set is positive. We define the weight function v_M by

$$v_M(x_1,y_1,x_2,y_2,T) := \int_{A_H\backslash A_M} \sigma_M(h_M(a),\mathcal{Y}_M(x_1,y_1,x_2,y_2,T))da \tag{2.12}$$

where σ_M is defined in (2.4).

We set

$$J^T(f) := \sum_{M\in\mathcal{L}(A_0)} c_M \sum_{S\in\mathcal{T}_M}\sum_{x_m\in\kappa_S} c_{S,x_m}\int_{S_\sigma} J^T(x_m,\gamma,f)d\gamma, \tag{2.13}$$

where

$$J^T(x_m,\gamma,f) = |\Delta_\sigma(x_m\gamma)|_{\mathrm{F}}^{1/2}\int_{diag(A_M)\backslash H\times H}\int_{diag(A_M)\backslash H\times H} f_1(y_1^{-1}x_m\gamma y_2) \tag{2.14}$$

$$\times f_2(x_1^{-1}x_m\gamma x_2)v_M(x_1,y_1,x_2,y_2,T)d\overline{(x_1,x_2)}d\overline{(y_1,y_2)}.$$

Our main result is the following. We will prove it in section 2.4.

2.3 Theorem. *Let $\delta>0$. Then, there are positive numbers C and ε such that for all T with $d(T)\geqslant\delta\|T\|$, one has*

$$|K^T(f)-J^T(f)|\leqslant Ce^{-\varepsilon\|T\|}. \tag{2.15}$$

2.3 Preliminaries to estimates

We fix a norm $\|\cdot\|$ on G as in (1.15). Let F' be a finite extension of F. We set $\underline{G}' := \underline{G}\times_{\mathrm{F}}\mathrm{F}'$ and $G' := \underline{G}'(\mathrm{F}')$. One can extend the absolute value $|\cdot|_{\mathrm{F}}$ to F', and the norm $\|\cdot\|$ to G'. For x,y in G', we set

$$\|(x,y)\| := \|x\|\|y\|.$$

To obtain our estimates, we will use notation of (1.18) and (1.19). Since the norm takes values greater or equal to 1, we will freely apply the properties (1.20).

2.4 Lemma. *Let S be a maximal torus of H and let M be the centralizer of A_S in H. We fix $x_m\in G\cap\underline{MS}_\sigma=\tilde{M}\cap\underline{MS}_\sigma$. Then, one has*

$$\inf_{s\in S}\|(sx_m^{-1}x_1,sx_2)\|\preccurlyeq\inf_{s'\in\underline{S}(\mathrm{F}')}\|(s'x_m^{-1}x_1,s'x_2)\|,\quad x_1,x_2\in H. \tag{2.16}$$

Proof :

Since H^1A_H is of finite index in H, using (1.21) we may assume that x_1,x_2 belong to H^1A_H. Since $A_G=A_H$, using the invariance of the property by the left action of $diag(A_H)$ on (x_1,x_2), it is enough to prove the result for $x_1\in H^1$ and $x_2=a_2y_2$ with $a_2\in A_H$ and $y_2\in H^1$.

To establish (2.16), we first assume that $A_S = A_H$ which implies that the quotient $A_H\backslash S$ is compact. By (1.21), there is a positive constant C such that

$$\inf_{s\in S} \|(sx_m^{-1}x_1, sx_2)\| \leqslant C \inf_{a\in A_H} \|(ax_m^{-1}x_1, ax_2)\|.$$

We deduce from (1.17) that

$$\|(ax_m^{-1}x_1, ax_2)\| \leqslant \|x_m^{-1}\|\|a\|^2\|a_2\|\|x_1\|\|y_2\|.$$

Taking the lower bound in $a \in A_H$, there is a positive constant C_1 such that

$$\inf_{s\in S} \|(sx_m^{-1}x_1, sx_2)\| \leqslant C_1\|x_1\|\|a_2\|\|y_2\|. \tag{2.17}$$

We now use the following Lemma of [Ar3] (Lemma 4.1):

If S_0 is a maximal torus of H with $A_H\backslash S$ compact, then there exists an element $s_0 \in S_0$ such that

$$\|y\| \leqslant \|y^{-1}s_0y\|, \quad y \in H^1. \tag{2.18}$$

We apply this Lemma to $S_0 = S$. Since $\underline{S}(\mathrm{F}')$ commutes with s_0, using the property (1.17) of the norm, one deduces

$$\|y_2\| \leqslant \|s'y_2\|^2\|s_0\|, \quad y_2 \in H^1, s' \in \underline{S}(\mathrm{F}'). \tag{2.19}$$

On the other hand $S_1 := x_mSx_m^{-1}$ is a maximal torus of H which satisfies $A_{S_1} = A_H$ since $x_m \in G \cap \underline{MS}_\sigma$. Applying (2.18) to $S_0 = S_1$, there exists $s_1 \in S$ such that

$$\|x_1\| \leqslant \|x_1^{-1}x_ms_1x_m^{-1}x_1\|, \quad x_1 \in H^1. \tag{2.20}$$

The same argument as above leads to

$$\|x_1\| \leqslant \|s'x_m^{-1}x_1\|^2\|s_1\|, \quad x_1 \in H^1, s' \in \underline{S}(\mathrm{F}'). \tag{2.21}$$

Then, by (2.17), (2.19) and (2.21), and applying the properties (1.20), we deduce that

$$\inf_{s\in S} \|(sx_m^{-1}x_1, sa_2y_2)\| \leqslant \|s'x_m^{-1}x_1\|\|s'y_2\|\|a_2\|, \quad s' \in \underline{S}(\mathrm{F}'), x_1, y_2 \in H^1, a_2 \in A_H. \tag{2.22}$$

To obtain our result, we have to prove that

$$\|s'x_m^{-1}x_1\|\|s'y_2\|\|a_2\| \leqslant \|(s'x_m^{-1}x_1, s'a_2y_2)\|, \quad s' \in \underline{S}(\mathrm{F}'), x_1, y_2 \in H^1, a_2 \in A_H. \tag{2.23}$$

We can write $\underline{S} = \underline{T}\underline{A}_H$ where $\underline{T}$ is a maximal torus of the derived group $\underline{H}_{der}$ of

$\underline{H}$. We set $T' := \underline{T}(\mathrm{F}')$ and $A'_H := \underline{A}_H(\mathrm{F}')$. Then T' is contained in H'^1. Moreover, the intersection of $\underline{T}$ and $\underline{A}_H$ is finite. Hence, one has the exact sequence

$$1 \to \underline{T} \cap \underline{A}_H \to \underline{T} \times \underline{A}_H \to \underline{S} \to 1.$$

Going to F'-points, the long exact sequence in cohomology implies that $T'A'_H$ is of finite index in $\underline{S}(\mathrm{F}')$. By (1.21), it is enough to prove (2.23) for $s' = t'a' \in \underline{S}(\mathrm{F}')$ with $t' \in T'$ and $a' \in A'_H$. By (1.5), if $x_1 \in H^1$ then $x_1 \in H'^1 \subset G'^1$ and $x_m^{-1}x_1x_m \in G'^1$. Since H is split, we have $A'_H = A'_G$. Then (1.23) gives

$$\|a't'x_m^{-1}x_1\| \approx \|a't'x_m^{-1}x_1x_m\| \approx \|a'\|\|t'x_m^{-1}x_1x_m\|, \quad a' \in A'_H, t' \in T', x_1 \in H^1,$$

and

$$\|a't'y_2\| \approx \|a'\|\|t'y_2\| \quad a' \in A'_H, t' \in T', y_2 \in H^1.$$

Applying (1.20), we deduce that

$$\|t'a'x_m^{-1}x_1\|\|a't'y_2\|\|a_2\| \approx \|a_2\|\|a'\|^2\|t'x_m^{-1}x_1x_m\|\|t'y_2\| \approx \|a_2\|\|a'\|\|t'x_m^{-1}x_1x_m\|\|t'y_2\|, \tag{2.24}$$

for $t' \in T', a' \in A'_H, x_1, y_2 \in H^1, a_2 \in A_H$.
Let us prove that

$$\|a'\|\|a'a_2\| \approx \|a'\|\|a_2\|, \quad a' \in A'_H, a_2 \in A_H. \tag{2.25}$$

We have $\|a'a_2\| \leqslant \|a'\|\|a_2\|$ by (1.17). Then $\|a'\|\|a'a_2\| \leqslant (\|a'\|\|a_2\|)^2$ since $1 \leqslant \|a_2\|$. As $\|a'\| = \|a'a_2a_2^{-1}\| \leqslant \|a'a_2\|\|a_2\|$, we have $\|a'\|\|a_2\| \leqslant (\|a'a_2\|\|a_2\|)^2$ and (2.25) follows. Applying (2.25) in (2.24), we deduce that

$$\|t'a'x_m^{-1}x_1\|\|a't'y_2\|\|a_2\| \leqslant \|a'\|\|t'x_m^{-1}x_1x_m\|\|a'a_2\|\|t'y_2\|, \tag{2.26}$$

for $t' \in T', a' \in A'_H, x_1, y_2 \in H^1, a_2 \in A_H$.

Since $x_m^{-1}H^1x_m \subset G'^1$ and $A'_H = A'_G$, we obtain from (1.23)

$$\|a'\|\|t'x_m^{-1}x_1x_m\| \approx \|a't'x_m^{-1}x_1x_m\| \approx \|a't'x_m^{-1}x_1\|, \quad a' \in A'_H, t' \in T', x_1 \in H^1,$$

and

$$\|a'a_2\|\|t'y_2\| \approx \|a'a_2t'y_2\|, \quad a' \in A'_H, t' \in T', a_2 \in A_H, y_2 \in H^1.$$

Applying this in (2.26) and using (1.20), we deduce that

$$\|t'a'x_m^{-1}x_1\|\|a't'y_2\|\|a_2\| \leqslant \|a't'x_m^{-1}x_1\|\|a't'a_2y_2\| \tag{2.27}$$

for $a' \in A'_H$, $t' \in T'$, $x_1, y_2 \in H^1$.

Then, the property (2.23) follows. This finishes the proof of the Lemma when $A_H\backslash S$ is compact.

We now prove (2.16) for any maximal torus S of H. Let A_S be the maximal split torus of S and M be the centralizer of A_S in H. Thus we have $A_M = A_S$ and $A_M\backslash S$ is compact. Let $P = MN_P \in \mathcal{P}(M)$ and let K be a compact subgroup of H such that $H = PK$. Each $x \in H$ can be written $x = m_P(x)n_P(x)k(x)$ with $m_P(x) \in M, n_P(x) \in N_P$ and $k(x) \in K$. Then, there is a positive constant C such that

$$\inf_{s\in S} \|(sx_m^{-1}x_1, sx_2)\| \leqslant C \inf_{s\in S} \big(\|sx_m^{-1}m_P(x_1)\|\|sm_P(x_2)\|\big)\|n_P(x_1)\|\|n_P(x_2)\|, \tag{2.28}$$

for $x_1, x_2 \in H$. By assumption on x_m, there are $h_m \in \underline{M}$ and $a_m \in \underline{S}_\sigma$ such that $x_m = h_m a_m \in \tilde{M}$. Hence, we can applied the first part of the proof to (M, S) instead of (H, S). Therefore, we obtain

$$\inf_{s\in S} \|(sx_m^{-1}x_1, sx_2)\| \leqslant \inf_{s'\in \underline{S}(\mathrm{F}')} \big(\|s'x_m^{-1}m_P(x_1)\|\|s'm_P(x_2)\|\big)\|n_P(x_1)\|\|n_P(x_2)\|, \quad x_1, x_2 \in H.$$

To compare the right hand-side of this inequality to those of (2.16), we will use the Iwasawa decomposition (1.12) of H'. Let K' be a compact subgroup of H' such that $H' = \underline{P}(\mathrm{F}')K' = \underline{M}(\mathrm{F}')\underline{N}_P(\mathrm{F}')K'$. According to (1.13), each y in H' can be written $y = m'_P(y)n'_P(y)k'$ with $m'_P(y) \in \underline{M}(\mathrm{F}')$, $n'_P(y) \in \underline{N}_P(\mathrm{F}')$ and $k' \in K'$. Then for $x \in H$ and $z \in \underline{M}(\mathrm{F}')$, we have $zx = zm_P(x)n_P(x)k = m'_P(zx)n'_P(zx)k'$ with $k \in$ and $k' \in K'$. Hence, since K and K' are compact subsets, there is a positive constant C' such that

$$\|n'_P(zx)^{-1}m'_P(zx)^{-1}zm_P(x)n_P(x)\| \leqslant C', \quad z \in \underline{M}(\mathrm{F}'), x \in H.$$

Since $zm_p(x) \in \underline{M}(\mathrm{F}')$ for $z \in \underline{M}(\mathrm{F}')$ and $x \in H$, we deduce from (1.22) that there is a positive constant C_1 such that for $x \in H$ and $z \in \underline{M}(\mathrm{F}')$, one has

$$\|n'_P(zx)^{-1}m'_P(zx)^{-1}zm_P(x)n'_P(zx)\| \leqslant C_1 \quad \text{and} \quad \|n'_P(zx)^{-1}n_P(x)\| \leqslant C_1.$$

By (1.17), we obtain

$$\|zm_P(x)\| \leqslant C_1\|m'_P(zx)\|\|n'_P(zx)\|^2 \quad \text{and} \quad \|n_P(x)\| \leqslant C_1\|n'_P(zx)\|.$$

Using (1.22) again, it follows that

$$\|zm_P(x)\| \leqslant \|zx\|, \quad \text{and } \|n_P(x)| \leqslant \|zx\|, \quad z \in \underline{M}(\mathrm{F}'), x \in H,$$

hence by (1.20)

$$\|zm_P(x)\|\|n_P(x)| \leqslant \|zx\|, \quad z \in \underline{M}(\mathrm{F}'), x \in H. \tag{2.29}$$

We deduce that

$$\|s'm_P(x_2)\|\|n_P(x_2)| \leqslant \|s'x_2\|, \quad s' \in \underline{S}(\mathrm{F}'), x_2 \in H. \tag{2.30}$$

Since $x_m = h_m a_m$ with $h_m \in \underline{M}$ and $a_m \in \underline{S}_\sigma$, one has $x_m s' x_m^{-1} \in \underline{M} \cap H' = \underline{M}(\mathrm{F}')$ for $s' \in \underline{S}(\mathrm{F}')$. Therefore, we deduce from (2.29) that

$$\|x_m s' x_m^{-1} m_P(x_1)\| |n_P(x_1)| \leqslant \|x_m s' x_m^{-1} x_1\|, \quad s' \in \underline{S}(\mathrm{F}'), x_1 \in H. \tag{2.31}$$

Since $\|s' x_m^{-1} m_P(x_1)\| \leqslant \|x_m^{-1}\| \|x_m s' x_m^{-1} m_P(x_1)\|$ and $\|x_m s' x_m^{-1} x_1\| \leqslant \|x_m\| \|s' x_m^{-1} x_1\|$, we deduce the estimate (2.16) from (2.28), (2.30) and (2.31). This finishes the proof of the Lemma. □

The following Lemma is the analogue of Lemma 4.2 of ([Ar3]).

2.5 Lemma. *Let S be a maximal torus of H and let $x_m \in \kappa_S$. Then, there is a positive integer k with the property that, for any given compact subset Ω of G, there exists a positive constant C_Ω such that, for all $\gamma \in S_\sigma$ with $x_m \gamma \in G^{\sigma-reg}$, and all x_1, x_2 in H satisfying $x_1^{-1} x_m \gamma x_2 \in \Omega$, one has*

$$\inf_{s \in S} \|(s x_m^{-1} x_1, s x_2)\| \leqslant C_\Omega |\Delta_\sigma(x_m \gamma)|_{\mathrm{F}}^{-k}.$$

Proof :

Let F' be a finite extension of E such that $\underline{\tilde{S}}$ splits over F'. Recall that we can write $x_m = h_m a_m$ with $h_m \in \underline{H}$ and $a_m \in \underline{S}_\sigma$. Thus we may and will choose F' such that $h_m \in \underline{H}(\mathrm{F}')$ and $a_m \in \underline{S}_\sigma(\mathrm{F}')$. For convenience of lecture, if $\underline{J}$ is an algebraic variety defined over F, we set $J' := \underline{J}(\mathrm{F}')$.

By the previous Lemma 2.4, it is enough to prove the existence of a positive integer k satisfying the property that for any compact subset Ω' of $G'^{\sigma-reg}$, there exists $C_{\Omega'} > 0$ such that

$$\inf_{s' \in S'} \left(\|s' x_m^{-1} x_1\| \|s' x_2\| \right) \leqslant C_{\Omega'} |\Delta_\sigma(x_m \gamma)|_{\mathrm{F}}^{-k} \tag{2.32}$$

for all $x_1, x_2 \in H'$ and $\gamma \in S_\sigma$ satisfying $x_m \gamma \in G^{\sigma-reg}$ and $x_1^{-1} x_m \gamma x_2 \in \Omega'$.

Let $B' = S'N'$ be a Borel subgroup of H' containing S' and K' be a compact subgroup of H' such that $H' = S'N'K' = N'S'K'$. We can also write $H' = (h_m S' h_m^{-1})(h_m N' h_m^{-1})(h_m K' h_m^{-1})$. By (1.21), one can reduce the proof of the statement for $x_1 \in (h_m S' h_m^{-1})(h_m N' h_m^{-1})$ and $x_2 \in S'N'$.

Let $x_1 = h_m s_1 n_1 h_m^{-1}$ and $x_2 = s_1 s_2 n_2$ with $s_1, s_2 \in S'$ and $n_1, n_2 \in N'$. Since $x_m = h_m a_m$, we have $x_m s_1 x_m^{-1} = h_m s_1 h_m^{-1}$, hence for $s' \in S'$, we have $s' x_m^{-1} x_1 = s' x_m^{-1} x_m s_1 x_m^{-1} h_m n_1 h_m^{-1} = s' s_1 x_m^{-1} h_m n_1 h_m^{-1}$. We obtain

$$\inf_{s' \in S'} \left(\|s' x_m^{-1} x_1\| \|s' x_2\| \right) = \inf_{s' \in S'} \left(\|s' x_m^{-1} h_m n_1 h_m^{-1}\| \|s' s_2\| \right).$$

Notice that $x_1^{-1} x_m \gamma x_2 = h_m n_1^{-1} h_m^{-1} x_m s_1^{-1} x_m^{-1} x_m \gamma s_1 s_2 n_2 = h_m n_1^{-1} h_m^{-1} x_m \gamma s_2 n_2$.

Therefore, we are reduced to prove (2.32) for $x_1 = h_m n_1 h_m^{-1}$ with $n_1 \in N'$, $x_2 = s_2 n_2$ with $n_2 \in N'$, $s_2 \in S'$ and $\gamma \in S_\sigma$ such that $x_m \gamma$ is σ-regular and

$x_1^{-1}x_m\gamma x_2 \in \Omega'$. By the properties of the norm, there is some positive constant C' such that

$$\inf_{s'\in S'} \big(\|s'x_m^{-1}x_1\|\|s'x_2\|\big) \leqslant C'\|n_1\|\|s_2\|\|n_2\|, \quad x_1 = h_m n_1 h_m^{-1}, x_2 = s_2 n_2. \tag{2.33}$$

We want to estimate $\|n_1\|\|s_2\|\|n_2\|$ when $x_1 = h_m n_1 h_m^{-1}$ and $x_2 = s_2 n_2$ satisfy $x_1^{-1}x_m\gamma x_2 \in \Omega'$. For this, we use the isomorphism Ψ from G' to $H' \times H'$ defined in (1.26). If $x \in H'$ then $\Psi(x) = (x, x)$ and if $y \in G$ satisfies $y^{-1} = \sigma(y)$ then $\Psi(y) = (y, y^{-1})$. We set $(y_1, y_2) := \Psi(x_1^{-1}x_m\gamma x_2)$. Then, we have

$$y_1 = h_m n_1^{-1} a_m \gamma n_2 s_2 = h_m (n_1^{-1} a_m \gamma n_2 (a_m\gamma)^{-1})(a_m\gamma s_2),$$

and

$$y_2 = h_m n_1^{-1} a_m^{-1}\gamma^{-1} n_2 s_2 = h_m \big(n_1^{-1} a_m^{-1}\gamma^{-1} n_2 \gamma a_m\big)(a_m\gamma)^{-1} s_2.$$

Since $H' = N'S'K'$, the condition $x_1^{-1}x_m\gamma x_2 \in \Omega'$ implies that there exist two compact subsets $\Omega_N \subset N'$ and $\Omega_S \subset S'$ depending only on Ω' such that

$$n_1^{-1} a_m \gamma n_2 (a_m\gamma)^{-1} \in \Omega_N, \quad \text{and} \quad n_1^{-1} a_m^{-1}\gamma^{-1} n_2 \gamma a_m \in \Omega_N,$$

$$a_m\gamma s_2 \in \Omega_S, \quad \text{and} \quad (a_m\gamma)^{-1} s_2 \in \Omega_S.$$

We deduce from the second property that s_2 and γ must lie in compact subsets of S'. We set

$$\nu_1(\gamma, n_1, n_2) := n_1^{-1} a_m \gamma n_2 (a_m\gamma)^{-1} \quad \text{and} \quad \nu_2(\gamma, n_1, n_2) := n_1^{-1} (a_m\gamma)^{-1} n_2 a_m \gamma.$$

We consider the map ψ from $N' \times N'$ into itself defined by $\psi(n_1, n_2) = (\nu_1, \nu_2)$. Recall that $\Phi(S', \mathfrak{h}')$ denotes the set of roots of S' in the Lie algebra $\mathfrak{h}'$ of H' (cf. 1.27). Let $\mathfrak{n}'$ be the Lie algebra of N'. For $\alpha \in \Phi(S', \mathfrak{h}')$, we denote by $X_\alpha \in \mathfrak{n}'$ the root vector in $\mathfrak{h}'$ corresponding to α. Then $(a_m\gamma)$ acts on X_α by $a_\alpha := (a_m\gamma)^\alpha$. The differential $d_{(n_1,n_2)}\psi$ of ψ at $(n_1, n_2) \in N' \times N'$ is given by $d_{(n_1,n_2)}\psi(X_1, X_2) = (\mathrm{Ad}(a_m\gamma n_2^{-1}(a_m\gamma)^{-1})Y_1, \mathrm{Ad}((a_m\gamma)^{-1}n_2^{-1}a_m\gamma)Y_2)$ where

$$Y_1 = -\mathrm{Ad}(n_1)X_1 + \mathrm{Ad}(a_m\gamma)\mathrm{Ad}(n_2)X_2$$

and

$$Y_2 = -\mathrm{Ad}(n_1)X_1 + \mathrm{Ad}(a_m\gamma)^{-1}\mathrm{Ad}(n_2)X_2.$$

The map $(X_1, X_2) \mapsto (Y_1, Y_2)$ is the composition of the map $(X_1, X_2) \mapsto (\mathrm{Ad}(n_1)X_1, \mathrm{Ad}(n_2)X_2)$, whose determinant is equal to 1, with $d_e\psi$ where e is the neutral point of $N' \times N'$. We deduce that the jacobian of ψ at (n_1, n_2) is independent of (n_1, n_2). At the neutral point $e \in N' \times N'$, we have $d_e\psi(X_\alpha, 0) = (-X_\alpha, -X_\alpha)$ and $d_e\psi(0, X_\alpha) = (a_\alpha X_\alpha, a_{-\alpha}X_\alpha)$. Hence, the jacobian of ψ is equal to

$$\Big|\prod_{\alpha\in\Phi(S',\mathfrak{h}')} a_\alpha(1-a_{-2\alpha})\Big|_{\mathrm{F}'} = |\det(\mathrm{Ad}(a_m\gamma))_{\mathfrak{h}'/\mathfrak{s}'}|_{\mathrm{F}'}|\det(1-\mathrm{Ad}(a_m\gamma)^{-2})_{\mathfrak{h}'/\mathfrak{s}'}|_{\mathrm{F}'} = |D_{H'}((a_m\gamma)^{-2})|_{\mathrm{F}'}.$$

Recall that $x_m\gamma$ is assumed to be σ-regular. Thus, by (1.29), one has $\Delta_\sigma(x_m\gamma) = D_{H'}(a_m^{-2}\gamma^{-2}) \neq 0$. Then, arguing as in ([HC2] proof of Lemma 10 and Lemma 11), we deduce that the map ψ is an F′-rational isomorphism of $\underline{N} \times \underline{N}$ to itself whose inverse $(\nu_1, \nu_2) \mapsto (n_1, n_2) := (n_1(\gamma, \nu_1, \nu_2), n_2(\gamma, \nu_1, \nu_2))$ is rational. Moreover, there is a positive integer k such that the map

$$(y, \nu_1, \nu_2) \mapsto D_{\underline{H}}(y)^k(n_1(y, \nu_1, \nu_2), n_2(y, \nu_1, \nu_2))$$

is defined by an F′-rational morphism between the algebraic varieties $\underline{S} \times \underline{N} \times \underline{N}$ and $\underline{N} \times \underline{N}$. Since ν_1, ν_2 and γ lie in compact subsets depending only on Ω', one deduces that there exists a constant $C_{\Omega'} > 0$ such that

$$\|(n_1(\gamma, \nu_1, \nu_2), n_2(\gamma, \nu_1, \nu_2))\| \leqslant C_{\Omega'}|D_{H'}(a_m^{-2}\gamma^{-2})|_{\mathrm{F}'}^{-k} = C_{\Omega'}|\Delta_\sigma(x_m\gamma)|_{\mathrm{F}}^{-k}.$$

The Lemma follows from (2.33) and the fact that s_2 lies in a compact set. □

2.4 Proof of Theorem 2.3

Our goal is to prove that $|K^T(f) - J^T(f)|$ is bounded by a function which approaches 0 as T approaches infinity. By definition, $K^T(f)$ and $J^T(f)$ are finite linear combinations of $\int_{S_\sigma} K^T(x_m, \gamma, f)d\gamma$ and $\int_{S_\sigma} J^T(x_m, \gamma, f)d\gamma$ respectively, where $M \in \mathcal{L}(A_0)$, S is a maximal torus of M satisfying $A_S = A_M$ and $x_m \in \kappa_S$ (cf. (2.8) and (2.13)).

We fix $M \in \mathcal{L}(A_0)$ and a maximal torus S of M such that $A_S = A_M$. Let $x_m \in \kappa_S$. To obtain our result, it is enough to establish the estimate (2.15) for $\int_{S_\sigma} |K^T(x_m, \gamma, f) - J^T(x_m, \gamma, T)|d\gamma$. This will be done in the Corollary 2.9 below.

For $\varepsilon > 0$, we define

$$S_\sigma(\varepsilon, T) := \{\gamma \in S_\sigma; 0 < |\Delta_\sigma(x_m\gamma)|_{\mathrm{F}} \leqslant e^{-\varepsilon\|T\|}\}. \tag{2.34}$$

2.6 Lemma. *1. There exists $\varepsilon_0 > 0$ such that the map $\gamma \mapsto |\Delta_\sigma(x_m\gamma)|_{\mathrm{F}}^{-\varepsilon_0}$ is locally integrable on S_σ.*

2. Let $\varepsilon > 0$. Let B be a bounded subset of S_σ and p be a nonnegative integer. Then, there is a positive constant $C_{B,p}$ depending on B and p, such that

$$\int_{S_\sigma(\varepsilon,T)\cap B} |\log|\Delta_\sigma(x_m\gamma)|_{\mathrm{F}}^{-1}|^p d\gamma \leqslant C_{B,p}e^{-\frac{\varepsilon\varepsilon_0\|T\|}{2}}.$$

Proof :

1. The proof follows those of the group case, we use the similar statement on Lie algebras and the exponential map. We denote by $\mathfrak{s}$ the Lie algebra of S. For $X \in \mathfrak{s}$, we set $\eta(X) = |det(\mathrm{ad}X_{|\mathfrak{h}/\mathfrak{s}})|_{\mathrm{F}}$. By ([HC3] Lemma 44), there exists $\varepsilon_0 > 0$ such that

$X \mapsto \eta(X)^{-2\varepsilon_0}$ is locally integrable on $\mathfrak{s}$. To obtain the result, it is sufficient to prove that

$$\text{for each } \gamma_0 \in S_\sigma, \text{ there exists a compact neighborhood } U_0 \text{ of } 1 \text{ such that the integral } \int_{U_0} |\Delta_\sigma(x_m\gamma_0\gamma)|_{\mathrm{F}}^{-\varepsilon_0} d\gamma \text{ converges.} \tag{2.35}$$

If $x_m\gamma_0$ is σ-regular, there is a compact neighborhood U_0 of 1 in S_σ such that $|\Delta_\sigma(x_m\gamma_0\gamma)|_{\mathrm{F}} = |\Delta_\sigma(x_m\gamma_0)|_{\mathrm{F}} \neq 0$ for all $\gamma \in U_0$. Hence (2.35) is clear.

We assume that $x_m\gamma_0$ is not σ-regular. We choose an extension F' of E such that $\tilde{S}$ splits over F' and $\underline{p}(x_m) \in \underline{\tilde{S}}_\sigma(\mathrm{F}')$. We use notation of (1.27). Let Φ_0 be the set of root α in $\Phi(S'_\sigma, \mathfrak{g}')$ such that $\underline{p}(x_m\gamma_0)^\alpha = 1$. We set

$$\nu(\gamma) = \prod_{\alpha \in \Phi(S'_\sigma, \mathfrak{g}') - \Phi_0} |1 - \underline{p}(x_m\gamma_0)^\alpha \gamma^{-2\alpha}|_{\mathrm{F}'}^2.$$

We have $\Delta_\sigma(x_m\gamma_0\gamma) = D_{G'}(\underline{p}(x_m\gamma_0)\gamma^{-2}) = det(1 - \mathrm{Ad}\underline{p}(x_m\gamma_0)\gamma^{-2})_{|\mathfrak{g}/\tilde{\mathfrak{s}}}$ and each root of $\Phi(S'_\sigma, \mathfrak{g}')$ has multiplicity 2. Hence, we obtain

$$|\Delta_\sigma(x_m\gamma_0\gamma)|_{\mathrm{F}'} = \nu(\gamma) \prod_{\alpha \in \Phi_0} |1 - \gamma^{-2\alpha}|_{\mathrm{F}'}^2.$$

We choose a compact neighborhood W of 1 in S_σ such that $\nu(\gamma) = \nu(1) \neq 0$ for $\gamma \in W$. Let $\beta = \sup_{\gamma \in W} \prod_{\alpha \in \Phi(S'_\sigma, \mathfrak{g}') - \Phi_0} |1 - \gamma^{-2\alpha}|_{\mathrm{F}'}^2$. Then, for $\gamma \in W$, we have

$$\beta|\Delta_\sigma(x_m\gamma_0\gamma)|_{\mathrm{F}'} = \beta\nu(1) \prod_{\alpha \in \Phi_0} |1 - \gamma^{-2\alpha}|_{\mathrm{F}'}^2 \geqslant \nu(1)|\Delta_\sigma(\gamma)|_{\mathrm{F}'}.$$

Consider the exponential map, there exist two open neighborhoods ω and U of 0 and 1 in $\mathfrak{s}$ and S_σ respectively, such that the map $X \mapsto \exp(\tau X)$ is well-defined on ω and is an isomorphism and an homeomorphism onto U. For $X \in \omega$, we have

$$\frac{|\Delta_\sigma(\exp(\tau X)|_{\mathrm{F}'}^{1/2}}{\eta(X)} = \prod_{\alpha \in \Phi(S'_\sigma, \mathfrak{g}')} \frac{|1 - e^{2\tau\alpha(X)}|_{\mathrm{F}'}}{|\alpha(X)|_{\mathrm{F}'}}.$$

We can choose a compact neighborhood $\omega_0 \subset \omega$ of 0 in $\mathfrak{s}$ such that this product is a positive constant c and $U_0 := \exp(\tau\omega_0)$ is contained in W. We deduce that

$$\int_{U_0} |\Delta_\sigma(x_m\gamma_0\gamma)|_{\mathrm{F}}^{-\varepsilon_0} d\gamma \leqslant \left(\frac{\nu(1)}{\beta}\right)^{-\varepsilon_0} \int_{U_0} |\Delta_\sigma(\gamma)|_{\mathrm{F}}^{-\varepsilon_0} d\gamma = \left(\frac{\nu(1)}{\beta}\right)^{-\varepsilon_0} c \int_{\omega_0} \eta(X)^{-2\varepsilon_0} dX.$$

The right hand side of this inequality is finite by our choice of ε_0. Hence, we have proved (2.35).

2. Let $\varepsilon_0 > 0$ as in *1.* We set $I_p = \int_{S_\sigma(\varepsilon,T)\cap B} |\log|\Delta_\sigma(x_m\gamma)|_{\mathrm{F}}^{-1}|^p d\gamma$.

If p is a positive integer, then there is positive constant C' such that $|\log y|^p \leqslant C' y^{\varepsilon_0/2}$ for all $y \geqslant 1$. Since $|\Delta_\sigma(x_m\gamma)|_{\mathrm{F}}^{-1} \geqslant e^{\varepsilon\|T\|} \geqslant 1$ for all $\gamma \in S_\sigma(\varepsilon, T)$, we obtain

$$I_p \leqslant C' \int_{S_\sigma(\varepsilon,T)\cap B} |\Delta_\sigma(x_m\gamma)|_{\mathrm{F}}^{-\varepsilon_0/2} d\gamma \leqslant C' e^{-\frac{\varepsilon\varepsilon_0\|T\|}{2}} \int_B |\Delta_\sigma(x_m\gamma)|_{\mathrm{F}}^{-\varepsilon_0} d\gamma.$$

If $p = 0$ then by definition of $S_\sigma(\varepsilon, T)$, one has

$$I_0 = \int_{S_\sigma(\varepsilon,T)\cap B} |\Delta_\sigma(x_m\gamma)|_{\mathrm{F}}^{-\varepsilon_0} |\Delta_\sigma(x_m\gamma)|_{\mathrm{F}}^{\varepsilon_0} d\gamma \leqslant e^{-\varepsilon\varepsilon_0\|T\|} \int_B |\Delta_\sigma(x_m\gamma)|_{\mathrm{F}}^{-\varepsilon_0} d\gamma.$$

In the two cases, the result follows from *1.* □

2.7 Lemma. *Let $\varepsilon_0 > 0$ as in Lemma 2.6. Given $\varepsilon > 0$, we can choose a constant $c > 0$ such that for any $T \in a_{0,\mathrm{F}}$, one has*

$$\int_{S_\sigma(\varepsilon,T)} \big(|K^T(x_m, \gamma, f)| + |J^T(x_m, \gamma, f)|\big) d\gamma \leqslant c e^{-\frac{\varepsilon\varepsilon_0\|T\|}{4}}.$$

Proof :

We recall that

$$K^T(x_m, \gamma, f) = |\Delta_\sigma(x_m\gamma)|^{1/2} \int_{diag(A_M)\backslash H\times H} \int_{diag(A_M)\backslash H\times H} f_1(y_1^{-1} x_m \gamma y_2)$$

$$\times f_2(x_1^{-1} x_m \gamma x_2) u_M(x_1, y_1, x_2, y_2, T) d\overline{(x_1, x_2)} d\overline{(y_1, y_2)}$$

where

$$u_M(x_1, y_1, x_2, y_2, T) = \int_{A_H\backslash A_M} u(y_1^{-1} a x_1, T) u(y_2^{-1} a x_2, T) da.$$

We first establish an estimate of u_M. Let $x, y \in H$ and $a \in A_M$. According to (1.11) applied to H, we can write $y^{-1}ax = k_1 a_0 k_2$ with $k_1, k_2 \in K$ and $a_0 \in A_0$. By definition of the norm, there is a positive constant C_0 such that

$$\log \|y^{-1}ax\| \leqslant C_0(\|h_{A_0}(a_0)\| + 1).$$

If $u(y^{-1}ax) \neq 0$, then, by definition of $u(\cdot, T)$ (cf. (2.7)), the projection of $h_{A_0}(a_0)$ in $a_H\backslash a_M$ belongs to the convex hull in $a_H\backslash a_M$ of the $W(H, A_0)$-translates of T. Thus, there is a constant $C_1 > 0$ such that

$$\inf_{z\in A_H} \log \|y^{-1} z a x\| \leqslant C_1(\|T\| + 1). \tag{2.36}$$

We assume that $\|T\| \geqslant 1$. Taking $C_2 = \max(2C_1, 1)$ and using the property (1.17) of the norm, we obtain

$$\inf_{z\in A_H} \log\|za\| \leqslant C_2(\|T\| + \log\|x\| + \log\|y\|). \tag{2.37}$$

We apply this to (x_1, y_1) and (x_2, y_2) such that $u(y_1^{-1}ax_1, T)u(y_2^{-1}ax_2, T) \neq 0$. Hence, we deduce that

$$\inf_{z\in A_H} \log\|za\| \leqslant C_2(\|T\| + \log\|x_1\| + \log\|y_1\| + \log\|x_2\| + \log\|y_2\|).$$

As $\|x\| \leqslant \|x_m\|\|x_m^{-1}x\|$ and $1 \leqslant \|T\|$, taking the integral over $a \in A_H\backslash A_M$, we deduce the following inequality

$$u_M(x_1, y_1, x_2, y_2, T) \leqslant (\|T\|+\log\|x_m^{-1}x_1\|+\log\|x_m^{-1}y_1\|+\log\|x_2\|+\log\|y_2\|), \tag{2.38}$$

for all x_1, y_1, x_2 and y_2 in H.

The function $u_M(x_1, y_1, x_2, y_2, T)$ is invariant by the diagonal (left) action of A_M on (x_1, x_2) and (y_1, y_2) respectively. Since x_m commutes with $A_S = A_M$ (cf. Lemma 1.1), we can replace $\log\|x_m^{-1}x_1\| + \log\|x_2\|$ and $\log\|x_m^{-1}y_1\| + \log\|y_2\|$ by $\inf_{a\in A_M} \log\|(ax_m^{-1}x_1, ax_2)\|$ and $\inf_{a\in A_M} \log\|(ax_m^{-1}y_1, ay_2)\|$ respectively. By assumption, the quotient $A_M\backslash S$ is compact, then, using (1.21), one has

$$\inf_{a\in A_M} \|(ax_m^{-1}x, ax')\| \approx \inf_{s\in S} \|(sx_m^{-1}x, sx')\|, \quad x, x' \in H.$$

Therefore, as $\|T\| \geqslant 1$, the inequality (2.38) gives

$$u_M(x_1, y_1, x_2, y_2, T) \leqslant \|T\|+\log\inf_{s\in S}\|(sx_m^{-1}x_1, sx_2)\|+\log\inf_{s\in S}\|(sx_m^{-1}y_1, sy_2)\|, x_1, y_1, x_2, y_2 \in H.$$

In other words, this means that there are a positive constant C_3 and a positive integer d such that, for all x_1, y_1, x_2 and $y_2 \in H$, one has

$$u_M(x_1, y_1, x_2, y_2, T) \leqslant C_3(\|T\| + \log\inf_{s\in S}\|(sx_m^{-1}x_1, sx_2)\| + \log\inf_{s\in S}\|(sx_m^{-1}y_1, sy_2)\|)^d.$$

Let Ω be a compact set containing the support of f_1 and f_2. By Lemma 2.5, there is a positive integer k (independent of Ω) and a positive constant C_Ω such that, if $x_m\gamma \in x_m S_\sigma$ is a σ-regular point with $f_1(y_1^{-1}x_m\gamma y_2)f_2(x_1^{-1}x_m\gamma x_2) \neq 0$ for some x_1, x_2, y_1 and y_2 in H then

$$u_M(x_1, y_1, x_2, y_2, T) \leqslant C_\Omega(\|T\| + \log|\Delta_\sigma(x_m\gamma)|^{-k})^d.$$

This inequality and the expression of $K^T(x_m, \gamma, f)$ give

$$|K^T(x_m, \gamma, f)| \leqslant C_\Omega(\|T\| + \log|\Delta_\sigma(x_m\gamma)|^{-k})^d|\mathcal{M}(f_1)(x_m\gamma)\mathcal{M}(f_2)(x_m\gamma)|, \tag{2.39}$$

where $\mathcal{M}(f_j)$ is the orbital integral of f_j defined in (1.34). By Theorem 1.2, these orbital integrals are bounded by a constant C_4 on $(x_m S_\sigma) \cap G^{\sigma-reg}$. Hence, we obtain

$$|K^T(x_m, \gamma, f)| \leqslant C_\Omega C_4^2 (\|T\| + \log |\Delta_\sigma(x_m\gamma)|^{-k})^d.$$

Let B be the set of γ in S_σ such that $K^T(x_m, \gamma, f) \neq 0$. Then B is bounded by Theorem 1.2 and (2.39). Using Lemma 2.6, we can find a constant $C > 0$ such that

$$\int_{S_\sigma(\varepsilon,T)} |K^T(x_m, \gamma, f)| d\gamma \leqslant C e^{-\frac{\varepsilon\varepsilon_0\|T\|}{4}}. \tag{2.40}$$

If $\|T\| \leqslant 1$, then (2.36) implies that if $u(x^{-1}ay) \neq 0$ then

$$\inf_{z\in A_H} \log \|za\| \leqslant 2C_1 + \log \|x\| + \log \|y\|.$$

The same arguments to obtain (2.38) imply that there is a positive constant C_1' such that

$$u_M(x_1, y_1, x_2, y_2, T) \leqslant (C_1' + \log \|x_m^{-1}x_1\| + \log \|x_m^{-1}y_1\| + \log \|x_2\| + \log \|y_2\|), \tag{2.41}$$

for x_1, y_1, x_2 and y_2 in H. Replacing $\|T\|$ by C_1' in the reasoning after (2.38), we deduce that $\int_{S_\sigma(\varepsilon,T)} |K^T(x_m, \gamma, f)| d\gamma$ is bounded. Hence, one obtains (2.40) for $\|T\| \leqslant 1$.

We will now establish a similar estimate when K^T is replaced by J^T. For this, it is enough to prove that the weight function v_M have an estimate like (2.38). We will see that this follows easily from the definition of v_M. Indeed, for x_1, y_1, x_2 and y_2 in H, one has by definition

$$v_M(x_1, y_1, x_2, y_2, T) := \int_{A_H\backslash A_M} \sigma_M(h_M(a), \mathcal{Y}_M(x_1, y_1, x_2, y_2, T)) da$$

where $\sigma_M(\cdot, \mathcal{Y}_M(x_1, y_1, x_2, y_2, T))$ is a bounded function which vanishes in the complement of the convex hull $\mathcal{S}_M(\mathcal{Y}_M(x_1, y_1, x_2, y_2, T))$ of the (H, M)-orthogonal set $\mathcal{Y}_M(x_1, y_1, x_2, y_2, T)$ (cf. (2.5)). Since $\mathcal{Y}_M(x_1, y_1, x_2, y_2, T)$ is the set of points $Z_P = \inf^P(T_P + h_P(y_1) - h_{\overline{P}}(x_1), T_P + h_P(y_2) - h_{\overline{P}}(x_2))$ for $P \in \mathcal{P}(M)$ (cf. (2.11)), if $\sigma_M(X, \mathcal{Y}_M(x_1, y_1, x_2, y_2, T)) \neq 0$ then $\|X\| \leqslant \|Z_P\|$ for $P \in \mathcal{P}(M)$. By definition of T_P, one has $\|T_P\| \leqslant \|T\|$.

Let us prove that for $P \in \mathcal{P}(M)$, one has

$$\|h_P(x)\| \leqslant 1 + \log \|x\|, \quad x \in H. \tag{2.42}$$

We first compare $\|m\|$ and $\|h_M(m)\|$ for $m \in M$. Let $M = K_M A_0 K_M$ be the Cartan decomposition of M where K_M is a suitable compact subgroup of M. Then, each $m \in M$ can be written $m = ka(m)k'$ with $k, k' \in K_M$ and $a(m) \in A_0$. Since K_M is

compact, the property (1.21) gives $\|m\| \approx \|a(m)\|$, $m \in M$ and this property does not depend on our choice of $a(m)$. By (1.25), we have $\|a\| \approx e^{\|h_{A_0}(a)\|}, a \in A_0$. Hence, there are a positive constant C and a nonnegative integer d such that $e^{\|h_{A_0}(a(m))\|} \leqslant C\|m\|^d$ for all $m \in M$. By (1.8) applied to (M, A_0), if $a \in A_0$ then $h_M(a)$ is the orthogonal projection of $h_{A_0}(a)$ onto a_M, thus $\|h_M(a)\| \leqslant \|h_{A_0}(a)\|$. Since $h_M(m) = h_M(a(m))$ for $m \in M$, we obtain that there is a positive constant C' such that

$$\|h_M(m)\| \leqslant \|h_{A_0}(a(m))\| \leqslant C'(1 + \log \|m\|), \quad m \in M. \tag{2.43}$$

By definition (cf. (1.13), (1.14)), we have $h_P(x) = h_M(m_P(x))$ for $x \in H$ and by (1.22), we have $\|m_P(x)\| \leqslant \|x\|, x \in H$. Thus, our claim (2.42) follows from (2.43).

Therefore, there are a positive C_1 and a positive integer d such that if $\sigma_M(h_M(a), \mathcal{Y}_M(x_1, y_1, x_2, y_2, T)) \neq 0$, then

$$\|h_M(a)\| \leqslant \|Z_P\| \leqslant C_1(\|T\| + \log \|x_1\| + \log \|y_1\| + \log \|x_2\| + \log \|y_2\|)^d.$$

Since $\|x\| \leqslant \|x_m\|\|x_m^{-1}x\|$ for $x \in H$, this gives the following estimates of v_M analogous to (2.38) and (2.41):

If $\|T\| > 1$ then
$v_M(x_1, y_1, x_2, y_2, T) \leqslant \|T\| + \log \|x_m^{-1}x_1\| + \log \|x_m^{-1}y_1\| + \log \|x_2\| + \log \|y_2\|,$ (2.44)
$x_1, y_1, x_2, y_2 \in H$,

and

there is a positive constant C_2' such that for $\|T\| \leqslant 1$, one has
$v_M(x_1, y_1, x_2, y_2, T) \leqslant C_2' + \log \|x_m^{-1}x_1\| + \log \|x_m^{-1}y_1\| + \log \|x_2\| + \log \|y_2\|,$ (2.45)
$x_1, y_1, x_2, y_2 \in H$.

Arguing exactly as above for K^T, we deduce that there is a positive constant C' such that

$$\int_{S_\sigma(\varepsilon,T)} |J^T(x_m, \gamma, f)| d\gamma \leqslant C' e^{-\frac{\varepsilon\varepsilon_0\|T\|}{4}}.$$

This finishes the proof of the Lemma. □

2.8 Lemma. *Fix $\delta > 0$. Then, there exist positive numbers C, ε_1 and ε_2 such that, for all T with $d(T) \geqslant \delta\|T\|$, and for all x_1, y_1, x_2 and y_2 in the set $H_{\varepsilon_2} := \{x \in H; \|x\| \leqslant e^{\varepsilon_2\|T\|}\}$, one has*

$$|u_M(x_1, y_1, x_2, y_2, T) - v_M(x_1, y_1, x_2, y_2, T)| \leqslant Ce^{-\varepsilon_1\|T\|}. \tag{2.46}$$

Proof :

If $\|T\|$ remains bounded then, by (2.38), (2.41), (2.44) and (2.45), the functions u_M and v_M are bounded and the result (2.46) is trivial. Thus we have to prove the Lemma for $\|T\|$ sufficiently large and $d(T) \geqslant \delta\|T\|$.

By (5.8) of [Ar3], we can choose ε_2 such that $d(\mathcal{Y}_M(x,y,T)) > 0$ for all $x, y \in H_{\varepsilon_2}$. By the discussion of l.c. bottom of page 38 and top of page 39, there is a constant $C_0 > 0$ such that, for T with $d(T) \geqslant \delta\|T\|$ and $\|T\| > C_0$, for $x, y \in H_{\varepsilon_2}$ and $a \in A_H \backslash A_M$, one has

$$u(y^{-1}ax, T) = \sigma_M(h_M(a), \mathcal{Y}_M(x,y,T)).$$

By Lemma 2.2, for $X \in a_M$, we have

$$\sigma_M(X, \mathcal{Y}_M(x_1,y_1,x_2,y_2,T)) = \sigma_M(X, \mathcal{Y}_M(x_1,y_1,T))\sigma_M(X, \mathcal{Y}_M(x_2,y_2,T)).$$

Thus, one deduces that

$$\sigma_M(h_M(a), \mathcal{Y}_M(x_1,y_1,x_2,y_2,T)) = u(y_1^{-1}ax_1, T)u(y_2^{-1}ax_2, T),$$

for $a \in A_H \backslash A_M$. Hence, for $d(T) \geqslant \delta\|T\| \geqslant \delta C_0$, and x_i, y_i in H_{ε_2}, we have

$$u_M(x_1,y_1,x_2,y_2,T) = v_M(x_1,y_1,x_2,y_2,T).$$

This finishes the proof of the Lemma. □

Theorem 2.3 follows from the corollary below.

2.9 Corollary. *Fix $\delta > 0$. There exist two positive numbers ε and $c > 0$ such that, for all T with $d(T) \geqslant \delta\|T\|$, one has*

$$\int_{\gamma \in S_\sigma} |K^T(x_m, \gamma, f) - J^T(x_m\gamma, f)|\, d\gamma \leqslant ce^{-\varepsilon\|T\|}. \tag{2.47}$$

Proof :

By Lemma 2.7, it is enough to prove that we can find positive numbers ε, ε' and C_0 such that

$$\int_{\gamma \in S_\sigma - S_\sigma(\varepsilon,T)} |K^T(x_m,\gamma,f) - J^T(x_m,\gamma,f)| d\gamma \leqslant C_0 e^{-\varepsilon'\|T\|} \tag{2.48}$$

where $S_\sigma(\varepsilon, T)$ is defined in (2.34).

Let $\varepsilon > 0$. Let Ω be a compact subset of G which contains the support of f_1 and f_2. We will estimate $|u_M(x_1,y_1,x_2,y_2,T) - v_M(x_1,y_1,x_2,y_2,T)|$ for x_1, x_2, y_1 and y_2 in H satisfying $x_1^{-1}x_m\gamma x_2 \in \Omega$ and $y_1^{-1}x_m\gamma y_2 \in \Omega$ for some $\gamma \in S_\sigma - S_\sigma(\varepsilon, T)$ with $x_m\gamma \in G^{\sigma-reg}$. For this, we will use the invariance of the functions u_M and v_M by the diagonal left action of A_M on (x_1, x_2) and (y_1, y_2) respectively.

By Lemma 2.5, there are a positive integer k and a positive constant C_Ω, (depending only on Ω) such that, for all $\gamma \in S_\sigma - S_\sigma(\varepsilon, T)$ with $x_m\gamma \in G^{\sigma-reg}$ and for all x_i, y_i in H, $i = 1, 2$ with $x_1^{-1}x_m\gamma x_2$ and $y_1^{-1}x_m\gamma y_2$ in Ω, we have

$$\inf_{s\in S} \|(sx_m^{-1}x_1, sx_2)\| \leqslant C_\Omega \Delta_\sigma(x_m\gamma)^{-k} \leqslant C_\Omega e^{k\varepsilon\|T\|} \tag{2.49}$$

and

$$\inf_{s\in S} \|(sx_m^{-1}y_1, sy_2)\| \leqslant C_\Omega \Delta_\sigma(x_m\gamma)^{-k} \leqslant C_\Omega e^{k\varepsilon\|T\|}.$$

Since $A_M\backslash S$ is compact, we deduce from (1.21) and (2.49) that there is a constant $C'_\Omega > 0$ such that

$$\inf_{a\in A_M} \|(ax_m^{-1}x_1, ax_2)\| \leqslant C'_\Omega e^{k\varepsilon\|T\|}.$$

Thus, for $\eta > 0$, there is $a_0 \in A_M$ such that

$$\|a_0x_m^{-1}x_1\|\|a_0x_2\| \leqslant C_\Omega e^{k\varepsilon\|T\|} + \eta. \tag{2.50}$$

Since $A_M = A_S$, the point a_0 commutes with x_m by (1.28) and we have $\|a_0x_1\| \leqslant \|x_m\|\|x_m^{-1}a_0x_1\|$.

If $\|T\|$ remains bounded, then $\|a_0x_i\|, i = 1, 2$ are bounded by a constant independent of $\|T\|$. By the same arguments, there is $a_1 \in A_M$ such that $\|a_1y_i\|, i = 1, 2$ are bounded by a constant independent of $\|T\|$. Using the invariance of u_M and v_M by the left action of $diag(A_M)$ on (x_1, x_2) and (y_1, y_2) respectively and the estimates (2.38), (2.41) , (2.44) and (2.45) for u_M and v_M, we deduce that $|u_M(x_1, y_1, x_2, y_2, T) - v_M(x_1, y_1, x_2, y_2, T)|$ is bounded by a constant independent of T and of x_i, y_i. Recall that by Theorem 1.2, the constant

$$C_1 := \int_{S_\sigma} \mathcal{M}(|f_1|)(x_m\gamma)\mathcal{M}(|f_2|)(x_m\gamma)d\gamma$$

is finite. We deduce that $\int_{\gamma\in S_\sigma - S_\sigma(\varepsilon,T)} |K^T(x_m, \gamma, f) - J^T(x_m, \gamma, f)|d\gamma$ is bounded, hence we obtain (2.48).

We assume that $\|T\|$ is not bounded. Let $\varepsilon_1, \varepsilon_2$ and C as in Lemma 2.8. Taking $\|T\|$ to be sufficiently large and ε such that $k\varepsilon$ is smaller than the constant ε_2, we can assume by (2.50) that

$$\|a_0x_i\| \leqslant e^{\varepsilon_2\|T\|}, \quad i = 1, 2.$$

The same arguments are valid for $\|y_i\|$, $i = 1, 2$. Thus, there is $a_1 \in A_M$ such that

$$\|a_1y_i\| \leqslant e^{\varepsilon_2\|T\|}, \quad i = 1, 2.$$

Using Lemma 2.8 and the invariance of u_M and v_M by the left action of the diagonal of A_M on (x_1, x_2) and (y_1, y_2) respectively, we deduce that, for all T with $d(T) \geqslant \delta\|T\|$, one has

$$|u_M(x_1, y_1, x_2, y_2, T) - v_M(x_1, y_1, x_2, y_2, T)| \leqslant Ce^{-\varepsilon_1\|T\|}.$$

Hence, we obtain

$$\int_{S-S_\sigma(\varepsilon,T)} |K^T(x_m, \gamma, f) - J^T(x_m, \gamma, T)| \leqslant CC_1 e^{-\varepsilon_1\|T|},$$

where $C_1 := \int_{S_\sigma} \mathcal{M}(|f_1|)(x_m\gamma)\mathcal{M}(|f_2|)(x_m\gamma)d\gamma$. This finishes the proof of the Corollary. □

2.5 The function $J^T(f)$

The goal of this section is to prove that $J^T(f)$ is of the form

$$\sum_{k=0}^{N} p_k(T, f)e^{\xi_k(T)}, \tag{2.51}$$

where $\xi_0 = 0, \xi_1, \ldots, \xi_N$ are distinct points in ia_0^* and each $p_k(T, f)$ is a polynomial function of T. Moreover, the constant term $\tilde{J}(f) := p_0(0, f)$ is well-defined and is uniquely determined by $K^T(f)$. Except for one detail, our arguments and calculations are the same as those of section 6 of [Ar3]. We give the details of proof for convenience of the reader.

Recall that $J^T(f)$ is a finite sum of the distributions

$$J^T(x_m, \gamma, f) = |\Delta_\sigma(x_m\gamma)|_{\mathrm{F}}^{1/2} \int_{diag(A_M)\backslash H\times H} \int_{diag(A_M)\backslash H\times H} f_1(y_1^{-1}x_m\gamma y_2)$$

$$\times f_2(x_1^{-1}x_m\gamma x_2)v_M(x_1, y_1, x_2, y_2, T)d\overline{(x_1, x_2)}d\overline{(y_1, y_2)}$$

where $M \in \mathcal{L}(A_0)$, S is a maximal torus of M such that $A_S = A_M$, $x_m \in \kappa_S$ and $v_M(x_1, y_1, x_2, y_2, T) := \int_{A_H\backslash A_M} \sigma_M(h_M(a), \mathcal{Y}_M(x_1, y_1, x_2, y_2, T))da$ where $\mathcal{Y}_M(x_1, y_1, x_2, y_2, T)$ is defined in (2.11).

We first study the weight function v_M as a function of T. We fix $M \in \mathcal{L}(A_0)$ and x_1, y_1, x_2 and y_2 in H.

Let $\mathscr{L}_M := (a_{M,\mathrm{F}} + a_H)/a_H$ and $\widetilde{\mathscr{L}}_M := (\tilde{a}_{M,\mathrm{F}} + a_H)/a_H$ be the projection in a_M/a_H of the lattices $a_{M,\mathrm{F}}$ and $\tilde{a}_{M,\mathrm{F}}$ respectively. By (1.10), one has

$$\tilde{a}_{M,\mathrm{F}}/\tilde{a}_{H,\mathrm{F}} = \tilde{a}_{M,\mathrm{F}}/\tilde{a}_{M,\mathrm{F}} \cap a_H \simeq \widetilde{\mathscr{L}}_M. \tag{2.52}$$

For $M = A_0$, we replace the subscript A_0 by 0. We denote by $\mathscr{L}^\vee := \mathrm{Hom}(\mathscr{L}, 2\pi i\mathbb{Z})$ the dual lattice of a lattice $\mathscr{L}$.

Let $P \in \mathcal{P}(M)$. We introduce the following sublattice of $\mathscr{L}_M$. For $k \in \mathbb{N}$, we set

$$\mu_{\alpha,k} := k\log(q)\check{\alpha}, \alpha \in \Delta_P,$$

where q is the order of the residual field of F, and

$$\mathscr{L}_{M,k} := \sum_{\alpha\in\Delta_P} \mathbb{Z}\mu_{\alpha,k}.$$

Then $\mathscr{L}_{M,k}$ is a lattice in $a_M^H \simeq a_M/a_H$ independent of P and by ([Ar2] §4), one can find $k \in \mathbb{N}^*$ such that for all $M \in \mathcal{L}(A_0)$, one has

$$\mathscr{L}_{M,k} \subset \widetilde{\mathscr{L}}_M.$$

The set of points $\sum_{\alpha\in\Delta_P} y_\alpha\mu_{\alpha,k}$ with $y_\alpha \in]-1, 0]$ is a fundamental domain of $\mathscr{L}_{M,k}$ which we denote by $\mathcal{D}_{M,k}$.

For $X \in \mathscr{L}_M/\mathscr{L}_{M,k}$ and $Y \in a_M/a_H$, we denote by $\bar{X}_P(Y)$ the representative of X in $\mathscr{L}_M$ such that $\bar{X}_P(Y) - Y \in \mathcal{D}_{M,k}$. (2.53)

For $\lambda \in a^*_{M,\mathbb{C}}$, we set

$$\theta_{P,k}(\lambda) = vol(a_M^H/\mathscr{L}_{M,k})^{-1} \prod_{\alpha\in\Delta_P} (1 - e^{-\lambda(\mu_{\alpha,k})}). \tag{2.54}$$

We fix $T \in a_{0,\mathrm{F}}$. By definition of σ_M (cf. (2.4)), the function v_M depends only on the image of T_P in $\mathscr{L}_M$. Hence we can assume that T lies in the lattice $\mathscr{L}_0$. For $P \in \mathcal{P}(M)$, the map $T \mapsto T_P$ sends surjectively $\mathscr{L}_0$ onto the intersection of $\mathscr{L}_M$ with the closure $\overline{a_P^+}$ of the chamber associated to P. Thus, we may restrict T to lie in the intersection of $\mathscr{L}_0$ with suitable regular points in some positive chamber a_0^+ of $a_H\backslash a_0$. Then the points T_P range over a suitable regular points in $\mathscr{L}_M \cap a_P^+$.

We recall that $\mathcal{Y}_M(x_1, y_1, x_2, y_2, T)$ is the set of points $Z_P := Z_P(x_1, y_1, x_2, y_2, T)$ defined in (2.10). Thus, we can write

$$Z_P = T_P + Z_P^0 \text{ with } Z_P^0 := \inf{}^P(h_P(y_1) - h_{\overline{P}}(x_1), h_P(y_2) - h_{\overline{P}}(x_2)). \tag{2.55}$$

Notice that the points Z_P^0 do not necessarily belong to the lattice $\mathscr{L}_M$. It is the only difference with [Ar3] section 6 in what follows.

2.10 Lemma. *There is a positive integer N independent of M and polynomial functions $q_\xi(T)$ for $\xi \in (\frac{1}{N}\mathscr{L}_0^\vee)/\mathscr{L}_0^\vee$ (depending on x_1, y_1, x_2 and y_2), such that*

$$v_M(x_1,y_1,x_2,y_2,T)=\sum_{\xi\in(\frac{1}{N}\mathscr{L}_0^\vee)/\mathscr{L}_0^\vee} q_\xi(T)e^{\xi(T)}.$$

Moreover, the constant term $\tilde{v}_M(x_1,y_1,x_2,y_2) := q_0(0)$ *of* $v_M(x_1,y_1,x_2,y_2,T)$ *is given by*

$$\tilde{v}_M(x_1,y_1,x_2,y_2)=\lim_{\Lambda\to 0}\Big(\sum_{P\in\mathcal{P}(M)}|\mathscr{L}_M/\mathscr{L}_{M,k}|^{-1}\sum_{X\in\mathscr{L}_M/\mathscr{L}_{M,k}}e^{\langle\Lambda,\bar{X}_P(Z_P^0)\rangle}\theta_{P,k}(\Lambda)^{-1}\Big).$$

Proof :

The kernel of the surjective map $h_M : A_H\backslash A_M \to \tilde{a}_{M,\mathrm{F}}/\tilde{a}_{H,\mathrm{F}}$ is a compact group which has volume 1 by our convention of choice of measure. Thus, using (2.52), we can write

$$v_M(x_1,y_1,x_2,y_2,T):=\sum_{X\in\widetilde{\mathscr{L}}_M}\sigma_M(X,\mathcal{Y}_M(x_1,y_1,x_2,y_2,T)).$$

For our study, it is convenient to take a sum over $\mathscr{L}_M$. The finite quotient $\widetilde{\mathscr{L}}_M^\vee/\mathscr{L}_M^\vee$ can be identified with the character group of $\mathscr{L}_M/\widetilde{\mathscr{L}}_M$ under the pairing

$$(\nu,X)\in\widetilde{\mathscr{L}}_M^\vee/\mathscr{L}_M^\vee\times\mathscr{L}_M/\widetilde{\mathscr{L}}_M\mapsto e^{\nu(X)}.$$

Hence, by inversion formula on finite abelian groups, we obtain

$$v_M(x_1,y_1,x_2,y_2,T)=|\mathscr{L}_M/\widetilde{\mathscr{L}}_M|^{-1}\sum_{\nu\in\widetilde{\mathscr{L}}_M^\vee/\mathscr{L}_M^\vee}\sum_{X\in\mathscr{L}_M}\sigma_M(X,\mathcal{Y}_M(x_1,y_1,x_2,y_2,T))e^{\nu(X)}.$$

Coming back to the definition of σ_M (cf. (2.4)), we fix a small point $\Lambda\in(a_M/a_H)^*_{\mathbb{C}}$ whose real part Λ_R is in general position. One has

$$\sigma_M(X,\mathcal{Y}_M(x_1,y_1,x_2,y_2,T))=\sum_{P\in\mathcal{P}(M)}(-1)^{|\Delta_P^\Lambda|}\varphi_P^\Lambda(X-Z_P)$$

$$=\lim_{\Lambda\to 0}\sum_{P\in\mathcal{P}(M)}(-1)^{|\Delta_P^\Lambda|}\varphi_P^\Lambda(X-Z_P)e^{\Lambda(X)}.$$

By definition of φ_P^Λ, the function $X\mapsto e^{\Lambda(X)}$ is rapidly decreasing on the support of $X\mapsto\varphi_P^\Lambda(X-Z_P)$. Hence, the product of these two functions is summable over $X\in\mathscr{L}_M$. Therefore, we can write

$$v_M(x_1,y_1,x_2,y_2,T)=\sum_{\nu\in\widetilde{\mathscr{L}}_M^\vee/\mathscr{L}_M^\vee}\lim_{\Lambda\to 0}\sum_{P\in\mathcal{P}(M)}F_P^T(\Lambda)\tag{2.56}$$

where

$$F_P^T(\Lambda):=|\mathscr{L}_M/\widetilde{\mathscr{L}}_M|^{-1}\sum_{X\in\mathscr{L}_M}(-1)^{|\Delta_P^\Lambda|}\varphi_P^\Lambda(X-Z_P)e^{(\Lambda+\nu)(X)}.$$

The above discussion implies that

$$\text{the map } \Lambda \mapsto \sum_{P\in\mathcal{P}(M)} F_P^T(\Lambda) \text{ is analytic at } \Lambda = 0. \tag{2.57}$$

We fix $P \in \mathcal{P}(M)$. We want to express $F_P^T(\Lambda)$ in terms of a product of geometric series. For this, we write

$$F_P^T(\Lambda) := |\mathscr{L}_M/\widetilde{\mathscr{L}}_M|^{-1} \sum_{X\in\mathscr{L}_M/\mathscr{L}_{M,k}} \sum_{X'\in\mathscr{L}_{M,k}} (-1)^{|\Delta_P^\Lambda|}\varphi_P^\Lambda(X+X'-Z_P)e^{(\Lambda+\nu)(X+X')}. \tag{2.58}$$

Let $X \in \mathscr{L}_M/\mathscr{L}_{M,k}$. Recall that $\bar{X}_P(Y)$ is the representative of X in $\mathscr{L}_M$ such that $\bar{X}_P(Y) - Y \in \mathcal{D}_{M,k}$. We set

$$\bar{X}_P^\Lambda(Y) := \bar{X}_P(Y) + \sum_{\alpha\in\Delta_P^\Lambda} \mu_{\alpha,k}.$$

Thus $\bar{X}_P^\Lambda(Y)$ is also a representative of X in $\mathscr{L}_M$. Taking $Y := Z_P$, we can set

$$\varphi_P^\Lambda(X+X'-Z_P) = \varphi_P^\Lambda(\bar{X}_P^\Lambda(Z_P)+X'-Z_P)$$

in (2.58). The set of points $X' \in \mathscr{L}_{M,k}$ such that this characteristic function equals to 1 is exactly the set

$$\{\sum_{\alpha\in\Delta_P^\Lambda} n_\alpha\mu_{\alpha,k} - \sum_{\alpha\in\Delta_P-\Delta_P^\Lambda} n_\alpha\mu_{\alpha,k}; n_\alpha \in \mathbb{N}\}.$$

Therefore, a simple calculation as in [Ar3] top of page 45 gives

$$\begin{aligned}(-1)^{|\Delta_P^\Lambda|} &\sum_{X'\in\mathscr{L}_{M,k}} \varphi_P^\Lambda(X+X'-Z_P)e^{(\Lambda+\nu)(X+X')} \\ &= e^{(\Lambda+\nu)(\bar{X}_P(Z_P))}\Big(\prod_{\alpha\in\Delta_P}(1-e^{-(\Lambda+\nu)(\mu_{\alpha,k})}\Big)^{-1}.\end{aligned} \tag{2.59}$$

We have fixed the Haar measure on $a_M^H \simeq a_M/a_G$ with the property that the quotient of a_M/a_H by the lattice $\widetilde{\mathscr{L}}_M$ has volume 1. Thus, we have

$$|\mathscr{L}_M/\widetilde{\mathscr{L}}_M|^{-1}\prod_{\alpha\in\Delta_P}(1-e^{-(\Lambda+\nu)(\mu_{\alpha,k})})^{-1} = |\mathscr{L}_M/\widetilde{\mathscr{L}}_{M,k}|^{-1}\theta_{P,k}(\Lambda+\nu)^{-1}.$$

By the above equality, (2.58) and (2.59), we obtain

$$F_P^T(\Lambda) = |\mathscr{L}_M/\mathscr{L}_{M,k}|^{-1} \sum_{X\in\mathscr{L}_M/\mathscr{L}_{M,k}} e^{<\Lambda+\nu,\bar{X}_P(Z_P)>}\theta_{P,k}(\Lambda+\nu)^{-1}. \tag{2.60}$$

Let $X \in \mathscr{L}_M/\mathscr{L}_{M,k}$. We recall that T_P belongs to $\mathscr{L}_M$ for $P \in \mathcal{P}(M)$ and $Z_P = T_P + Z_P^0$ (cf. (2.55)). By definition (cf. (2.53)), the point $\bar{X}_P(Z_P)$ is the unique representative of X in $\mathcal{L}_M$ such that $\bar{X}_P(Z_P) - T_P - Z_P^0 \in \mathcal{D}_{M,k}$ and $\overline{(X-T_P)}_P(Z_P^0)$ is the unique representative of $X - T_P$ in $\mathscr{L}_M$ such that $\overline{(X-T_P)}_P(Z_P^0) - Z_P^0 \in \mathcal{D}_{M,k}$. Hence, we deduce that

$$\bar{X}_P(Z_P) = \overline{(X-T_P)}_P(Z_P^0) + T_P. \tag{2.61}$$

Replacing X by $X - T_P$ in (2.60), we obtain

$$F_P(\Lambda)^T = |\mathscr{L}_M/\mathscr{L}_{M,k}|^{-1} \sum_{X \in \mathscr{L}_M/\mathscr{L}_{M,k}} e^{<\Lambda+\nu, T_P + \bar{X}_P(Z_P^0)>} \theta_{P,k}(\Lambda+\nu)^{-1} \tag{2.62}$$

where $\bar{X}_P(Z_P^0)$ is independent of T. Thus by (2.56), we have established that $v_M(x_1, y_1, x_2, y_2, T)$ is equal to

$$\sum_{\nu \in \widetilde{\mathscr{L}}_M^\vee/\mathscr{L}_M^\vee} \lim_{\Lambda \to 0} \Big(\sum_{P \in \mathcal{P}(M)} |\mathscr{L}_M/\mathscr{L}_{M,k}|^{-1} \sum_{X \in \mathscr{L}_M/\mathscr{L}_{M,k}} e^{<\Lambda+\nu, T_P + \bar{X}_P(Z_P^0)>} \theta_{P,k}(\Lambda+\nu)^{-1} \Big). \tag{2.63}$$

Recall that the expression in the brackets is analytic at $\Lambda = 0$ (cf. (2.57)). To analyze this expression as function of T, we argue as in ([W1] p.315). We give the details for convenience of lecture. We replace Λ by $z\Lambda$. The map $z \mapsto \theta_{P,k}(z\Lambda+\nu)^{-1}$ may have a pole at $z = 0$. Let r denotes the biggest order of this pole when P runs $\mathcal{P}(M)$. Then, using Taylor expansions, one deduces that

$$\lim_{\Lambda \to 0} \Big(\sum_{P \in \mathcal{P}(M)} |\mathscr{L}_M/\mathscr{L}_{M,k}|^{-1} \sum_{X \in \mathscr{L}_M/\mathscr{L}_{M,k}} e^{<\Lambda+\nu, T_P + \bar{X}_P(Z_P^0)>} \theta_{P,k}(\Lambda+\nu)^{-1} \Big) =$$

$$\sum_{m=0}^{r} \sum_{P \in \mathcal{P}(M)} C_m \sum_{X \in \mathscr{L}_M/\mathscr{L}_{M,k}} \frac{\partial^m}{\partial z^m} (e^{<z\Lambda+\nu, T_P + \bar{X}_P(Z_P^0)>})_{[z=0]} \frac{\partial^{r-m}}{\partial z^{r-m}} (z^r \theta_{P,k}(z\Lambda+\nu)^{-1})_{[z=0]},$$

where $C_m = \frac{1}{m!(r-m)!} |\mathscr{L}_M/\mathscr{L}_{M,k}|^{-1}$.

But we have

$$\frac{\partial^m}{\partial z^m} (e^{<z\Lambda+\nu, T_P + \bar{X}_P(Z_P^0)>})_{[z=0]} = (<\Lambda, T_P + \bar{X}_P(Z_P^0)>)^m e^{<\nu, T_P + \bar{X}_P(Z_P^0)>},$$

and $\dfrac{\partial^{r-m}}{\partial z^{r-m}} (z^r \theta_{P,k}(z\Lambda+\nu)^{-1})_{[z=0]}$ is independent of T_P.

Therefore, we deduce that $v_M(x_1, y_1, x_2, y_2, T)$ is a finite sum of functions

$$q_{P,\nu}(T_P) e^{\nu(T_P)}, \quad \nu \in \widetilde{\mathscr{L}}_M^\vee/\mathscr{L}_M^\vee, P \in \mathcal{P}(M),$$

where $q_{P,\nu}$ is a polynomial function on a_M.

Since $\mathscr{L}_0^\vee \subset \widetilde{\mathscr{L}_0}^\vee$ are lattices of same rank, one can find a positive integer N such that $N\widetilde{\mathscr{L}_0}^\vee \subset \mathscr{L}_0^\vee$. Therefore, by our choice of T and the above expression, we can write

$$v_M(x_1, y_1, x_2, y_2, T) = \sum_{\xi \in (\frac{1}{N}\mathscr{L}_0^\vee)/\mathscr{L}_0^\vee} q_\xi(T) e^{\xi(T)},$$

where $q_\xi(T)$ is a polynomial function of T. This gives the first part of the Lemma.

Since the polynomials $q_\xi(T)$ are obviously uniquely determined, the constant term $\tilde{v}_M(x_1, y_1, x_2, y_2) := q_0(0)$ is well defined. To calculate it, we take the summand corresponding to $\nu = 0$ in (2.63) and then set $T = 0$. We obtain

$$\tilde{v}_M(x_1, y_1, x_2, y_2) = \lim_{\Lambda \to 0} \Big(\sum_{P \in \mathcal{P}(M)} |\mathscr{L}_M/\mathscr{L}_{M,k}|^{-1} \sum_{X \in \mathscr{L}_M/\mathscr{L}_{M,k}} e^{\langle \Lambda, \bar{X}_P(Z_P^0)\rangle} \theta_{P,k}(\Lambda)^{-1} \Big).$$

This finishes the proof of the Lemma. □

We substitute the expression we have obtained for v_M in Lemma 2.10 into the expression for $J^T(x_m, \gamma, f)$. Hence, we obtain the following similar decomposition for $J^T(f)$.

2.11 Corollary. *There is a decomposition*

$$J^T(f) = \sum_{\xi \in (\frac{1}{N}\mathscr{L}_0^\vee)/\mathscr{L}_0^\vee} p_\xi(T, f) e^{\xi(T)}, \quad T \in \mathscr{L}_0 \cap a_0^+,$$

where N is positive integer and each $p_\xi(T, f)$ is a polynomial function of T. Moreover, the constant term $\tilde{J}(f) := p_0(0, f)$ of $J^T(f)$ is given by

$$\tilde{J}(f) := \sum_{M \in \mathcal{L}(A_0)} c_M \sum_{S \in \mathcal{T}_M} \sum_{x_m \in \kappa_S} c_{S,x_m} \int_{S_\sigma} \tilde{J}(x_m, \gamma, f) d\gamma,$$

where

$$\tilde{J}(x_m, \gamma, f) = |\Delta_\sigma(x_m\gamma)|^{1/2} \int_{diag(A_M)\backslash H \times H} \int_{diag(A_M)\backslash H \times H} f_1(y_1^{-1} x_m \gamma y_2)$$

$$\times f_2(x_1^{-1} x_m \gamma x_2) \tilde{v}_M(x_1, y_1, x_2, y_2) d\overline{(x_1, x_2)} d\overline{(y_1, y_2)}.$$

A Appendix. Spherical character of a supercuspidal representation as weighted orbital integral

Let (π, V) be a unitary irreducible admissible representation of G. We say that π is H-distinguished if the space $V^{*H} = \mathrm{Hom}_H(\pi, \mathbb{C})$ of H-invariant linear forms on V is nonzero. In that case, a distribution $m_{\xi,\xi'}$, called spherical character, can be

associated to two H-invariant linear forms ξ, ξ' on V (cf. definition below). By ([Ha] Theorem 1), spherical characters are locally integrable functions on G, which are smooth on the set of σ-regular points of G.

From now, we assume that $A_H = \{1\}$. We fix a H-distinguished supercuspidal representation (τ, V) of G. We denote by $d(\tau)$ its formal degree.

The aim of this appendix is to deduce from our main results, the value $m_{\xi,\xi'}(g)$, when $g \in G$ is σ-regular and $\xi, \xi' \in V^{*H}$, in terms of weighted orbital integrals of a matrix coefficient of τ (cf. Theorem A.2). This result is analogous to that of J. Arthur in the group case ([Ar2]). Notice that this result of J. Arthur can be deduced from his local trace formula ([Ar3]) which was obtained later.

Let $(\cdot, \cdot)$ be a G-invariant hermitian inner product on V. Since τ is unitary, it induces an isomorphism $\iota : v \mapsto (\cdot, v)$ from the conjugate complex vector space $\overline{V}$ of V and the smooth dual $\check{V}$ of V, which intertwines the complex conjugate of τ and its contragredient $\check{\tau}$. If ξ is a linear form on V, we define the linear form $\overline{\xi}$ on $\overline{V}$ by $\overline{\xi}(u) := \overline{\xi(u)}$.
For ξ_1 and ξ_2 two nonzero H-invariant linear forms on V, we associate the spherical character m_{ξ_1,ξ_2} defined to be the distribution on G given by

$$m_{\xi_1,\xi_2}(f) := \sum_{u \in \mathcal{B}} \xi_1\big(\tau(f)u\big)\overline{\xi_2(u)},$$

where $\mathcal{B}$ is an orthonormal basis of V. Since $\tau(f)$ is of finite rank, this sum is finite. Moreover, this sum does not depend on the choice of $\mathcal{B}$. Indeed, let (τ^*, V^*) be the dual representation of τ. For $f \in C_c^\infty(G)$, we set $\check{f}(g) := f(g^{-1})$. By ([R] Théorème III.3.4 and I.1.2), the linear form $\tau^*(\check{f})\xi$ belongs to $\check{V}$. Hence we can write $\iota^{-1}(\tau^*(\check{f})\xi) = \sum_{v \in \mathcal{B}} \big(\tau^*(\check{f})\xi\big)(v) \cdot v$ where $(\lambda, v) \mapsto \lambda \cdot v$ is the action of $\mathbb{C}$ on $\overline{V}$. Therefore we deduce easily that one has

$$m_{\xi_1,\xi_2}(f) = \overline{\xi}_2\big(\iota^{-1}(\tau^*(\check{f})\xi_1)\big). \tag{1.1}$$

Since τ is a supercuspidal representation, we can define the $H \times H$-invariant pairing $\mathcal{L}$ on $V \times \overline{V}$ by

$$\mathcal{L}(u, v) := \int_H (\tau(h)u, v)dh.$$

By ([Z] Theorem 1.5),

$$\text{the map } v \mapsto \xi_v : u \mapsto \mathcal{L}(u, v) \text{ is a surjective linear map from } \overline{V} \text{ onto } V^{*H}. \tag{1.2}$$

For $v, w \in V$, we denote by $c_{v,w}$ the corresponding matrix coefficient defined by $c_{v,w}(g) := (\tau(g)v, w)$ for $g \in G$.

A.1 Lemma. *Let $\xi_1, \xi_2 \in V^{*H}$ and $v, w \in V$. Then we have*

$$m_{\xi_1,\xi_2}(\check{c}_{v,w}) = d(\tau)^{-1}\xi_1(v)\overline{\xi_2(w)}.$$

Proof :
By (1.2), there exist v_1 and v_2 in V such that $\xi_j = \xi_{v_j}$ for $j = 1, 2$. By definition of the spherical character, for $f \in C_c^\infty(G)$ and $\mathcal{B}$ an orthonormal basis of V, one has

$$\begin{aligned} m_{\xi_1,\xi_2}(f) &= \sum_{u\in\mathcal{B}} \int_H (\tau(h)\tau(f)u, v_1)dh \int_H \overline{(\tau(h)u, v_2)}dh \\ &= \sum_{u\in\mathcal{B}} \int_{H\times H} (u, \tau(\check{f})\tau(h_1)v_1)(\tau(h_2)v_2, u)dh_1dh_2 \\ &= \int_{H\times H} (\tau(h_2)v_2, \tau(\check{f})\tau(h_1)v_1)dh_1dh_2 \end{aligned}$$

Hence we obtain

$$m_{\xi_1,\xi_2}(f) = \int_{H\times H}\int_G f(g)(\tau(h_1gh_2)v_2, v_1)dgdh_1dh_2. \tag{1.3}$$

Let $f(g) := \check{c}_{v,w}(g) = \overline{(\tau(g)w, v)}$. By the orthogonality relation of Schur, for $h_1, h_2 \in H$, one has

$$\int_G (\tau(g)\tau(h_2)v_2, \tau(h_1)v_1)\overline{(\tau(g)w, v)}dg = d(\tau)^{-1}(\tau(h_2)v_2, w)(v, \tau(h_1)v_1).$$

Thus, we deduce that

$$m_{\xi_1,\xi_2}(f) = d(\tau)^{-1}\xi_w(v_2)\xi_{v_1}(v) = d(\tau)^{-1}\xi_1(v)\overline{\xi_2(w)}. \qquad \square$$

For $M \in \mathcal{L}(A_0)$, we define the weight function w_M on $H \times H$ by

$$w_M(y_1, y_2) := \tilde{v}_M(1, y_1, 1, y_2),$$

where $\tilde{v}_M$ is defined in Lemma 2.10 and 1 is the neutral element of H. For $f \in C_c^\infty(G)$, we define the weighted orbital integral of f by

$$\mathcal{WM}(f)(g) := |\Delta_\sigma(g)|^{1/2} \int_{H\times H} f(y_1gy_2)w_M(y_1, y_2)dy_1dy_2, \quad g \in G^{\sigma-reg} \cap \tilde{M}.$$

A.2 Theorem. *Let $M \in \mathcal{L}(A_0)$ and $S \in \mathcal{T}_M$. Let $x_M \in \kappa_S$ and $\gamma \in S_\sigma$ be such that $x_m\gamma$ is σ-regular. Then, for $v, w \in V$, we have*

$$c_M c_{S,x_m}\mathcal{WM}(c_{v,w})(x_m\gamma) = m_{\xi_w,\xi_v}(x_m\gamma).$$

Proof :

Let f_1 be a matrix coefficient of τ and $f_2 \in C_c^\infty(G)$. We set $f := f_1 \otimes f_2$. For $x \in G$, we define

$$h(g) := \int_G f_1(xu) f_2(ugx) du,$$

so that

$$K_f(x,y) = \big[\rho(yx^{-1})h\big](e).$$

If π is a unitary irreducible admissible representation of G, one has

$$\pi\big(\rho(yx^{-1})h\big) = \int_{G\times G} f_1(xu) f_2(ugy) \pi(g) du dg$$

$$= \int_{G\times G} f_1(xu) f_2(u_2) \pi(u^{-1}u_2 y^{-1}) du du_2 = \int_{G\times G} f_1(u_1^{-1}) f_2(u_2) \pi(u_1 x u_2 y^{-1}) du_1 du_2$$

$$= \pi(\check{f}_1)\pi(x)\pi(f_2)\pi(y^{-1}).$$

Since τ is supercuspidal and f_1 is a matrix coefficient of τ, we deduce that $\pi\big(\rho(yx^{-1})h\big)$ is equal to 0 if π is not equivalent to τ. Therefore, applying the Plancherel formula ([W2] Théorème VIII.1.1.) to $\big[\rho(yx^{-1})h\big]$, we obtain

$$K_f(x,y) = d(\tau)\mathrm{tr}\big(\tau(\check{f}_1)\tau(x)\tau(f_2)\tau(y^{-1})\big).$$

We identify $\check{V} \otimes V$ with a subspace of Hilbert-Schmidt operators on V. Taking an orthonormal basis $\mathcal{B}_{HS}(V)$ of $\check{V} \otimes V$ for the scalar product $(S, S') := \mathrm{tr}(SS'^*)$, one obtains

$$K_f(x,y) = d(\tau)\mathrm{tr}\Big(\tau(\check{f}_1)\tau(x)\tau(f_2)\tau(y)^*\Big) = d(\tau)(\tau(\check{f}_1)\tau(x)\tau(f_2), \tau(y))$$

$$= d(\tau) \sum_{S\in\mathcal{B}_{HS}(V)} (\tau(\check{f}_1)\tau(x)\tau(f_2), S^*)\overline{(\tau(y), S^*)}$$

$$= d(\tau) \sum_{S\in\mathcal{B}_{HS}(V)} \mathrm{tr}\big(\tau(x)\tau(f_2)S\tau(\check{f}_1)\big)\mathrm{tr}\overline{\big(\tau(y)S\big)},$$

where the sums over S are finite since $\tau(f_2)$ and $\tau(\check{f}_1)$ are of finite rank. Therefore, the truncated kernel is equal to

$$K^T(f) = d(\tau) \sum_{S\in\mathcal{B}_{HS}(V)} P_\tau^T(\check{\tau} \otimes \tau(f)S)\overline{P_\tau^T(S)}$$

where

$$P_\tau^T(S) = \int_H \mathrm{tr}\big(\tau(h)S\big)u(h,T)dh, \quad S \in \check{V} \otimes V.$$

For $\check{v}\otimes v \in \check{V}\otimes V$, one has $\mathrm{tr}\big(\tau(h)(\check{v}\otimes v)\big) = c_{\check{v},v}(h)$. Since $c_{\check{v},v}$ is compactly supported, the truncated local period $P_\tau^T(S)$ converges when $\|T\|$ approaches infinity to

$$P_\tau(S) = \int_H \mathrm{tr}\big(\tau(h)S\big)dh.$$

Therefore, we obtain

$$\lim_{\|T\|\to+\infty} K^T(f) = d(\tau)m_{P_\tau,P_\tau}(f), \tag{1.4}$$

where m_{P_τ,P_τ} is the spherical character of the representation $\check{\tau}\otimes\tau$ associated to the $H\times H$-invariant linear form P_τ on $\check{V}\otimes V$.

Recall that $\tilde{J}(f)$ is the constant term of $J^T(f)$. We deduce from Theorem 2.15 that

$$d(\tau)m_{P_\tau,P_\tau}(f) = \tilde{J}(f). \tag{1.5}$$

We now express m_{P_τ,P_τ} in terms of H-invariant linear forms on V. Let V_H be the orthogonal of V^{*H} in V. Since $\xi_u(v) = \overline{\xi_v(u)}$ for $u,v\in V$, the space $\overline{V}_H$ is the kernel of $v\mapsto\xi_v$. Let W be a complementary subspace of V_H in V. Then, the map $v\mapsto\xi_v$ is an isomorphism from $\overline{W}$ to V^{*H} and $(u,v)\mapsto\xi_v(u)$ is a nondegenerate hermitian form on W. Let $(e_1,\dots,e_n)$ be an orthogonal basis of W for this hermitian form. We set $\xi_i := \xi_{e_i}$ for $i=1,\dots,n$. Thus we have $\xi_i(e_i)\neq 0$.
We identify $\overline{V}$ and $\check{V}$ by the isomorphism ι. We claim that

$$P_\tau = \sum_{i=1}^n \frac{1}{\xi_i(e_i)}\overline{\xi_i}\otimes\xi_i \tag{1.6}$$

Indeed, we have $P_\tau(v\otimes u) = \xi_v(u) = \overline{\xi_u(v)}$. Hence, the two sides are equal to 0 on $\overline{V}\otimes V_H + \overline{V}_H\otimes V + \overline{V}_H\otimes V_H$ and take the same value $\xi_k(e_l)$ on $e_k\otimes e_l$ for $k,l\in\{1,\dots n\}$. Hence, by definition of spherical characters, we deduce that

$$m_{P_\tau,P_\tau}(f_1\otimes f_2) = \sum_{u\otimes v\in\, o.b.(\bar{V}\otimes V)} P_\tau\Big(\bar{\tau}(f_1)\otimes\tau(f_2)(u\otimes v)\Big)\overline{P_\tau(u\otimes v)}$$

$$= \sum_{u\otimes v\in\, o.b.(\bar{V}\otimes V)}\ \sum_{i,j=1}^n \frac{1}{\xi_i(e_i)\xi_j(e_j)}\overline{\xi_i}(\bar{\tau}(f_1)u)\xi_i(\tau(f_2)v)\overline{\overline{\xi_j}(u)\xi_j(v)},$$

where $o.b.(\bar{V}\otimes V)$ is an orthonormal basis of $\bar{V}\otimes V$. By definition of $\bar{\xi}$ for $\xi\in V^{*H}$, one has $\bar{\xi}(\bar{\tau}(f_1)u) = \overline{\xi(\tau(\bar{f}_1))}$. Therefore, we obtain

$$m_{P_\tau,P_\tau}(f_1\otimes f_2) = \sum_{i,j=1}^n \frac{1}{\xi_i(e_i)\xi_j(e_j)}\overline{m_{\xi_i,\xi_j}(\bar{f}_1)}m_{\xi_i,\xi_j}(f_2). \tag{1.7}$$

Let v and w in V. Let $f_1 := c_{v,w}$ so that $\bar{f}_1 = \check{c}_{v,w}$. . If $v \in V_H$ or $w \in V_H$, it follows from Lemma A.1 that $m_{\xi_i,\xi_j}(\bar{f}_1) = 0$ for $i, j \in \{1, \dots, n\}$, hence $m_{P_\tau,P_\tau}(f_1 \otimes f_2) = 0$. Thus, we deduce from (1.5) that

$$\tilde{J}(c_{v,w} \otimes f_2) = 0, \quad v \in V_H \text{ or } w \in V_H. \tag{1.8}$$

Let $k, l \in \{1, \dots, n\}$. We set $f_1 := c_{e_k,e_l}$, hence $\bar{f}_1 = \check{c}_{e_l,e_k}$. By Lemma A.1, one has $m_{\xi_i,\xi_j}(\bar{f}_1) = d(\tau)^{-1}\xi_i(e_l)\xi_j(e_k)$. Therefore, by (1.5) and (1.7) we obtain

$$\tilde{J}(c_{e_k,e_l} \otimes f_2) = m_{\xi_l,\xi_k}(f_2). \tag{1.9}$$

By sesquilinearity, ones deduces from (1.8) and (1.9) that one has

$$\tilde{J}(c_{v,w} \otimes f_2) = m_{\xi_w,\xi_v}(f_2) \quad v, w \in V. \tag{1.10}$$

Let $(J_n)_n$ be a sequence of compact open sugroups whose intersection is equal to the neutral element of G. The characteristic function g_n of $J_n x_m \gamma J_n$ approaches the Dirac measure at $x_m\gamma$ as n approaches $+\infty$. Thus, if $v, w \in V$ then $m_{\xi_w,\xi_v}(g_n)$ converges to $m_{\xi_w,\xi_v}(x_m\gamma)$. By Corollary 2.11, the constant term $\tilde{J}(c_{v,w} \otimes g_n)$ converges to $c_M c_{S,x_m} \mathcal{WM}(c_{v,w})(x_m\gamma)$. We deduce the Theorem from (1.10). □

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P. Delorme, Aix-Marseille Université, CNRS, Centrale Marseille, I2M, UMR 7373, 13453 Marseille, France.
E-mail address: patrick.delorme@univ-amu.fr

P. Harinck, Ecole polytechnique, CMLS - CNRS, UMR 7640, Route de Saclay, 91128 Palaiseau Cédex, France.
E-mail address: pascale.harinck@ polytechnique.edu

S. Souaifi, Université de Strasbourg, IRMA CNRS, UMR 7501, 7 rue Descartes, 67084 Strasbourg Cédex, France.
E-mail address: sofiane.souaifi@math.unistra.fr