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The distribution of prime numbers: overview of $n.\ln(n)$

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Abstract

The empirical formula giving the n th prime number $p(n)$ is $p(n) = n.\ln(n)$ (from ROSSER (2)). Other studies have been performed (from DUSART for example (1)) in order to better estimate the n th prime number. Unfortunately these formulas don't work since there is a significant difference between the real n th prime number and the number given by the formulas. Here we propose a new model in which the difference is effectively reduced compared to the empirical formula. We discuss about the results and hypothesize that $p(n)$ can be approximated with a constant defined in this work. As prime numbers are important to cryptography and other fields, a better knowledge of the distribution of prime numbers would be very useful. Further investigations are needed to understand the behavior of this constant and therefore to determine the n th prime number with a basic formula that could be used in both theoretical and practical research.

Preliminaries

We define the difference Δ as below:

$$\Delta = N - (n.\ln(n)) \quad n \in N^* \quad (1)$$

if N is the real n th prime number and $n.ln(n)$ the approximated n th prime number (see abstract).

We define ζ as below:

$$\zeta = (n.ln(n)) - \Delta \quad n \in N^* \quad (2)$$

We define ϵ as below:

$$\epsilon = \frac{\zeta}{\Delta} \quad \Delta \neq 0 \quad (3)$$

The aim is to know Δ to find the real n th prime number. In fact Δ is the difference between the real n th prime number and the number given by the empirical formula. According to (2) we have:

$$\zeta = (n.ln(n)) - \Delta \quad n \in N^*$$

$$\zeta = (n.ln(n)) - \frac{\zeta}{\epsilon} \quad \epsilon \neq 0$$

$$\zeta = \frac{\epsilon(n.ln(n)) - \zeta}{\epsilon} \quad \epsilon \neq 0$$

$$\epsilon\zeta + \zeta = \epsilon(n.ln(n))$$

$$\zeta(\epsilon + 1) = \epsilon(n.ln(n))$$

$$\zeta = \frac{\epsilon(n.ln(n))}{\epsilon + 1}$$

According to (3) we have:

$$\epsilon = \frac{\zeta}{\Delta} \quad \Delta \neq 0$$

$$\Delta = \frac{\zeta}{\epsilon} \quad \epsilon \neq 0$$

$$\Delta = \frac{\epsilon(n.\ln(n))}{\epsilon^2 + \epsilon} \quad \epsilon \neq 0 \quad (4)$$

Finally the real nth prime number is given by the following formula:

$$N = (n.\ln(n)) + \Delta \quad n \in N^*$$

$$N = (n.\ln(n)) \frac{2+\epsilon}{1+\epsilon} \quad \epsilon \neq -1 \quad (5)$$

Consequently, we must to know ϵ to find Δ and the real nth prime number. In this work we discuss about the value of ϵ that is associated with four intervals and two values for n (see Methods and results). Although there are variations for ϵ we define it as a constant because ϵ seems to show small variations (its value is -0.14092488 for $n=2$ and 7.271015283 for $n = 10^6$).

Methods and results

Methods

In this work we use four intervals and two values for n as described below:

$$n \in [2, 200] \quad n \in [1000, 1195] \quad n \in [1800, 1999] \quad n \in [2600, 2799] \quad n = 100000 \quad n = 10^6$$

Note that the gap between each interval is about 600 (except for the first).

By using Microsoft Excel 2016 and a list of known prime numbers we define, for each n of each interval, the corresponding prime number. For example the prime number N corresponding to $n=2$ is 3. If $n=5$, the prime number N that is associated is 11.

Then, for each nth prime number, we calculate $n.\ln(n)$ that is the approximated nth prime number, Δ , ζ and ϵ . Finally, for each interval, we determine the average value for ϵ . We define ϵ' as the average value for ϵ . Our results suggest that ϵ' shows small variations even

if n is comprised between 2 and 10^6 .

Finally, we calculate for each n of each interval, the corresponding n th prime number given by our method with the formula (5). We define this number as N' and we determine the difference between N' and the real n th prime number as Δ' . Our results show that the average value for Δ' is significantly smaller than the average value for Δ .

Results

Note the corresponding values for ϵ' for each interval:

$$n \in [2.200] \quad \epsilon' = 5.1504379$$

$$n \in [1000.1195] \quad \epsilon' = 5.81488845$$

$$n \in [1800.1999] \quad \epsilon' = 6.09402715$$

$$n \in [2600.2799] \quad \epsilon' = 6.19761627$$

$$n = 100000 \quad \epsilon = 6.757176176$$

$$n = 10^6 \quad \epsilon = 7.271015283$$

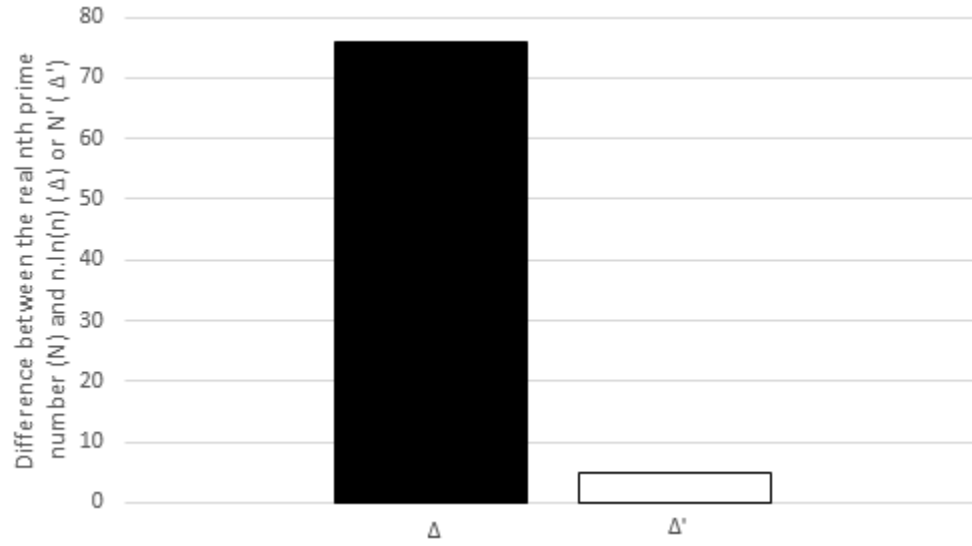


Figure 1: Δ' (4.87377461) is significantly smaller than Δ (75.819752) (average value). Results for the interval $[2, 200]$

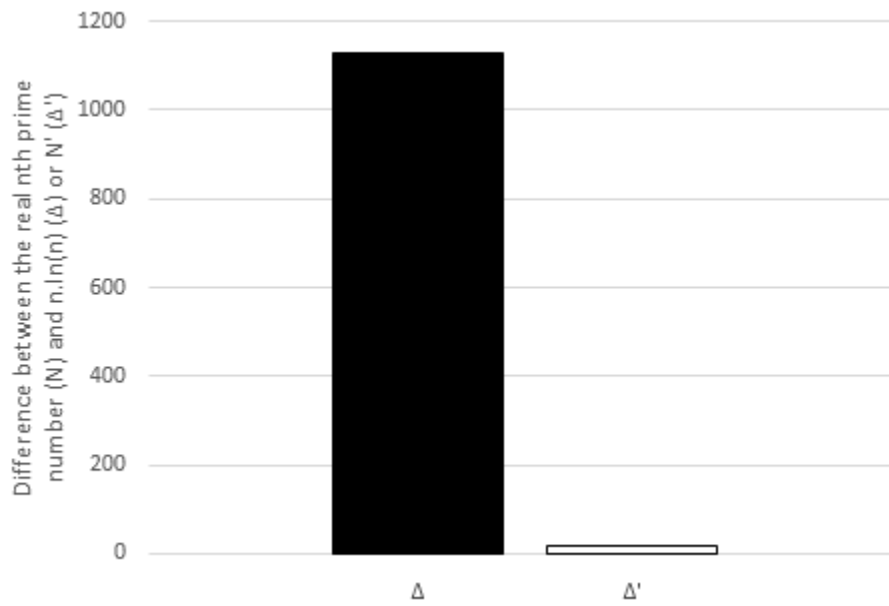


Figure 2: Δ' (15.0668754) is significantly smaller than Δ (1127.19363) (average value). Results for the interval $[1000, 1195]$

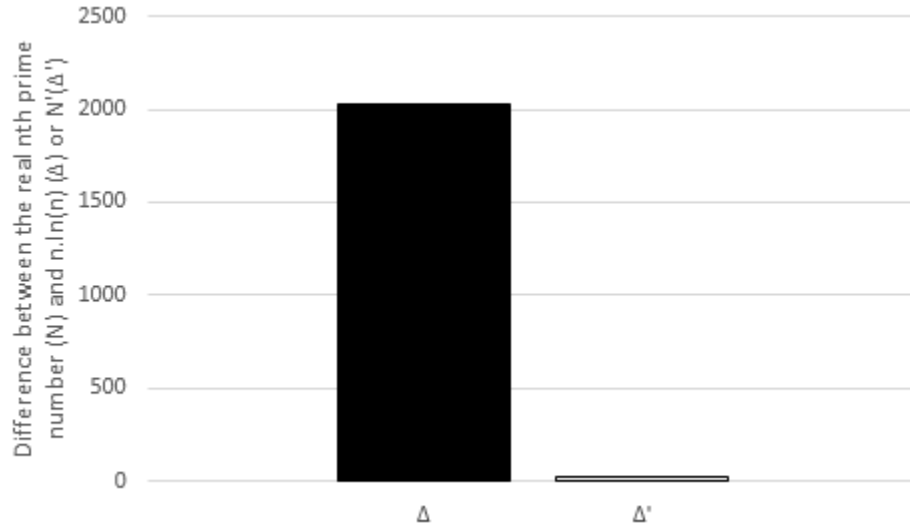


Figure 3: Δ' (19.3845647) is significantly smaller than Δ (2022.39968) (average value). Results for the interval [1800, 1999]

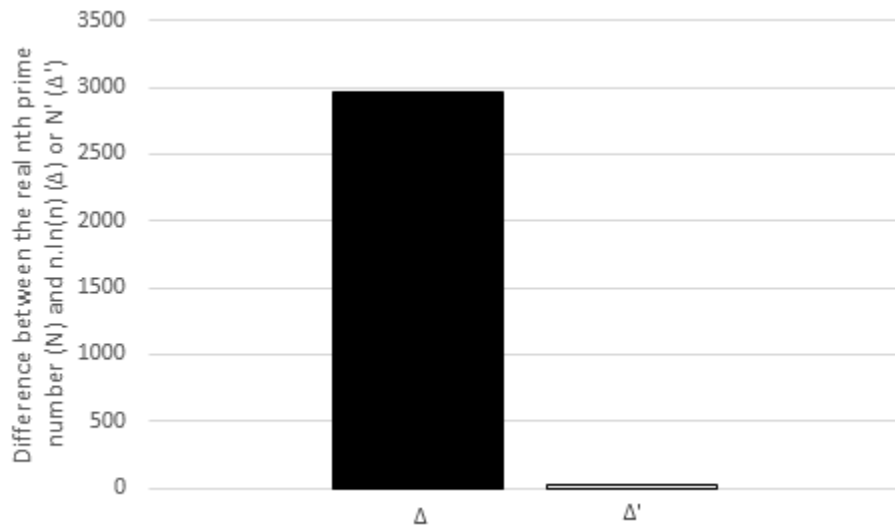


Figure 4: Δ' (29.0238221) is significantly smaller than Δ (2963.99395) (average value). Results for the interval [2600, 2799]

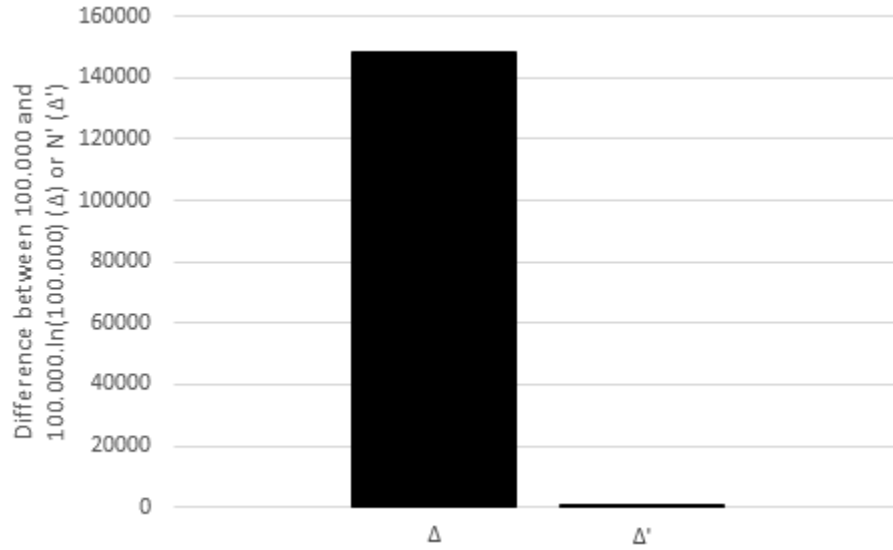


Figure 5: $\Delta' = 0$ is significantly smaller than Δ (148416). Results for the value $n=100000$

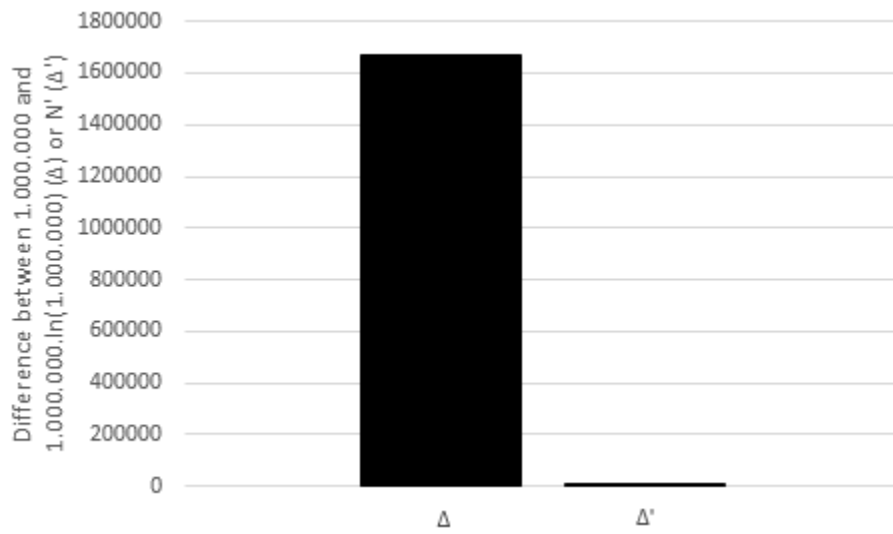


Figure 6: $\Delta' = 0$ is significantly smaller than Δ (1670352). Results for the value $n = 10^6$

Discussion

We have established a relationship between the real n th prime number and a constant ϵ . The use of our method shows results much more effective compared to the empirical formula. For certain values of n , we found the real n th prime number (i.e $\Delta' = 0$). For example, if

n=100000 the real nth prime number that is associated is 1299709. The best value for ϵ for n=100000 is 6.757176176. With this value, we find the real prime number ($\Delta' = 0$). Unfortunately the best value for ϵ is found only if we know the real nth prime number. If the real nth prime number is unknown, the best value for ϵ remains to be elucidated and the best value is required to find the real nth prime number. If we know ϵ it is possible to determine the precise nth prime number ($\Delta' = 0$ since there is a link between Δ and ϵ (see (4)).

Interestingly ϵ seems to show very small variations whereas n is comprised between 2 and 10^6 . For this reason, we hypothesize that ϵ could be a very good constant in order to find a formula establishing a more precise link with the nth prime number. Nevertheless small variations for ϵ are responsible for greater variations for Δ' .

ϵ seems to exhibit very small variations while n allows high values. However ϵ shows high variations between the first and the second interval ($5.81488845-5.1504379 = 0.66445055$ for a difference of n=800 between the two intervals). ϵ shows smaller variations between the last interval and 100000 ($6.757176176-6.19761627=0.559559906$) and between 100000 and 10^6 ($7.271015283-6.757176176=0.513839107$). These results suggest that ϵ is not defined by the length of the gap between two intervals because in this case ϵ would be much more greater than 7.271015283 for n= 10^6 . Furthermore, these results suggest that

$$\epsilon \xrightarrow[n \rightarrow +\infty]{} k$$

if $k \in R$ and $k > 7.271015283$

If this is the case, ϵ is a constant when $n \rightarrow +\infty$ and it will be easier to find a formula to determine nth prime numbers when n tends to infinity. But the most likely hypothesis is that ϵ doesn't converge when n tends to infinity and it will be more difficult to find a formula to determine nth prime numbers.

Interestingly we notice that if we consider a specific n th prime number (in the third interval of this study for example), the value of ϵ of the next or the previous prime number is very close to the value of ϵ for the specific n th prime number considered. This is very useful because we could imagine a formula giving the next prime number if the previous is known. However this is an observation with exceptions and there is no evidence when $n \rightarrow +\infty$. Moreover, even if the value of ϵ of a next prime number seems to be close to the value of ϵ of a previous prime number, there are unpredictable variations and a little mistake for the value of ϵ results in a big error in the determination of the n th prime number. Note that the value of a specific ϵ can be smaller than the value of ϵ of a previous prime number. Moreover, for certain values of n we observe exceptions, for example for $n=30$ (the real n th prime number corresponding is 113) we found $\epsilon = 8.30638366$, a value that is greater than the value of ϵ that is associated with $n=10^6$ (7.271015283). These observations suggest there are several exceptions for the value of ϵ . However the average value for ϵ (ϵ') seems to be greater than the value for ϵ' that is associated with smaller values for n . Therefore the exact relationship between n and ϵ is unknown and may not exist. Thus, the behavior of ϵ remains to be elucidated (see below).

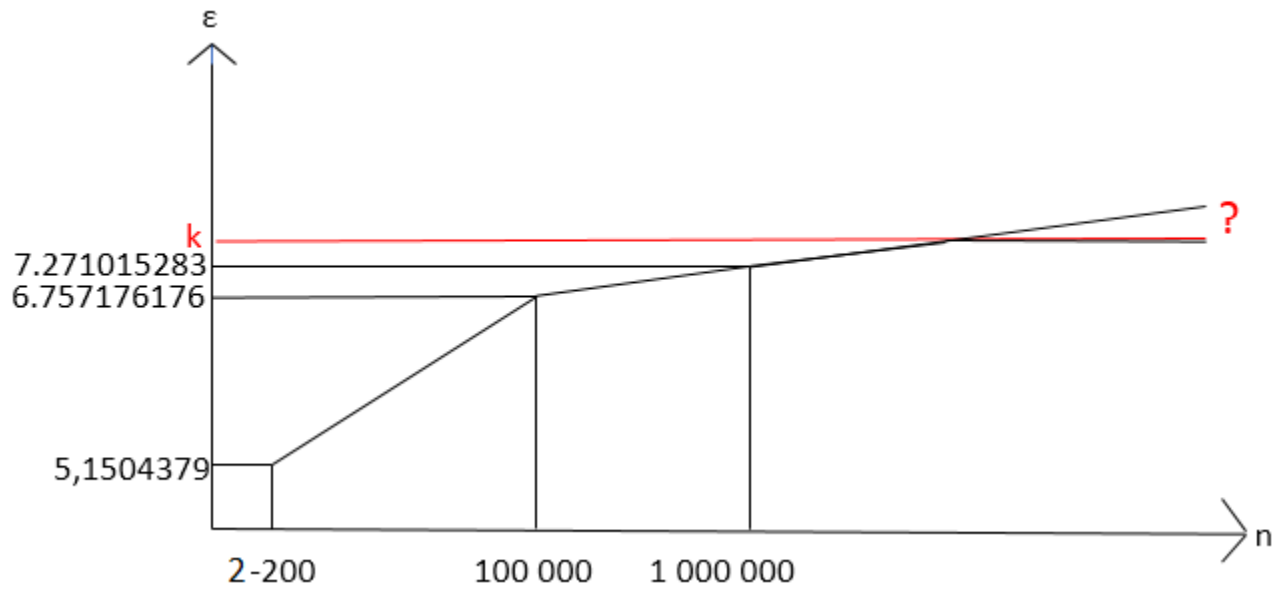


Figure 7: The behavior of ϵ remains to be elucidated when $n > 10^6$. ϵ may converge to k when $n \rightarrow +\infty$ or not.

Conjecture

For $n > 2$ ϵ seems to be comprised between 0 and $+\infty$ (if ϵ doesn't converge when $n \rightarrow +\infty$).

So we hypothesize that:

$$N = (n \cdot \ln(n))^{\frac{2+\epsilon}{1+\epsilon}}$$

$$N = (n \cdot \ln(n))^p \text{ with } 1 < p < 2, p = \frac{2+\epsilon}{1+\epsilon} \text{ and } n > 2$$

In fact -1 (for $n=1$) and -0.14092488 (for $n=2$) seem to be the only negative values for ϵ .

When $n \rightarrow +\infty$ and if ϵ doesn't converge we have $p \rightarrow 1$ because:

$$\lim_{\epsilon \rightarrow +\infty} \frac{2+\epsilon}{1+\epsilon} = 1$$

But p is not exactly 1 and a small variation of p (for example 1.00000005 instead of 1.00000004) is responsible for a big error in the determination of big n th prime numbers.

Now, when n is smaller ϵ is also smaller and we have:

$$\lim_{\epsilon \rightarrow 0} \frac{2+\epsilon}{1+\epsilon} = 2$$

This is the reason for which we hypothesized that $1 < p < 2$. However this is a conjecture because it is not sure if -1 and -0.14092488 are the only negative values for ϵ , even if it is very likely.

Conclusion

We established a model in which we were able to find the real n th prime number by using a new constant called ϵ . When this constant is known precisely, the n th prime number is known precisely. If this constant is imprecise, the n th prime number will be imprecise. Surprisingly we found that ϵ would be comprised between -0.14092488 (for $n=2$) and 7.271015283 (for $n=10^6$), even if n undergoes high variations (from 2 to 10^6). But small variations in ϵ result in an imprecise formula and there are several exceptions with values of ϵ that are greater than 7.271015283 . For this reason, further investigations are needed to understand the behavior of ϵ and to establish a potential relationship between n and ϵ . For example a new work is necessary to study all prime numbers between 3 and 10^6 (not shown in this study because we worked with only four intervals and two values for n).

In all cases, even if ϵ isn't known precisely, our formula seems to be more precise than the empirical formula $n \cdot \ln(n)$.

Tools

Statistics. Statistics were performed using Microsoft Excel 2016.

The list of prime numbers used in this study. http://compoasso.free.fr/primelistweb/page/prime/liste_online.php

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