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► To cite this version:

Matteo Cicuttin, Alexandre Ern, Simon Lemaire. A Multiscale Hybrid High-Order method. 2017.
hal-01467434v2

HAL Id: hal-01467434

<https://hal.science/hal-01467434v2>

Preprint submitted on 14 Sep 2017 (v2), last revised 15 Feb 2018 (v3)

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A Multiscale Hybrid High-Order method

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September 14, 2017

Abstract

We devise a multiscale Hybrid High-Order (HHO) method. The method hinges on discrete unknowns that are polynomials attached to the mesh faces and cells; those attached to the mesh cells can be eliminated locally using static condensation. The main building ingredient is a multiscale reconstruction operator that maps onto a fine-scale space spanned by oscillatory basis functions. The method handles arbitrary orders of approximation $k \geq 0$, and is applicable on general meshes. For face unknowns that are polynomials of degree k , we devise two versions of the method, depending on the polynomial degree $(k - 1)$ or k of the cell unknowns. We prove, in the case of periodic coefficients, an energy-error estimate of the form $(\varepsilon^{1/2} + H^{k+1} + (\varepsilon/H)^{1/2})$. We illustrate the theoretical results on periodic and locally periodic test-cases.

1 Introduction

Over the last few years, many advances have been accomplished in the design of arbitrary-order polytopal discretization methods. Such methods are capable of handling meshes with polytopal cells, and possibly including hanging nodes. The use of polytopal meshes can be motivated by the increased flexibility when meshing complex geometries, and when using agglomeration techniques for mesh coarsening (see, e.g., [8]). Classical examples of polytopal methods are the (polytopal) Finite Element Method (FEM) [43, 41], which typically uses non-polynomial basis functions to enforce continuity requirements, and non-conforming methods such as the Discontinuous Galerkin (DG) [6, 17, 11] and the Hybridizable Discontinuous Galerkin (HDG) [16] methods. We also mention the Weak Galerkin (WG) [44] method (see [14] for its links to HDG).

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More recently, new paradigms have emerged. One salient example is the Virtual Element Method (VEM) [10], which is formulated in terms of virtual (i.e., non-computable) conforming functions. The key idea is that the virtual space contains those polynomial functions leading to optimal approximation properties, whereas the remaining functions need not be computed (only their degrees of freedom need to be) provided some suitable local stabilization is introduced. The degrees of freedom in VEM are attached to the mesh vertices, and, as the order of the approximation is increased, also to the mesh edges, faces, and cells. Another recent polytopal method is the Hybrid High-Order (HHO) method, which has been introduced for locking-free linear elasticity in [18], and for diffusion in [20]. The HHO method has been originally formulated as a non-conforming method, using polynomial unknowns attached to the mesh faces and cells. The HHO method has been bridged in [15] both to HDG (by identifying a suitable numerical flux trace), and to the non-conforming VEM considered in [7] (by identifying an isomorphism between the HHO degrees of freedom and a local virtual finite-dimensional space, which again contains those polynomial functions leading to optimal approximation properties). The focus here is on HHO methods. These methods offer several assets, including a dimension-independent construction, local conservativity, and attractive computational costs, especially in 3D. Indeed, the stencil is more compact than for methods involving degrees of freedom attached to the mesh vertices, and static condensation allows one to eliminate cell degrees of freedom, leaving only face degrees of freedom, whose number grows quadratically with the polynomial order, whereas the growth of globally coupled degrees of freedom is cubic for DG methods.

In this work, we are interested in diffusion problems featuring heterogeneous/anisotropic coefficients that are highly oscillatory. The case of slowly varying coefficients has already been treated in [19, 21], where error estimates tracking the dependency of the approximation with respect to the local heterogeneity/anisotropy ratios have been derived. Let Ω be an open, bounded, connected polytopal subset of \mathbb{R}^d , $d \in \{2, 3\}$. Let $\varepsilon > 0$, supposedly much smaller than the diameter of the domain Ω , encode the highly oscillatory nature of the coefficients. We consider the model problem

$$\begin{cases} -\operatorname{div}(\mathbb{A}_\varepsilon \nabla u_\varepsilon) = f & \text{in } \Omega, \\ u_\varepsilon = 0 & \text{on } \partial\Omega, \end{cases} \quad (1)$$

where $f \in L^2(\Omega)$ is non-oscillatory, and \mathbb{A}_ε is an oscillatory, uniformly elliptic and bounded matrix-valued field on Ω . It is well-known that the H^{k+2} -norm of the solution u_ε to Problem (1) scales as $\varepsilon^{-(k+1)}$, meaning that mono-scale methods (including the mono-scale HHO method of order $k \geq 0$ of [19, 21]) provide an energy-norm decay of the error of order $(h/\varepsilon)^{k+1}$. To be accurate, such methods must hence rely on a mesh resolving the fine scale, i.e. with size $h \ll \varepsilon$. Since ε is supposedly much smaller than the diameter of Ω , an accurate approximation necessarily implies an overwhelming number of degrees of freedom. In a multi-query context, where the solution is needed for a large number of right-hand sides (think, e.g., of a time-dependent model), a mono-scale solve is hence unaffordable. In that context, multiscale methods may be preferred. Multiscale methods aim at resolving

the fine scale in an offline step, reducing the online step to the solution of a system of small size (based on an approximation on a coarse mesh with size $H \gg \varepsilon$, using oscillatory basis functions computed in the offline step). In the single-query context, multiscale methods are also interesting since they allow one to organize computations in a more efficient way.

Multiscale approximation methods on classical element shapes (such as simplices or quadrangles/hexahedra) have been extensively analyzed in the literature. Examples include, e.g., the Multiscale Finite Element Method (MsFEM) [31, 32, 23] (with error bound of the form $(\varepsilon^{1/2} + H + (\varepsilon/H)^{1/2})$ in the periodic case), its variant using oversampling [31, 24] (with improved error bound of the form $(\varepsilon^{1/2} + H + \varepsilon/H)$ in the periodic case), the Petrov-Galerkin variant of MsFEM using oversampling [33], and more recently, the MsFEM à la Crouzeix–Raviart of [36, 37] (with upper bound of the form $(\varepsilon^{1/2} + H + (\varepsilon/H)^{1/2})$ in the periodic case). We also mention [4] which is an extension to arbitrary orders of approximation of the classical MsFEM (with error bound of the form $(\varepsilon^{1/2} + H^k + (\varepsilon/H)^{1/2})$ in the periodic case using H^1 -conforming finite elements of degree $k \geq 1$). Multiscale (arbitrary-order) methods supporting polytopal meshes include [26, 25, 12] in the HDG context (see also [39] in the WG context), and [5, 40] for the so-called Multiscale Hybrid-Mixed (MHM) methods. Present research directions focus on reducing and possibly eliminating the cell resonance error by introducing adequate local decompositions (see, e.g., [38, 35]). Note also that there exist other paradigms to approximate oscillatory problems, like the Heterogeneous Multiscale Method (HMM) [22, 1], whose focus is more on computing an approximation of the homogenized solution instead of computing the oscillatory one.

In this work, we devise a multiscale HHO (MsHHO) method, which can be seen as a generalization to arbitrary order and general element shapes of the MsFEM à la Crouzeix–Raviart of [36, 37]. This method already applies to general element shapes, but the analysis in [36, 37] only deals with simplicial element shapes. Thus, our goal is to introduce and analyze (under the classical assumption of periodic coefficients) an *arbitrary-order* and *polytopal* multiscale method, using the quite general framework of HHO methods. Two MsHHO methods are proposed, both employing polynomials of arbitrary order $k \geq 0$ for the face unknowns. For the mixed-order MsHHO method, the cell unknowns are polynomials of order $(k - 1)$ (if $k \geq 1$), whereas they are polynomials of order $k \geq 0$ for the equal-order MsHHO method. The mixed-order MsHHO method does not require stabilization, whereas a simple stabilization is introduced in the equal-order case. We prove for both methods an energy-error estimate of the form $(\varepsilon^{1/2} + H^{k+1} + (\varepsilon/H)^{1/2})$ in the periodic case. To motivate the use of a high-order method, we note, as it was already pointed out in [4], that this upper bound, say $g_k(H)$, is minimal for $H_k = (\varepsilon^{1/2}/2(k+1))^{2/(2k+3)}$, and as $k \geq 0$ increases, H_k increases while $g_k(H_k)$ decreases. Our fine-scale space construction bears some resemblance with the (polynomial-based) one advocated in [26] in the HDG context. However, the two methods differ, both in the construction and in the analysis (in the latter reference, the analysis is sharp only for $H \ll \varepsilon$). Moreover, the analysis of the MsHHO method differs from that of the monoscale HHO method since the local fine-scale space does not contain polynomial functions up to order $(k + 1)$; in this respect, our key approximation result is Lemma 4.5 below. Our analysis also sheds new light on the relationship between the

non-computed functions of the local virtual space and the associated local stabilization.

The article is organized as follows. In Sections 2 and 3 we introduce, respectively, the continuous and discrete settings. In Section 4, we introduce the fine-scale approximation space, exhibiting its (oscillatory) basis functions and studying, locally, its approximation properties. In Section 5, we introduce the two versions of the MsHHO method, analyze their stability, and derive energy-error estimates. In Section 6, we present some numerical illustrations in the periodic and locally periodic settings. Finally, in Appendix A we collect some useful estimates on the first-order two-scale expansion.

2 Continuous setting

From now on, and in order to lead the analysis, we assume that the diffusion matrix \mathbb{A}_ε satisfies $\mathbb{A}_\varepsilon(\cdot) = \mathbb{A}(\cdot/\varepsilon)$ in Ω , where \mathbb{A} is a symmetric and \mathbb{Z}^d -periodic matrix field on \mathbb{R}^d . Letting $Q := (0, 1)^d$, we define, for $1 \leq p \leq +\infty$ and $m \in \mathbb{N}^*$, the following periodic spaces:

$$\begin{aligned} L_{\text{per}}^p(Q) &:= \{v \in L_{\text{loc}}^p(\mathbb{R}^d) \mid v \text{ is } \mathbb{Z}^d\text{-periodic}\}, \\ W_{\text{per}}^{m,p}(Q) &:= \{v \in W_{\text{loc}}^{m,p}(\mathbb{R}^d) \mid v \text{ is } \mathbb{Z}^d\text{-periodic}\}, \end{aligned}$$

with the classical conventions that $W_{\text{per}}^{m,2}(Q)$ is denoted $H_{\text{per}}^m(Q)$ and that the subscript “loc” can be omitted for $p = +\infty$. Letting $\mathcal{S}_d(\mathbb{R})$ denote the set of real-valued $d \times d$ symmetric matrices, we also define, for real numbers $0 < a \leq b$,

$$\mathcal{S}_a^b := \{\mathbb{M} \in \mathcal{S}_d(\mathbb{R}) \mid \forall \boldsymbol{\xi} \in \mathbb{R}^d, a|\boldsymbol{\xi}|^2 \leq \mathbb{M}\boldsymbol{\xi} \cdot \boldsymbol{\xi} \leq b|\boldsymbol{\xi}|^2\}.$$

We assume that there exist real numbers $0 < \alpha \leq \beta$ such that

$$\mathbb{A}(\cdot) \in \mathcal{S}_\alpha^\beta \text{ a.e. in } \mathbb{R}^d. \quad (2)$$

Assumption (2) ensures that $\mathbb{A}_\varepsilon \in L^\infty(\Omega; \mathbb{R}^{d \times d})$ is such that $\mathbb{A}_\varepsilon(\cdot) \in \mathcal{S}_\alpha^\beta$ a.e. in Ω for any $\varepsilon > 0$, and hence guarantees the existence and uniqueness of the solution to (1) in $H_0^1(\Omega)$ for any $\varepsilon > 0$. More importantly, the assumption (2) ensures that the (whole) family $(\mathbb{A}_\varepsilon)_{\varepsilon>0}$ G-converges [3, Section 1.3.2] to some constant symmetric matrix $\mathbb{A}_0 \in \mathcal{S}_\alpha^\beta$. Henceforth, we denote $\rho := \beta/\alpha \geq 1$ the (global) heterogeneity/anisotropy ratio of both $(\mathbb{A}_\varepsilon)_{\varepsilon>0}$ and \mathbb{A}_0 . Letting $(\mathbf{e}_1, \dots, \mathbf{e}_d)$ denote the canonical basis of \mathbb{R}^d , the expression of \mathbb{A}_0 is known to read, for integers $1 \leq i, j \leq d$,

$$[\mathbb{A}_0]_{ij} = \int_Q \mathbb{A}(\mathbf{e}_j + \nabla \mu_j) \cdot (\mathbf{e}_i + \nabla \mu_i) = \int_Q \mathbb{A}(\mathbf{e}_j + \nabla \mu_j) \cdot \mathbf{e}_i, \quad (3)$$

where, for any integer $1 \leq l \leq d$, the so-called corrector $\mu_l \in H_{\text{per}}^1(Q)$ is the solution with zero mean-value on Q to the problem

$$\begin{cases} -\operatorname{div}(\mathbb{A}(\nabla \mu_l + \mathbf{e}_l)) = 0 & \text{in } \mathbb{R}^d, \\ \mu_l \text{ is } \mathbb{Z}^d\text{-periodic.} \end{cases} \quad (4)$$

For further use, we also define the linear operator $\mathcal{R}_\varepsilon : L^p_{\text{per}}(Q) \rightarrow L^p(\Omega)$, $1 \leq p \leq +\infty$, such that, for any function $\chi \in L^p_{\text{per}}(Q)$, $\mathcal{R}_\varepsilon(\chi) \in L^p(\Omega)$ satisfies $\mathcal{R}_\varepsilon(\chi)(\cdot) = \chi(\cdot/\varepsilon)$ in Ω . In particular, for any integers $1 \leq i, j \leq d$, we have $[\mathbb{A}_\varepsilon]_{ij} = \mathcal{R}_\varepsilon(\mathbb{A}_{ij})$. A useful property of \mathcal{R}_ε is the relation $\partial_l(\mathcal{R}_\varepsilon(\chi)) = \frac{1}{\varepsilon} \mathcal{R}_\varepsilon(\partial_l \chi)$, valid for any function $\chi \in W^{1,p}_{\text{per}}(Q)$ and any integer $1 \leq l \leq d$.

The homogenized problem reads

$$\begin{cases} -\operatorname{div}(\mathbb{A}_0 \nabla u_0) = f & \text{in } \Omega, \\ u_0 = 0 & \text{on } \partial\Omega. \end{cases} \quad (5)$$

We introduce the so-called first-order two-scale expansion

$$\mathcal{L}_\varepsilon^1(u_0) := u_0 + \varepsilon \sum_{l=1}^d \mathcal{R}_\varepsilon(\mu_l) \partial_l u_0. \quad (6)$$

Note that $(u_\varepsilon - \mathcal{L}_\varepsilon^1(u_0))$ does not a priori vanish on the boundary of Ω .

3 Discrete setting

We denote by $\mathcal{H} \subset \mathbb{R}_+^*$ a countable set of meshsizes having 0 as its unique accumulation point, and we consider mesh sequences of the form $(\mathcal{T}_H)_{H \in \mathcal{H}}$. For any $H \in \mathcal{H}$, a *mesh* \mathcal{T}_H is a finite collection of nonempty disjoint open polytopes (polygons/polyhedra) T , called *elements* or *cells*, such that $\bar{\Omega} = \bigcup_{T \in \mathcal{T}_H} \bar{T}$ and $H = \max_{T \in \mathcal{T}_H} H_T$, H_T standing for the diameter of the cell T . The mesh cells being polytopal, their boundary is composed of a finite union of portions of affine hyperplanes in \mathbb{R}^d called *facets* (each facet has positive $(d-1)$ -dimensional measure). A closed subset F of $\bar{\Omega}$ is called a *face* if either (i) there exist $T_1, T_2 \in \mathcal{T}_H$ such that $F = \partial T_1 \cap \partial T_2 \cap Z$ where Z is an affine hyperplane supporting a facet of both T_1 and T_2 (and F is termed *interface*), or (ii) there exists $T \in \mathcal{T}_H$ such that $F = \partial T \cap \partial\Omega \cap Z$ where Z is an affine hyperplane supporting a facet of both T and Ω (and F is termed *boundary face*). Interfaces are collected in the set \mathcal{F}_H^i , boundary faces in \mathcal{F}_H^b , and we let $\mathcal{F}_H := \mathcal{F}_H^i \cup \mathcal{F}_H^b$. The diameter of a face $F \in \mathcal{F}_H$ is denoted H_F . For all $T \in \mathcal{T}_H$, we define $\mathcal{F}_T := \{F \in \mathcal{F}_H \mid F \subset \partial T\}$ the set of faces lying on the boundary of T ; note that the faces in \mathcal{F}_T compose the boundary of T . For any $T \in \mathcal{T}_H$, we denote by $\mathbf{n}_{\partial T}$ the unit normal vector to ∂T pointing outward T , and for any $F \in \mathcal{F}_T$, we let $\mathbf{n}_{T,F} := \mathbf{n}_{\partial T|F}$ (by definition, $\mathbf{n}_{T,F}$ is a constant vector on F).

We adopt the following notion of admissible mesh sequence; cf. [17, Section 1.4] and [21, Definition 2.1].

Definition 3.1 (Admissible mesh sequence). *The mesh sequence $(\mathcal{T}_H)_{H \in \mathcal{H}}$ is admissible if, for all $H \in \mathcal{H}$, \mathcal{T}_H admits a matching simplicial sub-mesh \mathfrak{T}_H (meaning that the cells in \mathfrak{T}_H are sub-cells of the cells in \mathcal{T}_H and that the faces of these sub-cells belonging to the skeleton of \mathcal{T}_H are sub-faces of the faces in \mathcal{F}_H) such that there exists a real number $\gamma > 0$, called mesh regularity parameter, such that, for all $H \in \mathcal{H}$, the following holds:*

(i) For all simplex $S \in \mathfrak{T}_H$ of diameter H_S and inradius R_S , $\gamma H_S \leq R_S$;

(ii) For all $T \in \mathcal{T}_H$, and all $S \in \mathfrak{T}_T := \{S \in \mathfrak{T}_H \mid S \subseteq T\}$, $\gamma H_T \leq H_S$.

Two classical consequences of Definition 3.1 are that, for any mesh \mathcal{T}_H belonging to an admissible mesh sequence, (i) the quantity $\text{card}(\mathcal{F}_T)$ is bounded independently of the diameter H_T for all $T \in \mathcal{T}_H$ [17, Lemma 1.41], and (ii) mesh faces have a comparable diameter to the diameter of the cells to which they belong [17, Lemma 1.42].

For any $q \in \mathbb{N}$, and any integer $1 \leq l \leq d$, we denote by \mathbb{P}_l^q the linear space spanned by l -variate polynomial functions of total degree less or equal to q . We let

$$N_l^q := \dim(\mathbb{P}_l^q) = \binom{q+l}{q}.$$

Let a mesh \mathcal{T}_H be given. For any $T \in \mathcal{T}_H$, $\mathbb{P}_d^q(T)$ is composed of the restriction to T of polynomials in \mathbb{P}_d^q , and for any $F \in \mathcal{F}_H$, $\mathbb{P}_{d-1}^q(F)$ is composed of the restriction to F of polynomials in \mathbb{P}_d^q (this space can also be described as the restriction to F of polynomials in $\mathbb{P}_{d-1}^q \circ \Theta^{-1}$, where Θ is any affine bijective mapping from \mathbb{R}^{d-1} to the affine hyperplane supporting F). We also introduce, for any $T \in \mathcal{T}_H$, the following broken polynomial space:

$$\mathbb{P}_{d-1}^q(\mathcal{F}_T) := \{v \in L^2(\partial T) \mid v|_F \in \mathbb{P}_{d-1}^q(F) \ \forall F \in \mathcal{F}_T\}.$$

The term ‘broken’ refers to the fact that no continuity is required between adjacent faces for functions in $\mathbb{P}_{d-1}^q(\mathcal{F}_T)$. For any $T \in \mathcal{T}_H$, we denote by $(\Phi_T^{q,i})_{1 \leq i \leq N_d^q}$ a set of basis functions of the space $\mathbb{P}_d^q(T)$, and for any $F \in \mathcal{F}_H$, we denote by $(\Phi_F^{q,j})_{1 \leq j \leq N_{d-1}^q}$ a set of basis functions of the space $\mathbb{P}_{d-1}^q(F)$. We define, for any $T \in \mathcal{T}_H$ and $F \in \mathcal{F}_H$, Π_T^q and Π_F^q as the L^2 -orthogonal projectors onto $\mathbb{P}_d^q(T)$ and $\mathbb{P}_{d-1}^q(F)$, respectively. Whenever no confusion can arise, we write, for all $T \in \mathcal{T}_H$, all $F \in \mathcal{F}_T$, and all $v \in H^1(T)$, $\Pi_F^q(v)$ instead of $\Pi_F^q(v|_F)$.

We conclude this section by recalling some classical results, that are valid for any mesh \mathcal{T}_H belonging to an admissible mesh sequence in the sense of Definition 3.1. For any $T \in \mathcal{T}_H$ and $F \in \mathcal{F}_T$, the trace inequalities

$$\|v\|_{L^2(F)} \leq c_{\text{tr,d}} H_F^{-1/2} \|v\|_{L^2(T)} \quad \forall v \in \mathbb{P}_d^q(T), \quad (7)$$

$$\|v\|_{L^2(F)} \leq c_{\text{tr,c}} \left(H_T^{-1} \|v\|_{L^2(T)}^2 + H_T \|\nabla v\|_{L^2(T)^d}^2 \right)^{1/2} \quad \forall v \in H^1(T), \quad (8)$$

hold [17, Lemmas 1.46 and 1.49], as well as the local Poincaré inequality

$$\|v\|_{L^2(T)} \leq c_P H_T \|\nabla v\|_{L^2(T)^d} \quad \forall v \in H^1(T) \text{ such that } \int_T v = 0, \quad (9)$$

where $c_P = \pi^{-1}$ for convex elements [9]; estimates in the nonconvex case can be found, e.g., in [42]. Finally, proceeding as in [27, Lemma 5.6], one can prove using the above trace and Poincaré inequalities that

$$|v - \Pi_T^q(v)|_{H^m(T)} + H_T^{1/2} |v - \Pi_T^q(v)|_{H^m(F)} \leq c_{\text{app}} H_T^{s-m} |v|_{H^s(T)} \quad \forall v \in H^s(T), \quad (10)$$

for integers $1 \leq s \leq q+1$ and $0 \leq m \leq (s-1)$. All of the above constants are independent of the meshsize and can depend on the underlying polynomial degree q , the space dimension d , and on the mesh regularity parameter γ .

Henceforth, we use the symbol c to denote a generic positive constant, whose value can change at each occurrence, provided it is independent of the micro-scale ε , any meshsize H_T or H , the homogenized solution u_0 , and the parameters α, β characterizing the spectrum of the diffusion matrices; the value of c can depend on the space dimension d , the underlying polynomial degree, the mesh regularity parameter γ , and some higher-order norms of the diffusion matrix \mathbb{A} or the correctors μ_l that will be made clear from the context. A recent hp -analysis of the monoscale HHO method can be found in [2].

4 Fine-scale approximation space

Let $k \in \mathbb{N}$ and let \mathcal{T}_H be a member of an admissible mesh sequence in the sense of Definition 3.1. In this section, we introduce the fine-scale approximation space on which we will base our multiscale HHO method. We first construct in Section 4.1 a set of cell-based and face-based basis functions, then we provide in Section 4.2 a local characterization of the underlying space, finally we study its approximation properties in Section 4.3.

4.1 Oscillatory basis functions

The oscillatory basis functions consist of cell- and face-based basis functions.

4.1.1 Cell-based basis functions

Let $T \in \mathcal{T}_H$. If $k = 0$, we do not define cell-based basis functions. Assume now that $k \geq 1$. For all $1 \leq i \leq N_d^{k-1}$, we consider the problem

$$\inf \left\{ \int_T \left[\frac{1}{2} \mathbb{A}_\varepsilon \nabla \varphi \cdot \nabla \varphi - \Phi_T^{k-1,i} \varphi \right], \varphi \in H^1(T), \Pi_F^k(\varphi) = 0 \ \forall F \in \mathcal{F}_T \right\}. \quad (11)$$

Problem (11) admits a unique minimizer. This minimizer, that we will denote $\varphi_{\varepsilon,T}^{k+1,i} \in H^1(T)$, can be proved to solve, for real numbers $(\lambda_{F,j}^T)_{F \in \mathcal{F}_T, 1 \leq j \leq N_{d-1}^k}$ satisfying the compatibility condition

$$\sum_{F \in \mathcal{F}_T} \int_F \sum_{j=1}^{N_{d-1}^k} \lambda_{F,j}^T \Phi_F^{k,j} = - \int_T \Phi_T^{k-1,i},$$

the continuous problem

$$\begin{cases} -\operatorname{div}(\mathbb{A}_\varepsilon \nabla \varphi_{\varepsilon,T}^{k+1,i}) = \Phi_T^{k-1,i} & \text{in } T, \\ \mathbb{A}_\varepsilon \nabla \varphi_{\varepsilon,T}^{k+1,i} \cdot \mathbf{n}_{T,F} = \sum_{j=1}^{N_{d-1}^k} \lambda_{F,j}^T \Phi_F^{k,j} & \text{on all } F \in \mathcal{F}_T, \\ \Pi_F^k(\varphi_{\varepsilon,T}^{k+1,i}) = 0 & \text{on all } F \in \mathcal{F}_T. \end{cases} \quad (12)$$

The superscript $k + 1$ is meant to remind us that the functions $\varphi_{\varepsilon,T}^{k+1,i}$ are used to generate a linear space which has the same approximation capacity as the polynomial space of order at most $k + 1$, as will be shown in Section 4.3.

Remark 4.1 (Practical computation). *To compute $\varphi_{\varepsilon,T}^{k+1,i}$ for all $1 \leq i \leq N_d^{k-1}$, one considers in practice a (shape-regular) matching simplicial mesh \mathcal{T}_h^T of the cell T , with size h smaller than ε . Then, one can solve Problem (12) approximately by using a classical (equal-order) monoscale HHO method (or any other monoscale approximation method). For the implementation of the monoscale HHO method, we refer to [13]. One can either consider a weak formulation in $\{\varphi \in H^1(T), \Pi_F^k(\varphi) = 0 \forall F \in \mathcal{F}_T\}$, which leads to a coercive problem, or a weak formulation in $H^1(T)$, which leads to a saddle-point system with Lagrange multipliers. Equivalent considerations apply below to the computation of the face-based basis functions. Note that the error estimates we provide in this work for our approach do not take into account the local approximations of size h and assume that (12) and (14) below are solved exactly.*

4.1.2 Face-based basis functions

Let $T \in \mathcal{T}_H$. For all $F \in \mathcal{F}_T$ and all $1 \leq j \leq N_{d-1}^k$, we consider the problem

$$\inf \left\{ \int_T \left[\frac{1}{2} \mathbb{A}_\varepsilon \nabla \varphi \cdot \nabla \varphi \right], \varphi \in H^1(T), \Pi_F^k(\varphi) = \Phi_F^{k,j}, \Pi_\sigma^k(\varphi) = 0 \forall \sigma \in \mathcal{F}_T \setminus \{F\} \right\}. \quad (13)$$

Problem (13) admits a unique minimizer. This minimizer, that we will denote $\varphi_{\varepsilon,T,F}^{k+1,j} \in H^1(T)$, can be proved to solve, for real numbers $(\lambda_{\sigma,q}^{T,F})_{\sigma \in \mathcal{F}_T, 1 \leq q \leq N_{d-1}^k}$ satisfying the compatibility condition

$$\sum_{\sigma \in \mathcal{F}_T} \int_\sigma \sum_{q=1}^{N_{d-1}^k} \lambda_{\sigma,q}^{T,F} \Phi_\sigma^{k,q} = 0,$$

the continuous problem

$$\left\{ \begin{array}{ll} -\operatorname{div}(\mathbb{A}_\varepsilon \nabla \varphi_{\varepsilon,T,F}^{k+1,j}) = 0 & \text{in } T, \\ \mathbb{A}_\varepsilon \nabla \varphi_{\varepsilon,T,F}^{k+1,j} \cdot \mathbf{n}_{T,\sigma} = \sum_{q=1}^{N_{d-1}^k} \lambda_{\sigma,q}^{T,F} \Phi_\sigma^{k,q} & \text{on all } \sigma \in \mathcal{F}_T, \\ \Pi_F^k(\varphi_{\varepsilon,T,F}^{k+1,j}) = \Phi_F^{k,j}, & \text{on } F, \\ \Pi_\sigma^k(\varphi_{\varepsilon,T,F}^{k+1,j}) = 0 & \text{on all } \sigma \in \mathcal{F}_T \setminus \{F\}. \end{array} \right. \quad (14)$$

4.2 Discrete space

We introduce, for any $T \in \mathcal{T}_H$, the space

$$V_{\varepsilon,T}^{k+1} := \{v_\varepsilon \in H^1(T) \mid \operatorname{div}(\mathbb{A}_\varepsilon \nabla v_\varepsilon) \in \mathbb{P}_d^{k-1}(T), \mathbb{A}_\varepsilon \nabla v_\varepsilon \cdot \mathbf{n}_{\partial T} \in \mathbb{P}_{d-1}^k(\mathcal{F}_T)\}, \quad (15)$$

with the convention that $\mathbb{P}_d^{-1}(T) := \{0\}$. We recall that the condition $\mathbb{A}_\varepsilon \nabla v_\varepsilon \cdot \mathbf{n}_{\partial T} \in \mathbb{P}_{d-1}^k(\mathcal{F}_T)$ is equivalent to $\mathbb{A}_\varepsilon \nabla v_\varepsilon \cdot \mathbf{n}_{T,F} \in \mathbb{P}_{d-1}^k(F)$ for all $F \in \mathcal{F}_T$. Proceeding as in [15, Section 2.4], it can be easily shown that the dimension of $V_{\varepsilon,T}^{k+1}$ is $(N_d^{k-1} + \text{card}(\mathcal{F}_T) \times N_{d-1}^k)$ (or $\text{card}(\mathcal{F}_T)$ if $k = 0$).

Proposition 4.2 (Characterization of $V_{\varepsilon,T}^{k+1}$). *For any $T \in \mathcal{T}_H$, the following holds:*

$$V_{\varepsilon,T}^{k+1} = \text{Span} \left\{ (\varphi_{\varepsilon,T}^{k+1,i})_{1 \leq i \leq N_d^{k-1}}, (\varphi_{\varepsilon,T,F}^{k+1,j})_{F \in \mathcal{F}_T, 1 \leq j \leq N_{d-1}^k} \right\}. \quad (16)$$

Proof. To establish (16), we only need to prove that

$$V_{\varepsilon,T}^{k+1} \subset \text{Span} \left\{ (\varphi_{\varepsilon,T}^{k+1,i})_{1 \leq i \leq N_d^{k-1}}, (\varphi_{\varepsilon,T,F}^{k+1,j})_{F \in \mathcal{F}_T, 1 \leq j \leq N_{d-1}^k} \right\},$$

since the converse inclusion follows from the definition of the oscillatory basis functions. Let $v_\varepsilon \in V_{\varepsilon,T}^{k+1}$. Then, there exist real numbers $(\theta_T^i)_{1 \leq i \leq N_d^{k-1}}$ (only if $k \geq 1$) and $(\theta_{T,F}^j)_{F \in \mathcal{F}_T, 1 \leq j \leq N_{d-1}^k}$, satisfying the compatibility condition

$$\sum_{F \in \mathcal{F}_T} \int_F \sum_{j=1}^{N_{d-1}^k} \theta_{T,F}^j \Phi_F^{k,j} = - \int_T \sum_{i=1}^{N_d^{k-1}} \theta_T^i \Phi_T^{k-1,i} (= 0 \text{ if } k = 0),$$

such that

$$\begin{cases} -\text{div}(\mathbb{A}_\varepsilon \nabla v_\varepsilon) = \sum_{i=1}^{N_d^{k-1}} \theta_T^i \Phi_T^{k-1,i} (= 0 \text{ if } k = 0) & \text{in } T, \\ \mathbb{A}_\varepsilon \nabla v_\varepsilon \cdot \mathbf{n}_{T,F} = \sum_{j=1}^{N_{d-1}^k} \theta_{T,F}^j \Phi_F^{k,j} & \text{on all } F \in \mathcal{F}_T. \end{cases}$$

Let us now introduce

$$\zeta := v_\varepsilon - \sum_{i=1}^{N_d^{k-1}} \theta_T^i \varphi_{\varepsilon,T}^{k+1,i} - \sum_{\sigma \in \mathcal{F}_T} \sum_{j=1}^{N_{d-1}^k} x_\sigma^{k,j}(v_\varepsilon) \varphi_{\varepsilon,T,\sigma}^{k+1,j},$$

where, for all $\sigma \in \mathcal{F}_T$, the real numbers $(x_\sigma^{k,j}(v_\varepsilon))_{1 \leq j \leq N_{d-1}^k}$ solve the linear system

$$\sum_{j=1}^{N_{d-1}^k} \left(\int_\sigma \Phi_\sigma^{k,j} \Phi_\sigma^{k,q} \right) x_\sigma^{k,j}(v_\varepsilon) = \int_\sigma v_\varepsilon \Phi_\sigma^{k,q} \quad \text{for all } 1 \leq q \leq N_{d-1}^k.$$

It can be easily checked that $-\text{div}(\mathbb{A}_\varepsilon \nabla \zeta) = 0$ in T and that $\mathbb{A}_\varepsilon \nabla \zeta \cdot \mathbf{n}_{T,F} \in \mathbb{P}_{d-1}^k(F)$ and $\Pi_F^k(\zeta) = 0$ on all $F \in \mathcal{F}_T$. Using the compatibility conditions, we also infer that $\int_{\partial T} \mathbb{A}_\varepsilon \nabla \zeta \cdot \mathbf{n}_{\partial T} = 0$, which means that the previous system for ζ is compatible. Hence, $\zeta \equiv 0$, which concludes the proof. \square

Remark 4.3 (Space $V_{\varepsilon,T}^{k+1}$). *The definition of the space $V_{\varepsilon,T}^{k+1}$ is reminiscent of that considered in the non-conforming VEM in the case where $\mathbb{A}_\varepsilon = \mathbb{I}_d$; see [7] and also [15].*

We define $H_{\partial T} \in \mathbb{P}_{d-1}^0(\mathcal{F}_T)$ such that, for any $F \in \mathcal{F}_T$, $H_{\partial T|F} := H_F$. We will need the following inverse inequality on the normal component of $\mathbb{A}_\varepsilon \nabla v_\varepsilon$ for a function $v_\varepsilon \in V_{\varepsilon,T}^{k+1}$; for completeness, we also establish a bound on the divergence.

Lemma 4.4 (Inverse inequalities). *The following holds for all $v_\varepsilon \in V_{\varepsilon,T}^{k+1}$:*

$$H_T \|\operatorname{div}(\mathbb{A}_\varepsilon \nabla v_\varepsilon)\|_{L^2(T)} + \left\| H_{\partial T}^{1/2} \mathbb{A}_\varepsilon \nabla v_\varepsilon \cdot \mathbf{n}_{\partial T} \right\|_{L^2(\partial T)} \leq c \beta^{1/2} \|\mathbb{A}_\varepsilon^{1/2} \nabla v_\varepsilon\|_{L^2(T)^d}, \quad (17)$$

with c independent of ε , H_T , α and β .

Proof. Note that the functions on the left-hand side are (piecewise) polynomials, but the function on the right-hand side is not a polynomial in general. Let us first bound the divergence. Let $d_\varepsilon := \operatorname{div}(\mathbb{A}_\varepsilon \nabla v_\varepsilon) \in \mathbb{P}_d^{k-1}(T)$. Let S be a simplicial sub-cell of T . Considering the standard bubble function $b_S \in H_0^1(S)$ (equal to the scaled product of the barycentric coordinates in S taking the value one at the barycenter of S), we infer using integration by parts that, for some $c > 0$ depending on mesh regularity,

$$\begin{aligned} c \|d_\varepsilon\|_{L^2(S)}^2 &\leq \int_S d_\varepsilon b_S d_\varepsilon = \int_S \operatorname{div}(\mathbb{A}_\varepsilon \nabla v_\varepsilon) b_S d_\varepsilon \\ &= - \int_S \mathbb{A}_\varepsilon \nabla v_\varepsilon \cdot \nabla (b_S d_\varepsilon) \leq \beta^{1/2} \|\mathbb{A}_\varepsilon^{1/2} \nabla v_\varepsilon\|_{L^2(S)^d} H_S^{-1} \|d_\varepsilon\|_{L^2(S)}, \end{aligned}$$

where the last bound follows by applying an inverse inequality to the polynomial function $b_S d_\varepsilon$. Summing over all the simplicial sub-cells and invoking mesh regularity, we conclude that $\|\operatorname{div}(\mathbb{A}_\varepsilon \nabla v_\varepsilon)\|_{L^2(T)} \leq c \beta^{1/2} H_T^{-1} \|\mathbb{A}_\varepsilon^{1/2} \nabla v_\varepsilon\|_{L^2(T)^d}$. Let us now bound the normal component at the boundary. Let σ be a sub-face of a face $F \in \mathcal{F}_T$, and let $S \subseteq T$ be the simplex of the sub-mesh such that σ is a face of S . Then, $r_S := [\operatorname{div}(\mathbb{A}_\varepsilon \nabla v_\varepsilon)]|_S \in \mathbb{P}_d^{k-1}(S) \subset \mathbb{P}_d^k(S)$ and $r_\sigma := [\mathbb{A}_\varepsilon \nabla v_\varepsilon \cdot \mathbf{n}_{\partial T}]|_\sigma \in \mathbb{P}_{d-1}^k(\sigma)$. Note that $\mathbf{n}_{\partial T|_\sigma} = \mathbf{n}_{\partial S|_\sigma}$. Invoking [28, Lemma A.3], we infer that there is a vector-valued polynomial function \mathbf{q} in the Raviart–Thomas–Nédélec (RTN) finite element space of order k in S so that $\operatorname{div}(\mathbf{q}) = r_S$ in S , $\mathbf{q} \cdot \mathbf{n}_{\partial T|_\sigma} = r_\sigma$ on σ , and

$$\|\mathbf{q}\|_{L^2(S)^d} \leq c' \min_{\substack{\mathbf{z} \in \mathbf{H}(\operatorname{div}; S) \\ \operatorname{div}(\mathbf{z}) = r_S \text{ in } S \\ \mathbf{z} \cdot \mathbf{n}_{\partial T|_\sigma} = r_\sigma \text{ on } \sigma}} \|\mathbf{z}\|_{L^2(S)^d},$$

with c' depending on γ (but not on k) and $\mathbf{H}(\operatorname{div}; S) := \{\mathbf{z} \in L^2(S)^d \mid \operatorname{div}(\mathbf{z}) \in L^2(S)\}$. Since the function $[\mathbb{A}_\varepsilon \nabla v_\varepsilon]|_S$ is in $\mathbf{H}(\operatorname{div}; S)$ and satisfies the requested conditions on the divergence in S and the normal component on σ , we conclude that $\|\mathbf{q}\|_{L^2(S)^d} \leq c' \|\mathbb{A}_\varepsilon \nabla v_\varepsilon\|_{L^2(S)^d}$. A discrete trace inequality in the RTN finite element space shows that

$$\|\mathbb{A}_\varepsilon \nabla v_\varepsilon \cdot \mathbf{n}_{\partial T}\|_{L^2(\sigma)} = \|\mathbf{q} \cdot \mathbf{n}_{\partial T}\|_{L^2(\sigma)} \leq c H_\sigma^{-1/2} \|\mathbf{q}\|_{L^2(S)^d} \leq c H_\sigma^{-1/2} \|\mathbb{A}_\varepsilon \nabla v_\varepsilon\|_{L^2(S)^d},$$

where c depends on γ and k . We conclude by invoking mesh regularity. \square

4.3 Approximation properties

We now investigate the approximation properties of the space $V_{\varepsilon,T}^{k+1}$, for all $T \in \mathcal{T}_H$. Our aim is to study how well the first-order two-scale expansion $\mathcal{L}_\varepsilon^1(u_0)$ can be approximated in the discrete space $V_{\varepsilon,T}^{k+1}$. Let us define $\pi_{\varepsilon,T}^{k+1}(u_0) \in V_{\varepsilon,T}^{k+1}$ such that $\int_T \pi_{\varepsilon,T}^{k+1}(u_0) = \int_T \mathcal{L}_\varepsilon^1(u_0)$ and

$$\begin{cases} -\operatorname{div}(\mathbb{A}_\varepsilon \nabla \pi_{\varepsilon,T}^{k+1}(u_0)) = -\operatorname{div}(\mathbb{A}_0 \nabla \Pi_T^{k+1}(u_0)) \in \mathbb{P}_d^{k-1}(T) & \text{in } T, \\ \mathbb{A}_\varepsilon \nabla \pi_{\varepsilon,T}^{k+1}(u_0) \cdot \mathbf{n}_{\partial T} = \mathbb{A}_0 \nabla \Pi_T^{k+1}(u_0) \cdot \mathbf{n}_{\partial T} \in \mathbb{P}_{d-1}^k(\mathcal{F}_T) & \text{on } \partial T. \end{cases} \quad (18)$$

Note that the data in (18) are compatible. From (18) we infer that, for any $w \in H^1(T)$,

$$\int_T \mathbb{A}_\varepsilon \nabla \pi_{\varepsilon,T}^{k+1}(u_0) \cdot \nabla w = \int_T \mathbb{A}_0 \nabla \Pi_T^{k+1}(u_0) \cdot \nabla w. \quad (19)$$

Lemma 4.5 (Approximation in $V_{\varepsilon,T}^{k+1}$). *Assume that the correctors μ_l are in $W^{1,\infty}(\mathbb{R}^d)$ for any $1 \leq l \leq d$, and that $u_0 \in H^{k+2}(T) \cap W^{1,\infty}(T)$. Then, the following holds:*

$$\begin{aligned} \|\mathbb{A}_\varepsilon^{1/2} \nabla (\mathcal{L}_\varepsilon^1(u_0) - \pi_{\varepsilon,T}^{k+1}(u_0))\|_{L^2(T)^d} &\leq c \beta^{1/2} \rho^{1/2} \left(H_T^{k+1} |u_0|_{H^{k+2}(T)} \right. \\ &\quad \left. + \varepsilon |u_0|_{H^2(T)} + \varepsilon^{1/2} |\partial T|^{1/2} |u_0|_{W^{1,\infty}(T)} \right), \end{aligned} \quad (20)$$

with c independent of ε , H_T , u_0 , α and β , and possibly depending on d , k , γ and $\max_{1 \leq l \leq d} \|\mu_l\|_{W^{1,\infty}(\mathbb{R}^d)}$.

Proof. Subtracting/adding $\mathbb{A}_0 \nabla u_0$ and using (19) with $w = \mathcal{L}_\varepsilon^1(u_0)|_T - \pi_{\varepsilon,T}^{k+1}(u_0)$ which is in $H^1(T)$, we infer that

$$\begin{aligned} \|\mathbb{A}_\varepsilon^{1/2} \nabla (\mathcal{L}_\varepsilon^1(u_0) - \pi_{\varepsilon,T}^{k+1}(u_0))\|_{L^2(T)^d}^2 &= \int_T (\mathbb{A}_\varepsilon \nabla \mathcal{L}_\varepsilon^1(u_0) - \mathbb{A}_0 \nabla u_0) \cdot \nabla (\mathcal{L}_\varepsilon^1(u_0) - \pi_{\varepsilon,T}^{k+1}(u_0)) \\ &\quad + \int_T \mathbb{A}_0 \nabla (u_0 - \Pi_T^{k+1}(u_0)) \cdot \nabla (\mathcal{L}_\varepsilon^1(u_0) - \pi_{\varepsilon,T}^{k+1}(u_0)). \end{aligned}$$

Using the Cauchy–Schwarz inequality and the fact that $\mathcal{L}_\varepsilon^1(u_0)|_T - \pi_{\varepsilon,T}^{k+1}(u_0)$ has zero mean-value on T by construction, we infer that

$$\begin{aligned} \|\mathbb{A}_\varepsilon^{1/2} \nabla (\mathcal{L}_\varepsilon^1(u_0) - \pi_{\varepsilon,T}^{k+1}(u_0))\|_{L^2(T)^d} &\leq \beta^{1/2} \rho^{1/2} \|\nabla (u_0 - \Pi_T^{k+1}(u_0))\|_{L^2(T)^d} \\ &\quad + \alpha^{-1/2} \sup_{w \in H_\star^1(T)} \frac{|\mathcal{F}_\varepsilon(w)|}{\|\nabla w\|_{L^2(T)^d}}, \end{aligned}$$

with $\mathcal{F}_\varepsilon(w) = \int_T (\mathbb{A}_\varepsilon \nabla \mathcal{L}_\varepsilon^1(u_0) - \mathbb{A}_0 \nabla u_0) \cdot \nabla w$ and $H_\star^1(T) = \{w \in H^1(T) \mid \int_T w = 0\}$. The first term in the right-hand side is bounded using the approximation properties (10) of Π_T^{k+1} with $m = 1$ and $s = k + 2$, and the second term is bounded in Lemma A.2 (take $D = T$). \square

Remark 4.6 (Alternative estimate). *An alternative estimate to (20) can be derived under the slightly stronger regularity assumptions that there is $\kappa > 0$ so that $\mathbb{A} \in C^{0,\kappa}(\mathbb{R}^d; \mathbb{R}^{d \times d})$, and that $u_0 \in H^{\max(k+2,3)}(T)$. The proof of this estimate follows the strategy advocated in [36], where one invokes Lemma A.4 instead of Lemma A.2 at the end of the proof of Lemma 4.5 to infer that*

$$\begin{aligned} \|\mathbb{A}_\varepsilon^{1/2} \nabla(\mathcal{L}_\varepsilon^1(u_0) - \pi_{\varepsilon,T}^{k+1}(u_0))\|_{L^2(T)^d} &\leq c \beta^{1/2} \rho^{1/2} \left(H_T^{k+1} |u_0|_{H^{k+2}(T)} \right. \\ &\quad \left. + (\varepsilon + (\varepsilon H_T)^{1/2}) |u_0|_{H^2(T)} + \varepsilon H_T |u_0|_{H^3(T)} + \varepsilon^{1/2} H_T^{-1/2} |u_0|_{H^1(T)} \right), \end{aligned}$$

with c independent of ε , H_T , u_0 , α and β , and possibly depending on d , k , γ and $\|\mathbb{A}\|_{C^{0,\kappa}(\mathbb{R}^d; \mathbb{R}^{d \times d})}$. This local estimate leads to the same global error estimate for (both versions of) the MsHHO method described hereafter than (20); see in particular the end of the proof of Theorem 5.6.

5 The MsHHO method

In this section, we introduce and analyze the multiscale HHO (MsHHO) method. We consider first in Section 5.1 a mixed-order version and then in Section 5.2 an equal-order version concerning the polynomial degree used for the cell and face unknowns. Let \mathcal{T}_H be a member of an admissible mesh sequence in the sense of Definition 3.1.

5.1 The mixed-order case

Let $k \geq 1$. For all $T \in \mathcal{T}_H$, we consider the following local set of discrete unknowns:

$$\underline{U}_T^k := \mathbb{P}_d^{k-1}(T) \times \mathbb{P}_{d-1}^k(\mathcal{F}_T). \quad (21)$$

Any element $\underline{v}_T \in \underline{U}_T^k$ is decomposed as $\underline{v}_T := (v_T, v_{\mathcal{F}_T})$. For any $F \in \mathcal{F}_T$, we denote $v_F := v_{\mathcal{F}_T|F} \in \mathbb{P}_{d-1}^k(F)$. We do not consider the case $k = 0$ since this corresponds to the method already analyzed in [36] (up to a slightly different treatment of the right-hand side; cf. Remark 5.4). We introduce the local reduction operator $\underline{I}_T^k : H^1(T) \rightarrow \underline{U}_T^k$ such that, for any $v \in H^1(T)$, $\underline{I}_T^k v := (\Pi_T^{k-1}(v), \Pi_{\partial T}^k(v))$, where $\Pi_{\partial T}^k(v) \in \mathbb{P}_{d-1}^k(\mathcal{F}_T)$ is defined, for any $F \in \mathcal{F}_T$, by $\Pi_{\partial T}^k(v)|_F := \Pi_F^k(v)$. Reasoning as in [15, Section 2.4], it can be proved that, for all $T \in \mathcal{T}_H$, the restriction of \underline{I}_T^k to $V_{\varepsilon,T}^{k+1}$ is an isomorphism from $V_{\varepsilon,T}^{k+1}$ to \underline{U}_T^k . Thus, the triple $(T, V_{\varepsilon,T}^{k+1}, \underline{I}_T^k)$ defines a finite element in the sense of Ciarlet.

We define the local multiscale reconstruction operator $p_{\varepsilon,T}^{k+1} : \underline{U}_T^k \rightarrow V_{\varepsilon,T}^{k+1}$ such that, for any $\underline{v}_T = (v_T, v_{\mathcal{F}_T}) \in \underline{U}_T^k$, $p_{\varepsilon,T}^{k+1}(\underline{v}_T) \in V_{\varepsilon,T}^{k+1}$ satisfies $\int_T p_{\varepsilon,T}^{k+1}(\underline{v}_T) = \int_T v_T$ and solves the well-posed local Neumann problem

$$\int_T \mathbb{A}_\varepsilon \nabla p_{\varepsilon,T}^{k+1}(\underline{v}_T) \cdot \nabla w_\varepsilon = - \int_T v_T \operatorname{div}(\mathbb{A}_\varepsilon \nabla w_\varepsilon) + \int_{\partial T} v_{\mathcal{F}_T} \mathbb{A}_\varepsilon \nabla w_\varepsilon \cdot \mathbf{n}_{\partial T} \quad \forall w_\varepsilon \in V_{\varepsilon,T}^{k+1}. \quad (22)$$

Note that (22) can be equivalently rewritten

$$\int_T \mathbb{A}_\varepsilon \nabla p_{\varepsilon,T}^{k+1}(\underline{v}_T) \cdot \nabla w_\varepsilon = \int_T \nabla \underline{v}_T \cdot \mathbb{A}_\varepsilon \nabla w_\varepsilon - \int_{\partial T} (\underline{v}_T - \underline{v}_{\mathcal{F}_T}) \mathbb{A}_\varepsilon \nabla w_\varepsilon \cdot \mathbf{n}_{\partial T} \quad \forall w_\varepsilon \in V_{\varepsilon,T}^{k+1}. \quad (23)$$

Integrating by parts the left-hand side of (22) and exploiting the definition (15) of the space $V_{\varepsilon,T}^{k+1}$, one can see that, for any $\underline{v}_T \in \underline{U}_T^k$,

$$\Pi_T^{k-1}(p_{\varepsilon,T}^{k+1}(\underline{v}_T)) = \Pi_T^{k-1}(\underline{v}_T) = \underline{v}_T, \quad \Pi_{\partial T}^k(p_{\varepsilon,T}^{k+1}(\underline{v}_T)) = \Pi_{\partial T}^k(\underline{v}_{\mathcal{F}_T}) = \underline{v}_{\mathcal{F}_T}. \quad (24)$$

Owing to (15) and (22), we infer that, for all $v \in H^1(T)$,

$$\int_T \mathbb{A}_\varepsilon \nabla (v - p_{\varepsilon,T}^{k+1}(\underline{I}_T^k v)) \cdot \nabla w_\varepsilon = 0 \quad \forall w_\varepsilon \in V_{\varepsilon,T}^{k+1}, \quad (25)$$

so that $p_{\varepsilon,T}^{k+1} \circ \underline{I}_T^k : H^1(T) \rightarrow V_{\varepsilon,T}^{k+1}$ is the \mathbb{A}_ε -weighted elliptic projection. As a consequence, we have, for all $v \in H^1(T)$,

$$\|\mathbb{A}_\varepsilon^{1/2} \nabla (v - p_{\varepsilon,T}^{k+1}(\underline{I}_T^k v))\|_{L^2(T)^d} = \inf_{w_\varepsilon \in V_{\varepsilon,T}^{k+1}} \|\mathbb{A}_\varepsilon^{1/2} \nabla (v - w_\varepsilon)\|_{L^2(T)^d}. \quad (26)$$

Since the operator $p_{\varepsilon,T}^{k+1} \circ \underline{I}_T^k$ preserves the mean value, its restriction to $V_{\varepsilon,T}^{k+1}$ is the identity operator.

Remark 5.1 (Comparison with the monoscale HHO method). *In the monoscale HHO method, the reconstruction operator is simpler to construct since it maps onto $\mathbb{P}_d^{k+1}(T)$ (which is a proper subspace of $V_{\varepsilon,T}^{k+1}$ whenever \mathbb{A}_ε is a constant matrix on T), whereas in the multiscale context, we explore the whole space $V_{\varepsilon,T}^{k+1}$ to build the reconstruction. One advantage of doing this is that we no longer need to consider stabilization in the present case. Another advantage is that we recover the characterization of $p_{\varepsilon,T}^{k+1} \circ \underline{I}_T^k$ as the \mathbb{A}_ε -weighted elliptic projector onto $V_{\varepsilon,T}^{k+1}$, that is lost in the monoscale case as soon as \mathbb{A}_ε is not a constant matrix on T .*

The local bilinear form $a_{\varepsilon,T} : \underline{U}_T^k \times \underline{U}_T^k \rightarrow \mathbb{R}$ is defined as

$$a_{\varepsilon,T}(\underline{u}_T, \underline{v}_T) := \int_T \mathbb{A}_\varepsilon \nabla p_{\varepsilon,T}^{k+1}(\underline{u}_T) \cdot \nabla p_{\varepsilon,T}^{k+1}(\underline{v}_T).$$

We introduce the following semi-norm on \underline{U}_T^k :

$$\|\underline{v}_T\|_T^2 := \|\nabla \underline{v}_T\|_{L^2(T)^d}^2 + \left\| H_{\partial T}^{-1/2} (\underline{v}_T - \underline{v}_{\mathcal{F}_T}) \right\|_{L^2(\partial T)}^2. \quad (27)$$

Lemma 5.2 (Local stability). *The following holds:*

$$a_{\varepsilon,T}(\underline{v}_T, \underline{v}_T) \geq c \alpha \|\underline{v}_T\|_T^2 \quad \forall \underline{v}_T \in \underline{U}_T^k, \quad (28)$$

with constant c independent of ε , H_T , α and β .

Proof. Let $\underline{v}_T \in \underline{U}_T^k$. To derive an estimate on $\|\nabla v_T\|_{L^2(T)^d}$, we define $v_\varepsilon \in V_{\varepsilon,T}^{k+1}$ such that

$$\begin{cases} -\operatorname{div}(\mathbb{A}_\varepsilon \nabla v_\varepsilon) = -\Delta v_T \in \mathbb{P}_d^{k-1}(T) & \text{in } T, \\ \mathbb{A}_\varepsilon \nabla v_\varepsilon \cdot \mathbf{n}_{\partial T} = \nabla v_T \cdot \mathbf{n}_{\partial T} \in \mathbb{P}_{d-1}^k(\mathcal{F}_T) & \text{on } \partial T, \end{cases} \quad (29)$$

and satisfying, e.g., $\int_T v_\varepsilon = 0$ (the way the constant is fixed is unimportant here). Note that data in (29) are compatible. Then, the following holds:

$$\int_T \mathbb{A}_\varepsilon \nabla v_\varepsilon \cdot \nabla z = \int_T \nabla v_T \cdot \nabla z \quad \forall z \in H^1(T).$$

Using this last relation where we take $z = p_{\varepsilon,T}^{k+1}(\underline{v}_T)$, and using (23) where we take $w_\varepsilon = v_\varepsilon \in V_{\varepsilon,T}^{k+1}$ defined in (29), we infer that

$$\begin{aligned} -\int_T v_T \Delta v_T + \int_{\partial T} v_{\mathcal{F}_T} \nabla v_T \cdot \mathbf{n}_{\partial T} &= -\int_T v_T \operatorname{div}(\mathbb{A}_\varepsilon \nabla v_\varepsilon) + \int_{\partial T} v_{\mathcal{F}_T} \mathbb{A}_\varepsilon \nabla v_\varepsilon \cdot \mathbf{n}_{\partial T} \\ &= \int_T \mathbb{A}_\varepsilon \nabla v_\varepsilon \cdot \nabla v_T - \int_{\partial T} (v_T - v_{\mathcal{F}_T}) \mathbb{A}_\varepsilon \nabla v_\varepsilon \cdot \mathbf{n}_{\partial T} \\ &= \int_T \mathbb{A}_\varepsilon \nabla v_\varepsilon \cdot \nabla p_{\varepsilon,T}^{k+1}(\underline{v}_T) = \int_T \nabla v_T \cdot \nabla p_{\varepsilon,T}^{k+1}(\underline{v}_T). \end{aligned}$$

After an integration by parts, this yields

$$\|\nabla v_T\|_{L^2(T)^d}^2 = \int_T \nabla p_{\varepsilon,T}^{k+1}(\underline{v}_T) \cdot \nabla v_T + \int_{\partial T} (v_T - v_{\mathcal{F}_T}) \nabla v_T \cdot \mathbf{n}_{\partial T}.$$

By the Cauchy–Schwarz inequality and the discrete trace inequality (7), we then obtain

$$\|\nabla v_T\|_{L^2(T)^d} \leq c \left(\alpha^{-1/2} \|\mathbb{A}_\varepsilon^{1/2} \nabla p_{\varepsilon,T}^{k+1}(\underline{v}_T)\|_{L^2(T)^d} + \|H_{\partial T}^{-1/2}(v_T - v_{\mathcal{F}_T})\|_{L^2(\partial T)} \right). \quad (30)$$

To bound the second term in the right-hand side, we use (24) to infer that

$$\begin{aligned} [v_T - v_{\mathcal{F}_T}]_{|\partial T} &= [\Pi_T^{k-1}(p_{\varepsilon,T}^{k+1}(\underline{v}_T))]_{|\partial T} - \Pi_{\partial T}^k(p_{\varepsilon,T}^{k+1}(\underline{v}_T)) \\ &= \Pi_{\partial T}^k(\Pi_T^{k-1}(p_{\varepsilon,T}^{k+1}(\underline{v}_T)) - p_{\varepsilon,T}^{k+1}(\underline{v}_T)). \end{aligned}$$

Using the L^2 -stability of $\Pi_{\partial T}^k$, the continuous trace inequality (8), the local Poincaré inequality (9) (since $p_{\varepsilon,T}^{k+1}(\underline{v}_T) - \Pi_T^{k-1}(p_{\varepsilon,T}^{k+1}(\underline{v}_T))$ has zero mean-value on T), and the H^1 -stability of Π_T^{k-1} , we infer that

$$\|H_{\partial T}^{-1/2}(v_T - v_{\mathcal{F}_T})\|_{L^2(\partial T)} \leq c \alpha^{-1/2} \|\mathbb{A}_\varepsilon^{1/2} \nabla p_{\varepsilon,T}^{k+1}(\underline{v}_T)\|_{L^2(T)^d}. \quad (31)$$

This concludes the proof. \square

We define the skeleton $\partial\mathcal{T}_H$ of the mesh \mathcal{T}_H as $\partial\mathcal{T}_H := \bigcup_{F \in \mathcal{F}_H} F$. We introduce the broken polynomial spaces

$$\mathbb{P}_d^{k-1}(\mathcal{T}_H) := \{v \in L^2(\Omega) \mid v|_T \in \mathbb{P}_d^{k-1}(T) \ \forall T \in \mathcal{T}_H\}, \quad (32)$$

$$\mathbb{P}_{d-1}^k(\mathcal{F}_H) := \{v \in L^2(\partial\mathcal{T}_H) \mid v|_F \in \mathbb{P}_{d-1}^k(F) \ \forall F \in \mathcal{F}_H\}. \quad (33)$$

The global set of discrete unknowns is defined to be

$$\underline{\mathbb{U}}_H^k := \mathbb{P}_d^{k-1}(\mathcal{T}_H) \times \mathbb{P}_{d-1}^k(\mathcal{F}_H), \quad (34)$$

so that any $\underline{v}_H \in \underline{\mathbb{U}}_H^k$ can be decomposed as $\underline{v}_H := (v_{\mathcal{T}_H}, v_{\mathcal{F}_H})$. For any given $\underline{v}_H \in \underline{\mathbb{U}}_H^k$, we denote $\underline{v}_T := (v_T, v_{\mathcal{F}_T}) \in \underline{\mathbb{U}}_T^k$ its restriction to the mesh cell $T \in \mathcal{T}_H$. Note that unknowns attached to mesh interfaces are single-valued, in the sense that, for any $F \in \mathcal{F}_H^i$ such that $F = \partial T_1 \cap \partial T_2 \cap Z$ for $T_1, T_2 \in \mathcal{T}_H$, $v_F := v_{\mathcal{F}_H}|_F \in \mathbb{P}_{d-1}^k(F)$ is such that $v_F = v_{\mathcal{F}_{T_1}}|_F = v_{\mathcal{F}_{T_2}}|_F$. To take into account homogeneous Dirichlet boundary conditions, we further introduce the subspace $\underline{\mathbb{U}}_{H,0}^k := \{\underline{v}_H \in \underline{\mathbb{U}}_H^k \mid v_F \equiv 0 \ \forall F \in \mathcal{F}_H^b\}$. We define the global bilinear form $a_{\varepsilon,H} : \underline{\mathbb{U}}_H^k \times \underline{\mathbb{U}}_H^k \rightarrow \mathbb{R}$ such that

$$a_{\varepsilon,H}(\underline{u}_H, \underline{v}_H) := \sum_{T \in \mathcal{T}_H} a_{\varepsilon,T}(\underline{u}_T, \underline{v}_T) = \sum_{T \in \mathcal{T}_H} \int_T \mathbb{A}_\varepsilon \nabla p_{\varepsilon,T}^{k+1}(\underline{u}_T) \cdot \nabla p_{\varepsilon,T}^{k+1}(\underline{v}_T).$$

Then, the discrete problem reads: Find $\underline{u}_{\varepsilon,H} \in \underline{\mathbb{U}}_{H,0}^k$ such that

$$a_{\varepsilon,H}(\underline{u}_{\varepsilon,H}, \underline{v}_H) = \int_\Omega f v_{\mathcal{T}_H} \quad \forall \underline{v}_H \in \underline{\mathbb{U}}_{H,0}^k. \quad (35)$$

Setting $\|\underline{v}_H\|_H^2 := \sum_{T \in \mathcal{T}_H} \|\underline{v}_T\|_T^2$ on $\underline{\mathbb{U}}_H^k$, with $\|\cdot\|_T$ introduced in (27), we define a norm on $\underline{\mathbb{U}}_{H,0}^k$ since elements in $\underline{\mathbb{U}}_{H,0}^k$ are such that $v_F \equiv 0$ for all $F \in \mathcal{F}_H^b$.

Lemma 5.3 (Well-posedness). *The following holds:*

$$a_{\varepsilon,H}(\underline{v}_H, \underline{v}_H) = \sum_{T \in \mathcal{T}_H} \|\mathbb{A}_\varepsilon^{1/2} \nabla p_{\varepsilon,T}^{k+1}(\underline{v}_T)\|_{L^2(T)^d}^2 =: \|\underline{v}_H\|_{\varepsilon,H}^2 \geq c \alpha \|\underline{v}_H\|_H^2 \quad \forall \underline{v}_H \in \underline{\mathbb{U}}_H^k, \quad (36)$$

with constant c independent of ε , H , α and β . As a consequence, the discrete problem (35) is well-posed.

Proof. This is a direct consequence of Lemma 5.2. \square

Remark 5.4 (Non-conforming Finite Element (ncFE) formulation). *Consider the discrete space*

$$V_{\varepsilon,H,0}^{k+1} := \{v_{\varepsilon,H} \in L^2(\Omega) \mid v_{\varepsilon,H}|_T \in V_{\varepsilon,T}^{k+1} \ \forall T \in \mathcal{T}_H \text{ and } \Pi_F^k(\llbracket v_{\varepsilon,H} \rrbracket_F) = 0 \ \forall F \in \mathcal{F}_H\},$$

where $\llbracket \cdot \rrbracket_F$ denotes the jump operator for all interfaces $F \in \mathcal{F}_H^i$ (the sign is irrelevant) and the actual trace for all boundary faces $F \in \mathcal{F}_H^b$. Consider the following ncFE method: Find $u_{\varepsilon,H} \in V_{\varepsilon,H,0}^{k+1}$ such that

$$\tilde{a}_{\varepsilon,H}(u_{\varepsilon,H}, v_{\varepsilon,H}) = \sum_{T \in \mathcal{T}_H} \int_T f \Pi_T^{k-1}(v_{\varepsilon,H}) \quad \forall v_{\varepsilon,H} \in V_{\varepsilon,H,0}^{k+1}, \quad (37)$$

where $\tilde{a}_{\varepsilon,H}(u_{\varepsilon,H}, v_{\varepsilon,H}) := \sum_{T \in \mathcal{T}_H} \int_T \mathbb{A}_\varepsilon \nabla u_{\varepsilon,H} \cdot \nabla v_{\varepsilon,H}$. Then, using that the restriction of $\underline{\mathbb{I}}_T^k$ to $V_{\varepsilon,T}^{k+1}$ is an isomorphism from $V_{\varepsilon,T}^{k+1}$ to $\underline{\mathbb{U}}_T^k$ and that the restriction of $p_{\varepsilon,T}^{k+1} \circ \underline{\mathbb{I}}_T^k$ to $V_{\varepsilon,T}^{k+1}$ is the identity operator, it can be shown that $\underline{u}_{\varepsilon,H}$ solves (35) if and only if $\underline{u}_{\varepsilon,T} = \underline{\mathbb{I}}_T^k(u_{\varepsilon,H}|_T)$ for all $T \in \mathcal{T}_H$ where $u_{\varepsilon,H}$ solves (37). This proves that (35) is indeed a high-order (and polytopal) extension of the method in [36], up to a slightly different treatment of the right-hand side ($\Pi_T^{k-1}(v_{\varepsilon,H})$ instead of $v_{\varepsilon,H}$).

Let u_ε be the oscillatory solution to (1) and let $\underline{u}_{\varepsilon,H}$ be the discrete MsHHO solution to (35). Let us define the discrete error such that

$$\underline{e}_{\varepsilon,H} \in \underline{\mathbb{U}}_{H,0}^k, \quad \underline{e}_{\varepsilon,T} := \underline{\mathbb{I}}_T^k u_\varepsilon - \underline{u}_{\varepsilon,T} \quad \forall T \in \mathcal{T}_H. \quad (38)$$

Note that $\underline{e}_{\varepsilon,H}$ is well-defined as a member of $\underline{\mathbb{U}}_{H,0}^k$ since the oscillatory solution u_ε is in $H_0^1(\Omega)$ and functions in $H_0^1(\Omega)$ are single-valued at interfaces and vanish at the boundary.

Lemma 5.5 (Discrete energy-error estimate). *Let the discrete error $\underline{e}_{\varepsilon,H}$ be defined by (38). Assume that $u_0 \in H^{k+2}(\Omega)$. Then, the following holds:*

$$\|\underline{e}_{\varepsilon,H}\|_{\varepsilon,H} \leq c \rho^{1/2} \left(\beta \sum_{T \in \mathcal{T}_H} H_T^{2(k+1)} |u_0|_{H^{k+2}(T)}^2 + \sum_{T \in \mathcal{T}_H} \|\mathbb{A}_\varepsilon^{1/2} \nabla (u_\varepsilon - \pi_{\varepsilon,T}^{k+1}(u_0))\|_{L^2(T)^d}^2 \right)^{1/2}, \quad (39)$$

with constant c independent of ε , H , u_0 , α and β .

Proof. Lemma 5.3 implies that

$$\|\underline{e}_{\varepsilon,H}\|_{\varepsilon,H} = \sup_{\underline{v}_H \in \underline{\mathbb{U}}_{H,0}^k} \frac{a_{\varepsilon,H}(\underline{e}_{\varepsilon,H}, \underline{v}_H)}{\|\underline{v}_H\|_{\varepsilon,H}}. \quad (40)$$

Let $\underline{v}_H \in \underline{\mathbb{U}}_{H,0}^k$. Performing an integration by parts, and using the facts that the flux $\mathbb{A}_0 \nabla u_0 \cdot \mathbf{n}_F$ is continuous across any interface $F \in \mathcal{F}_H^i$ since $u_0 \in H^2(\Omega)$, and that $\underline{v}_H \in \underline{\mathbb{U}}_{H,0}^k$, we infer that

$$a_{\varepsilon,H}(\underline{u}_{\varepsilon,H}, \underline{v}_H) = \int_\Omega f v_{\mathcal{T}_H} = \sum_{T \in \mathcal{T}_H} \int_T \mathbb{A}_0 \nabla u_0 \cdot \nabla v_T - \sum_{T \in \mathcal{T}_H} \int_{\partial T} (v_T - v_{\mathcal{F}_T}) \mathbb{A}_0 \nabla u_0 \cdot \mathbf{n}_{\partial T}. \quad (41)$$

Using (23) with $w_\varepsilon = p_{\varepsilon,T}^{k+1}(\mathbb{I}_T^k u_\varepsilon)$, we then infer that

$$\begin{aligned} a_{\varepsilon,H}(\mathbf{e}_{\varepsilon,H}, \mathbf{v}_H) &= \sum_{T \in \mathcal{T}_H} \int_T (\mathbb{A}_\varepsilon \nabla p_{\varepsilon,T}^{k+1}(\mathbb{I}_T^k u_\varepsilon) - \mathbb{A}_0 \nabla u_0) \cdot \nabla \mathbf{v}_T \\ &\quad - \sum_{T \in \mathcal{T}_H} \int_{\partial T} (\mathbb{A}_\varepsilon \nabla p_{\varepsilon,T}^{k+1}(\mathbb{I}_T^k u_\varepsilon) - \mathbb{A}_0 \nabla u_0) \cdot \mathbf{n}_{\partial T} (\mathbf{v}_T - \mathbf{v}_{\mathcal{F}_T}). \end{aligned}$$

Adding/subtracting $\Pi_T^{k+1}(u_0)$ in the right-hand side yields $a_{\varepsilon,H}(\mathbf{e}_{\varepsilon,H}, \mathbf{v}_H) = \mathfrak{T}_1 + \mathfrak{T}_2$ with

$$\begin{aligned} \mathfrak{T}_1 &= \sum_{T \in \mathcal{T}_H} \int_T \mathbb{A}_0 \nabla (\Pi_T^{k+1}(u_0) - u_0) \cdot \nabla \mathbf{v}_T \\ &\quad - \sum_{T \in \mathcal{T}_H} \int_{\partial T} \mathbb{A}_0 \nabla (\Pi_T^{k+1}(u_0) - u_0) \cdot \mathbf{n}_{\partial T} (\mathbf{v}_T - \mathbf{v}_{\mathcal{F}_T}), \\ \mathfrak{T}_2 &= \sum_{T \in \mathcal{T}_H} \int_T (\mathbb{A}_\varepsilon \nabla p_{\varepsilon,T}^{k+1}(\mathbb{I}_T^k u_\varepsilon) - \mathbb{A}_0 \nabla \Pi_T^{k+1}(u_0)) \cdot \nabla \mathbf{v}_T \\ &\quad - \sum_{T \in \mathcal{T}_H} \int_{\partial T} (\mathbb{A}_\varepsilon \nabla p_{\varepsilon,T}^{k+1}(\mathbb{I}_T^k u_\varepsilon) - \mathbb{A}_0 \nabla \Pi_T^{k+1}(u_0)) \cdot \mathbf{n}_{\partial T} (\mathbf{v}_T - \mathbf{v}_{\mathcal{F}_T}). \end{aligned}$$

The term \mathfrak{T}_1 is estimated using Cauchy–Schwarz inequality and the approximation properties (10) of the projector Π_T^{k+1} for $m = 1$ and $s = k + 2$, yielding

$$|\mathfrak{T}_1| \leq c \beta \left(\sum_{T \in \mathcal{T}_H} H_T^{2(k+1)} |u_0|_{H^{k+2}(T)}^2 \right)^{1/2} \|\mathbf{v}_H\|_H.$$

Considering now \mathfrak{T}_2 , we use the definition (18) of $\pi_{\varepsilon,T}^{k+1}(u_0)$ and the relation (19) to infer that

$$\begin{aligned} \mathfrak{T}_2 &= \sum_{T \in \mathcal{T}_H} \int_T \mathbb{A}_\varepsilon \nabla (p_{\varepsilon,T}^{k+1}(\mathbb{I}_T^k u_\varepsilon) - \pi_{\varepsilon,T}^{k+1}(u_0)) \cdot \nabla \mathbf{v}_T \\ &\quad - \sum_{T \in \mathcal{T}_H} \int_{\partial T} \mathbb{A}_\varepsilon \nabla (p_{\varepsilon,T}^{k+1}(\mathbb{I}_T^k u_\varepsilon) - \pi_{\varepsilon,T}^{k+1}(u_0)) \cdot \mathbf{n}_{\partial T} (\mathbf{v}_T - \mathbf{v}_{\mathcal{F}_T}). \end{aligned}$$

The first term in the right-hand side can be bounded using the Cauchy–Schwarz inequality, whereas the second term is estimated by means of the inverse inequality from Lemma 4.4 since $(p_{\varepsilon,T}^{k+1}(\mathbb{I}_T^k u_\varepsilon) - \pi_{\varepsilon,T}^{k+1}(u_0)) \in V_{\varepsilon,T}^{k+1}$. This yields

$$\begin{aligned} |\mathfrak{T}_2| &\leq c \beta^{1/2} \left(\sum_{T \in \mathcal{T}_H} \|\mathbb{A}_\varepsilon^{1/2} \nabla (p_{\varepsilon,T}^{k+1}(\mathbb{I}_T^k u_\varepsilon) - \pi_{\varepsilon,T}^{k+1}(u_0))\|_{L^2(T)^d}^2 \right)^{1/2} \|\mathbf{v}_H\|_H \\ &\leq c \beta^{1/2} \left(\sum_{T \in \mathcal{T}_H} \|\mathbb{A}_\varepsilon^{1/2} \nabla (u_\varepsilon - \pi_{\varepsilon,T}^{k+1}(u_0))\|_{L^2(T)^d}^2 \right)^{1/2} \|\mathbf{v}_H\|_H, \end{aligned}$$

where the last bound follows from (26) since $\pi_{\varepsilon,T}^{k+1}(u_0) \in V_{\varepsilon,T}^{k+1}$. Since $\|\underline{v}_H\|_{\varepsilon,H}^2 \geq c\alpha\|\underline{v}_H\|_H^2$ owing to Lemma 5.3, we obtain the expected bound. \square

Theorem 5.6 (Energy-error estimate). *Assume that the correctors μ_l are in $W^{1,\infty}(\mathbb{R}^d)$ for any $1 \leq l \leq d$, and that $u_0 \in H^{k+2}(\Omega) \cap W^{1,\infty}(\Omega)$. Then, the following holds:*

$$\left(\sum_{T \in \mathcal{T}_H} \|\mathbb{A}_\varepsilon^{1/2} \nabla (u_\varepsilon - p_{\varepsilon,T}^{k+1}(\underline{u}_{\varepsilon,T}))\|_{L^2(T)^d}^2 \right)^{1/2} \leq c\beta^{1/2} \rho \left(\sum_{T \in \mathcal{T}_H} H_T^{2(k+1)} |u_0|_{H^{k+2}(T)}^2 \right. \\ \left. + \varepsilon |\partial\Omega| |u_0|_{W^{1,\infty}(\Omega)}^2 + \sum_{T \in \mathcal{T}_H} \left[\varepsilon^2 |u_0|_{H^2(T)}^2 + \varepsilon |\partial T| |u_0|_{W^{1,\infty}(T)}^2 \right] \right)^{1/2}, \quad (42)$$

with c independent of ε , H , u_0 , α and β . In particular, if the mesh \mathcal{T}_H is quasi-uniform, and tracking for simplicity only the dependency on ε and H with $\varepsilon \leq H \leq \ell_\Omega$ (ℓ_Ω denotes the diameter of Ω), we obtain an energy-error upper bound of the form $(\varepsilon^{1/2} + H^{k+1} + (\varepsilon/H)^{1/2})$.

Proof. Using the shorthand notation $e_{\varepsilon,T} := u_\varepsilon|_T - p_{\varepsilon,T}^{k+1}(\underline{u}_{\varepsilon,T})$ for all $T \in \mathcal{T}_H$, the triangle inequality implies that

$$\left(\sum_{T \in \mathcal{T}_H} \|\mathbb{A}_\varepsilon^{1/2} \nabla e_{\varepsilon,T}\|_{L^2(T)^d}^2 \right)^{1/2} \leq \left(\sum_{T \in \mathcal{T}_H} \|\mathbb{A}_\varepsilon^{1/2} \nabla (u_\varepsilon - p_{\varepsilon,T}^{k+1}(\mathbb{I}_T^k u_\varepsilon))\|_{L^2(T)^d}^2 \right)^{1/2} + \|\underline{e}_{\varepsilon,H}\|_{\varepsilon,H},$$

and owing to (26), we infer that

$$\left(\sum_{T \in \mathcal{T}_H} \|\mathbb{A}_\varepsilon^{1/2} \nabla e_{\varepsilon,T}\|_{L^2(T)^d}^2 \right)^{1/2} \leq \left(\sum_{T \in \mathcal{T}_H} \|\mathbb{A}_\varepsilon^{1/2} \nabla (u_\varepsilon - \pi_{\varepsilon,T}^{k+1}(u_0))\|_{L^2(T)^d}^2 \right)^{1/2} + \|\underline{e}_{\varepsilon,H}\|_{\varepsilon,H}.$$

Lemma 5.5 then implies that

$$\left(\sum_{T \in \mathcal{T}_H} \|\mathbb{A}_\varepsilon^{1/2} \nabla e_{\varepsilon,T}\|_{L^2(T)^d}^2 \right)^{1/2} \leq c\rho^{1/2} \left(\beta \sum_{T \in \mathcal{T}_H} H_T^{2(k+1)} |u_0|_{H^{k+2}(T)}^2 + \sum_{T \in \mathcal{T}_H} \|\mathbb{A}_\varepsilon^{1/2} \nabla (u_\varepsilon - \pi_{\varepsilon,T}^{k+1}(u_0))\|_{L^2(T)^d}^2 \right)^{1/2}.$$

To conclude the proof of (42), we add/subtract $\mathcal{L}_\varepsilon^1(u_0)$ in the last term in the right-hand side, and invoke the triangle inequality together with Lemma A.5 to bound $(u_\varepsilon - \mathcal{L}_\varepsilon^1(u_0))$ globally on Ω and Lemma 4.5 to bound $(\mathcal{L}_\varepsilon^1(u_0) - \pi_{\varepsilon,T}^{k+1}(u_0))$ locally on all $T \in \mathcal{T}_H$. Finally, to derive the upper bound for quasi-uniform meshes, we observe that the last term in (42) can be estimated as $\sum_{T \in \mathcal{T}_H} \varepsilon |\partial T| |u_0|_{W^{1,\infty}(T)}^2 \leq c\varepsilon H^{-1} |u_0|_{W^{1,\infty}(\Omega)}^2 \sum_{T \in \mathcal{T}_H} |\partial T| H_T \leq c'\varepsilon H^{-1} |u_0|_{W^{1,\infty}(\Omega)}^2$ with c' proportional to $|\Omega|$. \square

Remark 5.7 (Dependency on ρ). *The estimate (42) has a linear dependency with respect to the (global) heterogeneity/anisotropy ratio ρ (a close inspection of the proof shows that the term $\varepsilon^{1/2}|\partial\Omega|^{1/2}|u_0|_{W^{1,\infty}(\Omega)}$ only scales with $\rho^{1/2}$). This linear scaling is also obtained with the monoscale HHO method when the diffusivity is non-constant in each mesh cell; cf. [21, Theorem 3.1].*

Remark 5.8 (Discretization of the right-hand side). *Note that we could also integrate the right-hand side in (35) using $p_{\varepsilon,T}^{k+1}(\underline{v}_T)$ instead of v_T on each $T \in \mathcal{T}_H$, up to the addition in the right-hand sides of the bounds (39) and (42) of the optimally converging term $c\alpha^{-1/2}\left(\sum_{T \in \mathcal{T}_H} H_T^{2(k+1)}|f|_{H^k(T)}^2\right)^{1/2}$. Indeed, owing to (24), we have*

$$\sum_{T \in \mathcal{T}_H} \int_T f(v_T - p_{\varepsilon,T}^{k+1}(\underline{v}_T)) = \sum_{T \in \mathcal{T}_H} \int_T (f - \Pi_T^{k-1}(f))(v_T - p_{\varepsilon,T}^{k+1}(\underline{v}_T)),$$

which can be estimated by applying Cauchy–Schwarz inequality on each T , and (i) the approximation properties (10) of Π_T^{k-1} with $m = 0$ and $s = k$ for the first factor, (ii) the Poincaré inequality (9) (recall that $(v_T - p_{\varepsilon,T}^{k+1}(\underline{v}_T))$ has zero-mean on T) and the triangle inequality combined with Lemma 5.2 for the second factor. This alternative approach necessitates an integration against an oscillatory test function and is hence computationally more expensive (recall that f is assumed to be non-oscillatory).

5.2 The equal-order case

Let $k \geq 0$. For all $T \in \mathcal{T}_H$, we consider now the following local set of discrete unknowns:

$$\underline{U}_T^k := \mathbb{P}_d^k(T) \times \mathbb{P}_{d-1}^k(\mathcal{F}_T). \quad (43)$$

Any element $\underline{v}_T \in \underline{U}_T^k$ is again decomposed as $\underline{v}_T := (v_T, v_{\mathcal{F}_T})$, and for any $F \in \mathcal{F}_T$, we denote $v_F := v_{\mathcal{F}_T|F} \in \mathbb{P}_{d-1}^k(F)$. We redefine the local reduction operator $\underline{I}_T^k : H^1(T) \rightarrow \underline{U}_T^k$ so that, for any $v \in H^1(T)$, $\underline{I}_T^k v := (\Pi_T^k(v), \Pi_{\partial T}^k(v))$. Reasoning as in [15, Section 2.4], it can be proved that, for all $T \in \mathcal{T}_H$, the restriction of \underline{I}_T^k to $\tilde{V}_{\varepsilon,T}^{k+1}$ is an isomorphism from $\tilde{V}_{\varepsilon,T}^{k+1}$ to \underline{U}_T^k , where

$$\tilde{V}_{\varepsilon,T}^{k+1} := \{v_\varepsilon \in H^1(T) \mid \operatorname{div}(\mathbb{A}_\varepsilon \nabla v_\varepsilon) \in \mathbb{P}_d^k(T), \mathbb{A}_\varepsilon \nabla v_\varepsilon \cdot \mathbf{n}_{\partial T} \in \mathbb{P}_{d-1}^k(\mathcal{F}_T)\}. \quad (44)$$

Thus, the triple $(T, \tilde{V}_{\varepsilon,T}^{k+1}, \underline{I}_T^k)$ defines a finite element in the sense of Ciarlet.

The local multiscale reconstruction operator $p_{\varepsilon,T}^{k+1} : \underline{U}_T^k \rightarrow V_{\varepsilon,T}^{k+1}$ is still defined as in (22), so that the key relations (25) and (26) still hold. In particular, $p_{\varepsilon,T}^{k+1} \circ \underline{I}_T^k : H^1(T) \rightarrow V_{\varepsilon,T}^{k+1}$ is the \mathbb{A}_ε -weighted elliptic projection. However, the restriction of $p_{\varepsilon,T}^{k+1} \circ \underline{I}_T^k$ to the larger space $\tilde{V}_{\varepsilon,T}^{k+1}$ is *not* the identity operator since $p_{\varepsilon,T}^{k+1}$ maps onto the smaller space $V_{\varepsilon,T}^{k+1}$. Concerning (24), we still have $\Pi_{\partial T}^k(p_{\varepsilon,T}^{k+1}(\underline{v}_T)) = v_{\mathcal{F}_T}$, but now $\Pi_T^{k-1}(p_{\varepsilon,T}^{k+1}(\underline{v}_T)) = \Pi_T^{k-1}(v_T)$

is in general different from \mathbf{v}_T . This leads us to introduce the symmetric, positive semi-definite stabilization

$$j_{\varepsilon,T}(\underline{\mathbf{u}}_T, \underline{\mathbf{v}}_T) := \alpha \int_{\partial T} H_{\partial T}^{-1} (\mathbf{u}_T - \Pi_T^k(p_{\varepsilon,T}^{k+1}(\underline{\mathbf{u}}_T))) (\mathbf{v}_T - \Pi_T^k(p_{\varepsilon,T}^{k+1}(\underline{\mathbf{v}}_T))). \quad (45)$$

The local bilinear form $a_{\varepsilon,T} : \underline{\mathbf{U}}_T^k \times \underline{\mathbf{U}}_T^k \rightarrow \mathbb{R}$ is then defined as

$$a_{\varepsilon,T}(\underline{\mathbf{u}}_T, \underline{\mathbf{v}}_T) := \int_T \mathbb{A}_{\varepsilon} \nabla p_{\varepsilon,T}^{k+1}(\underline{\mathbf{u}}_T) \cdot \nabla p_{\varepsilon,T}^{k+1}(\underline{\mathbf{v}}_T) + j_{\varepsilon,T}(\underline{\mathbf{u}}_T, \underline{\mathbf{v}}_T).$$

Remark 5.9 (Variant). *Alternatively, one can discard the stabilization at the prize of computing additional oscillatory cell-based basis functions, using the basis functions $(\Phi_T^{k,i})_{1 \leq i \leq N_d^k}$ instead of $(\Phi_T^{k-1,i})_{1 \leq i \leq N_d^{k-1}}$ as proposed in Section 4.1.1. This is the approach pursued in [37] (up to a slightly different treatment of the right-hand side) for $k = 0$ where one oscillatory cell-based basis function is added (in the slightly different context of perforated domains). The analysis for polynomial degrees $k \geq 1$ is similar to the one presented in Section 5.1 and is omitted for brevity.*

Recall the local stability semi-norm $\|\cdot\|_T$ defined by (27).

Lemma 5.10 (Local stability and approximation). *The following holds:*

$$a_{\varepsilon,T}(\underline{\mathbf{v}}_T, \underline{\mathbf{v}}_T) \geq c \alpha \|\underline{\mathbf{v}}_T\|_T^2 \quad \forall \underline{\mathbf{v}}_T \in \underline{\mathbf{U}}_T^k. \quad (46)$$

Moreover, for all $v \in H^1(T)$,

$$j_{\varepsilon,T}(\underline{\mathbf{I}}_T^k v, \underline{\mathbf{I}}_T^k v)^{1/2} \leq c \|\mathbb{A}_{\varepsilon}^{1/2} \nabla (v - p_{\varepsilon,T}^{k+1}(\underline{\mathbf{I}}_T^k v))\|_{L^2(T)^d}, \quad (47)$$

with (distinct) constants c independent of ε , H_T , α and β .

Proof. To prove stability, we adapt the proof of Lemma 5.2. Let $\underline{\mathbf{v}}_T \in \underline{\mathbf{U}}_T^k$. The bound (30) on $\|\nabla \mathbf{v}_T\|_{L^2(T)^d}$ still holds, so that we only need to bound $\|H_{\partial T}^{-1/2}(\mathbf{v}_T - \mathbf{v}_{\mathcal{F}_T})\|_{L^2(\partial T)}$. Since $\Pi_{\partial T}^k(p_{\varepsilon,T}^{k+1}(\underline{\mathbf{v}}_T)) = \mathbf{v}_{\mathcal{F}_T}$, we infer that $(\mathbf{v}_T - \mathbf{v}_{\mathcal{F}_T}) = \Pi_{\partial T}^k(\mathbf{v}_T - p_{\varepsilon,T}^{k+1}(\underline{\mathbf{v}}_T))$, so that invoking the L^2 -stability of $\Pi_{\partial T}^k$ and the triangle inequality while adding/subtracting $\Pi_T^k(p_{\varepsilon,T}^{k+1}(\underline{\mathbf{v}}_T))$, we obtain

$$\begin{aligned} \|H_{\partial T}^{-1/2}(\mathbf{v}_T - \mathbf{v}_{\mathcal{F}_T})\|_{L^2(\partial T)} &\leq \|H_{\partial T}^{-1/2}(\mathbf{v}_T - \Pi_T^k(p_{\varepsilon,T}^{k+1}(\underline{\mathbf{v}}_T)))\|_{L^2(\partial T)} \\ &\quad + \|H_{\partial T}^{-1/2}(p_{\varepsilon,T}^{k+1}(\underline{\mathbf{v}}_T) - \Pi_T^k(p_{\varepsilon,T}^{k+1}(\underline{\mathbf{v}}_T)))\|_{L^2(\partial T)}. \end{aligned}$$

The first term in the right-hand side is bounded by $\alpha^{-1/2} j_{\varepsilon,T}(\underline{\mathbf{v}}_T, \underline{\mathbf{v}}_T)^{1/2}$, and the second one has been bounded (with the use of Π_T^{k-1} instead of Π_T^k) in the proof of Lemma 5.2 (see (31)) by $c \alpha^{-1/2} \|\mathbb{A}_{\varepsilon}^{1/2} \nabla p_{\varepsilon,T}^{k+1}(\underline{\mathbf{v}}_T)\|_{L^2(T)^d}$. To prove (47), we start from

$$j_{\varepsilon,T}(\underline{\mathbf{I}}_T^k v, \underline{\mathbf{I}}_T^k v) = \alpha \|H_{\partial T}^{-1/2} \Pi_T^k(v - p_{\varepsilon,T}^{k+1}(\underline{\mathbf{I}}_T^k v))\|_{L^2(\partial T)}^2.$$

The result then follows from the application of the discrete trace inequality (7), the L^2 -stability property of Π_T^k , and the local Poincaré inequality (9) (since $\int_T p_{\varepsilon,T}^{k+1}(\mathbb{I}_T^k v) = \int_T v$). \square

We define the broken polynomial space

$$\mathbb{P}_d^k(\mathcal{T}_H) := \{v \in L^2(\Omega) \mid v|_T \in \mathbb{P}_d^k(T) \ \forall T \in \mathcal{T}_H\},$$

and the global set of discrete unknowns is defined to be

$$\underline{\mathbb{U}}_H^k := \mathbb{P}_d^k(\mathcal{T}_H) \times \mathbb{P}_{d-1}^k(\mathcal{F}_H), \quad (48)$$

where $\mathbb{P}_{d-1}^k(\mathcal{F}_H)$ is still defined by (33). To take into account homogeneous Dirichlet boundary conditions, we consider again the subspace $\underline{\mathbb{U}}_{H,0}^k := \{\underline{\mathbf{v}}_H \in \underline{\mathbb{U}}_H^k \mid \mathbf{v}_F \equiv 0 \ \forall F \in \mathcal{F}_H^b\}$. We define the global bilinear form $a_{\varepsilon,H} : \underline{\mathbb{U}}_H^k \times \underline{\mathbb{U}}_H^k \rightarrow \mathbb{R}$ such that

$$a_{\varepsilon,H}(\underline{\mathbf{u}}_H, \underline{\mathbf{v}}_H) := \sum_{T \in \mathcal{T}_H} a_{\varepsilon,T}(\underline{\mathbf{u}}_T, \underline{\mathbf{v}}_T) = \sum_{T \in \mathcal{T}_H} \left(\int_T \mathbb{A}_\varepsilon \nabla p_{\varepsilon,T}^{k+1}(\underline{\mathbf{u}}_T) \cdot \nabla p_{\varepsilon,T}^{k+1}(\underline{\mathbf{v}}_T) + j_{\varepsilon,T}(\underline{\mathbf{u}}_T, \underline{\mathbf{v}}_T) \right).$$

Then, the discrete problem reads: Find $\underline{\mathbf{u}}_{\varepsilon,H} \in \underline{\mathbb{U}}_{H,0}^k$ such that

$$a_{\varepsilon,H}(\underline{\mathbf{u}}_{\varepsilon,H}, \underline{\mathbf{v}}_H) = \int_\Omega f \mathbf{v}_{\mathcal{T}_H} \quad \forall \underline{\mathbf{v}}_H \in \underline{\mathbb{U}}_{H,0}^k. \quad (49)$$

Recalling the norm $\|\underline{\mathbf{v}}_H\|_H^2 := \sum_{T \in \mathcal{T}_H} \|\mathbf{v}_T\|_T^2$ on $\underline{\mathbb{U}}_{H,0}^k$, we readily infer from Lemma 5.10 the following well-posedness result.

Lemma 5.11 (Well-posedness). *The following holds:*

$$\begin{aligned} a_{\varepsilon,H}(\underline{\mathbf{v}}_H, \underline{\mathbf{v}}_H) &= \sum_{T \in \mathcal{T}_H} \left(\|\mathbb{A}_\varepsilon^{1/2} \nabla p_{\varepsilon,T}^{k+1}(\underline{\mathbf{v}}_T)\|_{L^2(T)^d}^2 + j_{\varepsilon,T}(\underline{\mathbf{v}}_T, \underline{\mathbf{v}}_T) \right) \\ &=: \|\underline{\mathbf{v}}_H\|_{\varepsilon,H}^2 \geq c \alpha \|\underline{\mathbf{v}}_H\|_H^2 \quad \forall \underline{\mathbf{v}}_H \in \underline{\mathbb{U}}_H^k, \end{aligned} \quad (50)$$

with constant c independent of ε , H , α and β . As a consequence, the discrete problem (49) is well-posed.

Remark 5.12 (ncFE interpretation). *As in Remark 5.4, it is possible to give a ncFE interpretation of the scheme (49). Let*

$$\tilde{V}_{\varepsilon,H,0}^{k+1} := \left\{ v_{\varepsilon,H} \in L^2(\Omega) \mid v_{\varepsilon,H}|_T \in \tilde{V}_{\varepsilon,T}^{k+1} \ \forall T \in \mathcal{T}_H \text{ and } \Pi_F^k(\llbracket v_{\varepsilon,H} \rrbracket_F) = 0 \ \forall F \in \mathcal{F}_H \right\},$$

and consider the following ncFE method: Find $u_{\varepsilon,H} \in \tilde{V}_{\varepsilon,H,0}^{k+1}$ such that

$$\tilde{a}_{\varepsilon,H}(u_{\varepsilon,H}, v_{\varepsilon,H}) = \sum_{T \in \mathcal{T}_H} \int_T f \Pi_T^k(v_{\varepsilon,H}) \quad \forall v_{\varepsilon,H} \in \tilde{V}_{\varepsilon,H,0}^{k+1}, \quad (51)$$

where $\tilde{a}_{\varepsilon,H}(u_{\varepsilon,H}, v_{\varepsilon,H}) := \sum_{T \in \mathcal{T}_H} a_{\varepsilon,T}(\mathbb{I}_T^k(u_{\varepsilon,H|T}), \mathbb{I}_T^k(v_{\varepsilon,H|T}))$. Then, it can be shown that $\underline{u}_{\varepsilon,H}$ solves (49) if and only if $\underline{u}_{\varepsilon,T} = \mathbb{I}_T^k(u_{\varepsilon,H|T})$ for all $T \in \mathcal{T}_H$ where $u_{\varepsilon,H}$ solves (51). The main difference with respect to the mixed-order case is that it is no longer possible to simplify the expression of the bilinear form $\tilde{a}_{\varepsilon,H}$ since the restriction of $p_{\varepsilon,T}^{k+1} \circ \mathbb{I}_T^k$ to $\tilde{V}_{\varepsilon,T}^{k+1}$ is not the identity operator. As in the monoscale HHO method, the operator $p_{\varepsilon,T}^{k+1}$, which maps onto the smaller space $V_{\varepsilon,T}^{k+1}$, allows one to restrict the number of computed basis functions while maintaining optimal (and here also ε -robust) approximation properties. The functions (from the discrete space $\tilde{V}_{\varepsilon,T}^{k+1}$) that are eliminated (not computed) are handled by the stabilization term.

Lemma 5.13 (Discrete energy-error estimate). *Let the discrete error $\underline{e}_{\varepsilon,H}$ be defined by (38). Assume that $u_0 \in H^{k+2}(\Omega)$. Then, the following holds:*

$$\|\underline{e}_{\varepsilon,H}\|_{\varepsilon,H} \leq c \rho^{1/2} \left(\beta \sum_{T \in \mathcal{T}_H} H_T^{2(k+1)} |u_0|_{H^{k+2}(T)}^2 + \sum_{T \in \mathcal{T}_H} \|\mathbb{A}_{\varepsilon}^{1/2} \nabla (u_{\varepsilon} - \pi_{\varepsilon,T}^{k+1}(u_0))\|_{L^2(T)^d}^2 \right)^{1/2}, \quad (52)$$

with constant c independent of ε , H , u_0 , α and β .

Proof. The only difference with the proof of Lemma 5.5 is that we now have $a_{\varepsilon,H}(\underline{e}_{\varepsilon,H}, \underline{v}_H) = \mathfrak{T}_1 + \mathfrak{T}_2 + \mathfrak{T}_3$, where $\mathfrak{T}_1, \mathfrak{T}_2$ are defined and bounded in that proof and where

$$\mathfrak{T}_3 := \sum_{T \in \mathcal{T}_H} j_{\varepsilon,T}(\mathbb{I}_T^k u_{\varepsilon}, \underline{v}_T).$$

Since $j_{\varepsilon,T}$ is symmetric, positive semi-definite, we infer that

$$\begin{aligned} |\mathfrak{T}_3| &\leq \left(\sum_{T \in \mathcal{T}_H} j_{\varepsilon,T}(\mathbb{I}_T^k u_{\varepsilon}, \mathbb{I}_T^k u_{\varepsilon}) \right)^{1/2} \left(\sum_{T \in \mathcal{T}_H} j_{\varepsilon,T}(\underline{v}_T, \underline{v}_T) \right)^{1/2} \\ &\leq c \left(\sum_{T \in \mathcal{T}_H} \|\mathbb{A}_{\varepsilon}^{1/2} \nabla (u_{\varepsilon} - p_{\varepsilon,T}^{k+1}(\mathbb{I}_T^k u_{\varepsilon}))\|_{L^2(T)^d}^2 \right)^{1/2} \|\underline{v}_H\|_{\varepsilon,H}, \end{aligned}$$

where we have used (47). We can now conclude as before. \square

Theorem 5.14 (Energy-error estimate). *Assume that the correctors μ_l are in $W^{1,\infty}(\mathbb{R}^d)$ for any $1 \leq l \leq d$, and that $u_0 \in H^{k+2}(\Omega) \cap W^{1,\infty}(\Omega)$. Then, the following holds:*

$$\begin{aligned} \left(\sum_{T \in \mathcal{T}_H} \|\mathbb{A}_{\varepsilon}^{1/2} \nabla (u_{\varepsilon} - p_{\varepsilon,T}^{k+1}(\underline{u}_{\varepsilon,T}))\|_{L^2(T)^d}^2 \right)^{1/2} &\leq c \beta^{1/2} \rho \left(\sum_{T \in \mathcal{T}_H} H_T^{2(k+1)} |u_0|_{H^{k+2}(T)}^2 \right. \\ &\quad \left. + \varepsilon |\partial\Omega| |u_0|_{W^{1,\infty}(\Omega)}^2 + \sum_{T \in \mathcal{T}_H} \left[\varepsilon^2 |u_0|_{H^2(T)}^2 + \varepsilon |\partial T| |u_0|_{W^{1,\infty}(T)}^2 \right] \right)^{1/2}, \quad (53) \end{aligned}$$

with c independent of ε , H , u_0 , α and β . In particular, if the mesh \mathcal{T}_H is quasi-uniform, and tracking for simplicity only the dependency on ε and H with $\varepsilon \leq H \leq \ell_{\Omega}$, we obtain an energy-error upper bound of the form $(\varepsilon^{1/2} + H^{k+1} + (\varepsilon/H)^{1/2})$.

Proof. Identical to that of Theorem 5.6. □

Remark 5.15 (Dependency on ρ). *As in the mixed-order case (cf. Remark 5.7), the estimate (53) has a linear dependency with respect to the (global) heterogeneity/anisotropy ratio ρ .*

Remark 5.16 (Discretization of the right-hand side). *The same observation as in Remark 5.8 concerning the discretization of the right-hand side in (49) is still valid for the equal-order case.*

6 Numerical results

In this section, we discuss the organization of the computations and we present some numerical results illustrating the above analysis for both the mixed-order and equal-order MsHHO methods. Our numerical results have been obtained using the `Disk++` library, which is available as open-source under MPL license at the address <https://github.com/datafl4sh/diskpp>. The numerical core of the library is described in [13].

6.1 Offline/online solution strategy

Let us consider the mixed-order version ($k \geq 1$) of the MsHHO method introduced in Section 5.1. Similar considerations carry over to the equal-order case ($k \geq 0$) of Section 5.2. To solve (35), we adopt an offline/online strategy.

- In the offline step, all the computations are local, and independent of the right-hand side f . We first compute the cell-based and face-based basis functions, i.e., for all $T \in \mathcal{T}_H$, we compute the N_d^{k-1} functions $\varphi_{\varepsilon,T}^{k+1,i}$ solution to (12) (cf. Remark 4.1), and the $\text{card}(\mathcal{F}_T) \times N_{d-1}^k$ functions $\varphi_{\varepsilon,T,F}^{k+1,j}$ solution to (14). This first substep is fully parallelizable. In a second time, we compute the multiscale reconstruction operators $p_{\varepsilon,T}^{k+1}$, by solving (22) for all $T \in \mathcal{T}_H$. Each computation requires to invert a symmetric positive-definite matrix of size $(N_d^{k-1} + \text{card}(\mathcal{F}_T) \times N_{d-1}^k)$, which can be performed effectively via Cholesky factorization. This second substep is as well fully parallelizable. Finally, we perform static condensation locally in each cell of \mathcal{T}_H , to eliminate the cell unknowns. Details can be found in [21, Section 3.3.1]. Basically, in each cell, this substep consists in inverting a symmetric positive-definite matrix of size N_d^{k-1} . This last substep is also fully parallelizable.
- In the online step, we compute the L^2 -orthogonal projection of the right-hand side f onto $\mathbb{P}_d^{k-1}(\mathcal{T}_H)$, and we then solve a symmetric positive-definite global problem, posed in terms of the face unknowns only. The size of this problem is $\text{card}(\mathcal{F}_H^i) \times N_{d-1}^k$. If one wants to compute an approximation of the solution to (1) for another f (or for other boundary conditions), only the online step must be rerun.

6.2 Periodic test-case

We consider the periodic test-case studied in [36] (and also in [40]). We let $d = 2$, and Ω be the unit square. We consider Problem (1), with right-hand side $f(x, y) = \sin(x) \sin(y)$, and oscillatory coefficient

$$\mathbb{A}_\varepsilon(x, y) = a(x/\varepsilon, y/\varepsilon) \mathbb{I}_2, \quad a(x_1, x_2) = 1 + 100 \cos^2(\pi x_1) \sin^2(\pi x_2). \quad (54)$$

For the coefficient (54), the homogenized tensor is given by $\mathbb{A}_0 \approx 6.72071 \mathbb{I}_2$. We fix $\varepsilon = \pi/150 \approx 0.021$.

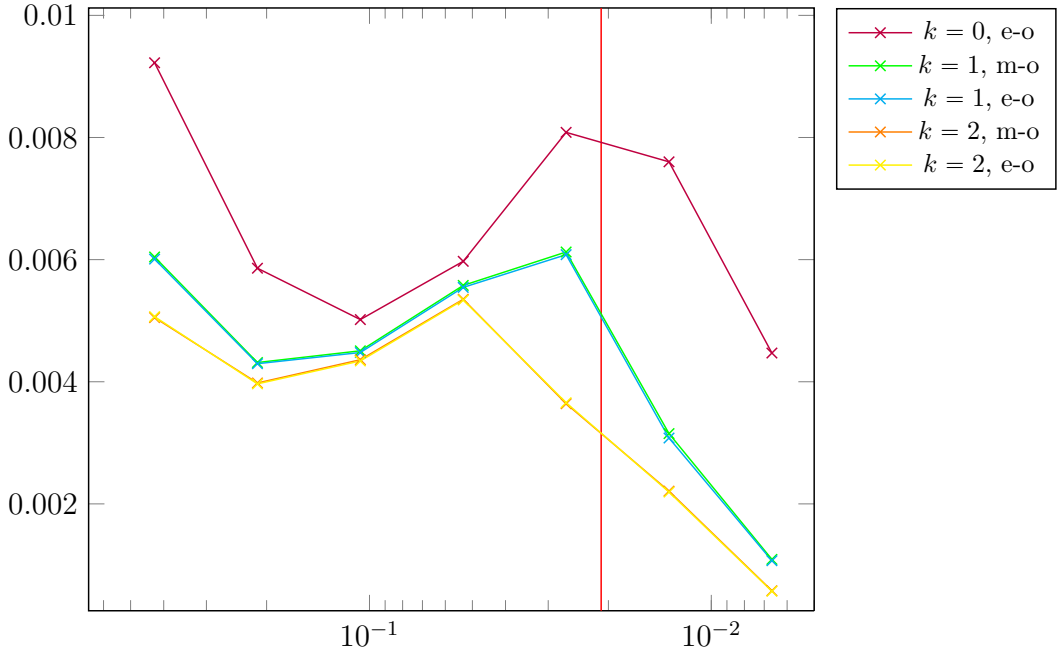


Figure 1: Periodic test-case: convergence results in energy-norm for mesh levels $l \in \{0, \dots, 6\}$; mixed-order MsHHO method with polynomial degrees $k \in \{1, 2\}$ and equal-order MsHHO method with polynomial degrees $k \in \{0, 1, 2\}$. The red vertical line indicates the value of ε .

We consider a sequence of hierarchical triangular meshes of size $H_l = 0.43 \times 2^{-l}$ with $l \in \{0, \dots, 9\}$, so that $H_5 < \varepsilon < H_4$. A reference solution is computed by solving (1) with the (equal-order) monoscale HHO method on the mesh of level $l_{\text{ref}} = 9$ with polynomial degree $k_{\text{ref}} = 2$. In Figure 1, we present the (absolute) energy-norm errors obtained with the MsHHO method on the meshes \mathcal{T}_{H_l} with $l \in \{0, \dots, 6\}$. We consider both the mixed-order MsHHO method with polynomial degrees $k \in \{1, 2\}$ and the equal-order MsHHO method with polynomial degrees $k \in \{0, 1, 2\}$. In all cases, the cell- and face-based oscillatory basis functions are pre-computed using the (equal-order) monoscale HHO method on the mesh of level $l_{\text{osc}} = 8$ with polynomial degree $k_{\text{osc}} = 1$. We have verified that the oscillatory basis functions are sufficiently well resolved by comparing our results to those obtained with

$k_{\text{osc}} = 2$ and obtaining only very marginal differences. The first observation we draw from Figure 1 is that the mixed-order and equal-order MsHHO methods employing the same polynomial order for the face unknowns deliver very similar results; indeed, the error curves are barely distinguishable both for $k = 1$ and $k = 2$. Moreover, we can observe all the main features expected from the error analysis: a pre-asymptotic regime where the term H^{k+1} essentially dominates (meshes of levels $l \in \{0, 1\}$), the resonance regime (meshes of levels $l \in \{2, 3, 4\}$ essentially), and the asymptotic regime where the mesh actually resolves the fine scale of the model coefficients (meshes of levels $l \in \{5, 6\}$). We can also see the advantages of using a higher polynomial order for the face unknowns: the error is overall smaller, the minimal error in the resonance regime is reached at a larger value of H and takes a smaller value (incidentally, the maximal error in the resonance regime takes a smaller value as well), and the asymptotic regime starts for larger values of H .

6.3 Locally periodic test-case

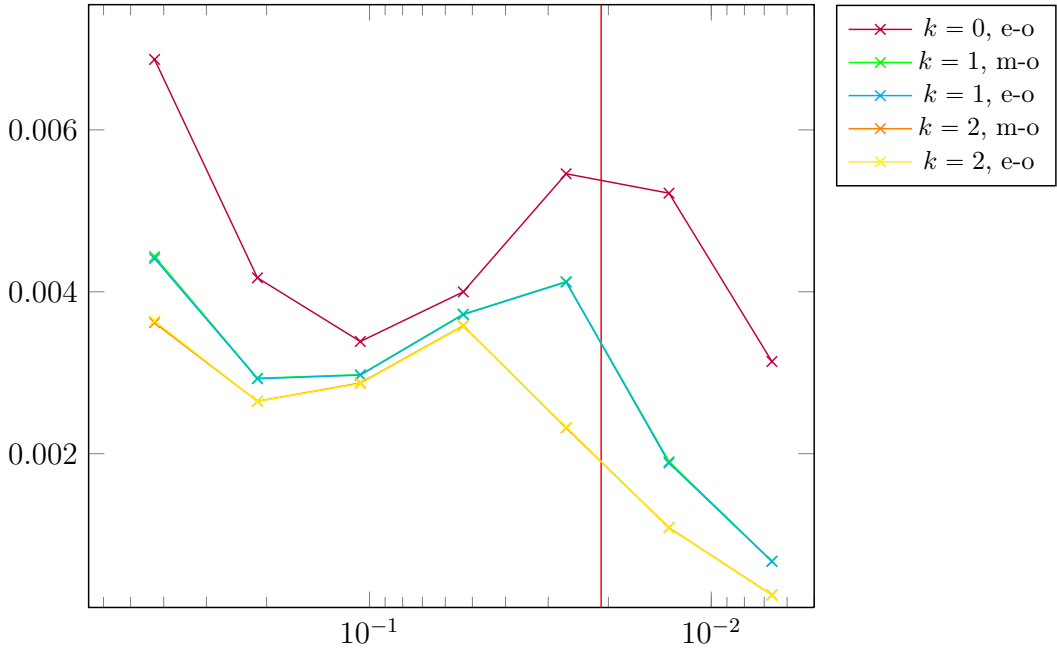


Figure 2: Locally periodic test-case: convergence results in energy-norm for mesh levels $l \in \{0, \dots, 6\}$; mixed-order MsHHO method with polynomial degrees $k \in \{1, 2\}$ and equal-order MsHHO method with polynomial degrees $k \in \{0, 1, 2\}$. The red vertical line indicates the value of ε .

Keeping the same two-dimensional domain Ω as in the periodic test-case of Section 6.2, we consider now a locally periodic test-case where we solve Problem (1) with unchanged right-hand side $f(x, y) = \sin(x) \sin(y)$, but with oscillatory coefficient

$$\mathbb{A}_\varepsilon(x, y) = \left(a(x/\varepsilon, y/\varepsilon) + e^{(x^2+y^2)/2} \right) \mathbb{I}_2, \quad \text{with } a \text{ given in (54),} \quad (55)$$

Table 1: Offline and online computational times

	Energy-error	Offline time (s)	Online time (s)	#DoFs
$k = 0$ ($l = 2$)	0.00338612	254	0.026	408
$k = 2$ ($l = 1$)	0.00264648	520	0.018	288

and with unchanged value of ε . We perform the same numerical experiments as in Section 6.2 using the same mesh level and polynomial order parameters for computing the reference solution and the oscillatory basis functions (we verified similarly the adequate resolution of the oscillatory basis functions). Results are reported in Figure 2. We can draw the same conclusions as in the periodic test-case: similarity of the results delivered by the mixed-order and the equal-order MsHHO methods for both $k = 1$ and $k = 2$, presence of the pre-asymptotic, resonance, and asymptotic regimes, and advantages of using a higher polynomial order for the face unknowns.

To briefly assess computational costs, we compute, for those mesh levels in the pre-asymptotic or resonance regimes for which the error is minimal, the computational times to perform the offline and online steps. We report the results in Table 1. We also report the number of degrees of freedom in the global system solved in the online step. We make the experiment for the equal-order MsHHO method of orders $k = 0$ and $k = 2$, for respective mesh levels $l = 2$ and $l = 1$. The implementation uses a processor Intel i7-3615QM and 16Gb of RAM. The solver is PARDISO of the Intel MKL library, and Eigen3 is used for linear algebra. We do not make use of parallelism in our implementation to compute the results. The results of Table 1 clearly show the interest of higher order approximations, since a better accuracy is reached at a smaller online computational cost.

Acknowledgments The authors are thankful to Alexei Lozinski (LMB, Université de Franche-Comté) for fruitful discussions on the topic.

A Estimates on the first-order two-scale expansion

In this appendix, we derive various useful estimates on the first-order two-scale expansion $\mathcal{L}_\varepsilon^1(u_0)$ defined by (6). Except for Lemma A.4, these estimates are classical; we provide (short) proofs since we additionally track the dependency of the constants on the parameters α and β characterizing the spectrum of \mathbb{A} and on the various length scales present in the problem.

A.1 Dual-norm estimates

Let D be an open, connected, polytopal subset of Ω ; in this work, we will need the cases where $D = \Omega$ or where $D = T \in \mathcal{T}_H$. Let ℓ_D be a length scale associated with D , e.g., its

diameter. Our goal is to bound the dual norm of the linear map such that

$$w \mapsto \mathcal{F}_\varepsilon(w) := \int_D (\mathbb{A}_\varepsilon \nabla \mathcal{L}_\varepsilon^1(u_0) - \mathbb{A}_0 \nabla u_0) \cdot \nabla w, \quad (56)$$

for all $w \in H_0^1(D)$ (Dirichlet case), or for all $w \in H_\star^1(D) := \{w \in H^1(D) \mid \int_D w = 0\}$ (Neumann case); note that $\mathcal{F}_\varepsilon(w)$ does not change if the values of w are shifted by a constant.

Lemma A.1 (Dual norm, Dirichlet case). *Assume that the homogenized solution u_0 belongs to $H^2(D)$ and that, for any $1 \leq l \leq d$, the corrector μ_l belongs to $W^{1,\infty}(\mathbb{R}^d)$. Then,*

$$\sup_{w \in H_0^1(D)} \frac{|\mathcal{F}_\varepsilon(w)|}{\|\nabla w\|_{L^2(D)^d}} \leq c \beta \varepsilon |u_0|_{H^2(D)}, \quad (57)$$

with c independent of ε , D , u_0 , α and β , and possibly depending on d and $\max_{1 \leq l \leq d} \|\mu_l\|_{W^{1,\infty}(\mathbb{R}^d)}$.

Proof. For any integer $1 \leq i \leq d$, we have

$$\begin{aligned} [\mathbb{A}_\varepsilon \nabla \mathcal{L}_\varepsilon^1(u_0)]_i &= \sum_{j=1}^d [\mathbb{A}_\varepsilon]_{ij} \partial_j \mathcal{L}_\varepsilon^1(u_0) \\ &= \sum_{j=1}^d [\mathbb{A}_\varepsilon]_{ij} \left(\partial_j u_0 + \varepsilon \sum_{l=1}^d \left(\frac{1}{\varepsilon} \mathcal{R}_\varepsilon(\partial_j \mu_l) \partial_l u_0 + \mathcal{R}_\varepsilon(\mu_l) \partial_{j,l}^2 u_0 \right) \right) \\ &= [\mathbb{A}_0 \nabla u_0]_i + \sum_{l=1}^d \mathcal{R}_\varepsilon(\theta_i^l) \partial_l u_0 + \varepsilon \sum_{l,j=1}^d [\mathbb{A}_\varepsilon]_{ij} \mathcal{R}_\varepsilon(\mu_l) \partial_{j,l}^2 u_0, \end{aligned} \quad (58)$$

with $\theta_i^l := \mathbb{A}_{il} + \sum_{j=1}^d \mathbb{A}_{ij} \partial_j \mu_l - [\mathbb{A}_0]_{il}$ satisfying the following properties:

- $\theta_i^l \in L_{\text{per}}^\infty(Q)$ by assumption on \mathbb{A} and on the correctors μ_l ;
- $\int_Q \theta_i^l = 0$ as a consequence of (3);
- $\sum_{i=1}^d \partial_i \theta_i^l = 0$ in \mathbb{R}^d as a consequence of (4).

Adapting [34, Equation (1.11)] (see also [30, Sections I.3.1 and I.3.3]), we infer that, for any integer $1 \leq l \leq d$, there exists a skew-symmetric matrix $\mathbb{T}^l \in W_{\text{per}}^{1,\infty}(Q)^{d \times d}$, satisfying $\int_Q \mathbb{T}^l = 0$ and such that, for any integer $1 \leq i \leq d$,

$$\theta_i^l = \sum_{q=1}^d \partial_q \mathbb{T}_{qi}^l. \quad (59)$$

Plugging (59) into (58), we infer that, for any integer $1 \leq i \leq d$,

$$[\mathbb{A}_\varepsilon \nabla \mathcal{L}_\varepsilon^1(u_0)]_i - [\mathbb{A}_0 \nabla u_0]_i = \varepsilon \left(\sum_{l,q=1}^d \partial_q (\mathcal{R}_\varepsilon(\mathbb{T}_{qi}^l)) \partial_l u_0 + \sum_{l,j=1}^d [\mathbb{A}_\varepsilon]_{ij} \mathcal{R}_\varepsilon(\mu_l) \partial_{j,l}^2 u_0 \right).$$

Since $\partial_q(\mathcal{R}_\varepsilon(\mathbb{T}_{qi}^l))\partial_l u_0 = \partial_q(\mathcal{R}_\varepsilon(\mathbb{T}_{qi}^l)\partial_l u_0) - \mathcal{R}_\varepsilon(\mathbb{T}_{qi}^l)\partial_{q,l}^2 u_0$, and recalling the definition (56) of \mathcal{F}_ε , this yields

$$\begin{aligned}\mathcal{F}_\varepsilon(w) = & \varepsilon \left(\sum_{i,l,j=1}^d \int_D [\mathbb{A}_\varepsilon]_{ij} \mathcal{R}_\varepsilon(\mu_l) \partial_{j,l}^2 u_0 \partial_i w - \sum_{i,l,q=1}^d \int_D \mathcal{R}_\varepsilon(\mathbb{T}_{qi}^l) \partial_{q,l}^2 u_0 \partial_i w \right) \\ & + \varepsilon \sum_{i,l,q=1}^d \int_D \partial_q (\mathcal{R}_\varepsilon(\mathbb{T}_{qi}^l) \partial_l u_0) \partial_i w.\end{aligned}\quad (60)$$

Since $\mathbb{T}_{qi}^l = -\mathbb{T}_{iq}^l$ for any integers $1 \leq i, q \leq d$, we infer by integration by parts of the last term that

$$\begin{aligned}\mathcal{F}_\varepsilon(w) = & \varepsilon \left(\sum_{i,l,j=1}^d \int_D [\mathbb{A}_\varepsilon]_{ij} \mathcal{R}_\varepsilon(\mu_l) \partial_{j,l}^2 u_0 \partial_i w - \sum_{i,l,q=1}^d \int_D \mathcal{R}_\varepsilon(\mathbb{T}_{qi}^l) \partial_{q,l}^2 u_0 \partial_i w \right) \\ & + \varepsilon \sum_{i,l,q=1}^d \int_{\partial D} \partial_q (\mathcal{R}_\varepsilon(\mathbb{T}_{qi}^l) \partial_l u_0) n_{\partial D, i} w,\end{aligned}\quad (61)$$

where $\mathbf{n}_{\partial D}$ is the unit outward normal to D . Since $w \in H_0^1(D)$, we obtain

$$\mathcal{F}_\varepsilon(w) = \varepsilon \left(\sum_{i,l,j=1}^d \int_D [\mathbb{A}_\varepsilon]_{ij} \mathcal{R}_\varepsilon(\mu_l) \partial_{j,l}^2 u_0 \partial_i w - \sum_{i,l,q=1}^d \int_D \mathcal{R}_\varepsilon(\mathbb{T}_{qi}^l) \partial_{q,l}^2 u_0 \partial_i w \right).$$

Using the Cauchy–Schwarz inequality, we finally deduce that

$$\sup_{w \in H_0^1(D)} \frac{|\mathcal{F}_\varepsilon(w)|}{\|\nabla w\|_{L^2(D)^d}} \leq c \beta \varepsilon \max_{1 \leq l \leq d} \left(\|\mu_l\|_{L^\infty(\mathbb{R}^d)}, \beta^{-1} \|\mathbb{T}^l\|_{L^\infty(\mathbb{R}^d)^{d \times d}} \right) |u_0|_{H^2(D)}.$$

We conclude by observing that $\|\mathbb{T}^l\|_{L^\infty(\mathbb{R}^d)^{d \times d}} \leq c \max_{1 \leq i \leq d} \|\theta_i^l\|_{L^\infty(\mathbb{R}^d)} \leq c \beta$. \square

Lemma A.2 (Dual norm, Neumann case (i)). *Assume that the homogenized solution u_0 belongs to $H^2(D) \cap W^{1,\infty}(D)$ and that, for any $1 \leq l \leq d$, the corrector μ_l belongs to $W^{1,\infty}(\mathbb{R}^d)$. Then,*

$$\sup_{w \in H_\star^1(D)} \frac{|\mathcal{F}_\varepsilon(w)|}{\|\nabla w\|_{L^2(D)^d}} \leq c \beta \left(\varepsilon |u_0|_{H^2(D)} + |\partial D|^{1/2} \varepsilon^{1/2} |u_0|_{W^{1,\infty}(D)} \right), \quad (62)$$

with c independent of ε , D , u_0 , α and β , and possibly depending on d and $\max_{1 \leq l \leq d} \|\mu_l\|_{W^{1,\infty}(\mathbb{R}^d)}$.

Proof. Our starting point is (60). The first two terms in the right-hand side are responsible for a contribution of order $\beta \varepsilon |u_0|_{H^2(D)}$, and it only remains to bound the last term. Following the ideas of [34, p. 29], we define, for $\eta > 0$, the domain $D_\eta := \{\mathbf{x} \in D \mid \text{dist}(\mathbf{x}, \partial D) < \eta\}$. If η is above a critical value (which scales as ℓ_D), $D_\eta = D$, otherwise $D_\eta \subsetneq D$. We introduce the cut-off function $\zeta_\eta \in C^0(\overline{D})$ such that $\zeta_\eta \equiv 0$ on

∂D , defined by $\zeta_\eta(\mathbf{x}) = \text{dist}(\mathbf{x}, \partial D)/\eta$ if $\mathbf{x} \in D_\eta$, and $\zeta_\eta(\mathbf{x}) = 1$ if $\mathbf{x} \in D \setminus D_\eta$. We have $0 \leq \zeta_\eta \leq 1$ and $\max_{1 \leq q \leq d} \|\partial_q \zeta_\eta\|_{L^\infty(D)} \leq \eta^{-1}$. We first infer that

$$\varepsilon \sum_{i,l,q=1}^d \int_D \partial_q (\mathcal{R}_\varepsilon(\mathbb{T}_{qi}^l) \partial_l u_0) \partial_i w = \varepsilon \sum_{i,l,q=1}^d \int_{D_\eta} \partial_q ((1 - \zeta_\eta) \mathcal{R}_\varepsilon(\mathbb{T}_{qi}^l) \partial_l u_0) \partial_i w,$$

since $(1 - \zeta_\eta)$ vanishes identically on $D \setminus D_\eta$ and since $\sum_{i,l,q=1}^d \int_D \partial_q (\zeta_\eta \mathcal{R}_\varepsilon(\mathbb{T}_{qi}^l) \partial_l u_0) \partial_i w = 0$ as can be seen by integration by parts, using the fact that $\mathbb{T}_{qi}^l = -\mathbb{T}_{iq}^l$ for any integers $1 \leq i, q \leq d$, and the fact that ζ_η vanishes identically on ∂D . Then, accounting for the fact that

$$\begin{aligned} \varepsilon \partial_q ((1 - \zeta_\eta) \mathcal{R}_\varepsilon(\mathbb{T}_{qi}^l) \partial_l u_0) &= -\varepsilon \partial_q \zeta_\eta \mathcal{R}_\varepsilon(\mathbb{T}_{qi}^l) \partial_l u_0 \\ &\quad + (1 - \zeta_\eta) \mathcal{R}_\varepsilon(\partial_q \mathbb{T}_{qi}^l) \partial_l u_0 + \varepsilon (1 - \zeta_\eta) \mathcal{R}_\varepsilon(\mathbb{T}_{qi}^l) \partial_{q,l}^2 u_0, \end{aligned}$$

we infer that

$$\begin{aligned} \left| \varepsilon \sum_{i,l,q=1}^d \int_D \partial_q (\mathcal{R}_\varepsilon(\mathbb{T}_{qi}^l) \partial_l u_0) \partial_i w \right| &\leq c \left[|D_\eta|^{1/2} \left(\frac{\varepsilon}{\eta} + 1 \right) \left(\max_{1 \leq l \leq d} \|\mathbb{T}^l\|_{W^{1,\infty}(\mathbb{R}^d)^{d \times d}} \right) |u_0|_{W^{1,\infty}(D)} \right. \\ &\quad \left. + \varepsilon \left(\max_{1 \leq l \leq d} \|\mathbb{T}^l\|_{L^\infty(\mathbb{R}^d)^{d \times d}} \right) |u_0|_{H^2(D)} \right] \|\nabla w\|_{L^2(D)^d}. \end{aligned}$$

Using the estimate $|D_\eta| \leq \eta |\partial D|$, the fact that $\max_{1 \leq l \leq d} \|\mathbb{T}^l\|_{W^{1,\infty}(\mathbb{R}^d)^{d \times d}} \leq c\beta$, and since the function $\eta \mapsto \frac{\varepsilon}{\sqrt{\eta}} + \sqrt{\eta}$ is minimal for $\eta = \varepsilon$, we finally infer the bound (62). \square

Remark A.3 (Weaker regularity assumption). *Without the regularity assumption $u_0 \in W^{1,\infty}(D)$, one can still invoke a Sobolev embedding since $u_0 \in H^2(D)$. The second term between the parentheses in the right-hand side of (62) becomes*

$$c(p) (|\partial D| \varepsilon \ell_D^{-d})^{1/2-1/p} (|u_0|_{H^1(D)} + \ell_D |u_0|_{H^2(D)}),$$

where $p = 6$ for $d = 3$ and p can be taken as large as wanted for $d = 2$ (note that $c(p) \rightarrow +\infty$ when $p \rightarrow +\infty$ in that case). The derivation of estimates in this setting is considered in [40]. However, it is unclear to us how the bound [40, eq. (27)] (which does not exhibit the resonant contribution $(\varepsilon/\ell_D)^{1/2-1/p} |u_0|_{H^1(D)}$) is obtained.

Lemma A.4 (Dual norm, Neumann case (ii)). *Assume that $D = T \in \mathcal{T}_H$ where \mathcal{T}_H is a member of an admissible mesh sequence in the sense of Definition 3.1; set $\ell_D = H_T$. Assume that the homogenized solution u_0 belongs to $H^3(D)$ and that there is $\kappa > 0$ so that $\mathbb{A} \in C^{0,\kappa}(\mathbb{R}^d; \mathbb{R}^{d \times d})$. Then,*

$$\sup_{w \in H_\star^1(D)} \frac{|\mathcal{F}_\varepsilon(w)|}{\|\nabla w\|_{L^2(D)^d}} \leq c\beta \left((\varepsilon + (\varepsilon \ell_D)^{1/2}) |u_0|_{H^2(D)} + \varepsilon \ell_D |u_0|_{H^3(D)} + \varepsilon^{1/2} \ell_D^{-1/2} |u_0|_{H^1(D)} \right), \quad (63)$$

with c independent of ε , D , u_0 , α and β , and possibly depending on d , γ and $\|\mathbb{A}\|_{C^{0,\kappa}(\mathbb{R}^d; \mathbb{R}^{d \times d})}$.

Proof. We proceed as in the proof of Lemma A.1. Concerning the regularity of θ_i^l , we now have $\theta_i^l \in C^{0,\iota}(\mathbb{R}^d)$ for some $\iota > 0$ since the Hölder continuity of \mathbb{A} on \mathbb{R}^d implies the Hölder continuity of μ_l and $\nabla \mu_l$ on \mathbb{R}^d for any $1 \leq l \leq d$; cf., e.g., [29, Theorem 8.22 and Corollary 8.36]. Following [34, p. 6-7] and [36, p. 131-132], we infer that the skew-symmetric matrix \mathbb{T}^l is such that $\mathbb{T}^l \in C^1(\mathbb{R}^d)^{d \times d}$. Our starting point is (61). The first two terms in the right-hand side are responsible for a contribution of order $\beta \varepsilon |u_0|_{H^2(D)}$, and it only remains to bound the last term. We have

$$\begin{aligned} \varepsilon \sum_{i,l,q=1}^d \int_{\partial D} \partial_q (\mathcal{R}_\varepsilon(\mathbb{T}_{qi}^l) \partial_l u_0) n_{\partial D,i} w &= \varepsilon \sum_{i,l,q=1}^d \int_{\partial D} \mathcal{R}_\varepsilon(\mathbb{T}_{qi}^l) \partial_{q,l}^2 u_0 n_{\partial D,i} w \\ &+ \sum_{i,l,q=1}^d \int_{\partial D} \mathcal{R}_\varepsilon(\partial_q \mathbb{T}_{qi}^l) \partial_l u_0 n_{\partial D,i} w =: \mathfrak{T}_1 + \mathfrak{T}_2. \end{aligned}$$

Using the Cauchy–Schwarz inequality and the trace inequality (8), the first term in the right-hand side can be estimated as

$$|\mathfrak{T}_1| \leq c \beta \varepsilon \ell_D^{-1} \left(|u_0|_{H^2(D)} + \ell_D |u_0|_{H^3(D)} \right) \left(\|w\|_{L^2(D)} + \ell_D \|\nabla w\|_{L^2(D)^d} \right),$$

since $\max_{1 \leq l \leq d} \|\mathbb{T}^l\|_{C^0(\mathbb{R}^d)^{d \times d}} \leq c \beta$. Observing that $\int_D w = 0$, we can use the Poincaré inequality (9) to infer that

$$|\mathfrak{T}_1| \leq c \beta \varepsilon \left(|u_0|_{H^2(D)} + \ell_D |u_0|_{H^3(D)} \right) \|\nabla w\|_{L^2(D)^d}.$$

To estimate the second term in the right-hand side, we adapt the ideas from [36, Lemma 4.6]. Considering the matching simplicial sub-mesh of D , let us collect in the set \mathfrak{F}_D all the sub-faces composing the boundary of D . Then, we can write

$$\mathfrak{T}_2 = \sum_{\sigma \in \mathfrak{F}_D} \sum_{l=1}^d \sum_{q=1}^d \sum_{q < i \leq d} \int_{\sigma} \mathcal{R}_\varepsilon(\nabla \mathbb{T}_{qi}^l) \cdot \boldsymbol{\tau}_\sigma^{qi} \partial_l u_0 w,$$

where the vectors $\boldsymbol{\tau}_\sigma^{qi}$ are such that $\|\boldsymbol{\tau}_\sigma^{qi}\|_{\ell^2} \leq 1$ and $\boldsymbol{\tau}_\sigma^{qi} \cdot \mathbf{n}_{\partial D|\sigma} = 0$. Then, using a straight-forward adaptation of the result in [36, Lemma 4.6], and since $\max_{1 \leq l \leq d} \|\mathbb{T}^l\|_{C^1(\mathbb{R}^d)^{d \times d}} \leq c \beta$, we infer that

$$\begin{aligned} \left| \int_{\sigma} \mathcal{R}_\varepsilon(\nabla \mathbb{T}_{qi}^l) \cdot \boldsymbol{\tau}_\sigma^{qi} \partial_l u_0 w \right| &\leq c \beta \varepsilon^{1/2} H_S^{-3/2} \left(|u_0|_{H^1(S)} + H_S |u_0|_{H^2(S)} \right) \\ &\quad \left(\|w\|_{L^2(S)} + H_S \|\nabla w\|_{L^2(S)^d} \right), \end{aligned}$$

where S is the simplicial sub-cell of D having σ as face. Collecting the contributions of all the sub-faces $\sigma \in \mathfrak{F}_D$ and using the mesh regularity assumptions on D , we infer that

$$|\mathfrak{T}_2| \leq c \beta \varepsilon^{1/2} \ell_D^{-3/2} \left(|u_0|_{H^1(D)} + \ell_D |u_0|_{H^2(D)} \right) \left(\|w\|_{L^2(D)} + \ell_D \|\nabla w\|_{L^2(D)^d} \right).$$

Finally, invoking the Poincaré inequality (9) since w has zero mean-value in D yields

$$|\mathfrak{T}_2| \leq c \beta \varepsilon^{1/2} \ell_D^{-1/2} \left(|u_0|_{H^1(D)} + \ell_D |u_0|_{H^2(D)} \right) \|\nabla w\|_{L^2(D)^d}.$$

Collecting the above bounds on \mathfrak{T}_1 and \mathfrak{T}_2 concludes the proof. \square

A.2 Global energy-norm estimate

Lemma A.5 (Energy-norm estimate). *Assume that the homogenized solution u_0 belongs to $H^2(\Omega) \cap W^{1,\infty}(\Omega)$, and that, for any $1 \leq l \leq d$, the corrector μ_l belongs to $W^{1,\infty}(\mathbb{R}^d)$. Then,*

$$\|\mathbb{A}_\varepsilon^{1/2} \nabla (u_\varepsilon - \mathcal{L}_\varepsilon^1(u_0))\|_{L^2(\Omega)^d} \leq c \beta^{1/2} \left(\rho^{1/2} \varepsilon |u_0|_{H^2(\Omega)} + |\partial\Omega|^{1/2} \varepsilon^{1/2} |u_0|_{W^{1,\infty}(\Omega)} \right), \quad (64)$$

with c independent of ε , Ω , u_0 , α and β , and possibly depending on d and $\max_{1 \leq l \leq d} \|\mu_l\|_{W^{1,\infty}(\mathbb{R}^d)}$.

Proof. The regularity assumptions on u_0 and the correctors imply $(u_\varepsilon - \mathcal{L}_\varepsilon^1(u_0)) \in H^1(\Omega)$; however, we do not have $(u_\varepsilon - \mathcal{L}_\varepsilon^1(u_0)) \in H_0^1(\Omega)$. Following the ideas in [34, p. 28], we define, for $\eta > 0$, the domain $\Omega_\eta := \{\mathbf{x} \in \Omega \mid \text{dist}(\mathbf{x}, \partial\Omega) < \eta\}$. If η is above a critical value, $\Omega_\eta = \Omega$, otherwise $\Omega_\eta \subsetneq \Omega$. We introduce the cut-off function $\zeta_\eta \in C^0(\overline{\Omega})$ such that $\zeta_\eta \equiv 0$ on $\partial\Omega$, defined by $\zeta_\eta(\mathbf{x}) = \text{dist}(\mathbf{x}, \partial\Omega)/\eta$ if $\mathbf{x} \in \Omega_\eta$, and $\zeta_\eta(\mathbf{x}) = 1$ if $\mathbf{x} \in \Omega \setminus \Omega_\eta$. We have $0 \leq \zeta_\eta \leq 1$ and $\max_{1 \leq i \leq d} \|\partial_i \zeta_\eta\|_{L^\infty(\Omega)} \leq \eta^{-1}$. The function ζ_η allows us to define a corrected first-order two-scale expansion $\mathcal{L}_\varepsilon^{1,0}(u_0) := u_0 + \varepsilon \zeta_\eta \sum_{l=1}^d \mathcal{R}_\varepsilon(\mu_l) \partial_l u_0$ such that $(u_\varepsilon - \mathcal{L}_\varepsilon^{1,0}(u_0)) \in H_0^1(\Omega)$. We start with the triangle inequality:

$$\begin{aligned} \|\mathbb{A}_\varepsilon^{1/2} \nabla (u_\varepsilon - \mathcal{L}_\varepsilon^1(u_0))\|_{L^2(\Omega)^d} &\leq \|\mathbb{A}_\varepsilon^{1/2} \nabla (u_\varepsilon - \mathcal{L}_\varepsilon^{1,0}(u_0))\|_{L^2(\Omega)^d} \\ &\quad + \|\mathbb{A}_\varepsilon^{1/2} \nabla (\mathcal{L}_\varepsilon^1(u_0) - \mathcal{L}_\varepsilon^{1,0}(u_0))\|_{L^2(\Omega)^d}. \end{aligned} \quad (65)$$

Let us focus on the first term in the right-hand side of (65). We have

$$\begin{aligned} \|\mathbb{A}_\varepsilon^{1/2} \nabla (u_\varepsilon - \mathcal{L}_\varepsilon^{1,0}(u_0))\|_{L^2(\Omega)^d}^2 &= \int_\Omega \mathbb{A}_\varepsilon \nabla (u_\varepsilon - \mathcal{L}_\varepsilon^{1,0}(u_0)) \cdot \nabla (u_\varepsilon - \mathcal{L}_\varepsilon^{1,0}(u_0)) \\ &\quad + \int_\Omega \mathbb{A}_\varepsilon \nabla (\mathcal{L}_\varepsilon^1(u_0) - \mathcal{L}_\varepsilon^{1,0}(u_0)) \cdot \nabla (u_\varepsilon - \mathcal{L}_\varepsilon^{1,0}(u_0)). \end{aligned}$$

Since $(u_\varepsilon - \mathcal{L}_\varepsilon^{1,0}(u_0)) \in H_0^1(\Omega)$, we infer that

$$\begin{aligned} \|\mathbb{A}_\varepsilon^{1/2} \nabla (u_\varepsilon - \mathcal{L}_\varepsilon^{1,0}(u_0))\|_{L^2(\Omega)^d} &\leq \alpha^{-1/2} \sup_{w \in H_0^1(\Omega)} \frac{\left| \int_\Omega \mathbb{A}_\varepsilon \nabla (u_\varepsilon - \mathcal{L}_\varepsilon^{1,0}(u_0)) \cdot \nabla w \right|}{\|\nabla w\|_{L^2(\Omega)^d}} \\ &\quad + \|\mathbb{A}_\varepsilon^{1/2} \nabla (\mathcal{L}_\varepsilon^1(u_0) - \mathcal{L}_\varepsilon^{1,0}(u_0))\|_{L^2(\Omega)^d}. \end{aligned} \quad (66)$$

Since $\int_{\Omega} \mathbb{A}_{\varepsilon} \nabla u_{\varepsilon} \cdot \nabla w = \int_{\Omega} \mathbb{A}_0 \nabla u_0 \cdot \nabla w$ for any $w \in H_0^1(\Omega)$ in view of (1) and (5), the estimates (65) and (66) lead to

$$\begin{aligned} \|\mathbb{A}_{\varepsilon}^{1/2} \nabla(u_{\varepsilon} - \mathcal{L}_{\varepsilon}^1(u_0))\|_{L^2(\Omega)^d} &\leq \alpha^{-1/2} \sup_{w \in H_0^1(\Omega)} \frac{|\mathcal{F}_{\varepsilon}(w)|}{\|\nabla w\|_{L^2(\Omega)^d}} \\ &\quad + 2\beta^{1/2} \|\nabla(\mathcal{L}_{\varepsilon}^1(u_0) - \mathcal{L}_{\varepsilon}^{1,0}(u_0))\|_{L^2(\Omega)^d}, \end{aligned} \quad (67)$$

recalling that $\mathcal{F}_{\varepsilon}(w) = \int_{\Omega} (\mathbb{A}_{\varepsilon} \nabla \mathcal{L}_{\varepsilon}^1(u_0) - \mathbb{A}_0 \nabla u_0) \cdot \nabla w$. Since we can bound the first term in the right-hand side of (67) using Lemma A.1 (with $D = \Omega$), it remains to estimate the second term. Owing to the definition of ζ_{η} , we infer that

$$\|\nabla(\mathcal{L}_{\varepsilon}^1(u_0) - \mathcal{L}_{\varepsilon}^{1,0}(u_0))\|_{L^2(\Omega)^d} = \varepsilon \left\| \nabla \left((1 - \zeta_{\eta}) \sum_{l=1}^d \mathcal{R}_{\varepsilon}(\mu_l) \partial_l u_0 \right) \right\|_{L^2(\Omega_{\eta})^d}. \quad (68)$$

For any integer $1 \leq i \leq d$, we have

$$\begin{aligned} \partial_i \left((1 - \zeta_{\eta}) \sum_{l=1}^d \mathcal{R}_{\varepsilon}(\mu_l) \partial_l u_0 \right) &= -\partial_i \zeta_{\eta} \sum_{l=1}^d \mathcal{R}_{\varepsilon}(\mu_l) \partial_l u_0 + \frac{(1 - \zeta_{\eta})}{\varepsilon} \sum_{l=1}^d \mathcal{R}_{\varepsilon}(\partial_i \mu_l) \partial_l u_0 \\ &\quad + (1 - \zeta_{\eta}) \sum_{l=1}^d \mathcal{R}_{\varepsilon}(\mu_l) \partial_{i,l}^2 u_0, \end{aligned}$$

and using the properties of the cut-off function ζ_{η} , we infer that

$$\varepsilon \left\| \nabla \left((1 - \zeta_{\eta}) \sum_{l=1}^d \mathcal{R}_{\varepsilon}(\mu_l) \partial_l u_0 \right) \right\|_{L^2(\Omega_{\eta})^d} \leq c \left(|\Omega_{\eta}|^{1/2} \left(\frac{\varepsilon}{\eta} + 1 \right) |u_0|_{W^{1,\infty}(\Omega)} + \varepsilon |u_0|_{H^2(\Omega)} \right).$$

Since $|\Omega_{\eta}| \leq |\partial\Omega|\eta$, and choosing $\eta = \varepsilon$ to minimize the function $\eta \mapsto \frac{\varepsilon}{\sqrt{\eta}} + \sqrt{\eta}$, we can conclude the proof (note that $\rho \geq 1$ by definition). \square

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