



HAL
open science

Reassigning and synchrosqueezing the Stockwell Transform: Complementary proofs

Dominique Fourer, François Auger, Jiabin Hu

► **To cite this version:**

Dominique Fourer, François Auger, Jiabin Hu. Reassigning and synchrosqueezing the Stockwell Transform: Complementary proofs. 2015. hal-01467244

HAL Id: hal-01467244

<https://hal.science/hal-01467244>

Preprint submitted on 14 Feb 2017

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

Reassigning and synchrosqueezing the Stockwell Transform: Complementary proofs

Dominique Fourer, François Auger and Jiabin Hu

January 23, 2017

Contents

1	Relationship between the squared modulus of the S-transform and the Wigner-Ville distribution	1
1.1	Wigner-Ville distribution of a Gaussian analysis window	2
2	Frequency domain expression of the S-transform	2
3	S-transform of particular signals	3
3.1	S-transform of a time-delayed signal	3
3.2	S-transform of a frequency-shifted signal	3
3.3	S-transform of a rescaled signal	4
3.4	S-transform of a sinusoid	4
3.5	Tuning $\omega_0 T$ to compute the S-transform of a sinusoid	4
3.6	S-transform of an impulse signal	5
3.7	Tuning $\omega_0 T$ to compute the S-transform of an impulse	5
4	Marginalization of the S-transform over time	6
5	S-transform simplified reconstruction formula	6
5.1	Proof	6
5.2	Relationship with the Morlet wavelet synthesis formula	6
6	Energy conservation of the S-transform	7
7	Frequency derivative of the phase of the S-transform	8
8	Computation of the reassignment operators using S-transforms	8
8.1	time coordinate	8
8.2	Frequency coordinate	9
9	Computation of the Levenberg-Marquardt reassignment operators using S-transforms	9
9.1	Computation of the partial time derivatives of $R_x(t, \omega)$	10
9.2	Computation of the partial frequency derivatives of $R_x(t, \omega)$	10
10	Definition of the S-transform local instantaneous modulation operator	11
11	Implementation of the reassigned Gabor spectrogram and of the synchrosqueezed STFT	12
11.1	Computation of the spectrogram and the reassigned spectrogram	13
11.2	Computation of the synchrosqueezed STFT	13
11.3	Computation of the vertical synchrosqueezed STFT	14
12	Fourier transform of the Morlet Wavelet	14

1 Relationship between the squared modulus of the S-transform and the Wigner-Ville distribution

$$|\text{ST}_x(t, \omega)|^2 = \text{ST}_x(t, \omega) \cdot \text{ST}_x(t, \omega)^* \quad (1)$$

$$= \frac{\omega^2}{2\pi(\omega_0 T)^2} \iint_{\mathbb{R}^2} x(\tau') x(\tau'')^* e^{-\frac{\omega^2(t-\tau')^2}{2(\omega_0 T)^2}} e^{-\frac{\omega^2(t-\tau'')^2}{2(\omega_0 T)^2}} e^{-j\omega(\tau'-\tau'')} d\tau' d\tau'' \quad (2)$$

As the Wigner-Ville distribution can be seen as the Fourier transform of the instantaneous autocorrelation function of a signal x

$$\text{WV}_x(t, \omega) = \int_{\mathbb{R}} x(t + \frac{\tau}{2}) x(t - \frac{\tau}{2})^* e^{-j\omega\tau} d\tau. \quad (3)$$

Hence, using the inverse Fourier transform we obtain

$$x(t + \frac{\tau}{2}) x(t - \frac{\tau}{2})^* = \int_{\mathbb{R}} \text{WV}_x(t, \Omega) e^{j\Omega\tau} \frac{d\Omega}{2\pi}. \quad (4)$$

By replacing $\tau' = t + \frac{\tau}{2}$ and $\tau'' = t - \frac{\tau}{2}$ we obtain $t = \frac{\tau' + \tau''}{2}$ and $\tau = \tau' - \tau''$ thus we have

$$x(\tau') x(\tau'')^* = \int_{\mathbb{R}} \text{WV}_x\left(\frac{\tau' + \tau''}{2}, \Omega\right) e^{j\Omega(\tau' - \tau'')} \frac{d\Omega}{2\pi}. \quad (5)$$

Similarly with $h(\lambda) = \frac{1}{\sqrt{2\pi}T} e^{-\frac{\lambda^2}{2T^2}}$ we have

$$\text{WV}_h(\lambda, \omega) = \frac{1}{2\pi T^2} \int_{\mathbb{R}} e^{-\frac{(\lambda + \frac{\tau}{2})^2}{2T^2}} e^{-\frac{(\lambda - \frac{\tau}{2})^2}{2T^2}} e^{-j\omega\tau} d\tau \quad (6)$$

($\text{WV}_h(\lambda, \omega)$ is further expressed in Section 1.1) and

$$e^{-\frac{(\lambda + \frac{\tau}{2})^2}{2T^2}} e^{-\frac{(\lambda - \frac{\tau}{2})^2}{2T^2}} = 2\pi T^2 \int_{\mathbb{R}} \text{WV}_h(\lambda, \Omega') e^{j\Omega'\tau} \frac{d\Omega'}{2\pi}. \quad (7)$$

If we define

$$\left(\lambda + \frac{\tau}{2}\right)^2 = \frac{\omega^2}{\omega_0^2} (t - \tau')^2 \quad (8)$$

$$\left(\lambda - \frac{\tau}{2}\right)^2 = \frac{\omega^2}{\omega_0^2} (t - \tau'')^2 \quad (9)$$

we obtain $\lambda = \frac{\omega}{\omega_0} (t - \frac{\tau' + \tau''}{2})$ and $\tau = \frac{\omega}{\omega_0} (\tau'' - \tau')$ thus we have

$$e^{-\frac{\omega^2(t-\tau')^2}{2(\omega_0 T)^2}} e^{-\frac{\omega^2(t-\tau'')^2}{2(\omega_0 T)^2}} = 2\pi T^2 \int_{\mathbb{R}} \text{WV}_h\left(\frac{\omega}{\omega_0} \left(t - \frac{\tau' + \tau''}{2}\right), \Omega'\right) e^{j\Omega' \frac{\omega}{\omega_0} (\tau'' - \tau')} \frac{d\Omega'}{2\pi}. \quad (10)$$

So, (2) can be expressed using Eqs. (5) and (10) as

$$|\text{ST}_x(t, \omega)|^2 = \frac{\omega^2}{\omega_0^2} \iiint_{\mathbb{R}^4} \text{WV}_x\left(\frac{\tau' + \tau''}{2}, \Omega\right) e^{j\Omega(\tau' - \tau'')} \text{WV}_h\left(\frac{\omega}{\omega_0} \left(t - \frac{\tau' + \tau''}{2}\right), \Omega'\right) e^{j\frac{\omega\Omega'}{\omega_0} (\tau'' - \tau') - j\omega(\tau' - \tau'')} \frac{d\Omega}{2\pi} \frac{d\Omega'}{2\pi} d\tau' d\tau''. \quad (11)$$

If we define $\tau_1 = \frac{\tau' + \tau''}{2}$ and $\tau_2 = \tau' - \tau''$, we obtain

$$\begin{pmatrix} \tau' \\ \tau'' \end{pmatrix} = \Phi(\tau_1, \tau_2) = \begin{pmatrix} \tau_1 + \frac{\tau_2}{2} \\ \tau_1 - \frac{\tau_2}{2} \end{pmatrix}, \quad (12)$$

as a result for a multiple substitution integration we obtain

$$d\tau' d\tau'' = |\det J_{\Phi}| d\tau_1 d\tau_2 = d\tau_1 d\tau_2 \quad (13)$$

where $\det J_{\Phi}$ is the determinant of the Jacobian of matrix Φ . As a result, (11) can be expressed as

$$|\text{ST}_x(t, \omega)|^2 = \frac{\omega^2}{\omega_0^2} \iiint_{\mathbb{R}^4} \text{WV}_x(\tau_1, \Omega) \text{WV}_h\left(\frac{\omega}{\omega_0} (t - \tau_1), \Omega'\right) e^{j\tau_2(\Omega - \omega - \frac{\omega\Omega'}{\omega_0})} \frac{d\Omega}{2\pi} \frac{d\Omega'}{2\pi} d\tau_1 d\tau_2 \quad (14)$$

$$= \frac{\omega^2}{\omega_0^2} \iiint_{\mathbb{R}^3} \text{WV}_x(\tau_1, \Omega) \text{WV}_h\left(\frac{\omega}{\omega_0} (t - \tau_1), \Omega'\right) \delta\left(\Omega - \omega - \frac{\omega\Omega'}{\omega_0}\right) \frac{d\Omega}{2\pi} d\Omega' d\tau_1 \quad (15)$$

$$= \frac{|\omega|}{\omega_0} \iiint_{\mathbb{R}^3} \text{WV}_x(\tau_1, \Omega) \text{WV}_h\left(\frac{\omega}{\omega_0} (t - \tau_1), \Omega'\right) \delta\left(\frac{\omega_0\Omega}{\omega} - \omega_0 - \Omega'\right) \frac{d\Omega}{2\pi} d\Omega' d\tau_1 \quad (16)$$

$$= \frac{|\omega|}{\omega_0} \iint_{\mathbb{R}^2} \text{WV}_x(\tau_1, \Omega) \text{WV}_h\left(\frac{\omega}{\omega_0} (t - \tau_1), \frac{\omega_0}{\omega} (\Omega - \omega)\right) \frac{d\Omega}{2\pi} d\tau_1 \quad (17)$$

1.1 Wigner-Ville distribution of a Gaussian analysis window

If we define $h(t) = \frac{1}{\sqrt{2\pi}T} e^{-\frac{t^2}{2T^2}}$, its Wigner-Ville distribution can be expressed as

$$\text{WV}_h(t, \omega) = \int_{\mathbb{R}} h\left(t + \frac{\tau}{2}\right) h\left(t - \frac{\tau}{2}\right)^* e^{-j\omega\tau} d\tau \quad (18)$$

$$= \frac{1}{2\pi T^2} \int_{\mathbb{R}} e^{-\frac{(t+\frac{\tau}{2})^2}{2T^2}} e^{-\frac{(t-\frac{\tau}{2})^2}{2T^2}} e^{-j\omega\tau} d\tau \quad (19)$$

$$= \frac{1}{2\pi T^2} \int_{\mathbb{R}} e^{-\frac{t^2 + \frac{\tau^2}{4}}{T^2}} e^{-j\omega\tau} d\tau \quad (20)$$

$$= \frac{1}{2\pi T^2} e^{-\frac{t^2}{T^2}} \int_{\mathbb{R}} e^{-\frac{\tau^2}{4T^2}} e^{-j\omega\tau} d\tau \quad (21)$$

$$= \frac{1}{\sqrt{\pi}T} e^{-\frac{t^2}{T^2}} e^{-\omega^2 T^2} \quad (22)$$

Thus, from (17) we finally obtain

$$|\text{ST}_x(t, \omega)|^2 = \frac{|\omega|}{\sqrt{\pi}\omega_0 T} \iint_{\mathbb{R}^2} \text{WV}_x(\tau, \Omega) e^{-\frac{\omega^2(t-\tau)^2}{(\omega_0 T)^2}} e^{-\frac{(\omega_0 T)^2(\Omega-\omega)^2}{\omega^2}} d\tau \frac{d\Omega}{2\pi} \quad (23)$$

2 Frequency domain expression of the S-transform

As a signal x can be expressed as the inverse of its Fourier transform

$$x(t) = \int_{-\infty}^{+\infty} F_x(\xi) e^{+j\xi t} \frac{d\xi}{2\pi} \quad (24)$$

thus, $\text{ST}_x(t, \omega)$ can now be expressed as

$$\text{ST}_x(t, \omega) = \frac{|\omega|}{\sqrt{2\pi}\omega_0 T} \iint_{\mathbb{R}^2} F_x(\xi) e^{-\frac{(t-\tau)^2 \omega^2}{2(\omega_0 T)^2}} e^{-j(\omega-\xi)\tau} d\tau \frac{d\xi}{2\pi} \quad (25)$$

$$= \frac{|\omega|}{\omega_0} \iint_{\mathbb{R}^2} F_x(\xi) h\left(\frac{\omega}{\omega_0}(t-\tau)\right) e^{-j(\omega-\xi)\tau} d\tau \frac{d\xi}{2\pi} \quad (26)$$

where $h(t) = \frac{1}{\sqrt{2\pi}T} e^{-\frac{t^2}{2T^2}}$.

The Fourier transform of $g(t, \frac{\omega_0 T}{|\omega|}) = \frac{|\omega|}{\omega_0} h\left(\frac{\omega}{\omega_0}(t-\tau)\right)$ can be expressed as

$$F_g(\Omega) = \frac{|\omega|}{\omega_0} \int_{-\infty}^{+\infty} h\left(\frac{\omega}{\omega_0}(t-\tau)\right) e^{-j\Omega t} dt \quad (27)$$

$$= e^{-j\Omega\tau} \int_{-\infty}^{+\infty} h(\alpha) e^{-j\frac{\omega_0}{\omega}\Omega\alpha} d\alpha = e^{-j\Omega\tau} F_h\left(\frac{\omega_0}{\omega}\Omega\right) \quad (28)$$

Thus (26) can now be expressed as

$$\text{ST}_x(t, \omega) = \iiint_{\mathbb{R}^3} F_x(\xi) F_h\left(\frac{\omega_0}{\omega}\Omega\right) e^{j(\Omega(t-\tau))} e^{j(\xi-\omega)\tau} d\tau \frac{d\xi}{2\pi} \frac{d\Omega}{2\pi} \quad (29)$$

$$= \iint_{\mathbb{R}^2} F_x(\xi) F_h\left(\frac{\omega_0}{\omega}\Omega\right) e^{j\Omega t} \int_{-\infty}^{+\infty} e^{j(\xi-\Omega-\omega)\tau} d\tau \frac{d\xi}{2\pi} \frac{d\Omega}{2\pi} \quad (30)$$

$$= \iint_{\mathbb{R}^2} F_x(\xi) F_h\left(\frac{\omega_0}{\omega}\Omega\right) e^{j\Omega t} \delta(\xi - \Omega - \omega) d\xi \frac{d\Omega}{2\pi} \quad (31)$$

$$= e^{-j\omega t} \int_{-\infty}^{+\infty} F_x(\xi) F_h\left(\frac{\omega_0}{\omega}(\xi - \omega)\right) e^{j\xi t} \frac{d\xi}{2\pi} \quad (32)$$

$$= \int_{-\infty}^{+\infty} F_x(\Omega + \omega) F_h\left(\frac{\omega_0}{\omega}\Omega\right) e^{j\Omega t} \frac{d\Omega}{2\pi} \quad (33)$$

3 S-transform of particular signals

3.1 S-transform of a time-delayed signal

If $y(t) = x(t - t_1)$, then we obtain

$$\text{ST}_y(t, \omega) = \frac{|\omega| e^{-j\omega t}}{\sqrt{2\pi\omega_0 T}} \int_{-\infty}^{+\infty} x(t + \tau - t_1) e^{-\frac{\omega^2 \tau^2}{2(\omega_0 T)^2}} e^{-j\omega\tau} d\tau$$

and

$$\text{ST}_x(t - t_1, \omega) = \frac{|\omega| e^{-j\omega(t-t_1)}}{\sqrt{2\pi\omega_0 T}} \int_{-\infty}^{+\infty} x(t - t_1 + \tau) e^{-\frac{\omega^2 \tau^2}{2(\omega_0 T)^2}} e^{-j\omega\tau} d\tau \quad (34)$$

$$= \text{ST}_y(t, \omega) e^{-j\omega t_1}. \quad (35)$$

Thus, we finally deduce

$$\text{ST}_y(t, \omega) = \text{ST}_x(t - t_1, \omega) e^{j\omega t_1}. \quad (36)$$

3.2 S-transform of a frequency-shifted signal

If we define $y(t) = x(t) e^{j\omega_1 t}$, then using $F_y(\omega) = F_x(\omega - \omega_1)$ we obtain

$$\text{ST}_y(t, \omega) = \int_{-\infty}^{+\infty} F_x(\omega - \omega_1 + \Omega) F_h\left(\frac{\omega_0 \Omega}{\omega}\right) e^{j\Omega t} \frac{d\Omega}{2\pi} \quad (37)$$

As,

$$\text{ST}_x(t, \omega - \omega_1) = \int_{-\infty}^{+\infty} F_x(\omega - \omega_1 + \Omega) F_h\left(\frac{\omega_0 \Omega}{\omega - \omega_1}\right) e^{j\Omega t} \frac{d\Omega}{2\pi} \quad (38)$$

we finally deduce that

$$\text{ST}_y(t, \omega) \neq \text{ST}_x(t, \omega - \omega_1). \quad (39)$$

3.3 S-transform of a rescaled signal

If we define $z(t) = \frac{1}{\sqrt{s}} x\left(\frac{t}{s}\right)$, then we have

$$\text{ST}_z(t, \omega) = \frac{|\omega|}{\sqrt{2\pi\omega_0 T}} \int_{-\infty}^{+\infty} \frac{1}{\sqrt{s}} x\left(\frac{\tau}{s}\right) e^{-\frac{\omega^2 (t-\tau)^2}{2(\omega_0 T)^2}} e^{-j\omega\tau} d\tau \quad (40)$$

$$= \frac{|\omega|}{\sqrt{2\pi\omega_0 T}} \int_{-\infty}^{+\infty} \frac{1}{\sqrt{s}} x(\tau') e^{-\frac{\omega^2 (t-s\tau')^2}{2(\omega_0 T)^2}} e^{-j\omega s\tau'} s d\tau' \quad (41)$$

$$= \frac{1}{\sqrt{s}} \frac{s|\omega|}{\sqrt{2\pi\omega_0 T}} \int_{-\infty}^{+\infty} x(\tau') e^{-\frac{s^2 \omega^2 (t-s\tau')^2}{2(\omega_0 T)^2}} e^{-js\omega\tau'} d\tau' \quad (42)$$

$$= \frac{1}{\sqrt{s}} \text{ST}_x\left(\frac{t}{s}, s\omega\right). \quad (43)$$

3.4 S-transform of a sinusoid

If we consider a signal $x(t) = A e^{j\omega_1 t}$, we have $F_x(\omega) = 2\pi A \delta(\omega - \omega_1)$. Hence, the S-transform of x can be expressed as

$$\text{ST}_x(t, \omega) = A \int_{-\infty}^{+\infty} \delta(\Omega + \omega - \omega_1) F_h\left(\frac{\omega_0 \Omega}{\omega}\right) e^{j\Omega t} d\Omega \quad (44)$$

$$= A F_h\left(\frac{\omega_0}{\omega} (\omega_1 - \omega)\right) e^{j(\omega_1 - \omega)t} \quad (45)$$

$$= A e^{-\frac{(\omega_1 - \omega)^2 (\omega_0 T)^2}{2\omega^2}} e^{j(\omega_1 - \omega)t}. \quad (46)$$

Thus, we finally deduce

$$\text{ST}_x(t, \omega_1) = A \quad (47)$$

3.5 Tuning $\omega_0 T$ to compute the S-transform of a sinusoid

Let's begin with the S-transform of a sinusoid expressed as $ST_x(t, \omega) = A e^{-\frac{(\omega_1 - \omega)^2 (\omega_0 T)^2}{2\omega^2}} e^{j(\omega_1 - \omega)t}$. If one wants the set $\{\omega, |ST_x(t, \omega)| > \Gamma\}$ with $\Gamma < 1$, to have a width $\Delta\omega_1$ around ω_1 , we have to find the two values ω and ω' that verifies $\Delta\omega_1 = \omega - \omega' > 0$ (with $\omega > \omega'$) obtained by solving the equation

$$e^{-\frac{(\omega_1 - \omega)^2 (\omega_0 T)^2}{2\omega^2}} = \Gamma \quad (48)$$

$$-\frac{(\omega_1 - \omega)^2 (\omega_0 T)^2}{2\omega^2} = \log(\Gamma) \quad (49)$$

$$(\omega_1 - \omega)^2 = \frac{2 \log(1/\Gamma)}{(\omega_0 T)^2} \omega^2 \quad (50)$$

$$\left(1 - \frac{2 \log(1/\Gamma)}{(\omega_0 T)^2}\right) \omega^2 - 2\omega_1 \omega + \omega_1^2 = 0 \quad (51)$$

Thus we obtain the solutions

$$\omega = \frac{\omega_1}{1 - \frac{\sqrt{2 \log(1/\Gamma)}}{\omega_0 T}} = \frac{\omega_1}{1 - \alpha} \quad \text{with } \alpha = \frac{\sqrt{2 \log(1/\Gamma)}}{\omega_0 T} \quad (52)$$

$$\omega' = \frac{\omega_1}{1 + \frac{\sqrt{2 \log(1/\Gamma)}}{\omega_0 T}} = \frac{\omega_1}{1 + \alpha}. \quad (53)$$

Thus we have,

$$\Delta\omega_1 = \omega - \omega' = \frac{\omega_1}{1 - \alpha} - \frac{\omega_1}{1 + \alpha} = \frac{2\alpha}{1 - \alpha^2} \omega_1 = \frac{2 \sqrt{2 \log(1/\Gamma)}}{1 - \frac{2 \log(1/\Gamma)}{\omega_0 T^2}} \omega_1. \quad (54)$$

that leads to

$$1 - \frac{2 \log(1/\Gamma)}{(\omega_0 T)^2} = \frac{\omega_1}{\Delta\omega_1} 2 \frac{\sqrt{2 \log(1/\Gamma)}}{\omega_0 T} \quad (55)$$

$$(\omega_0 T)^2 - 2 \log(1/\Gamma) = \frac{\omega_1}{\Delta\omega_1} 2 \sqrt{2 \log(1/\Gamma)} \omega_0 T \quad (56)$$

$$(\omega_0 T)^2 - \frac{\omega_1}{\Delta\omega_1} 2 \sqrt{2 \log(1/\Gamma)} \omega_0 T - 2 \log(1/\Gamma) = 0 \quad (57)$$

So finally, as $\omega_0 T > 0$, we obtain a unique solution

$$\omega_0 T = \sqrt{2 \log(1/\Gamma)} \left(\sqrt{1 + \frac{\omega_1^2}{\Delta\omega_1^2}} + \frac{\omega_1}{\Delta\omega_1} \right) \quad (58)$$

3.6 S-transform of an impulse signal

If $x(t) = \delta(t - t_1)$ then we obtain

$$ST_x(t, \omega) = \frac{|\omega|}{\omega_0} \int_{-\infty}^{+\infty} \delta(\tau - t_1) h \left(\frac{\omega}{\omega_0} (t - \tau) \right) e^{-j\omega\tau} d\tau \quad (59)$$

$$= \frac{|\omega|}{\omega_0} h \left(\frac{\omega}{\omega_0} (t - t_1) \right) e^{-j\omega t_1} \quad (60)$$

$$= \frac{|\omega|}{\sqrt{2\pi}\omega_0 T} e^{-\frac{\omega^2 (t - t_1)^2}{2(\omega_0 T)^2}} e^{-j\omega t_1} \quad (61)$$

Thus, we finally deduce

$$ST_x(t_1, \omega) = \frac{|\omega|}{\sqrt{2\pi}\omega_0 T} e^{-j\omega t_1} \quad (62)$$

3.7 Tuning $\omega_0 T$ to compute the S-transform of an impulse

Let's begin with the S-transform of an impulse located at instant t_1 expressed as $ST_x(t, \omega) = \frac{|\omega|}{\sqrt{2\pi\omega_0 T}} e^{-\frac{\omega^2(t-t_1)^2}{2(\omega_0 T)^2}} e^{-j\omega t_1}$. If one wants the set $\{t, \frac{ST_x(t, \omega)}{ST_x(t_1, \omega)} > \Gamma\}$ with $\Gamma < 1$, to have a width Δt_1 around t_1 , we have to find the two values t and t' that verifies $\Delta t_1 = t - t'$ obtained by solving the equation

$$\frac{ST_x(t, \omega)}{ST_x(t_1, \omega)} = e^{-\frac{\omega^2(t-t_1)^2}{2(\omega_0 T)^2}} = \Gamma \quad (63)$$

that leads to

$$-\frac{\omega^2(t-t_1)^2}{2(\omega_0 T)^2} = \log(\Gamma) \quad (64)$$

$$(t-t_1)^2 = 2\log(1/\Gamma) \frac{(\omega_0 T)^2}{\omega^2} \quad (65)$$

$$t^2 - 2tt_1 + t_1^2 - 2\log(1/\Gamma) \frac{(\omega_0 T)^2}{\omega^2} = 0. \quad (66)$$

From the solutions

$$t = \frac{t + \alpha}{2} \quad \text{with } \alpha = 2\sqrt{2\log(1/\Gamma)} \frac{\omega_0 T}{\omega} \quad (67)$$

$$t' = \frac{t - \alpha}{2} \quad (68)$$

$$(69)$$

one can deduce

$$\Delta t_1 = t - t' = 2\sqrt{2\log(1/\Gamma)} \frac{\omega_0 T}{\omega} \quad (70)$$

So finally we have

$$\omega_0 T = \frac{\omega \Delta t_1}{2\sqrt{2\log(1/\Gamma)}} \quad (71)$$

4 Marginalization of the S-transform over time

$$\int_{-\infty}^{+\infty} ST(t, \omega) dt = \int_{-\infty}^{+\infty} x(\tau) e^{-j\omega\tau} \underbrace{\left[\frac{|\omega|}{\sqrt{2\pi\omega_0 T}} \int_{-\infty}^{+\infty} e^{-\frac{(t-\tau)^2\omega^2}{2(\omega_0 T)^2}} dt \right]}_1 d\tau \quad (72)$$

$$= \int_{-\infty}^{+\infty} x(\tau) e^{-j\omega\tau} d\tau = F_x(\omega) \quad (73)$$

5 S-transform simplified reconstruction formula

5.1 Proof

The S-transform simplified reconstruction formula leads to the following result

$$\int_{-\infty}^{+\infty} ST_x(t, \omega) e^{j\omega t} \frac{d\omega}{|\omega|} = \iint_{\mathbb{R}^2} F_x(\Omega + \omega) F_h\left(\frac{\omega_0}{\omega} \Omega\right) e^{j\Omega t} e^{j\omega t} \frac{d\Omega}{2\pi} \frac{d\omega}{|\omega|} \quad (74)$$

$$= \iint_{\mathbb{R}^2} F_x(\Omega') e^{j\Omega' t} F_h\left(\frac{\omega_0}{\omega} (\Omega' - \omega)\right) \frac{d\Omega'}{2\pi} \frac{d\omega}{|\omega|} \quad (75)$$

Using the following variables substitution with the corresponding Jacobian matrix

$$\Phi(\xi, \Omega') = \begin{pmatrix} \frac{\omega_0 \Omega'}{\xi} \\ \Omega' \end{pmatrix} \quad (76)$$

$$J_{\Phi}(\xi, \Omega') = \begin{pmatrix} -\frac{\omega_0 \Omega'}{\xi^2} & \frac{\omega_0}{\xi} \\ 0 & 1 \end{pmatrix} \quad (77)$$

we obtain

$$\int_{-\infty}^{+\infty} \text{ST}_x(t, \omega) \mathbf{e}^{j\omega t} \frac{d\omega}{|\omega|} = \iint_{\mathbb{R}^2} F_x(\Omega') \mathbf{e}^{j\Omega' t} F_h(\xi - \omega_0) \left(\frac{|\omega_0 \Omega'|}{\xi^2} \frac{|\xi|}{|\omega_0 \Omega'|} \right) \frac{d\Omega'}{2\pi} d\xi \quad (78)$$

$$= \int_{-\infty}^{+\infty} F_x(\Omega') \mathbf{e}^{j\Omega' t} \frac{d\Omega'}{2\pi} \cdot \int_{-\infty}^{+\infty} F_h(\xi - \omega_0) \frac{d\xi}{|\xi|} \quad (79)$$

$$= x(t) \cdot C_h(\omega_0 T) \quad (80)$$

$$\text{with } C_h(\omega_0 T) = \int_{-\infty}^{+\infty} \mathbf{e}^{-\frac{(\xi T - \omega_0 T)^2}{2}} \frac{d\xi}{|\xi|} = \int_{-\infty}^{+\infty} \mathbf{e}^{-\frac{(x - \omega_0 T)^2}{2}} \frac{dx}{|x|}. \quad (81)$$

Hence, x can be recovered from ST_x using the simplified S-transform inversion formula

$$x(t) = \frac{1}{C_h(\omega_0 T)} \int_{-\infty}^{+\infty} \text{ST}_x(t, \omega) \mathbf{e}^{j\omega t} \frac{d\omega}{|\omega|}. \quad (82)$$

5.2 Relationship with the Morlet wavelet synthesis formula

For the wavelet transform of a signal x expressed as

$$W_x(t, s) = \frac{1}{\sqrt{|s|}} \int_{\mathbb{R}} x(\tau) \Psi\left(\frac{\tau - t}{s}\right)^* d\tau \quad (83)$$

the simplified wavelet reconstruction formula is given by

$$\hat{x}(t) = \frac{1}{C_\Psi} \int_{\mathbb{R}} W_x(t, s) |s|^{-3/2} ds \quad (84)$$

$$= \frac{1}{\sqrt{\omega_0} C_\Psi} \int_{\mathbb{R}} \text{CW}_x(t, \omega) \frac{d\omega}{\sqrt{|\omega|}} \quad (85)$$

with $s = \frac{\omega_0}{\omega}$ for $\omega_0 > 0$ and $\text{CW}_x(t, \omega) = W_x(t, \frac{\omega_0}{\omega})$.

For the Morlet wavelet transform denoted $\text{MW}_x(t, \omega)$ (*i.e.* $\text{MW}_x(t, \omega) = \text{CW}_x(t, \omega)$ when $\Psi(t) = \frac{\pi^{-1/4}}{\sqrt{T}} \mathbf{e}^{-\frac{t^2}{2T^2}} \mathbf{e}^{j\omega_0 t}$), thus using $F_\Psi(\omega) = \sqrt{2T} \pi^{1/4} \mathbf{e}^{-\frac{(\omega - \omega_0)^2 T^2}{2}}$ (see Section 12) we have

$$C_\Psi = \sqrt{2T} \pi^{1/4} \int_{\mathbb{R}} \mathbf{e}^{-\frac{(\omega - \omega_0)^2 T^2}{2}} \frac{d\omega}{|\omega|} \quad (86)$$

$$= \sqrt{2T} \pi^{1/4} \int_{\mathbb{R}} \mathbf{e}^{-\frac{(\omega T - \omega_0 T)^2}{2}} \frac{d\omega}{|\omega|} \quad (87)$$

$$= \sqrt{2T} \pi^{1/4} \int_{\mathbb{R}} \mathbf{e}^{-\frac{(x - \omega_0 T)^2}{2}} \frac{dx}{|x|} \quad (88)$$

$$= \sqrt{2T} \pi^{1/4} C_h(\omega_0 T) \quad (89)$$

Using the relation between the S-transform and the Morlet wavelet transform expressed as $\text{MW}_x(t, \omega) = \sqrt{2\omega_0 T} \pi^{1/4} \frac{1}{\sqrt{|\omega|}} \mathbf{e}^{j\omega t} \text{ST}_x(t, \omega)$ (see the main article) and using Eq. (89), we obtain from Eq. (85)

$$\hat{x}(t) = \frac{1}{\sqrt{2\omega_0 T} \pi^{1/4} C_h(\omega_0 T)} \int_{\mathbb{R}} \text{MW}_x(t, \omega) \frac{d\omega}{\sqrt{|\omega|}} \quad (90)$$

$$= \frac{1}{C_h(\omega_0 T)} \int_{\mathbb{R}} \text{ST}_x(t, \omega) \mathbf{e}^{j\omega t} \frac{d\omega}{|\omega|} \quad (91)$$

that is the simplified S-transform reconstruction formula given by Eq. (82).

As the Morlet wavelet is considered to be approximately analytic ($F_\Psi(\omega) \approx 0$ for $\omega < 0$ when $\omega_0 T \gg 0$), thus we have $|C_\Psi s i| < \infty$. Using the relationship of $C_h(\omega_0 T)$ and $B_h(\omega_0 T)$ with C_Ψ , one can deduce that $|C_h(\omega_0 T)| < \infty$ and $|B_h(\omega_0 T)| < \infty$.

6 Energy conservation of the S-transform

Starting from the frequency domain expression of the S-transform given by Eq.(33)

$$\text{ST}_x(t, \omega) = \int_{-\infty}^{+\infty} F_x(\omega + \Omega) F_h\left(\frac{\omega_0}{\omega} \Omega\right) e^{j\Omega t} \frac{d\Omega}{2\pi}, \quad (92)$$

the Stockwellogram can be expressed as

$$|\text{ST}_x(t, \omega)|^2 = \iint_{\mathbb{R}^2} F_x(\omega + \Omega_1) F_x(\omega + \Omega_2)^* F_h\left(\frac{\omega_0}{\omega} \Omega_1\right) F_h\left(\frac{\omega_0}{\omega} \Omega_2\right)^* e^{-j(\Omega_2 - \Omega_1)t} \frac{d\Omega_1}{2\pi} \frac{d\Omega_2}{2\pi} \quad (93)$$

Hence, when the Stockwellogram is marginalized over time, we obtain

$$\int_{-\infty}^{+\infty} |\text{ST}_x(t, \omega)|^2 dt = \iint_{\mathbb{R}^2} F_x(\omega + \Omega_1) F_x(\omega + \Omega_2)^* F_h\left(\frac{\omega_0}{\omega} \Omega_1\right) F_h\left(\frac{\omega_0}{\omega} \Omega_2\right)^* \delta(\Omega_2 - \Omega_1) \frac{d\Omega_1}{2\pi} d\Omega_2 \quad (94)$$

$$= \int_{-\infty}^{+\infty} |F_x(\omega + \Omega')|^2 |F_h\left(\frac{\omega_0}{\omega} \Omega'\right)|^2 \frac{d\Omega'}{2\pi} \quad (95)$$

$$= \int_{-\infty}^{+\infty} |F_x(\Omega)|^2 |F_h\left(\frac{\omega_0}{\omega} (\Omega - \omega)\right)|^2 \frac{d\Omega}{2\pi} \quad (96)$$

$$= \int_{-\infty}^{+\infty} |F_x(\Omega)|^2 e^{-\frac{(\omega_0 T)^2 (\Omega - \omega)^2}{\omega^2}} \frac{d\Omega}{2\pi} \quad (97)$$

that corresponds to a smoothed version of the energy spectral density of the analyzed signal x .

Now, one can calculate

$$\iint_{\mathbb{R}^2} |\text{ST}_x(t, \omega)|^2 dt \frac{d\omega}{|\omega|} = \iint_{\mathbb{R}^2} |F_x(\Omega)|^2 |F_h\left(\frac{\omega_0 \Omega}{\omega} - \omega_0\right)|^2 \frac{d\Omega}{2\pi} \frac{d\omega}{|\omega|} \quad (98)$$

Using the following variables substitution with the corresponding Jacobian matrix

$$\Phi(\omega', \Omega') = \begin{pmatrix} \frac{\Omega_0 \Omega'}{\omega'} \\ \Omega' \end{pmatrix} \quad (99)$$

$$J_{\Phi}(\omega', \Omega') = \begin{pmatrix} \frac{-\omega_0 \Omega'}{\omega'^2} & \frac{\omega_0}{\omega'} \\ 0 & 1 \end{pmatrix}, \quad (100)$$

we obtain

$$\iint_{\mathbb{R}^2} |\text{ST}_x(t, \omega)|^2 dt \frac{d\omega}{|\omega|} = \int_{-\infty}^{+\infty} |F_h(\omega' - \omega_0)|^2 \frac{d\omega'}{|\omega'|} \int_{-\infty}^{+\infty} |F_x(\Omega)|^2 \frac{d\Omega}{2\pi} \quad (101)$$

$$= B_h(\omega_0 T) E_x \quad (102)$$

So finally, the Parseval's theorem leads to

$$E_x = \int_{-\infty}^{+\infty} |x(t)|^2 dt = \int_{-\infty}^{+\infty} |F_x(\omega)|^2 \frac{d\omega}{2\pi} = \frac{1}{B_h(\omega_0 T)} \iint_{\mathbb{R}^2} |\text{ST}_x(t, \omega)|^2 dt \frac{d\omega}{|\omega|} \quad (103)$$

where the proportionality factor $B_h(\omega_0 T)$ is given by

$$B_h(\omega_0 T) = \int_{-\infty}^{+\infty} |F_h(\omega - \omega_0)|^2 \frac{d\omega}{|\omega|} = \int_{-\infty}^{+\infty} e^{-(\omega - \omega_0)^2 T^2} \frac{d\omega}{|\omega|} = \int_{-\infty}^{+\infty} e^{-\omega_0 (x-1)^2 T^2} \frac{dx}{|x|} = \int_{-\infty}^{+\infty} |F_h(\omega_0 (x-1))|^2 \frac{dx}{|x|} \quad (104)$$

7 Frequency derivative of the phase of the S-transform

Let define the S-transform as a particular case of the Short-Time Fourier Transform (STFT) using the window

$$g(t, \omega) = \frac{|\omega|}{\sqrt{2\pi\omega_0 T}} e^{-\frac{\omega^2 t^2}{2(\omega_0 T)^2}}$$

$$\text{ST}_x(t, \omega) = M_x(t, \omega) e^{j\Phi_x(t, \omega)} = \int_{\mathbb{R}} x(\tau) g(t - \tau, \omega) e^{-j\omega\tau} d\tau \quad (105)$$

When $\omega \neq 0$, we have

$$\frac{\partial \text{ST}_x}{\partial \omega}(t, \omega) = \frac{\partial M_x}{\partial \omega}(t, \omega) e^{j\Phi_x(t, \omega)} + j \frac{\partial \Phi_x}{\partial \omega}(t, \omega) \text{ST}_x(t, \omega) \quad (106)$$

$$= \frac{\partial}{\partial \omega} \left[e^{-j\omega t} \int_{\mathbb{R}} x(\tau) g(t - \tau, \frac{\omega_0}{|\omega|} T) e^{j\omega(t-\tau)} d\tau \right] \quad (107)$$

$$= -jt \text{ST}_x(t, \omega) + e^{-j\omega t} \left[\int_{\mathbb{R}} x(\tau) \frac{\partial g}{\partial \omega}(t - \tau, \frac{\omega_0}{|\omega|} T) e^{j\omega(t-\tau)} d\tau + j \int_{\mathbb{R}} x(\tau) (t - \tau) g(t - \tau, \frac{\omega_0}{|\omega|} T) e^{j\omega(t-\tau)} d\tau \right] \quad (108)$$

$$= -jt \text{ST}_x(t, \omega) + j \text{ST}_x^{\mathcal{T}g}(t, \omega) + \frac{\omega}{|\omega| \sqrt{2\pi} \omega_0 T} \int_{\mathbb{R}} x(\tau) e^{-\frac{\omega^2(t-\tau)^2}{2(\omega_0 T)^2}} e^{-j\omega\tau} d\tau - \frac{|\omega| \omega}{\sqrt{2\pi} (\omega_0 T)^3} \int_{\mathbb{R}} x(\tau) (t - \tau)^2 e^{-\frac{\omega^2(t-\tau)^2}{2(\omega_0 T)^2} - j\omega\tau} d\tau \quad (109)$$

$$= -jt \text{ST}_x(t, \omega) + j \text{ST}_x^{\mathcal{T}g}(t, \omega) + \frac{\omega}{|\omega|^2} \text{ST}_x(t, \omega) - \frac{\omega}{(\omega_0 T)^2} \text{ST}_x^{\mathcal{T}^2g}(t, \omega). \quad (110)$$

Thus, we have

$$\frac{\frac{\partial \text{ST}_x}{\partial \omega}(t, \omega)}{\text{ST}_x(t, \omega)} = \frac{1}{M_x(t, \omega)} \frac{\partial M_x}{\partial \omega}(t, \omega) + j \frac{\partial \Phi_x}{\partial \omega}(t, \omega) = -jt + j \frac{\text{ST}_x^{\mathcal{T}g}(t, \omega)}{\text{ST}_x(t, \omega)} + \frac{\omega}{|\omega|^2} - \frac{\omega}{(\omega_0 T)^2} \frac{\text{ST}_x^{\mathcal{T}^2g}(t, \omega)}{\text{ST}_x(t, \omega)} \quad (111)$$

so finally we deduce

$$\frac{\partial \Phi_x}{\partial \omega}(t, \omega) = \text{Im} \left(\frac{\frac{\partial \text{ST}_x}{\partial \omega}(t, \omega)}{\text{ST}_x(t, \omega)} \right) = -t + \text{Re} \left(\frac{\text{ST}_x^{\mathcal{T}g}(t, \omega)}{\text{ST}_x(t, \omega)} \right) - \text{Im} \left(\frac{\omega}{(\omega_0 T)^2} \frac{\text{ST}_x^{\mathcal{T}^2g}(t, \omega)}{\text{ST}_x(t, \omega)} \right) \quad (112)$$

8 Computation of the reassignment operators using S-transforms

8.1 time coordinate

We consider first the definition of the reassigned time coordinates

$$\hat{t}(t, \omega) = \frac{\iint_{\mathbb{R}^2} \tau \text{WV}_x(\tau, \Omega) \text{WV}_h(\frac{\omega}{\omega_0}(t - \tau), \frac{\omega_0}{\omega}(\Omega - \omega)) d\tau d\Omega}{\iint_{\mathbb{R}^2} \text{WV}_x(\tau, \Omega) \text{WV}_h(\frac{\omega}{\omega_0}(t - \tau), \frac{\omega_0}{\omega}(\Omega - \omega)) d\tau d\Omega} \quad (113)$$

$$= \text{Re} \left(\frac{\iint_{\mathbb{R}^2} \tau \text{Ri}_x(\tau, \Omega) \text{Ri}_h(\frac{\omega}{\omega_0}(t - \tau), \frac{\omega_0}{\omega}(\Omega - \omega)) d\tau d\Omega}{\iint_{\mathbb{R}^2} \text{Ri}_x(\tau, \Omega) \text{Ri}_h(\frac{\omega}{\omega_0}(t - \tau), \frac{\omega_0}{\omega}(\Omega - \omega)) d\tau d\Omega} \right) \quad (114)$$

where $\text{Ri}(t, \omega) = x(t) F_x(\omega)^* e^{-j\omega t}$ is the Rihaczek distribution. Thus we have

$$\hat{t}(t, \omega) = \frac{1}{\frac{\omega_0}{|\omega|} |\text{ST}_x(t, \omega)|^2} \text{Re} \left(\int_{\mathbb{R}} x(\tau) \tau h(\frac{\omega}{\omega_0}(t - \tau)) e^{-j\omega\tau} d\tau \cdot \int_{\mathbb{R}} F_x(\Omega)^* F_h(\frac{\omega_0}{\omega}(\Omega - \omega))^* e^{-j(\Omega - \omega)t} \frac{d\Omega}{2\pi} \right) \quad (115)$$

$$= t - \text{Re} \left(\frac{\int_{\mathbb{R}} x(\tau) (t - \tau) h(\frac{\omega}{\omega_0}(t - \tau)) e^{-j\omega\tau} d\tau \text{ST}_x(t, \omega)^*}{\frac{\omega_0}{|\omega|} |\text{ST}_x(t, \omega)|^2} \right) \quad (116)$$

$$= t - \text{Re} \left(\frac{\text{ST}_x^{\mathcal{T}g}(t, \omega)}{\text{ST}_x(t, \omega)} \right) \quad (117)$$

8.2 Frequency coordinate

We consider first the definition of the reassigned frequency coordinates

$$\hat{\omega}(t, \omega) = \frac{\iint_{\mathbb{R}^2} \Omega \text{WV}_x(\tau, \Omega) \text{WV}_h\left(\frac{\omega}{\omega_0}(t - \tau), \frac{\omega_0}{\omega}(\Omega - \omega)\right) d\tau d\Omega}{\iint_{\mathbb{R}^2} \text{WV}_x(\tau, \Omega) \text{WV}_h\left(\frac{\omega}{\omega_0}(t - \tau), \frac{\omega_0}{\omega}(\Omega - \omega)\right) d\tau d\Omega} \quad (118)$$

$$= \text{Re} \left(\frac{\iint_{\mathbb{R}^2} \Omega \text{Ri}_x(\tau, \Omega) \text{Ri}_h\left(\frac{\omega}{\omega_0}(t - \tau), \frac{\omega_0}{\omega}(\Omega - \omega)\right) d\tau d\Omega}{\iint_{\mathbb{R}^2} \text{Ri}_x(\tau, \Omega) \text{Ri}_h\left(\frac{\omega}{\omega_0}(t - \tau), \frac{\omega_0}{\omega}(\Omega - \omega)\right) d\tau d\Omega} \right) \quad (119)$$

$$= \frac{1}{|\omega| |\text{ST}_x(t, \omega)|^2} \text{Re} \left(\int_{\mathbb{R}} x(\tau) h\left(\frac{\omega}{\omega_0}(t - \tau)\right) e^{-j\omega\tau} d\tau \cdot \int_{\mathbb{R}} \Omega F_x(\Omega)^* F_h\left(\frac{\omega_0}{\omega}(\Omega - \omega)\right) e^{-(\Omega - \omega)t} \frac{d\Omega}{2\pi} \right) \quad (120)$$

$$= \omega + \text{Re} \left(\frac{\text{ST}_x(t, \omega)}{|\text{ST}_x(t, \omega)|^2} \cdot \int_{\mathbb{R}} (\Omega - \omega) F_x(\Omega)^* F_h\left(\frac{\omega_0}{\omega}(\Omega - \omega)\right)^* e^{-(\Omega - \omega)t} \frac{d\Omega}{2\pi} \right) \quad (121)$$

$$= \omega + \text{Re} \left(\frac{\text{ST}_x(t, \omega)}{|\text{ST}_x(t, \omega)|^2} \cdot \int_{\mathbb{R}} (\Omega - \omega) F_x(\Omega)^* F_g(\Omega - \omega)^* e^{-(\Omega - \omega)t} \frac{d\Omega}{2\pi} \right) \quad (122)$$

Using the following property

$$F_h\left(\frac{\omega_0}{\omega}(\Omega - \omega)\right) = \int_{\mathbb{R}} h(t) e^{-j\frac{\omega_0}{\omega}(\Omega - \omega)t} dt = \frac{|\omega|}{\omega_0} \int_{\mathbb{R}} h\left(\frac{\omega}{\omega_0}\tau\right) e^{-j(\Omega - \omega)\tau} d\tau \quad (123)$$

$$= \int_{\mathbb{R}} g(\tau, \sigma_t) e^{-j(\Omega - \omega)\tau} d\tau = F_g(\Omega - \omega) \quad (124)$$

If we define $\mathcal{D}g(t) = \frac{dg}{dt}(t) = \frac{-t}{\sigma_t^2} g(t)$ and we have $F_{\mathcal{D}g}(\omega) = j\omega F_g(\omega)$

Thus using $F_g(\omega) = -j\frac{1}{\omega} F_{\mathcal{D}g}(\omega)$, we obtain

$$\hat{\omega}(t, \omega) = \omega - \text{Re} \left(j \frac{\text{ST}_x(t, \omega)}{|\text{ST}_x(t, \omega)|^2} \cdot \int_{\mathbb{R}} F_x(\Omega)^* F_{\mathcal{D}g}(\Omega - \omega)^* e^{-(\Omega - \omega)t} \frac{d\Omega}{2\pi} \right) \quad (125)$$

$$= \omega - \text{Im} \left(\frac{\text{ST}_x^{\mathcal{D}g}(t, \omega)^*}{\text{ST}_x(t, \omega)^*} \right) \quad (126)$$

$$= \omega + \text{Im} \left(\frac{\text{ST}_x^{\mathcal{D}g}(t, \omega)}{\text{ST}_x(t, \omega)} \right) \quad (127)$$

9 Computation of the Levenberg-Marquardt reassignment operators using S-transforms

Starting from the Stockwellogram classical reassignment operators given by

$$\hat{t}(t, \omega) = t - \text{Re} \left(\frac{\text{ST}_x^{\mathcal{T}g}(t, \omega)}{\text{ST}_x(t, \omega)} \right) \quad (128)$$

$$\hat{\omega}(t, \omega) = \omega + \text{Im} \left(\frac{\text{ST}_x^{\mathcal{D}g}(t, \omega)}{\text{ST}_x(t, \omega)} \right) \quad (129)$$

and the definition of the Levenberg-Marquardt reassignment operators

$$\begin{pmatrix} \hat{t}(t, \omega) \\ \hat{\omega}(t, \omega) \end{pmatrix} = \begin{pmatrix} t \\ \omega \end{pmatrix} - (\nabla^t R_x(t, \omega) + \mu I_2)^{-1} R_x(t, \omega) \quad (130)$$

$$\text{with } R_x(t, \omega) = \begin{pmatrix} t \\ \omega \end{pmatrix} - \begin{pmatrix} \hat{t}(t, \omega) \\ \hat{\omega}(t, \omega) \end{pmatrix} \quad (131)$$

$$\nabla^t R_x(t, \omega) = \left(\frac{\partial R_x}{\partial t}(t, \omega) \quad \frac{\partial R_x}{\partial \omega}(t, \omega) \right) \quad (132)$$

we directly obtain

$$R_x(t, \omega) = \begin{pmatrix} \operatorname{Re} \left(\frac{\operatorname{ST}_x^{\mathcal{T}g}(t, \omega)}{\operatorname{ST}_x(t, \omega)} \right) \\ -\operatorname{Im} \left(\frac{\operatorname{ST}_x^{\mathcal{D}g}(t, \omega)}{\operatorname{ST}_x(t, \omega)} \right) \end{pmatrix} \quad (133)$$

$$\begin{aligned} \nabla^t R_x(t, \omega) = & \\ & \begin{pmatrix} \frac{\partial}{\partial t} \left[\operatorname{Re} \left(\frac{\operatorname{ST}_x^{\mathcal{T}g}(t, \omega)}{\operatorname{ST}_x(t, \omega)} \right) \right] & \frac{\partial}{\partial \omega} \left[\operatorname{Re} \left(\frac{\operatorname{ST}_x^{\mathcal{T}g}(t, \omega)}{\operatorname{ST}_x(t, \omega)} \right) \right] \\ -\frac{\partial}{\partial t} \left[\operatorname{Im} \left(\frac{\operatorname{ST}_x^{\mathcal{D}g}(t, \omega)}{\operatorname{ST}_x(t, \omega)} \right) \right] & -\frac{\partial}{\partial \omega} \left[\operatorname{Im} \left(\frac{\operatorname{ST}_x^{\mathcal{D}g}(t, \omega)}{\operatorname{ST}_x(t, \omega)} \right) \right] \end{pmatrix} \end{aligned} \quad (134)$$

The coefficients of matrix $\nabla^t R_x(t, \omega)$ can be detailed as below.

9.1 Computation of the partial time derivatives of $R_x(t, \omega)$

Using $\frac{\partial \operatorname{ST}_x}{\partial t} = \operatorname{ST}_x^{\mathcal{D}g}(t, \omega)$ and $\frac{\partial \operatorname{ST}_x^{\mathcal{T}g}}{\partial t} = \operatorname{ST}_x(t, \omega) + \operatorname{ST}_x^{\mathcal{T}g}(t, \omega)$ we obtain

$$\frac{\partial}{\partial t} \left[\operatorname{Re} \left(\frac{\operatorname{ST}_x^{\mathcal{T}g}(t, \omega)}{\operatorname{ST}_x(t, \omega)} \right) \right] = \operatorname{Re} \left(\frac{\frac{\partial \operatorname{ST}_x^{\mathcal{T}g}}{\partial t}(t, \omega) \operatorname{ST}_x(t, \omega) - \operatorname{ST}_x^{\mathcal{T}g}(t, \omega) \frac{\partial \operatorname{ST}_x}{\partial t}(t, \omega)}{\operatorname{ST}_x(t, \omega)^2} \right) \quad (135)$$

$$= 1 + \operatorname{Re} \left(\frac{\operatorname{ST}_x^{\mathcal{T}g}(t, \omega) \operatorname{ST}_x(t, \omega) - \operatorname{ST}_x^{\mathcal{T}g}(t, \omega) \operatorname{ST}_x^{\mathcal{D}g}(t, \omega)}{\operatorname{ST}_x(t, \omega)^2} \right) \quad (136)$$

$$= \boxed{1 + \operatorname{Re} \left(\frac{\operatorname{ST}_x^{\mathcal{T}g}(t, \omega)}{\operatorname{ST}_x(t, \omega)} - \frac{\operatorname{ST}_x^{\mathcal{T}g}(t, \omega) \operatorname{ST}_x^{\mathcal{D}g}(t, \omega)}{\operatorname{ST}_x(t, \omega)^2} \right)}. \quad (137)$$

Using $\frac{\partial \operatorname{ST}_x^{\mathcal{D}g}}{\partial t}(t, \omega) = \operatorname{ST}_x^{\mathcal{D}^2g}(t, \omega)$ we obtain

$$-\frac{\partial}{\partial t} \left[\operatorname{Im} \left(\frac{\operatorname{ST}_x^{\mathcal{D}g}(t, \omega)}{\operatorname{ST}_x(t, \omega)} \right) \right] = -\operatorname{Im} \left(\frac{\frac{\partial \operatorname{ST}_x^{\mathcal{D}g}}{\partial t}(t, \omega) \operatorname{ST}_x(t, \omega) - \operatorname{ST}_x^{\mathcal{D}g}(t, \omega) \frac{\partial \operatorname{ST}_x}{\partial t}(t, \omega)}{\operatorname{ST}_x(t, \omega)^2} \right) \quad (138)$$

$$= \boxed{-\operatorname{Im} \left(\frac{\operatorname{ST}_x^{\mathcal{D}^2g}(t, \omega)}{\operatorname{ST}_x(t, \omega)} - \left(\frac{\operatorname{ST}_x^{\mathcal{D}g}(t, \omega)}{\operatorname{ST}_x(t, \omega)} \right)^2 \right)}. \quad (139)$$

9.2 Computation of the partial frequency derivatives of $R_x(t, \omega)$

Using the following equalities (computation details in Section 7)

$$\frac{\partial \operatorname{ST}_x}{\partial \omega}(t, \omega) = \frac{1}{\omega} \operatorname{ST}_x(t, \omega) - \frac{\omega}{(\omega_0 T)^2} \operatorname{ST}_x^{\mathcal{T}^2g}(t, \omega) + j(\operatorname{ST}_x^{\mathcal{T}g}(t, \omega) - t \operatorname{ST}_x(t, \omega)) \quad (140)$$

$$\text{and } \frac{\partial \operatorname{ST}_x^{\mathcal{T}g}}{\partial \omega}(t, \omega) = \frac{1}{\omega} \operatorname{ST}_x^{\mathcal{T}g}(t, \omega) - \frac{\omega}{(\omega_0 T)^2} \operatorname{ST}_x^{\mathcal{T}^3g}(t, \omega) + j(\operatorname{ST}_x^{\mathcal{T}^2g}(t, \omega) - t \operatorname{ST}_x^{\mathcal{T}g}(t, \omega)) \quad (141)$$

we obtain

$$\begin{aligned} & \frac{\partial}{\partial \omega} \left[\operatorname{Re} \left(\frac{\operatorname{ST}_x^{\mathcal{T}g}(t, \omega)}{\operatorname{ST}_x(t, \omega)} \right) \right] \\ &= \operatorname{Re} \left(\frac{\frac{\partial \operatorname{ST}_x^{\mathcal{T}g}}{\partial \omega}(t, \omega) \operatorname{ST}_x(t, \omega) - \operatorname{ST}_x^{\mathcal{T}g}(t, \omega) \frac{\partial \operatorname{ST}_x}{\partial \omega}(t, \omega)}{\operatorname{ST}_x(t, \omega)^2} \right) \end{aligned} \quad (142)$$

$$= \operatorname{Re} \left(j \frac{\operatorname{ST}_x^{\mathcal{T}^2g}(t, \omega) \operatorname{ST}_x(t, \omega) - \operatorname{ST}_x^{\mathcal{T}g}(t, \omega)^2}{\operatorname{ST}_x(t, \omega)^2} + \frac{\omega}{(\omega_0 T)^2} \frac{\operatorname{ST}_x^{\mathcal{T}^2g}(t, \omega) \operatorname{ST}_x^{\mathcal{T}g}(t, \omega) - \operatorname{ST}_x^{\mathcal{T}^3g}(t, \omega) \operatorname{ST}_x(t, \omega)}{\operatorname{ST}_x(t, \omega)^2} \right) \quad (143)$$

$$= \boxed{-\operatorname{Im} \left(\frac{\operatorname{ST}_x^{\mathcal{T}^2g}(t, \omega)}{\operatorname{ST}_x(t, \omega)} - \left(\frac{\operatorname{ST}_x^{\mathcal{T}g}(t, \omega)}{\operatorname{ST}_x(t, \omega)} \right)^2 \right) + \frac{\omega}{(\omega_0 T)^2} \operatorname{Re} \left(\frac{\operatorname{ST}_x^{\mathcal{T}^2g}(t, \omega) \operatorname{ST}_x^{\mathcal{T}g}(t, \omega) - \operatorname{ST}_x^{\mathcal{T}^3g}(t, \omega) \operatorname{ST}_x(t, \omega)}{\operatorname{ST}_x(t, \omega)^2} \right)} \quad (144)$$

Using $\text{ST}_x^{\mathcal{D}g}(t, \omega) = -\frac{\omega^2}{(\omega_0 T)^2} \text{ST}_x^{\mathcal{T}g}(t, \omega)$ we have

$$\frac{\partial \text{ST}_x^{\mathcal{D}g}}{\partial \omega}(t, \omega) = -\frac{\partial}{\partial \omega} \left[\frac{\omega^2}{(\omega_0 T)^2} \text{ST}_x^{\mathcal{T}g} \right] \quad (145)$$

$$= -\frac{2\omega}{(\omega_0 T)^2} \text{ST}_x^{\mathcal{T}g}(t, \omega) - \frac{\omega^2}{(\omega_0 T)^2} \left(\frac{\partial \text{ST}_x^{\mathcal{T}g}}{\partial \omega}(t, \omega) \right) \quad (146)$$

$$= -\frac{-3\omega}{(\omega_0 T)^2} \text{ST}_x^{\mathcal{T}g}(t, \omega) + \frac{\omega^3}{(\omega_0 T)^4} \text{ST}_x^{\mathcal{T}^3g}(t, \omega) - j \frac{\omega^2}{(\omega_0 T)^2} \text{ST}_x^{\mathcal{T}^2g}(t, \omega) + jt \frac{\omega^2}{(\omega_0 T)^2} \text{ST}_x^{\mathcal{T}g}(t, \omega) \quad (147)$$

we obtain

$$-\frac{\partial}{\partial \omega} \left[\text{Im} \left(\frac{\text{ST}_x^{\mathcal{D}g}(t, \omega)}{\text{ST}_x(t, \omega)} \right) \right] \quad (148)$$

$$= -\text{Im} \left(\frac{\frac{\partial \text{ST}_x^{\mathcal{D}g}}{\partial \omega}(t, \omega) \text{ST}_x - \text{ST}_x^{\mathcal{D}g} \frac{\partial \text{ST}_x}{\partial \omega}(t, \omega)}{\text{ST}_x(t, \omega)^2} \right) \quad (149)$$

$$= -\text{Im} \left(\frac{\frac{-2\omega}{(\omega_0 T)^2} \text{ST}_x^{\mathcal{T}g}(t, \omega) \text{ST}_x(t, \omega)}{\text{ST}_x^2} + \frac{\omega^3}{(\omega_0 T)^4} \frac{\text{ST}_x^{\mathcal{T}^3g}(t, \omega) \text{ST}_x(t, \omega) - \text{ST}_x^{\mathcal{T}^2g}(t, \omega) \text{ST}_x^{\mathcal{T}g}(t, \omega)}{\text{ST}_x(t, \omega)^2} \right. \\ \left. - j \frac{\omega^2}{(\omega_0 T)^2} \frac{\text{ST}_x^{\mathcal{T}^2g}(t, \omega) \text{ST}_x(t, \omega) - (\text{ST}_x^{\mathcal{T}g})^2}{\text{ST}_x(t, \omega)^2} \right) \quad (150)$$

$$= \boxed{-\text{Re} \left(\frac{\text{ST}_x^{\mathcal{D}g}(t, \omega)}{\text{ST}_x(t, \omega)} - \frac{\text{ST}_x^{\mathcal{T}g}(t, \omega) \text{ST}_x^{\mathcal{D}g}(t, \omega)}{\text{ST}_x(t, \omega)^2} \right) - \frac{\omega^3}{(\omega_0 T)^4} \text{Im} \left(\frac{\text{ST}_x^{\mathcal{T}^3g}(t, \omega)}{\text{ST}_x(t, \omega)} - \frac{\text{ST}_x^{\mathcal{T}^2g}(t, \omega) \text{ST}_x^{\mathcal{T}g}(t, \omega)}{\text{ST}_x(t, \omega)^2} \right)} \\ + \frac{2\omega}{(\omega_0 T)^2} \text{Im} \left(\frac{\text{ST}_x^{\mathcal{T}g}(t, \omega)}{\text{ST}_x(t, \omega)} \right)} \quad (151)$$

10 Definition of the S-transform local instantaneous modulation operator

Let be $x(t) = a(t) e^{j\phi(t)}$ a Gaussian-modulated linear chirp (*i.e.* $\phi(t)$ and $\log(a(t))$ functions are quadratic). Thus, the frequency modulation \hat{q}_x can be defined as

$$\hat{q}_x(t, \omega) = \frac{d^2 \phi}{dt^2}(t). \quad (152)$$

The proof of this result follows. By definition, $x(t)$ is differentiable and its derivative is

$$\frac{dx}{dt}(t) = \left(\frac{da}{dt}(t) + j \frac{d\phi}{dt}(t) a(t) \right) e^{j\phi(t)} \quad (153)$$

$$= \underbrace{\left(\frac{d}{dt} (\log(a(t))) + j \frac{d\phi}{dt}(t) \right)}_{l(t)} x(t) \quad (154)$$

where l is an affine complex-valued function since $\log(a(t))$ and $\phi(t)$ are quadratic. Thus, there exists complex numbers α and β such that $l(t) = \alpha t + \beta$ with

$$\alpha = \frac{dl}{dt}(t) = \frac{d^2}{dt^2} (\log(a(t))) + j \frac{d^2 \phi}{dt^2}(t). \quad (155)$$

Using $\text{ST}_x(t, \omega) = e^{-j\omega t} \int_{\mathbb{R}} x(t + \tau) g(-\tau, \sigma_t) e^{-j\omega \tau} d\tau$ as the expression of the S-transform of a signal x , we

obtain

$$\frac{\partial \text{ST}_x}{\partial t}(t, \omega) = -j\omega e^{-j\omega t} \int_{\mathbb{R}} x(t+\tau)g(-\tau, \sigma_t) e^{-j\omega\tau} d\tau + e^{-j\omega t} \int_{\mathbb{R}} \frac{dx}{dt}(t+\tau)g(-\tau, \sigma_t) e^{-j\omega\tau} d\tau \quad (156)$$

$$= -j\omega \text{ST}_x(t, \omega) + e^{-j\omega t} \int_{\mathbb{R}} l(t+\tau)x(t+\tau)g(-\tau, \sigma_t) e^{-j\omega\tau} d\tau \quad (157)$$

$$= -j\omega \text{ST}_x(t, \omega) + e^{-j\omega t} \alpha \int_{\mathbb{R}} x(t+\tau)(t+\tau)g(-\tau, \sigma_t) e^{-j\omega\tau} d\tau + \beta \text{ST}_x(t, \omega) \quad (158)$$

$$= -j\omega \text{ST}_x(t, \omega) - \alpha \int_{\mathbb{R}} x(\tau)((t-\tau)-t)g(t-\tau, \sigma_t) e^{-j\omega\tau} d\tau + \beta \text{ST}_x(t, \omega) \quad (159)$$

$$= -j\omega \text{ST}_x(t, \omega) - \alpha (\text{ST}_x^{\mathcal{T}g}(t, \omega) - t \text{ST}_x(t, \omega)) + \beta \text{ST}_x(t, \omega). \quad (160)$$

Dividing by $\text{ST}_x(t, \omega)$ we obtain

$$\frac{\frac{\partial \text{ST}_x}{\partial t}(t, \omega)}{\text{ST}_x(t, \omega)} = \frac{\text{ST}_x^{\mathcal{D}g}(t, \omega)}{\text{ST}_x(t, \omega)} = -j\omega - \alpha \left(\frac{\text{ST}_x^{\mathcal{T}g}(t, \omega)}{\text{ST}_x(t, \omega)} - t \right) + \beta. \quad (161)$$

Differentiating with respect to t , we obtain

$$\frac{\partial}{\partial t} \left(\frac{\text{ST}_x^{\mathcal{D}g}(t, \omega)}{\text{ST}_x(t, \omega)} \right) = -\alpha \left(\frac{\partial}{\partial t} \left(\frac{\text{ST}_x^{\mathcal{T}g}(t, \omega)}{\text{ST}_x(t, \omega)} \right) - 1 \right) \quad (162)$$

Thus, we have

$$\alpha = \frac{d^2}{dt^2} (\log(a(t))) + j \frac{d^2 \phi}{dt^2}(t) = -\frac{\frac{\partial}{\partial t} \left(\frac{\text{ST}_x^{\mathcal{D}g}(t, \omega)}{\text{ST}_x(t, \omega)} \right)}{\frac{\partial}{\partial t} \left(\frac{\text{ST}_x^{\mathcal{T}g}(t, \omega)}{\text{ST}_x(t, \omega)} \right) - 1} \quad (163)$$

that leads to

$$\hat{q}_x(t, \omega) = -\text{Im} \left(\frac{\text{ST}_x^{\mathcal{D}^2g}(t, \omega) \text{ST}_x(t, \omega) - \text{ST}_x^{\mathcal{D}g}(t, \omega)^2}{\text{ST}_x^{\mathcal{T}^{\mathcal{D}g}}(t, \omega) \text{ST}_x(t, \omega) - \text{ST}_x^{\mathcal{T}g}(t, \omega) \text{ST}_x^{\mathcal{D}g}(t, \omega)} \right). \quad (164)$$

11 Implementation of the reassigned Gabor spectrogram and of the synchrosqueezed STFT

The discrete-time Gabor transform of a signal x can be approximated by $F_x^g(nT_s, \frac{2\pi m}{MT_s}) \approx F_x^g[n, m]$ with $n \in \mathcal{Z}$ and $m \in \mathcal{M}$ as

$$F_x^g[n, m] = e^{-j\frac{2\pi mn}{M}} \sum_{k=-K/2}^{K/2} x[n+k]g[-k, L] e^{-j\frac{2j\pi mk}{M}} \quad (165)$$

$$= \frac{e^{-j\frac{2\pi mn}{M}}}{\sqrt{2\pi L}} \sum_{k=-K/2}^{K/2} x[n+k] e^{-\frac{k^2}{2L^2}} e^{-j\frac{2j\pi mk}{M}} \quad (166)$$

with $g[k, L] = T_s g(kT_s, \frac{T}{T_s}) = \frac{1}{\sqrt{2\pi L}} e^{-\frac{n^2}{2L^2}}$, where $K = 2L\sqrt{2\log(1/\Gamma)}$ is obtained by setting a threshold Γ thus we have

$$e^{-\frac{(K/2)^2}{2L^2}} \leq \Gamma. \quad (167)$$

The signal x can be recovered using the STFT synthesis formula as

$$\hat{x}[n] = \frac{1}{g(0, T)} \sum_{m \in \mathcal{M}} F_x^g[n, m] e^{\frac{2j\pi mn}{M}} \Delta\omega \quad \text{with } \Delta\omega = \frac{2\pi}{MT_s} \quad (168)$$

$$= \frac{\sqrt{2\pi}}{ML} \sum_{m \in \mathcal{M}} F_x^g[n, m] e^{\frac{2j\pi mn}{M}}. \quad (169)$$

11.1 Computation of the spectrogram and the reassigned spectrogram

The spectrogram is simply computed as $|F_x^g[n, m]|^2$ and the reassigned spectrogram is computed as

$$\text{RV}_x[n, m] = \sum_{n \in \mathcal{Z}} \sum_{m \in \mathcal{Z}} |F_x^g[n, m]|^2 \delta[n - \hat{n}[n, m]] \delta[m - \hat{m}[n, m]] \quad (170)$$

where \hat{n} and \hat{m} correspond to the reassigned time-frequency coordinates computed as

$$\hat{n}[n, m] = n - \left\lfloor \text{Re} \left(\frac{T_s^{-1} F_x^{\mathcal{T}g}[n, m]}{F_x^g[n, m]} \right) \right\rfloor \quad (171)$$

$$\hat{m}[n, m] = m + \left\lfloor \text{Im} \left(\frac{T_s F_x^{\mathcal{D}g}[n, m]}{F_x^g[n, m]} \right) \right\rfloor \quad (172)$$

$$(173)$$

where $\mathcal{T}g(t, T) = tg(t, T)$ and $\mathcal{D}g(t, T) = \frac{dg}{dt}(t, T) = \frac{-1}{T^2} \mathcal{T}g(t, T)$.

The Levenberg-Marquardt reassigned time-frequency coordinates are computed as

$$\begin{pmatrix} \tilde{n}[n, m] \\ \tilde{m}[n, m] \end{pmatrix} = \begin{pmatrix} n \\ m \end{pmatrix} - \left[(\nabla^t R_x[n, m] + \mu I_2)^{-1} R_x[n, m] \right] \quad (174)$$

$$\text{with } R_x[n, m] = \begin{pmatrix} n - \tilde{n}[n, m] \\ m - \tilde{m}[n, m] \end{pmatrix} \quad (175)$$

$$\nabla^t R_x[n, m] = \begin{pmatrix} 1 + \text{Re} \left(\frac{F_x^{\mathcal{T}^2 g}[n, m]}{F_x^g[n, m]} - \frac{F_x^{\mathcal{T}g}[n, m] F_x^{\mathcal{D}g}[n, m]}{F_x^g[n, m]^2} \right) & -\text{Im} \left(\frac{T_s^{-2} F_x^{\mathcal{T}^2 g}[n, m]}{F_x^g[n, m]} - \left(\frac{T_s^{-1} F_x^{\mathcal{T}g}[n, m]}{F_x^g[n, m]} \right)^2 \right) \\ -\text{Im} \left(\frac{T_s^2 F_x^{\mathcal{D}^2 g}[n, m]}{F_x^g[n, m]} - \left(\frac{T_s F_x^{\mathcal{D}g}[n, m]}{F_x^g[n, m]} \right)^2 \right) & -\text{Re} \left(\frac{F_x^{\mathcal{T}^2 g}[n, m]}{F_x^g[n, m]} - \frac{F_x^{\mathcal{T}g}[n, m] F_x^{\mathcal{D}g}[n, m]}{F_x^g[n, m]^2} \right) \end{pmatrix} \quad (176)$$

where $\mathcal{T}^2 g(t, T) = t \frac{dg}{dt}(t, T)$, $\mathcal{T}^2 g(t, T) = t^2 g(t, T)$ and $\mathcal{D}^2 g(t, T) = \frac{d^2 g}{dt^2}(t, T)$. Thus, the Levenberg-Marquardt spectrogram is simply obtained by replacing (\hat{n}, \hat{m}) by (\tilde{n}, \tilde{m}) in Eq. (170).

11.2 Computation of the synchrosqueezed STFT

The discrete-time synchrosqueezed STFT can be defined as

$$T_x[n, m] = \frac{1}{2\pi} \sum_{m' \in \mathcal{M}} F_x^g[n, m'] e^{\frac{2j\pi m'n}{M}} \delta[m' - \hat{m}[n, m']] \quad (177)$$

where (\hat{n}, \hat{m}) can be replaced by (\tilde{n}, \tilde{m}) to obtain the Levenberg-Marquardt synchrosqueezed STFT.

The signal x can be recovered from the synchrosqueezed STFT as

$$\hat{x}[n] = \frac{1}{g(0, T)} \sum_{m' \in \mathcal{M}} T_x[n, m] \Delta\omega \quad (178)$$

$$= \frac{(2\pi)^{3/2} L}{M} \sum_{m' \in \mathcal{M}} T_x[n, m]. \quad (179)$$

11.3 Computation of the vertical synchrosqueezed STFT

Let be a signal $x(t) = A(t) e^{j\phi(t)}$ with $\phi(t) = \phi_0 + \omega t + \frac{qt^2}{2}$. Thus we have

$$\frac{dx}{dt}(t) = \underbrace{\left(\frac{dA}{dt}(t) + j \frac{d\phi}{dt}(t) \right)}_{l(t)} x(t) \quad (180)$$

thus, there exists $(\alpha, \beta) \in \mathbb{C}^2$ thus we have $l(t) = \alpha t + \beta$, $\frac{dl}{dt}(t) = \frac{d^2}{dt^2}(\log(A(t))) + j \frac{d^2 \phi}{dt^2}(t)$, hence $q = \text{Im}(\alpha)$.

Thus, as $F_x^g(t, \omega) = \mathbf{e}^{-j\omega t} \int_{\mathbb{R}} x(t + \tau)g(-\tau, T) \mathbf{e}^{-j\omega\tau}$ we have

$$\frac{\partial}{\partial t} (F_x^g(t, \omega)) = -j\omega F_x^g(t, \omega) + \mathbf{e}^{-j\omega t} \int_{\mathbb{R}} \frac{dx}{dt}(t + \tau)g(-\tau, T) \mathbf{e}^{-j\omega\tau} d\tau \quad (181)$$

$$= -j\omega F_x^g(t, \omega) + \mathbf{e}^{-j\omega t} \int_{\mathbb{R}} (\alpha(t + \tau) + \beta)x(t + \tau)g(-\tau, T) \mathbf{e}^{-j\omega\tau} d\tau \quad (182)$$

$$= (\beta - j\omega + \alpha t) F_x^g(t, \omega) - \alpha \mathbf{e}^{-j\omega t} \int_{\mathbb{R}} x(t + \tau)(-\tau)g(-\tau, T) \mathbf{e}^{-j\omega\tau} d\tau \quad (183)$$

$$= (\beta - j\omega + \alpha t) F_x^g(t, \omega) - \alpha F_x^{\mathcal{T}g}(t, \omega) \quad (184)$$

then,

$$\frac{\frac{\partial}{\partial t} (F_x^g(t, \omega))}{F_x^g(t, \omega)} = (\beta - j\omega + \alpha t) - \alpha \frac{F_x^{\mathcal{T}g}(t, \omega)}{F_x^g(t, \omega)} \quad (185)$$

then,

$$\frac{\partial}{\partial t} \left(\frac{\frac{\partial}{\partial t} (F_x^g(t, \omega))}{F_x^g(t, \omega)} \right) = \alpha \left(1 - \frac{\partial}{\partial t} \left(\frac{F_x^{\mathcal{T}g}(t, \omega)}{F_x^g(t, \omega)} \right) \right) \quad (186)$$

$$\frac{F_x^{\mathcal{D}^2g}(t, \omega) F_x^g(t, \omega) - F_x^{\mathcal{D}g}(t, \omega)}{F_x^g(t, \omega)^2} = \alpha \left(1 - \frac{(F_x^g(t, \omega) + F_x^{\mathcal{T}Dg}(t, \omega)) F_x^g(t, \omega) - F_x^{\mathcal{T}g}(t, \omega) F_x^{\mathcal{D}g}(t, \omega)}{F_x^g(t, \omega)^2} \right). \quad (187)$$

Hence we have

$$\alpha = \frac{F_x^{\mathcal{D}^2g}(t, \omega) F_x^g(t, \omega) - F_x^{\mathcal{D}g}(t, \omega)^2}{F_x^{\mathcal{T}g}(t, \omega) F_x^{\mathcal{D}g}(t, \omega) - F_x^{\mathcal{T}Dg}(t, \omega) F_x^g(t, \omega)}. \quad (188)$$

As a result, the discrete-time local frequency modulation can be estimated as

$$\hat{q}_x[n, m] = T_s^{-2} \text{Im} \left(\frac{T_s^2 F_x^{\mathcal{D}^2g}[n, m] F_x^g[n, m] - (T_s F_x^{\mathcal{D}g}[n, m])^2}{F_x^{\mathcal{T}g}[n, m] F_x^{\mathcal{D}g}[n, m] - F_x^{\mathcal{T}Dg}[n, m] F_x^g[n, m]} \right). \quad (189)$$

thus an enhanced frequency estimator is given by

$$\bar{m}_q[n, m] = \hat{m}[n, m] + \frac{M}{2\pi} T_s^2 \hat{q}_x[n, m] (n - \hat{n}[n, m]). \quad (190)$$

So finally, the vertical synchrosqueezed STFT is obtained by replacing $\hat{m}[n, m]$ by $\bar{m}_q[n, m]$ in Eq. (177).

12 Fourier transform of the Morlet Wavelet

According to the definition of the Morlet mother wavelet, we have

$$F_{\Psi}(\omega) = \int_{-\infty}^{+\infty} \Psi(t) \mathbf{e}^{-j\omega t} dt \quad (191)$$

$$= \frac{\pi^{-1/4}}{\sqrt{T}} \int_{-\infty}^{+\infty} \mathbf{e}^{-\frac{t^2}{2T^2} + j(\omega_0 - \omega)t} dt \quad (192)$$

$$= \sqrt{2T} \pi^{1/4} \mathbf{e}^{-\frac{(\omega - \omega_0)^2 T^2}{2}} \quad (193)$$