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# Topological classification of Morse-Smale diffeomorphisms without heteroclinic curves on 3-manifolds

Ch. Bonatti      V. Grines      F. Laudenbach      O. Pochinka \*

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## Abstract

We show that, up to topological conjugation, the equivalence class of a Morse-Smale diffeomorphism without heteroclinic curves on 3-manifold is completely defined by an embedding of two-dimensional stable and unstable heteroclinic laminations to a characteristic space.

**Key words:** Morse-Smale diffeomorphism, topological classification, heteroclinic lamination  
**MSC:** 37C05, 37C15, 37C29, 37D15.

## 1 Introduction and formulation of the result

In 1937 A. Andronov and L. Pontryagin [2] introduced the notion of *rough system* of differential equation given in a bounded part of the plane, that is a system which keeps its qualitative properties after small changes in the right-hand side. They proved that the flows generated by such systems are exactly the flows having the next properties:

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- the set of fixed points and periodic orbits is finite and all its elements are hyperbolic;
- there are no separatrices going from one saddle to itself or to another one;
- all  $\omega$ - and  $\alpha$ -limit sets are contained in the union of fixed points and periodic orbits (limit cycles).

The above description characterizes the rough flows on the two-dimensional sphere also. Similar result for flows with closed section and without equilibrium states on two-dimension tori follows from paper of A. Mayer [16] in 1939. A. Andronov and L. Pontryagin have shown also in [2] that the set of the rough flows is dense in the space of  $C^1$ -flows<sup>1</sup>. Similar criterion and density were proved by M. Peixoto [22], [23] in 1962 for *structural stable flows* on orientable surfaces of genus greater than zero. Direct generalization of the properties of rough flows on surfaces leads to the following class of dynamical systems (continuous and discrete — flows and diffeomorphisms (cascades)).

**Definition 1.1** *A smooth dynamical system given on an  $n$ -dimensional manifold ( $n \geq 1$ )  $M^n$  is called Morse-Smale if: 1) its non-wandering set consists of a finite number of fixed points and periodic orbits where each of them is hyperbolic; 2) the stable and unstable manifolds  $W_p^s$ ,  $W_q^u$  of any pair of non-wandering points  $p$  and  $q$  intersect transversely<sup>2</sup>.*

Let  $M$  be a given closed 3-dimensional manifold and  $f : M \rightarrow M$  be a Morse-Smale diffeomorphism.

For  $q = 0, 1, 2, 3$  denote by  $\Omega_q$  the set of all periodic points of  $f$  with  $q$ -dimensional unstable manifold. Let  $\Omega_f$  be the union of all periodic points. Let us represent the dynamics of  $f$  in the form “source-sink” in the following way. Set

$$A_f = W_{\Omega_0 \cup \Omega_1}^u, \quad R_f = W_{\Omega_2 \cup \Omega_3}^s, \quad V_f = M \setminus (A_f \cup R_f).$$

Due to Theorem 1.1 in [11] the set  $A_f$  (resp.  $R_f$ ) is an attractor (resp. a repeller)<sup>3</sup> of  $f$  whose topological dimension is equal to 0 or 1. Due to Theorem 1.2 in [11] the set  $V_f$  is a connected 3-manifold and  $V_f = W_{A_f \cap \Omega_f}^s \setminus A_f = W_{R_f \cap \Omega_f}^u \setminus R_f$ . Moreover, the quotient  $\hat{V}_f = V_f/f$  is a closed connected 3-manifold and when  $\hat{V}_f$  is orientable, then it is either irreducible or diffeomorphic to  $\mathbb{S}^2 \times \mathbb{S}^1$ . The natural projection  $p_f : V_f \rightarrow \hat{V}_f$  is an infinite cyclic covering. Therefore, there is a natural epimorphism from the the first homology group of  $\hat{V}_f$  to  $\mathbb{Z}$ ,

$$\eta_f : H_1(\hat{V}_f; \mathbb{Z}) \rightarrow \mathbb{Z},$$

---

<sup>1</sup>This statement was not explicitly formulated in [2] and was mentioned first time in papers of E. Leontovich [15] and M. Peixoto [21]. G. Baggis [3] in 1955 made explicit some details of the proofs in [2], which were not published.

<sup>2</sup>Two smooth submanifolds  $X_1, X_2$  of an  $n$ -manifold  $X$  intersect transversely if either  $X_1 \cap X_2 = \emptyset$  or  $T_x X_1 + T_x X_2 = T_x X$  for each point  $x \in (X_1 \cap X_2)$  (here  $T_x A$  stands for tangent space to the manifold  $A$  at the point  $x$ ).

<sup>3</sup>A compact set  $A \subset M$  is an attractor of a diffeomorphism  $f : M \rightarrow M$  if there is a neighborhood  $U$  of the set  $A$  such that  $f(U) \subset \text{int } U$  and  $A = \bigcap_{n \in \mathbb{N}} f^n(U)$ . A set  $R \subset M$  is called a repeller of  $f$  if it is an attractor of  $f^{-1}$ .

defined as follows: if  $\gamma$  is a path in  $V_f$  joining  $x$  to  $f^n(x)$ ,  $n \in \mathbb{Z}$ , then  $\eta_f$  maps the homology class of the cycle  $p_f \circ \gamma$  to  $n$ . Set:

$$\begin{aligned}\Gamma_f^s &:= W_{\Omega_1}^s \setminus A_f \text{ and } \hat{\Gamma}_f^s := p_f(\Gamma_f^s); \\ \Gamma_f^u &:= W_{\Omega_2}^u \setminus R_f \text{ and } \hat{\Gamma}_f^u := p_f(\Gamma_f^u).\end{aligned}$$

**Definition 1.2** *The sets  $\hat{\Gamma}_f^s$  and  $\hat{\Gamma}_f^u$  are called the two-dimensional stable and unstable laminations associated with the diffeomorphism  $f$ .*

A precise definition of what is a lamination will be given in Definition 2.1.

**Definition 1.3** *The collection  $S_f = (\hat{V}_f, \eta_f, \hat{\Gamma}_f^s, \hat{\Gamma}_f^u)$  is called the scheme of the diffeomorphism  $f$ .*

**Definition 1.4** *The schemes  $S_f$  and  $S_{f'}$  of two Morse-Smale diffeomorphisms  $f, f' : M \rightarrow M$  are said to be equivalent if there is a homeomorphism  $\hat{\varphi} : \hat{V}_f \rightarrow \hat{V}_{f'}$  with following properties:*

- 1)  $\eta_f = \eta_{f'} \hat{\varphi}_*$ ;
- 2)  $\hat{\varphi}(\hat{\Gamma}_f^s) = \hat{\Gamma}_{f'}^s$  and  $\hat{\varphi}(\hat{\Gamma}_f^u) = \hat{\Gamma}_{f'}^u$ .

Using the notion of the scheme above in a series of papers by Ch. Bonatti, V. Grines, V. Medvedev, E. Pecou, O. Pochinka [5], [7], [8], [9], the problem of topological classification<sup>4</sup> of Morse-Smale diffeomorphisms on 3-manifolds was solved in different particular cases.

In the present article, we give the topological classification of the Morse-Smale diffeomorphisms belonging to the subset  $G(M)$  of the Morse-Smale diffeomorphisms  $f : M \rightarrow M$  which have no heteroclinic curves (see Section 2). According to [6], when the ambient manifold is orientable, then it is either sphere  $\mathbb{S}^3$  or the connected sum of a finite number copies of  $\mathbb{S}^2 \times \mathbb{S}^1$ .

**Theorem 1** *Two Morse-Smale diffeomorphisms in  $G(M)$  are topologically conjugate if and only if their schemes are equivalent.*

## 2 Dynamics of diffeomorphisms in the class $G(M)$

In this section we introduce some notions connected with Morse-Smale diffeomorphisms on 3-manifold  $M$ . More detailed information on Morse-Smale diffeomorphisms is contained in [13] for example.

Let  $f : M \rightarrow M$  be a Morse-Smale diffeomorphism. If  $x$  is a periodic point its *Morse index* is the dimension of its unstable manifold  $W_x^u$ ; the point  $x$  is called a *saddle point* when its two invariant manifolds have positive dimension, that is, its Morse index is not extremely. *Sink point* has Morse index 0 and *source point* has Morse index 3. The following notions are key concepts for describing the structure of intersection of stable and unstable manifolds of saddle periodic points. If  $x, y$  are distinct periodic saddle points of  $f$  and  $W_x^u \cap W_y^s \neq \emptyset$ , then:

---

<sup>4</sup>Recall that two diffeomorphisms  $f, f'$  given on  $M$  are said to be *topologically conjugate* if there is a homeomorphism  $h : M \rightarrow M$  which satisfies  $f'h = hf$ .

- if  $\dim W_x^s < \dim W_y^s$ , any connected component of  $W_x^u \cap W_y^s$  is 1-dimensional and called a *heteroclinic curve* (see figure 1);
- if  $\dim W_x^s = \dim W_y^s$ , the set  $W_x^u \cap W_y^s$  is countable; each of its points is called a *heteroclinic point*; the orbit of a heteroclinic point is called a *heteroclinic orbit*.

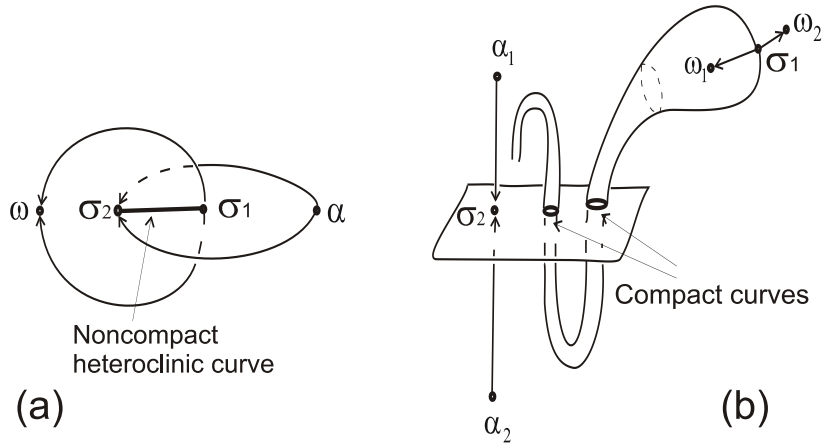


Figure 1: Heteroclinic curves

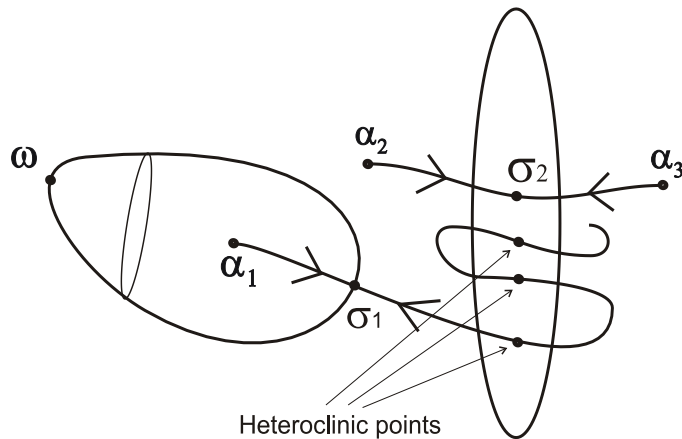


Figure 2: Heteroclinic points

According to S. Smale [24], it is possible to define a partial order in the set of saddle points of a given Morse-Smale diffeomorphism  $f$  as follows: for different periodic orbits  $p \neq q$ , one sets  $p \prec q$  if and only if  $W_q^u \cap W_p^s \neq \emptyset$ . Smale proved that this relation is a partial order. In that case, it follows from lemma 1.5 of [18] that there is a sequence of different periodic orbits  $p_0, \dots, p_n$  satisfying the following conditions:  $p_0 = p$ ,  $p_n = q$  and  $p_i \prec p_{i+1}$ . The sequence  $p_0, \dots, p_n$  is said to be an *n-chain connecting p to q*. The length of the longest chain connecting  $p$  to  $q$  is denoted by  $beh(q|p)$ . If  $W_q^u \cap W_p^s = \emptyset$ , we pose  $beh(q|p) = 0$ . For a subset  $P$  of the

periodic orbits let us set  $beh(q|P) = \max_{p \in P} \{beh(q|p)\}$ . The present paper is devoted to studying Morse-Smale diffeomorphisms in dimension 3 which have no heteroclinic curve. We recall from the introduction that this class of diffeomorphisms is denoted by  $G(M)$ . Let  $f \in G(M)$ . It follows from [11] that if the set  $\Omega_2$  is empty then  $R_f$  consists of unique source. If  $\Omega_2 \neq \emptyset$ , denote by  $n$  the length of the longest chain connecting two points of  $\Omega_2$ . Divide the set  $\Omega_2$  into  $f$ -invariant disjoint parts  $\Sigma_0, \Sigma_1, \dots, \Sigma_n$  using the rule:  $beh(q|(\Omega_2 \setminus q)) = 0$  for each orbit  $q \in \Sigma_0$  and  $beh(q|\Sigma_i) = 1$  for each orbit  $q \in \Sigma_{i+1}$ ,  $i \in \{0, \dots, n-1\}$ . Since  $\Omega_1$  for  $f$  is  $\Omega_2$  for  $f^{-1}$ , then it is possible to divide the periodic orbits of the set  $\Omega_1$  into parts in a similar way. The absence of heteroclinic curves means that there are no chains connecting a saddle from  $\Omega_2$  with a saddle from  $\Omega_1$ . Thus we explain all material for  $\Omega_2$  and say that all is similar for  $\Omega_1$ .

Set  $W_i^u := W_{\Sigma_i}^u$ ,  $W_i^s := W_{\Sigma_i}^s$ . Then,  $R_f := \bigcup_{i=0}^n cl(W_i^s)$ , where  $cl(\cdot)$  stands for the closure of  $(\cdot)$ . We now specify what a lamination is and which sort of regularity it may have.

**Definition 2.1** *Let  $X$  be a  $n$ -dimensional and  $Y \subset X$  be a closed subset. Let  $q$  be an integer  $0 < q < n$ . A codimension- $q$  lamination with support  $Y$  is a decomposition  $Y = \bigcup_{j \in J} L_j$  into pairwise disjoint smooth  $(n-q)$ -dimensional connected manifolds  $L_j$ , which are called the leaves. The family  $L = \{L_j, j \in J\}$  is said to be a  $C^{1,0}$ -lamination<sup>5</sup> if for every point  $x \in Y$  the following conditions hold:*

- 1) *There are an open neighborhood  $U_x \subset X$  of  $x$  and a homeomorphism  $\psi : U_x \rightarrow \mathbb{R}^n$  such that  $\psi$  maps every plaque, that is a connected component of  $U_x \cap L_j$ , into a codimension- $q$  subspace  $\{(x^1, \dots, x^n) \in \mathbb{R}^n \mid x^{n-q+1} = c^{n-q+1}, \dots, x^n = c^n\}$ . If  $Y = X$  one says that  $\mathcal{L}$  is foliation.*
- 2) *The tangent plane field  $TY := \bigcup_{j \in J} TL_j$  exists on  $Y$  and is continuous.*

By abuse, the lamination and its support are generally denoted in the same way. We recall the  $\lambda$ -Lemma in the strong form which is proved in [19, Remarks p. 85].

**Lemma 2.2 ( $\lambda$ -lemma.)** *Let  $f : X \rightarrow X$  be a diffeomorphism of an  $n$ -manifold, and let  $p$  be a fixed point of  $f$ . Denote  $W_p^u$  and  $W_p^s$  be the unstable and stable manifold respectively; say  $\dim W_p^u = m$ ,  $0 < m < n$ . Let  $B^s$  be a compact subset of  $W_p^s$  (containing  $p$  or not) and let  $F : B^s \rightarrow C^1(\mathbb{D}^m, X)$  be a continuous family of embedded closed  $m$ -disks of class  $C^1$  transverse to  $W_p^s$ ; set  $F(x) := D_x^u$ . Let  $D^u \subset W_p^u$  be a compact  $m$ -disk and let  $V \subset X$  be a compact  $n$ -ball such that  $D^u$  is a connected component of  $W_p^u \cap V$ . Then, when  $k$  goes to  $+\infty$ , the sequence  $f^k(D_x^u) \cap V$  converges to  $D^u$  in the  $C^1$  topology uniformly for  $x \in B^s$ .*

Notice that it is important for applications that  $B^s$  may not contained the point  $p$ . Going back to our setting, a first application of the  $\lambda$ -lemma is that we have  $W_i^u \subset cl(W_{i+1}^u)$  and the closure of  $cl(W_0^u) = W_0^u \cup \Omega_0$ . Moreover,  $cl(W_n^u) \cap (M \setminus \Omega_0)$  is a  $C^{1,0}$ -lamination of codimension one. From this one derives that  $\hat{L}_f^u$  is also a  $C^{1,0}$ -lamination. Here is a typical example.

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<sup>5</sup>There are different possible notations. Here, we use the one which is given in [10, Definition 1.1.22].

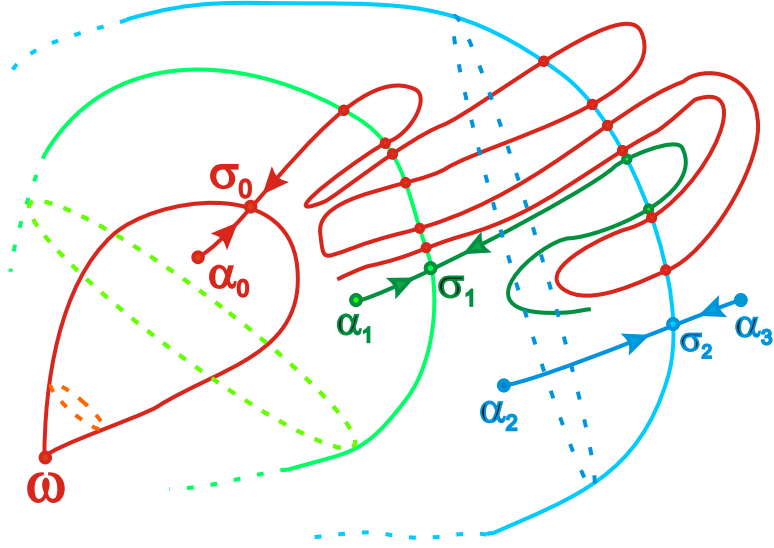


Figure 3: A phase portrait of a diffeomorphism from the class  $G(M)$ .

On Figure 3 there is a phase portrait of a diffeomorphism  $f \in G(M)$  whose non-wandering set  $\Omega_f$  consists of fixed points: one sink  $\omega$ , three saddle points  $\Sigma_0 = \sigma_0, \Sigma_1 = \sigma_1, \Sigma_2 = \sigma_2$  with two-dimensional unstable manifolds and four sources  $\alpha_0, \alpha_1, \alpha_2, \alpha_3$ . All further proofs we will illustrate on this diffeomorphism. For this case  $V_f := W_\omega^s \setminus \{\omega\}$ . As the restriction of  $f$  to the basin  $W_\omega^s$  of  $\omega$  is topologically conjugate to any homothety,  $\hat{V}_f$  is diffeomorphic to  $\mathbb{S}^2 \times \mathbb{S}^1$ . As  $f|_{W_i^u}$  is topologically conjugate to a homothetic then  $(W_i^u \setminus \Sigma_i)/f$  is diffeomorphic to the 2-torus; but this torus does not embed to  $\hat{V}_f$ , except when  $i = 0$ . On Figure 4 there is the lamination associated with the diffeomorphism  $f \in G(M)$  whose phase portrait is on Figure 3. On the left, the lamination is embedded in  $\mathbb{S}^2 \times \mathbb{S}^1$ . We represent the manifold  $\mathbb{S}^2 \times \mathbb{S}^1$  as 3-dimensional annulus, suggesting that its two boundary components are *radially* identified. On the right, the lamination is embedded in  $\mathbb{R}^3$ . We are going to show that the topological classification of diffeomorphisms in the class  $G(M)$  reduces to classifying some appropriate laminations  $\hat{\Gamma}_f^u$  and  $\hat{\Gamma}_f^s$ . The technical key to the proof consists of constructing special foliations in some neighborhoods of the laminations.

### 3 Compatible foliations

Let  $f \in G(M)$ . Recall that we divided the set  $\Omega_2$  into the  $f$ -invariant parts  $\Sigma_0, \dots, \Sigma_n$ . Using this partition, we explain how to construct compatible foliations (see definition 3.3) around  $W_{\Omega_2}^u \cup W_{\Omega_2}^s$ . Similarly, it is possible to construct compatible foliations around  $W_{\Omega_1}^s \cup W_{\Omega_1}^u$ . In what follows, we give ourselves four models of direct hyperbolic linear isomorphisms  $a_{\mu,\nu} \in$

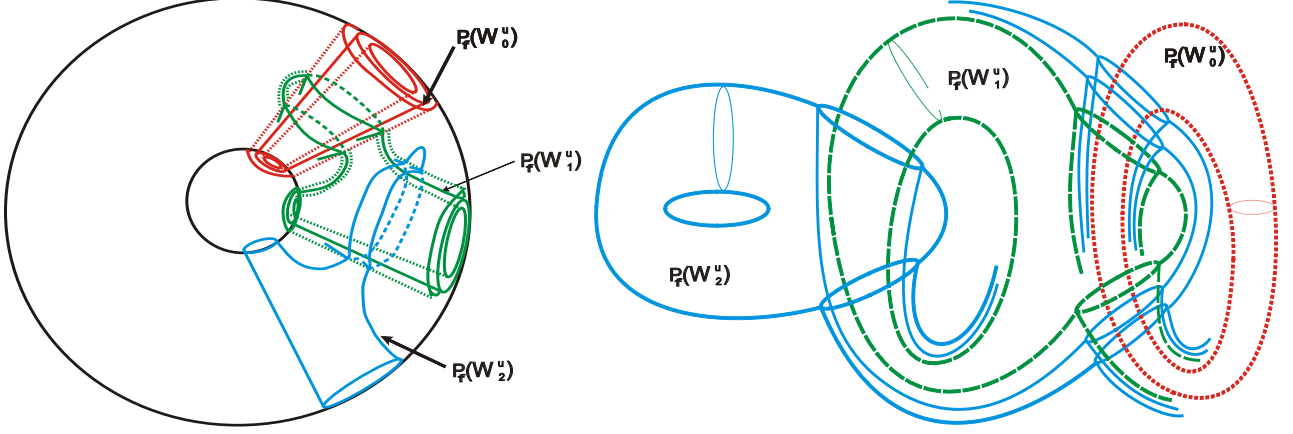


Figure 4: A lamination associated with the diffeomorphism  $f \in G(M)$  whose phase portrait is pictured in Figure 3.

$GL(\mathbb{R}^3)$ ,  $\mu, \nu \in \{-, +\}$  given by the following formula:

$$a_{\mu, \nu}(x_1, x_2, x_3) = (\mu 2x_1, 2x_2, \nu \frac{x_3}{4}).$$

The origin  $O$  is the unique fixed point which is a saddle point with unstable manifold  $W_O^u = Ox_1x_2$  and stable manifold  $W_O^s = Ox_3$ . If  $\mu = +$  (resp.  $-$ ), the orientation of the unstable manifold is preserved (resp. reversed), and similarly for the orientation of the stable manifold with respect to  $\nu$ . We refer to each of them as the *canonical diffeomorphism*, noted by  $a$  without taking  $\pm$  into account. For  $p \in \Omega_2$ , let  $per(p)$  denote the period of  $f$  at  $p$ .

**Definition 3.1** A neighborhood  $N_p$  of a saddle point  $p \in \Omega_2$  is called linearizable if there is a homeomorphism  $\mu_p : N_p \rightarrow \mathcal{N}$  which conjugates the diffeomorphism  $f^{per(p)}|_{N_p}$  to the canonical diffeomorphism  $a|_{\mathcal{N}}$ .

According to the local topological classification of hyperbolic fixed point [19, Theorem 5.5], every  $p \in \Omega_2$  has a linearizable neighbourhood  $N_p$ . For  $t \in (0, 1)$ , set  $\mathcal{N}^t := \{(x_1, x_2, x_3) \in \mathbb{R}^3 \mid -t < (x_1^2 + x_2^2)x_3 < t\}$  and  $\mathcal{N} := \mathcal{N}^1$ . The set  $\mathcal{N}^t$  is invariant by the canonical diffeomorphism  $a$ . By [24],  $W_p^s$  and  $W_p^u$  are smooth submanifolds of  $M$ . The *boundary* of  $\mathcal{N}$  is the surface in  $\mathbb{R}^3$  defined by the equations  $(x_1^2 + x_2^2)x_3 = \pm 1$ . The open manifold  $N_p$  has a similar boundary in  $M$  denoted by  $\partial N_p$ . This boundary is formed by points which are not in  $N_p$  but limit points of arcs in  $N_p$ ; it is distinct from its closure as a subset of  $M$ . Clearly, the linearizing homeomorphism  $\mu_p$  extends to  $\partial N_p$ . For each  $i \in \{0, \dots, n\}$ , choose some  $p \in \Sigma_i$  and  $\mu_p$  conjugating  $f^{per(p)}$  to  $a|_{\mathcal{N}}$ . Then, for  $k \in \{1, \dots, per(p) - 1\}$  define  $\mu_{f^k(p)}$  so that the next formula holds for every  $x \in N_{f^{k-1}(p)}$ :

$$\mu_{f^k(p)}(f(x)) = \mu_{f^{k-1}(p)}(x).$$



We define a pair of transverse foliations  $(\mathcal{F}^u, \mathcal{F}^s)$  in  $\mathcal{N}$  in the following way: – the leaves of  $\mathcal{F}^u$  are the fibres in  $\mathcal{N}$  of the projection  $(x_1, x_2, x_3) \mapsto x_3$ ; – the leaves of  $\mathcal{F}^s$  are the fibres in  $\mathcal{N}$  of the projection  $(x_1, x_2, x_3) \mapsto (x_1, x_2)$ . By construction,  $W_O^u$  and  $W_O^s$  are leaves of  $\mathcal{F}^u$  and  $\mathcal{F}^s$  respectively. Let  $N_i$  denote the union  $\bigcup_{p \in \Sigma_i} N_p$ . This is an  $f$ -invariant neighbourhood of  $\Sigma_i$ .

Let  $\mu_i : N_i \rightarrow \mathcal{N}$  be the map whose restriction to  $N_p$  is  $\mu_p$ . Thus, taking the pullback of them by  $\mu_i$  gives a pair of  $f$ -invariant foliations  $(F_i^u, F_i^s)$  on  $N_i$  which are said to be *linearizable*. By construction,  $W_i^u$  and  $W_i^s$  are made of leaves of  $F_i^u$  and  $F_i^s$  respectively. Sometimes we want to deform the linearizable neighborhood  $N_p$  by *shrinking*. Observe that the homotheties of ratio  $\rho \in (0, 1)$  act on  $\mathcal{N}$  preserving  $\mathcal{F}^u$  and  $\mathcal{F}^s$  and map  $\mathcal{N}$  to  $\mathcal{N}^{\rho^3}$ . By conjugation, similar contractions  $c_\rho$  are available in  $N_p$  for every  $p \in \Omega_2$ . The neighborhood  $c_\rho(N_p)$  is said to be obtained from  $N_p$  by *shrinking*.

**Lemma 3.2** *For every  $\rho \in (0, 1)$ , the shrunk neighbourhood  $c_\rho(N_p)$  is linearizable. Generically, the boundary of  $c_\rho(N_p)$  does not contain any heteroclinic point.*

**Proof:** For a given  $\mu_p$ , we define  $\mu_p^\rho$  as follows: its domain is  $\mu_p^{-1}(\mathcal{N}^{\rho^3})$  and, on this domain, it is defined by  $\mu_p^\rho = c_\rho^{-1} \circ \mu_p$ . Its range is  $\mathcal{N}$ . Since the heteroclinic points form a countable set, for almost every  $\rho \in (0, 1)$  the boundary of the domain of  $\mu_p^\rho$  avoids the heteroclinic points.  $\diamond$

Observe that the canonical diffeomorphism and the contraction  $c_\rho$  keep both foliations invariant. Recall the  $f$ -invariant partition  $\Omega_2 = \Sigma_0 \sqcup \Sigma_1 \sqcup \dots \sqcup \Sigma_n$ . Let us introduce the following notations:

- for any  $t \in (0, 1)$ , set  $N_p^t := \mu_p^{-1}(\mathcal{N}^t)$  and  $N_i^t := \bigcup_{p \in \Sigma_i} N_p^t$ ;
- for any point  $x \in N_i$ , denote  $F_{i,x}^u$  (resp.  $F_{i,x}^s$ ) the leaf of the foliation  $F_i^u$  (resp.  $F_i^s$ ) passing through  $x$ ;
- for each point  $x \in N_i$ , set  $x_i^u = W_i^u \cap F_{i,x}^s$  and  $x_i^s = W_i^s \cap F_{i,x}^u$ . Thus, we have  $x = (x_i^u, x_i^s)$  in the coordinates defined by  $\mu_i$ .

We also introduce the *radial functions*  $r_i^u, r_i^s : N_i \rightarrow [0, +\infty)$  defined by:

$$r_i^u(x) = \|\mu_i(x_i^u)\|^2 \quad \text{and} \quad r_i^s(x) = |\mu_i(x_i^s)|.$$

With this definition at hand, the neighborhood  $N_i^t$  of  $\Sigma_i$  is defined by the inequality

$$r_i^u(x) \cdot r_i^s(x) < t.$$

Observe that the radial function  $r_i^s$  endows each stable separatrix of  $p \in \Sigma_p$  with a natural order which will be used later in the proof of Theorem 1.

**Definition 3.3** *The linearizable neighborhoods  $N_0, \dots, N_n$  are called compatible if, for any  $0 \leq i < j \leq n$  and  $x \in N_i \cap N_j$ , the following holds:*

$$F_{j,x}^s \cap N_i \subset F_{i,x}^s \quad \text{and} \quad F_{i,x}^u \cap N_j \subset F_{j,x}^u.$$

If linearizable neighbourhoods are compatible, they remain so after some of them are shrunk.

**Remark 3.4** The notion of compatible foliations is a modification of the admissible systems of tubular families introduced by J. Palis and S. Smale in [18] and [20].

We introduce the following notations:

- For  $i \in \{0, \dots, n\}$ , set  $A_i := A_f \cup \bigcup_{j=0}^i W_j^u$ ,  
 $V_i := W_{A_i \cap \Omega_f}^s \setminus A_i$ ,  $\hat{V}_i := V_i/f$ . Observe that  $f$  acts freely on  $V_i$  and denote the natural projection by  $p_i : V_i \rightarrow \hat{V}_i$ .
- For  $j, k \in \{0, \dots, n\}$  and  $t \in (0, 1)$ , set  $\hat{W}_{j,k}^s = p_k(W_j^s \cap V_k)$ ,  $\hat{W}_{j,k}^u = p_k(W_j^u \cap V_k)$ ,  
 $\hat{N}_{j,k}^t = p_k(N_j^t \cap V_k)$ .
- $L^u := \bigcup_{i=0}^n W_i^u$ ,  $L^s := \bigcup_{i=0}^n W_i^s$ ,  $L_i^u := L^u \cap V_i$ ,  $L_i^s := L^s \cap V_i$ ,  $\hat{L}_i^u := p_i(L_i^u)$ ,  $\hat{L}_i^s := p_i(L_i^s)$ .

**Theorem 2** For each diffeomorphism  $f \in G(M)$  there exist compatible linearizable neighborhoods of all saddle points whose Morse index is 2.

**Proof:** The proof consists of three steps.

**Step 1.** Here, we prove the following claim.

**Lemma 3.5** There exist  $f$ -invariant neighborhoods  $U_0^s, \dots, U_n^s$  of the sets  $\Sigma_0, \dots, \Sigma_n$  respectively, equipped with two-dimensional  $f$ -invariant foliations  $F_0^u, \dots, F_n^u$  of class  $C^{1,0}$  such that the following properties hold for each  $i \in \{0, \dots, n\}$ :

- (i) the unstable manifolds  $W_i^u$  are leaves of the foliation  $F_i^u$  and each leaf of the foliation  $F_i^u$  is transverse to  $L_i^s$ ;
- (ii) for any  $0 \leq i < k \leq n$  and  $x \in U_i^s \cap U_k^s$ , we have the inclusion  $F_{k,x}^u \cap U_i^s \subset F_{i,x}^u$ .

**Proof:** Let us prove this by a decreasing induction on  $i$  from  $i = n$  to  $i = 0$ . For  $i = n$ , it follows from the definition of  $V_n$  that  $(W_n^s \setminus \Sigma_n) \subset V_n$ . Since  $f$  acts freely and properly on  $W_n^s$ , the quotient  $\hat{W}_{n,n}^s$  is a smooth submanifold of  $\hat{V}_n$ ; it consists of finitely many circles. The lamination  $\hat{L}_n^s$  accumulates on  $\hat{W}_{n,n}^s$ . Choose an open tubular neighborhood  $\hat{N}_n^s$  of  $\hat{W}_{n,n}^s$  in  $\hat{V}_n$ ; denote its projection by  $\pi_n^u : \hat{N}_n^s \rightarrow \hat{W}_{n,n}^s$ . Its fibers form a 2-disc foliation  $\{d_{n,x}^u \mid x \in \hat{W}_{n,n}^s\}$  transverse to  $\hat{W}_{n,n}^s$ . Since  $\hat{L}_n^s$  is a  $C^{1,0}$ -lamination containing  $\hat{W}_{n,n}^s$ , each plaque of  $\hat{W}_{n,n}^s$  is the  $C^1$ -limit of any sequence of plaques approaching it  $C^0$ . Therefore, if the tube  $\hat{N}_n^s$  is small enough, its fibers are transverse to  $\hat{L}_n^s$ .

Set  $U_n^s := p_n^{-1}(\hat{N}_n^s) \cup W_n^u$ . This is an open set of  $M$  which carries a foliation  $F_n^u$  defined by taking the preimage of the fibers of  $\pi_n^u$  and by adding  $W_n^u$  as extra leaves. This is the wanted foliation satisfying (i) and (ii) for  $i = n$ . Notice that the plaques of  $F_n^u$  are smooth and by the  $\lambda$ -lemma, for any compact disc  $B$  in  $W_n^u$  there is  $\varepsilon > 0$  such that every plaque of  $F_n^u$  which is

$\varepsilon$ -close to  $B$  in topology  $C^0$  is also  $\varepsilon$ -close to  $B$  in topology  $C^1$ . Hence,  $F_n^u$  is a  $C^{1,0}$ -foliation. For the induction, we assume the construction is done for every  $j > i$  and we have to construct an  $f$ -invariant neighborhood  $U_i^s$  of the saddle points in  $\Sigma_i$  carrying an  $f$ -invariant foliation  $F_i^u$  satisfying (i) and (ii). Moreover, by genericity the boundary  $\partial U_j^s$ ,  $j > i$ , is assumed to avoid all heteroclinic points. For  $j > i$ , let  $\hat{U}_{j,i}^s := p_i(U_j^s \cap V_i)$  and  $\hat{F}_{j,i}^u := p_i(F_j^u \cap V_i)$ . For the same reason as in the case  $i = n$ , the set  $\hat{W}_{i,i}^s$  is a smooth submanifold of  $\hat{V}_i$  consisting of circles. Choose a tubular neighborhood  $\hat{N}_i^s$  of  $\hat{W}_{i,i}^s$  with a projection  $\pi_i^s : \hat{N}_i^s \rightarrow \hat{W}_{i,i}^s$  whose fibers are 2-discs. Similarly,  $\hat{W}_{i+1,i}^u$  is a compact submanifold, consisting of finitely many tori or Klein bottles. The set  $\hat{L}_i^u$  is a compact lamination and its intersection with  $\hat{W}_{i,i}^s$  consists of a countable set of points which are the projections of the heteroclinic points belonging to the stable manifolds  $W_i^s$ . Actually, there is a hierarchy in  $\hat{L}_i^u \cap \hat{W}_{i,i}^s$  which we are going to describe in more details.

Set  $H_k := \hat{W}_{i+k,i}^u \cap \hat{W}_{i,i}^s$  for  $k > 0$ . Since  $\hat{W}_{i+1,i}^u$  is compact,  $H_1$  is a finite set:  $H_1 = \{h_1^1, \dots, h_{t(1)}^1\}$ . We are given neighborhoods, called *boxes*,  $B_\ell^1$ ,  $\ell = 1, \dots, t(1)$ , about these points, namely, the connected components of  $\hat{U}_{i+1,i}^s \cap \hat{N}_i^s$ . Due to the fact that  $\partial \hat{U}_{i+1,i}^s$  contains no heteroclinic point,  $\partial \hat{U}_{i+1,i}^s \cap \hat{W}_{i,i}^s$  is isolated from  $\hat{L}_i^u$ . Therefore, if the tube  $\hat{N}_i^s$  is small enough,  $\hat{L}_i^u$  does not intersect  $\partial \hat{U}_{i+1,i}^s \cap \hat{N}_i^s$ . Then, by shrinking  $U_j^s$ ,  $j > i + 1$  (in the sense of Lemma 3.2) if necessary, we may guarantee that  $\hat{U}_{j,i}^s \cap \hat{N}_i^s$  is disjoint from  $\partial \hat{U}_{i+1,i}^s \cap \hat{N}_i^s$ .

Since  $\hat{W}_{i+2,i}^u$  accumulates on  $\hat{W}_{i+1,i}^u$ , there are only finitely many points of  $H_2$  outside of all boxes  $B_\ell^1$ ,  $\ell = 1, \dots, t(1)$ . Let  $\bar{H}_2 := \{h_1^2, \dots, h_{t(2)}^2\}$  be this finite set. The open set  $\hat{U}_{i+2,i}^s$  is a neighborhood of  $\bar{H}_2$ . The connected components of  $\hat{U}_{i+2,i}^s \cap \hat{N}_i^s$  which contain points of  $\bar{H}_2$  will be the box  $B_\ell^2$  for  $\ell = 1, \dots, t(2)$ . We argue with  $B_\ell^2$  with respect to  $\hat{L}_i^u$  and the neighborhoods  $\hat{U}_{j,i}$ ,  $j > i + 1$ , in a similar manner as we do with  $B_\ell^1$ . And so on, until  $\bar{H}_n$ .

Due to the induction hypothesis, each above-mentioned box is foliated. Namely,  $B_\ell^1$  is foliated by  $\hat{F}_{i+1,i}^u$ ; the box  $B_\ell^2$  is foliated by  $\hat{F}_{i+2,i}^u$ , and so on. But the leaves are not contained in fibres of  $\hat{N}_i^s$ ; even more, not every leaf intersects  $\hat{W}_{i,i}^s$ . We have to correct this situation in order to construct the foliation  $F_i^u$  satisfying the wanted conditions (i) and (ii). For every  $j > i$ , the foliation  $F_j^u$  may be extended to the boundary  $\partial U_j^s$  and a bit beyond. Once this is done, if  $\hat{N}_i^s$  is enough shrunk, each leaf of  $\hat{F}_{i+k,i}^u$  through  $x \in B_\ell^k$  intersects  $\hat{W}_{i,i}^s$  (it is understood that the boxes are intersected with the shrunk tube without changing their names). Thus, we have a projection along the leaves  $\pi_{k,\ell} : B_\ell^k \rightarrow \hat{W}_{i,i}^s$ ; but, the image of  $\pi_{k,\ell}$  is larger than  $B_\ell^k \cap \hat{W}_{i,i}^s$ . Then, we choose a small enlargement  $B_\ell^{k'}$  of  $B_\ell^k$  such that  $B_\ell^{k'} \setminus B_\ell^k$  is foliated by  $\hat{F}_{i+k,i}^u$  and avoids the lamination  $\hat{L}_i^u$ . On  $B_\ell^{k'} \setminus B_\ell^k$  we have two projections: one is  $\hat{\pi}_i^u$  and the other one is  $\pi_{k,\ell}$ . We are going to interpolate between both using a partition of unity (we do it for  $B_\ell^k$  but it is understood that it is done for all boxes). Let  $\phi : \hat{N}_i^s \rightarrow [0, 1]$  be a smooth function which equals 1 near  $B_\ell^k$  and whose support is contained in  $B_\ell^{k'}$ . Define a global  $C^1$  retraction  $\hat{q} : \hat{N}_i^s \rightarrow \hat{W}_{i,i}^s$  by the formula

$$\hat{q}(x) = (1 - \phi(x))\hat{\pi}_i^u(x) + \phi(x)(\pi_{k,\ell}(x)).$$

Here, we use an affine manifold structure on each component of  $\hat{W}_{i,i}^s$  by identifying it with the 1-torus  $\mathbb{T} := \mathbb{R}/\mathbb{Z}$ . So, any positively weighted barycentric combination makes sense for a pair

of points sufficiently close. When  $x \in \hat{W}_{i,i}^s$ , we have  $\hat{q}(x) = x$ . Then, by shrinking the tube  $\hat{N}_i^s$  once more if necessary we make  $\hat{q}$  be a fibration whose fibres are transverse to the lamination  $\hat{L}_i^s$  and we make each leaf of  $\hat{F}_{j,i}^u$ ,  $j > i$ , in every box  $B_k^\ell$  be contained in a fibre of  $q$ . Henceforth, taking the preimage of that tube (and its fibration) by  $p_i$  and adding the unstable manifold  $W_i^u$  provide the wanted  $U_i^s$  and its foliation  $F_i^u$  satisfying the required properties. Thus, the induction is proved.  $\diamond$

We also have the following statement.

**Lemma 3.6** *There exist  $f$ -invariant neighborhoods  $U_0^u, \dots, U_n^u$  of the sets  $\Sigma_0, \dots, \Sigma_n$  respectively, equipped with one-dimensional  $f$ -invariant foliations  $F_0^s, \dots, F_n^s$  of class  $C^{1,0}$  such that the following properties hold for each  $i \in \{0, \dots, n\}$ : (iii) the stable manifold  $W_i^s$  is a leaf of the foliation  $F_i^s$  and each leaf of the foliation  $F_i^s$  is transverse to  $L_i^u$ ;*

(iv) *for any  $0 \leq j < i$  and  $x \in U_i^u \cap U_j^u$ , we have the inclusion  $(F_{j,x}^s \cap U_i^u) \subset F_{i,x}^s$ .*

**Proof:** The proof is done by an increasing induction from  $i = 0$ ; it is skipped due to similarity to the previous one.  $\diamond$

**Step 2.** We prove the following statement for each  $i = 0, \dots, n$ .

**Lemma 3.7** (v) *There exists an  $f$ -invariant neighborhood  $\tilde{N}_i$  of the set  $\Sigma_i$  contained in  $U_i^s \cap U_i^u$  and such that the restrictions of the foliations  $F_i^u$  and  $F_i^s$  to  $\tilde{N}_i$  are transverse.*

**Proof:** For this aim, let us choose a fundamental domain<sup>6</sup>  $K_i^s$  of the restriction of  $f$  to  $W_i^s \setminus \Sigma_i$  and take a tubular neighborhood  $N(K_i^s)$  of  $K_i^s$  whose disc fibres are contained in leaves of  $F_i^u$ . By construction,  $F_i^u$  is transverse to  $W_i^s$  and, according to the Lemma 3.6,  $F_i^s$  is a  $C^{1,0}$ -foliation. Therefore, if the tube  $N(K_i^s)$  is small enough,  $F_i^u$  is transverse to  $F_i^s$  in  $N(K_i^s)$ . Set

$$\tilde{N}_i := W_i^u \bigcup_{k \in \mathbb{Z}} f^k(N(K_i^s)).$$

This is a neighborhood of  $\Sigma_i$ ; it satisfies condition (v) and the previous properties (i)–(iv) still hold. A priori the boundary of  $\tilde{N}_i$  is only piecewise smooth; but, by choosing  $N(K_i^s)$  correctly at its corners we may arrange that  $\partial\tilde{N}_i$  be smooth.  $\diamond$

**Step 3.** For proving Theorem 2 it remains to show the existence of linearizable neighborhoods  $N_i \subset \tilde{N}_i$ ,  $i = 0, \dots, n$ , for which the required foliations are the restriction to  $N_i$  of the foliations  $F_i^u$  and  $F_i^s$ . For each orbit of  $f$  in  $\Sigma_i$ , choose one  $p$ . Let  $\tilde{N}_p$  be a connected component of  $\tilde{N}_i$  containing  $p$ . There is a homeomorphism  $\varphi_p^u : W_p^u \rightarrow W_O^u$  (resp.  $\varphi_p^s : W_p^s \rightarrow W_O^s$ ) conjugating the diffeomorphisms  $f^{per(p)}|_{W_p^u}$  and  $a|_{W_O^u}$  (resp.  $f^{per(p)}|_{W_p^s}$  and  $a|_{W_O^s}$ ). In addition, for any point  $z \in \tilde{N}_p$  there is unique pair of points  $z_s \in W_p^s$ ,  $z_u \in W_p^u$

<sup>6</sup>Let us recall that a *fundamental domain of action  $g : X \rightarrow X$  on  $X$*  is a closed set  $D_g \subset X$  such that there is a set  $\tilde{D}_g$  with the following properties:

- 1)  $cl(\tilde{D}_g) = D_g$ ;
- 2)  $g^k(\tilde{D}_g) \cap \tilde{D}_g = \emptyset$  for all  $k \in (\mathbb{Z} \setminus \{0\})$ ;
- 3)  $\bigcup_{k \in \mathbb{Z}} g^k(\tilde{D}_g) = X$ .

such that  $z = F_{i,z_u}^s \cap F_{i,z_s}^u$ . We define a topological embedding  $\tilde{\mu}_p : \tilde{N}_p \rightarrow \mathbb{R}^3$  by the formula  $\tilde{\mu}_p(z) = (x_1, x_2, x_3)$  where  $(x_1, x_2) = \varphi_p^u(z_u)$  and  $x_3 = \varphi_p^s(z_s)$ . Since the foliations  $F_i^u$  and  $F_i^s$  are  $f$ -invariant, this definition makes  $\tilde{\mu}_p$  conjugate the restriction  $f^{per(p)}|_{\tilde{N}_p}$  to  $a^{per(p)}$ . For  $k = 1, \dots, per(p) - 1$ , set  $\tilde{N}_{f^k(p)} := f^k(\tilde{N}_p)$  and define  $\tilde{\mu}_{f^k(p)}$  so that the equivariance formula holds:  $\tilde{\mu}_{f^k(p)}(f^k(x)) = a^k \tilde{\mu}_p(x)$  for every  $x \in \tilde{N}_p$ . Choose  $t_0 \in (0, 1]$  such that  $\mathcal{N}^{t_0} \subset \tilde{\mu}_p(\tilde{N}_p)$  for every  $p \in \Sigma_i$ . Observe that  $a|_{\mathcal{N}^{t_0}}$  is conjugate to  $a|_{\mathcal{N}}$  by the suitable homothety  $h$ . Set  $N_p = \tilde{\mu}_p^{-1}(\mathcal{N}^{t_0})$  and  $\mu_p = h\tilde{\mu}_p : N_p \rightarrow \mathcal{N}$ . Then,  $N_p$  is the wanted neighborhood with its linearizing homeomorphism  $\mu_p$ . This finishes the proof of Theorem 2.  $\diamond$

## 4 Proof of the classification theorem

Let us prove that the diffeomorphisms  $f$  and  $f'$  in  $G(M)$  are topologically conjugate if and only if there is a homeomorphism  $\hat{\varphi} : \hat{V}_f \rightarrow \hat{V}_{f'}$  such that 1)  $\eta_f = \eta_{f'} \hat{\varphi}_*$ ; 2)  $\hat{\varphi}(\hat{\Gamma}_f^s) = \hat{\Gamma}_{f'}^s$  and  $\hat{\varphi}(\hat{\Gamma}_f^u) = \hat{\Gamma}_{f'}^u$ .

### 4.1 Necessity

Let  $f : M \rightarrow M$  and  $f' : M \rightarrow M$  be two elements in  $G(M)$  which are topologically conjugated by some homeomorphism  $h : M \rightarrow M$ . Then  $h$  conjugates the invariant manifolds of periodic points of  $f$  and  $f'$ . More precisely, if  $p$  is a periodic point of  $f$ , then  $h(p)$  is a periodic point of  $f'$  and  $h(W^u(p)) = W^u(h(p))$ ,  $h(W^s(p)) = W^s(h(p))$ . In particular,  $h$  maps  $V_f$  to  $V_{f'}$  by a homeomorphism noted  $\varphi$ . Moreover, if  $x$  is any points of  $V_f$ , for every  $n \in \mathbb{Z}$  the following holds:

$$\varphi(f^n(x)) = f'^n(\varphi(x)).$$

This formula says exactly that  $\varphi$  is the lift of a map  $\hat{\varphi} : \hat{V}_f \rightarrow \hat{V}_{f'}$ . By construction of  $\eta_f$ , the same formula says that  $\eta_f = \eta_{f'} \circ \hat{\varphi}_*$ , where  $\hat{\varphi}_* : H_1(\hat{V}_f; \mathbb{Z}) \rightarrow H_1(\hat{V}_{f'}; \mathbb{Z})$  denotes the map induced in homology. By definition of the quotient topology,  $\hat{\varphi}$  is continuous. Since the same holds for  $\varphi^{-1}$ , one checks that  $\hat{\varphi}$  is a homeomorphism. As  $\varphi$  conjugates the laminations  $\Gamma_f^s$  (resp.  $\Gamma_f^u$ ) to  $\Gamma_{f'}^s$  (resp.  $\Gamma_{f'}^u$ ), the same holds for  $\hat{\varphi}$  in the quotient spaces with respect the projections of the laminations.

### 4.2 Sufficiency

For proving the sufficiency of the conditions in Theorem 1, let us consider a homeomorphism  $\hat{\varphi} : \hat{V}_f \rightarrow \hat{V}_{f'}$  such that:

- (1)  $\eta_f = \eta_{f'} \hat{\varphi}_*$ ;
- (2)  $\hat{\varphi}(\hat{\Gamma}_f^s) = \hat{\Gamma}_{f'}^s$  and  $\hat{\varphi}(\hat{\Gamma}_f^u) = \hat{\Gamma}_{f'}^u$ .

From now on, the dynamical objects attached to  $f'$  will be denoted by  $L'^u, L'^s, \Sigma'_i, \dots$  with the same meaning as  $L^u, L^s, \Sigma_i, \dots$  have with respect to  $f$ . Due to property (1),  $\hat{\varphi}$  lifts to an *equivariant*<sup>7</sup> homeomorphism  $\varphi : V_f \rightarrow V_{f'}$ , that is:  $f'|_{V_{f'}} = \varphi f \varphi^{-1}|_{V_f}$ . Due to property (2),

<sup>7</sup>For brevity, equivariance stands for  $(f, f')$ -equivariance.

$\varphi$  maps  $\Gamma_f^u$  to  $\Gamma_{f'}^u$  and  $\Gamma_f^s$  to  $\Gamma_{f'}^s$ . Thanks to Theorem 2 we may use compatible linearizable neighbourhoods of the saddle points of  $f$  (resp.  $f'$ ).

An idea of the proof is the following: we modify the homeomorphism  $\varphi$  in a neighborhood of  $\Gamma_f^u$  such that the final homeomorphism preserves the compatible foliations, then we do similar modification near  $\Gamma_f^s$ . So we get a homeomorphism  $h : M \setminus (\Omega_0 \cup \Omega_3) \rightarrow M \setminus (\Omega'_0 \cup \Omega'_3)$  conjugating  $f|_{M \setminus (\Omega_0 \cup \Omega_3)}$  with  $f'|_{M \setminus (\Omega'_0 \cup \Omega'_3)}$ . Notice that  $M \setminus (W_{\Omega_1}^s \cup W_{\Omega_2}^s \cup \Omega_3) = W_{\Omega_0}^s$  and  $M \setminus (W_{\Omega'_1}^s \cup W_{\Omega'_2}^s \cup \Omega'_3) = W_{\Omega'_0}^s$ . Since  $h(W_{\Omega_1}^s) = W_{\Omega'_1}^s$  and  $h(W_{\Omega_2}^s) = W_{\Omega'_2}^s$  then  $h(W_{\Omega_0}^s \setminus \Omega_0) = W_{\Omega'_0}^s \setminus \Omega'_0$ . Thus for each connected component  $A$  of  $W_{\Omega_0}^s \setminus \Omega_0$  there is a sink  $\omega \in \Omega_0$  such that  $A = W_{\omega}^s \setminus \omega$ . Similarly  $h(A)$  is a connected component of  $W_{\Omega'_0}^s \setminus \Omega'_0$  such that  $h(A) = W_{\omega'}^s \setminus \omega'$  for a sink  $\omega' \in \Omega'_0$ . Then we can continuously extend  $h$  to  $\Omega_0$  assuming  $h(\omega) = \omega'$  for every  $\omega \in \Omega_0$ . A similar extension of  $h$  to  $\Omega_3$  finishes the proof. Thus below in a sequence of lemmas we explain only how to modify the homeomorphism  $\varphi$  in a neighborhood of  $\Gamma_f^u$  such that the final homeomorphism preserves the compatible foliations.

Recall the partition  $\Sigma_0 \sqcup \dots \sqcup \Sigma_n$  associated with the Smale order on the periodic points of index 2.

**Lemma 4.1** *For every  $i = 0, \dots, n$  the following equality holds  $\varphi(W_i^u \cap V_f) = W_i'^u \cap V_{f'}$  and there is a unique continuous extension of  $\varphi|_{W_i^u \cap V_f}$  to  $\Sigma_i$  which is equivariant and bijective from  $\Sigma_i$  to  $\Sigma_i'$ .*

**Proof:** Let  $p \in \Sigma_0$ . Denote its orbit by  $orb_f(p)$ . The punctured unstable manifold  $W^u(p) \setminus \{p\}$  projects by  $p_f$  to one compact leaf  $\ell(p)$ . Both sides of the next equality are  $f$ -invariant and project to the same leaf, thus:

$$p_f^{-1}(\ell(p)) = W^u(orb(p)) \setminus \{orb(p)\}.$$

Then, the number of connected components of  $p_f^{-1}(\ell(p))$  is  $per(p)$ , the period of  $p$ . The image  $\hat{\varphi}(\ell(p))$  is a compact leaf of  $\hat{\Gamma}_{f'}^u$ . By the previous argument, it is  $\ell'(p')$  for some  $p' \in \Sigma'_0$ . Since  $\hat{\varphi}$  lifts to  $\varphi$ , then  $\varphi[p_f^{-1}(\ell(p))] = p_{f'}^{-1}(\ell'(p'))$  which implies the equality of the number of connected components. Thus  $per(p) = per(p')$ . From this, we can deduce that, up to replacing  $p'$  with  $f'^k(p')$  for some integer  $k$ , we have  $\varphi(W^u(p) \setminus \{p\}) = W^u(p') \setminus \{p'\}$ . Using the property  $p = \lim_{n \rightarrow -\infty} f^n(x)$  for every  $x \in W^u(p)$  and the similar property for  $p'$  in addition to the equivariance of  $\varphi$ , one extends continuously  $\varphi|_{W_p^u}$  by defining  $\varphi(p) = p'$ . Doing the same for every orbit of  $\Sigma_0$ , we get a continuous extension of  $\varphi|_{W_0^u}$  to  $\Sigma_0$  which is still equivariant. One easily checks that this extension is continuous, unique, and hence equivariant. Then, arguing similarly with  $\hat{\varphi}^{-1}$ , we derive that the extension of  $\varphi$  maps  $\Sigma_0$  bijectively onto  $\Sigma'_0$ .

Denote  $\ell_0 := \bigcup_{p \in \Sigma_0} \ell(p)$ . We have  $\hat{\varphi}(\ell_0) = \ell'_0$ . Let now  $p \in \Sigma_1$ . The closure in  $\hat{V}_f$  of  $\ell(p) := p_f(W^u(p) \setminus \{p\})$  is contained in  $\ell(p) \cup \ell_0$ . We deduce that  $\hat{\varphi}(\ell(p))$  is a leaf of  $\hat{\Gamma}_{f'}^u$  of the form  $\ell'(p')$  for some  $p' \in \Sigma'_1$  and we can continue inductively. Thus, there is a continuous extension of  $\varphi|_{W_i^u}$  to every  $\Sigma_i$  for  $i = 0, 1, \dots, n$  which is a bijection  $\Sigma_i \rightarrow \Sigma'_i$ . Arguing with  $\hat{\varphi}^{-1}$ , we derive that  $n' = n$ .  $\diamond$

Recall the radial functions  $r_i^u, r_i^s : N_i \rightarrow [0, +\infty)$  which are introduced above Definition 3.3; recall also the order which is defined by  $r_i^s$  on each stable separatrix  $\gamma_p$  of  $p \in \Sigma_i$ . Similar functions are associated with the dynamics of  $f'$ .

**Lemma 4.2** *There is a unique continuous extension of  $\varphi|_{\Gamma_f^u}$*

$$\varphi^{us} : \Gamma_f^u \cup (L^u \cap L^s) \longrightarrow \Gamma_{f'}^u \cup (L'^u \cap L'^s)$$

such that the following holds:

- 1) If  $x \in W_j^u \cap W_i^s$ ,  $j > i$ , then  $\varphi^{us}(x) \in W_j'^u \cap W_i'^s$ .
- 2) If  $x$  and  $y$  lie in  $\gamma_p \cap L^u$  with  $r_p^s(x) < r_p^s(y)$ , then  $\varphi^{us}(x)$  and  $\varphi^{us}(y)$  lie in  $\gamma'_{\varphi(p)} \cap L'^u$  with  $r'_{\varphi(p)}(\varphi^{us}(x)) < r'_{\varphi(p)}(\varphi^{us}(y))$ .

Notice that  $\varphi|_{\Gamma_f^u}$  being equivariant its continuous extension is also equivariant.

**Proof:** This statement is proved by induction on  $i$ . We recall that  $V_i \setminus L_i^s = V_f \setminus cl(W_i^u)$  is a dense open set in  $V_f$  (and similarly with  $'$ ), and according to Lemma 4.1,  $\varphi$  maps  $V_i \setminus L_i^s$  to  $V_i' \setminus L_i'^s$  homeomorphically and conjugates  $f$  to  $f'$ . Thus, for every  $i = 0, \dots, n$ , we have an equivariant homeomorphism  $\varphi_i : V_i \setminus L_i^s \rightarrow V_i' \setminus L_i'^s$  which maps  $W_j^u \setminus L_i^s$  to  $W_j'^u \setminus L_i'^s$  for every  $j > i$ , again as a consequence of Lemma 4.1.

First, take  $i = 0$ . The manifold  $\hat{V}_0$  is closed and three-dimensional. We have  $\hat{L}_0^s = \hat{W}_{0,0}^s$ , which consists of finite number disjoint smooth circles, and similarly with  $'$ .

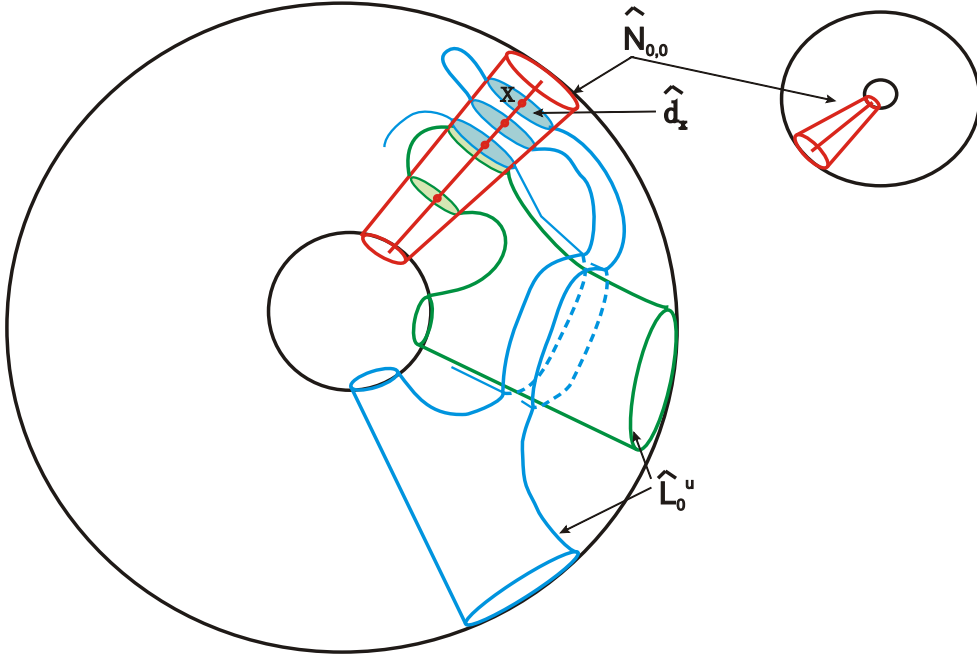


Figure 5: Case  $i = 0$  in proof of Lemma 4.2 for the diffeomorphism from Figure 3.

We look for an extension  $\hat{\varphi}_0^{us}$  of  $\hat{\varphi}_0|_{\hat{L}_0^u}$  to  $\hat{L}_0^s \cap \hat{L}_0^u$ . If  $N_0$  is the neighborhood of  $\Sigma_0$  extracted from a compatible system given by Theorem 2 and if  $\hat{N}_{0,0}$  denotes the corresponding tubular neighborhood of  $\hat{L}_0^s$  in  $\hat{V}_0$ , the trace of  $\hat{L}_0^u$  in that tube is a lamination by disks:

$$\hat{L}_0^u \cap \hat{N}_{0,0} = \{\hat{d}_x \mid x \in \hat{L}_0^u \cap \hat{L}_0^s\},$$

where  $\hat{d}_x$  denotes the fiber of the tube over  $x \in \hat{L}_0^s$  (see figure 5).

The complement in  $\hat{V}'_0$  of the interior of  $\hat{N}'_{0,0}$  is a compact set contained in  $\hat{V}'_0 \setminus \hat{L}'_0^s$ . Then its preimage  $K$  by the homeomorphism  $\hat{\varphi}_0$  is a compact set contained in  $\hat{V}_0 \setminus \hat{L}_0^s$ . When  $t = 0$ , we have  $\hat{N}_{0,0}^t = \hat{L}_0^s$  and hence disjoint from  $K$ . Then, if  $t$  is small enough,  $\hat{\varphi}_0(\hat{N}_{0,0}^t \setminus \hat{L}_0^s) \subset \hat{N}'_{0,0} \setminus \hat{L}'_0^s$ . Finally, the map  $\hat{\varphi}_0$  (which is not defined on  $\hat{L}_0^s$ ) possesses the two following properties:

1. If  $N_0$  is shrunk enough, we have  $(\hat{\varphi}_0(\hat{N}_{0,0}) \cap \hat{L}_0^u) \subset (\hat{N}'_{0,0} \cap \hat{L}'_0^u)$ , where  $\hat{N}'_{0,0}$  denotes the tube associated with the chosen linearizable neighborhood  $N'_0$  of  $\Sigma'_0$ .
2. If  $\hat{d}_x$  is a plaque of  $\hat{L}_0^u \cap \hat{N}_{0,0}$ , the image  $\hat{\varphi}_0(\hat{d}_x \setminus \{x\})$  is contained in some fiber  $\hat{d}_{x'}$ , with  $x' \in \hat{L}'_0^u \cap \hat{L}'_0^s$ .

As a consequence, the wanted extension may be defined by  $\hat{\varphi}_0^{us}(x) = x'$ . As the considered plaques are arcwise connected, the construction lifts to the cover and yields a continuous map  $\varphi_0^{us} : \Gamma_f^u \cup (L_0^s \cap L_0^u) \rightarrow \Gamma_{f'}^u \cup (L_0'^s \cap L_0'^u)$  which is a continuous equivariant extension of  $\varphi|_{\Gamma_f^u}$ .

We are left to prove that  $\varphi_0^{us}$  is increasing on its domain in each separatrix of  $\Sigma_0$ . For this aim, consider a point  $p \in \Sigma_0$ , one of its separatrices  $\gamma_p$  and a connected component  $N_{\gamma_p}$  of  $N_p \setminus W_p^u$  containing  $\gamma_p$ . Take an infinite proper arc  $C$  in  $N_{\gamma_p} \setminus W_p^s$  which crosses transversely each leaf of foliation  $F_0^u$  and whose one end is  $p$ . It is ordered as its projection onto  $\gamma_p$ . Its image through  $\varphi_0$  is a proper arc  $C'$  contained in  $N'_{\varphi(p)} \setminus W'_{\varphi(p)u}$ . Moreover,  $\varphi(p)$  is one end of  $C'$ . The property  $r_{\varphi(p)}'^s(\varphi_0^{us}(x)) < r_{\varphi(p)}'^s(\varphi_0^{us}(y))$  if  $r_p^s(x) < r_p^s(y)$  for  $x, y \in \gamma_p \cap L^u$  follows if we are sure that  $C'$  intersects each plaque of  $L_0'^u \cap N'_{\varphi(p)}$  in one point at most. This is true since  $\varphi_0$  is a homeomorphism on its image from  $N_0 \setminus W_0^s$  to  $N'_0 \setminus W_0'^s$  mapping  $L_0^u$  into  $L_0'^u$ .

For the induction, let  $i \in \{1, \dots, n\}$  and let us assume that there is a continuous extension

$$\varphi_{i-1}^{us} : \Gamma_f^u \cup \bigcup_{j=0}^{i-1} (L_j^s \cap L_j^u) \rightarrow \Gamma_{f'}^u \cup \bigcup_{j=0}^{i-1} (L_j'^s \cap L_j'^u),$$

which is monotone on each separatrix of  $\Sigma_j$ ,  $j < i$ . The image  $\hat{W}_{i,i}^s$  of  $W_i^s$  by the projection  $p_i : V_i \rightarrow \hat{V}_i$  is made of finitely many disjoint circles which are the images of the stable separatrices of  $\Sigma_i$ . About  $\hat{W}_{i,i}^s$ , there is a tube  $\hat{N}_{i,i}$  which is the projection by  $p_i$  of a neighborhood  $N_i$  of  $\Sigma_i$  extracted from a compatible system given by Theorem 2 (see Figure 6). The trace of  $\hat{L}_i^u$  in that tube is a lamination by disks:

$$\hat{L}_i^u \cap \hat{N}_{i,i} = \{\hat{d}_x \mid x \in \hat{L}_i^u \cap \hat{W}_{i,i}^s\},$$

where  $\hat{d}_x$  denotes the fiber of the tube over  $x \in \hat{W}_{i,i}^s$ .



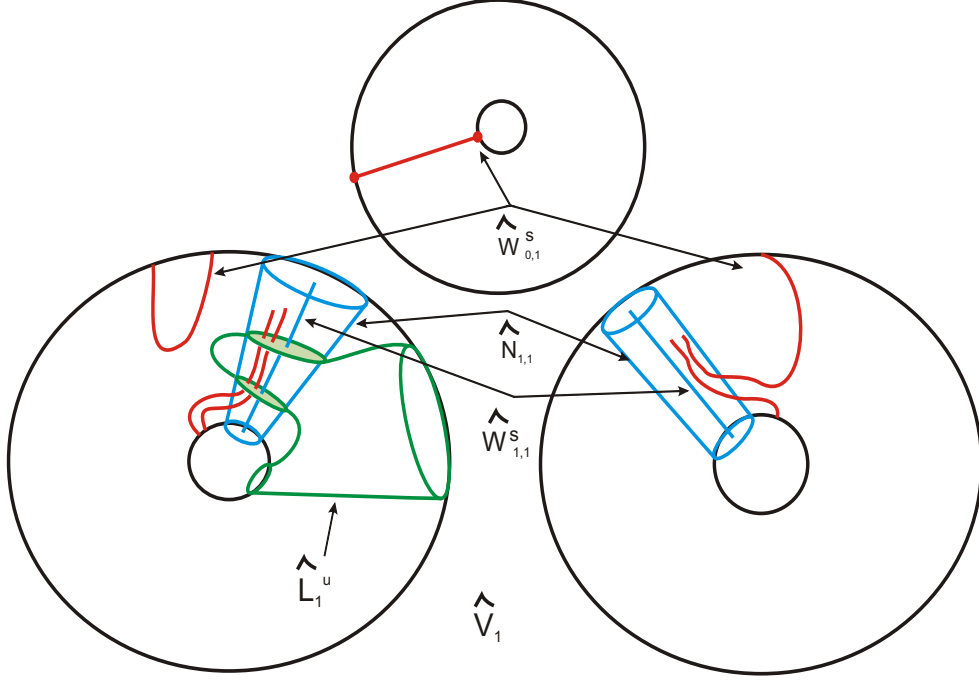


Figure 6: Illustration of induction in Lemma 4.2 for the diffeomorphism from Figure 3.

In  $V_i$ , there are two laminations  $L_i^u$  and  $L_i^s$  (and the corresponding objects with  $'$ ). The map  $\varphi_i$ , not defined on  $L_i^s$ , sends  $L_i^u \setminus L_i^s$  homeomorphically onto  $L_i'^u \setminus L_i'^s$ . Thanks to the induction hypothesis, the restriction  $\varphi_i|_{(L_i^u \setminus L_i^s)}$  extends continuously to  $L_i^u \setminus W_i^s$ ; this extension, automatically equivariant, is denoted  $\psi_i$ . This induces on the factor space  $\hat{V}_i$  a homeomorphism

$$\hat{\psi}_i : \hat{L}_i^u \setminus \hat{W}_{i,i}^s \rightarrow \hat{L}_i'^u \setminus \hat{W}_{i,i}'^s.$$

In order to extend  $\hat{\psi}_i$  to  $\hat{L}_i^u \cap \hat{L}_i^s$ , we use the fact that  $\hat{L}_i^u$  is compact for arguing as in the case  $i = 0$ . Consider the tube  $\hat{N}_{i,i}^t$  depending on  $t \in (0, 1)$  and look at its compact lamination by disks  $\hat{L}_i^u \cap \hat{N}_{i,i}^t$ . After removing  $\hat{W}_{i,i}^s$  which marks one puncture on each leaf, it is leaf-wise mapped by  $\hat{\psi}_i$  into  $\hat{V}_i' \setminus \hat{W}_{i,i}'^s$ . The mentioned compactness allows us, as in case  $i = 0$ , to conclude that there exists some  $t \in (0, 1)$  such that  $\hat{L}_i^u \cap (\hat{N}_{i,i}^t \setminus \hat{W}_{i,i}^s)$  is mapped into  $\hat{N}_{i,i}'^t$  where  $\hat{N}_{i,i}'^t$  denotes the tube associated with the chosen linearizable neighborhood  $N_i'$  of  $\Sigma_i'$ . Finally, the map  $\hat{\psi}_i$  possesses the two following properties:

1. If  $N_i$  is shrunk enough, we have  $(\hat{\psi}_i(\hat{N}_{i,i}) \cap \hat{L}_i^u) \subset (\hat{N}_{i,i}' \cap \hat{L}_i'^u)$ .
2. If  $\hat{d}_x$  is a plaque of  $\hat{L}_i^u \cap \hat{N}_{i,i}$ , the image  $\hat{\psi}_i(\hat{d}_x \setminus \{x\})$  is contained in some fiber  $\hat{d}_{x'}$ , with  $x' \in \hat{L}_i'^u \cap \hat{W}_{i,i}'^s$ .

Now, the extension  $\hat{\varphi}_i^{us}$  of  $\hat{\psi}_i$  is defined by  $x \mapsto x'$ . One checks it is a continuous extension. The wanted  $\varphi_i^{us}$  is the lift of  $\hat{\varphi}_i^{us}$  to  $V_i$ . It has the required properties allowing us to finish the induction.  $\diamond$

**Remark 4.3** Due to Lemma 3.2 we may assume that in all lemmas below the chosen values  $t = \beta_i, a_i, \dots$  are such that the boundary of the linearizable neighbourhood  $N_i^t$  does not contain any heteroclinic point.

**Lemma 4.4** *There are numbers  $\beta_0, \dots, \beta_n \in (0, 1)$  such that, for every  $i \in \{0, \dots, n\}$ , for every point  $p \in \Sigma_i$  and  $x \in N_p^{\beta_i} \cap L^u$ , the next inequality holds:*

$$r_i^{u'}(\varphi^{us}(x_i^u)) \cdot r_i^{s'}(\varphi^{us}(x_i^s)) < \frac{1}{2}.$$

**Proof:** As  $N_n \cap L^u = W_n^u$  and  $\varphi^{us}(W_n^u) = W_n^{s'}$ , it is possible to choose any  $\beta_n \in (0, 1)$ .

Indeed, for  $p \in \Sigma_n$  and  $x \in W_p^u$ , we have  $r_{\varphi(p)}^{s'}(\varphi^{us}(x_i^s)) = 0$ . For  $i \in \{0, \dots, n-1\}$  and  $p \in \Sigma_i$ , choose some heteroclinic point  $y \in W_p^s \cap L^u$  arbitrarily. Set:

$$\lambda_p^{u'}(t) = \sup_x \{r_{\varphi(p)}^{u'}(\varphi^{us}(x_i^u)) \mid x \in N_p^t \cap F_{i,y}^u\} \quad \text{and} \quad \lambda_p^{s'} = r_p^{s'}(\varphi^{us}(y)).$$

When  $t$  goes to 0, the arc  $N_p^t \cap F_{i,y}^u$  shrinks to the point  $y$ . Then, according to Lemma 4.2,  $\lambda_p^{u'}(t)$  also goes to 0. Therefore, there exists some  $\beta_p \in (0, 1)$  such that  $\lambda_p^{u'}(\beta_p) \cdot \lambda_p^{s'} < \frac{1}{8}$ . Denote by  $Q_p$  the compact subset of  $M$  bounded by  $\partial N_p^{\beta_p}$ ,  $F_{i,y}^u$  and  $f^{per(p)}(F_{i,y}^u)$ . Notice that  $Q_p$  is a fundamental domain for the restriction of  $f^{per(p)}$  to the connected component of  $N_p^{\beta_p} \setminus W_p^u$  containing  $y$ . For every  $x \in Q_p$ , we have  $r_{\varphi(p)}^{u'}(\varphi^{us}(x_i^u)) \leq 4\lambda_p^{u'}(\beta_p)$  and  $r_{\varphi(p)}^{s'}(\varphi^{us}(x_i^s)) \leq \lambda_p^{s'}$ . Then, for every  $x \in Q_p \cap L^u$  we have:

$$r_p^{u'}(\varphi^{us}(x_i^u)) \cdot r_p^{s'}(\varphi^{us}(x_i^s)) \leq 4\lambda_p^{u'}(\beta_p) \cdot \lambda_p^{s'} < \frac{1}{2}.$$

Set  $\beta_i = \min_{p \in \Sigma_i} \{\beta_p\}$ . Hence,  $\beta_i$  is the required number. ◊

**Lemma 4.5** *When  $n > 0$ , there exist real numbers  $a_j \in (0, \beta_j]$  fulfilling the following property: for every  $j = 1, \dots, n$  and every integer  $i < j$ , each connected component of  $\hat{W}_{i,i}^{s'} \cap \hat{N}_{j,i}^{a_j}$  is an open interval which is either disjoint from  $A_j^i := \bigcup_{k=i+1}^{j-1} \hat{N}_{k,i}^{a_k}$  or included in  $A_j^i$ . Moreover, only finitely many of these intervals are not covered by  $A_j^i$ .*

**Proof:** The proof is done by induction on  $j$  from 1 to  $n$ . For  $j = 1$ , one is allowed to take  $a_1 = \beta_1$ . Indeed,  $\hat{W}_{0,0}^{s'}$  is a smooth curve and  $\hat{W}_{1,0}^u$  is a smooth closed surface which is transverse to  $\hat{W}_{0,0}^{s'}$ . Therefore, there are finitely many intersection points. By the choice of  $\beta_1$ , the projection in  $\hat{V}_0$  of  $N_1^{a_1}$  is a tubular neighborhood of  $\hat{W}_{1,0}^u$ . Moreover, each component of  $\hat{W}_{0,0}^{s'} \cap \hat{N}_{1,0}^{a_1}$  is a fiber of this tube.

For the induction, assume the numbers  $a_1, \dots, a_{j-1}$  are given with the required properties and let us find  $a_j$ . In particular, the subset  $A_j^i$  is assumed to be defined. According to Remark 4.3, the boundary of  $A_j^i$  contains no heteroclinic point.

First, fix  $i < j$ . Consider the projection  $\hat{W}_{j,i}^u$  of  $W_j^u$  in  $\hat{V}_i$ . This is a union of leaves in the lamination  $\hat{L}_i^u$ . The following is a well-known fact (see, for example, Statement 1.1 in [12]): if  $x$  is a point from  $\hat{L}_i^u$  which is accumulated by a sequence of plaques from  $\hat{W}_{j,i}^u$ , then  $x$  does not lie in  $\hat{W}_{j,i}^u$  but belongs to some  $\hat{W}_{k,i}^u$  with  $k < j$ . Then the part of  $\hat{W}_{j,i}^u$  which is covered by  $A_j^i$  contains every intersection points  $\hat{W}_{i,i}^s \cap \hat{W}_{j,i}^u$  except finitely many of them. From this finiteness and the fact that  $A_j^i \cap \hat{W}_{i,i}^s \cap \hat{W}_{j,i}^u$  is actually contained in  $A_j^i$ , an easy compactness argument<sup>8</sup> allows us to find a positive number  $a_j^i$  such that the collection of disjoint intervals made by  $\hat{W}_{i,i}^s \cap \hat{N}_j^{a_j^i}$  fulfills the wanted property with respect to the considered  $i$ . By defining  $a_j := \inf\{a_j^0, \dots, a_j^{j-1}\}$ , we are sure that  $\hat{N}_j^{a_j}$  satisfies all the wanted properties.  $\diamond$

The corollary below follows from Lemma 4.5 immediately.

**Corollary 4.6** *For each  $i \in \{0, \dots, n-1\}$  the intersection  $\hat{W}_{i,i}^s \cap (\bigcup_{j=i+1}^n \hat{N}_{j,i}^{a_j^i})$  consists of finitely many open arcs  $\hat{I}_1^i, \dots, \hat{I}_{r_i}^i$  such that, for each  $l = 1, \dots, r_i$ , the arc  $\hat{I}_l^i$  is a connected component of  $\hat{W}_{i,i}^s \cap \hat{N}_{j,i}^{a_j^i}$  for some  $j > i$ .*

For brevity, for  $i = 0, \dots, n$ , we denote by  $\varphi_i^u$  the restriction  $\varphi^{us}|_{W_i^u}$  in the rest of the proof of Theorem 1. Let  $\psi_i^s : W_i^s \rightarrow W_i^{s'}$  be any equivariant homeomorphism which extends  $\varphi^{us}|_{W_i^s \cap L^u}$  and let  $t_i \in (0, 1)$  be a small enough number so that, for every  $x \in N_i^{t_i}$ , the next inequality holds:

$$(*)_i \quad r^{s'}(\varphi_i^u(x_i^u)) \cdot r^{u'}(\psi_i^s(x_i^s)) < 1.$$

In this setting, one derives an equivariant embedding  $\phi_{\varphi_i^u, \psi_i^s} : N_i^{t_i} \rightarrow N_i^{t_i'}$  which is defined by sending  $x \in N_i^{t_i}$  to  $(\varphi_i^u(x_i^u), \psi_i^s(x_i^s))$ .

**Lemma 4.7** *There is an equivariant homeomorphism  $\psi^s : L^s \rightarrow L^{s'}$  consisting of conjugating homeomorphisms  $\psi_0^s : W_0^s \rightarrow W_0^{s'}, \dots, \psi_n^s : W_n^s \rightarrow W_n^{s'}$  such that for each  $i \in \{0, \dots, n\}$ :*

- 1)  $\psi_i^s|_{W_i^s \cap L^u} = \varphi_i^u|_{W_i^s \cap L^u}$ ;
- 2) the topological embedding  $\phi_{\varphi_i^u, \psi_i^s}$  is well-defined on  $N_i^{a_i}$ ;
- 3) if  $x \in (W_i^s \cap N_j^{a_j^i})$ ,  $j > i$ , then  $\psi_i^s(x) = \phi_{\varphi_j^u, \psi_j^s}(x)$ .

**Proof:** We are going to construct  $\psi_i^s$  by a decreasing induction on  $i$  from  $i = n$  to  $i = 0$ . The stable manifolds of the saddles in  $\Sigma_n$  have no heteroclinic points. Therefore, the only constraints on  $\psi_n^s$  imposed by the first item is its value on  $\Sigma_n$ . In particular, we are allowed to change  $\psi_n^s$  to  $f^{lk} \circ \psi_n^s$  if  $k$  is *admissible* in the sense that  $k$  is a multiple of all periods  $per(p)$ ,  $p \in \Sigma_n$ .

This remark is used in the following way. One starts with any equivariant homeomorphism  $\psi_n^s$  such that for any  $p \in \Sigma_n$  the stable manifold  $W_p^s$  is mapped to the stable manifold of  $\varphi_n^u(p)$ ; hence, item 1 is fulfilled. Choose a fundamental domain  $I$  of  $f|_{W_n^s \setminus \Sigma_n}$ . Consider the fundamental

<sup>8</sup>Let  $B(t)$  be the closure of  $\partial A_j^i \cap \hat{W}_{i,i}^s \cap \hat{N}_j^t$ . The intersection  $\bigcap_{k \in \mathbb{N}} B(\frac{1}{k})$  is empty. Then  $B(t)$  is empty when  $t$  is small enough.

domain of  $f|_{N_n^{an} \setminus W_n^u}$  defined by  $N_I := \{x \in N_n^{an} \mid x_n^s \in I\}$ ; set  $\lambda_n'^u := \sup\{r'^u(\varphi_n^u(x_n^u)) \mid x \in N_I\}$  and  $\lambda_n'^s := \sup\{r'^s(\psi_n^s(x_n^s)) \mid x \in N_I\}$ . If the product  $\lambda_n'^u \lambda_n'^s$  is less than 1, the inequality  $(*)_n$  is fulfilled by the pair  $(\varphi_n^u, \psi_n^s)$  and hence, the embedding  $\phi_{\varphi_n^u, \psi_n^s}$  is well-defined on  $N_n^{an}$ .

If not, we replace  $\psi_n^s$  with  $f^{rk} \circ \psi_n^s$  with  $k$  admissible and large enough. Indeed, the effect of this change is to multiply  $\lambda_n'^s$  by some positive factor bounded by  $(\frac{1}{4})^k$  while  $\lambda_n'^u$  is kept fixed and hence,  $(*)_n$  becomes fulfilled when  $k$  is large enough. Since the third item is empty for  $i = n$ , we have built some  $\psi_n^s$  as desired.

For the induction, let us build  $\psi_i^s$ ,  $i < n$ , with the required properties assuming that the homeomorphisms  $\psi_n^s, \dots, \psi_{i+1}^s$  have been already built. The stable manifolds of saddles in  $\Sigma_i$  have heteroclinic intersections with unstable manifolds of saddles in  $\Sigma_j$  with  $j > i$  only. The image  $\hat{W}_{i,i}^s$  of  $W_i^s$  through  $p_i : V_i \rightarrow \hat{V}_i$  is a closed smooth 1-dimensional submanifold. According to Corollary 4.6, the intersection  $\hat{W}_{i,i}^s \cap (\bigcup_{j=i+1}^n \hat{N}_j^{aj})$  consists of finitely many open arcs  $\hat{I}_1^i, \dots, \hat{I}_{r_i}^i$  such that  $\hat{I}_l^i$  for each  $l = 1, \dots, r_i$  is a connected component of  $\hat{W}_{i,i}^s \cap \hat{N}_{j,i}^{aj}$  for some  $j > i$ .

In order to satisfy the third item of the statement,  $\psi_i^s$  is defined on  $p_i^{-1}(\hat{I}_l^i)$  in an equivariant way. Denote by  $\psi_{i,l}^s$  this partial definition of  $\psi_i^s$ ; its image is contained in  $W_{i,i}^s$ .

More precisely, if  $I_{l,\alpha}^i$  is a connected component of  $p_i^{-1}(\hat{I}_l^i)$  it is a proper arc in some  $N_i^{aj}$  and it intersects  $W_j^u$  in a unique point  $x_{l,\alpha}^i$ . Set  $x_{l,\alpha}^i = \varphi^{us}(x_{l,\alpha}^i)$  and denote  $I_{l,\alpha}^i$  the connected component of  $W_i^s \cap N_j'$  passing through the point  $x_{l,\alpha}^i$ . Then, the restriction of  $\psi_{i,l}^s$  to the arc  $I_{l,\alpha}^i$  reads:

$$\psi_{i,l,\alpha}^s = \phi_{\varphi_j^u, \psi_j^s}|_{I_{l,\alpha}^i} : I_{l,\alpha}^i \rightarrow I_{l,\alpha}^i.$$

Due to lemma 4.2, the map  $\varphi^{us}$  sends  $W_i^s \cap L^u$  to  $W_i'^s \cap L'^u$  by preserving the order on each separatrix of  $W_i^s \setminus \Sigma_i$  and  $W_i'^s \setminus \Sigma_i'$ . On the other hand,  $\psi_{i,l,\alpha}^s$  is also order preserving. Both together imply that  $\psi_{i,l}^s$  is order preserving since we know that it is an injective map. And even, the union of all  $\psi_{i,l}^s$  – which makes sense as their respective domains are mutually disjoint – are order preserving. Therefore, there is an equivariant homeomorphism  $\psi_i^s : W_i^s \setminus \Sigma_i \rightarrow W_i'^s \setminus \Sigma_i$  which extends all  $\psi_{i,l}^s$ .

Since  $\varphi^{us}$  is continuous, the above homeomorphism extends continuously to  $\psi_i^s : W_i^s \rightarrow W_i'^s$ . At this point of the construction items 1 and 3 of the statement are satisfied. The condition of item 2 follows from Lemma 4.4 for stable separatrices which contain heteroclinic points. If some stable separatrix has no heteroclinic points, one changes  $\psi_i^s$  to  $f^{rk} \circ \psi_i^s$  on the separatrix where  $k$  is a large common multiple of the period of the separatrix, like to the construction made in the case  $i = n$ .  $\diamond$

**PROOF OF THEOREM 1 CONTINUED.** Let us recall that we denoted by  $a : \mathbb{R}^3 \rightarrow \mathbb{R}^3$  the canonical linear diffeomorphism with the unique fixed point  $O = (0,0,0)$  which is a saddle point with the plane  $Ox_1x_2$  as the unstable manifold and the axis  $Ox_3$  as the stable manifold; for simplicity, we assume that  $a$  has a sign  $\nu = +$  (see the beginning of Section 3). Let

$$N = \{(x_1, x_2, x_3) \in \mathbb{R}^3 : 0 \leq (x_1^2 + x_2^2)x_3 \leq 1\}.$$

Let  $\rho > 0$ ,  $\delta \in (0, \frac{\rho}{4})$  and

$$d = \{(x_1, x_2, x_3) \in \mathbb{R}^3 : x_1^2 + x_2^2 \leq \rho^2, x_3 = 0\},$$

$$\begin{aligned}
U &= \{(x_1, x_2, x_3) \in \mathbb{R}^3 : (\rho - \delta)^2 \leq x_1^2 + x_2^2 \leq \rho^2, x_3 = 0\}, \\
c &= \{(x_1, x_2, x_3) \in \mathbb{R}^3 : x_1^2 + x_2^2 = \rho^2, x_3 = 0\}, \\
c^0 &= \{(x_1, x_2, x_3) \in \mathbb{R}^3 : x_1^2 + x_2^2 = (\rho - \frac{\delta}{2})^2, x_3 = 0\}, \\
c^1 &= \{(x_1, x_2, x_3) \in \mathbb{R}^3 : x_1^2 + x_2^2 = (\rho - \delta)^2, x_3 = 0\}.
\end{aligned}$$

Let  $K = d \setminus \text{int } a^{-1}(d)$ ,  $V = (K \cup a(K)) \cap \{(x_1, x_2, x_3) \in \mathbb{R}^3 : x_1 \geq 0, x_3 = 0\}$  and  $\beta = U \cap Ox_1^+$ , where  $Ox_1^+ = \{(x_1, x_2, x_3) \in \mathbb{R}^3 : x_1^2 + x_2^2 = 0, x_1 > 0\}$ .

Choose a point  $Z^0 = (0, 0, z^0) \in Ox_3^+$  such that  $\rho^2 \cdot z^0 < \frac{1}{4}$  (see Figure 7). Then, choose a point  $Z^1 = (0, 0, z^1)$  in  $Ox_3^+$  so that  $z^0 > z^1 > \frac{z^0}{4}$ . Let  $\Pi(z) = \{(x_1, x_2, x_3) \in \mathbb{R}^3 : x_3 = z\}$ . For every set  $A \subset Ox_1x_2$ , let  $\tilde{A} = A \times [0, z^0]$ . Denote by  $\mathcal{W}$  the 3-ball bounded by the surface  $\tilde{c}$  and the two planes  $\Pi(z^0)$  and  $\Pi(\frac{z^0}{4})$ . Let  $\Delta$  be a closed 3-ball bounded by the surface  $\tilde{c}^1$  and the two planes  $Ox_1x_2$  and  $\Pi(z^1)$ . Let

$$\mathcal{T} = \bigcup_{k \in \mathbb{Z}} a^k(\tilde{d}) \quad \text{and} \quad \mathcal{H} = \bigcup_{k \in \mathbb{Z}} a^k(\Delta).$$

Notice that the construction yields  $\mathcal{H} \subset \text{int } \mathcal{T}$  and makes  $\mathcal{W}$  a fundamental domain for the action of  $a$  on  $\mathcal{T}$ .

Now, we come back to  $f$ . Let  $n \geq 1$ ,  $i \in \{0, \dots, n-1\}$  and let  $G_i$  be the union of all stable separatrices of saddle points in  $\Sigma_i$  which contains heteroclinic points. Let  $\check{G}_i \subset G_i$  be the union of separatrices in  $G_i$  such that  $G_i = \bigcup_{\gamma \in \check{G}_i} \text{orb}(\gamma)$  and, for every pair  $(\gamma_1, \gamma_2)$  of

distinct separatrices in  $\check{G}_i$  and every  $k \in \mathbb{Z}$ , one has  $\gamma_2 \neq f^k(\gamma_1)$ . For  $\gamma \in G_i$  with the end point  $p \in \Sigma_i$  and a point  $q \in \Sigma_j$ ,  $j > i$ , let us consider a sequence of different periodic orbits  $p = p_0 \prec p_1 \prec \dots \prec p_k = q$  such that  $\gamma \cap W_{p_1}^u \neq \emptyset$ , the length of the longest such chain is denoted by  $\text{beh}(q|\gamma)$ .

Let  $\gamma \in \check{G}_i$  be a separatrix of  $p \in \Sigma_i$  and let  $N_\gamma^t$  be the connected component of  $N_p^t \setminus W_p^u$  which contains  $\gamma$ . We endow with the index  $\gamma$  (resp.  $p$ ) the preimages in  $M$  (through the linearizing map  $\mu_p$ ) of all objects from the linear model  $\mathcal{N}$  associated with the separatrix  $\gamma$  (resp.  $p$ ); for being precise we decide that  $\mu_p(\gamma) = Ox_3^+$ . For a separatrix  $\gamma$  in  $\check{G}_i$ , let us fix a saddle point  $q_\gamma$  such that  $\text{beh}(q_\gamma|\gamma) = 1$ . Notice that the intersection  $\gamma \cap W_{q_\gamma}^u$  consists of a finite number of heteroclinic orbits.

**Lemma 4.8** *Let  $n \geq 1$ ,  $i \in \{0, \dots, n-1\}$ . For every  $\gamma \in \check{G}_i$  there are positive numbers  $\rho$ ,  $\delta$  and  $\varepsilon$  (depending on  $\gamma$ ) such that for every heteroclinic point  $Z_\gamma^0 \in (\gamma \cap W_{q_\gamma}^u)$  with  $z^0 < \varepsilon$  the following properties hold:*

- (1)  $U_p$  avoids all heteroclinic points;
- (2)  $\varphi(\tilde{d}_p) \subset \phi_{\varphi_i^u, \psi_i^s}(N_i^{a_i})$ ;
- (3)  $\varphi(\tilde{c}_p) \cap \phi_{\varphi_i^u, \psi_i^s}(\tilde{c}_p^0) = \emptyset$ ,  $\varphi(\tilde{c}_p^1) \cap \phi_{\varphi_i^u, \psi_i^s}(\tilde{c}_p^0) = \emptyset$  and  $\varphi(\tilde{\beta}_\gamma) \subset \phi_{\varphi_i^u, \psi_i^s}(\tilde{V}_\gamma)$ .

**Proof:** Let  $\gamma \in \check{G}_i$ ,  $i \in \{0, \dots, n-1\}$ . Due to Lemma 3.2, there is a generic  $\rho > 0$  such that the curve  $c_\gamma$  avoids all heteroclinic points. Since  $W_l^s$  accumulates on  $W_k^s$  for every  $l < k$ ,

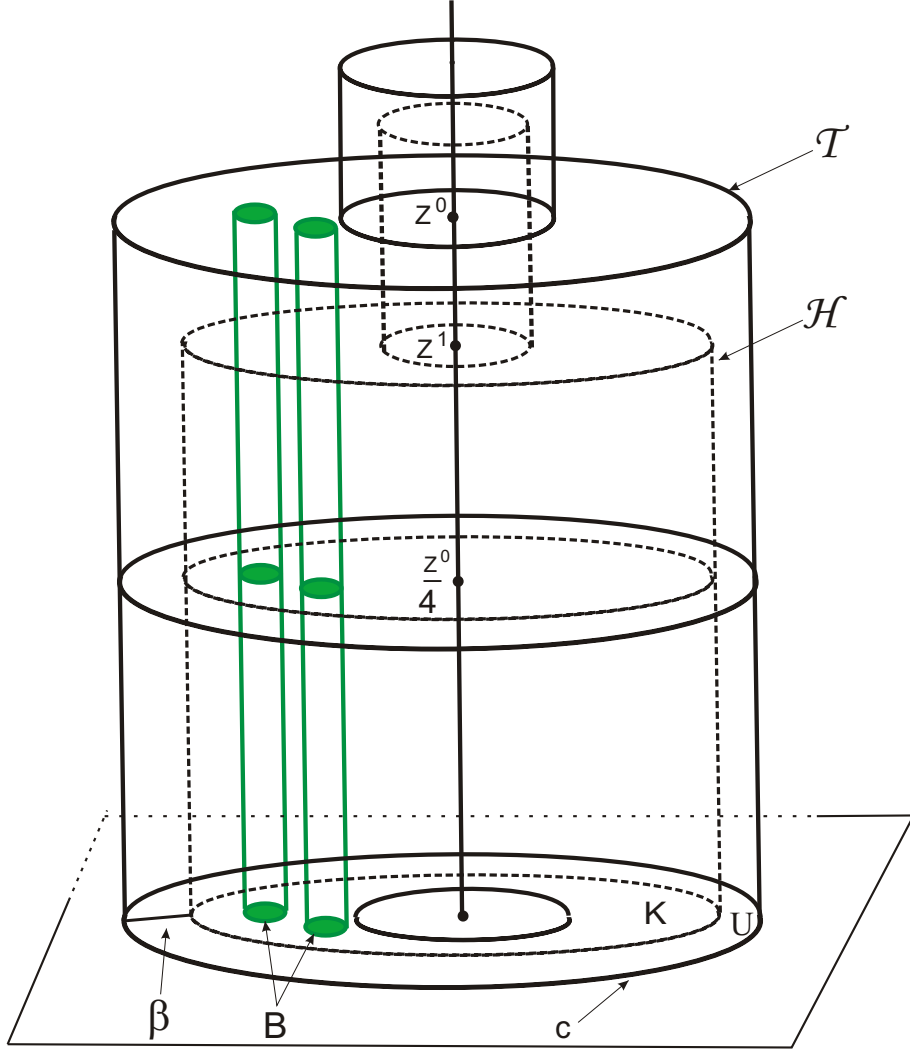


Figure 7: A linear model

then  $K_p \cap W_{i-1}^s$  is made of a finite number of heteroclinic points  $y_1, \dots, y_r$  which we can cover by closed 2-discs  $b_1, \dots, b_r \subset \text{int } K_p$ . In  $K_p \setminus \text{int}(b_1 \cup \dots \cup b_r)$  there is a finite number of heteroclinic points from  $W_{i-2}^s$  which we cover by the union of a finite number of closed 2-discs, and so on. Thus we get that all heteroclinic points in  $K_p$  belong to the union of finitely many closed 2-discs avoiding  $\partial K_p$ . Therefore, there is  $\delta \in (0, \frac{\rho}{4})$  such that  $U_p$  avoids heteroclinic points. This proves item (1).

By assumption of Theorem 1,  $\varphi$  is defined on the complement of the stable manifolds and, by definition,  $\phi_{\varphi_i^u, \psi_i^s}$  coincides with  $\varphi$  on  $W_i^u \setminus L^s$ , and hence on  $U_p$ . As  $\varphi$  and  $\phi_{\varphi_i^u, \psi_i^s}$  are continuous, we can choose  $\varepsilon > 0$  sufficiently small so that, if  $Z_\gamma^0$  is any heteroclinic point in the intersection  $\gamma \cap W_{q_\gamma}^u$  with  $z^0 < \varepsilon$ , the requirements of (2) and (3) are fulfilled.  $\diamond$

Let us fix  $U_p$  satisfying item (1) of Lemma 4.8 and let us define

$$U_i = \bigcup_{p \in \Sigma_i} \left( \bigcup_{k=0}^{per(p)-1} f^k(U_p) \right), \quad K_i = \bigcup_{p \in \Sigma_i} \left( \bigcup_{k=0}^{per(p)-1} f^k(K_p) \right).$$

**Lemma 4.9** *Let  $n \geq 2$ . For every  $i \in \{0, \dots, n-2\}$  and  $\gamma \in \check{G}_i$ , there is a heteroclinic point  $Z_\gamma^0 \in \gamma$  satisfying the conditions of Lemma 4.8 and in addition:*

$$\mathcal{T}_\gamma \cap \tilde{U}_j = \emptyset \quad \text{for } j \in \{i+1, \dots, n-1\}.$$

In this statement, it is meant that  $\tilde{U}_{n-1}$  is associated with the points  $Z_{\gamma'}, \gamma' \in \check{G}_{n-1}$  given by Lemma 4.8 and  $\tilde{U}_j$  is associated with the points  $Z_{\gamma''}, \gamma'' \in \check{G}_j$  given by Lemma 4.9 for every  $j > i$ . Therefore, it makes sense to prove that lemma by decreasing induction on  $i$  from  $i = n-2$  to 0. That is what is done below. It is also worth noticing that nothing is required with respect to  $\Sigma_n$ ; the reason why is that the stable separatrices of  $\Sigma_n$  have no heteroclinic points.

**Proof:** Let us first prove the lemma for  $i = n-2$ . Let  $\gamma \in \check{G}_{n-2}$  and let  $p$  be the saddle end point of  $\gamma$ . Notice that the intersection  $\gamma \cap K_{n-1}$  consists of a finite number points  $a_1, \dots, a_l$  avoiding  $U_{n-1}$ . Let  $d_1, \dots, d_l \subset K_{n-1}$  be compact discs with centres  $a_1, \dots, a_l$  and radius  $r_*$  (in linear coordinates of  $N_p$ ) avoiding  $U_{n-1}$ . Let us choose a number  $n^* \in \mathbb{N}$  such that  $\frac{\rho}{2^{n^*}} < r_*$ . Let  $Z_\gamma^* \subset \gamma$  be a point such that the segment  $[p, Z_\gamma^*]$  of  $\gamma$  avoids  $\tilde{K}_{n-1}$  and  $\mu_p(Z_\gamma^*) = Z^* = (0, 0, z^*)$  where  $z^* < \varepsilon$ . Then every heteroclinic point  $z_\gamma^0$  so that  $z^0 < \frac{z^*}{2^{n^*}}$  possesses the property:  $\mathcal{T}_\gamma \cap \tilde{K}_{n-1}$  avoids  $\tilde{U}_{n-1}$ .

For the induction, let us assume now that the construction of the desired heteroclinic points is done for  $i+1, i+2, \dots, n-2$ . Let us do it for  $i$ . Let  $\gamma \in \check{G}_i$ . By assumption of the induction  $(\bigcup_{k=i+1}^{j-1} \mathcal{T}_k) \cap \tilde{U}_j = \emptyset$  for  $j \in \{i+2, \dots, n-1\}$ . Since  $W_{k-1}^s$  accumulates on  $W_k^s$  for every  $k \in \{0, \dots, n\}$ , then  $(\bigcup_{k=i+1}^{j-1} \mathcal{T}_k) \cap K_j$  is a compact subset of  $K_j$  and the intersection  $(\gamma \setminus (\bigcup_{k=i+1}^{j-1} \mathcal{T}_k)) \cap K_j$  consists of a finite number points  $a_1, \dots, a_l$  avoiding  $U_j$ . Let  $d_1, \dots, d_l \subset K_j$  be compact discs with centres  $a_1, \dots, a_l$  and radius  $r_*$  (in linear coordinates of  $N_p$ ) avoiding  $U_j$  and such that  $r_*$  is less than the distance between  $\partial(K_j \setminus U_j)$  and  $(\bigcup_{k=i+1}^{j-1} \mathcal{T}_k) \cap K_j$ . Similar to the case  $i = n-2$  it is possible to choose a heteroclinic point  $Z_\gamma^0$  sufficiently close to the saddle  $p$  where  $\gamma$  ends such that the set  $(\mathcal{T}_\gamma \setminus (\bigcup_{k=i+1}^{j-1} \mathcal{T}_k)) \cap \tilde{K}_j$  avoids  $\tilde{U}_j$ .  $\diamond$

Everywhere below, we assume that for every  $\gamma \subset \check{G}_i$  the neighborhood  $\mathcal{T}_\gamma$  satisfies to Lemmas 4.8 and 4.9. Let

$$\mathcal{T}_i = \bigcup_{\gamma \subset \check{G}_i} \left( \bigcup_{k=0}^{per(\gamma)-1} f^k(\mathcal{T}_\gamma) \right).$$

For  $\gamma \subset \check{G}_i$ ,  $j > i$ , let us denote by  $\mathcal{J}_{\gamma,j}$  the union of all connected components of  $W_j^u \cap \mathcal{T}_\gamma$  which do not lie in  $\text{int } \mathcal{T}_k$  with  $i < k < j$ . Let  $\mathcal{J}_\gamma = \bigcup_{j=i+1}^n \mathcal{J}_{\gamma,j}$  and

$$\mathcal{J}_i = \bigcup_{\gamma \subset \check{G}_i} \left( \bigcup_{k=0}^{\text{per}(\gamma)-1} f^k(\mathcal{J}_\gamma) \right).$$

Let  $\mathcal{W}_\gamma$  be the fundamental domain of  $f^{\text{per}(\gamma)}|_{\mathcal{T}_\gamma \setminus W_p^u}$  limited by the plaques of the two heteroclinic points  $Z_\gamma^0$  and  $f^{\text{per}(\gamma)}Z_\gamma^0$ . Notice that  $\gamma \cap \mathcal{W}_\gamma$  is a fundamental domain of  $f^{\text{per}(\gamma)}|_\gamma$ . Since  $W_k^u$  accumulates on  $W_l^u$  only when  $l < k$ , then the set  $\mathcal{J}_{\gamma,j} \cap \mathcal{W}_\gamma$  consists of a finite number of closed 2-discs. Hence, the set  $\mathcal{J}_\gamma \cap \gamma \cap \mathcal{W}_\gamma$  consists of a finite number of heteroclinic points; denote them  $Z_\gamma^2, \dots, Z_\gamma^m$  ( $m$  depends on  $\gamma$ ). Finally, choose an arbitrary point  $Z_\gamma^1 \in \gamma$  so that the arc  $(z_\gamma^0, z_\gamma^1) \subset \gamma$  does not contain heteroclinic points from  $\mathcal{J}_\gamma$ . Let us construct  $\mathcal{H}_\gamma$  using the point  $Z^1 = \mu_p(Z_\gamma^1)$ . Without loss of generality we will assume that  $\mu_p(Z_\gamma^i) = Z^i = (0, 0, z^i)$  for  $z^0 > z^1 > \dots > z^m > \frac{z^0}{4}$ . For  $i = 0, \dots, n-1$  let

$$\mathcal{H}_i = \bigcup_{\gamma \subset \check{G}_i} \left( \bigcup_{k=0}^{\text{per}(\gamma)-1} f^k(\mathcal{H}_\gamma) \right) \quad \text{and} \quad \mathcal{M}_i = V_f \bigcup_{k=0}^i (G_k \cup \Sigma_k).$$

**Lemma 4.10** *There is an equivariant topological embedding  $\varphi_0 : \mathcal{M}_0 \rightarrow M'$  with following properties:*

- 1)  $\varphi_0$  coincides with  $\varphi$  out of  $\mathcal{T}_0$ ;
- 2)  $\varphi_0|_{\mathcal{H}_0} = \phi_{\psi_0^u, \psi_0^s}|_{\mathcal{H}_0}$ , where  $\psi_0^u = \varphi|_{W_0^u}$ ;
- 3)  $\varphi_0(W_1^u) = W_1^{lu}$  and  $\varphi_0(W_k^u \setminus \bigcup_{j=1}^{k-1} \text{int } \mathcal{T}_j) \subset W_k^{lu}$  for every  $k \in \{2, \dots, n\}$ .

**Proof:** The desired  $\varphi_0$  should be an interpolation between  $\varphi : V_f \setminus \mathcal{T}_0 \rightarrow M'$  and  $\phi_{\varphi_0^u, \psi_0^s}|_{\mathcal{H}_0}$ . Due to Lemma 4.8 (2) and the equivariance of the considered maps, the embedding

$$\xi_0 = \phi_{\psi_0^u, \psi_0^s}^{-1} \varphi : \mathcal{T}_0 \setminus W_0^s \rightarrow M$$

is well-defined. Let  $\gamma \subset \check{G}_0$  be a separatrix ending at  $p \in \Sigma_0$  and  $\xi_\gamma = \xi_0|_{\mathcal{T}_\gamma}$ . By construction, the topological embedding  $\xi = \mu_p \xi_\gamma \mu_p^{-1} : \mathcal{T} \rightarrow N$  has the following properties:

- 1)  $\xi a = a \xi$  (as  $a \mu_p = \mu_p f^{\text{per}(\gamma)}$  and  $\xi_\gamma f^{\text{per}(\gamma)} = f^{\text{per}(\gamma)} \xi_\gamma$ );
- 2)  $\xi$  is the identity on  $Ox_1x_2$  (as  $\phi_{\psi_0^u, \psi_0^s}|_{W_0^u} = \varphi|_{W_0^u}$ );
- 3)  $\xi(\Pi(z^0) \cap \mathcal{T}) \subset \Pi(z^0)$  and  $\xi(\Pi(z^i) \cap \partial \mathcal{T}) \subset \Pi(z^i)$ ,  $i \in \{2, \dots, m\}$  (as  $\xi_\gamma(L^u \cap \mathcal{T}_\gamma \setminus \gamma) \subset L^u$ );
- 4)  $\xi(\tilde{c}) \cap \tilde{c}^0 = \emptyset$ ,  $\xi(\tilde{c}^1) \cap \tilde{c}^0 = \emptyset$  and  $\xi(\tilde{\beta}) \subset \tilde{V}$  (due to Lemma 4.8 (3)).

Thus,  $\xi$  satisfies all conditions of Proposition 5.1 and, hence, there is an embedding  $\zeta : \mathcal{T} \rightarrow N$  such that:

- 1)  $\zeta a = a \zeta$ ;
- 2)  $\zeta$  is the identity on  $\mathcal{H}$ ;



- 3)  $\zeta(\Pi(z^i) \cap \mathcal{T}) \subset \Pi(z^i)$ ,  $i \in \{0, 2, \dots, m\}$
- 4)  $\zeta$  is  $\xi$  on  $\partial\mathcal{T}$ .

Then the embedding  $\zeta_\gamma = \mu_p^{-1}\zeta\mu_p : \mathcal{T}_\gamma \rightarrow N_\gamma$  satisfies the properties:

- 1)  $\zeta_\gamma f^{per(\gamma)} = f^{per(\gamma)}\zeta_\gamma$ ;
- 2)  $\zeta_\gamma$  is the identity on  $\mathcal{H}_\gamma$ ;
- 3)  $\zeta(\mathcal{J}_\gamma) \subset L^u$
- 4)  $\zeta_\gamma$  is  $\xi_\gamma$  on  $\partial\mathcal{T}_\gamma$ .

Independently, one does the same for every separatrix  $\gamma \in \check{G}_0$ . Then, it is extended to all separatrices in  $G_0$  by equivariance. As a result, we get a homeomorphism  $\zeta_0$  of  $\mathcal{T}_0$  onto  $\xi_0(\mathcal{T}_0)$  which coincides with  $\xi_0$  on  $\partial\mathcal{T}_0$ . Now, define the embedding  $\varphi_0 : \mathcal{M}_0 \rightarrow M'$  to be equal to  $\phi_{\psi_0^u, \psi_0^s}\zeta_0$  on  $\mathcal{T}_0$  and to  $\varphi$  on  $\mathcal{M}_0 \setminus \mathcal{T}_0$ . One checks the next properties:

- 1)  $\varphi_0$  coincides with  $\varphi$  out of  $\mathcal{T}_0$ ;
- 2)  $\varphi_0|_{\mathcal{H}_0} = \phi_{\psi_0^u, \psi_0^s}|_{\mathcal{H}_0}$ ;
- 3)  $\varphi_0(\mathcal{J}_0) \subset L^u$ .

The last property and the definition of the set  $\mathcal{J}_\gamma$  imply that  $\varphi_0(W_1^u) = W_1^u$  and  $\varphi_0(W_k^u \setminus \bigcup_{j=1}^{k-1} \text{int } \mathcal{T}_j) \subset W_k^u$  for every  $k \in \{2, \dots, n\}$ . Thus  $\varphi_0$  satisfies all required conditions of the lemma.  $\diamond$

**Lemma 4.11** *Assume  $n \geq 2$ ,  $i \in \{0, \dots, n-2\}$ , and assume there is an equivariant topological embedding  $\varphi_i : \mathcal{M}_i \rightarrow M'$  with following properties:*

- 1)  $\varphi_i$  coincides with  $\varphi_{i-1}$  out of  $\mathcal{T}_i$ ;
- 2)  $\varphi_i|_{\mathcal{H}_i} = \phi_{\psi_i^u, \psi_i^s}$ , where  $\psi_i^u = \varphi_{i-1}|_{W_i^u}$  and  $\varphi_{-1} = \varphi$ ;
- 3) there is an  $f$ -invariant union of tubes  $\mathcal{B}_i \subset (\mathcal{T}_i \cap \bigcup_{j=0}^{i-1} \mathcal{H}_j)$  containing  $(\mathcal{T}_i \cap (\bigcup_{j=0}^{i-1} W_j^s))$  where  $\varphi_i$  coincides with  $\varphi_{i-1}$  (we assume  $\mathcal{B}_0 = \emptyset$ );

- 4)  $\varphi_i(W_{i+1}^u) = W_{i+1}^u$  and  $\varphi_i(W_k^u \setminus \bigcup_{j=i+1}^{k-1} \text{int } \mathcal{T}_j) \subset W_k^u$  for every  $k \in \{i+2, \dots, n\}$ .

Then there is a homeomorphism  $\varphi_{i+1}$  with the same properties 1)-4)

**Proof:** The desired  $\varphi_{i+1}$  should be an interpolation between  $\varphi_i : \mathcal{M}_{i+1} \setminus \mathcal{T}_{i+1} \rightarrow M'$  and  $\phi_{\psi_{i+1}^u, \psi_{i+1}^s}|_{\mathcal{H}_{i+1}}$  where  $\psi_{i+1}^u = \varphi_i|_{W_{i+1}^u}$ . Let  $\gamma \in \check{G}_{i+1}$  be a separatrix ending at  $p \in \Sigma_{i+1}$ . It follows from the definition of the set  $\mathcal{J}_i$  and the choice of the point  $q_\gamma$  that  $(W_{q_\gamma}^u \cap \mathcal{T}_i) \subset \mathcal{J}_i$ . Then, due to condition 4) for  $\varphi_i$  we have  $\varphi_i(W_{q_\gamma}^u \cap \mathcal{T}_i) \subset W_{q_\gamma}^u$ . By the property 1) of the homeomorphism  $\varphi_i$  and the properties of  $\mathcal{T}_{i+1}$  from Lemmas 4.8 (1) and 4.9, we get that  $\varphi_i|_{\tilde{U}_p} = \varphi|_{\tilde{U}_p}$ . Then  $\phi_{\varphi_{i+1}^u, \psi_{i+1}^s}|_{\tilde{U}_p} = \phi_{\psi_{i+1}^u, \psi_{i+1}^s}|_{\tilde{U}_p}$ . Thus it follows from the property (2) in Lemma 4.8 that the following embedding is well-defined:  $\xi_\gamma = \phi_{\psi_{i+1}^u, \psi_{i+1}^s}^{-1}\varphi_i : \mathcal{T}_\gamma \setminus (\gamma \cup p) \rightarrow M'$ .

By construction, the topological embedding  $\xi = \mu_p \xi_\gamma \mu_p^{-1}$  satisfies to all conditions of Proposition 5.1. Let  $\zeta$  be the embedding which is yielded by that proposition. Define  $\zeta_\gamma = \mu_p^{-1}\zeta\mu_p$ . Notice that by the property 3) of the homeomorphism  $\psi^s$  in Lemma 4.7 and by the properties  $\psi_{i+1}^u = \varphi_i|_{W_i^u}$ , we have that  $\zeta_\gamma$  is the identity on a neighborhood  $\tilde{B}_\gamma \subset (\mathcal{T}_\gamma \cap \bigcup_{j=0}^i \mathcal{H}_j)$  of

$\mathcal{T}_\gamma \cap (\bigcup_{j=0}^i W_j^s)$ . Independently, one does the same for every separatrix  $\gamma \in \check{G}_{i+1}$ . Assuming that

$\zeta_{f(\gamma)} = f' \zeta_\gamma f^{-1}$  and  $\tilde{B}_{i+1} = \bigcup_{\gamma \in \check{G}_{i+1}} \left( \bigcup_{k=0}^{per(\gamma)-1} f^k(\tilde{B}_\gamma) \right)$  we get a homeomorphism  $\zeta_{i+1}$  on  $\mathcal{T}_{i+1}$ .

Thus the required homeomorphism coincides with  $\phi_{\psi_{i+1}^u, \psi_{i+1}^s}$  on  $\mathcal{H}_{i+1}$  and with  $\varphi_i$  out of  $\mathcal{T}_{i+1}$ .  $\diamond$

Let  $G$  be the union of all stable one-dimensional separatrices which do not contain heteroclinic points,  $N_G^t = \bigcup_{\gamma \in G} N_\gamma^t$  and

$$\mathcal{M} = \mathcal{M}_{n-1} \cup G.$$

**Lemma 4.12** *There are numbers  $0 < \tau_1 < \tau_2 < 1$  and an equivariant embedding  $h_{\mathcal{M}} : \mathcal{M} \rightarrow M'$  with the following properties:*

- 1)  $h_{\mathcal{M}}$  coincides with  $\varphi_{n-1}$  out of  $N_G^{\tau_2}$ ;
- 2)  $h_{\mathcal{M}}$  coincides with  $\phi_{\varphi_{n-1}, \psi^s}$  on  $|_{N_G^{\tau_1}}$ , where  $\psi^s : L^s \rightarrow L^s$  is yielded by Lemma 4.7;
- 3) there is an  $f$ -invariant neighborhood of the set  $N_G \cap (G_0 \cup \dots \cup G_{n-1})$  where  $h_{\mathcal{M}}$  coincides with  $\varphi_{n-1}$ .

**Proof:** Let  $\check{G} \subset G$  be a union of separatrices from  $G$  such that  $\gamma_2 \neq f^k(\gamma_1)$  for every  $\gamma_1, \gamma_2 \in \check{G}$ ,  $k \in \mathbb{Z} \setminus \{0\}$  and  $G = \bigcup_{\gamma \in \check{G}} orb(\gamma)$ . Let  $i \in \{0, \dots, n\}$ ,  $p \in \Sigma_i$  and  $\gamma \in \check{G}$ .

Notice that  $(N_\gamma \setminus (\gamma \cup p)) / f^{per(\gamma)}$  is homeomorphic to  $X \times [0, 1]$  where  $X$  is 2-torus and the natural projection  $\pi_\gamma : N_\gamma \setminus (\gamma \cup p) \rightarrow X \times [0, 1]$  sends  $\partial N_\gamma^t$  to  $X \times \{t\}$  for each  $t \in (0, 1)$  and sends  $W_p^u \setminus p$  to  $X \times \{0\}$ . Let  $\xi_\gamma = \phi_{\varphi_{n-1}|_{W_i^u, \psi_i^s}}^{-1} \varphi_{n-1}|_{N_\gamma^i \setminus (\gamma \cup p)}$  and  $\hat{\xi}_\gamma = \pi_\gamma \xi_\gamma \pi_\gamma^{-1}|_{X \times [0, a_i]}$ . Due to item 3) of Lemma 4.11, the homeomorphism  $\hat{\xi}_\gamma$  coincides with the identity in some neighborhood of  $\pi_\gamma(N_\gamma^{a_i} \cap (G_0 \cup \dots \cup G_{n-1}))$ . Let us choose this neighborhood of the form  $B_\gamma \times [0, a_i]$ . Let us choose numbers  $0 < \tau_{1,\gamma} < \tau_{2,\gamma} < a_i$  such that  $\hat{\xi}_\gamma(X \times [0, \tau_{2,\gamma}]) \subset X \times [0, \tau_{1,\gamma}]$ . By construction,  $\hat{\xi}_\gamma : X \times [0, \tau_{2,\gamma}] \rightarrow X \times [0, 1]$  is a topological embedding which is the identity on  $X \times \{0\}$  and  $\hat{\xi}_\gamma|_{B_\gamma \times [0, \tau_{2,\gamma}]} = id|_{B_\gamma \times [0, \tau_{2,\gamma}]}$ . Then, due to Proposition 5.2,

1. there is a homeomorphism  $\hat{\zeta}_\gamma : X \times [0, \tau_{2,\gamma}] \rightarrow \hat{\xi}_\gamma(X \times [0, \tau_{2,\gamma}])$  such that  $\hat{\zeta}_\gamma$  is identity on  $X \times [0, \tau_{1,\gamma}]$  and is  $\hat{\xi}_\gamma$  on  $X \times \{\tau_{2,\gamma}\}$ .

2.  $\hat{\zeta}_\gamma|_{B_\gamma \times [0, \tau_{2,\gamma}]} = id|_{B_\gamma \times [0, \tau_{2,\gamma}]}$ .

Let  $\zeta_\gamma$  be a lift of  $\hat{\zeta}_\gamma$  on  $N_\gamma^{\tau_{2,\gamma}}$  which  $\xi_\gamma$  on  $\partial N_\gamma^{\tau_{2,\gamma}}$ . Thus  $\varphi_\gamma = \phi_{\varphi_{n-1}|_{W_i^u, \psi_i^s}} \zeta_\gamma$  is the desired extension of  $\varphi_{n-1}$  to  $N_\gamma$ . Doing the same for every separatrix  $\gamma \in \check{G}$  and extending it to the other separatrices from  $G$  by equivariance, we get the required homeomorphism  $h_{\mathcal{M}}$  for  $\tau_1 = \min_{\gamma \in \check{G}} \{\tau_{1,\gamma}\}$  and  $\tau_2 = \min_{\gamma \in \check{G}} \{\tau_{2,\gamma}\}$ .  $\diamond$

So far in this section, we have modified the homeomorphism  $\varphi$  in a union of suitable linearizable neighborhoods of  $\Omega_2$  (with their 1-dimensional separatrices removed) so that the modified homeomorphism extends equivariantly to  $W^s(\Omega_2)$ . At the same time, we can perform a similar modification about  $\Omega_1$  since the involved linearizable neighborhoods of  $\Omega_2$  and  $\Omega_1$  are

mutually disjoint. Thus, we get a homeomorphism  $h : M \setminus (\Omega_0 \cup \Omega_3) \rightarrow M \setminus (\Omega'_0 \cup \Omega'_3)$  conjugating  $f|_{M \setminus (\Omega_0 \cup \Omega_3)}$  with  $f'|_{M \setminus (\Omega'_0 \cup \Omega'_3)}$ . Notice that  $M \setminus (W_{\Omega_1}^s \cup W_{\Omega_2}^s \cup \Omega_3) = W_{\Omega_0}^s$  and  $M \setminus (W_{\Omega'_1}^s \cup W_{\Omega'_2}^s \cup \Omega'_3) = W_{\Omega'_0}^s$ . Since  $h(W_{\Omega_1}^s) = W_{\Omega'_1}^s$  and  $h(W_{\Omega_2}^s) = W_{\Omega'_2}^s$ , then  $h(W_{\Omega_0}^s \setminus \Omega_0) = W_{\Omega'_0}^s \setminus \Omega'_0$ . Thus for each connected component  $A$  of  $W_{\Omega_0}^s \setminus \Omega_0$ , there is a sink  $\omega \in \Omega_0$  such that  $A = W_{\omega}^s \setminus \omega$ . Similarly,  $h(A)$  is a connected component of  $W_{\Omega'_0}^s \setminus \Omega'_0$  such that  $h(A) = W_{\omega'}^s \setminus \omega'$  for a sink  $\omega' \in \Omega'_0$ . Then we can continuously extend  $h$  to  $\Omega_0$  by defining  $h(\omega) = \omega'$  for every  $\omega \in \Omega_0$ . A similar extension of  $h$  to  $\Omega_3$  finishes the proof of Theorem 1.

## 5 Topological background

We use below the notations which have been introduced before Lemma 4.8.

**Proposition 5.1** *Let  $z^0 > z^1 > \dots > z^m > \frac{z^0}{4} > 0$  and  $\xi : \mathcal{T} \setminus Ox_3 \rightarrow N$  be a topological embedding with the following properties:*

- (i)  $\xi a = a\xi$ ;
- (ii)  $\xi$  is the identity on  $Ox_1x_2$ ;
- (iii)  $\xi(\Pi(z^0 \cap \mathcal{T})) = \Pi(z^0)$  and  $\xi(\Pi(z^i) \cap \partial\mathcal{T}) \subset \Pi(z^i)$ ,  $i \in \{2, \dots, m\}$ ;
- (iv)  $\xi(\tilde{c}) \cap \tilde{c}^0 = \emptyset$ ,  $\xi(\tilde{c}^1) \cap \tilde{c}^0 = \emptyset$  and  $\xi(\tilde{\beta}) \subset \tilde{V}$ .

Then there is a homeomorphism  $\zeta : \mathcal{T} \rightarrow N$  such that

- 1)  $\zeta a = a\zeta$ ;
- 2)  $\zeta$  is the identity on  $\mathcal{H}$  – and consequentially on  $Ox_1x_2$ ;
- 3)  $\zeta(\Pi(z^i) \cap \mathcal{T}) \subset \Pi(z^i)$ ,  $i \in \{0, 2, \dots, m\}$
- 4)  $\zeta$  is  $\xi$  on  $\partial\mathcal{T}$ .

Moreover, if  $\xi$  is identity on  $\tilde{B}$  for a set  $B \subset (K \setminus U)$  then  $\zeta$  is also identity on  $\tilde{B}$ .

**Proof:** For  $j = 0, \dots, m$ , we denote by  $E_j$  the domain of  $\mathbb{R}^3$  located between the horizontal planes  $\Pi(z_j)$  and  $\Pi(z_{j+1})$ , with  $z_{m+1} = \frac{z_0}{4}$ . Since the wanted  $\zeta$  has to be equivariant with respect to  $a$ , it is useful to describe a fundamental domain  $\mathcal{V}$  for the action of  $a$  on the closure of  $\mathcal{T} \setminus \mathcal{H}$ ; the natural one is

$$\mathcal{V} = cl(\mathcal{T} \setminus \mathcal{H}) \cap \left( \bigcup_{j=0}^m E_j \right),$$

where  $cl(-)$  stands for *closure of*  $(-)$ . The domain  $\mathcal{V}$  is sliced by the horizontal planes  $\Pi(z_j)$ ,  $j = 2, \dots, m$ , and the vertical cylinders  $a^{-1}(\tilde{c})$  and  $\tilde{c}^1$ , yielding the decomposition  $\mathcal{V} = R_0 \cup R_1 \cup Q_0 \cup Q_2 \cup \dots \cup Q_m$  into solid tori whose interiors are mutually disjoint. Notice that the plane  $\Pi(z_1)$  is not used in this decomposition.

More precisely,  $R_0 \subset E_0$  is limited by the cylinders  $a^{-1}(\tilde{c}^1)$  and  $a^{-1}(\tilde{c})$ ; then,  $R_1 \subset E_0$  is limited by the cylinders  $a^{-1}(\tilde{c})$  and  $\tilde{c}^1$ . The others of the list are obtained from  $\tilde{U}$  by slicing  $\mathcal{V}$  with horizontal planes. The first of the latter, namely  $Q_0$ , is special as it is bounded by  $\Pi(z_0)$  and  $\Pi(z_2)$ ; then,  $Q_j$  is bounded by  $\Pi(z_j)$  and  $\Pi(z_{j+1})$  for  $j = 2, \dots, m$ . The vertical parts in the boundaries of the above-mentioned solid tori are provided by the vertical slices or the vertical parts of  $\partial\mathcal{T} \cup \partial\mathcal{H}$ .

For  $j = 0, 2, \dots, m$ , let  $U(z_j) := \tilde{U} \cap \Pi(z_j)$ . By construction, the top face of  $R_0$  is  $U'(z_0) := a^{-1}(U(z_{m+1})) = \Pi(z_0) \cap a^{-1}(\tilde{U})$ ; its bottom is  $U'(z_1) := \Pi(z_1) \cap a^{-1}(\tilde{U})$ . Similarly, the top of  $R_1$  is  $U''(z_0) := \Pi(z_0) \cap \tilde{K}$  and its bottom is  $U''(z_1) := \Pi(z_1) \cap \tilde{K}$ .

Very important is that each horizontal or vertical annulus  $Ann$  from the previous list is marked with a special arc noted  $\beta(Ann)$  linking the two boundary components of  $Ann$  and defined as follows:

$$\beta(Ann) = Ann \cap \{x_1 > 0, x_2 = 0\}.$$

According to assumption (iv), all these arcs (except when  $Ann = U''(z_0)$  or  $U''(z_1)$ ) fulfill the next condition, referred to as the  $\beta$ -condition, namely: they are mapped by  $\xi$  into  $\{x_1 > 0\}$ .

First of all, we define  $\zeta|_{R_1}$  by rescaling  $\zeta|_{\tilde{K}}$  in the next way. There is a homeomorphism  $\kappa : \tilde{K} \rightarrow R_1$  of the form:  $(x_1, x_2, x_3) \mapsto (x_1, x_2, \rho(x_3))$  where  $\rho : [0, z_0] \rightarrow [z_1, z_0]$  is any increasing continuous function. Then, we define  $\zeta|_{R_1} = \kappa \xi|_{\tilde{K}} \kappa^{-1}$ . Observe that  $\zeta$  equals  $\xi$  on  $U''(z_0)$  and coincide with the identity on  $U''(z_1)$ . As a consequence, the complement part of the statement follows directly. Indeed, if  $B$  lies in  $K$  and  $\xi|_{\tilde{B}} = Id$  then its conjugate by  $\kappa$  is the identity on  $\tilde{B} \cap R_1$ .

We continue by defining  $\zeta$  on the other horizontal annuli from the previous list. As required,  $\zeta$  is the identity when this annulus lies in  $\mathcal{H}$ . For the others, that is,  $U'(z_0)$  and  $U(z_j)$ ,  $j = 0, 2, \dots, m$ , Lemma 5.4 is applicable as it is explained right below.

Each of these annuli is bounded by two curves; one of the two lies in the frontier of  $\mathcal{T}$  on which  $\zeta$  has to coincide with  $\xi$  and is mapped in the respective plane  $\Pi(z_j)$  – according to (iii); and the other lies in  $\mathcal{H}$  where  $\zeta$  has to coincide with  $Id|_{\mathcal{H}}$ . In order to satisfy the equivariance property 3), we choose

$$(*) \quad \zeta|_{U(z_{m+1})} = a \zeta|_{U'(z_0)} a^{-1}.$$

Moreover, due to the  $\beta$ -condition, Lemma 5.4 holds and yields  $\zeta$  on each of the listed horizontal annuli.

We continue by defining  $\zeta$  on the vertical annuli in the above splitting of  $\mathcal{V}$ . When such an annulus lies in  $\partial H$  (resp.  $\partial \mathcal{T}$ ), we must take  $\zeta = Id$  (resp.  $\zeta = \xi$ ) over there. The last two annuli are  $R_0 \cap R_1$  and  $R_1 \cap Q_0$  on which  $\zeta$  is already defined by conjugating by  $\kappa$ . Notice that the  $\beta$ -condition holds for these two annuli because conjugating by  $\kappa$  preserves the  $\beta$ -condition.

Let us look more precisely to  $\partial Q_0$ . It is made of the following: two horizontal annuli  $U(z_0)$  and  $U(z_2)$ , and three vertical ones  $R_1 \cap Q_0$ ,  $\tilde{c}^1 \cap E_1$  (lying in  $\mathcal{H}$ ) and  $\tilde{c} \cap (E_0 \cup E_1)$ . The  $\beta$ -condition holds for each of these latter annuli.

For finishing the proof, it remains to extend the  $\zeta$  which we have defined right above on the tori  $\partial R_0$ ,  $\partial Q_0$  and  $\partial Q_j$ ,  $j = 2, \dots, m$  to embeddings of the solid tori  $R_0, Q_0, Q_2, \dots, Q_m$  with values in  $E_0, E_0 \cup E_1, E_2, \dots, E_m$  respectively. The boundary data consists of annuli where  $\zeta$  fulfils the  $\beta$ -condition. Therefore, the assumptions of Proposition 5.5 are fulfilled; then the conclusion holds true and yields the desired extension of  $\zeta$  to the listed solid tori.

According to (\*), we can extend the  $\zeta$  which is built above on a fundamental domain to  $\mathcal{T} \setminus \mathcal{H}$  equivariantly. Since this extension coincides with the identity on  $\mathcal{H}$ , it extends by  $Id|_{Ox_1x_2}$ . This is a continuous extension because any point of the plane  $Ox_1x_2$  adheres only to  $\mathcal{H} \setminus Ox_1x_2$  when considering the closure of  $\mathcal{T} \setminus Ox_1x_2$ .

◇

**Proposition 5.2** *Let  $X$  be a compact topological space,  $0 < \tau_1 < \tau_2 < 1$  and  $\hat{\xi} : X \times [0, \tau_2] \rightarrow X \times [0, 1]$  be a topological embedding which is the identity on  $X \times \{0\}$ ,  $X \times [0, \tau_1] \subset \hat{\xi}(X \times [0, \tau_2])$ . Then*

1. *there is a homeomorphism  $\hat{\zeta} : X \times [0, \tau_2] \rightarrow \xi(X \times [0, \tau_2])$  such that  $\hat{\zeta}$  is identity on  $X \times [0, \tau_1]$  and is  $\hat{\xi}$  on  $X \times \{\tau_2\}$ .*
2. *if for a set  $B \subset X$  the equality  $\hat{\xi}|_{B \times [0, \tau_2]} = id|_{B \times [0, \tau_2]}$  is true then  $\hat{\zeta}|_{B \times [0, \tau_2]} = id|_{B \times [0, \tau_2]}$ .*

**Proof:** Let us choose  $l \in (\tau_1, \tau_2)$  such that  $X \times [0, l] \subset \hat{\xi}(X \times [0, \tau_2])$ . Define a homeomorphism  $\kappa : [\tau_1, 1] \rightarrow [0, 1]$  by the formula

$$\kappa(t) = \begin{cases} (x, \frac{l(t-\tau_1)}{l-\tau_1}), & t \in [\tau_1, l]; \\ (x, t), & t \in [l, 1]. \end{cases}$$

Let  $\mathcal{K}(x, t) = (x, \kappa(t))$  on  $X \times [\tau_1, 1]$ . Then the required homeomorphism can be defined by the formula

$$\hat{\zeta}(x, t) = \begin{cases} (x, t), & t \in [0, \tau_1]; \\ \mathcal{K}^{-1}\xi(\mathcal{K}((x, s))), & s \in [\tau_1, \tau_2]. \end{cases}$$

Property 2 automatically follows from this formula. ◇

We now collect some facts of geometric topology in dimension 2 and 3 on which the proof of Proposition 5.1 is based. We begin with the so-called Schönflies Theorem (see Theorem 10.4 in [17]).

**Proposition 5.3** *Every topological embedding of  $S^1$  into  $\mathbb{R}^2$  is the restriction of a global homeomorphism of  $\mathbb{R}^2$  which is the identity map outside some compact set of the plane.*

One can derive the Annulus Theorem in dimension 2; we state and prove it in the only case which we use. The coordinates of  $\mathbb{R}^2$  are  $(x_1, x_2)$ . The unit closed disc in  $\mathbb{R}^2$  is denoted by  $\mathbb{D}^2$ ; its boundary is  $S^1$ . The annulus  $2\mathbb{D}^2 \setminus \text{int}(\mathbb{D}^2)$  is denoted by  $\mathbb{A}$ . Finally,  $\mathbb{I}$  denotes the arc  $\{1 \leq x_1 \leq 2, x_2 = 0\}$ .

**Lemma 5.4** *Let  $g : 2S^1 \cup \mathbb{I} \rightarrow \mathbb{R}^2 \setminus (0, 0)$  be a topological embedding which surrounds the origin in the direct sense and has the next properties:  $g(\mathbb{I}) \subset \{x_1 > 0\}$ , the image  $g(2S^1)$  avoids the circle  $C$  of radius  $\frac{3}{2}$  and  $g(1, 0)$  lies inside  $\frac{3}{2}\mathbb{D}^2$ . Then  $g|_{2\mathbb{D}^2}$  extends to an embedding  $G : \mathbb{A} \rightarrow \mathbb{R}^2 \setminus \text{int}(\mathbb{D}^2)$  which coincides with the identity on  $S^1$  and maps  $\mathbb{I}$  into  $\{x_1 > 0\}$ .*

**Proof:** Let  $p$  be the last point on  $g(\mathbb{I})$  starting from  $g(1, 0)$  which belongs to  $C$ . Let  $q$  be its inverse image in  $\mathbb{I}$ . Define  $G$  on the segment  $[(1, 0), q]$  as the affine map whose image is  $[(1, 0), p]$  and take  $G$  coinciding with  $g$  on  $[q, (2, 0)]$ . The image  $G(\mathbb{I})$  is a simple arc in  $\{x_1 > 0\}$ . Because, any simple arc is tame in the plane, this definition of  $G$  on  $\mathbb{I}$  and the values which are imposed on the two circles  $S$  and  $2S^1$  extends to a neighborhood  $N$  of  $S^1 \cup \mathbb{I} \cup 2S^1$  in  $\mathbb{R}^2$ . By taking one boundary component of  $N$  one derives a parametrized simple curve  $C'$  in  $\mathbb{R}^2 \setminus \mathbb{D}^2$  which does

not surround the origin. Therefore, by the Schönflies Theorem,  $C'$  bounds a disc  $D$  in  $\mathbb{R}^2 \setminus \mathbb{D}^2$  and the parametrization of  $C'$  extends to a parametrization of  $D$ . This yields the complete definition of  $G$ .  $\diamond$

We are now going to apply deep theorems of geometric topology in dimension 3 to a concrete situation emanating from the problem we are facing in Proposition 5.1. The setting is the following. We look at the 3-space

$$Y = \mathbb{A} \times [0, 1] = \{(x_1, x_2, x_3) \in \mathbb{R}^3 \mid 1 \leq x_1^2 + x_2^2 \leq 4, 0 \leq x_3 \leq 1\}.$$

Denote  $Q_0$  the solid torus in  $Y$  limited by the next two annuli:

- the vertical annulus  $A_v := \{x_1^2 + x_2^2 = 1, 0 \leq x_3 \leq 1\}$ ;
- the *standard curved* annulus  $A_0 := \{x_1^2 + x_2^2 + (x_3 - \frac{1}{2})^2 = \frac{5}{4}\} \cap Y$ .

Observe that  $A_0$  is contained in  $Y$  and contains the two horizontal circles forming  $\partial A_v$ .

**Proposition 5.5** *Let  $g : A_0 \rightarrow Y$  a bi-collared<sup>9</sup> topological embedding. It is assumed that  $g$  coincides with the identity on  $\partial A_0$  and maps the arc  $\Gamma_0 := A_0 \cap \{x_1 > 0, x_2 = 0\}$  to an arc  $\Gamma$  in  $Y \cap \{x_1 > 0\}$ . Then,  $g$  extends to an embedding  $G : Q_0 \rightarrow Y$  which coincides with the identity on  $A_v$ .*

**Proof:** The image  $A := g(A_0)$  separates  $Y$  in two components  $X$  and  $X^*$ . Since  $A$  is bi-collared, these two domains of  $Y$  are topological 3-manifolds. Therefore, we are allowed to invoke one of the deepest theorems in 3-dimensional topological geometry, namely E. Moise's theorem (See [17, Chap.23 & Theorem 35.3] for existence and [17, 36.2] for uniqueness):

*Every 3-manifold has a unique PL-structure, up to an arbitrarily small topological isotopy. Moreover, its topological boundary is a PL-submanifold.*

As a consequence,  $X$  and  $X^*$  have  $PL$ -structures which agree on their intersection  $A$ . By uniqueness applied for  $X \cup X^*$ , the  $PL$ -structure on the union is the standard one after some  $C^0$ -small ambient isotopy. Denote by  $P$  the planar surface  $\{x_2 = 0, x_1 > 0\} \cap Y$ . After a new  $C^0$ -small ambient isotopy in  $Y$ , we may assume that  $A$  and  $P$  are in general position. In what follows, we borrow the idea of proof from [14, Theorem 3.1]<sup>10</sup>.

Since  $P$  intersects each connected component of  $\partial A$  in one point only, we are sure that in general position  $P \cap A$  is made of finitely many simple closed curves  $c_1, \dots, c_k$  in  $\text{int}A$  and one arc  $\gamma$  which links the two components of  $\partial A$ . One of the above curves is *innermost* in  $P$ , meaning that it bounds a disc in  $P$  whose interior avoids  $A$ ; let say that  $c_1$  is so. More precisely,  $c_1$  bounds a disc  $d$  in  $P$  and a disc  $\delta$  in  $A$ . By innermost position,  $d \cup \delta$  is an embedded  $PL$  2-sphere  $\sigma$ . As  $Y$  lies in  $\mathbb{R}^3$ , this sphere bounds a 3-ball  $\Delta \subset Y$ . We are going to use these data in two ways.

First, we use  $\delta$  for finding an isotopy  $h_t$  of  $A$  in itself from  $Id|_A$  to  $h_1 : A \rightarrow A$  such that  $h_1(\Gamma) \cap \delta = \emptyset$ . This is easily done as  $\Gamma \cap \delta$  avoids one point  $z_\delta$  in  $\delta$ : one pushes  $\Gamma \cap \delta$  along

<sup>9</sup>This means that  $g$  extends to an embedding  $A_0 \times (-\varepsilon, +\varepsilon)$  for some  $\varepsilon > 0$ .

<sup>10</sup>In this article, it should be meant that the  $PL$ -category (or smooth category) is used. Indeed, there is no *general position* statement in topological geometry without more specific assumption.

the rays of  $\delta$  issued  $z_\delta$ . Notice that  $h_1(\Gamma)$  still lies in  $\{x_1 > 0\}$ , but this could be not true for  $h_t(\Gamma)$ ,  $t \neq 0, 1$ , when  $\delta$  is not contained in  $\{x_1 > 0\}$ .

Once, this is done, the ball  $\Delta$  is used for finding an ambient isotopy of  $Y$  which is supported in a neighborhood of  $B$ , small enough so that  $h_1(\Gamma)$  is kept fixed, and which moves  $A \cap \Delta$  to the complement of  $P$ . Hence, this isotopy cancels  $c_1$  from  $A \cap P$ ; all intersection curves contained in  $\text{int } \delta$  are cancelled at the same time. By repeating isotopies similar to the two previous ones, as many times as necessary, we get an embedding  $g' : A_0 \rightarrow Y$  which coincides with the identity on  $\partial A_0$ , still maps  $\Gamma_0$  into  $\{x_1 > 0\}$  and fulfils the next property:  $A' := g'(A_0) \cap P$  is made of one arc  $\gamma'$  only which links the two components of  $g'(\partial A_0)$ . The annulus  $A'$  divides  $Y$  into two (closed) domains  $X'$  and  $X'^*$  which come from the splitting  $X \cup_A X^*$  by an ambient isotopy fixing  $\{x_3 = 0, 1\}$  pointwise.

As  $\gamma'$  is the only intersection component of  $P \cap A'$ , one knows that  $\gamma'$  divides  $P$  into a disc  $\mu' \subset X'$  (meaning a *meridian* disc in a solid torus) and its complement in  $P$ . Removing from  $X'$  a regular neighborhood of  $\mu'$  yields a  $PL$  embedded 2-sphere  $S$ . According to the Alexander theorem [1], this sphere bounds a ball  $B_{X'}$  in  $\mathbb{R}^3$ , as  $Y \subset \mathbb{R}^3$ . It is not possible that  $B_{X'}$  contains  $\mu'$  in its interior; in the contrary,  $B_{X'}$  would get out of  $X'^*$  and have a non-bounded interior. As a consequence,  $X'$  is a *solid torus* since it is made of a ball and a 1-handle attached. The same holds for  $X$  as it is ambient isotopic to  $X'$  in  $Y$ .

This is not sufficient for concluding. It would be necessary to prove the same for the curve  $g'(\Gamma_0)$ , after making it a closed curve by adding the vertical arc  $\gamma_0^* \subset A_v$  which links the two points of  $g'(\Gamma_0) \cap A_v = \Gamma_0 \cap A_v = g(\Gamma_0) \cap A_v = \Gamma_0 \cap A_v = \gamma' \cap A_v$ . That is the place where the assumption about  $g(\Gamma_0)$  is used.

*CLAIM. There exists an ambient isotopy from  $Id|_Y$  to  $k_1$  which is stationary on the vertical annulus  $A_v$ , which maps  $A'$  into itself and moves  $g'(\Gamma_0)$  to  $\gamma'$ .*

**PROOF OF THE CLAIM.** Assume first that the arcs  $g'(\Gamma_0)$  and  $\gamma'$  meet in their end points only. Consider the closed curve  $\alpha$  which is made of  $\gamma \cup g'(\Gamma_0)$ ; it is contained in  $\{x_1 > 0\} \cap A'$ . By construction, the homological intersection of  $\alpha$  with  $\gamma'$  is zero. Therefore,  $\alpha$  bounds a disc  $\delta' \subset A'$ . Notice that it could be not contained in  $\{x_1 > 0\}$ . The disc  $\delta'$  allows one to move  $g'(\Gamma_0)$  to  $\gamma'$  by an isotopy of  $A'$  into itself with the required properties.

In case where  $g'(\Gamma_0) \cap \text{int } \gamma'$  is non-empty, in general position this intersection is made of finitely many points. Among them, choose the point  $x$  which is the closest to  $\gamma' \cap \{x_3 = 1\}$  when traversing  $\gamma'$  starting from bottom. Denote by  $x_0$  the point  $\gamma' \cap \{x_3 = 1\}$ . One forms a closed curve  $\tau \subset A'$  made of two arcs in  $A'$ , from  $x$  to  $x_0$  respectively in  $\gamma'$  and  $g'(\Gamma_0)$ . For the same reason as for  $\alpha$  above, the curve  $\tau$  bounds a disc in  $A'$  which allows one to cancel  $x$  from  $g'(\Gamma_0) \cap \gamma'$  by an isotopy of  $A'$  into itself with the required properties. Iterating this process reduces us to the first case. The claim is proved.  $\diamond$

Let  $g'' := k_1 g' : A_0 \rightarrow A'$ . As a consequence of the claim, the closed curve  $g''(\Gamma_0) \cup \gamma_0^*$  is the boundary of the *meridian* disc  $\mu'$ . We are going to show that  $g''$  extends to an embedding  $G'' : Q_0 \rightarrow X'$  which coincides with the identity on  $A_v$ . In this aim, we denote by  $\mu_0$  the meridian of  $Q_0$  defined by  $\mu_0 = Q_0 \cap P$ . For beginning with, we consider regular neighborhoods  $N(\mu_0)$  and

$N(\mu')$  of both meridians and we extend  $g''|_{N(\mu_0) \cap A_0}$  to a homeomorphism  $G''_0 : N(\mu_0) \rightarrow N(\mu')$  which is the identity on  $N(\mu_0) \cap A_v$ .

Let  $B_{Q_0}$  be the ball in  $Q_0$  which is the closure of  $Q_0 \setminus N(\mu_0)$ . The restricted map  $G''_0|_{\partial B_{Q_0}}$  glued with the restriction of  $g''$  to the closure of  $A_0 \setminus N(\mu_0)$  yields a homeomorphism

$$G''_1 : \partial B_{Q_0} \rightarrow \partial B_{X'}.$$

The desired  $G''$  is obtained by extending  $G''_1$  to  $B_{Q_0}$  by the cone construction (seeing a ball as the cone on its boundary). Since  $g$  and  $g''$  are related one to the other by an ambient isotopy fixing  $A_v$  pointwise, an extension of  $g$  follows from an extension of  $g''$ .  $\diamond$

## References

- [1] J.W. Alexander, On the subdivision of 3-spaces by polyhedron // Proc. Natl. Acad. Sci. USA 1924, t. 10, 6–8.
- [2] A. Andronov, L. Pontryagin. Rough systems // DAN. 1937. V. 14. N. 5. 247–250.
- [3] G.F. de Baggis. Rough systems of two differential equations // Uspehi Matem. Nauk. 1955. V.10. 4. 101–126.
- [4] R.H. Bing, The Geometric Topology of 3-Manifolds, Colloquium Publications 40, Amer. Math. Soc., 1983.
- [5] Bonatti Ch., Grines V. Knots as topological invariant for gradient-like diffeomorphisms of the sphere  $S^3$  // Journal of Dynamical and Control Systems (Plenum Press, New York and London). 2000. V. 6. N. 4. 579–602.
- [6] C. Bonatti, V. Grines, V. Medvedev, E. Pecou. Three-manifolds admitting Morse-Smale diffeomorphisms without heteroclinic curves // Topology and its Applications. 2002. V. 117. 335–344.
- [7] Bonatti Ch., Grines V., Medvedev V., Pecou E. Topological classification of gradient-like diffeomorphisms on 3-manifolds // Topology. 2004. 43. 369–391.
- [8] C. Bonatti, V. Grines, O. Pochinka. Classification of the Morse-Smale diffeomorphisms with the finite set of heteroclinic orbits on 3-manifolds // Trudy math. inst. im. V. A. Steklova. 2005. V. 250. 5–53.
- [9] Ch. Bonatti, V. Grines, O. Pochinka. Classification of Morse-Smale diffeomorphisms with the chain of saddles on 3-manifolds // Foliations 2005. World Scientific. Singapore. 2006. 121–147.
- [10] A. Candel. Laminations with transverse structure // Topology. 1999. Vol. 38, No. 1, 141–165.



- [11] V. Grines, V. Medvedev, O. Pochinka, E. Zhuzhoma. Global attractors and repellers for Morse-Smale diffeomorphisms // Trudy math. inst. im. V. A. Steklova. 2010. V. 271. 111–133.
- [12] Grines V., Pochinka O. Morse-Smale cascades on 3-manifolds // Russian Mathematical Surveys. 2013. Vol. 68. No. 1. P. 117–173.
- [13] Grines V., Medvedev T., Pochinka O. Dynamical Systems on 2- and 3-Manifolds. Springer International Publishing Switzerland, 2016.
- [14] V.Z. Grines, E.V. Zhuzhoma, V.S. Medvedev. New relations for Morse-Smale systems with trivially embedded one dimensional separatrices // Sbornik Math. 194 No 7 (2003), 979-1007.
- [15] E.A. Leontovich. Some mathematical works of Gorky school of A.A. Andronov // Proc. of the third All-Union mathematical congress. 1958. V. 3. 116–125.
- [16] A. Mayer. Rough map circle to circle // Uch. Zap. GGU. 1939. V. 12. P. 215–229.
- [17] E. Moise, Geometric Topology In dimensions 2 and 3// GTM 47, Springer, 1977.
- [18] J. Palis. On Morse-Smale dynamical systems // Topology. 1969. V. 8. N. 4. 385–404.
- [19] J. Palis, W. de Melo. Geometrical theory of dynamical systems // Moscow. Mir. 1998, Springer, 1982, ISBN: 978-1-4612-5703-5 (Online).
- [20] J. Palis, S. Smale. Structural stability Theorems // Proceedings of the Institute on Global Analysis. American Math. Society. 1970. V. 14. 223–231.
- [21] M.M. Peixoto. On structural stability // Ann. of Math. 1959. V. 69. N<sup>o</sup>1, 199–222.
- [22] M. Peixoto. Structural stability on two-dimensional manifolds // 1962. V. 1. N. 2. 101–120.
- [23] M. Peixoto. Structural stability on two-dimensional manifolds (a further remarks) // 1963. V. 2. N.2. 179–180.
- [24] S. Smale. Differentiable dynamical systems // Bull. Amer. Math. Soc. 1967. V. 73. 6. 747–817.

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