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# Structure at Infinity Revisited for Delay Systems

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## Abstract

Structure at infinity in a strong sense is considered for delay systems. A sufficient condition for the existence of this canonical form is given. The loss of generality is balanced by the fact that some solutions to control problems by static state feedback without anticipation can be obtained.

## 1 Introduction

The structure at infinity or the Smith-McMillan form at infinity are well known tools for the characterization of the solvability of some control problems such as model matching, disturbance rejection, row-by-row decoupling,... For linear finite dimensional systems see [1, 10] for instance. The notion of zeros at infinity has been generalized to non-linear systems [7] and several concepts are available for singular systems [4]. For linear infinite dimensional systems and in the particular case of bounded operators, the structure at infinity was introduced in [3], described in several equivalent ways and used to solve some control problems in [5]. The particular case of delay systems was studied in [6]. However the structure at infinity defined there is too weak to insure a good solution for control problems: indeed the potential compensators may be anticipative (see also [9]). In this paper we introduce the concept of strong structure at infinity which is more convenient to infinite dimensional systems (and to the delay systems as a particular case). This structure is only well defined for some classes of systems. The positive result is that if this structure at infinity is well available then all potential solutions of control problems are non-anticipative and may be realized by static state feedback.

## 2 Preliminaries

We consider delay systems described by

$$\begin{cases} \dot{x}(t) &= A_0x(t) + A_1x(t-1) + B_0u(t) \\ y(t) &= C_0x(t) \end{cases} \quad (1)$$

where  $x(t) \in \mathbb{R}^n$ ,  $y(t) \in \mathbb{R}^p$  and  $u(t) \in \mathbb{R}^m$ . Note that some results may be extended to systems with several

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commensurate delays in state, input and output. The transfer function matrix of the system is

$$T(s) = C_0(sI - A_0 - A_1e^{-s})^{-1}B_0$$

which may be decomposed as follows

$$T(s) = \sum_{j=0}^{\infty} T_j(s)e^{-js}, \quad (2)$$

where  $T_j(s) = C_0(sI - A_0)^{-1} [A_1(sI - A_0)^{-1}]^j B_0$ . Each matrix  $T_j(s)$  may be decomposed as a serie using the following constant matrices introduced by Kirillova and Churakova (see [11]):

$$\begin{aligned} Q_i(j) &= A_0Q_{i-1}(j) + A_1Q_{i-1}(j-1), \\ Q_0(0) &= I, Q_i(j) = 0, i < 0 \text{ or } j < 0. \end{aligned} \quad (3)$$

We have

$$T_j(s) = \sum_{i=0}^{\infty} C_0Q_i(j)B_0s^{-(i+1)}$$

Another expression which will be used in this paper is the following one

$$T(s) = \sum_{i=0}^{\infty} \left( \sum_{j=0}^i C_0Q_i(j)B_0e^{-js} \right) s^{-(i+1)}. \quad (4)$$

Both expression may be obtained by a simple calculation using the relations (3), see for example [9, 11]. Note that for the case of a system with simple delays we have  $Q_i(j) = 0$ , for  $i < j$ . The situation is different for delay systems of neutral type.

## 3 Structure at infinity

In order to define the structure at infinity (weak and strong) we need the notion of proper functions.

**Definition 3.1** *A complex valued function  $f(s)$  is called weak proper if  $\lim_{s \rightarrow \infty} f(s)$  is finite when  $s \in \mathbb{R}$  tends to  $\infty$ . It is called strictly weak proper if this limit is 0. A matrix  $B(s)$  is weak biproper if it is a weak proper and its inverse is also weak proper. Weak proper is replaced by strong proper if the same occurs when  $\Re(s) \rightarrow \infty$ .*

It is obvious that strong properness implies weak properness. The strong properness was used in [3] and [5] in the description of the structure at infinity for infinite dimensional systems. In [6] and [9] the weak notion was used in order to define the structure at infinity of delay system and to solve some control problem.

However, this structure at infinity cannot be used, in general, to solve control problems with non predictive control laws in an efficient way. For example, if the transfer function of the system is  $T(s) = s^{-3} + s^{-2}e^{-s}$ , the weak structure at infinity is  $s^{-3}$  since  $T(s) = s^{-3}(1 + se^{-s})$ , and since  $1 + se^{-s}$  is weak biproper. Suppose one has to solve the model matching problem for a given model  $T_m(s) = s^{-3}$ . As the structure at infinity of the plant and the model are the same, there exist a proper compensator  $C(s)$  such that  $T(s)C(s) = T_m(s)$ , see [6]. In fact the unique compensator solving this problem is  $C(s) = 1 + se^{-s}$  which is not realizable by linear static state feedback even if one uses distributed delays.

If we consider the notion of strong properness to define the structure at infinity, some difficulties occur in the construction of the canonical form at infinity. For the given example the structure at infinity is not  $s^{-3}$  because  $1 + se^{-s}$  is not proper in the strong sense and it is not possible, in fact, to define strong structure at infinity for the given example. Is there a structure at infinity in the strong sense for every delay system? For the moment it is not possible to give a positive answer.

The importance of the strong properness is given by the following considerations. Consider the problem of row-by-row decoupling for a square system ( $m = p$ ). If the problem is solvable by static state feedback (see Definition 4.1), then the corresponding compensator is given by  $C(s) = (W_0 + W(s))$  with a strictly strong proper matrix  $W(s)$ . The same consideration may be made for other control problems: disturbance rejection, model matching.

As the matrices  $T_i(s)$  are rational functions with the degree of the denominator greater than the degree of the numerator,  $T_i(s)$  are strictly proper in the strong sense which is, in this case, the same as the weak sense and the structure at infinity is well defined in the classical (finite dimensional) sense. The orders of the zeros at infinity of the matrices  $T_i(s)$  may be compared in the following sense. We say that the orders of  $T_i(s)$  are *increasing* if the maximal order of the zeros at infinity of  $T_i(s)$  is less than or equal to the minimal order of those of  $T_{i+1}(s)$ , for all  $i \geq 0$ . This hypothesis excludes the example given above. The counterpart of this loss of generality allows us to solve control problem with *good* compensators. We have the following main result.

**Theorem 3.2** *If the structure at infinity of each matrix  $T_i(s)$  are increasing, there then exist two strong biproper matrices  $B_1(s)$  and  $B_2(s)$  such that*

$$B_1(s)T(s)B_2(s) = \begin{bmatrix} \Delta_0(s) & 0 & \cdots & 0 & 0 \\ 0 & \Delta_1(s)e^{-s} & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & \Delta_k(s)e^{-ks} & 0 \\ 0 & 0 & \cdots & 0 & 0 \end{bmatrix},$$

where  $\Delta_i(s) = \text{diag} [s^{-n_{i,1}}, \dots, s^{-n_{i,j_i}}]$  and  $n_{i,j} \leq n_{i,j+1}$ ,  $i = 1, \dots, k$  and  $n_{i,j_i} \leq n_{i+1,j_{i+1}}$ .

PROOF: Let  $\Lambda_0(s)$  be the canonical form at infinity of the matrix  $T_0(s)$ , there then exist two rational strong biproper matrices  $B_1^0(s)$  and  $B_2^0(s)$  such that

$$B_1^0(s)T(s)B_2^0(s) = \Lambda_0(s) + \sum_{i=1}^{\infty} B_1^0(s)T_i(s)B_2^0(s)e^{-is},$$

and

$$\Lambda_0(s) = \begin{bmatrix} \Delta_0(s) & 0 \\ 0 & 0 \end{bmatrix}.$$

This may be written as:

$$\bar{T}(s) = B_1^0(s)T(s)B_2^0(s) = \begin{bmatrix} \Delta_0(s) + \theta_{11}(s) & \theta_{12}(s) \\ \theta_{21}(s) & \theta_{22}(s) \end{bmatrix},$$

moreover by our assumption,

$$\Delta_0^{-1}(s)\theta_{11}(s), \quad \Delta_0^{-1}(s)\theta_{12}(s) \quad \text{and} \quad \theta_{21}(s)\Delta_0^{-1}(s)$$

are strictly proper in the strong sense. This gives

$$\bar{T}(s)B(s) = \begin{bmatrix} \Delta_0(s) & \theta_{12}(s) \\ \theta_{21}(s)\beta^{-1}(s) & \theta_{22}(s) \end{bmatrix},$$

where  $\beta(s) = (I + \Delta_0^{-1}(s)\theta_{11}(s))$  and  $B(s)$  being the strong biproper matrix:

$$B(s) = \begin{bmatrix} \beta^{-1}(s) & 0 \\ 0 & I \end{bmatrix}.$$

Here and in what follows  $I$  denote the identity matrix with adequate dimension which is not precised in each case. Let us define  $B_1^1$  and  $B_2^1$  as follows:

$$B_1^1(s) = \begin{bmatrix} I & 0 \\ \theta_{21}(s)\beta^{-1}(s)\Delta_0^{-1}(s) & I \end{bmatrix},$$

$$B_2^1(s) = \begin{bmatrix} I & -\Delta_0^{-1}(s)\theta_{12}(s) \\ 0 & I \end{bmatrix}.$$

Then

$$B_1^1(s)T(s)B_2^1(s) = \begin{bmatrix} \Delta_0(s) & 0 \\ 0 & \theta_{22}(s) \end{bmatrix} = \tilde{T}(s),$$

i. e.

$$\tilde{T}(s) = \Lambda_0(s) + \sum_{i=1}^{\infty} \tilde{T}_i(s)e^{-is}, \quad (5)$$

where

$$\tilde{T}_i(s) = \begin{bmatrix} 0 & 0 \\ 0 & R_i(s) \end{bmatrix},$$

the matrices  $R_i(s)$  being strictly proper. The same procedure may then be applied to the matrix

$$\sum_{i=1}^{\infty} R_i(s)e^{-is} = e^{-s} \sum_{i=1}^{\infty} R_i(s)e^{-(i-1)s}$$

because the rational matrices  $R_i(s)$  have also increasing orders. ■

## 4 Control problems

We consider two control problems: row-by-row decoupling and disturbance rejection with static state feedback of the form:

$$u(t) = \sum_{i=0}^k F_i x(t-i) + \sum_{i=0}^k G_i v(t-i), \quad t \in [0, k+1]$$

with  $G_0$  invertible for the row-by-row decoupling problem and  $G_i = 0$  when we consider only the disturbance rejection problem.

**Definition 4.1** For  $m = p$  we say that the row-by-row decoupling problem is solvable iff there exist a feedback of the given type such that the transfer function matrix of the closed loop system is of the form

$$T_{F,G}(s) = \text{diag} [h_1(s), \dots, h_p(s)]$$

where  $h_i(s)$  are non zero strictly strong proper functions.

The result is

**Theorem 4.2** If the zeros at infinity of  $T_i(s)$  are increasing, then the row-by-row decoupling problem is solvable if and only if

$$\Sigma_{\infty}(C_0, A_0, A_1, B_0) = \{\Sigma_{\infty}(c_1, A_0, A_1, B_0), \dots, \Sigma_{\infty}(c_p, A_0, A_1, B_0)\},$$

where  $c_i$ 's are the rows of the matrix  $C_0$ , and  $\Sigma_{\infty}$  denotes the strong structure at infinity.

PROOF: Assume that the zeros at infinity of  $T_j$  are increasing and that the row-by-row decoupling problem is solvable. Then

$$T_{F,G}(s) = \text{diag} [h_1(s), \dots, h_p(s)],$$

which means that the global structure of  $T_{F,G}$  is equal to the union of the rows' structures at infinity. On the other hand, as  $G_0$  is regular, one can see that

$$T_{F,G}(s) = T(s)B(s),$$

with a strong biproper matrix  $B(s)$ . This means that  $T$  and  $T_{F,G}$  have the same structure at infinity.

For the converse, assume now that  $T_i(s)$  have increasing orders and the condition of the theorem is satisfied. Let  $s^{-n_i}e^{-j_i}$  be the strong structure at infinity of the row  $i$ ,  $i = 1, \dots, m$ . Our assumption gives

$$c_i Q_i(j) B_0 = 0 \quad \text{for } l < n_i - 1, \quad j \geq j_i.$$

Then the first non zero moment of the decomposition (4) is

$$\left[ \sum_{j=0}^{n_i-1} c_i Q_{n_i-1}(j) B_0 e^{-js} \right]_{i=1}^m, \quad (6)$$

where  $c_i$  is a row of  $C_0$  and we have

$$c_i Q_{n_i-1}(j) B_0 = 0 \quad \text{for } j < j_i.$$

Let us denote

$$E_l = [c_i Q_{n_i-1}(j_i + l) B_0]_{i=1}^m, \quad l = 0, \dots, n_i - 1 - j_i.$$

The condition of the theorem yields to the invertibility of the matrix  $E_0$ . The first moment of the decomposition (4) given in (6) is expressed by:

$$E = E_0 + E_1 e^{-s} + \dots + E_k e^{-ks},$$

where  $k = \sup\{n_i - 1, i = 1, \dots, m\}$ .

Let us denote by  $F$  and  $G$  the needed feedback:

$$F = F_0 + F_1 e^{-s} + \dots, \quad G = G_0 + G_1 e^{-s} + \dots,$$

and by  $A$  the matrix  $A_0 + A_1 e^{-s}$ . A formal calculation gives, as in the classical paper [2]:

$$c_i (A + B_0 F)^l = c_i A^l, \quad l \leq n_i - 1,$$

and then

$$c_i (A + B_0 F)^l B_0 = c_i A^l B_0 = 0 \quad l < n_i - 1,$$

and

$$\begin{aligned} [c_i (A + B_0 F)^{n_i-1} B_0]_{i=1}^m &= [c_i A^{n_i-1} B_0]_{i=1}^m = \\ &E \text{diag} \{e^{-j_1 s}, \dots, e^{-j_m s}\}. \end{aligned}$$

Let us remark that  $E$  is invertible. If  $G = E^{-1}$ , then

$$\begin{aligned} [c_i (A + B_0 F)^{n_i-1} B_0]_{i=1}^m G &= \\ \text{diag} \{e^{-j_1 s}, \dots, e^{-j_m s}\}. \end{aligned}$$

In the same way we can write:

$$\begin{aligned} [c_i (A + B_0 F)^{n_i}]_{i=1}^m &= [c_i A^{n_i-1} (A + B_0 F)]_{i=1}^m = \\ [c_i A^{n_i}]_{i=1}^m + [c_i A^{n_i-1} B_0]_{i=1}^m F &= [c_i A^{n_i}]_{i=1}^m + E F. \end{aligned}$$

Then  $F$  may be calculated from this relation:

$$F = -G [c_i A^{n_i}]_{i=1}^m.$$

■

For the disturbance rejection problem the definition is

**Definition 4.3** The disturbance rejection problem for the system

$$\begin{cases} \dot{x}(t) &= A_0 x(t) + A_1 x(t-1) + B_0 u(t) \\ &+ D_0 q(t) \\ y(t) &= C_0 x(t) \end{cases} \quad (7)$$

is solvable if there exist a feedback

$$u(t) = \sum_{i=0}^k F_i x(t-i), \quad t \in [0, k+1]$$

such that the output of the system is not affected by the disturbance  $q(t)$ .

Our result is

**Theorem 4.4** If the zeros at infinity of  $T_i(s)$  are increasing, then the disturbance rejection problem is solvable if and only if

$$\Sigma_{\infty} [s^{-1} T(s) \quad T^D(s)] = \Sigma_{\infty} [s^{-1} T(s) \quad 0]$$

where  $T(s)$  is the transfer function matrix of the control:

$$T(s) = C_0(sI - A_0 - A_1e^{-s})^{-1}B_0$$

and  $T^D(s)$  is the transfer function matrix of the disturbance:

$$T^D(s) = C_0(sI - A_0 - A_1e^{-s})^{-1}D_0.$$

PROOF: Suppose that the problem is solvable. Then, for all  $k$ , the problem is solvable for the systems (see [6])

$$\begin{cases} \dot{z}_k(t) &= \mathcal{A}_k z_k(t) + \mathcal{B}_k u(t) + \mathcal{D}_k q_k(t), \\ w_k(t) &= \mathcal{C}_k z_k(t), \end{cases} \quad (8)$$

where  $\mathcal{A}_k$ ,  $\mathcal{B}_k$ ,  $\mathcal{D}_k$  and  $\mathcal{C}_k$  are given as in [6]:

$$\mathcal{A}_k = \begin{bmatrix} A_0 & 0 & \cdots & 0 \\ A_1 & A_0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & A_0 \end{bmatrix}, \quad \mathcal{B}_k = \begin{bmatrix} B_0 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & B_0 \end{bmatrix},$$

$$\mathcal{C}_k = \begin{bmatrix} C_0 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & C_0 \end{bmatrix}, \quad \mathcal{D}_k = \begin{bmatrix} D_0 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & D_0 \end{bmatrix}.$$

Let  $\Phi_k$  and  $\Phi_k^D$  be the transfer function matrices of the control and disturbance of the systems (8). Then,

$$\Sigma_\infty[s^{-1}\Phi_k(s) \quad \Phi_k^D(s)] = \Sigma_\infty[s^{-1}\Phi_k(s) \quad 0].$$

As

$$\Phi_k = \begin{bmatrix} T_0 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ T_k & \cdots & T_0 \end{bmatrix}, \quad \Phi_k^D = \begin{bmatrix} T_0^D & \cdots & 0 \\ \vdots & \ddots & \vdots \\ T_k^D & \cdots & T_0^D \end{bmatrix},$$

and according to the assumption on the zeros of  $T_j(s)$ , this gives  $\Sigma_\infty[s^{-1}T(s) \quad T^D(s)] = \Sigma_\infty[s^{-1}T(s) \quad 0]$ .

The converse may be obtained in the same way and using the fact that, for the systems (8), we can choose the feedback with a lower triangular form which insures that for the system (7) the feedback is without anticipation [6]. ■

Note that the assumption on increasing zeros orders at infinity concerns the matrices  $T_j$ , but are verified for the matrices  $T_j^D$  in the context of the theorem. Other problems may be considered as invertibility, model matching. The result given in [6] may be adapted to the strong structure at infinity.

**Remark 4.5** For systems without strong structure at infinity defined here, further investigations are needed to characterize the solution of control problems (see [8] for another approach). Consider the above mentioned example:  $T(s) = s^{-3} + s^{-2}e^{-s}$ . Let  $T^m(s) = \sum_{j=0}^{\infty} T_j^m e^{-js}$  be a model. What are the conditions which insure the existence of a strong proper compensator  $C(s) = \sum_{j=0}^{\infty} C_j(s)e^{-js}$  such that  $TC = T^m$  (it is the classical model matching problem). Suppose that  $T^m(s) = s^{-5} - s^{-3}e^{-2s}$ . A simple calculation yields to the following strong proper compensator:  $s^{-2} - s^{-1}e^{-s}$ . We can see that  $[T(s) \quad T_m(s)]B(s) = [T(s) \quad 0]$ , with strong biproper matrix  $B(s)$ . Hence,  $[T(s) \quad T_m(s)]$  and  $[T(s) \quad 0]$  are equivalent at infinity. The problem which needs some investigation is: how to define the canonical form corresponding to this equivalence relation?

## 5 Conclusion

For a class of delay systems we define the *strong* structure at infinity which allows to solve control problems by static state feedback without anticipation. However, it is not clear how to do when this structure at infinity is not defined. Further investigations on the structure at infinity in every time interval  $[k, k+1]$  may give solution in a general case. The weak structure at infinity is always available in solving such problems but the solution must be taken with care.

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