



**HAL**  
open science

# Disturbance rejection and row-by-row decoupling for linear delay systems of neutral type

Rabah Rabah, Michel Malabre

► **To cite this version:**

Rabah Rabah, Michel Malabre. Disturbance rejection and row-by-row decoupling for linear delay systems of neutral type. 14th international conference on Mathematical theory of Networks and Systems (MTNS 2000), Université de Perpignan, Jun 2000, Perpignan, France. hal-01465741

**HAL Id: hal-01465741**

**<https://hal.science/hal-01465741>**

Submitted on 13 Feb 2017

**HAL** is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

# Disturbance rejection and row-by-row decoupling for linear delay systems of neutral type

Rabah Rabah, Michel Malabre

Institut de Recherche en Communications et Cybernétique de Nantes, UMR 6597

1, rue de la Noë, B. P. 92101

F-44321 Nantes Cedex 03, France

Rabah.Rabah@emn.fr, Michel.Malabre@ircyn.ec-nantes.fr

**Keywords:** Neutral-type time-delay systems, Structure at infinity, Decoupling.

## 1 Introduction

The present paper is concerned with delay systems of neutral type, i.e. systems described by equations

$$\begin{cases} \dot{x}(t) - A_{-1}\dot{x}(t-1) &= A_0x(t) + A_1x(t-1) \\ &+ B_0u(t) + D_0d(t) \\ y(t) &= C_0x(t) \end{cases} \quad (1)$$

where  $x(t) \in \mathbb{R}^n$  is the state,  $u(t) \in \mathbb{R}^p$  the input,  $d(t) \in \mathbb{R}^q$  is a disturbance and  $y(t) \in \mathbb{R}^m$  the output. The matrices  $A_0, A_1, A_{-1}, B_0, C_0$  and  $D_0$  are of suitable dimensions. In this paper we assume that matrices  $B_0$  and  $C_0$  are of full rank.

There are more general forms of delay systems of neutral type: with several delays, with distributed delays... Our considerations may be formulated for systems with several commensurate delays in the state, input or output. For the sake of clarity, we limit our presentation here to a single delay in the state and its derivative. For other approaches concerning systems with delay of neutral type see for example [1, 4].

The first part is devoted to the description of solutions of the system and transfer function matrix in several ways. The new approach is the generalization of the description given by Olbrot [4] and Zmood [17] for systems with simple delay in state (see [4, 14] for more details on different approaches). Then, we describe the structure at infinity of the system (1) and related questions. Applications to control problems are considered in the last part. The main results are the characterization of the solvability of the row-by-row decoupling problem and the disturbance rejection for linear time-delay systems of neutral type. The precompensator solving the given problem may be realized by generalized static state feedback, i. e. feedback which contains the delayed derivative of the new control (for the decoupling problem) or the delayed derivative of the disturbance. An important property is that the formal stability of the neutral-type system is not affected by such feedbacks.

## 2 Preliminaries

The solution may be expressed using the fundamental matrix of solutions  $\Phi(t)$ , which satisfies the differential equation

$$\dot{\Phi}(t) = A_0\Phi(t) + A_1\Phi(t-h) + A_{-1}\dot{\Phi}(t-h)$$

with the initials conditions:

$$\Phi(0) = I, \quad \Phi(t) = 0 \quad \text{for } t < 0.$$

Let us give the expression of this matrix. Let  $Q_i(j)$  be the matrices defined by the relations [2, 3]:

$$Q_i(j) = A_0Q_{i-1}(j) + A_1Q_{i-1}(j-1) + A_{-1}Q_i(j-1)$$

with initial conditions:

$$Q_0(0) = I, \quad Q_i(j) = 0 \quad \text{if } ij < 0.$$

Then the matrix  $\Phi(t)$  may be written as (see [15])

$$\Phi(t) = \sum_{i=0}^{\infty} \sum_{j=0}^k Q_i(j) \frac{(t-j)^i}{i!} \quad \text{for } t \in [k, k+1[.$$

This relation may be verified by induction. Another expression of the fundamental matrix may be obtained via the solutions of the system

$$\begin{cases} \mathbf{E}_k \dot{z}_k(t) &= \mathbf{A}_k z_k(t) + \mathbf{B}_k v_k(t) + \mathbf{D}_k w_k(t) \\ w_k(t) &= \mathbf{C}_k z(t) \end{cases} \quad (2)$$

where the matrices  $\mathbf{E}_k, \mathbf{A}_k, \mathbf{B}_k$  and  $\mathbf{C}_k, k \in \mathbb{N}$  are composed by  $(k+1)^2$  blocks as follows.

$$\mathbf{E}_k = \begin{bmatrix} I & 0 & 0 & \dots & 0 \\ -A_{-1} & I & 0 & \dots & 0 \\ 0 & -A_{-1} & I & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & I \end{bmatrix},$$

$$\mathbf{A}_k = \begin{bmatrix} A_0 & 0 & 0 & \dots & 0 \\ A_1 & A_0 & 0 & \dots & 0 \\ 0 & A_1 & A_0 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & A_0 \end{bmatrix},$$

$$\mathbf{B}_k = \begin{bmatrix} B_0 & 0 & 0 & \dots & 0 \\ 0 & B_0 & 0 & \dots & 0 \\ 0 & 0 & B_0 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & B_0 \end{bmatrix},$$

$$\mathbf{C}_k = \begin{bmatrix} C_0 & 0 & 0 & \dots & 0 \\ 0 & C_0 & 0 & \dots & 0 \\ 0 & 0 & C_0 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & C_0 \end{bmatrix},$$

$$\mathbf{D}_k = \begin{bmatrix} D_0 & 0 & 0 & \dots & 0 \\ 0 & D_0 & 0 & \dots & 0 \\ 0 & 0 & D_0 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & D_0 \end{bmatrix}.$$

As  $\mathbf{E}_k$  is invertible, the system (2) is regular for all integers  $k$ . This system allows to give another expression for the fundamental solution [9]:

$$\Phi(t) = [0 \quad \dots \quad 0 \quad I] e^{\mathbf{E}_k^{-1} \mathbf{A}_k (t-k)} Z_k, \quad t \in [k, k+1[,$$

where the matrices  $Z_k$  are given by the relations

$$Z_0 = I, \quad Z_k = \begin{bmatrix} I \\ e^{\mathbf{E}_{k-1}^{-1} \mathbf{A}_{k-1}} Z_{k-1} \end{bmatrix}.$$

The transfer function matrix of the system (1) between the  $u$  and  $y$  is given by the expression

$$T(s, e^{-s}) = C_0 (sI - A_0 - A_1 e^{-s} - A_{-1} s e^{-s})^{-1} B_0.$$

The transfer function matrix between the disturbance and the output is

$$T^{D_0}(s, e^{-s}) = C_0 (sI - A_0 - A_1 e^{-s} - A_{-1} s e^{-s})^{-1} D_0$$

And  $T(s, e^{-s})$  may be decomposed as follows (the same holds for  $T^{D_0}(s, e^{-s})$ ):

$$T(s, e^{-s}) = \sum_{j=0}^{\infty} C_0 R_0(s) R_1^j(s) B_0 e^{-js},$$

where

$$R_0(s) = (sI - A_0)^{-1} \quad R_1(s) = (A_1 + A_{-1}s)(sI - A_0)^{-1}.$$

Let  $T_j(s) = C_0 R_0(s) R_1(s)^j B_0$  and  $\Theta_k(s)$  be the transfer function matrix between the input and the output for the system (2). As in the case of a linear system with delay [7], one has the following equality:

$$\Theta_k(s) = \begin{bmatrix} T_0(s) & 0 & \dots & 0 \\ T_1(s) & T_0(s) & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ T_k(s) & T_{k-1}(s) & \dots & T_0(s) \end{bmatrix}. \quad (3)$$

In a similar way the the transfer function matrix between the disturbance and the output may be written as

$$\Theta_k^{D_0}(s) = \begin{bmatrix} T_0^{D_0}(s) & 0 & \dots & 0 \\ T_1^{D_0}(s) & T_0^{D_0}(s) & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ T_k^{D_0}(s) & T_{k-1}^{D_0}(s) & \dots & T_0^{D_0}(s) \end{bmatrix}. \quad (4)$$

where  $T_j^{D_0}(s) = C_0 R_0(s) R_1(s)^j D_0$ . As for the fundamental solution one can write the transfer function through the matrices  $Q_i(j)$ , more precisely, the rational matrices  $T_j(s)$  and  $T_j^{D_0}(s)$  verify

$$T_j(s) = \sum_{i=0}^{\infty} C_0 Q_i(j) B_0 s^{-(i+1)},$$

and

$$T_j^{D_0}(s) = \sum_{i=0}^{\infty} C_0 Q_i(j) D_0 s^{-(i+1)}.$$

This gives a relation between the approach of Gabasov and Kirillova based on matrices  $Q_i(j)$  [2, 3, 15] and that of Olbrot, Zmood and other authors [8, 14, 17] developed for single delay system. This relation for delay systems of neutral type was pointed out in [9].

### 3 Structure at infinity

The transfer function matrix of a delay system is not rational. Moreover, it is not analytical at infinity. The notions of properness must be precised.

**Definition 3.1** A complex valued function  $f(s)$  is called weak proper if  $\lim_{s \rightarrow \infty} f(s)$  is finite when  $s \in \mathbb{R}$  tends to  $\infty$ . It is called strictly weak proper if this limit is 0. A matrix  $B(s)$  is weak biproper if it is weak proper and its inverse is also weak proper. Weak proper is replaced by strong proper if the same occurs when  $\Re(s) \rightarrow \infty$ .

Let us put  $A(e^{-s}) = A_0 + A_1 e^{-s} + A_{-1} s e^{-s}$ . We have the following result for delay system of neutral type (see [9] and [12] for the case of a simple system with delay)

**Theorem 3.2** There exist weak biproper matrices  $B_1(s, e^{-s})$  and  $B_2(s, e^{-s})$  such that

$$B_1(s, e^{-s}) T(s, e^{-s}) B_2(s, e^{-s}) =$$

$$\begin{bmatrix} \Delta_0(s) & 0 & \dots & 0 & 0 \\ 0 & \Delta_1(s) e^{-s} & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & \Delta_k(s) e^{-ks} & 0 \\ 0 & 0 & \dots & 0 & 0 \end{bmatrix},$$

where

$$\Delta_i(s) = \text{diag} [s^{-n_{i,1}}, \dots, s^{-n_{i,j_i}}]$$

and  $n_{i,j} \leq n_{i,j+1}$ ,  $i = 1, \dots, k$ . The list of integers

$$\{n_{i,j}, i = 1, \dots, k; j = j_1, \dots, j_i\}$$

is called the weak structure at infinity of the system  $T(s, e^{-s})$  and is noted by  $\Sigma_\infty^w T(s, e^{-s})$  or  $\Sigma_\infty^w(C_0, A(e^{-s}), B_0)$ .

This structure at infinity allows to characterize some control problems as disturbance rejection, row-by-row decoupling, model matching, etc.

## 4 Main results

### 4.1 The row-by-row decoupling problem

The main result for the row-by-row decoupling problem is given by the following theorem where the system is considered without disturbance and with  $p = m$ .

**Theorem 4.1** *The following properties are equivalent*

1. *The row-by-row decoupling problem for the delay neutral-type system (1) is solvable by a weak biproper precompensator:*

$$T(s, e^{-s})K(s, e^{-s}) = \text{diag} \{h_1(s, e^{-s}), \dots, h_m(s, e^{-s})\}.$$

2. *The weak structure at infinity verifies:*

$$\Sigma_\infty^w(C_0, A(e^{-s}), B_0) = \begin{bmatrix} \Sigma_\infty^w(c_1, A(e^{-s}), B_0) \\ \vdots \\ \Sigma_\infty^w(c_p, A(e^{-s}), B_0) \end{bmatrix},$$

where  $c_i$ 's are the rows of the matrix  $C_0$ .

3. *The matrix*

$$W_0 = \begin{bmatrix} c_1 Q_{n_1-1}(k_1) B_0 \\ \vdots \\ c_m Q_{n_m-1}(k_m) B_0 \end{bmatrix},$$

is invertible, where for each row  $i$  the integers  $n_i$  and  $k_i$  are such that:  $c_i Q_{n_i-1}(k_i) B_0 \neq 0$  and  $c_i Q_l(j) B_0 = 0$  for  $l < n_i - 1$  and  $j < k_i$ .

4. *The decoupling problem is solvable by generalized static state feedback*

$$u = F(e^{-s})x + G(s, e^{-s})v,$$

where

$$\begin{aligned} F(e^{-s}) &= F_0 + F_1 e^{-s} + \dots, \\ G(s, e^{-s}) &= G_0 + G_1(s) e^{-s} + \dots, \end{aligned}$$

with (possible) polynomial matrices  $G_i(s)$ ,  $i \geq 1$ ,  $G_0 = W_0^{-1}$  and constant matrices  $F_i$ ,  $i \in \mathbb{N}$ . The relation between the precompensator  $K(s, e^{-s})$  and the feedback law

$$u = F(e^{-s})x + G(s, e^{-s})v$$

is given by

$$K(s, e^{-s}) = (I - F(e^{-s})(sI - A(e^{-s}))^{-1} B_0)^{-1} G(s, e^{-s}).$$

5. *For  $\mathfrak{C}_i = \bigcap_{j \neq i}^m \text{Ker } c_j$ , let*

$$\mathcal{V}_\Sigma(\mathfrak{C}_i, A(e^{-s}), B_0), \quad i = 1, \dots, m$$

be the subspaces

$$\{x \in \mathfrak{C}_i : x = (sI - A(e^{-s}))\xi(s, e^{-s}) - B_0\omega(s, e^{-s})\},$$

with strictly weak proper  $\xi$  and  $\omega$  such that

$$\xi(s, e^{-s}) \in \mathfrak{C}_i \quad \text{for } s > s_0.$$

Then

$$\text{Im } B_0 = \sum_{i=1}^m \text{Im } B_0 \cap \mathcal{V}_\Sigma(\mathfrak{C}_i, A(e^{-s}), B_0), \quad (5)$$

with  $\text{Im } B_0 \cap \mathcal{V}_\Sigma(\mathfrak{C}_i, A(e^{-s}), B_0) \neq 0$ .

**Proof.** The implication  $1 \Rightarrow 2 \Rightarrow 3$  and  $4 \Rightarrow 5 \Rightarrow 1$  may be shown as for the time-delay system (not of neutral type, that is  $A_{-1} = 0$  as in [11]). The formal calculations are similar to the case of system without delay (see also [9]). The only crucial implication is  $3 \Rightarrow 4$ .

Assume that  $W_0$  is invertible. Then

$$T(s, e^{-s}) = \text{diag} \{s^{-n_1} e^{-k_1 s}, \dots, s^{-n_m} e^{-k_m s}\} (W_0 + W(s, e^{-s})),$$

and  $W_0 + W(s, e^{-s})$  is weak biproper because

$$\lim_{\mathbb{R} \ni s \rightarrow \infty} (W_0 + W(s, e^{-s})) = W_0.$$

Then  $K(s, e^{-s}) \stackrel{\text{def}}{=} (W_0 + W(s, e^{-s}))^{-1}$  is also biproper and

$$T(s, e^{-s})K(s, e^{-s}) = \text{diag} \{s^{-n_1} e^{-k_1 s}, \dots, s^{-n_m} e^{-k_m s}\},$$

Note that  $W(s, e^{-s})$  is strictly weak proper and may be decomposed as:

$$W(s, e^{-s}) = W_1(s, e^{-s}) + W_2(s, e^{-s}),$$

with

$$W_1(s, e^{-s}) = \sum_{j=0}^{\infty} \sum_{i=0}^{\infty} C_0 Q_{i+\nu-1}(\kappa + j) B_0 s^{-i} e^{-j s} - W_0,$$

The integers  $\nu$  and  $\kappa$  denote for each row  $i$  the integers  $n_i$  and  $k_i$  respectively, that is

$$W_0 = C_0 Q_{\nu-1}(\kappa) B_0.$$

The matrix  $W_1(s, e^{-s})$  is strictly strong proper. This implies that the precompensator

$$K_1(s, e^{-s}) = (W_0 + W_1(s, e^{-s}))^{-1}$$

is strong biproper. As for a system without neutral term [10, 11, 13], this precompensator may be realized with static state feedback:

$$\begin{aligned} F^1(e^{-s}) &= F_0 + F_1 e^{-s} + F_2 e^{-2s} + \dots, \\ G^1(e^{-s}) &= G_0^1 + G_1^1 e^{-s} + G_2^1 e^{-2s} + \dots, \end{aligned}$$

where, for example,  $G_0^1 = W_0^{-1}$  and

$$F_0 = -G_0^1 \begin{bmatrix} c_1 Q_{n_1}(k_1) \\ \vdots \\ c_m Q_{n_m}(k_m) \end{bmatrix}.$$

the other matrices are computed as in [10] and [13].

This gives

$$\begin{aligned} K_1(s, e^{-s}) &= \\ &= (I - F^1(e^{-s})(sI - A(e^{-s}))^{-1} B_0)^{-1} G^1(e^{-s}). \end{aligned}$$

Note that in [10, 13] additional conditions are required in order to insure strong biproperness. Here those conditions are verified by construction.

Let now define  $K_2(s, e^{-s}) \stackrel{\text{def}}{=} K(s, e^{-s}) - K_1(s, e^{-s})$ , which gives:

$$K(s, e^{-s}) = K_1(s, e^{-s}) + K_2(s, e^{-s}),$$

Taking  $F(e^{-s}) = F^1(e^{-s})$  and

$$\begin{aligned} G^2(s, e^{-s}) &= \\ &= [I - F(e^{-s})(sI - A(e^{-s}))^{-1} B_0] K_2(s, e^{-s}), \end{aligned}$$

one obtains

$$\begin{aligned} K_2(s, e^{-s}) &= \\ &= [I - F(e^{-s})(sI - A(e^{-s}))^{-1} B_0]^{-1} G^2(s, e^{-s}) \end{aligned}$$

And then, from the expressions of  $K_1(s, e^{-s})$  and  $K_2(s, e^{-s})$ , we get

$$\begin{aligned} K(s, e^{-s}) &= \\ &= [I - F(e^{-s})(sI - A(e^{-s}))^{-1} B_0]^{-1} G(s, e^{-s}), \end{aligned}$$

with

$$G(s, e^{-s}) \stackrel{\text{def}}{=} G^1(e^{-s}) + G^2(s, e^{-s}).$$

If  $T_{F,G}(s, e^{-s})$  is the closed loop transfer matrix, we have:

$$T_{F,G}(s, e^{-s}) = T(s, e^{-s})K(s, e^{-s})$$

and

$$T_{F,G}(s, e^{-s}) = \text{diag} \{s^{-n_1} e^{-k_1 s}, \dots, s^{-n_m} e^{-k_m s}\}.$$

Hence 4 is satisfied.  $\blacksquare$

**Remark 4.2** The weak proper part of the precompensator  $K_2(s, e^{-s})$  is realized by the matrix  $G^2(s, e^{-s})$  which acts on the new control  $v$ . This means that this new control must be sufficiently smooth. If it is not the case, the decoupling problem is not solvable by this kind of feedback if  $K_2(s, e^{-s}) \neq 0$ .

## 4.2 The disturbance rejection problem

In order to show a similar result for the disturbance rejection problem, we need a preliminary result concerning the systems (2) which permits a representation of the system (1). Let  $V_*(\text{Ker } \mathbf{C}_k, \mathbf{A}_k, \mathbf{B}_k)$  be the maximal  $(\mathbf{A}_k, \mathbf{B}_k)$ -invariant subspace (see [16]) contained in  $\text{Ker } \mathbf{C}_k$ . This subspace is feedback invariant, that is, there exists a matrix  $\mathbf{F}_k$  such that  $V_*(\text{Ker } \mathbf{C}_k, \mathbf{A}_k, \mathbf{B}_k)$  is  $(\mathbf{A}_k + \mathbf{B}_k \mathbf{F}_k)$ -invariant. It is well known that the disturbance rejection problem for the system (2) with  $E_k = I_k$  (when  $A_{-1} = 0$ ) is solvable by feedback

$$v_k = \mathbf{F}_k z_k + \mathbf{G}_k w_k$$

iff

$$\text{Im } \mathbf{D}_k \subset V_*(\text{Ker } \mathbf{C}_k, \mathbf{A}_k, \mathbf{B}_k) + \text{Im } \mathbf{B}_k.$$

The matrix  $\mathbf{F}_k$  may be taken such that  $V_*(\text{Ker } \mathbf{C}_k, \mathbf{A}_k, \mathbf{B}_k)$  is  $(\mathbf{A}_k + \mathbf{B}_k \mathbf{F}_k)$ -invariant. In [7] it has been shown, in the case of systems without neutral term, that the matrices  $\mathbf{F}_k$  and  $\mathbf{G}_k$  are lower block triangular matrices. Let us show that in the case of system (2), the situation is the same.

The system (2) is equivalent to the system

$$\begin{cases} \dot{z}_k &= \mathbf{E}_k^{-1} \mathbf{A}_k z_k + \mathbf{E}_k^{-1} \mathbf{B}_k v_k + \mathbf{E}_k^{-1} \mathbf{D}_k w_k \\ w_k &= \mathbf{C}_k z_k \end{cases} \quad (6)$$

For this system the classical result holds and the corresponding feedback matrices  $\mathbf{F}_k$  and  $\mathbf{G}_k$  are lower triangular matrices as shown in [7]. A simple calculation shows that the same matrices give a feedback which solves the disturbance rejection problem for the system (2):

$$\begin{aligned} &\mathbf{C}_k (s\mathbf{I}_k - \mathbf{E}_k^{-1} \mathbf{A}_k - \mathbf{E}_k^{-1} \mathbf{B}_k \mathbf{F}_k)^{-1} \times \\ &\quad (\mathbf{E}_k^{-1} \mathbf{B}_k \mathbf{G}_k + \mathbf{E}_k^{-1} \mathbf{D}_k) = \\ &\quad \mathbf{C}_k (s\mathbf{E}_k - \mathbf{A}_k - \mathbf{B}_k \mathbf{F}_k)^{-1} (\mathbf{B}_k \mathbf{G}_k + \mathbf{D}_k) \end{aligned}$$

For the disturbance rejection problem we have the following result.

**Theorem 4.3** *The following propositions are equivalent:*

1. *The disturbance rejection problem for the delay system of neutral type (1) is solvable by a weak proper precompensator :*

$$T(s, e^{-s})K(s, e^{-s}) + T^{D_0}(s, e^{-s}) \equiv 0$$

2. *The weak structure at infinity verifies:*

$$\Sigma_\infty^w [T(s, e^{-s}) \quad T^{D_0}(s, e^{-s})] = \Sigma_\infty^w [T(s, e^{-s}) \quad 0]$$

3. *The disturbance rejection problem is solvable by generalized static state feedback*

$$u = F(e^{-s})x + G(s, e^{-s})d,$$

with

$$\begin{aligned} F(e^{-s}) &= F_0 + F_1 e^{-s} + F_2 e^{-2s} + \dots, \\ G(s, e^{-s}) &= G_0 + G_1(s) e^{-s} + G_2(s) e^{-2s} + \dots, \end{aligned}$$

with (possible) polynomial matrices  $G_i(s)$ ,  $i \geq 1$  and constant matrices  $F_i$ ,  $i \in \mathbb{N}$ .

4.  $\mathcal{D}_0 \subset \mathcal{V}_\Sigma(\text{Ker } C_0, A, B_0) + \mathcal{B}_0$ , where  $\mathcal{D}_0$  and  $\mathcal{B}_0$  are the images of  $D_0$  and  $B_0$  respectively, the subspace  $\mathcal{V}_\Sigma(\text{Ker } C_0, A, B_0)$  being given by

$$\{x \in \text{Ker } C_0 : \\ x = (sI - A(e^{-s}))\xi(s, e^{-s}) - B_0\omega(s, e^{-s})\},$$

with strictly weak proper  $\xi$  and  $\omega$  such that

$$\xi(s, e^{-s}) \in \text{Ker } C_0 \quad \text{for } s > s_0.$$

**Proof.** The equivalences 1)  $\Leftrightarrow$  2)  $\Leftrightarrow$  4) are obtained as for systems without neutral term (see [7, 12]). If 3) is satisfied, then 1) holds with

$$K = (I - F(sI - A)^{-1}B_0)^{-1} (F(sI - A)^{-1}D_0 + G),$$

the argument  $(s, e^{-s})$  being omitted for the sake of simplicity and, as noted before,

$$A(s, e^{-s}) = A_0 + A_1e^{-s} + A_{-1}se^{-s}$$

Suppose now that 1) is satisfied, the precompensator  $K(s, e^{-s})$  is decomposed into a strong proper part  $K_1(s, e^{-s})$  and a weak proper part  $K_2(s, e^{-s})$ . This gives:

$$T(s, e^{-s}) (K_1(s, e^{-s}) + K_2(s, e^{-s})) = -T^{D_0}(s, e^{-s}) \quad (7)$$

Let us now put

$$D_1(s, e^{-s}) \stackrel{\text{def}}{=} B_0K_2(s, e^{-s}) + D_0$$

Then (7) may be written as:

$$T(s, e^{-s})K_1(s, e^{-s}) + \\ C_0 (sI - A(s, e^{-s}))^{-1} D_1(s, e^{-s}) = 0$$

This means that the strong proper precompensator  $K_1(s, e^{-s})$  solves the disturbance rejection problem for a new disturbance  $D_1(s, e^{-s})$  (see Remark 4.4 for the state space representation of this disturbance).

In order to design the feedback which solves our problem, we first design the feedback which realizes the precompensator  $K_1(s, e^{-s})$  solving the new disturbance rejection problem (Step 1), then we deduce the feedback solving the original problem (Step 2).

STEP 1: The precompensator  $K_1(s, e^{-s})$  may be decomposed as:

$$K_1(s, e^{-s}) = K_1^0(s) + K_1^1(s)e^{-s} + \dots,$$

where  $K_1^i(s)$ ,  $i \in \mathbb{N}$  are rational and strictly (strong) proper. As  $K_1(s, e^{-s})$  solves the new disturbance problem, using the partial representation by the systems (2) and the corresponding transfer function matrices, we get, for all  $k \in \mathbb{N}$ :

$$\Theta_k(s)\Gamma_k(s) + \Theta_k^{D_1}(s) = 0,$$

with proper precompensator  $\Gamma_k(s)$  given by :

$$\Gamma_k(s) = \begin{bmatrix} K_1^0(s) & 0 & \cdots & 0 \\ K_1^1(s) & K_1^0(s) & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ K_1^k(s) & K_1^{k-1}(s) & \cdots & K_1^0(s) \end{bmatrix}.$$

The matrix  $\Theta_k(s)$  being given by (3) and  $\Theta_k^{D_1}(s)$  in a similar way from the decomposition of  $D_1(s, e^{-s})$  (see Remark 4.4). This means that, for each  $k$ , the disturbance decoupling problem is solvable for the finite dimensional systems (2) with the new disturbance. Using the geometric approach, one can design the corresponding matrices  $\mathbf{F}_k$  and  $\mathbf{G}_k$  which are of lower triangular forms (as pointed out at the beginning of this section):

$$\mathbf{F}_k = \begin{bmatrix} F_0 & 0 & \cdots & 0 \\ F_1 & F_0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ F_k & F_{k-1} & \cdots & F_0 \end{bmatrix} \quad (8)$$

and

$$\mathbf{G}_k = \begin{bmatrix} G_0 & 0 & \cdots & 0 \\ G_1 & G_0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ G_k & G_{k-1} & \cdots & G_0 \end{bmatrix} \quad (9)$$

Taking

$$F(e^{-s}) = F_0 + F_1e^{-s} + F_2e^{-2s} + \dots, \\ G^1(s, e^{-s}) = G_0 + G_1(s)e^{-s} + G_2(s)e^{-2s} + \dots,$$

using the relation between the system (2) and the delay system, we obtain

$$T(s, e^{-s})K_1(s, e^{-s}) = \\ C_0 (sI - A(s, e^{-s}) - B_0F(e^{-s}))^{-1} \times \\ (B_0G^1(s, e^{-s}) + D_1(s, e^{-s})).$$

That is,  $K_1(s, e^{-s})$  is realized by static feedback.

STEP 2: A formal calculation gives:

$$K_1 = (I - F(sI - A)^{-1}B_0)^{-1} (F(sI - A)^{-1}D_1 + G^1),$$

where the arguments are omitted for simplicity. This can be rewritten as:

$$K_1 = F(sI - A)^{-1}D_1 + G^1 + F(sI - A)^{-1}B_0K_1$$

Replacing  $D_1(s, e^{-s})$  by its expression  $B_0K_2(s, e^{-s}) + D_0$ , we obtain:

$$K_1 = \\ F(sI - A)^{-1} (B_0K_2 + D_0) + G^1 + F(sI - A)^{-1}B_0K_1$$

and then

$$K_1 = F(sI - A)^{-1}D_0 + F(sI - A)^{-1}B_0K + G^1.$$

Adding  $K_2(s, e^{-s})$  to both parts leads to

$$K = F(sI - A)^{-1}D_0 + F(sI - A)^{-1}B_0K + G^1 + K_2$$

Let us now put

$$G(s, e^{-s}) \stackrel{\text{def}}{=} G^1(e^{-s}) + K^2(s, e^{-s}),$$

we obtain:

$$K = F(sI - A)^{-1}D_0 + F(sI - A)^{-1}B_0K + G.$$

And this gives

$$K = (I - F(sI - A)^{-1}B_0)^{-1} (F(sI - A)^{-1}D_0 + G).$$

This means that  $K(s, e^{-s})$  is realizable by the feedback

$$u = F(e^{-s})x + G(s, e^{-s})d$$

which may contain the delayed derivative of the original disturbance. ■

**Remark 4.4** Let us precise in some example how the new disturbance is constructed using the initial one.

Suppose that the initial disturbance is one dimensional acting on the system as  $D_0d(t)$ . Assume now that for this system we obtain (see the proof of the Theorem 4.3) a weak proper part of the precompensator as:

$$K^2(s, e^{-s}) = s^2e^{-s} + se^{-2s}.$$

Then the new disturbance is

$$D_1(s, e^{-s}) = D_0 + B_0(s^2e^{-s} + se^{-2s})$$

and in the time domain, the disturbing term is given by:

$$D_0d(t) + B_0\dot{d}(t-1) + B_0\dot{d}(t-2).$$

Another possibility is to consider a new disturbance vector:

$$q = \begin{bmatrix} d \\ \dot{d} \\ \ddot{d} \end{bmatrix},$$

and then, in the time domain, we get

$$D_1q(t) = \bar{D}_0q(t) + \bar{D}_1q(t-1) + \bar{D}_2q(t-2),$$

with

$$\bar{D}_0 = [D \ 0 \ 0], \quad \bar{D}_1 = [0 \ 0 \ B_0]$$

and

$$\bar{D}_2 = [0 \ B_0 \ 0].$$

This expression allows to consider the system (2) with a new disturbance and to make use of this representation in order to solve the disturbance decoupling problem for the system (2) with a new disturbance:

$$D_k^1 = \begin{bmatrix} \bar{D}_0 & 0 & 0 & \dots & 0 & 0 \\ \bar{D}_1 & \bar{D}_0 & 0 & \dots & 0 & 0 \\ \bar{D}_2 & \bar{D}_1 & \bar{D}_0 & \dots & 0 & 0 \\ 0 & \bar{D}_2 & \bar{D}_1 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \\ 0 & 0 & 0 & \dots & \bar{D}_1 & \bar{D}_0 \end{bmatrix}.$$

Note that in the classical finite dimensional case, one can consider the disturbance decoupling problem when the disturbance is not measurable. In this case, the solution is a strictly proper compensator and the feedback is of the form  $u = Fx$ . Here, we can also consider the case when the precompensator is strictly weak proper. The structural condition may be reformulated in this context. However, the weak proper part (even if it is *strictly* proper in the weak sense) needs, for its realization, the disturbance. Hence, this problem, except for some classes of systems, cannot be solved. As for the row-by-row decoupling problem, if the disturbance is not smooth enough, the problem cannot be solved by this kind of generalized feedback.

Let us conclude this main section by the following general remark about the existence of stable solution for both considered problems.

**Remark 4.5** The given results allow to discuss the case of solving the mentioned problems with stability. According to the special form of the generalized state feedback, the neutral type of the closed loop systems is not affected by the feedback. In particular, the formal stability (or instability) of the original system is not changed by the feedback. This means that if the original system is not formally stable, the generalized state feedback solving the disturbance rejection or the row-by-row decoupling problems cannot stabilize the system. This result may be compared with the result by Loiseau et al. in this Conference [6].

## 5 Conclusion

The weak structure at infinity of time-delay system of neutral type is used to solve the disturbance rejection and the row-by-row decoupling problems. Delayed derivative of the disturbance or of the new control must be used in a general case. This is the counterpart of the generality. For practical use this means that if the disturbance or the new control are not smooth enough, we need in fact very high gain in approximation.

## References

- [1] A. Bensoussan, G. Da Prato, M. C. Delfour, S. J. Mitter. *Representation and control of infinite dimensional systems*. Birkhäuser, Boston, 1992.
- [2] R. G. Gabasov, F. M. Kirillova. *Qualitative theory of optimal processes*, (in Russian). Nauka, Moscow, 1971.
- [3] F. Kirillova, S. Churakova. *On controllability problem for linear systems with after-effect*. Diff. Uravneniya, **3**, No 3, 1967.
- [4] H. Górecki, S. Fuksa, P. Grabowski, A. Korytowski. *Analysis and synthesis of time delay systems*, J. Wiley Ed., 1989.

- [5] M. L. J. Hautus. *(A, B)-invariant and stabilizability subspace, a frequency domain description*. Automatica, **16**, pp. 703-707, 1980.
- [6] J.-J. Loiseau, M. Cardelli, X. Dusser. *A note on BIBO stabilisation of neutral-type time-delay systems*. MTNS2000, Perpignan, June 2000.
- [7] M. Malabre, R. Rabah. *Structure at infinity, model matching and disturbance rejection for linear systems with delays*. Kybernetika, **29**, No 5, pp. 485–498, 1993.
- [8] A. W. Olbrot. *Algebraic criteria of controllability to zero function for linear constant time-lag systems*. Control and Cybernetics, **2**, No 1/2, 1973, pp. 60-76.
- [9] R. Rabah. *Structural Properties and Controllability for Delay Systems of Neutral Type*. Proc. IFAC Conf. on System Structure and Control, Nantes, July 1995, pp. 354–359.
- [10] Rabah R., Malabre M. *Structure at infinity for delay systems revisited*. Proc. IMACS and IEEE-SMC Multi-conference CESA'96, Symposium on Modelling, Analysis and Simulation, Lille, France, July 9-12, pp. 87–90.
- [11] R. Rabah, M. Malabre. *The structure at infinity of linear delay systems and the row-by-row decoupling problem*. Proceedings of the 7th IEEE Mediterranean Conference on Control and Automation, Haifa, Israel, June 28-30, 1999, pp. 1845–1854.
- [12] R. Rabah, M. Malabre (1998). *On the structure at infinity of linear delay systems with application to decoupling problem*. Theory and practice of control and systems: Proc. 6th IEEE Mediterranean Conference, Alghero, Sardinia, Italy, June 1998, pp. 198–203. Extended version: Kybernetika (Praha), **35**, No 6, pp. 668–680, 1999.
- [13] O. Sename, R. Rabah, J.-F. Lafay, *Decoupling without prediction of linear systems with delays: a structural approach*. Systems and Control Letters, **25**, 1995, pp. 387–395.
- [14] A. C. Tsoi, *Recent advances in the algebraic system theory of delay differential equations*, in the book: Recent theoretical developments in control, M.J. Gregson Ed. Academic Press, 1978, pp. 67-127.
- [15] Shklyar B. S. *The relative controllability of systems of neutral type with delayed arguments*. Differentsialnye Uravnenia, **10**, pp. 1443–1450, 1974.
- [16] W. M. Wonham, *Linear multivariable control: a geometric approach*. Springer-Verlag, New York, 1979.
- [17] R. B. Zmood, *The Euclidean space controllability of control systems with delays*. SIAM J. on Control, **12**, No 4, 1974, pp. 609-623.