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On van der Corput's k-th derivative test for exponential sums

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Abstract

We give an overview of van der Corput's method for exponential sums, with a particular interest for the simplest estimates with the k-derivative test. We study the optimality of the results and we present recent improvements.
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1. Introduction

Let $a \in \mathbb{Z}$ and $M \in \mathbb{N}$. Let $f : [a+1, a+M] \to \mathbb{R}$ be a $\mathscr{C}^k$ function with $k \geq 1$. We are interested in bounding the modulus of the following exponential sum

$$\sum_{a<m\leq a+M} \mathrm{e}(f(m))$$

with the classical notation $\mathrm{e}(t) := \mathrm{e}^{2i\pi t}$, $(t \in \mathbb{R})$.

The current form of this sum occurs in many applications, and the particular case of dyadic summation ($a = M$) plays a prominent role in the applications. However, we may restrict to the

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case $a = 0$ without loss of generality by replacing f by $x \mapsto f(a + x)$. Thus, in the sequel, we shall reduce the study to the exponential sums

$$\sum_{m=1}^{M} \mathrm{e}(f(m)) \tag{1}$$

keeping in mind that the phase f may also depend on M and various other parameters.

The literature on the subject of such trigonometric sums is indeed abundant, and in particular goes back to 1916 with Weyl's results on the equidistribution of a real sequence modulo 1. Subsequently, Hardy and Littlewood used Weyl's work for Waring's problem.

The history on further works gave rise to essentially three methods, which we briefly introduce.

Van der Corput's method (1920) relies on two analytical transforms, and applies on various classical problems, including the Lindelöf problem and the Dirichlet divisor's problem. In particular, van der Corput introduced an ingenious hypothesis to ensure that the exponential sum has a nontrivial bound. Since van der Corput's estimates are the purpose of the present paper, we shall add no more at this point, and elaborate on the various aspects of the method in the sequel.

Later, Vinogradov (1930) developed a method using diophantine systems involving Newton sums. His work was motivated by estimates for the Riemann zeta function, and yields a zero-free region for ζ. Since, Vinogradov's diophantine system has been investigated on its own or for other applications, including Waring's problem, and has been the subject of major recent improvements, especially in Wooley's recent work (see [32,33]).

The next results that we describe here are due to Bombieri & Iwaniec (1985). Their ingenious method (see [2,1]) was a breakthrough in the Lindelöf problem, and was later transposed to classical problems (such as the Dirichlet divisor problem, see [10]). Their idea combines the philosophy of both van der Corput's and Vinogradov's methods: they introduce new analytical transforms of the initial exponential sum, and by means of a sieve inequality, the problem reduces to two spacing problems, involving a new diophantine system which is a variation of Vinogradov's system. It should be noticed that the latest results on the classical problems mentioned above (see [9]) still use Bombieri and Iwaniec's approach: indeed, the method involves various parameters to be optimised, and in the original proof, some parameters had to be constrained so that the two spacing problems could be handled. The remarkable task in the results that followed (by Huxley and Watt in particular) involves a better optimisation of the parameters (introducing higher moments) and a complication in the treatment of the spacing problems (see [9],...). It should also be noticed that the second spacing problem, namely the diophantine system, has been the subject of a very recent development by Bourgain, using a decoupling inequality for curves (see [3]).

The aim of this paper is to give a presentation of van der Corput's method and its main lemmas. In Section 2, we recall the context and notation through two classical examples. In Section 3, we present van der Corput's simplest estimates, under a minimal hypothesis on the phase of the exponential sum. In Sections 4 and 5, we study the case of a semi-monomial phase and the theory of exponent pairs. In Section 6, we give an overview of the tools that occur in the modern theory, including the techniques developed with Bombieri and Iwaniec's method. In the next sections, we go back to van der Corput's simplest estimates, and we investigate potential improvements with new tools. More precisely, in Sections 7–11 we study the optimality for small order derivatives. In Sections 11 and 12, we present recent and new improvements. Finally, in Section 13, we conclude with some open questions.

In the present paper we shall only consider unidimensional exponential sums. For the multidimensional exponential sums, we refer the reader to [6] for other aspects of the theory, and to [24] for recent results.

2. Context through a few historic examples

In many applications of exponential sums, the idea is to exploit cancellation in the corresponding sum (1), and in particular avoid the case where all the numbers $f(m)$ with $m \in \{1, \dots, M\}$ are congruent modulo 1. Surprisingly (and this is the key of van der Corput's method), this situation leading to the trivial bound cannot occur whenever the phase $x \mapsto f(x)$ is sufficiently regular and that for some $k \geq 2$ the function $|f^{(k)}|$ has a fixed order of magnitude $\lambda_k > 0$ small enough on the interval $[1, M]$.

Before we introduce more formal hypotheses, we shall present classical examples of such phases f. For instance, let $\alpha \in \mathbb{R} \setminus \mathbb{N}$, $\alpha \neq 0$ be fixed. Then for $T > 0$ and $M, N \in \mathbb{N}$ such that $M \leq N$, the monomial phase defined by

$$f(x) := T\Big(\frac{N+x}{N}\Big)^{\alpha} \quad (x \in [1, M])$$

has the following behaviour: for each natural number $k \geq 1$ there exist real numbers $c_1 = c_1(\alpha, k)$ and $c_2 = c_2(\alpha, k)$ such that $0 < c_1 < c_2$ and

$$c_1 \frac{T}{N^k} \leq |f^{(k)}(x)| \leq c_2 \frac{T}{N^k} \quad (x \in [1, M]). \tag{2}$$

In the case of a phase $x \mapsto T \log(N + x)$, one has similar estimates. For example, for $T \asymp N$, the second derivative $f''(x)$ has order of magnitude $1/N$ on $[1, M]$. Similarly, if T has order of magnitude a fixed power of N, one may find a small derivative $f^{(k)}$, at least for N large enough.

We shall now introduce such phases through two classical examples.

2.1. *Order of the Riemann ζ function in the critical strip*

For $\sigma \in [\frac{1}{2}, 1]$ fixed, one is interested in finding the infimum of the exponents $A(\sigma)$ such that

$$\zeta(\sigma + it) \ll t^{A(\sigma)} \quad (|t| \geq 3).$$

The case $\sigma = 1/2$ is of particular interest and is called the Lindelöf Problem. The Lindelöf Hypothesis asserts that for any $\varepsilon > 0$ one should have

$$\zeta\left(\tfrac{1}{2} + it\right) \ll_{\varepsilon} t^{\varepsilon} \quad (|t| \geq 3).$$

Following Lemma 2.11 of [6] along with dyadic and Abel summation, one has

$$\left|\zeta\left(\tfrac{1}{2} + it\right)\right| \ll (\log t) \max_{N \leq |t|} N^{-1/2} \max_{1 \leq M \leq N} \left|\sum_{m=1}^{M} \mathrm{e}\Big(\frac{t}{2\pi} \log(N+m)\Big)\right| \quad (|t| \geq 3). \tag{3}$$

The crucial size of parameters for $|t|$ large is $M \asymp N \asymp |t|^{1/2}$ for which the phase $f : x \mapsto \frac{t}{2\pi} \log(N + x)$ satisfies

$$|f'''(x)| \asymp \frac{|t|}{N^3} \asymp |t|^{-1/2} \quad (x \in [1, M]).$$

2.2. The Dirichlet divisor problem

Let $\tau(n)$ denote the number of divisors of the integer n. In order to study the asymptotic behaviour of the distribution function, Theorem 4.5 of [6] gives the estimate

$$\sum_{n\le t}\tau(n) = t(\log t + 2\gamma - 1) + \Delta(t)$$

with

$$\Delta(t) = -2\sum_{n\le t^{1/2}}\psi\Big(\frac{t}{n}\Big) + O(1). \tag{4}$$

Again, using dyadic summation, one is reduced to estimating

$$\sum_{m=1}^{M}\psi\Big(\frac{t}{N+m}\Big)$$

where M and N satisfy $1 \le M \le N \le t^{1/2}$, and where $\psi(y)$ is the normalised fractional part

$$\psi(y) := y - \lfloor y\rfloor - \tfrac{1}{2} \quad (y \in \mathbb{R}).$$

In order to introduce exponential sums, one uses the following inequality, based on a truncated Fourier expansion of $\psi(y)$,

$$\Big|\sum_{m=1}^{M}\psi\Big(\frac{t}{N+m}\Big)\Big| \ll \frac{M}{H} + \sum_{h=1}^{H}\min\Big(\frac{1}{h},\frac{H}{h^2}\Big)\Big|\sum_{m=1}^{M}\mathrm{e}\Big(\frac{ht}{N+m}\Big)\Big| \tag{5}$$

for any $H \ge 1$. (See [11]. See also [27] for a refined and optimal version.)

The crucial size of parameters for the Divisor Problem is $M \asymp N \asymp t^{1/2}$ for which H has to be chosen of size $(t/N^3)^{-1/3} \asymp t^{1/6}$ for t large.

2.3. A remark on the hypothesis for the phase

We recall that a real-valued sequence $(u_n)_{N\in\mathbb{N}}$ is said to be equidistributed modulo 1 if for each $0 \le \alpha < \beta \le 1$ the fractional parts $\{u_n\}$ satisfy

$$\lim_{M\to+\infty}\frac{1}{M}\#\big\{m \in \{1,\dots,M\}\colon \{u_m\} \in [\alpha,\beta]\big\} = \beta - \alpha.$$

Weyl's criterion states that this condition is equivalent to

$$\sum_{m=1}^{M} e(hu_m) = o(M) \quad (M \to +\infty)$$

for each $h \in \mathbb{Z} \smallsetminus \{0\}$ fixed.

It should be noticed that the problem of the equidistribution of the fractional part of $(3/2)^m$ for $m \in \mathbb{N}$ is still an open problem, completely out of reach through the classical method. This is consistent with the fact that in the corresponding exponential sums in Weyl's criterion, the phase $x \mapsto (3/2)^x$ has no derivative of small order of magnitude on intervals of the form $[1, M]$ and therefore, no cancellation can be exploited in the exponential sums.

3. Simplest van der Corput estimates

3.1. Van der Corput's second derivative test

We are now ready to state van der Corput's first result, also referred to as *van der Corput's inequality*.

Theorem 1 (*Van Der Corput, 1922*)**.** *Let $\alpha \geq 1$ be a real number. There exists a constant $C(\alpha) > 0$ such that for any integer $M \geq 1$, any real number $\lambda_2 > 0$ and any $\mathscr{C}^2$ function $f : [1, M] \to \mathbb{R}$ such that*

$$\lambda_2 \leq |f''(x)| \leq \alpha\lambda_2 \quad (x \in [1, M]) \tag{6}$$

one has

$$\Big|\sum_{m=1}^{M} \mathrm{e}(f(m))\Big| \leq C(\alpha)\big(M\lambda_2^{1/2} + \lambda_2^{-1/2}\big).$$

Remarks.
- The result is quite uniform in M, λ_2 and f. In particular, λ_2 may depend on M, the optimal choice being $\lambda_2 = 1/M$.
- The result is trivial for $\lambda_2 \geq 1$. However, as soon as $M \geq \lambda_2^{-1} > 4C(\alpha)^2$ the bound is nontrivial.

In order to point out the uniformity of the result, we give an immediate corollary: *Let $0 < \alpha < \beta$ be real numbers. Let $\mathcal{F}_2(\alpha, \beta)$ denote the set of $\mathscr{C}^2$ functions $F : [0, 1] \to \mathbb{R}$ such that $\alpha \leq |F''(x)| \leq \beta$ for $x \in [0, 1]$. There exists a constant $C(\alpha, \beta) > 0$ such that*

$$\sup_{F \in \mathscr{F}_2(\alpha,\beta)} \Big|\sum_{m=1}^{M} \mathrm{e}\Big(TF\Big(\frac{m}{M}\Big)\Big)\Big| \leq C(\alpha, \beta)\Big(M\Big(\frac{T}{M^2}\Big)^{1/2} + \Big(\frac{T}{M^2}\Big)^{-1/2}\Big)$$

for any integer $M \geq 1$ and any real number $T > 0$.

We may now apply Theorem 1 to the sums in the right hand side of (5). We get

$$\Big|\sum_{m=1}^{M} \mathrm{e}\Big(\frac{ht}{N+m}\Big)\Big| \ll M\Big(\frac{ht}{N^3}\Big)^{1/2} + \Big(\frac{ht}{N^3}\Big)^{-1/2}.$$

Summing up in (5) with the choice $H \asymp (t/N^3)^{-1/3}$, we recover Voronoï's result

$$\Delta(t) \ll t^{1/3} \log t \quad (t \geq 3),$$

without using Voronoï formula (see [30]).

3.2. Sketch of the proof of Theorem 1

We may clearly assume that $0 < \lambda_2 < 1$, otherwise the trivial bound gives the result. We may also assume that $f'' > 0$ (otherwise one considers $-f$).

The first ingredient is *Hilfssatz 1* of [29]: this is a truncated version of the Poisson formula, valid since f in $\mathscr{C}^1$ and f' is increasing. We have

$$\sum_{m=1}^{M} \mathrm{e}(f(m)) = \sum_{H_1 \leq \nu \leq H_2} \int_1^M \mathrm{e}(f(x) - \nu x)\mathrm{d}x + O\Big(\frac{1}{\eta} + \log\Big(2 + f'(M) - f'(1)\Big)\Big) \tag{7}$$

where $0 < \eta \leq 1$ and where $H_1 := f'(1) - \eta$ and $H_2 := f'(M) + \eta$ (see Lemma 3.5 of [6] for details).

To derive this result from the classical Poisson formula

$$\sum_{1\leq m\leq M}^{*} \mathrm{e}(f(m)) = \lim_{N\to+\infty} \sum_{|\nu|\leq N} \int_1^M \mathrm{e}(f(x) - \nu x)\mathrm{d}x \tag{8}$$

(where $\sum^*$ means that the terms for $m = 1$ and $m = M$ are halved), van der Corput uses estimates of oscillating integrals: the philosophy is that the summation over ν is dominated by the values ν such that $x \mapsto f(x) - \nu x$ has a critical point in $[1, M]$.

Namely, let $a, b, \lambda_1 \in \mathbb{R}$ such that $a < b$ and $\lambda_1 > 0$. Then for $\mathscr{C}^1$ functions $F : [a, b] \to \mathbb{R}$ and $G : [a, b] \to \mathbb{R}$ such that G/F' is monotonic and that $|F'(x)/G(x)| \geq \lambda_1$ for $x \in [a, b]$, one has

$$\int_a^b G(x)e(F(x))\mathrm{d}x \ll \lambda_1^{-1}, \tag{9}$$

(for details, see Lemma 3.1 of [6]).

The second ingredient is *Hilfssatz 2* of [29]: this is an analogue of (9) for the second derivative. For any $\mathscr{C}^2$ function $F : [a, b] \to \mathbb{R}$ such that $F''(x) \geq \lambda_2 > 0$ for $x \in [a, b]$, one has

$$\int_a^b e(F(x))\mathrm{d}x \ll \lambda_2^{-1/2} \tag{10}$$

(see Lemma 3.2 of [6] for details).

We are now ready to conclude the proof of Theorem 1. Using (10) to estimate each integral in (7) and choosing $\eta = 1$, we have

$$\sum_{m=1}^{M} \mathrm{e}(f(m)) \ll (H_2 - H_1)\lambda_2^{-1/2} + \log\big(H_2 - H_1\big)$$

and we conclude with the inequality $H_2 - H_1 = 2 + f'(M) - f'(1) \ll 1 + M\alpha\lambda_2$. $\square$

Remark. The proof above provides ingredients for an extra result: in the case of a $\mathscr{C}^1$ phase $f : [1, M] \to \mathbb{R}$ such that f' is monotonous and such that $0 < \lambda_1 \leq |f'(x)| \leq \alpha\lambda_1 < 1/2$, the formula (7) with $\eta = 1/4$ implies

$$\sum_{m=1}^{M} \mathrm{e}(f(m)) = \int_1^M \mathrm{e}(f(x))\mathrm{d}x + O(1)$$

and one deduces from (9) the following *first derivative test*

$$\sum_{m=1}^{M} \mathrm{e}(f(m)) \ll \lambda_1^{-1}. \tag{11}$$

One may also prove directly the result via the Kusmin–Landau theorem (see Theorem 2.1 of [6]).

3.3. Van der Corput's A-process

One may be tempted to apply Theorem 1 to the Lindelöf problem, namely to bound the exponential sum

$$\Big|\sum_{m=1}^{M} \mathrm{e}\Big(\frac{t}{2\pi}\log(N+m)\Big)\Big| \quad (M \le N \le t).$$

Unfortunately, here for the crucial size $M \asymp N \asymp t^{1/2}$, the phase $f : x \mapsto \frac{t}{2\pi}\log(N+x)$ satisfies $|f''(x)| \asymp 1$, $x \in [1, M]$ so that the bound given by Theorem 1 is essentially trivial.

However one would be content with an analogue of this result where the third derivative would be considered, since in the crucial case, one has $|f'''(x)| \asymp |t|^{-1/2}$, $x \in [1, M]$.

In order to introduce derivatives in the exponential sum (1), Weyl considered the following identity

$$\Big|\sum_{m=1}^{M} \mathrm{e}(f(m))\Big|^2 = \sum_{|h|<M}\sum_{m\in I_h} \mathrm{e}\big(f(m+h)-f(m)\big) \tag{12}$$

where $I_h = [1+\frac{|h|-h}{2}, M-\frac{|h|+h}{2}]$. Weyl's motivation was to study the case where f is polynomial of degree $k \ge 2$, for which the new polynomial $f(x+h)-f(x)$ in the inner sum has degree $k-1$. By repeating the argument $k-1$ times, one is reduced to studying the case of polynomials of degree 1, for which explicit values of the exponential sums are available.

However, for the case of a $\mathscr{C}^3$ function $f : [1, M] \to \mathbb{R}$ such that $f'''(x) \asymp \lambda_3 > 0$, each inner sum in (12) for $h \neq 0$ has a phase

$$g_h : x \mapsto f(x+h) - f(x) \tag{13}$$

that satisfies $|g_h''(x)| \asymp |h|\lambda_3$ for $x \in I_h$, and for which Theorem 1 may be applied.

Unfortunately, due to the range of summation for h the second derivative $|g_h''|$ may not be small enough, at least for $|h|$ large.

In order to control the size of $|h|$, van der Corput uses the following inequality, essentially due to Weyl, and well known as the Weyl–van der Corput inequality.

Lemma 1 (*Lemma 6.8 of [25]*)**.** *Let $M, H \ge 1$ be integers. Then for any sequence $(z_m)_{1\le m\le M} \in \mathbb{C}^M$ one has*

$$\Big|\sum_n z_n\Big|^2 \le \Big(1+\frac{M-1}{H}\Big)\sum_{|h|<H}\Big(1-\frac{|h|}{H}\Big)\sum_{1\le m, m+h\le M} z_{m+h}\overline{z_m}.$$

As an immediate consequence for exponential sums, we obtain the inequality

$$\Big|\sum_{m=1}^{M} \mathrm{e}(f(m))\Big|^2 \le \Big(1+\frac{M-1}{H}\Big)\sum_{|h|<H}\Big|\sum_{1\le m, m+h\le M} \mathrm{e}\big(f(m+h)-f(m)\big)\Big|. \tag{14}$$

In the case of a $\mathscr{C}^3$ function $f : [1, M] \to \mathbb{R}$ such that $0 < \lambda_3 \le |f'''(x)| \le \alpha\lambda_3$ on $x \in [1, M]$, one may apply Theorem 1 to the inner exponential sum with phase (13) and $h \neq 0$.

Choosing $1 \le H \le M$, we have

$$\Big|\sum_{m=1}^{M} \mathrm{e}(f(m))\Big|^2 \ll_\alpha \frac{M^2}{H} + \frac{M}{H} \sum_{1\le|h|<H} \Big(M(|h|\lambda_3)^{1/2} + (|h|\lambda_3)^{-1/2}\Big)$$
$$\ll_\alpha \frac{M^2}{H} + M^2(H\lambda_3)^{1/2} + M(H\lambda_3)^{-1/2}.$$

When $1 < \lambda_3^{-1/3} < M$, a relevant choice to optimise the right hand side is $H \asymp \min(M, \lambda_3^{-1/3})$. The result obtained still trivially holds for $\lambda_3 \ge 1$ or $M \le \lambda_3^{-1/3}$.

We may state van der Corput's result for the third derivative.

Theorem 2 (*Theorem 5.11 of [26]*)**.** *Let $\alpha \ge 1$ be a real number. There exists a constant $C(\alpha) > 0$ such that for any integer $M \ge 1$, any real number $\lambda_3 > 0$ and any $\mathscr{C}^3$ function $f : [1, M] \to \mathbb{R}$ satisfying*

$$\lambda_3 \le |f''(x)| \le \alpha\lambda_3 \quad (x \in [1, M]) \tag{15}$$

one has

$$\Big|\sum_{m=1}^{M} \mathrm{e}(f(m))\Big| \le C(\alpha)\big(M\lambda_3^{1/6} + M^{1/2}\lambda_3^{-1/6}\big).$$

We are now ready to return to the Lindelöf problem: applying this result to the sums in (3), we recover van der Corput's classical result

$$\zeta\left(\tfrac{1}{2} + it\right) \ll |t|^{1/6}\log t \quad (|t| \ge 3)$$

(see Theorem 5.12 of [26] for details).

3.4. Van der Corput's k-th derivative test

By a repeated use of (14), van der Corput's result for the k-th derivative, known as *van der Corput's k-th derivative test* follows by induction on k.

Theorem 3 (*Theorem 5.13 of [26]*)**.** *Let $\alpha \ge 1$ be a real number and $k \ge 2$ be an integer. There exists a constant $C(\alpha, k) > 0$ such that for any integer $M \ge 1$, any real number $\lambda_k > 0$ and any $\mathscr{C}^k$ function $f : [1, M] \to \mathbb{R}$ such that*

$$\lambda_k \le |f''(x)| \le \alpha\lambda_k \quad (x \in [1, M]) \tag{16}$$

one has

$$\Big|\sum_{m=1}^{M} \mathrm{e}(f(m))\Big| \le C(\alpha, k)\Big(M\lambda_k^{1/(2^k-2)} + M^{1-2^{2-k}}\lambda_k^{-1/(2^k-2)}\Big). \tag{17}$$

Remarks. • It may be of some interest in the sequel to point out the following trivial consequence of Theorem 3

$$\Big|\sum_{m=1}^{M} \mathrm{e}(f(m))\Big| \le 2C(\alpha, k)M\lambda_k^{1/(2^k-2)} \quad \text{for } M \ge \lambda_k^{-2^{k-2}/(2^{k-1}-1)} \tag{18}$$

where we emphasise the significant term in the upper bound and its domain of validity.

- In Theorems 1–3, we did not make explicit the dependence on α and k. For a more precise statement, see [26]. For an explicit constant $C(\alpha)$ in Theorem 1, see also the section below Theorem I.6.7 of [25].
- In Vinogradov's method, the explicit dependence on k is crucial. For example, in order to study $\sum_{m=1}^{M} \mathrm{e}(T\log(M+m))$, Vinogradov's results include the choice $k \asymp (\log T)/(\log M)$.

4. Van der Corput's B-process

We recall that the proof of Theorem 1 described earlier used a truncated Poisson formula of the type

$$\sum_{1\le m\le M} \mathrm{e}(f(m)) \simeq \sum_{\nu\in[f'(1),f'(M)]} \int_1^M \mathrm{e}(f(x)-\nu x)\mathrm{d}x$$

valid for a $\mathscr{C}^1$ function $f:[1,M]\to\mathbb{R}$ such that f' is monotonous. Moreover, if f is $\mathscr{C}^2$ such that $|f''(x)| \asymp \lambda_2 > 0$ $(x\in[1,M])$, then each of the integral satisfies

$$I(\nu) := \int_1^M \mathrm{e}(f(x)-\nu x)\mathrm{d}x \ll \lambda_2^{-1/2}.$$

However, with more hypotheses on the phase f, namely control on some derivatives $f^{(k)}$ for $k\ge 3$, the stationary phase method may improve the bound for these integrals. Indeed, whenever the method applies, $I(\nu)$ should be dominated by a neighbourhood of the critical point of $x\mapsto f(x)-\nu x$, namely the unique $x_\nu\in[1,M]$ such that $f'(x_\nu)-\nu=0$. Hence one should have

$$I(\nu) \simeq \int_{-\delta}^{\delta} \mathrm{e}\big(f(x_\nu+t)-\nu(x_\nu+t)\big)\mathrm{d}t \simeq \int_{\mathbb{R}} \mathrm{e}\big(f(x_\nu)-\nu x_\nu+\tfrac{1}{2}f''(x_\nu)t^2\big)\mathrm{d}t,$$

which can be explicitly computed.

Summing up all the contributions, the main term is now a new exponential sum. The process described here is known as van der Corput's B-transform. We present here the original version from *Satz 1* of [29].

Theorem 4 (*Van Der Corput, 1922*)**.** *Let $a,b\in\mathbb{R}$ such that $a<b$ and let $\lambda_2,\lambda_3>0$. Let $f:[a,b]\to\mathbb{R}$ be a $\mathscr{C}^3$ function such that*

$$|f''(x)| \asymp \lambda_2, \qquad |f'''(x)| \ll \lambda_3 \quad (x\in[a,b]).$$

Then setting $A_1 := \min(f'(a), f'(b))$ and $A_2 = \max(f'(a), f'(b))$, we have

$$\sum_{a\le m\le b} \mathrm{e}(f(m)) = \mathrm{e}^{i\sigma\pi/4} \sum_{A_1\le\nu\le A_2} \frac{\mathrm{e}\big(f(x_\nu)-\nu x_\nu\big)}{\sqrt{|f''(x_\nu)|}} + E$$

with

$$E \ll \lambda_2^{-1/2} + \log(2+(b-a)\lambda_2) + (b-a)\lambda_2^{1/5}\lambda_3^{1/5},$$

where, for any $\nu\in[A_1,A_2]$ the real number $x_\nu\in[a,b]$ is defined by the relation $f'\big(x_\nu\big)=\nu$, and where $\sigma\in\{-1,1\}$ is the sign of f''.

Remarks. • We recall that the B-transform is an involution.

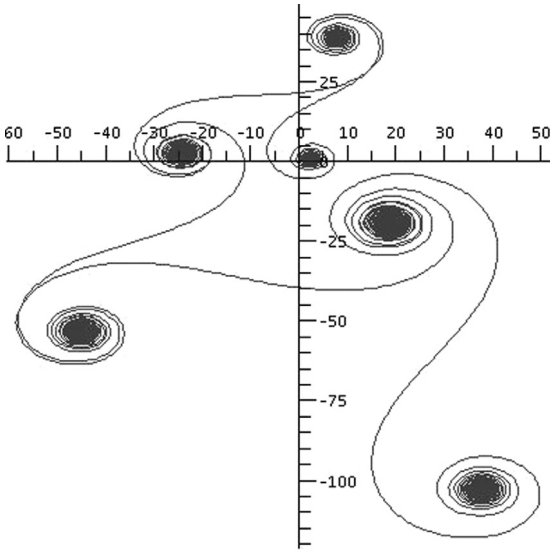
- The error term E has been subject to many improvements since, and more general exponential sums have been considered. For a complete history and a complete exposition of recent results, see [28].
- In order to exploit the new exponential, one needs some structure on the new phase $t \mapsto f^*(t) := f(x_t) - tx_t$. We shall elaborate on this point in the next section. However, we should mention right away that this transform is particularly adapted for a phase f that is monomial. For example, for $\alpha > 1$, $T > 0$ and $N \in \mathbb{N}$, the phase defined by

$$f(x) := T\left(\frac{x}{N}\right)^{\alpha} \quad (N \le x \le 2N)$$

satisfies

$$f^*(y) = c_\alpha T\left(\frac{y}{L}\right)^{\alpha/(\alpha-1)} \quad (\alpha L \le y \le \alpha 2^{\alpha-1} L)$$

where we have set $L := T/N$ and $c_\alpha := (1/\alpha)^{\alpha/(\alpha-1)} - (1/\alpha)^{-1/(\alpha-1)}$.



$$\sum_{m=1}^{M} \exp(2i\pi\sqrt{2}(m+1250)^{1,2}) \quad (M = 1, 2, \ldots, 20\,000).$$

Remark. Theorem 4 has been subject to several refinements, including an improvement of the error term when the phase is $\mathscr{C}^4$. See [28] for very recent improvements and an extensive history on the subject. See also [13] for the case of exponential sums on average with a sharp error term.

5. Semi-monomial functions and exponent pairs

We now have presented the two transforms introduced by van der Corput: the A-process and the B-process, which we may summarise as follows.

$$\sum_m \mathrm{e}(f(m)) \rightsquigarrow \sum_m \mathrm{e}(f(m+h) - f(m))$$

and

$$\sum_{m\in I} \mathrm{e}(f(m)) \rightsquigarrow \sum_{\nu\in f'(I)} \mathrm{e}(f^*(\nu))$$

where we have set

$$f^*(y) := f(z(y)) - yz(y) \quad \text{with } z(y) := (f')^{-1}(y) \quad (y \in f'(I)). \tag{19}$$

The principle of van der Corput's method is to combine these transforms and consider sequences such as $A^{n_1}BA^{n_2}\cdots B$ or $BA^{n_1}BA^{n_2}\cdots B$ where, for example, for the first sequence we apply to the exponential sum n_1 times the A-transform, then the B transform to the new exponential sum, then n_2 times the A transform....

In order to optimise such sequences to obtain the best bound possible for the exponential sum, one needs to make sure that all the phases that occur through the process are consistent with any of the transforms A or B. We start with two observations: the condition (2) for any $k \geq 1$ is not sufficient to provide information on the derivatives of the phase f^* after a B-transform. Similarly, considering only the case of a monomial phase f is not sufficient, since the difference functions $x \mapsto f(x+h) - f(x)$ are no longer monomial. Moreover, it is not even sufficient to consider any monomial phase to apply the B-transform: indeed, for the phase

$$f(x) := x^{3/2} \quad (N \leq x \leq 2N),$$

we have $f^*(y) = -y^3/3$ for $y \in f'([N, 2N])$ so that (2) is not satisfied by f^* for $k \geq 4$.

5.1. The set of semi-monomial functions

In order to apply the A- and B-transforms properly, we introduce a set of functions f that may written as $g + u$ where the function g is a monomial and the function u is a perturbation. To ensure stability for (2), for each $k \geq 1$, $u^{(k)}$ should be smaller that $g^{(k)}$ and $(g+u)^*$ should be $g^* + v$ for some perturbation v.

We now formalise the construction. For any real numbers $\alpha < 1$ and $x > 0$, we set

$$g_\alpha(x) := \frac{x^\alpha}{\alpha} \quad \text{for } \alpha \neq 0, \quad \text{and} \quad g_0(x) := \log x.$$

For any $N, T, \alpha, \varepsilon > 0$ such that $\varepsilon < 1/2$ and $Q \in \mathbb{N}$ we denote by $\mathscr{F}(N, Q, \alpha, T, \varepsilon)$ the set of the functions f defined on some segment $I_f \subseteq [N, 2N]$ and such that $f = \frac{T}{N^\alpha} g_\alpha + u$ for some function $u : [a, b] \to \mathbb{R}$ that satisfies

$$|u^{(q)}(x)| < \varepsilon \frac{T}{N^\alpha} |g_\alpha^{(q)}(x)| \quad (x \in I_f, 1 \leq q \leq Q).$$

We now have the required stability:

- If $f \in \mathscr{F}(N, Q, \alpha, T, \varepsilon)$ with f defined on $I_f = [a, b]$, then for $1 \leq h < \min(b-a, \frac{2\varepsilon N}{1-\alpha+Q})$, the function $f_1 : [a, b-h] \to \mathbb{R}$ defined by

  $$f_1(x) := f(x) - f(x+h) \quad (a \leq x \leq b-h)$$

  is in $\mathscr{F}\big(N, Q-1, \alpha-1, (1-\alpha)\frac{hT}{N}, 3\varepsilon\big)$.

- Similarly, if $f \in \mathscr{F}(N, Q, \alpha, T, \varepsilon)$, there exists a constant $C = C(\alpha, Q)$ such that for any $L \in f'(I_f)$ the restriction to the function $-f^*$ is in
$$\mathscr{F}\Big(L, Q, \overline{\alpha}, \widetilde{T}, C\varepsilon\Big)$$
where we have set $\overline{\alpha} := \frac{\alpha}{\alpha-1}$ and $\widetilde{T} := T\big(\frac{LN}{T}\big)^{\overline{\alpha}} \asymp T$.

(For details, see Lemmas 3.7 and 3.9 of [6].)

5.2. *Exponent pairs*

Let k, l be real numbers such that $0 \le k \le 1/2 \le l \le 1$. We say that (k, l) is an exponent pair if for any real number $\alpha < 1$ there exists $Q = Q(k, l, \alpha)$, $\varepsilon = \varepsilon(k, l, \alpha) < 1/2$ and a constant $C(k, l, \alpha) > 0$ such that for every $N > 0$, $T > 0$, and $f \in \mathscr{F}(N, Q, \alpha, T, \varepsilon)$
$$\Big|\sum_{m\in I_f} \mathrm{e}(f(m))\Big| \le C(k, l, \alpha)\Big(\Big(\frac{T}{N}\Big)^k N^l + \frac{N}{T}\Big). \tag{20}$$

In particular, if (k, l) is an exponent pair and $\alpha < 1$ is a real number, then
$$\max_{\substack{J \text{ interval}\\ J\subset[M,2M]}} \left|\sum_{m\in J} \mathrm{e}\left(T\left(\frac{m}{M}\right)^{\alpha}\right)\right| \ll_{\alpha,k,\ell} \left(\frac{T}{M}\right)^k M^{\ell} + \frac{M}{T}$$
for every $T > 0$ and $M \ge 1$.

Remarks. • The crucial case in the bound (20) occurs for $T \gg N$. Indeed, if T/N is sufficiently small so that $c_1 T/N \le |f'(x)| \le c_2 T/N < 1/2$, (11) immediately provides the bound $O(N/T)$ for the exponential sum.

- In his original work [29], van der Corput considered systems of exponent pairs (k_1, l_1), $(k_2, l_2), \ldots$ for which the right hand side of (20) is of the form $\sum_{1\le j\le J}(T/N)^{k_j} N^{l_j}$.

The theory of exponent pairs that we introduce here is due to Phillips [12]. For the exposition, we essentially follow the lines of [6]. However, the reader should be aware that, due to another choice of normalisation, the definition of our set $\mathscr{F}(N, Q, \alpha, T, \varepsilon)$ of semi-monomial functions is different from that of chapter 3 of [6] denoted by $\mathbf{F}(N, P, s, y, \varepsilon)$. We also modified our definition of exponent pair accordingly so that the couple (k, l) represents the same object in both cases.

Combining the A and B transforms, we may construct some new exponent pairs (see Theorem 3.8 and 3.10 of [6]):

- If (k, l) is an exponent pair, then
$$A(k, l) := \Big(\frac{k}{2k+2}, \frac{k+l+1}{2k+2}\Big)$$
is an exponent pair. This is the result obtained by applying the A-transform, and applying the exponent pair (k, l) to the new exponential sum.
- If (k, l) is an exponent pair, then
$$B(k, l) := \Big(l - \tfrac{1}{2}, k + \tfrac{1}{2}\Big)$$
is an exponent pair. This is the result obtained by applying the B-transform, and applying the exponent pair (k, l) to the new exponential sum.

We now give some examples of exponent pairs:

- The pair $(0, 1)$: this is the trivial bound.
- The pair $B(0, 1) = (\frac{1}{2}, \frac{1}{2})$: we recover the bound of Theorem 1.
- The pair $A^{k-2}B(0, 1)$: the main term corresponds to the bound of Theorem 3.

The *Exponent Pair Conjecture* asserts that $(\varepsilon, \frac{1}{2} + \varepsilon)$ should be an exponent pair for every $\varepsilon > 0$. This would in particular imply the Lindelöf hypothesis, and the bound $\Delta(x) \ll_\varepsilon x^{(1/4)+\varepsilon}$ for the Divisor Problem.

Since van der Corput's results and the refinement due to Phillips, new exponent pairs have been found. In particular, Bombieri and Iwaniec's method has created new exponent pairs that do not arise from sequences of transforms A and B. For an exposition of the method, see [6,9]. See also [3] for very recent refinements of the method.

To conclude this section, we give a short exposition of applications of exponent pairs for the Lindelöf problem and the Divisor problem. We shall denote by θ_L an admissible exponent for the Lindelöf problem, and by θ_D an admissible exponent for the Divisor problem, so that for any $\varepsilon > 0$ one has

$$\zeta\left(\tfrac{1}{2} + it\right) \ll_\varepsilon |t|^{\theta_L+\varepsilon} \quad (|t| \geq 3)$$

and

$$\sum_{n\leq t} \tau(n) = t(\log t + 2\gamma - 1) + O_\varepsilon(t^{\theta_D+\varepsilon}) \quad (t \geq 3).$$

Lindelöf problem:

- van der Corput (1920): $\theta_L = \frac{1}{6}$. (See also the exponent pair $AB(0, 1)$.)
- Exponent pair $ABA^3B(0, 1)$: $\theta_L = \frac{27}{164} = 0,16\,463\ldots$.
- Bombieri and Iwaniec (1985): $\theta_L = \frac{9}{56} = 0,1\,607\,142\ldots$.
- Improvements due to Huxley and Watt (1988 and 1989).
- Huxley (2005): $\theta_L = \frac{32}{205} = 0,15\,609\,756\ldots$.
- Current record: Bourgain (2014) $\theta_L = \frac{53}{342} = 0,1\,549\,707\ldots$.

Divisor problem:

- Voronoï (1903): $\theta_D = \frac{1}{3}$. (See also the exponent pair $B(0, 1)$.)
- Exponent pair $BA^3B(0, 1)$: $\theta_D = \frac{27}{82} = 0,32\,926\ldots$.
- Iwaniec & Mozzochi (1988): $\theta_D = \frac{7}{22} = 0,31\,818\,181\ldots$.
- Current record: Huxley (2003) $\theta_D = \frac{131}{416} = 0,314\,903\ldots$.

Remark. The Exponent Pair Conjecture would imply that $\theta_L = 0$ and $\theta_D = 1/4$ would be admissible choices.

6. A few tools in modern methods

6.1. Weyl shift

Repeating N times the exponential sum and shifting each sum by an index $n \leq N$ we have

$$N \sum_{m=1}^{M} \mathrm{e}(f(m)) = \sum_{n=1}^{N} \sum_{m=1}^{M-N} \mathrm{e}(f(m+n)) + O(N^2),$$

which yields the Weyl shifted sum

$$\sum_{m=1}^{M} \mathrm{e}(f(m)) \ll \frac{1}{N} \sum_{m} \Big| \sum_{n=1}^{N} \mathrm{e}\big(f(m+n) - f(m)\big) \Big| + N \quad (1 \le N < M). \tag{21}$$

In the case of a $\mathscr{C}^k$ function $f : [1, M] \to \mathbb{R}$ with a small k-th derivative, and provided that N is small enough, each phase $x \mapsto f(m+x) - f(m)$ may be replaced by the corresponding Taylor polynomial $P_{m,k}(x) := \sum_{1 \le j \le k-1} \frac{f^{(j)}(m)}{j!} x^j$, hence

$$\sum_{m=1}^{M} \mathrm{e}(f(m)) \ll \frac{1}{N} \sum_{m=1}^{M-N} \Big| \sum_{n=1}^{N} \mathrm{e}\big(P_{m,k}(n)\big) \Big| + N.$$

At this stage, we may describe briefly Weyl's method and Vinogradov's method.

- Weyl considers the inner sum $\sum_{n=1}^{N} \mathrm{e}\big(P_{m,k}(n)\big)$ and applies Weyl's van der Corput's inequality $k-2$ times so that the new phase deduced from $P_{m,k}$ is a polynomial of degree 1 and explicit bounds are available.
- Vinogradov's approach is the following: Applying Hölder's inequality we have, for any $s \in \mathbb{N}$

$$\Big| \sum_{m=1}^{M} \mathrm{e}(f(m)) \Big|^{2s} \ll \frac{M^{2s-1}}{N^{2s}} \sum_{m} \Big| \sum_{n=1}^{N} \mathrm{e}\big(P_{m,k}(n)\big) \Big|^{2s} + N^{2s}. \tag{22}$$

When the coefficients of $P_{m,k}(X)$ are well distributed modulo 1, we should have

$$\frac{1}{M} \sum_{m=1}^{M} \Big| \sum_{n=1}^{N} \mathrm{e}\big(P_{m,k}(n)\big) \Big|^{2s} \simeq J_{s,k}(N) \Delta(M) \tag{23}$$

where

$$J_{s,k}(N) := \int_{[0,1]^{k-1}} \Big| \sum_{n=1}^{N} \mathrm{e}\big(\alpha_1 n + \cdots + \alpha_{k-1} n^{k-1}\big) \Big|^{2s} \mathrm{d}\alpha_1 \ldots \mathrm{d}\alpha_{k-1}$$

and where $\Delta(M)$ controls the distribution of the coefficients $f^{(j)}(m)/j!$ $(1 \le j \le k-1)$ of $P_{m,k}(X)$.

Via Fourier orthogonality, the integral $J_{s,k}(N)$ counts the numbers of solutions of the diophantine system

$$\sum_{i=1}^{s} \big(x_i^j - y_i^j\big) = 0 \quad (1 \le j \le k-1) \tag{24}$$

such that $1 \le x_i, y_i \le N$ for $1 \le i \le s$. The estimate now reduces to a counting problem, where new arguments, including arithmetic, may be used.

6.2. *Variations of the A-transform*

There are several variations of Lemma 1: we shall mention three of them.

The following result is essentially Lemma 5.6.2 of [9]

Lemma 2. *Let $M \geq 2$ and let $(z_m)_{1\leq m\leq M}$ be a sequence of complex numbers. Let $H \leq M/2$ be a positive integer. Then*

$$\Big|\sum_{m=1}^{M} z_m\Big|^2 \leq \frac{M+2H-2}{H}\Big(\sum_{m=1}^{M}|z_m|^2 + 2\mathrm{Re}\sum_{h=1}^{H-1}\Big(1-\frac{h}{H}\Big)\sum_{m=1+h}^{M-h} z_{n+h}\overline{z_{n-h}}\Big).$$

In the case of the choice $z_m = \mathrm{e}(f(m))$, the right hand side provides an exponential sum with a phase of the form $f(m+h) - f(m-h)$. In some applications, when m is a fixed parameter, the phase $x \mapsto f(m+x) - f(m-x)$ has a simple Taylor expansion with only odd degree monomials.

The next result is dual, and involves symmetric sums:

Lemma 3 (*Lemma 1 of [22]*)**.** *Let $M \geq 1$. Let $(z_m)_{n\in\mathbb{Z}} \in \mathbb{C}^{\mathbb{Z}}$ such that $z_m = 0$ for $m \notin [1, M]$. Let $1 \leq H \leq M$. Then*

$$\Big|\sum_{m} z_m\Big|^2 \ll \frac{M}{H}\max_{1\leq H_1,H_2\leq H}\sum_{m}\Big(\Big|\sum_{|h|\leq 2H_1} z_{m+h}z_{m-h}\Big| + \Big|\sum_{|h|\leq 2H_2} z_{m+2h}z_{m-2h}\Big|\Big).$$

Again, for the choice $z_m = \mathrm{e}(f(m))$ the phase of the new exponential sum is of the form $f(m+n) + f(m-n)$, which proves crucial in some applications (see [22,23]).

The following lemma is a particular case of Lemma 6.1 of [6].

Lemma 4 (*Lemma 1 of [19]*)**.** *Let $M, H \geq \mathbb{N}$. Let $(a(m,h))_{(m,h)\in\mathbb{Z}^2} \in \mathbb{C}^{\mathbb{Z}^2}$ such that $a(m,h) = 0$ $(m,h) \notin [1, M] \times [1, H]$. We set $S := \sum_{(m,h)\in\mathbb{Z}^2} a(m,h)$ and we choose two integers Q and R such that $1 \leq Q \leq M$ and $1 \leq R \leq H$. We then have*

$$S^2 \ll \frac{MH}{QR}\sum_{|q|<Q}\sum_{|r|<R}\Big(1-\frac{|q|}{Q}\Big)\Big(1-\frac{|r|}{R}\Big)\sum_{(m,h)\in\mathbb{Z}^2} a(m+q,h)\overline{a(m,h+r)}.$$

6.3. Variation of the B-transform: the $\widetilde{B}$-transform

One should also mention a variation of the B-transform, named $\widetilde{B}$-transform in the case of a phase of the form $f(x) = \frac{r}{q}x + \frac{s}{q}x^2 + g(x)$ where g is $\mathscr{C}^3$ with some suitable control on the derivatives (see [1], chapter 7 of [6,21] and part II of [9]). This new transform is a crucial ingredient in the Bombieri and Iwaniec's method and in particular all the results about the Lindelöf problem since 1985.

The following formulation is a particular case of Theorem 3.3 of [21]. Let

$$S := \sum_{a<n\leq b}\mathrm{e}\Big(\frac{r}{q}n + \frac{s}{q}n^2 + \varphi(n)\Big)$$

where $\varphi : [a,b] \to \mathbb{R}$ is a $\mathscr{C}^3$ function such that

$$\varphi''(x) \asymp \lambda_2 > 0, \qquad |\varphi'''(x)| \leq \lambda_3 \quad (x \in [a,b]).$$

Let $[\alpha,\beta] := \varphi'([a,b])$, $z = (\varphi')^{-1} : [\alpha,\beta] \to [a,b]$ and $\varphi^*(y) := \varphi(z(y)) - yz(y)$ $(y \in [\alpha,\beta])$ and

$$G(r,s;q) := \sum_{k \bmod q}\mathrm{e}\Big(\frac{r}{q}k + \frac{s}{q}k^2\Big) \quad (q \in \mathbb{N}, r, s \in \mathbb{Z}).$$

Theorem 5. *With the notation above, one has*

$$S = \frac{e^{i\pi/4}}{q} \sum_{q\alpha \le \nu \le q\beta} \left| g''\big(z(\nu/q)\big) \right|^{-1/2} G(r+\nu, s; q) e\big(g^*(\nu/q)\big) + O(R)$$

where

$$R := q(1 + (b-a)\lambda_3\lambda_2^{-1}) \log(b - a + \lambda_2 + q + 1) + \lambda_2^{-1}.$$

6.4. Double large sieve

The double large sieve is an inequality that formalises the idea developed in (23). We give here the formulation of Lemma 7.5 of [6].

Let $\mathscr{X}$ and $\mathscr{Y}$ be two subsets of $\mathbb{R}^K$. Let $a(\mathbf{x})$ and $b(\mathbf{y})$ be arbitrary complex numbers for $\mathbf{x} \in \mathscr{X}$ and $\mathbf{y} \in \mathscr{Y}$. Let $X_1, \dots, X_K, Y_1, \dots, Y_K$ be positive numbers. Define the bilinear forms

$$B_1(b; \mathscr{Y}) := \sum_{\substack{\mathbf{x}\in\mathscr{X} \\ |x_k - x_k'| \le (2Y_k)^{-1}}} \sum_{\substack{\mathbf{x}'\in\mathscr{X} \\ 1\le k\le K}} |a(\mathbf{x})a(\mathbf{x}')|$$

$$B_2(a; \mathscr{X}) := \sum_{\substack{\mathbf{y}\in\mathscr{Y} \\ |y_k - y_k'| \le (2X_k)^{-1}}} \sum_{\substack{\mathbf{y}'\in\mathscr{Y} \\ 1\le k\le K}} |b(\mathbf{y})b(\mathbf{y}')|,$$

$$B(a, b; \mathscr{X}, \mathscr{Y}) := \sum_{\substack{\mathbf{x}\in\mathscr{X} \\ |x_k| \le X_k}} \sum_{\substack{\mathbf{y}\in\mathscr{Y} \\ |y_k| \le Y_k}} a(\mathbf{x})b(\mathbf{y})e(\mathbf{x}\cdot\mathbf{y}).$$

Then

$$|B(a, b; \mathscr{X}, \mathscr{Y})|^2 \le (2\pi^2)^K \prod_{k=1}^{K} (1 + X_k Y_k) B_1(b; \mathscr{Y}) B_2(a; \mathscr{X}). \tag{25}$$

Remark. Unlike the treatment of (23) by Vinogradov, the double large sieve contains Cauchy's inequality: we develop in the following toy example the consequence for the corresponding diophantine system. We consider the sum

$$\sum_m \Big| \sum_n e\Big(\{f'(m)\}n + \Big\{ \tfrac{1}{2} f''(m) \Big\} n^2 \Big) \Big|^3$$
$$= \sum_m a(m) \sum_{n_1, n_2, n_3} e\Big(\{f'(m)\} s_1(n_1, n_2, n_3) + \Big\{ \tfrac{1}{2} f''(m) \Big\} s_2(n_1, n_2, n_3) \Big)$$

with $s_j(n_1, n_2, n_3) := n_1^j + n_2^j + n_3^j$ and where the $a(m)$ have modulus 1. The sum is now written as a bilinear form of type $B(a, b; \mathscr{X}, \mathscr{Y})$ where $\mathscr{X}$ is the set of vectors of the form $\big(\{f'(m)\}, \{\frac{1}{2} f''(m)\}\big)$ and $\mathscr{Y}$ is the set of vectors of the form $\big(s_1(n_1, n_2, n_3), s_2(n_1, n_2, n_3)\big)$, so that here $X_1 = X_2 = 1,\ Y_1 = 3N$ and $Y_2 = 3N^2$.

The term $B_1(b; \mathscr{Y})$ counts the couples of integers $(m_1, m_2) \in [1, M]^2$ such that

$$\big| \{f'(m_1)\} - \{f'(m_2)\} \big| \le 1/(3N), \qquad \Big| \Big\{ \tfrac{1}{2} f''(m_1) \Big\} - \Big\{ \tfrac{1}{2} f''(m_2) \Big\} \Big| \le 1/(3N^2),$$

whereas the term $B_2(a;\mathscr{X})$ counts the number of solutions of a diophantine system

$$n_1^j+n_2^j+n_3^j=n_4^j+n_5^j+n_6^j \quad (j=1,2)$$

with $1\le n_i\le N$.

6.5. Examples of diophantine systems

The diophantine system (24) is a crucial ingredient in Vinogradov's method, and has been studied extensively in various directions such as the zero-free region for the Riemann ζ function and the Waring problem, but also more recently as a problem on its own. The number $J_{s,k}(N)$ of the solutions such that $1\le x_j, y_j\le N$ satisfies the heuristic

$$J_{s,k}(N)\ll_{s,k,\varepsilon} N^{\varepsilon}\Big(N^s+N^{2s-\frac{1}{2}k(k+1)}\Big)$$

for any $\varepsilon>0$. For $k=2$, the result is now classic for each $s\ge 1$. For $k=3$, this conjecture has been proven recently by Wooley for all $s\ge 1$ (see [32]).

For $k\ge 4$, the conjecture is currently proven for $s\le \frac{1}{2}k(k+1)-\frac{1}{3}k-8k^{2/3}$ and $s\ge k^2-k$ (see [33,32])

The introduction of the variations of A and B transforms in the preliminary treatment of the exponential sum, combined with the double large sieve, yields new spacing problems and interesting variations of the diophantine system. Here are a few examples that will occur in the sequel:

(1) The following system

$$\begin{cases} n_1+n_2+n_3+n_4 = n_5+n_6+n_7+n_8\\ n_1^2+n_2^2+n_3^2+n_4^2 = n_5^2+n_6^2+n_7^2+n_8^2\\ |n_1^{3/2}+n_2^{3/2}+n_3^{3/2}+n_4^{3/2} - (n_5^{3/2}+n_6^{3/2}+n_7^{3/2}+n_8^{3/2})|\le \delta N^{3/2}\\ N\le n_i\le 2N\end{cases} \tag{26}$$

has been introduced in [2]. The heuristic for the number $\mathscr{A}_1(N)$ of solutions of this system is

$$\mathscr{A}_1(N)\ll_{\varepsilon} N^{\varepsilon}\big(N^4+\delta N^5\big).$$

This conjecture has been proven in [2] and is a crucial ingredient in the original version of the Bombieri–Iwaniec's method. A generalisation of this system with ten variables and an extra condition with term $n_i^{1/2}$ has been later introduced by Watt (see [31]). The latest results for this system are due to Bourgain (see [3]).

(2) The system

$$\begin{cases} r_1 = r_2\\ r_1n_1+h_1q_1 = r_2n_2+h_2q_2\\ |r_1n_1^2+2h_1q_1n_1+h_1q_1^2 - (r_2n_2^2+2h_2q_2n_2+h_2q_2^2)|\le \delta HQ^2\end{cases} \tag{27}$$

where $(r_j, q_1h_1, n_1)\in[1,R]\times[Q,2Q]\times[H,2H]\times[1,N]$ (with $R\le H/2$) has been introduced in [19] and where it is proven that the number $\mathscr{A}_2(R,N,H,Q)$ of solutions satisfies

$$\mathscr{A}_2(R,N,H,Q)\ll_{\varepsilon}(RHQN)^{\varepsilon}\big(1+\delta Q\big)$$

for any $\varepsilon>0$.

(3) The system

$$\begin{cases} n_1+n_2+n_3+n_4 = n_5+n_6+n_7+n_8 \\ n_1^2+n_2^2+n_3^2+n_4^2 = n_5^2+n_6^2+n_7^2+n_8^2 \\ |n_1^3+n_2^3+n_3^3+n_4^3 - (n_5^3+n_6^3+n_7^3+n_8^3)| \le \delta N^3 \\ \qquad N \le n_i \le 2N \end{cases} \tag{28}$$

is also studied in [2], and the heuristic is identical to that of system (26).

(4) The system

$$\begin{cases} n_1^2+n_2^2+n_3^2 = n_4^2+n_5^2+n_6^2 \\ |n_1^4+n_2^4+n_3^4 - (n_4^4+n_5^4+n_6^4)| \le \delta N^4 \\ N \le n_i \le 2N \end{cases} \tag{29}$$

is studied in [18] and the number $\mathscr{A}_4(N)$ of solutions satisfies

$$\mathscr{A}_4(N) \ll_\varepsilon N^\varepsilon\big(N^3+\delta N^4\big)$$

for any $\varepsilon > 0$.

7. On van der Corput's hypothesis

We now go back to the simplest of van der Corput's estimates, namely Theorem 3, and more particularly the monomials in M and λ_k that occur in the upper bound. In order to ease the exposition in the next section, we no longer make explicit the dependency in α in (16) and (17). Moreover, we only keep the formulation (18), so that Theorem 3 now reads as follows:

For any integer $M \ge 1$, any real number $\lambda_k > 0$ and any $\mathscr{C}^k$ function $f : [1, M] \to \mathbb{R}$ such that

$$|f^{(k)}(x)| \asymp \lambda_k \quad (x \in [1, M]), \tag{30}$$

we have

$$\sum_{m=1}^{M} \mathrm{e}(f(m)) \ll M\lambda_k^{\theta_k} \quad \text{for } M \ge \lambda_k^{-\beta_k} \tag{31}$$

with the notation

$$\theta_k := \frac{1}{2^k-2}, \qquad \beta_k := \frac{2^{k-2}}{2^{k-1}-1}.$$

We are interested in studying the optimality of the exponents θ_k and β_k under the sole assumption (30). Although this problem is already quite interesting on its own, we may have some ulterior motive, in view of potential applications. For the moment, we are satisfied with a few observations:

(1) *Nonsemi-monomial phase.* In many applications, the phase may not allow the use of exponent pairs: for example, in the case of

$$\sum_{m=1}^{M} \mathrm{e}\big(M^{k-1}\log(M+m) + P(m)\big)$$

where P is a polynomial of degree at most $k-1$, the phase is not semi-monomial in general.

(2) *The case of short exponential sums*. We shall say that an exponential sum is short if, possibly after a change of variable, the range of summation is $N < m \le N + M$ where the length M of that sum is negligible with respect to the size N of the variable. For example, for $\alpha < 1$, we consider the sum

$$S(N) := \sum_{N \le m \le N+N^{2/3}} \mathrm{e}\left(N^2 \left(\frac{m}{N}\right)^{\alpha}\right), \quad (M = N^{2/3})$$

(where x^α is replaced by $\log x$ for $\alpha = 0$). In that case, the bound (31) with $k = 3$ yields $S(N) \ll N^{1/2}$ whereas the most powerful exponent pair for this sum gives

$$S(N) \ll \max_{I \subset [N,2N]} \left| \sum_{m \in I} \mathrm{e}\left(N^2 \left(\frac{m}{N}\right)^{\alpha}\right) \right| \ll N^{\frac{1}{2}+\frac{32}{205}}.$$

(3) *Forbidden exponents.* Even is the case of a large sum ($M = N$)

$$S(N) := \sum_{N \le m \le 2N} \mathrm{e}\left(N^2 \left(\frac{m}{N}\right)^{\alpha}\right)$$

exponent pairs do not apply for an exponent $\alpha \ge 1$, whereas Theorem 3 with $k = 3$ applies as soon as $(\alpha - 1)(\alpha - 2) \ne 0$.

In the sequel, we shall be interested in three questions, for k fixed, under the sole hypothesis (30):

Question 1. *May we replace the exponent β_k by a smaller exponent?*

Question 2. *May we replace the exponent θ_k by a larger exponent?*

Question 3. *Does there exist an increasing function $\varphi : \mathbb{R}^+ \to \mathbb{R}^+$ such that $\varphi(t) \to +\infty$ as $t \to +\infty$, and an exponent $\gamma_k > 0$ such that*

$$\frac{1}{H} \sum_{h \asymp H} \left| \sum_{m=1}^{M} \mathrm{e}\left(\frac{h}{H} f(m)\right) \right| \ll \frac{M \lambda_k^{\theta_k}}{\varphi(H)} \qquad \left(M \ge \lambda_k^{-\beta_k}, 1 \le H \le \lambda_k^{-\gamma_k}\right)? \tag{32}$$

One motivation for Question 3 is that the sum introduced in (32) occurs in many lattice problems, such as on the sum of fractional parts (see (5)), and also on counting integer points on a curve or close to curve. To illustrate some of the results in the sequel, we shall be interested in bounding

$$\mathscr{R}_M(f) := \#\{m \in [1, M] \cap \mathbb{Z} \colon f(m) \in \mathbb{Z}\} \tag{33}$$

where $f : [1, M] \to \mathbb{R}$ is a function.

Lemma 5.3.2 of [9] implies that for any $H \ge 2$ one has

$$\mathscr{R}_M(f) \le \frac{2M}{H} + \frac{4}{H} \mathrm{Re} \sum_{h=1}^{H-1} \left(1 - \frac{h}{H}\right) \sum_{m=1}^{M} \mathrm{e}(hf(m)). \tag{34}$$

8. New terminology related to van der Corput's hypothesis

8.1. Van der Corput k-couples

Definition 1. Let $k \geq 2$ and two real numbers $\theta, \beta > 0$. We say that (θ, β) is a van der Corput k-couple if for any $\mathscr{C}^k$ function $f : [1, M] \to \mathbb{R}$ such that

$$0 < \lambda_k \ll |f^{(k)}(x)| \ll \lambda_k \quad (1 \leq x \leq M)$$

one has

$$S \ll M\lambda_k^{\theta} \quad \text{for } M \geq \lambda_k^{-\beta}.$$

Remark. It follows trivially from (18) that for each $k \geq 2$, $\left(\frac{1}{2^k-2}, \frac{2^{k-2}}{2^k-2}\right)$ is a van der Corput k-couple.

We start by proving that whenever (θ, β) is a van der Corput k-couple, we have $\beta > 1/k$. Indeed, suppose that $\beta \leq 1/k$. Let

$$S(M) := \sum_{m=1}^{M} \mathrm{e}\left(\frac{m^k}{100k!M^k}\right) \quad (M \geq 1). \tag{35}$$

Then the phase f in $S(M)$ satisfies $|f^{(k)}(x)| = \lambda_k := M^{-k}$ for $1 \leq x \leq M$, so that

$$S(M) \ll M\lambda_k^{\theta} \asymp M^{1-k\theta}.$$

Now a direct computation shows that $|S(M) - M| \leq \frac{2\pi}{100}M$ so that $|S(M)| \gg M$. When $M \to +\infty$, this contradicts the fact that $1 - k\theta < 1$. This proves that $\beta > 1/k$.

Definition 2.
- If (θ, β) is a van der Corput k-couple, the real number

$$\eta(\theta, \beta) := \frac{k(\beta - \theta) - 1}{k\beta - 1} \tag{36}$$

  is called the *index* of (θ, β).
- A van der Corput k-couple (θ, β) is called *pure* if $\eta(\theta, \beta) = 0$.

Remarks. Since for a k-couple (θ, β) we have $\beta > 1/k$, the index $\eta(\theta, \beta)$ is well defined. The only pure couple in this sequence $\left(\frac{1}{2^k-2}, \frac{2^{k-2}}{2^k-2}\right)$ is the 2-couple $(\frac{1}{2}, \frac{1}{2})$.

8.2. Long sums and short sums

We recall a classical technique to increase the length of an exponential sum. Let (θ, β) be a van der Corput k-couple. Then for any $M_1 \geq 1, \lambda_k > 0$ such that $M_1 \geq \lambda_k^{-\beta}$ and any $f : [1, M] \to \mathbb{R}$ such that $\lambda_k \leq |f^{(k)}(x)| \ll \lambda_k$ $(1 \leq x \leq M_1)$, we have $\sum_{m=1}^{M_1} \mathrm{e}(f(m)) \ll M_1\lambda_k^{\theta}$.

Under the same hypotheses and notation, we are interested in bounding the subsum $\sum_{m=1}^{M} \mathrm{e}(f(m))$ with $1 \leq M < \lambda_k^{-\beta}$ for which the van der Corput k-couple (θ, β) does not apply directly. The classical technique is as follows: using Fourier orthogonality, one

has

$$\sum_{m=1}^{M} \mathrm{e}(f(m)) = \int_{-1/2}^{1/2} \Big(\sum_{\nu=1}^{M} \mathrm{e}(-x\nu)\Big)\Big(\sum_{m=1}^{M_1} \mathrm{e}\big(f(m)+xm\big)\Big)\mathrm{d}x$$
$$\ll \int_{-1/2}^{1/2} \min\Big(M, \frac{1}{|x|}\Big)\Big|\sum_{m=1}^{M_1} \mathrm{e}\big(f(m)+xm\big)\Big|\mathrm{d}x.$$

Since each phase in the inner exponential sum has a k-th derivative that does not depend on the parameter x, we may apply the previous bound, and we have

$$\sum_{m=1}^{M} \mathrm{e}(f(m)) \ll (\log M) M_1 \lambda_k^{\theta}.$$

By choosing $M_1 \asymp \lambda_k^{-\beta}$ we deduce a bound that combines both estimates: we have

$$\sum_{m=1}^{M} \mathrm{e}(f(m)) \ll M\lambda_k^{\theta} + \lambda_k^{\theta-\beta} \log M \tag{37}$$

for any choice of $M \geq 1$ and $\lambda_k > 0$.

Lemma 5. *Let $k \geq 2$, $\theta, \beta > 0$ and $\eta \in \mathbb{R}$ such that (θ, β) is a van der Corput k-couple of index η. Then:*

(i) *We have $\beta - \theta \geq 1/k$ and $0 \leq \eta < 1$.*
(ii) *We have*

$$\sum_{m=1}^{M} \mathrm{e}(f(m)) \ll M^{\eta} \lambda_k^{-\frac{\theta}{k\beta-1}} \quad (\lambda_k^{-1/k} \leq M \leq \lambda_k^{-\beta})$$

for any $\mathscr{C}^k$ function $f : [1, M] \to \mathbb{R}$ such that $|f^{(k)}(x)| \asymp \lambda_k$, $(1 \leq x \leq M)$.
(iii) *We have*

$$\sum_{m=1}^{M} \mathrm{e}(f(m)) \ll M\lambda_k^{\theta} + \lambda_k^{-(\beta-\theta)} \quad (M \geq 1, \lambda_k > 0)$$

uniformly for $M \geq 1$, $\lambda_k > 0$ and $\mathscr{C}^k$ functions $f : [1, M] \to \mathbb{R}$ such that $|f^{(k)}(x)| \asymp \lambda_k$, $(1 \leq x \leq M)$.
(iv) *For any $\frac{1}{k} < \gamma \leq \beta$, the couple $(\frac{k\gamma-1}{k\beta-1}\theta, \gamma)$ is a van der Corput k-couple of same index η.*
(v) *The pair $\big(\frac{\theta}{2\theta+2}, \frac{\beta}{\theta+1}\big)$ is a van der Corput $(k+1)$-couple.*

Proof. (i) We apply (37) to the sum (35) and we have $M \ll M^{1-k\theta} + M^{k(\beta-\theta)} \log M$. By letting M tend to $+\infty$, we have $k(\beta-\theta) \geq 1$. It follows immediately that $0 \leq \eta < 1$.

(ii) For any integer $1 \leq q \leq M/2$, we have

$$\sum_{1\leq m\leq M} \mathrm{e}(f(m)) = \sum_{r=1}^{q} \sum_{1\leq l\leq (M-r)/q} \mathrm{e}(f(ql+r)).$$

Setting $g_r(x) := f(qx+r)$, we have $g_r^{(k)}(x) \asymp q^k \lambda_k$ uniformly for $1 \leq r \leq q$ and $0 \leq x \leq (M-r)/q$. Since $\lambda_k^{-1/k} \leq M \leq \lambda_k^{-\beta}$, one easily checks that q may be chosen so

that $q \asymp Q$ with

$$\frac{M}{Q}(Q^k\lambda_k)^\beta = 1.$$

In particular, we have

$$\lfloor (M-r)/q \rfloor (q^k\lambda_k)^\beta \gg 1 \quad (0 \le r \le q)$$

so that

$$\sum_{1\le m\le M} \mathrm{e}(f(m)) = \sum_{r=1}^{q} \sum_{1\le l\le (M-r)/q} \mathrm{e}(f(ql+r)) \ll q\frac{M}{q}(q^k\lambda_k)^\theta \ll M(Q^k\lambda_k)^\theta$$

and the expected result follows by substituting the value of Q.

(iii) For $M < \lambda_k^{-1/k}$, the result follows from the trivial bound, and for $M > \lambda_k^{-\beta}$ the result follows from the definition of (θ, β). In the remaining case, we use the bound from (ii). Since $\eta \ge 0$, we have

$$M^\eta \lambda_k^{-\frac{\theta}{k\beta-1}} \le \lambda_k^{-\beta\eta}\lambda_k^{-\frac{\theta}{k\beta-1}} = \lambda_k^{\theta-\beta}.$$

(iv) We proceed as in the proof of (ii) by writing

$$\sum_{1\le m\le M} \mathrm{e}(f(m)) = \sum_{r=1}^{q} \sum_{1\le l\le (M-r)/q} \mathrm{e}(f(ql+r))$$

except that we now choose $q \asymp \lambda_k^{(\gamma-\beta)/(k\beta-1)}$.

(v) We apply the A-transform of (14) to an exponential sum of phase $f : [1, M] \to \mathbb{R}$ such that $|f^{(k+1)}(x)| \asymp \lambda_{k+1}$, and we apply the bound (iii) to the sums on the right hand side: we have

$$\Big|\sum_{m=1}^{M} \mathrm{e}(f(m))\Big|^2 \ll \frac{M^2}{H} + M\big(M(H\lambda_{k+1})^\theta + (H\lambda_{k+1})^{-\beta}\big).$$

The result now follows by choosing $H \asymp \lambda_{k+1}^{-\theta/(1+\theta)}$. $\square$

Remark. The result (ii) in the previous statement may apply to very short sums: writing $M = \lambda_k^{-1/k}R$, we have the following reformulation

$$\sum_{1\le m\le \lambda_k^{-1/k}R} \mathrm{e}(f(m)) \ll \lambda_k^{-1/k}R^\eta \quad (1 \le R \le \lambda_k^{-\beta+1/k}).$$

As an example, using the 3-couple $(\frac{1}{6}, \frac{2}{3})$ of index $\eta = 1/2$, we have, for any $\mathscr{C}^3$ function $F : [0, 1] \to \mathbb{R}$ such that $0 < A_1 \le F'''(x) \le A_2$ for $0 \le x \le 1$,

$$\sum_{N<n\le N+N^{1/3}\log N} \mathrm{e}\Big(N^2F\Big(\frac{n}{N}\Big)\Big) \ll_{A_1,A_2} N^{1/3}(\log N)^{1/2}.$$

9. Optimality: the case $k = 2$

We start with Questions 1 and 2 of Section 7. We recall that for any $\mathscr{C}^2$ function $f : [1, M] \to \mathbb{R}$ such that $|f''(x)| \asymp \lambda_2 > 0$ for $x \in [1, M]$ we have

$$\sum_{m=1}^{M} \mathrm{e}(f(m)) \ll M\lambda_2^{\theta_2} \quad \text{for } M \geq \lambda_2^{-\beta_2}$$

with $(\theta_2, \beta_2) = (\frac{1}{2}, \frac{1}{2})$. The example of the Gauss sums

$$\sum_{m=1}^{q} \mathrm{e}(m^2/q) = \frac{1}{2}(1 + i^{-q})(1 + i)q^{1/2}$$

shows that both θ_2 and β_2 are optimal.

We are now interested in Question 3. First, using Theorem 1 in (34) and choosing $H \asymp \lambda_2^{-1/3}$, we have

$$\mathscr{R}_M(f) \ll M\lambda_2^{1/3} \text{ for } M \gg \lambda_2^{-2/3}. \tag{38}$$

Our aim in the sequel of this section is to prove that the exponent $1/3$ in (38) is sharp in the case of the size $M \asymp \lambda_2^{-2/3}$.

Our argument is based on [7] where Grekos bounds the number of integer points on convex $\mathscr{C}^2$ curves. The bound of his Theorem I is of the form $O(LR^{-1/3})$ where L denotes the length of the curve and R denotes the lower bound of the radii of curvature. In Theorem II, Grekos constructs a curve that contains at least $LR^{-1/3}$. His proof generalises Linnik's construction (see chapter I.2 of [9] for details).

For our purpose, namely integer points on the particular curves of equation $y = f(x)$, Grekos' result would apply for a $\mathscr{C}^2$ function f such that $0 < f''(x) \leq 1/R$ for $x \in [1, M]$. However the construction in Theorem II of [7] is not sufficient here, since we aim to impose that $f''(x)$ has order of magnitude $1/R$ to conform to (6).

In the following lemma, we state an analogue of Lemme 1 of [7], with an appropriate control on the second derivative.

Lemma 6. *Let $Q \geq 19$ be an integer. For any integer N such that $Q/3 \leq N \leq Q$, there exists a function $F = F_{Q,N} \in \mathscr{C}^2([0, N], \mathbb{R})$ such that*

(1) $F''(0) = F''(N) = \frac{1}{Q^3}$
(2) $\frac{1}{2Q^3} \leq F''(x) \leq \frac{161}{Q^3}$ *for* $x \in [0, N]$
(3) $F'(0) = -\frac{1}{2N(N+1)}$, $F'(N) = \frac{1}{2N(N-1)}$
(4) $F(0) = F(N) = 0$.

Proof. It is sufficient to construct two functions $G, H \in \mathscr{C}^2([0, N], \mathbb{R})$ that satisfy conditions (1) to (3), $G(0) = H(0) = 0$ and such that $H(N) < 0 < G(N)$. Indeed, the function F defined by

$$F(x) := \frac{G(N)H(x) - H(N)G(x)}{G(N) - H(N)} \quad (x \in [0, N])$$

then satisfies conditions (1)–(4).

We now construct a function G. Let

$$A = A(Q, N) := \frac{3\big(Q^3 - N(N^2-1)\big)}{2N(N^2-1)} > 0.$$

Note that

$$0 < A(Q, N) \le 80 \quad \text{for } Q \ge 19,\ Q/3 \le N \le Q.$$

For such a choice of A, the function defined by

$$G(x) = -\frac{x}{2N(N+1)} + \int_0^x (x-t)\Big(\frac{1+A}{Q^3} - \frac{4A}{N^2Q^3}\Big(t - \frac{N}{2}\Big)^2\Big)\mathrm{d}t \quad (x \in [0, N])$$

satisfies conditions (1)–(3) and $G(0) = 0$. Moreover,

$$G(N) = \frac{1}{2(N^2-1)} > 0.$$

We now construct a function H. Let

$$B = B(Q, N) := \frac{3Q^3}{N(N^2-1)} - \frac{5}{2} > 0.$$

Note that

$$0 < A(Q, N) \le 160 \text{ for } Q \ge 19,\ Q/3 \le N \le Q$$

and

$$h(x) := \begin{cases} \dfrac{1}{2Q^3} + \dfrac{8}{N^2Q^3}\Big(x - \dfrac{N}{4}\Big)^2 & \text{if } 0 \le x \le N/2 \\ \dfrac{1+B}{Q^3} - \dfrac{16B}{N^2Q^3}\Big(x - \dfrac{3N}{4}\Big)^2 & \text{if } N/2 < x \le N. \end{cases}$$

For such a choice of B, the function defined by

$$H(x) := -\frac{x}{2N(N+1)} + \int_0^x (x-t)h(t)\mathrm{d}t \quad (x \in [0, N])$$

satisfies conditions (1)–(3) and $H(0) = 0$. Moreover,

$$H(N) = -\frac{1}{2(N+1)} + \frac{1}{4(N^2-1)} + \frac{3N^2}{16Q^3} < 0$$

for $Q \ge 3$ and $Q/3 \le N \le Q$, which completes the proof. □

Set

$$M_Q := \frac{(2Q - \lfloor Q/3 \rfloor)(\lfloor Q/3 \rfloor + 1)}{2} \quad (Q \ge 3). \tag{39}$$

Theorem 6. *Let $Q \ge 19$ be an integer. There exists a $\mathscr{C}^2$ function $f : [0, M_Q] \to \mathbb{R}$ such that*

$$\frac{1}{2Q^3} \le f''(x) \le \frac{161}{Q^3} \quad (x \in [0, M_Q])$$

and such that

$$\#\{m \in [0, M_Q] \cap \mathbb{Z} \colon f(m) \in \mathbb{Z}\} \ge \lfloor Q/3 \rfloor.$$

Proof. Let $Q \geq 19$ and $Q/3 \leq N \leq Q$. We denote by $F_{Q,N}$ the function constructed in Lemma 6. We set

$$\widetilde{M}_j := \tfrac{1}{2}(j+1)(2Q-j) \quad (j \in \mathbb{N} \cup \{-1, 0\})$$

so that in particular $\widetilde{M}_j - \widetilde{M}_{j-1} = Q - j$ for $j \geq 0$, and with the notation (39) $\widetilde{M}_j = M_Q$ for $j = \lfloor Q/3 \rfloor$.

The function f is now defined as follows: we set $f(0) := 0$ and for $0 \leq j \leq \lfloor Q/3 \rfloor$ we set

$$f(x) := j + \frac{x - \widetilde{M}_{j-1}}{Q-j} + F_{Q,Q-j}(x - \widetilde{M}_{j-1}) \quad (\widetilde{M}_{j-1} < x \leq \widetilde{M}_j).$$

It is now clear that $f(\widetilde{M}_{j-1}) = j$ for $0 \leq j \leq \lfloor Q/3 \rfloor$. Moreover, the function f has the expected regularity. For instance, using Lemma 6, one easily checks that

$$f'(\widetilde{M}_{j-1}) = \frac{2(Q-j)+1}{2(Q-j)(Q-j+1)} \qquad f''(\widetilde{M}_{j-1}) = \frac{1}{Q^3} \quad (0 \leq j \leq \lfloor Q/3 \rfloor). \quad \square$$

Remark. For any $Q \geq 19$, choosing $\lambda_2 := 1/Q^3$, the function $f = f_Q$ constructed above is $\mathscr{C}^2$ on $[1, M]$ for some $M \asymp \lambda_2^{-2/3}$, satisfies $f''(x) \asymp \lambda_2$ for $x \in [1, M]$ and

$$\mathscr{R}_M(f) \gg M\lambda_2^{1/3},$$

which confirms that the bound in (38) is sharp. We should mention that the hypothesis of Grekos's result resembles $\lambda_2^{-2/3} \leq M \leq \lambda_2^{-1}$. Here, we have only considered the case $M \asymp \lambda_2^{-2/3}$ for which the discussion of Farey sequence is straightforward. However, it would be of interest to study whether Grekos' proof can be adapted for $M \asymp \lambda_2^{-1}$, still under the assumption (6).

We shall now answer Question 3 of Section 7.

Corollary 1. *For infinitely many positive integers M, we have the following: there exists a $\mathscr{C}^2$ function $g : [1, M] \to \mathbb{R}$ such that*

$$g''(x) \asymp \lambda_2 := \frac{1}{M} \quad (1 \leq x \leq M)$$

and an integer $H \asymp M^{1/2}$ such that

$$\frac{1}{H}\Big|\sum_{h \asymp H}\Big(1 - \frac{h}{H}\Big)\sum_{m=1}^{M} \mathrm{e}\Big(\frac{h}{H}g(m)\Big)\Big| \gg M\lambda_2^{1/2},$$

where every constant implied is absolute.

It follows that the answer to Question 3 is NO. The result is even stronger: Question 3 is about sums of the type $\sum_h |\sum_m|$, but here we have proven that even in the case of a sum of the type $\sum_h c_h \sum_m$ with c_h smooth, there are cases where no further cancellation occurs.

Proof. For any $Q \geq 19$, we denote by f_Q the function constructed in Theorem 6. For any $H \geq 1$, (38) yields

$$\lfloor Q/3 \rfloor \leq \mathscr{R}_{M_Q}(f_Q) \leq \frac{2M_Q}{H} + \frac{4}{H}\mathrm{Re}\sum_{h=1}^{H-1}\Big(1 - \frac{h}{H}\Big)\sum_{m=1}^{M_Q} \mathrm{e}(hf_Q(m)).$$

Choosing absolute constants $0 < c_1 < c_2$ such that $2M_Q/H \le \frac{1}{3}\lfloor Q/3\rfloor$ and such that Theorem 1 implies

$$\frac{4}{H}\sum_{1\le hc_1Q}\Big|\sum_{m=1}^{M_Q}\mathrm{e}(hf_Q(m))\Big| \le \tfrac{1}{3}\lfloor Q/3\rfloor,$$

we have

$$\frac{1}{Q}\Big|\sum_{c_1Q<h\le c_2Q}\Big(1-\frac{h}{H}\Big)\sum_{m=1}^{M_Q}\mathrm{e}(hf_Q(m))\Big| \ge \lfloor Q/3\rfloor.$$

Hence, for any Q large enough, the function $g : x \mapsto Qf_Q(x)$ is $\mathscr{C}^2$ on $[1, M]$ with $M = M_Q \asymp Q^2$ and satisfies $g''(x) \asymp \lambda_2 := 1/M$ for $x \in [1, M]$ and

$$\frac{1}{Q}\Big|\sum_{c_1Q<h\le c_2Q}\Big(1-\frac{h}{H}\Big)\sum_{m=1}^{M}\mathrm{e}\Big(\frac{h}{Q}g(m)\Big)\Big| \gg Q \asymp M\lambda_2^{1/2}. \quad \square$$

10. Improvements in the case $k = 3$

We recall Theorem 2: For any $M \ge 1$ and $0 < \lambda_3 < 1$, and any $\mathscr{C}^3$ function $f : [1, M] \to \mathbb{R}$ such that $|f'''(x)| \asymp \lambda_3$ for $x \in [1, M]$, we have

$$\sum_{m=1}^{M}\mathrm{e}(f(m)) \ll M\lambda_3^{\theta_3} \quad \text{for } M \ge \lambda_3^{-\beta_3}$$

where $\theta_3 := 1/6$ and $\beta_3 = 2/3$.

Unlike the analogue for the second derivative test, it turns out that β_3 may be replaced by $1/2$, and thus the answer to Question 1 of Section 7 is YES. This result has been proven independently by Sargos and Gritsenko by different methods (see Corollaire 4.2 of [21] and Theorem of [8]). Their result is the following:

Theorem 7 (*Sargos (1995), Gritsenko (1996)*)**.** *For any $M \ge 1$, any $0 < \lambda_3 < 1$, and any $\mathscr{C}^3$ function $f : [1, M] \to \mathbb{R}$ such that $|f'''(x)| \asymp \lambda_3$ for $x \in [1, M]$, we have*

$$\sum_{m=1}^{M}\mathrm{e}(f(m)) \ll M\lambda_3^{1/6} + \lambda_3^{-1/3}.$$

Remark. The couple $(\frac{1}{6}, \frac{1}{2})$ is now a pure van der Corput 3-couple.

The sketch of the proof of Sargos is as follows: choose an appropriate rational approximation of $\frac{1}{2}f''(1)$ by $\frac{s}{q}$, say, and apply the $\widetilde{B}$-transform (Theorem 5) to the phase $x \mapsto f(x) - \frac{s}{q}x^2$.

We now study Question 2 of Section 7.

Lemma 7. *For each $M \ge 1$ we have*

$$\sum_{M<m\le 2M}\mathrm{e}\Big(\frac{2}{3\sqrt{3}}m^{3/2}\Big) = \frac{2^{3/2}(2^{3/4}-1)}{3^{5/4}}M^{3/4} + O(M^{3/5}).$$

Remark. In particular, for each $M \geq 1$, the function $f_M : x \mapsto \frac{2}{3\sqrt{3}}(M+x)^{3/2}$ satisfies $|f'''(x)| \asymp \lambda_3 := M^{-3/2}$ for $1 \leq x \leq M$ and

$$\sum_{m=1}^{M} \mathrm{e}(f(m)) \gg M\lambda_3^{1/6}.$$

Proof. Theorem 4 implies

$$\sum_{M<m\leq 2M} \mathrm{e}\Big(\frac{2}{3\sqrt{3}}m^{3/2}\Big) = \sum_{H_1\leq \nu\leq H_2} \sqrt{6\nu} + O(M^{3/5})$$

where $H_1 := M^{1/2}/\sqrt{3}$ and $H_2 := (2M)^{1/2}/\sqrt{3}$, and the classical estimate

$$\sum_{H_1\leq \nu\leq H_2} \sqrt{\nu} = \int_{H_1}^{H_2} \sqrt{t}\mathrm{d}t + O(M^{1/4})$$

gives the result. □

The previous lemma shows that the exponent θ_3 is optimal and thus, the answer to Question 2 is NO. However, one may refine the original question. Since the counter-example given here satisfies $M \asymp \lambda_3^{-2/3}$, it would be interesting to decide whether the exponent θ_3 is still optimal for larger sums, e.g. in the case $M \asymp \lambda_3^{-1}$. Indeed, large sums might be subject to some cancellation.

However, it has been conjectured by Sargos (see [21]) that even for large sums the exponent θ_3 should be optimal:

Conjecture 1 (*Sargos*)**.** *With the notation of Definition 1, if the pair* $(\theta, 1)$ *is a van der Corput* 3-*couple, then* $\theta \leq 1/6$.

Simultaneously, Sargos had conjectured that if one adds to the condition $|f'''(x)| \asymp \lambda_3$ ($x \in [1, M]$) the condition that f''' is monotonous, then the exponent θ_3 is no longer optimal for $M \asymp \lambda_3^{-1}$. With this extra condition, this second conjecture has been proven by the author.

Theorem 8 (*Théorème 1 of [17]*)**.** *Let* $M \geq 1$, $0 < \lambda_3 < 1$, *and let* $f : [1, M] \to \mathbb{R}$ *be* $\mathscr{C}^3$ *such that* $|f'''(x)| \asymp \lambda_3$ *for* $x \in [1, M]$. *If* f''' *is monotonous, then*

$$\sum_{m=1}^{M} \mathrm{e}(f(m)) \ll M\lambda_3^{\theta} \quad (M \geq \lambda_3^{-1})$$

where $\theta := \frac{1}{6} + \frac{1}{1354}$.

Remark. In the previous theorem, both the monotony of f''' and the size of M are crucial. Indeed, for $M \asymp \lambda_3^{-2/3}$, the phase in Lemma 7 has a monotonous third derivative, and yet the exponent θ_3 is optimal.

We now consider Question 3. This time, up to a term M^{ε}, the answer is YES.

Theorem 9 (*Theorem 1 of [20]*)**.** *Let* $M, H \geq 1$, $0 < \lambda_3 < 1$, *and let* $f : [1, M] \to \mathbb{R}$ *be* $\mathscr{C}^3$ *such that* $|f'''(x)| \asymp \lambda_3$ *for* $x \in [1, M]$. *Then for every* $\varepsilon > 0$, *we have*

$$\frac{1}{H}\sum_{H<h\leq 2H}\Big|\sum_{m=1}^{M} \mathrm{e}\Big(\frac{h}{H}f(m)\Big)\Big| \ll_{\varepsilon} M^{\varepsilon}\Big(\frac{M\lambda_3^{1/6}}{H^{1/9}} + M\lambda_3^{1/5} + M^{3/4}\Big) + \lambda_3^{-1/3}.$$

The proof uses Lemma 4 on the double sum $\sum_h \sum_m$, then the Weyl shift on the variable m, and the double large sieve. The corresponding diophantine system is (27).

11. Improvement for the case $k = 4$

11.1. Question 1

We recall that with Definition 1, Theorem 3 for $k = 4$ implies that $\left(\frac{1}{14}, \frac{4}{7}\right)$ is a van der Corput 4-couple.

By applying the A-transform and Theorem 7, it follows that $\left(\frac{1}{14}, \frac{3}{7}\right)$ is a van der Corput 4-couple, so that the answer to Question 1 is YES.

11.2. Question 2

This time, the answer to Question 2 is YES, and several results have been proven in that direction in the last two decades:

(1) Théorème 2.4 of [21] implies that $\left(\frac{3}{40} - \varepsilon, \frac{3}{5}\right)$ is a van der Corput 4-couple. The proof uses Weyl shift (22), then the double large sieve. The corresponding system is (28).
(2) Theorem 1 of [19] implies that $\left(\frac{1}{13} - \varepsilon, \frac{8}{13}\right)$ is a van der Corput 4-couple. The proof uses the A-transform of Lemma 2, followed by Lemma 4 on the double sum $\sum_h \sum_m$, the Weyl shift on the variable m, and the double large sieve. The corresponding diophantine system is (27).
(3) Theorem of [14] implies that $\left(\frac{1}{12} - \varepsilon, 1\right)$ is a van der Corput 4-couple. The proof uses Weyl shift and the double large sieve. The corresponding diophantine system is (24) for $k = 3$ with twelve variables.

Remark. The three last couples (θ, β) may not be compared directly to van der Corput's original result. Indeed, although the exponent θ has been improved, the couples correspond to larger values of β, hence to longer exponential sums.

12. Improvement for the case $k \geq 5$

It now clear that for $k \geq 5$, the answer to Questions 1 and 2 is YES. Indeed, using $(k - 4)$ times the A-process and any of the van der Corput 4-couple described in the previous section gives new k-couples.

We mention here a genuine new 5-couple which is not deduced from the previous 4-couples: indeed Theorem 1 of [22] implies that for any $\varepsilon > 0$, the pair $\left(\frac{7}{192} - \varepsilon, \frac{7}{16}\right)$ is a van der Corput 5-couple. The proof uses Lemma 3, the double large sieve and the results on the system (29).

As mentioned before, using iterations of the A-transform gives rise to many van der Corput k-couples. However, the A-transform involves Cauchy's inequality, so after the process A^r, the saving in the new sum has to be taken at a power $1/2^r$.

In this direction, analogues of the processes A^4 and A^5 have been investigated in [23,15], using diophantine systems. About the process in [15], we would like to mention that the result involves a power M^ε for any $\varepsilon > 0$ that may removed by using Theorem 1 of [4] instead of Theorem 1 of [15] and Theorem 1 of [5] instead of Lemma 2 of [19].

To conclude the exposition for general k, we generalise the proof in [14] by using new results for the Vinogradov system. Our results are the following:

Theorem 10. *Let $k \geq 4$, and let $f : [1, M] \to \mathbb{R}$ be a $\mathscr{C}^k$ function such that*

$$|f^{(k)}(x)| \asymp \lambda_k \quad \text{for } x \in [1, M].$$

Then

$$\sum_{m=1}^{M} \mathrm{e}(f(m)) \ll_\varepsilon M^{1+\varepsilon} \lambda_k^{\frac{1}{2(k-1)(k-2)}} + M^{1-\frac{1}{2(k-1)(k-2)}+\varepsilon} \quad \text{for } M \geq \lambda_k^{-(k-1)/(2k-3)}.$$

Proof. Use Lemma 1 of [16] and Theorem 1.2 of [32]. □

Remark. This result implies that for $k \geq 4$, the pair $\left(\frac{1}{2(k-1)(k-2)} - \varepsilon, 1\right)$ is a van der Corput k-couple for any $\varepsilon > 0$.

Theorem 11. *Let $k \geq 5$, and let $f : [1, M] \to \mathbb{R}$ be a $\mathscr{C}^k$ function such that*

$$|f^{(k)}(x)| \asymp \lambda_k \quad \text{for } x \in [1, M].$$

Then

$$\sum_{m=1}^{M} \mathrm{e}(f(m)) \ll_\varepsilon M^{1+\varepsilon} \lambda_k^{\frac{1}{2(2k-3)(k-3)}} \quad \text{for } M \geq \lambda_k^{-(k-1)/(2k-3)}.$$

Proof. Use Lemma 2 of [16] and Theorem 1.2 of [32]. □

Remark. This result implies that for $k \geq 5$, the pair $\left(\frac{1}{2(2k-3)(k-3)} - \varepsilon, \frac{k-1}{2k-3}\right)$ is a van der Corput k-couple for any $\varepsilon > 0$.

13. Reformulations and open questions

13.1. A reformulation towards Sargos' conjecture

Let $\alpha, \beta \in \mathbb{R}$ such that $0 < \alpha < \beta$. We consider the set $\mathcal{F}_3(\alpha, \beta)$ of the $\mathscr{C}^3$ functions $F : [0, 1] \to \mathbb{R}$ such that

$$\alpha \leq F'''(x) \leq \beta \quad \text{for } x \in [0, 1].$$

For each $\kappa \in \mathbb{R}$, $F \in \mathbb{R}^{[0,1]}$ and $M \geq 1$, we set

$$S_F(M; \kappa) := \sum_{1 \leq m \leq M} \mathrm{e}\Big(M^\kappa F\Big(\frac{m}{M}\Big)\Big).$$

We are interested in the sum $S_F(M; 2)$. Van der Corput's third derivative test asserts that for some $C(\alpha, \beta) > 0$ we have

$$\sup_{F \in \mathcal{F}_3(\alpha,\beta)} |S_F(M; 2)| \leq C(\alpha, \beta) M^{5/6} \quad (M \geq 1).$$

We consider the following real number

$$\mathscr{S}(\alpha, \beta) := \limsup_{M \to +\infty} M^{-5/6} \sup_{F \in \mathcal{F}_3(\alpha,\beta)} |S_F(M; 2)|.$$

Question. *Is $\mathscr{S}(\alpha, \beta)$ a positive number for some $0 < \alpha < \beta$?*

13.2. About the fourth derivative test

We consider the couples

$$V_k := \left(\frac{1}{k(k-1)}, \frac{1}{k-1}\right) \quad (k \geq 2).$$

We recall the following statements from the previous sections:

- The couple V_2 is a pure van der Corput 2-couple (deduced from van der Corput' inequality), and it is proven to be optimal.
- The couple V_3 is a pure van der Corput 3-couple (deduced from Theorem 7), and the exponents are optimal.

Question. *Is V_4 a van der Corput 4-couple? This would imply that for any $\mathscr{C}^4$ function $f : [1, M] \to \mathbb{R}$ such that $|f^{(4)}(x)| \asymp \lambda_4 > 0$ $(x \in [1, M])$ we would have*

$$\sum_{m=1}^{M} \mathrm{e}(f(m)) \ll M\lambda_4^{1/12} + \lambda_4^{-1/4}.$$

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