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# On the fourth derivative test for exponential sums 

Olivier Robert


#### Abstract

We give an upper bound for the exponential sum $\sum_{m=1}^{M} \exp (2 i \pi f(m))$ where $f$ is a real valued function whose fourth derivative has the order of magnitude $\lambda>0$ small. Van der Corput's classical bound, in terms of $M$ and $\lambda$ only, involves the exponent $1 / 14$. We show how this exponent may be replaced by any $\theta<1 / 12$ without further hypotheses. The proof uses a recent result by Wooley on the cubic Vinogradov system.


Let $C \geq 1$ be a real constant fixed once and for all. Let $a \in \mathbb{R}, M \in \mathbb{N}$ with $M \geq 10$. Let $0<\lambda_{4}<1 / 10$ be a real number and let $f:[a, a+M] \rightarrow \mathbb{R}$ be a $C^{4}$ function such that

$$
\begin{equation*}
\lambda_{4} \leq\left|f^{(4)}(x)\right| \leq C \lambda_{4} \text { for } x \in[a, a+M] \tag{1}
\end{equation*}
$$

We are interested in bounding the exponential sum

$$
\begin{equation*}
S_{a, M}(f):=\sum_{a<m \leq a+M} e(f(m)) \tag{2}
\end{equation*}
$$

with the usual notation $e(t):=e^{2 i \pi t}$ for $t \in \mathbb{R}$. Under the previous assumptions, van der Corput's result asserts that setting $\theta_{4}=1 / 14$ and $\beta_{4}=4 / 7$, one has

$$
\begin{equation*}
S_{a, M}(f) \ll_{C} M \lambda_{4}^{\theta_{4}} \tag{3}
\end{equation*}
$$

uniformly for $a \in \mathbb{R}, M \geq \lambda_{4}^{-\beta_{4}}$ and $f$ satisfying (1).
The purpose of this paper is to improve on the exponent $\theta_{4}$ in (3). Our result is the following.

Theorem 1 Let $a \in \mathbb{R}, M \in \mathbb{N}$ with $M \geq 10$. Let $0<\lambda_{4}<1 / 10$ be a real number and let $f:[a, a+M] \rightarrow \mathbb{R}$ be a $C^{4}$ function satisfying (1). Then for each $\varepsilon>0$, and with the notation (2), one has

$$
S_{a, M}(f) \ll_{C, \varepsilon} M^{1+\varepsilon} \lambda_{4}^{1 / 12}+M^{\frac{11}{12}+\varepsilon}
$$

uniformly for $a \in \mathbb{R}, M \geq \lambda_{4}^{-3 / 5}$ and $f$ satisfying (1).
Remark. Theorem 1 is not the first improvement of Van der Corput's result. Here is a brief history of these results under the hypothesis (1) alone :

- Sargos [4] proved that for $\theta_{4}=1 / 14, \beta_{4}$ may be replaced by $3 / 7$. Moreover, in the same paper, he also proved that $\theta_{4}$ could be replaced by any $\theta<\frac{3}{40}$ provided that $\beta_{4}$ is replaced by $3 / 5$.
- Robert \& Sargos [1] proved $\theta_{4}$ may be replaced by any $\theta<\frac{1}{13}$ provided that $\beta_{4}$ is replaced by $8 / 13$.

Note that our Theorem 1 implies that $\theta_{4}$ may be replaced by any $\theta<1 / 12$ provided that $\beta_{4}$ is replaced by 1 .

It should also be mentioned that more generally when $k \geq 2$ is fixed, van der Corput proved that if $f:[a, a+M] \rightarrow \mathbb{R}$ is $C^{k}$ such that $\lambda_{k} \leq\left|f^{(k)}(x)\right| \leq C \lambda_{k}$, then $S_{a, M}(f)<_{C} M \lambda_{k}^{\theta_{k}}$ for $M \geq \lambda_{k}^{-\beta_{k}}$ with $\theta_{k}:=1 /\left(2^{k}-2\right), \beta_{k}:=2^{k-1} /\left(2^{k}-\right.$ $2)$. For $k=2$, the example of the Gauss sums shows at once that $\theta_{2}=1 / 2$ is optimal. For $k=3$, the exponent $\theta_{3}=1 / 6$ is conjectured to be optimal even for $M \geq \lambda_{3}^{-1}$ (see [4]). For $k \geq 5$, explicit better exponents can be produced (see for example [3] for an exposition of the recent results for $4 \leq k \leq 10$ ).

Proof of Theorem 1. By a classical series of transforms, the proof reduces to estimating the number $J_{s}(N)$ of solutions of

$$
\begin{equation*}
\sum_{j=1}^{s}\left(x_{j}^{r}-y_{j}^{r}\right)=0 \quad(r=1,2,3) \quad 1 \leq x_{j}, y_{j} \leq N \tag{4}
\end{equation*}
$$

The only novelty in the proof of Theorem 1 is the use of Wooley's recent breakthrough [5] where he proved Vinogradov's cubic conjecture, namely that for each integer $s \geq 1$ and each $\varepsilon>0$ one has

$$
\begin{equation*}
J_{s}(N) \lll{ }_{\varepsilon} N^{s+\varepsilon}+N^{2 s-6+\varepsilon} \quad(N \geq 1) \tag{5}
\end{equation*}
$$

The preliminary transform of $S_{a, M}(f)$ is taken care of by the following result, which is a particular case of Lemme 1 of [3] with $k=4$.

Lemma 1 Let $a \in \mathbb{R}, M \in \mathbb{N}$ with $M \geq 10$. Let $0<\lambda_{4}<1 / 10$ be a real number and let $f:[a, a+M] \rightarrow \mathbb{R}$ be a $C^{4}$ function satisfying (1). Then setting $N:=\left\lfloor\lambda_{4}^{-1 / 5}\right\rfloor$, and with the notation (2), one has for each $s \geq 1$

$$
\left|S_{a, M}(f)\right|^{2 s} \ll s, C M^{2 s} \lambda_{4}\left(1+\frac{1}{M \lambda_{4}}\right)\left(\frac{J_{s}(N)}{N^{2 s-6}}\right) \log M
$$

uniformly for $a \in \mathbb{R}, M \geq \lambda_{4}^{-3 / 5}$ and $f$ satisfying (1), where $J_{s}(N)$ has been defined in (4).

The proof of Theorem 1 is now straightforward : we use Lemma 1 with $s=6$, and we bound $J_{6}(N)$ using (5). This completes the proof of Theorem 1.

## References

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