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| Jonathan Chappelon. Periodic balanced binary triangles. 2017. hal-01463665v1

HAL Id: hal-01463665

<https://hal.science/hal-01463665v1>

Preprint submitted on 9 Feb 2017 (v1), last revised 23 Nov 2017 (v3)

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Periodic balanced binary triangles

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February 9, 2017

Abstract

A binary triangle of size n is a triangle of zeroes and ones, with n rows, built with the same local rule as the standard Pascal triangle modulo 2. A binary triangle is said to be balanced if the absolute difference between the numbers of zeroes and ones that constitute this triangle is at most 1. In this paper, the existence of balanced binary triangles of size n , for all positive integers n , is shown. This is achieved by considering periodic balanced binary triangles.

Keywords: binary triangles, Steinhaus triangles, generalized Pascal triangles, balanced triangles, Steinhaus Problem, periodic triangles, periodic orbits.

MSC2010: 05B30, 11B75, 05A05, 11A99, 05A99.

1 Introduction

The *Steinhaus triangle* ∇S associated to the finite sequence $S = (a_0, a_1, \dots, a_{n-1})$, of length $n \geq 1$ in $\mathbb{Z}/2\mathbb{Z}$, is the triangle generated from S by the same local rule than the standard Pascal triangle modulo 2, that is the doubly indexed sequence $\nabla S = (a_{i,j})_{0 \leq i \leq j \leq n-1}$ defined by:

- i) $a_{0,j} = a_j$, for all $0 \leq j \leq n-1$,
- ii) $a_{i,j} = a_{i-1,j-1} + a_{i-1,j}$, for all $1 \leq i \leq j \leq n-1$.

Note that the sum in ii) is the sum modulo 2. This kind of binary triangles was introduced in [14] by Hugo Steinhaus himself. For example, the Steinhaus triangle ∇S associated to $S = 0010100$ is depicted in Figure 1.

The *generalized Pascal triangle* $\Delta(S_1, S_2)$ associated to the finite sequences $S_1 = (a_0, a_1, \dots, a_{n-1})$ and $S_2 = (b_0, b_1, \dots, b_{n-1})$, of length $n \geq 1$ in $\mathbb{Z}/2\mathbb{Z}$ and with $a_0 = b_0$, is the doubly indexed sequence $\Delta(S_1, S_2) = (a_{i,j})_{0 \leq j \leq i \leq n-1}$ defined by:

- i) $a_{i,0} = a_i$ and $a_{i,i} = b_i$, for all $0 \leq i \leq n-1$,
- ii) $a_{i,j} = a_{i-1,j-1} + a_{i-1,j}$, for all $1 \leq j < i \leq n-1$.

For example, the generalized Pascal triangle $\Delta(S_1, S_2)$ associated to $S_1 = 0000101$ and $S_2 = 0100001$ is depicted in Figure 1. Moreover, note that, for the constant binary sequences $S_1 = S_2 = 11 \cdots 1$ of size n , the triangle $\Delta(S_1, S_2)$ corresponds to the first n rows of the standard Pascal triangle modulo 2, the Sierpinski triangle.

In this paper, a *binary triangle* is either a Steinhaus triangle or a generalized Pascal triangle. The *size* of a binary triangle is the number of rows that constitute this triangle.

For any binary triangle T , let \mathbf{m}_T denote its multiplicity function, that is, the function $\mathbf{m}_T : \mathbb{Z}/2\mathbb{Z} \rightarrow \mathbb{N}$ that assigns to each element $x \in \mathbb{Z}/2\mathbb{Z}$ its multiplicity in T . The triangle T is said to be *balanced* if its multiplicity function is constant or almost constant, i.e., if the multiplicity difference

$$\delta \mathbf{m}_T := |\mathbf{m}_T(1) - \mathbf{m}_T(0)|$$

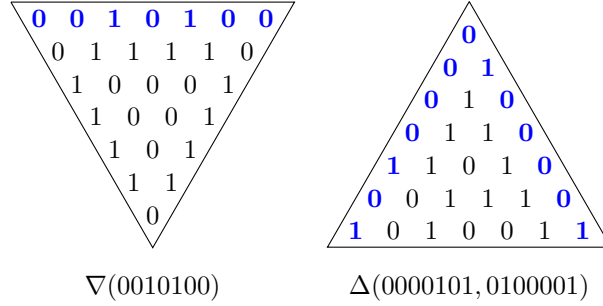


Figure 1: Binary triangles

is such that $\delta \mathbf{m}_T \in \{0, 1\}$. Since they contain 14 zeroes and 14 ones, the triangles depicted in Figure 1 are balanced binary triangles of size 7.

The goal of this paper is to prove that there exist balanced binary triangles of size n , for all positive integers n and for the both kinds of binary triangles. This completely solves a generalization of a problem posed in 1964 by Hugo Steinhaus [14].

Steinhaus Problem. Does there exist, for any positive integer $n \equiv 0$ or $3 \pmod{4}$, a binary sequence S of length n for which the associated triangle ∇S contains as many zeroes as ones?

Remark that a binary triangle of size n contains $\binom{n+1}{2}$ elements. Therefore the condition $n \equiv 0$ or $3 \pmod{4}$ is a necessary and sufficient condition for having a triangle of size n containing an even number of terms.

The Steinhaus Problem was solved for the first time by Heiko Harborth in 1972 [12]. In his paper, Harborth constructively showed that, for every positive integer $n \equiv 0$ or $3 \pmod{4}$, there exist at least four binary sequences S of length n such that ∇S is balanced. Since then, many solutions have appeared [9, 10, 11]. All of them are constructive and correspond to the search of sequences generating balanced triangles, that have some additional properties such as being antisymmetric or zero-sum.

The possible number of ones in a binary triangle was explored in [3]. The minimum number of ones is obviously 0 since the triangle of zeroes of size n is always a binary triangle. Since a Steinhaus triangle of size 2 contains either two ones and one zero, or three zeroes, it follows that the maximum number of ones in a binary triangle of size n is at least $\frac{2}{3} \binom{n+1}{2}$. The following result gives the average number of ones and zeroes in binary triangles.

Proposition 1.1. *The average number of ones and zeroes in a binary triangle of size n is exactly $\frac{1}{2} \binom{n+1}{2}$.*

Proof. By induction on $n \geq 1$.

First, for the Steinhaus triangles. For $n = 1$, the result is trivial. Suppose now that $n \geq 2$ and that the result is true for any Steinhaus triangle of size $n - 1$. Let ∇S be the Steinhaus triangle of size $n - 1$ generated from the sequence $S = (a_0, a_1, \dots, a_{n-2})$. There exist exactly two sequences S' of length n such that we retrieve ∇S as the subtriangle $\nabla S' \setminus S'$, that is, the last $n - 1$ rows of the Steinhaus triangle $\nabla S'$ of size n . These sequences S' are of the form

$$S' = \left(x, x + a_0, x + a_0 + a_1, \dots, x + \sum_{j=0}^{i-1} a_j, \dots, x + \sum_{j=0}^{n-2} a_j \right),$$

with $x \in \mathbb{Z}/2\mathbb{Z}$. Moreover, for all positive integers m , it is clear that there are 2^m binary sequences of length m and the same number of Steinhaus triangles of size m . It follows that, for all $x \in \mathbb{Z}/2\mathbb{Z}$, the total number of x in the set of all the Steinhaus triangles of size n is the sum of twice the total number of x in the set of Steinhaus triangles of size $n - 1$ and the total number of x in the set of sequences of length n . This leads to the result that the average number of x in a Steinhaus triangle of size n is

$$\frac{1}{2^n} \left(2 \times 2^{n-1} \times \frac{1}{2} \binom{n}{2} + 2^{n-1} n \right) = \frac{1}{2} \binom{n+1}{2},$$

for all $x \in \mathbb{Z}/2\mathbb{Z}$.

Now, for the generalized Pascal triangles. For $n = 1$ and $n = 2$, the result is clear. Suppose now that $n \geq 3$ and that the result is true for any generalized Pascal triangle of size $n - 2$. Let $\Delta(S_1, S_2)$

be the generalized Pascal triangle of size $n - 2$ generated from the sequences $S_1 = (a_0, a_1, \dots, a_{n-3})$ and $S_2 = (b_0, b_1, \dots, b_{n-3})$, with $a_0 = b_0$. There exist exactly 2^4 couples (S'_1, S'_2) of sequences of length n such that we retrieve $\Delta(S_1, S_2)$ as the subtriangle $\Delta(S'_1, S'_2) \setminus (S'_1 \cup S'_2 \setminus \{a_0\})$, that is, the generalized Pascal triangle obtained from $\Delta(S'_1, S'_2)$ by removing the left and right sides of the triangle. These couples of sequences (S'_1, S'_2) are of the form

$$\begin{aligned} S'_1 &= (x_1, x_2, a_0 + a_1, a_1 + a_2, \dots, a_{n-4} + a_{n-3}, x_3), \\ S'_2 &= (x_1, x_2 + a_0, b_0 + b_1, b_1 + b_2, \dots, b_{n-4} + b_{n-3}, x_4), \end{aligned}$$

where $x_1, x_2, x_3, x_4 \in \mathbb{Z}/2\mathbb{Z}$. Moreover, for all positive integers m , it is clear that there are 2^{2m-1} binary sequences of length $2m - 1$ and the same number of generalized Pascal triangles of size m . It follows that, for all $x \in \mathbb{Z}/2\mathbb{Z}$, the total number of x in the set of all the generalized Pascal triangles of size n is the sum of 2^4 times the total number of x in the set of generalized Pascal triangles of size $n - 2$ and the total number of x in the set of sequences of length $2n - 1$. This leads to the result that the average number of x in a generalized Pascal triangle of size n is

$$\frac{1}{2^{2n-1}} \left(2^4 \times 2^{2(n-2)-1} \times \frac{1}{2} \binom{n-1}{2} + 2^{2n-2}(2n-1) \right) = \frac{1}{2} \binom{n+1}{2},$$

for all $x \in \mathbb{Z}/2\mathbb{Z}$. This completes the proof. \square

This result shows that the Steinhaus Problem and the following generalization are *natural*.

Problem 1. Does there exist, for any positive integer n , a balanced Steinhaus triangle and a balanced generalized Pascal triangle of size n ?

As already announced before, this problem is positively solved in this paper. The solution presented here is constructive and based on the analysis of periodic balanced binary triangles.

Let us begin with some definitions and terminology. Let $S = (a_j)_{j \in \mathbb{Z}}$ be a doubly infinite sequence of $\mathbb{Z}/2\mathbb{Z}$. The *derived sequence* ∂S is the sequence obtained by pairwise adding consecutive terms of S , that is, the sequence defined by

$$\partial S = (a_{j-1} + a_j)_{j \in \mathbb{Z}}.$$

This derivation process can be iterated and, for every positive integer i , the i^{th} derived sequence $\partial^i S$ is recursively defined by $\partial^i = \partial(\partial^{i-1} S)$ with $\partial^0 S = S$. The *orbit* \mathcal{O}_S is the sequence of all the iterated derived sequences of S , that is,

$$\mathcal{O}_S = (\partial^i S)_{i \in \mathbb{N}}.$$

The orbit of S can also be seen as the doubly indexed sequence $\mathcal{O}_S = (a_{i,j})_{(i,j) \in \mathbb{N} \times \mathbb{Z}}$ defined by:

- i) $a_{0,j} = a_j$, for all $j \in \mathbb{Z}$,
- ii) $a_{i,j} = a_{i-1,j-1} + a_{i-1,j}$, for all $i \geq 1$ and for all $j \in \mathbb{Z}$.

An example of orbit \mathcal{O}_S associated with the sequence

$$S = (\dots, 0, 0, 1, 0, 1, 0, 1, 1, 0, 0, 0, 1, 1, 0, 0, 0, 1, 0, 0, 1, 1, 1, 0, \dots)$$

is depicted in Figure 2.

Binary triangles can then be considered as appearing in orbits of binary sequences. Let $\nabla S(i_0, j_0, n)$ denote the triangle build from the base to the top, whose *principal vertex* is at the position $(i_0, j_0) \in \mathbb{N} \times \mathbb{Z}$ in the orbit $\mathcal{O}_S = (a_{i,j})_{(i,j) \in \mathbb{N} \times \mathbb{Z}}$ and of *size* n , i.e., the Steinhaus triangle

$$\nabla S(i_0, j_0, n) = \nabla(a_{i_0, j_0}, a_{i_0, j_0+1}, \dots, a_{i_0, j_0+n-1}) = (a_{i_0+i, j_0+j})_{0 \leq i \leq j \leq n-1}.$$

Let $\Delta S(i_0, j_0, n)$ denote the triangle build from the top to the base, whose *principal vertex* is at the position $(i_0, j_0) \in \mathbb{N} \times \mathbb{Z}$ in the orbit \mathcal{O}_S and of *size* n , i.e., the generalized Pascal triangle

$$\begin{aligned} \Delta S(i_0, j_0, n) &= \Delta((a_{i_0, j_0}, a_{i_0+1, j_0}, \dots, a_{i_0+n-1, j_0}), (a_{i_0, j_0}, a_{i_0+1, j_0+1}, \dots, a_{i_0+n-1, j_0+n-1})) \\ &= (a_{i_0+i, j_0+j})_{0 \leq j \leq i \leq n-1}. \end{aligned}$$

Example of triangles appearing in an orbit \mathcal{O}_S is represented in Figure 2.

0	0	1	0	1	0	1	1	0	0	0	0	1	1	0	0	0	0	1	0	0	1	1	1	0
1	0	1	1	1	1	1	0	1	0	0	0	1	0	1	0	0	0	1	1	0	1	0	0	1
1	1	1	0	0	0	0	1	1	1	0	0	1	1	1	1	0	0	1	0	1	1	1	0	1
1	0	0	1	0	0	0	1	0	0	1	0	1	0	0	0	1	0	1	1	1	0	0	1	1
0	1	0	1	1	0	0	1	1	0	1	1	1	1	0	0	1	1	1	0	0	1	0	1	0
1	1	1	1	0	1	0	1	0	1	1	0	0	0	1	0	0	1	0	0	1	0	1	1	1
0	0	0	0	1	1	1	1	1	1	0	1	0	0	1	1	1	1	0	1	1	1	0	0	0
0	0	0	0	1	0	0	0	0	0	1	1	1	0	1	0	0	0	1	1	0	0	1	0	0
0	0	0	0	1	1	0	0	0	0	1	0	0	1	1	1	0	0	1	0	1	0	1	1	0
1	0	0	0	1	0	1	0	0	0	1	1	0	1	0	0	1	0	1	1	1	0	1	1	0
1	1	0	0	1	1	1	1	0	0	1	0	1	1	1	0	1	1	1	0	0	0	0	1	1
0	0	1	0	1	0	0	0	1	0	1	1	1	0	0	1	1	0	0	1	0	0	0	1	0
1	0	1	1	1	1	0	0	1	1	1	0	0	1	0	1	0	1	1	0	0	1	1	1	1
0	1	1	0	0	0	1	0	1	0	0	1	0	1	1	1	1	1	1	0	1	0	1	0	0

Figure 2: Binary triangles appearing in an orbit \mathcal{O}_S

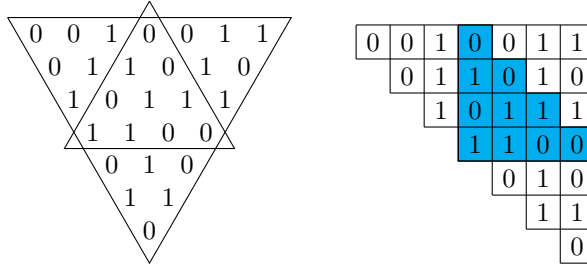


Figure 3: Isomorphism between \mathcal{PT}_n and \mathcal{ST}_{2n-1}

The sets of all the Steinhaus triangles of size n and of all the generalized Pascal triangles of size n are denoted by \mathcal{ST}_n and \mathcal{PT}_n , respectively. It is clear that these sets are $\mathbb{Z}/2\mathbb{Z}$ -vector spaces of dimension n and $2n-1$, respectively. Moreover, as depicted in Figure 3, there exists an obvious isomorphism between \mathcal{PT}_n and \mathcal{ST}_{2n-1} since a generalized Pascal triangle of size n can be seen as the center of a Steinhaus triangle of size $2n-1$.

A binary triangle of size n is constituted by $\binom{n+1}{2}$ elements, the n^{th} triangular number. Therefore, a binary triangle of size n contains an even number of terms for $n \equiv 0, 3 \pmod{4}$ and an odd number of terms for $n \equiv 1, 2 \pmod{4}$. It follows that a binary triangle T of size n is balanced if and only if

$$\delta \mathbf{m}_T = \begin{cases} 0 & \text{for } n \equiv 0, 3 \pmod{4}, \\ 1 & \text{for } n \equiv 1, 2 \pmod{4}. \end{cases}$$

In other words, a binary triangle T of size n is balanced if and only if either $\mathbf{m}_T(0) = \mathbf{m}_T(1)$, for $n \equiv 0, 3 \pmod{4}$, or $\mathbf{m}_T(0) = \mathbf{m}_T(1) \pm 1$, for $n \equiv 1, 2 \pmod{4}$. Then, the Steinhaus Problem only consists to determine whether there exist balanced Steinhaus triangles containing an even number of terms, for all the admissible sizes.

The main result of this paper is the following

Theorem 1.2. *There exists a binary doubly infinite sequence S such that its orbit \mathcal{O}_S contains balanced Steinhaus triangles and balanced generalized Pascal triangles of size n , for all positive integers n .*

This theorem completely and positively solves Problem 1, the generalization of the Steinhaus Problem for the two kinds of triangles, even when the triangles contain an odd number of terms. Note that the existence of balanced Steinhaus triangles with odd cardinality was first announced, without proof, in [9]. For the generalized Pascal triangles, the result is known by the community but never written before.

This paper is organized as follows. In the next section, the behavior of the p -periodic sequences under the action of the derivation process is studied and the set of p -tuples that generate p -periodic orbits is determined, for all values of p . An equivalence relation on the set of p -periodic orbits is given in

Section 3. This permits us to only consider the equivalence classes of p -periodic orbits and considerably reduce the number of orbits to analyse in the sequel. Let (i_0, j_0) be a fixed position in the orbit \mathcal{O}_S and $r \in \{0, 1, \dots, p-1\}$ be a fixed residue class modulo p . In Section 4, necessary and sufficient conditions on the family of Steinhaus triangles $\nabla S(i_0, j_0, pk+r)$, for being balanced for all non-negative integers k , are obtained. This leads to the proof of Theorem 1.2 in Section 5. Finally, we show in Section 6 that already known results on balanced triangles modulo m can also be expressed like periodic balanced triangles.

2 Periodic orbits

For any n_1 -tuple $X_1 = (a_0, a_1, \dots, a_{n_1-1})$ and any n_2 -tuple $X_2 = (b_0, b_1, \dots, b_{n_2-1})$ of elements in $\mathbb{Z}/2\mathbb{Z}$, the concatenation $X_1.X_2$ is the (n_1+n_2) -tuple $(a_0, a_1, \dots, a_{n_1-1}, b_0, b_1, \dots, b_{n_2-1})$. For any n -tuple X , the kn -tuple X^k is recursively defined by $X^k = X.X^{k-1}$ for all integers $k \geq 2$, with $X^1 = X$. For any n -tuple $X = (a_0, a_1, \dots, a_{n-1})$, the doubly infinite sequence $X^\infty = (b_j)_{j \in \mathbb{Z}}$ is defined by $b_{kn+j} = a_j$ for all $k \in \mathbb{Z}$ and for all $j \in \{0, 1, \dots, n-1\}$. For any doubly infinite sequence $S = (a_j)_{j \in \mathbb{Z}}$ and any positive integer n , we denote by $S[n]$ the initial segment of length n of S , that is, the n -tuple $S[n] = (a_0, a_1, \dots, a_{n-1})$.

Let p be a positive integer and let $S = (a_j)_{j \in \mathbb{Z}}$ be a doubly infinite sequence of elements of $\mathbb{Z}/2\mathbb{Z}$. The sequence S is said to be *periodic of period p* , or *p -periodic*, if $a_{j+p} = a_j$ for all $j \in \mathbb{Z}$. The p -periodicity of S is denoted by $S = (a_0, a_1, \dots, a_{p-1})^\infty$, where the p -tuple $(a_0, a_1, \dots, a_{p-1})$ is a *period* of length p of S .

First, it is clear that the periodicity of S is preserved under the derivation process.

Proposition 2.1. *For any p -tuple $(a_0, a_1, \dots, a_{p-1})$, we have*

$$\partial(a_0, a_1, \dots, a_{p-1})^\infty = (a_{p-1} + a_0, a_0 + a_1, \dots, a_{p-2} + a_{p-1})^\infty$$

An infinite sequence $(A_i)_{i \in \mathbb{N}}$ is said to be *pseudo-periodic of period p* if there exists $i_0 \in \mathbb{N}$ such that $A_{i+p} = A_i$ for all $i \geq i_0$.

Proposition 2.2. *The orbit of a periodic sequence is a pseudo-periodic sequence.*

Proof. Let S be a p -periodic sequence of $\mathbb{Z}/2\mathbb{Z}$ and let $\mathcal{O}_S = (\partial^i S)_{i \in \mathbb{N}}$ be its associated orbit. By Proposition 2.1, we know that, for every non-negative integer i , the derived sequence $\partial^i S$ is a p -periodic sequence. Since the number of p -tuples over $\mathbb{Z}/2\mathbb{Z}$ and thus the number of p -periodic sequences of $\mathbb{Z}/2\mathbb{Z}$ is finite, we deduce that there exist $0 \leq i_1 < i_2$ such that $\partial^{i_1} S = \partial^{i_2} S$. This leads to

$$\partial^{i+(i_2-i_1)} S = \partial^{i-i_1} \partial^{i_2} S = \partial^{i-i_1} \partial^{i_1} S = \partial^i S$$

for all $i \geq i_1$. The sequence \mathcal{O}_S is then a pseudo-periodic sequence of period $i_2 - i_1$. \square

We can retrieve the case where the orbit is pseudo-periodic and not periodic in the papers of Harborth [12] and of Eliahou and Hachez [9]. Here, we will study the special case where the orbit is fully periodic.

The orbit $\mathcal{O}_S = (a_{i,j})_{(i,j) \in \mathbb{N} \times \mathbb{Z}}$ is said to be *periodic of period p* , or *p -periodic*, if every row and every column is a p -periodic sequence, i.e., if the equalities

$$a_{i,j+p} = a_{i,j} \quad \text{and} \quad a_{i+p,j} = a_{i,j}$$

hold for all $i \in \mathbb{N}$ and all $j \in \mathbb{Z}$. In other words, the orbit $(a_{i,j})_{(i,j) \in \mathbb{N} \times \mathbb{Z}}$ is p -periodic if the equality

$$a_{i,j} = a_{\bar{i}, \bar{j}}$$

holds, for all $(i, j) \in \mathbb{N} \times \mathbb{Z}$, where \bar{x} is the rest in the euclidean division of x by p . For example, as depicted in Figure 4, the orbit \mathcal{O}_{X^∞} associated with the 6-tuple $X = 010100$ is 6-periodic.

Any square $P_{i_0, j_0} = (a_{i_0+i, j_0+j})_{0 \leq i, j \leq p-1}$ of size p is said to be a *period* of the p -periodic orbit \mathcal{O}_S . Remark that all the periods of a p -periodic orbit have the same multiplicity function, i.e., we have $\mathbf{m}_{P_{i_0, j_0}} = \mathbf{m}_{P_{0,0}}$ for all $(i_0, j_0) \in \mathbb{N} \times \mathbb{Z}$.

The set of p -tuples of $\mathbb{Z}/2\mathbb{Z}$ that generate p -periodic orbits is determined in the following result, that also appears in [12].

Theorem 2.3. *The orbit \mathcal{O}_{X^∞} associated with the p -tuple $X = (a_0, a_1, \dots, a_{p-1})$ is p -periodic if and only if the vector $v_X = (a_0, a_1, \dots, a_{p-1})^t$ is in the kernel of the matrix W_p , where M^t is the transposed*

0	1	0	1	0	0	0	1	0	1	0	0	0	1	0	1	0	0
0	1	1	1	1	0	0	1	1	1	1	0	0	1	1	1	1	0
0	1	0	0	0	1	0	1	0	0	0	1	0	1	0	0	0	1
1	1	1	0	0	1	1	1	1	0	0	1	1	1	1	0	0	1
0	0	0	1	0	1	0	0	0	1	0	1	0	0	0	1	0	1
1	0	0	1	1	1	1	0	0	1	1	1	1	0	0	1	1	1
0	1	0	1	0	0	0	1	0	1	0	0	0	1	0	1	0	0
0	1	1	1	1	0	0	1	1	1	1	0	0	1	1	1	1	0
0	1	0	0	0	1	0	1	0	0	0	1	0	1	0	0	0	1
1	1	1	0	0	1	1	1	1	0	0	1	1	1	1	0	0	1
0	0	0	1	0	1	0	0	0	1	0	1	0	0	0	1	0	1
1	0	0	1	1	1	1	0	0	1	1	1	1	0	0	1	1	1

Figure 4: The 6-periodic orbit $\mathcal{O}_{010100\infty}$

of the matrix M and W_p is the Wendt matrix of size p modulo 2, i.e., the circulant matrix of the binomial coefficients modulo 2

$$W_p = \begin{pmatrix} \binom{p}{p} & \binom{p}{p-1} & \binom{p}{p-2} & \cdots & \binom{p}{1} \\ \binom{p}{1} & \binom{p}{p} & \binom{p}{p-1} & \cdots & \binom{p}{2} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \binom{p}{p-1} & \binom{p}{p-2} & \binom{p}{p-3} & \cdots & \binom{p}{p} \end{pmatrix}.$$

For proving this result, we use the following lemma where it is shown that each term of the orbit \mathcal{O}_S can be expressed in function of the elements of the sequence S .

Lemma 2.4. *Let $\mathcal{O}_S = (a_{i,j})_{(i,j) \in \mathbb{N} \times \mathbb{Z}}$ be an orbit and let i_0 be a non-negative integer. Then,*

$$a_{i_0+i,j} = \sum_{k=0}^i \binom{i}{k} a_{i_0,j-k}$$

for all $(i,j) \in \mathbb{N} \times \mathbb{Z}$.

The proof of this lemma can be obtained by induction on $i \in \mathbb{N}$.

Proof of Theorem 2.3. Let $X = (a_0, a_1, \dots, a_{p-1})$ be a p -tuple and let $\mathcal{O}_{X^\infty} = (a_{i,j})_{(i,j) \in \mathbb{N} \times \mathbb{Z}}$ be its associated orbit. Since, by definition, the sequence X^∞ is p -periodic, we already know from Proposition 2.1 that the derived sequences $\partial^i X^\infty$ are p -periodic for all non-negative integers i . Therefore the equality $a_{i,j+p} = a_{i,j}$ is always verified for all $(i,j) \in \mathbb{N} \times \mathbb{Z}$. Thus, the orbit \mathcal{O}_{X^∞} is p -periodic if and only if $a_{i+p,j} = a_{i,j}$ for all $(i,j) \in \mathbb{N} \times \mathbb{Z}$. By Lemma 2.4, we have $a_{i+p,j} = \sum_{k=0}^i \binom{i}{k} a_{p,j-k}$ and $a_{i,j} = \sum_{k=0}^i \binom{i}{k} a_{0,j-k}$ for all $(i,j) \in \mathbb{N} \times \mathbb{Z}$. This is the reason why the equality $a_{i+p,j} = a_{i,j}$ holds for all $(i,j) \in \mathbb{N} \times \mathbb{Z}$ if and only if $a_{p,j} = a_{0,j}$ for all $j \in \mathbb{Z}$. Moreover, since the sequences $(a_{0,j})_{j \in \mathbb{Z}}$ and $(a_{p,j})_{j \in \mathbb{Z}}$ are p -periodic, we obtain that the orbit \mathcal{O}_{X^∞} is p -periodic if and only if $a_{p,j} = a_{0,j}$ for all $j \in \{0, 1, \dots, p-1\}$. From Lemma 2.4 again, we know that $a_{p,j} = \sum_{k=0}^p \binom{p}{k} a_{0,j-k}$ for all $j \in \{0, 1, \dots, p-1\}$. Therefore the orbit \mathcal{O}_{X^∞} is p -periodic if and only if

$$\begin{cases} \binom{p}{1} a_{0,-1} + \binom{p}{2} a_{0,-2} + \cdots + \binom{p}{p} a_{0,-p} & = 0 \\ \binom{p}{1} a_{0,0} + \binom{p}{2} a_{0,-1} + \cdots + \binom{p}{p} a_{0,-p+1} & = 0 \\ \vdots & \\ \binom{p}{1} a_{0,p-2} + \binom{p}{2} a_{0,p-3} + \cdots + \binom{p}{p} a_{0,-1} & = 0 \end{cases} \iff \begin{cases} \binom{p}{1} a_{p-1} + \binom{p}{2} a_{p-2} + \cdots + \binom{p}{p} a_0 & = 0 \\ \binom{p}{1} a_0 + \binom{p}{2} a_{p-1} + \cdots + \binom{p}{p} a_1 & = 0 \\ \vdots & \\ \binom{p}{1} a_{p-2} + \binom{p}{2} a_{p-3} + \cdots + \binom{p}{p} a_{p-1} & = 0 \end{cases},$$

i.e., if and only if the p -tuple X is in the kernel of the Wendt matrix $W_p = \left(\binom{p}{|i-j|} \right)_{1 \leq i,j \leq p}$ modulo 2. \square

The set of p -tuples X that generate p -periodic orbits \mathcal{O}_{X^∞} is denoted by \mathcal{PO}_p . It is then a $\mathbb{Z}/2\mathbb{Z}$ -vector space isomorphic to the kernel of the Wendt matrix W_p of size p modulo 2. Table 1 gives $\dim \ker(W_p)$ and then $|\mathcal{PO}_p| = 2^{\dim \ker(W_p)}$ for the first few values of p .

p	1	2	3	4	5	6	7	8	9	10	11	12
$\dim \ker(W_p)$	0	0	2	0	0	4	6	0	2	0	0	8
p	13	14	15	16	17	18	19	20	21	22	23	24
$\dim \ker(W_p)$	0	12	14	0	0	4	0	0	8	0	0	16

Table 1: The first few values of $\dim \ker(W_p)$

For example, for $p = 6$, we have $\dim \ker(W_6) = 4$ and $|\mathcal{PO}_6| = 2^4 = 16$. There are then 16 different 6-tuples that generate a 6-periodic orbit. More precisely, the set \mathcal{PO}_6 is given by

$$\begin{aligned} \mathcal{PO}_6 &= \langle 000101, 001010, 010001, 100010 \rangle \\ &= \{000000, 000101, 001010, 001111, 010001, 010100, 011011, 011110, \\ &\quad 100010, 100111, 101000, 101101, 110011, 110110, 111001, 111100\}. \end{aligned}$$

We retrieve here that the 6-tuple $X = 010100$ generates a 6-periodic orbit as depicted in Figure 4.

3 Symmetry group of \mathcal{PO}_p

In this section, a symmetry group on the set of p -tuples that generate p -periodic orbits is defined. First, the notion of translation and the action of the dihedral group D_3 on periodic orbits are introduced.

3.1 Translation

Let $\mathcal{O}_{X^\infty} = (a_{\bar{i}, \bar{j}})_{(i,j) \in \mathbb{N} \times \mathbb{Z}}$ be the p -periodic orbit associated with $X = (a_0, a_1, \dots, a_{p-1}) \in \mathcal{PO}_p$. The *translate of X by the vector $(u, v) \in \mathbb{Z}^2$* is the p -tuple $\mathbf{t}_{u,v}(X) = (a_{\overline{-u}, \overline{j-v}})_{0 \leq j \leq p-1}$. From Lemma 2.4, we know that

$$\mathbf{t}_{u,v}(X) = \left(\sum_{k=0}^{\overline{-u}} \binom{\overline{-u}}{k} a_{\overline{j-v-k}} \right)_{0 \leq j \leq p-1}.$$

From the definition of $\mathbf{t}_{u,v}(X)$, it is clear that

$$\mathcal{O}_{\mathbf{t}_{u,v}(X)^\infty} = \left(a_{\overline{-u}, \overline{j-v}} \right)_{(i,j) \in \mathbb{N} \times \mathbb{Z}}.$$

Therefore $\mathbf{t}_{u,v}$ is an automorphism of \mathcal{PO}_p . Moreover, the application

$$\begin{aligned} (\mathbb{Z}^2, +) &\longrightarrow (\text{Aut}(\mathcal{PO}_p), \circ) \\ (u, v) &\longmapsto \mathbf{t}_{u,v} \end{aligned}$$

is a group morphism.

For example, the translate of the 6-tuple $010100 \in \mathcal{PO}_6$ (Figure 4) by the vector $(2, 3)$ is $\mathbf{t}_{2,3}(010100) = 101000$, as we can see in its orbit $\mathcal{O}_{\mathbf{t}_{2,3}(010100)^\infty} = \mathcal{O}_{101000}^\infty$ depicted in Figure 5.

3.2 The dihedral group D_3

First, consider the Steinhaus triangles $\nabla S = (a_{i,j})_{1 \leq i \leq j \leq n-1}$ of size n . The *left and right sides* of ∇S are the sequences $l(S) = (a_{i,i})_{0 \leq i \leq n-1}$ and $r(S) = (a_{i,n-1})_{0 \leq i \leq n-1}$, respectively. From Lemma 2.4, we know that $l(S)$ and $r(S)$ can be expressed in function of the elements of $S = (a_j)_{0 \leq j \leq n-1}$

$$l(S) = \left(\sum_{k=0}^i \binom{i}{k} a_{i-k} \right)_{0 \leq i \leq n-1} \quad \text{and} \quad r(S) = \left(\sum_{k=0}^i \binom{i}{k} a_{n-1-k} \right)_{0 \leq i \leq n-1}.$$

The reversed sequence of S is the sequence read from the right to the left, that is $i(S) = (a_{n-1-j})_{0 \leq j \leq n-1}$.

Due to the symmetries involved in the local rule that generates ∇S , the Pascal local rule modulo 2, it is known that the Steinhaus triangles $\nabla l(S)$, $\nabla r(S)$ and $\nabla i(S)$ correspond to the rotations of ∓ 120 degrees around the center of the triangle ∇S and the reflection across the vertical line through the center of ∇S , respectively. More precisely, for all integers i and j such that $1 \leq i \leq j \leq n-1$, we have

$$a_{i-1,j-1} + a_{i-1,j} = a_{i,j} \iff a_{i-1,j} + a_{i,j} = a_{i-1,j-1} \iff a_{i-1,j} + a_{i-1,j-1} = a_{i,j}.$$

1	0	1	0	0	0	1	0	1	0	0	0	1	0	1	0	0	0
1	1	1	1	0	0	1	1	1	1	0	0	1	1	1	1	0	0
1	0	0	0	1	0	1	0	0	0	1	0	1	0	0	0	1	0
1	1	0	0	1	1	1	1	0	0	1	1	1	1	1	0	0	1
0	0	1	0	1	0	0	0	0	1	0	1	0	0	0	1	0	1
0	0	1	1	1	1	0	0	0	1	1	1	1	0	0	1	1	1
1	0	1	0	0	0	1	0	1	0	0	0	1	0	1	0	0	0
1	1	1	1	0	0	1	1	1	1	1	0	0	1	1	1	1	0
1	0	0	0	1	0	1	0	0	0	1	0	1	0	1	0	0	1
1	1	0	0	1	1	1	1	0	0	1	1	1	1	1	0	0	1
0	0	1	0	1	0	0	0	1	0	1	0	0	0	1	0	1	0
0	0	1	1	1	1	0	0	1	1	1	1	0	0	1	1	1	1

Figure 5: The translate $t_{2,3}(010100) = 101000$

Therefore

$$\nabla l(S) = \nabla(a_{j,j})_{1 \leq j \leq n-1} = (a_{j-i,j})_{0 \leq i \leq j \leq n-1},$$

$$\nabla r(S) = \nabla(a_{j,n-1})_{1 \leq j \leq n-1} = (a_{j-i,n-1-i})_{0 \leq i \leq j \leq n-1},$$

$$\nabla i(S) = \nabla(a_{0,n-1-j})_{0 \leq j \leq n-1} = (a_{i,n-1+i-j})_{0 \leq i \leq j \leq n-1}.$$

Since

$$r^3 = i^2 = (ir)^2 = id_{(\mathbb{Z}/2\mathbb{Z})^n},$$

the subgroup of $(\text{Aut}((\mathbb{Z}/2\mathbb{Z})^n), \circ)$, the group of automorphisms of the vector space of n -tuples over $\mathbb{Z}/2\mathbb{Z}$, generated by r and i is isomorphic to the dihedral group D_3

$$\langle r, i | r^3 = i^2 = (ir)^2 = id_{(\mathbb{Z}/2\mathbb{Z})^n} \rangle = D_3.$$

As depicted in Figure 6, it is easy to see that the multiplicity function of a Steinhaus triangle is invariant under the action of the dihedral group D_3 . Indeed, for any finite sequence S , we have

$$\mathbf{m}_{\nabla S} = \mathbf{m}_{\nabla r(S)} = \mathbf{m}_{\nabla i(S)}.$$

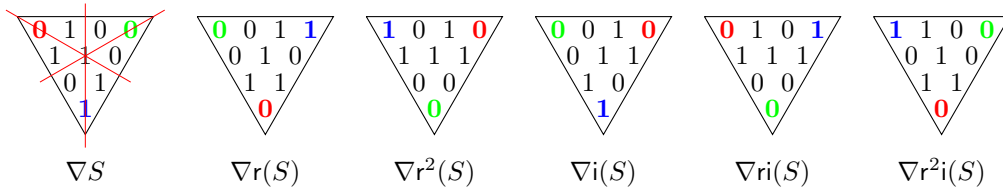


Figure 6: Action of D_3 on $\nabla(0100)$

The study of rotationally symmetric triangles and dihedrally symmetric triangles, that are triangles ∇S such that $S = r(S)$ and $S = r(S) = i(S)$, respectively, can be found in [1, 2].

Now, we consider the restrictions of r and i to the vector space \mathcal{PO}_p of p -tuples that generate p -periodic orbits. Since we only consider these restrictions, they are also denoted by r and i in the sequel.

Proposition 3.1. *For all positive integers p , we have*

$$r(\mathcal{PO}_p) = i(\mathcal{PO}_p) = \mathcal{PO}_p.$$

Proof. Let $X \in \mathcal{PO}_p$ and $\mathcal{O}_{X^\infty} = (a_{i,j})_{(i,j) \in \mathbb{N} \times \mathbb{Z}}$. Then, by definition, $r(X) = (a_{j,p-1})_{0 \leq j \leq p-1}$ and $i(X) = (a_{0,p-1-j})_{0 \leq j \leq p-1}$. Let $\mathcal{O}_{r(X)^\infty} = (b_{i,j})_{(i,j) \in \mathbb{N} \times \mathbb{Z}}$ and $\mathcal{O}_{i(X)^\infty} = (c_{i,j})_{(i,j) \in \mathbb{N} \times \mathbb{Z}}$. We will show that $b_{i,j} = a_{\overline{j-i}, \overline{-i-1}}$ and $c_{i,j} = a_{\overline{i}, \overline{i-j-1}}$ for all $i \in \mathbb{N}$ and all $j \in \mathbb{Z}$. We proceed by induction on $i \in \mathbb{N}$. For $i = 0$, by definition, we have that

$$b_{0,j} = b_{0,\overline{j}} = a_{\overline{j}, \overline{p-1}} = a_{\overline{j}, \overline{-1}}$$

and

$$c_{0,j} = c_{0,\overline{j}} = a_{0, \overline{p-1-j}} = a_{\overline{0}, \overline{-j-1}}$$

for all $j \in \mathbb{Z}$. Suppose that the formulas are verified for a certain value of $i - 1 \geq 0$ and for all $j \in \mathbb{Z}$. Then,

$$b_{i,j} = b_{i-1,j-1} + b_{i-1,j} = a_{\overline{j-i},\overline{-i}} + a_{\overline{j-i+1},\overline{-i}} = a_{\overline{j-i},\overline{-i}} + a_{\overline{j-i+1},\overline{-i}} = a_{\overline{j-i},\overline{-i-1}} = a_{\overline{j-i},\overline{-i-1}}$$

and

$$c_{i,j} = c_{i-1,j-1} + c_{i-1,j} = a_{\overline{i-1},\overline{i-j-1}} + a_{\overline{i-1},\overline{i-j-2}} = a_{\overline{i-1},\overline{i-j-1}} + a_{\overline{i-1},\overline{i-j-1-1}} = a_{\overline{i-1+1},\overline{i-j-1}} = a_{\overline{i},\overline{i-j-1}}$$

for all $j \in \mathbb{Z}$. The formulas are then proved and we deduce that $b_{i,j} = b_{\overline{i},\overline{j}}$ and $c_{i,j} = c_{\overline{i},\overline{j}}$ for all $i \in \mathbb{N}$ and all $j \in \mathbb{Z}$. Then the orbit $\mathcal{O}_{r(X)}^\infty$ and $\mathcal{O}_{i(X)}^\infty$ are p -periodic. Therefore $r(X)$ and $i(X)$ are in \mathcal{PO}_p . This proves that $r(\mathcal{PO}_p) \subset \mathcal{PO}_p$ and $i(\mathcal{PO}_p) \subset \mathcal{PO}_p$ and implies that

$$\mathcal{PO}_p = r^3(\mathcal{PO}_p) \subset r^2(\mathcal{PO}_p) \subset r(\mathcal{PO}_p) \subset \mathcal{PO}_p \quad \text{and} \quad \mathcal{PO}_p = i^2(\mathcal{PO}_p) \subset i(\mathcal{PO}_p) \subset \mathcal{PO}_p.$$

This concludes the proof. \square

It follows that r and i are automorphisms of the vector space \mathcal{PO}_p and the subgroup of $(\text{Aut}(\mathcal{PO}_p), \circ)$ generated by r and i is also isomorphic the dihedral group D_3

$$D_3 = \langle r, i \rangle = \{id_{\mathcal{PO}_p}, r, r^2, i, ri, r^2i\}.$$

More precisely, for any p -tuple X , we have

$$\begin{aligned} \mathcal{O}_{X^\infty} &= (a_{\overline{i},\overline{j}})_{(i,j) \in \mathbb{N} \times \mathbb{Z}} & \mathcal{O}_{i(X)^\infty} &= (a_{\overline{i},\overline{i-j-1}})_{(i,j) \in \mathbb{N} \times \mathbb{Z}} \\ \mathcal{O}_{r(X)^\infty} &= (a_{\overline{j-i},\overline{-i-1}})_{(i,j) \in \mathbb{N} \times \mathbb{Z}} & \mathcal{O}_{ri(X)^\infty} &= (a_{\overline{-j-1},\overline{-i-1}})_{(i,j) \in \mathbb{N} \times \mathbb{Z}} \\ \mathcal{O}_{r^2(X)^\infty} &= (a_{\overline{-j-1},\overline{i-j-1}})_{(i,j) \in \mathbb{N} \times \mathbb{Z}} & \mathcal{O}_{r^2i(X)^\infty} &= (a_{\overline{j-i},\overline{j}})_{(i,j) \in \mathbb{N} \times \mathbb{Z}} \end{aligned}$$

For instance, a representation of $\mathcal{O}_{g(010100)^\infty}$ for all $g \in D_3$ is given in Figure 7.

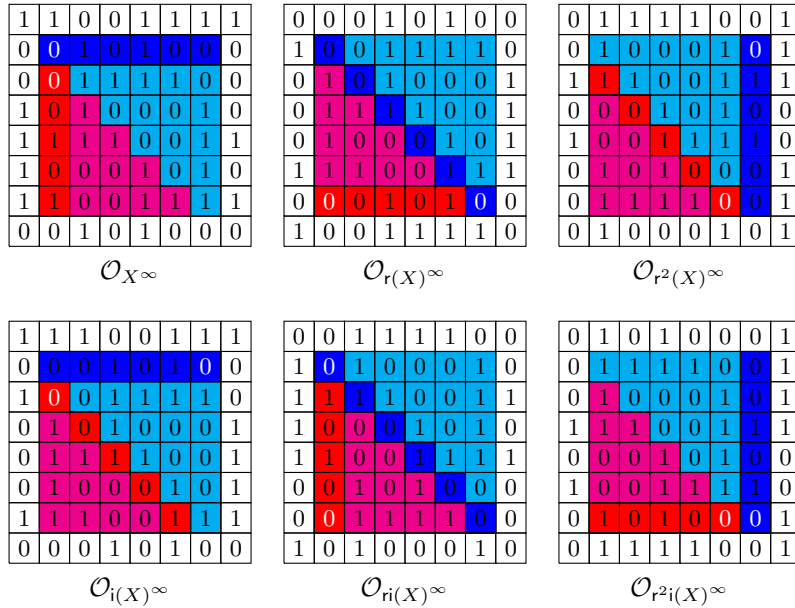


Figure 7: Action of D_3 on $\mathcal{O}_{010100}^\infty$

3.3 The symmetry group of \mathcal{PO}_p

Let G be the subgroup of $(\text{Aut}(\mathcal{PO}_p), \circ)$ generated by r , i , $\mathbf{t}_{1,0}$ and $\mathbf{t}_{0,1}$, that is,

$$G := \langle r, i, \mathbf{t}_{1,0}, \mathbf{t}_{0,1} \rangle.$$

As in D_3 , the equality $ir = r^2i$ holds in G . The equalities involving the translations are listed below.

Proposition 3.2. *For all $(u, v) \in \mathbb{Z}^2$, the equalities $r\mathbf{t}_{u,v} = \mathbf{t}_{v-u, -u}r$ and $i\mathbf{t}_{u,v} = \mathbf{t}_{u, u-v}i$ hold.*

Proof. Let $(u, v) \in \mathbb{Z}^2$ and $\mathcal{O}_X^\infty = (a_{\overline{i,j}})_{(i,j) \in \mathbb{N} \times \mathbb{Z}}$ be a p -periodic orbit. Then,

$$\mathcal{O}_{\mathfrak{t}_{u,v}(X)^\infty} = \left(a_{\overline{j-i+u-v, -i-1+u}} \right)_{(i,j) \in \mathbb{N} \times \mathbb{Z}} = \mathcal{O}_{\mathfrak{t}_{v-u, -u}r(X)^\infty}$$

and

$$\mathcal{O}_{\mathfrak{t}_{u,v}(X)^\infty} = \left(a_{\overline{i-u, i-j-1+v-u}} \right)_{(i,j) \in \mathbb{N} \times \mathbb{Z}} = \mathcal{O}_{\mathfrak{t}_{u, u-v}i(X)^\infty}.$$

□

From these equalities, it is clear that each element $g \in G$ can be uniquely written as

$$g = \mathfrak{t}_{u,v}r^{\alpha}i^{\beta}$$

with $u, v \in \{0, 1, \dots, p-1\}$, $\alpha \in \{0, 1, 2\}$ and $\beta \in \{0, 1\}$. Therefore G is a group of order $|G| = 6p^2$.

3.4 Equivalence classes of \mathcal{PO}_p

Now, we consider the binary relation \sim_G on the set \mathcal{PO}_p defined by $X_1 \sim_G X_2$ if and only if there exists $g \in G$ such that $X_2 = g(X_1)$. Since G is a subgroup of $(\text{Aut}(\mathcal{PO}_p), \circ)$, it is clear that \sim_G is an equivalence relation on \mathcal{PO}_p . Therefore, for searching balanced triangles, it is sufficient to examine only one representative of each equivalence classe in the set $\overline{\mathcal{PO}_p} := \mathcal{PO}_p / \sim_G$. In the sequel, the equivalence class of the tuple X is denoted by \overline{X} .

For example, for $p = 6$, we obtain that \mathcal{PO}_6 / \sim_G is constituted by 3 equivalence classes that contain the 16 tuples of \mathcal{PO}_6 generating 6-periodic orbits. More precisely,

$$\begin{aligned} \overline{\mathcal{PO}_6} = \mathcal{PO}_6 / \sim_G &= \{ \{000000\}, \{110110, 101101, 011011\}, \{010100, 101000, 010001, 100010, \\ &\quad 000101, 001010, 011110, 111100, 111001, 110011, 100111, 001111\} \\ &= \{ \overline{000000}, \overline{110110}, \overline{010100} \} \end{aligned}$$

since

$$\begin{array}{lll} 101101 = \mathfrak{t}_{0,5}(110110) & 101000 = \mathfrak{t}_{0,5}(010100) & 011110 = \mathfrak{t}_{5,0}(010100) \\ 011011 = \mathfrak{t}_{0,4}(110110) & 010001 = \mathfrak{t}_{0,4}(010100) & 111100 = \mathfrak{t}_{5,5}(010100) \\ & 100010 = \mathfrak{t}_{0,3}(010100) & 111001 = \mathfrak{t}_{5,4}(010100) \\ & 000101 = \mathfrak{t}_{0,2}(010100) & 110011 = \mathfrak{t}_{5,3}(010100) \\ & 001010 = \mathfrak{t}_{0,1}(010100) & 100111 = \mathfrak{t}_{5,2}(010100) \\ & & 001111 = \mathfrak{t}_{5,1}(010100) \end{array}$$

The 6-periodic orbits associated with these 3 equivalence classes are depicted in Figure 8.

0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0
1	0	1	1	0	1	1	0
0	1	1	0	1	1	0	1
1	1	0	1	1	0	1	1
1	0	1	1	0	1	1	0
0	1	1	0	1	1	0	1
1	1	0	1	1	0	1	1
1	0	1	1	0	1	1	0
0	1	1	0	1	1	0	1
1	1	0	0	1	1	1	1
0	0	1	0	1	0	0	0
0	0	1	1	1	1	0	0
1	0	1	0	0	0	1	0
1	1	1	1	0	0	1	1
1	0	0	0	1	0	1	0
1	1	0	0	1	1	1	1
0	0	1	0	1	0	0	0

Figure 8: The set $\overline{\mathcal{PO}_6} = \{ \overline{000000}, \overline{110110}, \overline{010100} \}$

Table 2 gives $|\overline{\mathcal{PO}_p}|$ for the first few values of p .

4 Family of periodic balanced Steinhaus triangles with the same principal vertex

In this section, we determine necessary and sufficient conditions for obtaining, in a p -periodic orbit, an infinite family of balanced Steinhaus triangles with the same principal vertex.

p	1	2	3	4	5	6	7	8	9	10	11	12
$ \mathcal{PO}_p $	1	1	2^2	1	1	2^4	2^6	1	2^2	1	1	2^8
$\overline{ \mathcal{PO}_p }$	1	1	2	1	1	3	3	1	2	1	1	7
p	13	14	15	16	17	18	19	20	21	22	23	24
$ \mathcal{PO}_p $	1	2^{12}	2^{14}	1	1	2^4	1	1	2^8	1	1	2^{16}
$\overline{ \mathcal{PO}_p }$	1	13	30	1	1	3	1	1	6	1	1	92

Table 2: The first few values of $|\overline{\mathcal{PO}_p}|$

Proposition 4.1. *Let $S = X^\infty$ with $X \in \mathcal{PO}_p$, $(i_0, j_0) \in \mathbb{N} \times \mathbb{Z}$ and $r \in \{0, 1, \dots, p-1\}$. The Steinhaus triangles*

$$T_k := \nabla S(i_0, j_0, kp + r)$$

are balanced for all non-negative integers k if and only if the triangle T_0 , the multiset difference $T_1 \setminus T_0$ and the period P are balanced, with p divisible by 4.

Proof. Suppose that $(i_0, j_0) \in \mathbb{N} \times \mathbb{Z}$ and $r \in \{0, 1, \dots, p-1\}$ are fixed. Let $k \in \mathbb{N}$. Then, from the periodicity of \mathcal{O}_S , we know that the Steinhaus triangle $T_k = \nabla S(i_0, j_0, kp + r)$ can be decomposed into elementary blocks T_0 , $T_1 \setminus T_0$ and P , as represented in Figure 9 for $k = 5$.

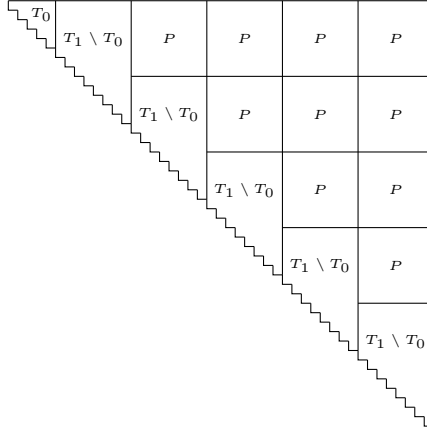


Figure 9: Decomposition of T_5

More precisely, the triangle T_k is constituted by one block T_0 , k blocks $T_1 \setminus T_0$ and $\binom{k}{2}$ blocks P . It follows that the multiplicity function \mathbf{m}_{T_k} of the triangle T_k must verify that

$$\mathbf{m}_{T_k} = \mathbf{m}_{T_0} + k\mathbf{m}_{T_1 \setminus T_0} + \binom{k}{2}\mathbf{m}_P.$$

First, suppose that T_0 , $T_1 \setminus T_0$ and P are balanced with p divisible by 4. Since p is divisible by 4, it is clear that the cardinalities $|T_1 \setminus T_0| = pr + \binom{p+1}{2}$ and $|P| = p^2$ are even. Therefore, since the multiset difference $T_1 \setminus T_0$ and the period P are balanced, the multiplicity functions $\mathbf{m}_{T_1 \setminus T_0}$ and \mathbf{m}_P are constant. It follows that $\delta_{T_k} = \delta_{T_0} \in \{0, 1\}$ and thus the triangles T_k are balanced for all non-negative integers k .

Conversely, suppose that the triangles T_k are balanced for all non-negative integers k . Thus, the value of $(\mathbf{m}_{T_k}(0) - \mathbf{m}_{T_k}(1)) - (\mathbf{m}_{T_0}(0) - \mathbf{m}_{T_0}(1))$ is in the set $\{-2, -1, 0, 1, 2\}$ for all $k \in \mathbb{N}$. Therefore,

$$\lim_{k \rightarrow +\infty} \frac{(\mathbf{m}_{T_k}(0) - \mathbf{m}_{T_k}(1)) - (\mathbf{m}_{T_0}(0) - \mathbf{m}_{T_0}(1))}{k} = 0.$$

It follows that

$$\lim_{k \rightarrow +\infty} (\mathbf{m}_{T_1 \setminus T_0}(0) - \mathbf{m}_{T_1 \setminus T_0}(1)) + \frac{k-1}{2}(\mathbf{m}_P(0) - \mathbf{m}_P(1)) = 0.$$

Then, we deduce that $\mathbf{m}_{T_1 \setminus T_0}(0) - \mathbf{m}_{T_1 \setminus T_0}(1) = \mathbf{m}_P(0) - \mathbf{m}_P(1) = 0$. Therefore the multiset difference $T_1 \setminus T_0$ and the period P are balanced and with even cardinalities. Finally, since $|P| = p^2$ and $|T_1 \setminus T_0| = pr + \binom{p+1}{2}$, we conclude that p must be divisible by 4 in this case. \square

This is the reason why, in the sequel of this paper, we only consider p -periodic orbits with a balanced period and where p is divisible by 4.

Note that the period of the orbit generated from every element of a same equivalence class of $\overline{\mathcal{PO}_p}$ has the same multiplicity function. Let us denote by $\overline{\mathcal{BPO}_p}$ the set of all the equivalence classes of $\overline{\mathcal{PO}_p}$ having a balanced period. Table 3 gives $|\overline{\mathcal{BPO}_p}|$ for the first few values of p divisible by 4.

p	4	8	12	16	20	24
$ \mathcal{PO}_p $	1	1	256	1	1	65536
$ \overline{\mathcal{PO}_p} $	1	1	7	1	1	92
$ \overline{\mathcal{BPO}_p} $	0	0	2	0	0	17

Table 3: The first few values of $|\overline{\mathcal{BPO}_p}|$

More precisely, the sets $\overline{\mathcal{BPO}_{12}}$ and $\overline{\mathcal{BPO}_{24}}$ are given below:

$$\overline{\mathcal{BPO}_{12}} = \{\overline{001010001010}, \overline{111000001110}\}$$

and

$$\overline{\mathcal{BPO}_{24}} = \left\{ \begin{array}{lll} \overline{11010110000000011010110}, & \overline{111010010000000011101001}, & \overline{011001001000000011100100}, \\ \overline{011101101000000011110110}, & \overline{001111101000000010111110}, & \overline{111110000100000010111000}, \\ \overline{100101000100000011010100}, & \overline{110111000100000010011100}, & \overline{111100010100000010110001}, \\ \overline{101110010100000011111001}, & \overline{101110010100000011111001}, & \overline{110110100010000011111010}, \\ \overline{110000010010000011100001}, & \overline{101111110010000010011111}, & \overline{100110000110000011111000}, \\ \overline{000011101110000011101110}, & \overline{100010100010100010100010} & \end{array} \right\}.$$

A representation of the orbits generated from the elements of $\overline{\mathcal{BPO}_{12}}$ and $\overline{\mathcal{BPO}_{24}}$ can be found in Appendix A.

5 Periodic balanced triangles

In this section we will prove Theorem 1.2, the main result of this paper.

Let X be a p -tuple of $\mathbb{Z}/2\mathbb{Z}$, with p divisible by 4, such that \overline{X} is in $\overline{\mathcal{BPO}_p}$ and let $S := X^\infty$. Now, for each remainder $r \in \{0, 1, \dots, p-1\}$ and for each position $(i_0, j_0) \in \{0, 1, \dots, p-1\}^2$, we test if the blocks $\nabla S(i_0, j_0, r)$ and $\nabla S(i_0, j_0, p+r) \setminus \nabla S(i_0, j_0, r)$ are balanced. If this is the case, we know from Proposition 4.1 that the Steinhaus triangles $\nabla S(i_0, j_0, kp+r)$ are balanced for all non-negative integers k .

Let $\mathcal{R}_{\overline{X}}$ denote the set of remainders $r \in \{0, 1, \dots, p-1\}$ for which there exists a position $(i_0, j_0) \in \{0, 1, \dots, p-1\}^2$ such that the Steinhaus triangles $\nabla S(i_0, j_0, kp+r)$ are balanced for all non-negative integers k .

As announced in Table 3, the first values of p , divisible by 4, for which $\overline{\mathcal{BPO}_p} \neq \emptyset$ are 12 and 24. For $p = 12$, we find that $\mathcal{R}_{\overline{X}} = \emptyset$ for each of the two equivalence classes \overline{X} of $\overline{\mathcal{BPO}_{12}}$. For $p = 24$, we find that $\mathcal{R}_{\overline{X}} \neq \emptyset$ for 15 of the 17 equivalence classes \overline{X} of $\overline{\mathcal{BPO}_{24}}$. Remark that the two equivalence classes \overline{X} of $\overline{\mathcal{BPO}_{24}}$ such that $\mathcal{R}_{\overline{X}} = \emptyset$ are exactly of the form $\overline{X} = \overline{Y^2}$ with $\overline{Y} \in \overline{\mathcal{BPO}_{12}}$. More precisely, Table 4 gives the exact number of remainders constituting $\mathcal{R}_{\overline{X}}$ for each $\overline{X} \in \overline{\mathcal{BPO}_{24}}$.

For six equivalence classes \overline{X} of $\overline{\mathcal{BPO}_{24}}$, we find that $|\mathcal{R}_{\overline{X}}| = 24$ and thus, from these 24-tuples, we obtain the proof of Theorem 1.2 for Steinhaus triangles, i.e., there exist periodic orbits containing balanced Steinhaus triangles of size n for all $n \geq 1$.

For instance, in the orbit \mathcal{O}_{X^∞} associated with the 24-tuple $X = 011101101000000011110110$, the existence of balanced Steinhaus triangles for all the possible sizes can be obtained from at least 5 positions. The following Steinhaus triangles $\nabla X^\infty(i_0, j_0, 24k+r)$ are balanced for all non-negative integers k :

(i_0, j_0)	r
(5, 8)	5, 6, 11, 12, 14, 19, 20, 21, 22
(3, 11)	0, 1, 2, 3, 9, 10, 17, 19
(1, 14)	0, 5, 8, 13, 16, 21, 22, 23
(2, 12)	0, 1, 15, 18, 23
(5, 15)	4, 7, 21, 22, 23

X	$ \mathcal{R}_{\overline{X}} $	X	$ \mathcal{R}_{\overline{X}} $
110101100000000011010110	18	101110010100000011111001	23
111010010000000011101001	16	101001000010000010000100	24
011001001000000011100100	23	110110100010000011111010	23
011101101000000011110110	24	110000010010000011100001	20
001111101000000010111110	17	101111110010000010011111	20
111110000100000010111000	24	100110000110000011111000	23
100101000100000011010100	24	000011101110000011101110	0
110111000100000010011100	24	100010100010100010100010	0
111100010100000010110001	24		

Table 4: $|\mathcal{R}_{\overline{X}}|$ for all $\overline{X} \in \overline{\mathcal{BPO}}_{24}$

More explicitly, the following Steinhaus triangles $\nabla X_0^\infty[24k+r]$ are balanced for all non-negative integers k :

X_0	r
000111001000010110011001	5, 6, 11, 12, 14, 19, 20, 21, 22
1000010101111000011110101	0, 1, 2, 3, 9, 10, 17, 19
001000110101001101110000	0, 5, 8, 13, 16, 21, 22, 23
000011001011111010110010	0, 1, 15, 18, 23
010000101100110010001110	4, 7, 21, 22, 23

The family of balanced Steinhaus triangles $\nabla X^\infty(5, 8, 24k+5)$, appearing in the orbit \mathcal{O}_{X^∞} for $X = 011101101000000011110110$, is depicted in Figure 10, where empty and full squares correspond to 0 and 1 respectively. Indeed, we can verify that the blocks $A := \nabla X^\infty(5, 8, 5)$, $B := \nabla X^\infty(5, 8, 29) \setminus \nabla X^\infty(5, 8, 5)$ and the period P are balanced, since their multiplicity functions verify:

x	$\mathbf{m}_A(x)$	$\mathbf{m}_B(x)$	$\mathbf{m}_P(x)$
0	8	210	288
1	7	210	288

The following proposition concludes the proof of Theorem 1.2 by showing that in an orbit \mathcal{O}_{X^∞} generated from a p -tuple X such that $\overline{X} \in \overline{\mathcal{BPO}}_p$, the existence of balanced Steinhaus triangles implies that of balanced generalized Pascal triangles.

Proposition 5.1. *Let $S = X^\infty$ with $\overline{X} \in \overline{\mathcal{BPO}}_p$, $(i_0, j_0) \in \mathbb{N} \times \mathbb{Z}$, $r \in \{0, 1, \dots, p-1\}$ and p divisible by 4. Then, the Steinhaus triangles $\nabla S(i_0, j_0, kp+r)$ are balanced for all non-negative integers k if and only if the generalized Pascal triangles $\Delta S(i_0+r+1, j_0+r, kp+(p-1-r))$ are balanced for all non-negative integers k .*

Proof. As depicted in Figure 11, the orbit $\mathcal{O}_S = (a_{i,j})_{(i,j) \in \mathbb{N} \times \mathbb{Z}}$ can be decomposed into elementary blocks U_0, U_1, V_0 and V_1 defined by

$$\begin{aligned}
U_0 &= \nabla S(i_0, j_0, r), \\
U_1 &= \nabla S(i_0, j_0, p+r) \setminus (\nabla S(i_0, j_0, r) \cup \nabla S(i_0+p, j_0+p, r)), \\
V_1 &= \Delta S(i_0+r+1, j_0+r, r'), \\
V_0 &= \Delta S(i_0+r+1, j_0+r, p+r') \setminus (\Delta S(i_0+r+1, j_0+r, r') \cup \Delta S(i_0+p+r+1, j_0+p+r, r')),
\end{aligned}$$

where $r' = p-1-r$, and the period P defined by

$$P = \{a_{p+r+i,j} \mid i, j \in \{0, 1, \dots, p-1\}\}.$$

Since \mathcal{O}_S is p -periodic with P balanced and p divisible by 4, we already know from Proposition 4.1 that the Steinhaus triangles $\nabla S(i_0, j_0, kp+r)$ are balanced for all non-negative integers k if and only if U_0 and $U_1 \cup U_0$ are balanced.

Similarly, the generalized Pascal triangles $\Delta S(i_0+r+1, j_0+r, kp+p-1-r)$ are balanced for all non-negative integers k if and only if V_1 and $V_0 \cup V_1$ are balanced.

First, since

$$U_0 \cup V_0 = U_1 \cup V_1 = P$$



Figure 10: The balanced Steinhaus triangles $\nabla 01110110100000011110110^\infty(5, 8, 24k + 5)$

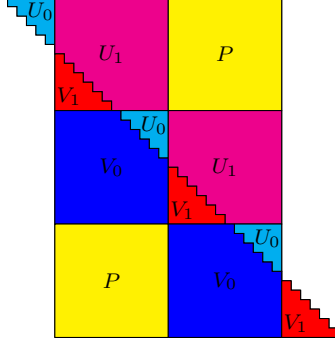


Figure 11: Decomposition of \mathcal{O}_S into elementary blocks U_0 , U_1 , V_0 and V_1

is balanced with an even cardinality, then we have $\mathbf{m}_{U_0 \cup V_0}(x) = \mathbf{m}_{U_1 \cup V_1}(x) = \frac{p^2}{2}$ for all $x \in \mathbb{Z}/2\mathbb{Z}$. It follows that

$$\mathbf{m}_{U_0 \cup U_1}(x) = \mathbf{m}_{U_0}(x) + \mathbf{m}_{U_1}(x) = (p^2/2 - \mathbf{m}_{V_0}(x)) + (p^2/2 - \mathbf{m}_{V_1}(x)) = p^2 - \mathbf{m}_{V_0 \cup V_1}(x)$$

for all $x \in \mathbb{Z}/2\mathbb{Z}$. Therefore $\delta \mathbf{m}_{U_0 \cup U_1} = \delta \mathbf{m}_{V_0 \cup V_1}$. Moreover, since

$$\mathbf{m}_{U_0}(x) - \mathbf{m}_{V_1}(x) = (\mathbf{m}_{U_0}(x) + \mathbf{m}_{U_1}(x)) - (\mathbf{m}_{U_1}(x) + \mathbf{m}_{V_1}(x)) = \mathbf{m}_{U_0 \cup U_1}(x) - \mathbf{m}_{U_1 \cup V_1}(x) = \mathbf{m}_{U_0 \cup U_1}(x) - p^2/2$$

for all $x \in \mathbb{Z}/2\mathbb{Z}$, it follows that

$$\delta \mathbf{m}_{U_0} - \delta \mathbf{m}_{V_1} = \delta \mathbf{m}_{U_0 \cup U_1} = \delta \mathbf{m}_{V_0 \cup V_1}.$$

Therefore the blocks U_0 and $U_1 \cup U_0$ are balanced if and only if the blocks V_1 and $V_0 \cup V_1$ are balanced. This concludes the proof. \square

For instance, in the orbit \mathcal{O}_{X^∞} associated with the 24-tuple $X = 011101101000000011110110$, the existence of balanced generalized Pascal triangles for all the possible sizes can then be obtained from at least 5 positions. The following generalized Pascal triangles $\Delta X^\infty(i_0, j_0, 24k + r)$ are balanced for all non-negative integers k :

(i_0, j_0)	r
(2, 11)	0, 1, 2, 16, 19
(6, 13)	4, 6, 13, 14, 20, 21, 22, 23
(1, 19)	1, 2, 3, 4, 9, 11, 12, 17, 18
(3, 15)	0, 1, 2, 7, 10, 15, 18, 23
(5, 14)	0, 5, 8, 22, 23

More explicitly, the following generalized Pascal triangles $\Delta(X_1^\infty[24k + r], X_2^\infty[24k + r])$ are balanced for all non-negative integers k :

X_1	X_2	r
010111100010110001101100	000001100100101001001100	0, 1, 2, 16, 19
001101011100011011000110	010010111101111010010101	4, 6, 13, 14, 20, 21, 22, 23
000100010011111000100011	011010100000010101101111	1, 2, 3, 4, 9, 11, 12, 17, 18
000010110111000000000000	010000110100101100001000	0, 1, 2, 7, 10, 15, 18, 23
011101100110001101100011	001100100101001001100000	0, 5, 8, 22, 23

6 Periodic balanced triangles modulo m

The definitions of Steinhaus and generalized Pascal triangles can be extended in $\mathbb{Z}/m\mathbb{Z}$ by considering the sum modulo m as the local rule, instead of the sum modulo 2. Examples of Steinhaus and generalized Pascal triangles modulo 7 are depicted in Figure 12.

The triangle T is said to be *balanced* if its multiplicity function is constant or almost constant, i.e., if

$$\delta \mathbf{m}_T := \max \{ |\mathbf{m}_T(x_1) - \mathbf{m}_T(x_2)| \mid x_1, x_2 \in \mathbb{Z}/m\mathbb{Z} \} \in \{0, 1\}.$$

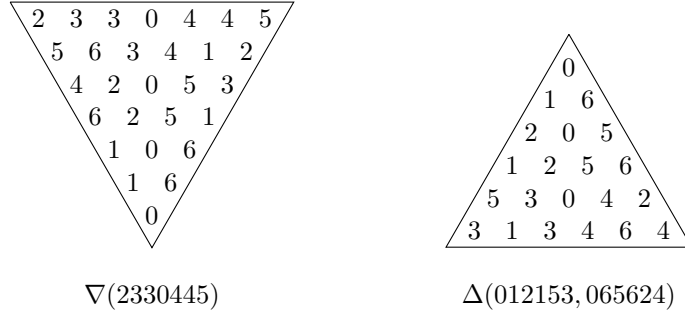


Figure 12: Steinhaus and generalized Pascal triangles in $\mathbb{Z}/7\mathbb{Z}$

Note that, when the triangle T is of size n such that the triangular number $\binom{n+1}{2}$ is divisible by m , the triangle T is balanced if $\delta \mathbf{m}_T = 0$, i.e., if the multiplicity function \mathbf{m}_T is constant, equal to $\frac{1}{m} \binom{n+1}{2}$. For example, the triangles in Figure 12, $\nabla(2330445)$ and $\Delta(012153, 065624)$, are balanced in $\mathbb{Z}/7\mathbb{Z}$ since they contain all the elements of $\mathbb{Z}/7\mathbb{Z}$ with the same multiplicity.

This generalization was introduced in [13], where the author posed the following problem.

Molluzzo Problem. Does there exist, for any positive integers m and n such that the triangular number $\binom{n+1}{m}$ is divisible by m , a balanced Steinhaus triangle modulo m of size n ?

This problem is still largely open. It is positively solved only for $m = 2$ (Steinhaus Problem), 4 [8], 5, 7 [4] and for all $m = 3^k$ with $k \in \mathbb{N}$ [4, 5]. It is also known [4] that there exist some values of m and n for which there do not exist balanced Steinhaus triangles: for $n = 5$ and $m = 15$ or $n = 6$ and $m = 21$.

In this section, some of these solutions are recalled because they involve balanced triangles that are also periodic.

First, in [5], it was proved that, for any odd number m , the Steinhaus triangles generated from an arithmetic progression with an invertible common difference in $\mathbb{Z}/m\mathbb{Z}$ and of length n is balanced for all $n \equiv 0$ or $-1 \pmod{\text{ord}_m(2^m)m}$, where $\text{ord}_m(2^m)$ is the multiplicative order of 2^m modulo m . For instance, for $(i \pmod m)_{i \in \mathbb{Z}}$, the sequence of the integers modulo m , the Steinhaus triangle $\nabla(0, 1, 2, \dots, n-1)$ is balanced in $\mathbb{Z}/m\mathbb{Z}$ for all $n \equiv 0$ or $-1 \pmod{\text{ord}_m(2^m)m}$. In the proof of this result, it appears that the orbit generated from any arithmetic progression is periodic of period $\text{ord}_m(2^m)m$. This implies that all these balanced Steinhaus triangles modulo an odd number m are also $\text{ord}_m(2^m)m$ -periodic. Remark that a generalization of this result in higher dimensions for balanced simplices can be found in [7] and these simplices are also of periodic structure.

In [6], the following integer sequence $S = (a_j)_{j \in \mathbb{Z}}$ defined by

$$\begin{cases} a_{3j} &= j, \\ a_{3j+1} &= -1 - 2j, \\ a_{3j+2} &= 1 + j, \end{cases}$$

for all $j \in \mathbb{Z}$, is considered. Note that this sequence is an interlacing of three arithmetic progressions. It was proved that, for every odd number m , the orbit of the projection of S in $\mathbb{Z}/m\mathbb{Z}$ contains balanced Steinhaus triangles of size n , for all n divisible by m and for all $n \equiv -1 \pmod{3m}$, and balanced generalized Pascal triangles of size n , for all $n \equiv -1 \pmod{m}$ and for all n divisible by $3m$. It was proved in [6] that the orbit of this special sequence modulo m is periodic of period $6m$. Thus, there exist periodic balanced triangles of these size modulo m odd.

All these results lead to consider the following subproblem of the Molluzzo Problem.

Problem 2. Does there exist, for any positive integers m , infinitely many periodic balanced (Steinhaus or generalized Pascal) triangles modulo m ?

This problem is positively solved for any odd number m , in [6, 5], and for $m = 2$ in the present paper. It remains to analyse the case where m is even and $m \geq 4$.

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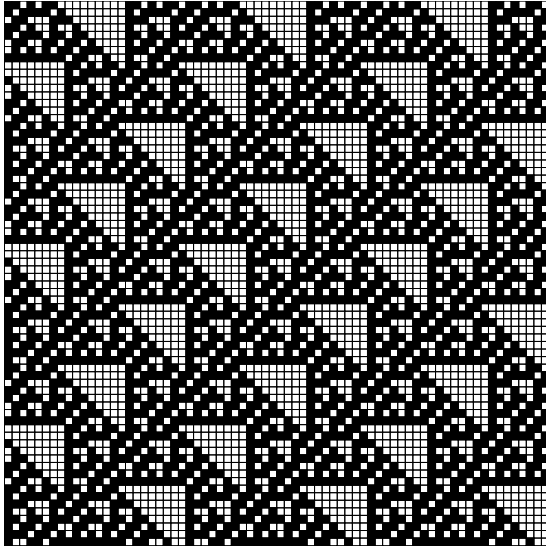
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A The 24-periodic orbits with balanced periods

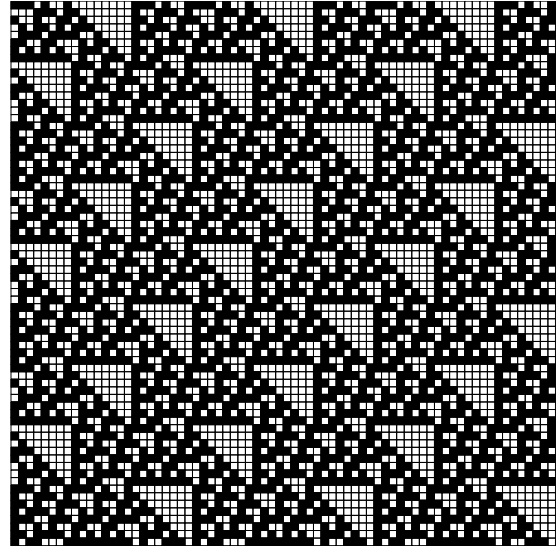
In this appendix, the orbits of representatives X_i , for all the 17 elements of $\overline{\mathcal{BPO}}_{24} = \{\overline{X_1}, \overline{X_2}, \dots, \overline{X_{17}}\}$, are depicted.

i	X_i	i	X_i
1	11010110000000011010110	10	101110010100000011111001
2	111010010000000011101001	11	101001000010000010000100
3	011001001000000011100100	12	110110100010000011111010
4	011101101000000011110110	13	110000010010000011100001
5	001111101000000010111110	14	101111110010000010011111
6	111110000100000010111000	15	100110000110000011111000
7	100101000100000011010100	16	000011101110000011101110
8	110111000100000010011100	17	100010100010100010100010
9	111100010100000010110001		

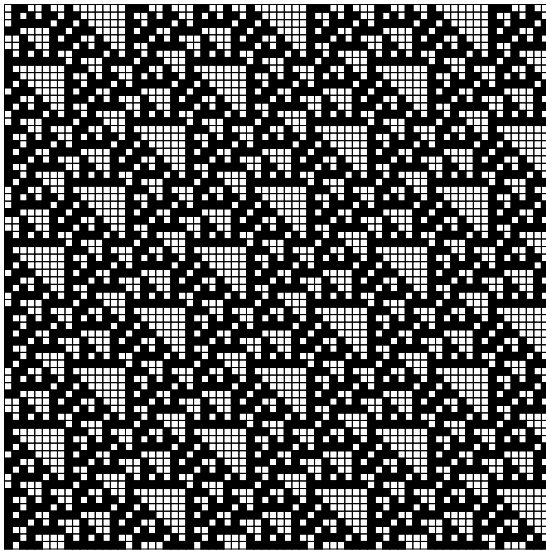
Moreover, we have also obtained the orbits of the elements of $\overline{\mathcal{BPO}}_{12} = \{\overline{Y_1}, \overline{Y_2}\}$. Indeed, as already remarked, we have $\overline{X_{16}} = Y_1^2$ and $\overline{X_{17}} = Y_2^2$. Therefore the orbits $\mathcal{O}_{X_{16}^\infty}$ and $\mathcal{O}_{X_{17}^\infty}$ correspond to $\mathcal{O}_{Y_1^\infty}$ and $\mathcal{O}_{Y_2^\infty}$, respectively.



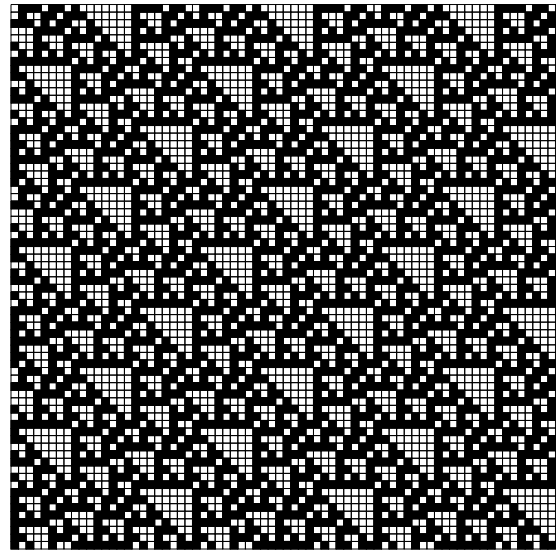
$\mathcal{O}_{X_1^\infty}$



$\mathcal{O}_{X_2^\infty}$



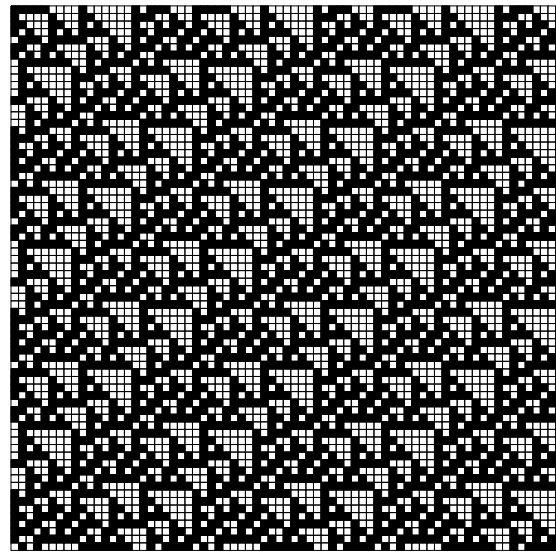
$\mathcal{O}_{X_3^\infty}$



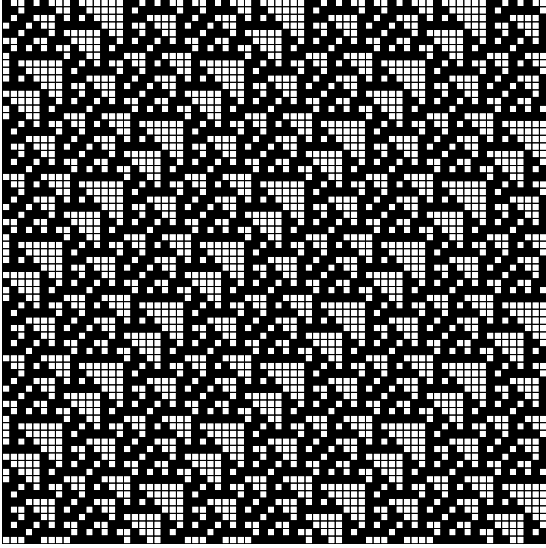
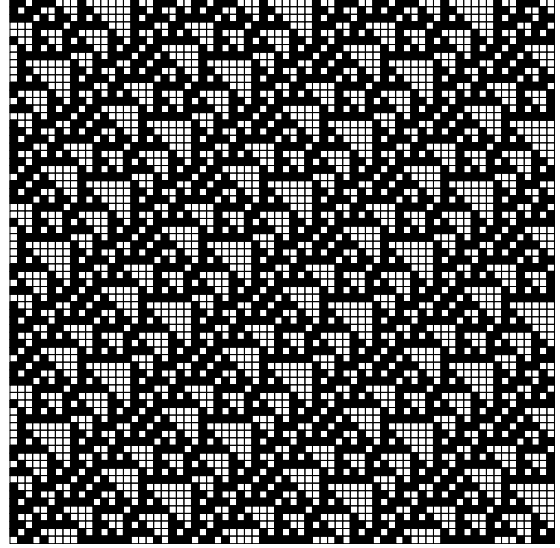
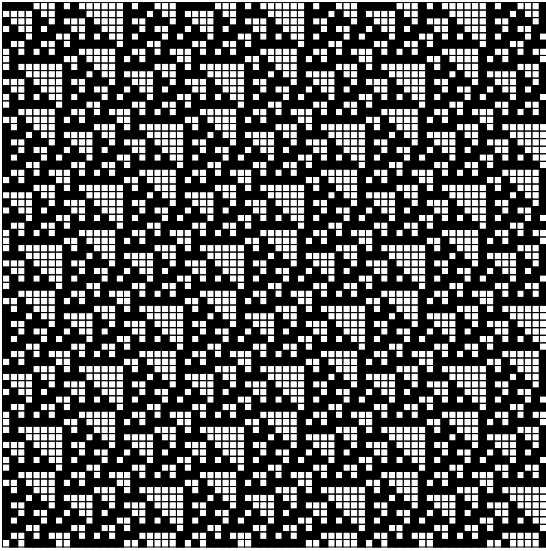
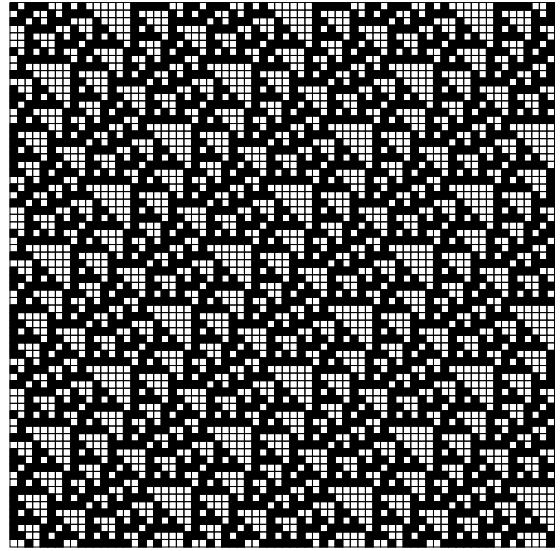
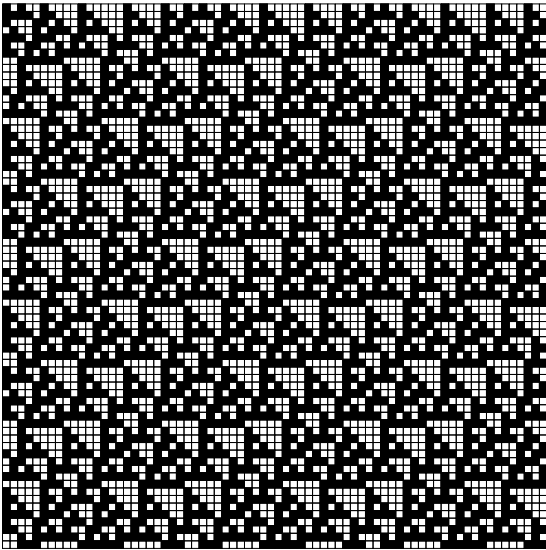
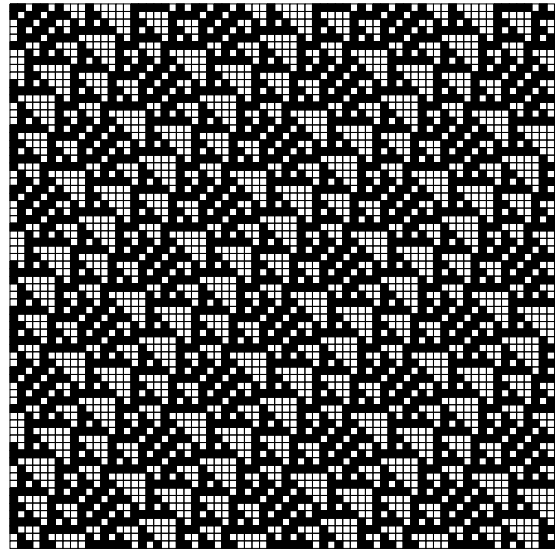
$\mathcal{O}_{X_4^\infty}$

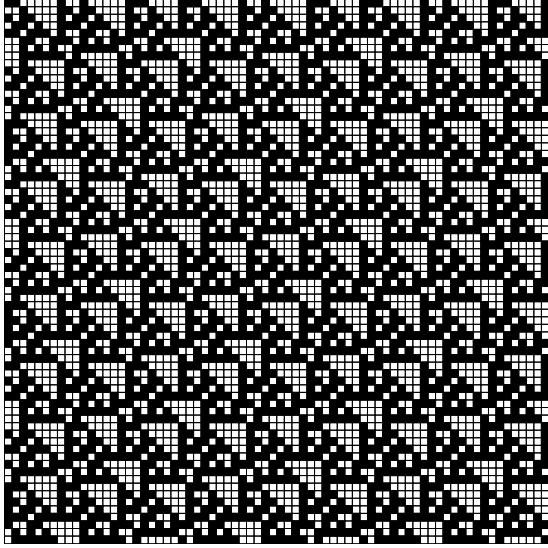


$\mathcal{O}_{X_5^\infty}$

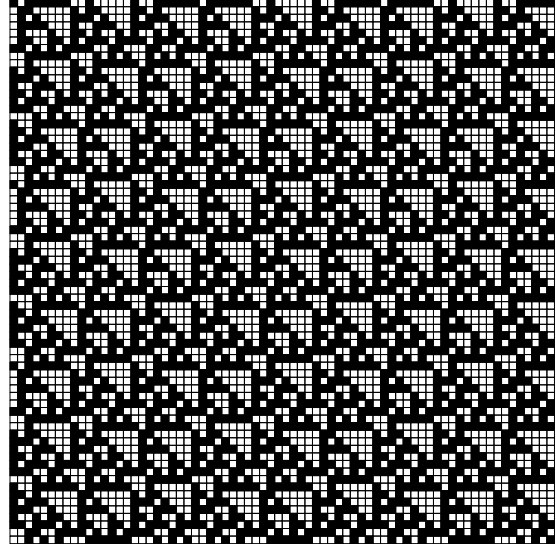


$\mathcal{O}_{X_6^\infty}$

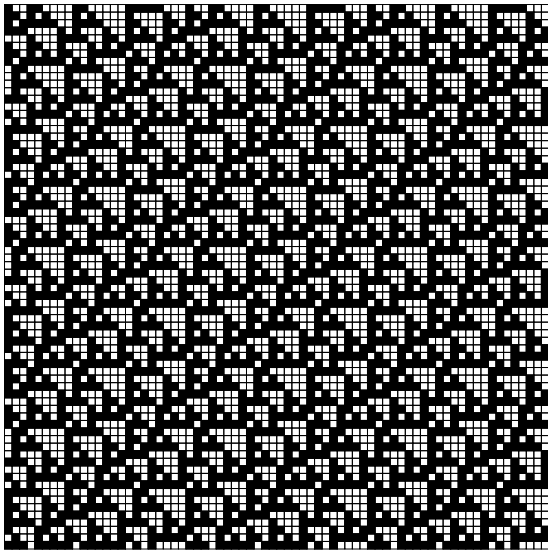
 $\mathcal{O}_{X_7^\infty}$  $\mathcal{O}_{X_8^\infty}$  $\mathcal{O}_{X_9^\infty}$  $\mathcal{O}_{X_{10}^\infty}$  $\mathcal{O}_{X_{11}^\infty}$  $\mathcal{O}_{X_{12}^\infty}$



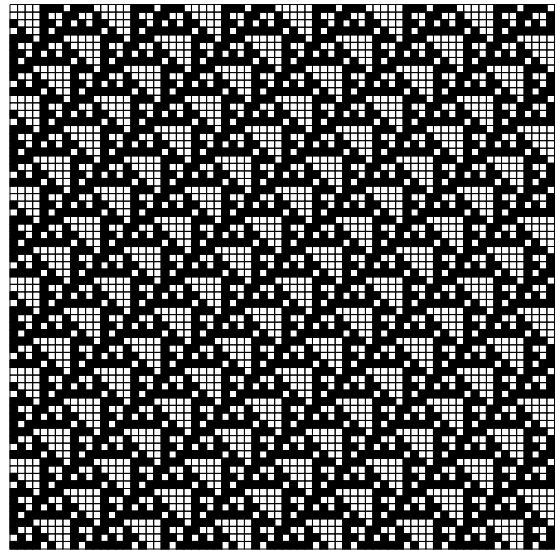
$\mathcal{O}_{X_{13}^\infty}$



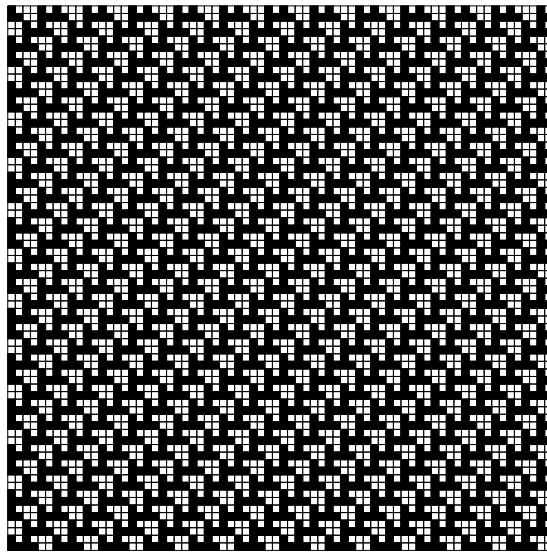
$\mathcal{O}_{X_{14}^\infty}$



$\mathcal{O}_{X_{15}^\infty}$



$\mathcal{O}_{X_{16}^\infty}$



$\mathcal{O}_{X_{17}^\infty}$