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# Spy Game in trees and grids <sup>\*</sup>

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## Abstract

In the Spy game, a *spy* is placed first at some vertex of a graph  $G$ . Then,  $k > 0$  *guards* are also occupying some vertices of  $G$ . At each turn, the spy moves at *speed*  $s \geq 2$ , i.e., along at most  $s$  edges and then, each guard moves at speed 1. The spy and any number of guards may occupy the same vertex. The goal of the guards is to *control* the spy at distance  $d \geq 0$ , i.e., to ensure that, at every turn (after the guards’ moves), at least one guard is at distance at most  $d$  from the spy, whatever be the strategy of the spy. We aim at determining a winning strategy for the guards using the smallest number of guards, denoted by  $gn_{s,d}(G)$  and called the *guard number* of  $G$  (for fixed  $s$  and  $d$ ).

In this paper, we study the Spy game through the framework of *fractional games*, where each vertex can have a fractional amount of guards and the moves of the guards are modeled by flows. This framework allows us to prove that  $gn_{s,d}(T)$  and a corresponding strategy for the guards can be computed in polynomial-time in the class of trees  $T$ . This algorithm is mainly based on a Linear Program. Using this framework, we also prove that there exists  $1 > \beta > 0$ , such that  $gn_{s,d}(G) = \Omega(n^{1+\beta})$  in any  $n \times n$  grid (or torus)  $G$ . This extends some known results on the Eternal Dominating Set in grids. Finally, we prove that there exists  $0 < \alpha < \log(3/2)$  such that there exists a fractional winning strategy using  $O(n^{2-\alpha})$  fractional guards in any  $n \times n$  grid (or torus), for any  $s \geq 2$  and  $d \geq 0$ . Note that, it is only known that  $gn_{s,d}(G) = O(n^2)$  in any  $n \times n$  grid (or torus).

Besides our results, we believe that the methods using fractional relaxation and Linear Programming are a promising way to better understand other combinatorial games in graphs.

**Keywords:** Cops and Robber games, Graphs, Linear Programming

## 1 Introduction

Many turn-by-turn 2-player games involve mobile agents moving in a graph to achieve some goals. A famous one is the *Cops and Robber* game in which a team of cops aims at capturing a robber in a graph [14, 13, 4]. In this game,  $k \in \mathbb{N}$  cops are first placed at the vertices of a graph  $G = (V, E)$ , then one robber is placed at one vertex. Turn-by-turn, each cop may move along one edge, and then, the robber may move along one edge. The main question is, given a

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graph  $G$ , what is the minimum number  $k$  of cops, denoted by  $cn(G)$  (for *cop-number*), required to capture the robber in  $G$ , i.e., to ensure that one cop eventually reaches the same vertex as the robber. In 1985, Meyniel conjectured that  $cn(G) = O(\sqrt{n})$  for any graph with  $n$  vertices [4]. In 2001, Schröder conjectured that  $cn(G) \leq g + 3$  for every graph with *genus* at most  $g$  [15]. Both conjectures are still open.

Meyniel’s conjecture has been proved in many graph classes [4]. In particular, it has been proved that  $cn(G) \leq \lfloor \frac{3}{2}g \rfloor + 3$  for any graph with genus at most  $g$  [15]. Moreover, it is easy to check that  $cn(G_{n \times m}) \leq 2$  for any  $n \times m$  grid  $G_{n \times m}$ . It has also been proved that  $cn(G) \leq 3$  for any planar graph  $G$  (i.e., of genus 0) [1]. Note, however, that it is not known whether  $cn(G) \leq 3$  for any graph  $G$  of genus 1.

To try to prove these conjectures, new variants of Cops and Robber games have been studied. In [9, 6], the Robber is *faster*, i.e., it can move along more than one edge at each turn. This variant seems to be very different from the original one (when the Robber has speed 1): the number of cops required to capture a robber with speed 2 in a grid  $G_{n \times n}$  is  $e^{\Omega(\log n / \log \log n)}$  but is still not known [2]. Another variant is the Cops and Robber with *radius of capture*. In this variant, all cops and the robber have speed one, and the goal of the cops is to reach a vertex at distance at most  $r \geq 0$  from the robber ( $r$  is the radius of capture) [3]. It is interesting to note that Meyniel’s conjecture extends to both the variants with speed or with radius.

## 1.1 Spy game

Recently, [7] defined and studied a new game, called *Spy game*, related to Cops and robber games. In this game, the *spy* is first placed at some vertex of a graph  $G = (V, E)$ . Then,  $k \in \mathbb{N}$  *guards* are placed at the vertices of  $G$ . Turn-by-turn, the spy may move along at most  $s \geq 1$  edges, and then each cop may move along one edge. The goal of the game is to minimize the number of guards, called *guard-number* and denoted by  $gn_{s,d}(G)$ , ensuring that, at every step after the guards’ turn, the spy is at distance at most  $d \geq 0$  from at least one guard (we say that the spy is *controlled* at distance  $d$ ). It has been shown that, for every  $d \geq 0$  and  $s \geq 2$ , computing  $gn_{s,d}(G)$  is NP-hard in a subclass of chordal graphs (precisely, graphs obtained from a clique and some paths, where one end of each path is connected to some vertices of the clique) [7]. The guard-number of paths is also characterized and almost tight lower and upper bounds are given in the case of cycles [7].

In this paper, we are considering the Spy game in trees and grids. All our results are based on a fractional relaxation of the game, which seems to be interesting by itself.

## 1.2 Eternal Dominating Set

The Spy game generalizes the *Eternal dominating set* Problem [11]. In the latter game, a team of mobile agents is occupying some vertices of a graph. At every turn, each of the agents is allowed to move to one of its neighbors or may stay idle (note that, in the original variant, only one agent was allowed to move at each turn [5]). The objective of the game is that, for any infinite sequence  $(v_1, v_2, \dots)$  of vertices, at the end of turn  $i$ , the vertex  $v_i$  is occupied. In other words, the agents must always occupy a dominating set  $D$ , such that, for any vertex  $v \notin D$ , the agents can move to another dominating set containing  $v$ . The minimum number of agents ensuring to win the game in a graph  $G$  is denoted by  $\gamma^m(G)$ . It is easy to see that the Eternal dominating set problem is equivalent to the Spy game when the spy is arbitrarily fast and for  $d = 0$ , i.e.,  $\gamma^m(G) = gn_{s,0}(G)$  for any  $s$  which is at least the diameter of the graph.

Eternal dominating set has been investigated in many graph classes. In grids, only few cases are known: for instance,  $1 + \lceil 4n/5 \rceil \leq \gamma^m(G) \leq 2 + \lceil 4n/5 \rceil$  in  $3 * n$  grids [8]. Our results provide new lower bounds for  $\gamma^m(G)$  in any  $n \times n$  grid  $G$ . In the class of trees  $T$ ,  $\gamma^m(T)$  can be computed in polynomial-time [12]. The key property in their algorithm is that, for maintaining

an Eternal dominating set in trees, it is sufficient to assign some area to each guard (roughly, two guards are responsible of a set of leaves adjacent to a common vertex). This property also holds for the Spy game (for any  $s \geq 2$  and  $d \geq 0$ ) in the class of paths where it has been proved that there is always an optimal solution where the guards are assigned disjoint subpaths. This property does not hold anymore for the Spy game (for any  $s \geq 2$  and  $d > 0$ ) in the class of trees as illustrated in the following example<sup>1</sup>. Roughly, this example illustrates the fact that, in trees, an optimal strategy for the guards cannot be obtained by dividing the tree into small subgraphs and assigning a constant number of guards to each part.

We present an example in the case  $s = 2$  and  $d = 1$  but it can be generalized to any  $s \geq 2$  and  $d > 0$ . Let  $S$  be the tree obtained from a star with three leaves by subdividing each edge exactly twice (i.e.,  $S$  has 10 vertices). Let  $(S_i)_{i \leq k}$  be  $k$  disjoint copies of  $S$  and let  $c_i$  be the unique vertex of degree 3 of  $S_i$ . Finally, let  $T$  be the tree obtained by adding one vertex  $c$  and making it adjacent to every  $c_i, i \leq k$ . It can be shown that  $gn_{2,1}(T) = k + 1$  and that, when the spy is in  $c$ , the guards have to occupy the vertices  $c, c_1, \dots, c_k$ . For any  $i \leq k$  and any non-leaf vertex  $v \in V(T)$ , there is a strategy of the spy that brings the guard initially at  $c_i$  to  $v$ . For this purpose, let  $j \neq i$  be such that  $v \in V(S_j)$ . The spy first goes (at full speed) to a leaf of  $S_i$ , then to another leaf of  $S_i$ , then it goes to a leaf of  $S_j$  that is not adjacent to  $v$  and finally the spy goes to the leaf of  $S_j$  that is adjacent to  $v$ . It can be checked that the guard that was initially at  $c_i$  must occupy  $v$ .

Part of this work is devoted to prove that, in trees, there always exists an optimal strategy with some other useful property (called spy-positional) that will allow us to derive a polynomial-time algorithm for computing  $gn_{s,d}$  and designing optimal strategies, hence extending previous work with  $d = 0$  to any  $d \geq 0$ .

### 1.3 Our results

In this paper, we consider the Spy game in the class of trees and grids. We prove that the guard number of any tree can be computed in polynomial-time and give non-trivial bounds on the guard number of grids. For this purpose, we develop a new method to analyze two-player games in graphs based on Linear Programming.

In the framework of fractional games, introduced in [10], the mobile agents are not integral entities anymore. Roughly, each vertex  $v \in V$  of a graph  $G = (V, E)$  can be occupied by an amount  $g_v \in \mathbb{R}^+$  of agents. Then, the total amount of mobile agents is  $\sum_{v \in V} g_v \in \mathbb{R}^+$ . Then, the moves of the agents are represented by some flow along the arcs. That is, at every turn, every vertex  $v$  sends some flow  $f_{vu} \in \mathbb{R}^+$  to each of its neighbors  $u \in N(v)$ , given that  $\sum_{u \in N(v)} f_{vu} \leq g_v$ , i.e., a vertex cannot send more than what it had.

Here, we adapt the Spy game to this framework. Roughly, the spy remains an integral entity (i.e., the rules do not change for the spy) while the guards are relaxed into fractional ones. The winning condition for the guards becomes that, at every turn, there must be a total amount of at least 1 guard in the ball of radius  $d$  around the position of the spy. The *fractional guard number*  $fgn_{s,d}(G)$  is the minimum total amount of fractional guards needed to control a spy with speed  $s$  at distance  $d$ . Since the fractional Spy game is a relaxation of the Spy game (roughly, the guards are more powerful), for any graph  $G$ ,  $fgn_{s,d}(G) \leq gn_{s,d}(G)$ .

Our contribution is twofold: for every  $s \geq 2$  and  $d \geq 0$ :

- We prove that there exists a Linear Program for computing  $fgn_{s,d}(T)$  and a corresponding strategy in the class of trees  $T$ . Moreover, such a Linear Program has size polynomial in the size of  $T$  and thus, runs in polynomial-time. Then, we show that any fractional strategy (winning for the guards) using  $k$  guards in a tree can be turned into a winning (integral) strategy using  $\lceil k \rceil$  guards. Altogether, this shows that, in any tree  $T$ ,  $fgn_{s,d}(T) = gn_{s,d}(T)$ ,

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<sup>1</sup>Thank you to Rudini Sampaio and Nicolas Martins for pointing out this example.

and that  $gn_{s,d}(T)$  and a corresponding winning strategy can be computed in polynomial-time.

- Then, we show that there is a constant  $0 < \beta < 1$  such that, for any  $n \times n$  grid  $G_{n \times n}$ ,  $\Omega(n^{1+\beta}) = fgn_{s,d}(G_{n \times n}) \leq gn_{s,d}(G_{n \times n})$ . This gives the first non trivial lower bound for the guard number (and also for  $\gamma^m$ ) in the class of grids. Finally, we show that there exists a constant  $0 < \alpha \leq \log(3/2) < 1$  such that  $fgn_{s,d}(G_{n \times n}) = O(n^{2-\alpha})$ . Note that, the best known upper bound for  $gn_{s,d}(G_{n \times n})$  is  $O(n^2)$ . The same bounds hold for the  $n \times n$  torus.

Finally, we believe that the methods using Linear Programming used in this paper are a promising way to better understand other combinatorial games in graphs.

## 2 Model and definitions

In this paper, all graphs are simple (without loops nor multi-edges), connected, and undirected. For any vertex  $v \in V$  in a graph  $G = (V, E)$ , let  $N(v)$  denote the set of neighbors of  $v$  and  $N[v] = N(v) \cup \{v\}$ . Moreover, for any integer  $s \geq 0$  and vertex  $v \in V$ , let  $N_s[v]$  be the set of vertices at distance at most  $s$  from  $v$ .

**Spy game.** Let  $s \geq 2$ ,  $d \geq 0$ , and  $k > 0$  be three integers. The Spy game in a graph  $G = (V, E)$  proceeds as follows. First, the spy is placed at a vertex. Then, the  $k$  guards are placed at some vertices. Possibly, one vertex may be occupied by several guards, and the guards and the spy may occupy the same vertex. Then, turn-by-turn, the spy first moves to any vertex at distance at most  $s$  from its current position (possibly, it may stay idle), and then each guard may move along one edge. The guards win if they ensure that, at every step, at least one guard is at distance at most  $d$  from the spy (after the guards' moves). Otherwise, the spy wins.

The parameter  $s$  is called the *speed* of the spy. Moreover, if there is a guard at distance at most  $d$  from the spy, we say that the guards control the spy at distance  $d$ . In what follows, let  $gn_{s,d}(G)$  denote the minimum number of guards needed to always control at distance  $d$  a spy with speed  $s$  in a graph  $G$ .

**Spy-positional strategies.** A strategy for the guards is a function describing the moves of the guards at every step. A strategy is *winning* if it allows the guards to perpetually control the spy. It is easy to show that there is always an optimal winning strategy (using the minimum number of guards) which is *positional*, i.e. such that the next move is only determined by the current position of both the spy and the guards, and not by the history of the game<sup>2</sup>. In other words, there is always an optimal winning strategy which is a function that takes the current positions of the spy and of the guards and returns the new positions of the guards (and so, their moves).

In this paper, we will also consider more constrained strategies. A winning strategy is said to be *Spy-positional* if it depends only on the position of the spy. That is, in a spy-positional strategy  $\sigma$ , the positions of the guards are only determined by the position of the spy. In particular, every time the spy occupies some vertex  $v$ , the set of vertices occupied by the guards is defined by some function  $\sigma_v : V(G) \rightarrow \mathbb{N}$  such that, for every  $u \in V$ ,  $\sigma_v(u)$  is the number of guards occupying  $u$  when the spy is occupying  $v$ . It is important to note that, in a spy-positional strategy, it is not required that the same guards occupy the same vertices when the spy is at some vertex. That is, assume that, at some step, the spy occupies some vertex  $v$ , some guard  $A$  occupies a vertex  $a$  and a guard  $B$  occupies a vertex  $b$ . It may happen that, after some steps, the spy goes back to  $v$  and now Guard  $A$  is at  $b$  and Guard  $B$  is at  $a$  (however, the set of vertices occupied by the guards is the same). The example presented in Section 1.2 illustrates this fact.

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<sup>2</sup>That can be easily shown by considering the configurations' graph of the game.

**Fractional Spy game.** In this paper, we will repeatedly use a fractional relaxation of the spy game, in which the guards are no longer integral entities. More formally, the fractional game proceeds as follows in a graph  $G = (V, E)$  and for parameters  $s \geq 2$ ,  $d \geq 0$  (two integers) and a positive real  $k \in \mathbb{R}^{+*}$ . First, the spy is placed at a vertex. Then, each vertex  $v$  receives some amount  $g_v \in \mathbb{R}^+$  (a non negative real) of guards such that the total amount  $\sum_{v \in V} g_v = k$ . Then, turn-by-turn, the spy may first move at distance at most  $s$  from its current position. Then, the moves of the "fractional" guards are defined as a flow. That is, for any vertex  $v \in V$ , and for any vertex  $u \in N[v]$ , let  $f(v, u) \in \mathbb{R}^+$  be the amount of guards going from  $v$  to  $u \in N[v]$ . We impose that  $\sum_{u \in N[v]} f(v, u) \leq g_v$ , i.e., the amount of guards leaving  $v$  is no more than what was at  $v$ . Finally, for any vertex  $v \in V$ , the amount of guards occupying  $v$  after the move is  $g'_v = \sum_{u \in N[v]} f(u, v)$ .

Let  $fgn_{s,d}(G)$  denote the minimum total amount of (fractional) guards needed to always control at distance  $d$  a spy with speed  $s$  in a graph  $G$ . Finally, let  $fgn_{s,d}^*(G)$  be the minimum total amount of (fractional) guards needed to always control at distance  $d$  a spy with speed  $s$  in a graph  $G$ , when the guards are constrained to play spy-positional strategies.

By definition,

**Claim 1.** For any graph  $G$  and any  $s \geq 2$ ,  $d \geq 0$ ,

$$fgn_{s,d}(G) \leq \min\{fgn_{s,d}^*(G), gn_{s,d}(G)\}.$$

### 3 Spy-positional fractional strategies in general graphs

This section is devoted to present a polynomial-time algorithm that computes optimal spy-positional fractional strategies in general graphs. Here, optimal means using the minimum total amount of guards with the extra constraint that guards are restricted to play spy-positional strategies. In other words, we prove that, for any graph  $G$ ,  $s \geq 2$ , and  $d \geq 0$ ,  $fgn_{s,d}^*(G)$  and a corresponding strategy can be computed in polynomial time.

We prove this result by describing a Linear Program with polynomial size that computes such strategies. In Section 4, we will show that in any tree  $T$ ,  $gn_{s,d}(T) = fgn_{s,d}^*(T)$ . More precisely, we will show that in trees, the Linear Program below can be used to compute optimal (integral) strategies in polynomial time. Let us now describe the Linear Program.

**Variables.** Let  $G = (V, E)$  be a connected  $n$ -node graph. Recall that a spy-positional strategy is defined by, for each position of the spy, the amount of guards that must occupy each vertex. Therefore, for any two vertices  $u, v \in V$ , let  $g_{u,v} \in \mathbb{R}^+$  be the non negative real variable representing the amount of guards occupying vertex  $v$  when the spy is at vertex  $u$ .

Moreover, for any  $x \in V$ ,  $y \in N_s[x]$  and for any  $u \in V$  and  $v \in N[u]$ , let  $f_{x,y,u,v} \in \mathbb{R}^+$  be the non negative real variable representing the amount of guards going from vertex  $u$  to  $v \in N[u]$  when the spy goes from  $x$  to  $y \in N_s[x]$ .

Finally, a variable  $k$  will represent the total amount of guards.

Overall, there are  $(|E| + 1)n^2 = O(n^4)$  real variables.

Note that they fully describe a strategy, since  $g$  encodes a distribution of cops for every position of the spy and  $f$  describes a feasible transition between two successive distributions.

**Objective function.** We aim at minimizing the total amount of guards.

$$\text{Minimize } k \tag{1}$$

**Constraints.** The first family of constraints states that, for every position  $v \in V$  of the spy, the total amount of guards is at most  $k$ .

$$\forall v \in V, \sum_{w \in V} g_{v,w} \leq k \quad (2)$$

The second family of constraints states that, for every position  $v \in V$  of the spy, the amount of guards at distance at most  $d$  from the spy is at least 1, i.e., the guards always control the spy at distance  $d$ .

$$\forall v \in V, \sum_{w \in N_d[v]} g_{v,w} \geq 1 \quad (3)$$

The third family of constraints states that, for any move of the spy (from  $x$  to  $y \in N_s[x]$ ), the corresponding moves of the guards ensure that the amount of guards leaving a vertex  $v \in V$  plus what remains at  $v$  equals the amount of guards that was at  $v$  before the move.

$$\forall x \in V, y \in N_s[x], v \in V, \sum_{w \in N[v]} f_{x,y,v,w} = g_{x,v} \quad (4)$$

The fourth family of constraints states that, for any move of the spy (from  $x$  to  $y \in N_s[x]$ ), the corresponding moves of the guards ensure that the amount of guards that are at a vertex  $w \in V$  after the moves equals the amount of guards arriving in  $w$  plus what remains at  $w$ .

$$\forall x \in V, y \in N_s[x], w \in V, \sum_{v \in N[w]} f_{x,y,v,w} = g_{y,w} \quad (5)$$

Finally, the last family of constraints expresses the definition domain of the variables:

$$k \geq 0 \quad (6)$$

$$\forall u, v \in V, g_{u,v} \geq 0 \quad (7)$$

$$\forall x \in V, y \in N_s[x], v \in V, w \in N[v], f_{x,y,v,w} \geq 0 \quad (8)$$

There are  $O(n^4)$  constraints and the above Linear Program has polynomial size and clearly computes an optimal spy-positional fractional strategy. Hence:

**Theorem 2.** *For any connected graph  $G$ , and any two integers  $s \geq 2$  and  $d \geq 0$ , the above Linear Program computes  $fgn_{s,d}^*(G)$  and a corresponding (spy-positional) strategy in polynomial time.*

## 4 Spy game is Polynomial in Trees

This section is devoted to prove that, in any tree  $T$  and for any  $s \geq 2$ ,  $d \geq 0$ ,  $gn_{s,d}(T) = fgn_{s,d}^*(T)$ . Therefore, using the Linear Program of Section 3, we prove that computing  $gn_{s,d}(T)$  can be done in polynomial time in trees.

The proof is twofold. First, we prove that  $gn_{s,d}(T) = fgn_{s,d}(T)$  for any  $s \geq 2$  and  $d \geq 0$  (i.e., the integral gap is null in trees), and then that  $fgn_{s,d}(T) = fgn_{s,d}^*(T)$ .

**Representation of strategies.** Let  $G = (V, E)$  be a graph,  $s \geq 2$  and  $d \geq 0$  be two integers. Let  $V = \{v_1, \dots, v_n\}$ . A winning strategy  $\sigma$  using  $k \in \mathbb{R}^+$  guards is defined as a set  $\sigma = \{\mathcal{C}_v\}_{v \in V}$  of sets of configurations. That is, for any  $v \in V$  (a possible position for the spy),  $\mathcal{C}_v$  is a non-empty set of functions, called *configurations*, that represent the possible positions of the guards when the spy is at  $v$ . More precisely, any  $\omega \in \mathcal{C}_v$  is a function  $\omega : V \rightarrow \mathbb{R}^+$ , where  $\omega(u) \in \mathbb{R}^+$  represents the amount of guards at vertex  $u \in V$ , that must satisfy  $\sum_{u \in V} \omega(u) = k$  and  $\sum_{u \in N_d[v]} \omega(u) \geq 1$ . Finally, for any  $v \in V$ , any  $\omega \in \mathcal{C}_v$ , and any  $v' \in N_s[v]$ , there must exist  $\omega' \in \mathcal{C}_{v'}$  such that the guards can go from  $\omega$  to  $\omega'$  in one step (i.e., a valid flow from  $\omega$  to  $\omega'$  must exist).

A strategy is integral if  $k \in \mathbb{N}^+$ , every of its configurations is a function  $V \rightarrow \mathbb{N}$ , and every move is an integral flow.

The *size* of a strategy is the number of different configurations necessary to describe the strategy. Note that, if the strategy is not spy-positional, a same position for the spy may correspond to different positions of the guards. Therefore, the size of an integral strategy using  $k$  guards in an  $n$ -node graph is  $O(n^k)$ . Moreover, the size of a fractional strategy is *a priori* unbounded. However, any (fractional or integral) spy-positional strategy has size  $O(n)$  ( $|\mathcal{C}_v| = 1$  for every  $v \in V$ ).

**Theorem 3.** *For any tree  $T$  and for any  $s \geq 2$ ,  $d \geq 0$ ,  $gn_{s,d}(T) = fgn_{s,d}(T)$ . More precisely, any fractional winning strategy using a total amount of  $k \in \mathbb{R}^+$  guards can be transformed into an integral winning strategy using  $\lfloor k \rfloor$  guards. Moreover, such a transformation can be done in polynomial time in the size of the fractional strategy.*

*Proof.* Let  $\sigma = \{\mathcal{C}_v\}_{v \in V}$  be any fractional winning strategy using a total amount of  $k \in \mathbb{R}^+$  guards to control a spy with speed  $s \geq 2$ , at distance  $d \geq 0$ , and in an  $n$ -node tree  $T = (V, E)$ .

For any configuration  $\omega$  of  $\sigma$ , we will define an integral configuration  $\omega^r$  (which we call a *rounding* of  $\omega$ ) using  $\lfloor k \rfloor$  guards, such that if the spy is controlled in  $\omega$  then it is also controlled in  $\omega^r$ . Moreover, for any two configurations  $\omega_1$  and  $\omega_2$  such that there is a feasible flow from  $\omega_1$  to  $\omega_2$ , we show that there is feasible integral flow from  $\omega_1^r$  to  $\omega_2^r$ . Altogether, this shows that the strategy  $\sigma^r$  obtained by rounding all configurations of  $\sigma$  is a winning integral strategy using  $\lfloor k \rfloor$  guards, which proves the theorem.

From now on, let us consider  $T$  to be rooted at some vertex  $r \in V$ .

Let us consider any configuration  $\omega : V \rightarrow \mathbb{R}^+$ . We first need some notations. Let  $\omega(T) = \sum_{v \in V} \omega(v)$  and let  $cont(T, \omega) = \{u \in V : \sum_{v \in N_d[u]} \omega(v) \geq 1\}$  (i.e.,  $cont(T, \omega)$  is the set of vertices  $u$  such that the spy on  $u$  is controlled at distance  $d$  by the guards in the configuration  $\omega$ ). For any vertex  $u$ , let  $T_u$  be the subtree of  $T$  rooted in  $u$  (i.e., the subtree that consists of  $u$  and all its descendants). Let  $\omega^+(u) = \sum_{v \in V(T_u)} \omega(v)$ . By definition,  $\omega^+(u) \geq \omega(u)$  for every  $u \in V$  and  $\omega^+(r) = \sum_{u \in V} \omega(u) = \omega(T)$ .

For  $x \in \mathbb{R}$ , let  $\{x\}$  denote the *fractional* part of  $x$ , i.e.  $\{x\} = x - \lfloor x \rfloor$ .

Let us define the *rounded* function  $\omega^r : V \rightarrow \mathbb{N}$  as, for every  $u \in V$ ,

$$\omega^r(u) = \lfloor \omega(u) + \sum_{v \text{ child of } u} \{\omega^+(v)\} \rfloor$$

Intuitively, the fractional part of guards that are in each of the subtrees rooted in the children of  $u$  is “pushed” to  $u$ . Then  $u$  “keeps” only the integral part of the sum of what it had plus what it received from its children.

**Claim 4.** *For any configuration  $\omega : V(T) \rightarrow \mathbb{R}^+$ ,  $\omega^r(T) = \lfloor \omega(T) \rfloor \leq \omega(T)$*

*Proof of the claim.* The proof is by induction on  $|V|$ . It clearly holds if  $|V| = 1$ . Let  $T_1, \dots, T_h$



be the subtrees of  $T \setminus r$  rooted in the children of  $r$ . By definition,

$$\begin{aligned}\omega^r(T) &= \omega^r(r) + \sum_{1 \leq i \leq h} \omega^r(T_i) \\ &= \lfloor \omega(r) + \sum_{1 \leq i \leq h} \{\omega(T_i)\} \rfloor + \sum_{1 \leq i \leq h} \omega^r(T_i)\end{aligned}$$

By induction,  $\omega^r(T_i) = \lfloor \omega(T_i) \rfloor$  for every  $1 \leq i \leq h$ , and so:

$$\begin{aligned}\omega^r(T) &= \lfloor \omega(r) + \sum_{1 \leq i \leq h} \{\omega(T_i)\} \rfloor + \sum_{1 \leq i \leq h} \lfloor \omega(T_i) \rfloor \\ &= \lfloor \omega(r) + \sum_{1 \leq i \leq h} \omega(T_i) \rfloor \\ &= \lfloor \omega(T) \rfloor\end{aligned}$$

◇

Therefore, Claim 4 proves that rounding a configuration using  $k$  guards provides an integral configuration using  $\lfloor k \rfloor$  guards.

**Claim 5.** For any configuration  $\omega : V(T) \rightarrow \mathbb{R}^+$ ,  $\text{cont}(T, \omega) \subseteq \text{cont}(T, \omega^r)$

*Proof of the claim.* Let  $u \in \text{cont}(T, \omega)$ . By definition,  $\sum_{v \in N_d[u]} \omega(v) \geq 1$ . Let  $r'$  be the vertex in  $N_d[u]$  that is closest to the root  $r$ , and let  $T'$  be the subtree of  $T$  rooted in  $r'$ . Finally, let  $T'_1, \dots, T'_h$  be the subtrees of  $T' \setminus N_d[u]$ . By Claim 4,  $\omega^r(T') = \lfloor \omega(T') \rfloor$  and  $\omega^r(T'_i) = \lfloor \omega(T'_i) \rfloor$  for any  $1 \leq i \leq h$ . Hence,

$$\omega^r(T') = \sum_{v \in N_d[u]} \omega^r(v) + \sum_{1 \leq i \leq h} \omega^r(T'_i) = \sum_{v \in N_d[u]} \omega^r(v) + \sum_{1 \leq i \leq h} \lfloor \omega(T'_i) \rfloor$$

and,

$$\omega^r(T') = \lfloor \omega(T') \rfloor = \left\lfloor \sum_{v \in N_d[u]} \omega(v) + \sum_{1 \leq i \leq h} \omega(T'_i) \right\rfloor$$

Since  $\sum_{v \in N_d[u]} \omega(v) \geq 1$ , it follows that

$$\left\lfloor \sum_{v \in N_d[u]} \omega(v) + \sum_{1 \leq i \leq h} \omega(T'_i) \right\rfloor \geq 1 + \left\lfloor \sum_{1 \leq i \leq h} \omega(T'_i) \right\rfloor \geq 1 + \sum_{1 \leq i \leq h} \lfloor \omega(T'_i) \rfloor$$

Altogether,  $1 + \sum_{1 \leq i \leq h} \lfloor \omega(T'_i) \rfloor \leq \omega^r(T') = \sum_{v \in N_d[u]} \omega^r(v) + \sum_{1 \leq i \leq h} \lfloor \omega(T'_i) \rfloor$ . Therefore,  $\sum_{v \in N_d[u]} \omega^r(v) \geq 1$  and  $u \in \text{cont}(T, \omega^r)$ . ◇

Claim 5 proves that every position of the spy that is controlled by the guards in a configuration  $\omega$  is also controlled by the guards in the configuration  $\omega^r$ .

**Claim 6.** Let  $\omega_1, \omega_2 : V(T) \mapsto \mathbb{R}^+$  be two configurations such that the guards can go from  $\omega_1$  to  $\omega_2$  in one step (there is feasible flow from  $\omega_1$  to  $\omega_2$ ). Then, the guards can go from  $\omega_1^r$  to  $\omega_2^r$  in one step (there is feasible integral flow from  $\omega_1^r$  to  $\omega_2^r$ ).

*Proof of the claim.* The proof is by induction on  $\omega_1^r(T) = \omega_2^r(T)$ , the result being trivial when it is null. Let  $f$  be the flow representing the move of the guards from  $\omega_1$  to  $\omega_2$ . Note that we may assume that, for every  $u, v \in V$ , at most one of  $f(u, v)$  and  $f(v, u)$  is non-null.

Among all vertices  $v \in V$  such that  $\omega_1(T_v) \geq 1$  or  $\omega_2(T_v) \geq 1$ , let  $x$  be a lowest one (such a vertex furthest from the root). By symmetry (there is a feasible flow from  $\omega_1$  to  $\omega_2$  if and only if there is a feasible flow from  $\omega_2$  to  $\omega_1$ ), up to exchanging  $\omega_1$  and  $\omega_2$ , we may assume that  $\omega_1(T_x) \geq 1$ . Note that, by minimality of  $x$ , for every descendant  $u \in V(T_x) \setminus \{x\}$  of  $x$ ,  $\omega_1(T_u) < 1$  and  $\omega_2(T_u) < 1$ .

Now, let  $\gamma_1$  be the function defined by  $\gamma_1(x) = \omega_1(T_x) - 1$ ,  $\gamma_1(u) = 0$  for every descendant  $u$  of  $x$ , and  $\gamma_1(v) = \omega_1(v)$  for every  $v \in V \setminus V(T_x)$ . Note that  $\gamma_1^r(v) = \omega_1^r(v)$  for every  $v \in V \setminus \{x\}$  and  $\gamma_1^r(x) = \omega_1^r(x) - 1$ . Now, there are two cases to be considered.

- First, assume that  $\omega_2(T_x) \geq 1$ . In this case, let  $\gamma_2$  be the function defined by  $\gamma_2(x) = \omega_2(T_x) - 1$ ,  $\gamma_2(u) = 0$  for every descendant  $u$  of  $x$ , and  $\gamma_2(v) = \omega_2(v)$  for every  $v \in V \setminus V(T_x)$ . Note that there is a feasible flow  $f'$  from  $\gamma_1$  to  $\gamma_2$ : for any  $u, v \in V(T_x)$ ,  $f'(u, v) = 0$  and for any  $u \in V$ ,  $v \in V \setminus V(T_x)$ ,  $f'(u, v) = f(u, v)$ . Note also that  $\gamma_2^r(v) = \omega_2^r(v)$  for every  $v \in V \setminus \{x\}$  and  $\gamma_2^r(x) = \omega_2^r(x) - 1$ .

By induction (since  $\gamma_1^r(T) = \gamma_2^r(T) = \omega_1^r(T) - 1$ ), there is a feasible integral flow  $f^*$  from  $\gamma_1^r$  to  $\gamma_2^r$ . Since  $\omega_1^r$  (resp.,  $\omega_2^r$ ) is obtained from  $\gamma_1^r$  (resp.,  $\gamma_2^r$ ) by adding 1 guard in  $x$ , this flow  $f^*$  is also a feasible integral flow from  $\omega_1^r$  to  $\omega_2^r$ .

- Second,  $\omega_2(T_x) < 1$ . Let  $p$  be the parent of  $x$  ( $x$  cannot be the root since  $\omega_2^r(T) \geq 1$ ). Note that, because there is flow from  $\omega_1$  to  $\omega_2$ , then  $\omega_2(p) + \omega_2(T_x) \geq \omega_1(T_x) \geq 1$ .

In this case, let  $\gamma_2$  be the function defined by  $\gamma_2(u) = 0$  for every  $u \in V(T_x)$ ,  $\gamma_2(v) = \omega_2(v)$  for every  $v \in V \setminus (V(T_x) \cup \{p\})$  and  $\gamma_2(p) = \omega_2(p) + \omega_2(T_x) - 1 \geq 0$ .

Note that there is a feasible flow  $f'$  from  $\gamma_1$  to  $\gamma_2$ : for any  $u, v \in V(T_x)$ ,  $f'(u, v) = 0$ , for any  $u, v \in V \setminus V(T_x)$ ,  $f'(u, v) = f(u, v)$ , and  $f'(x, p) = \gamma_1(x)$ . Note also that  $\gamma_2^r(v) = \omega_2^r(v)$  for every  $v \in V \setminus \{p\}$  and  $\gamma_2^r(p) = \omega_2^r(p) - 1$ .

By induction (since  $\gamma_1^r(T) = \gamma_2^r(T) = \omega_1^r(T) - 1$ ), there is a feasible integral flow  $f^*$  from  $\gamma_1^r$  to  $\gamma_2^r$ . Since  $\omega_1^r$  (resp.,  $\omega_2^r$ ) is obtained from  $\gamma_1^r$  (resp.,  $\gamma_2^r$ ) by adding 1 guard in  $x$  (resp., in  $p$ ), there is a feasible integral flow from  $\omega_1^r$  to  $\omega_2^r$  that can be obtained from  $f^*$  by adding to it one unit of flow from  $x$  to  $p$ .

◇

Claim 6 shows that the moves that were valid in  $\sigma$  still hold in the “rounded” strategy. This concludes the proof of the theorem. □

The second step in this section is to show that there is always an optimal fractional strategy which is spy-positional. For this purpose, we prove the following theorem.

**Theorem 7.** *For any tree  $T$  and for any  $s \geq 2$ ,  $d \geq 0$ ,  $fgn_{s,d}^*(T) = fgn_{s,d}(T)$ . More precisely, any fractional winning strategy using a total amount of  $k \in \mathbb{R}^+$  guards can be transformed into a spy-positional winning strategy using  $k$  guards.*

*Proof.* Let  $\sigma = \{\mathcal{C}_v\}_{v \in V}$  be any fractional winning strategy using a total amount of  $k \in \mathbb{R}^+$  guards to control a spy with speed  $s \geq 2$ , at distance  $d \geq 0$ , and in an  $n$ -node tree  $T = (V, E)$ . Recall that, for any vertex  $v \in V$ ,  $\mathcal{C}_v$  is the set of possible configurations  $\omega : V \rightarrow \mathbb{R}^+$  for the guards when the spy is at  $v$ .

Intuitively, the proof consists in defining a spy-positional strategy  $\sigma^{min}$  as follows. For any  $v \in V$ , we will define the function  $\omega_v^{min} : V \rightarrow \mathbb{R}^+$  to be the (unique) configuration of  $\sigma^{min}$  when the spy is at  $v$ , i.e.,  $\sigma^{min} = \{\{\omega_v^{min}\}_{v \in V}\}$ . We first prove that the spy at  $v \in V$  is controlled at distance  $d$  by the guards in the configuration  $\omega_v^{min}$ . Then, we prove that, for any move of the spy from  $v$  to  $v' \in V$ , the guards can move from  $\omega_v^{min}$  to  $\omega_{v'}^{min}$ .

From now on,  $T$  is rooted in an arbitrary vertex  $r \in V$ . For every  $u \in V$ , let  $T_u$  be the subtree of  $T$  rooted in  $u$  and let  $Children(u)$  be the set of children of  $u$ .

Let us first define  $\omega_v^{min} : V \rightarrow \mathbb{R}^+$  for every  $v \in V$ . For any weight function  $\omega : V \rightarrow \mathbb{R}^+$ , let  $\omega^+ : V \rightarrow \mathbb{R}^+$  be the *cumulative* function of  $\omega$ , defined by, for every  $u \in V$ ,  $\omega^+(u) = \sum_{v \in V(T_u)} \omega(v)$ .

Let  $v \in V$  and  $\mathcal{C}_v = \{\omega_1, \dots, \omega_h\} \in \sigma$  be the set of configurations of the guards, when the spy is in  $v$ . Let  $\alpha_v : V \rightarrow \mathbb{R}^+$  be such that, for every  $u \in V$ ,

$$\alpha_v(u) = \min_{1 \leq i \leq h} \omega_i^+(u)$$

Now,  $\omega_v^{min}$  is defined as the (unique) function such that  $\alpha$  is its cumulative function. Formally, for every  $u \in V$ :

$$\omega_v^{min}(u) = \alpha_v(u) - \sum_{w \in \text{Children}(u)} \alpha_v(w)$$

**Claim 8.** For every  $u \in V$ ,  $\omega_v^{min}(u) \geq 0$ .

*Proof of the claim.* Let  $1 \leq i \leq h$  be an integer such that  $\alpha_v(u) = \min_{1 \leq j \leq h} \omega_j^+(u) = \omega_i^+(u)$ . By definition of  $\alpha_v$ , for every  $w \in \text{Children}(u)$ ,  $\alpha_v(w) = \min_{1 \leq j \leq h} \omega_j^+(w) \geq \omega_i^+(w)$ . Hence,  $\omega_v^{min}(u) \geq \omega_i^+(u) - \sum_{w \in \text{Children}(u)} \omega_i^+(w) = \omega_i(u) \geq 0$ .  $\diamond$

Claim 8 proves that, for every  $v \in V$ ,  $\omega_v^{min} : V \rightarrow \mathbb{R}^+$  is a configuration.

**Claim 9.** For every  $v \in V$ ,  $\sum_{u \in V} \omega_v^{min}(u) = k$ .

*Proof of the claim.* For every  $1 \leq i \leq h$ ,  $\omega_i^+(r) = k$ . Hence,  $\alpha_v(r) = \min_{1 \leq i \leq h} \omega_i^+(r) = k$ .  $\sum_{u \in V} \omega_v^{min}(u) = (\omega_v^{min})^+(r) = \alpha_v(r) = k$  (since  $\alpha_v$  is the cumulative function of  $\omega_v^{min}$ ).  $\diamond$

Claim 9 proves that, for every  $v \in V$ , the configuration  $\omega_v^{min}$  uses  $k$  guards.

**Claim 10.** For every  $v \in V$ ,  $\sum_{u \in N_d[v]} \omega_v^{min}(u) \geq 1$ .

*Proof of the claim.* Let  $v^*$  be the vertex of  $N_d[v]$  that is closest to the root  $r$ . Let  $v_1, \dots, v_p$  be the descendants of  $v^*$  that are at distance exactly  $d+1$  from  $v$ . Since  $\alpha_v$  is the cumulative function of  $\omega_v^{min}$ , we have that  $\sum_{u \in N_d[v]} \omega_v^{min}(u) = \alpha_v(v^*) - \sum_{1 \leq j \leq p} \alpha_v(v_j)$ .

Let  $1 \leq i \leq h$  be an integer such that  $\alpha_v(v^*) = \min_{1 \leq j \leq h} \omega_j^+(v^*) = \omega_i^+(v^*)$ .

Since the guards in configuration  $\omega_i$  control the spy in  $v$  at distance  $d$ , we have that  $\sum_{u \in N_d[v]} \omega_i(u) = \omega_i^+(v^*) - \sum_{1 \leq j \leq p} \omega_i^+(v_j) \geq 1$ . Hence,

$$\begin{aligned} \sum_{u \in N_d[v]} \omega_v^{min}(u) &= \alpha_v(v^*) - \sum_{1 \leq j \leq p} \alpha_v(v_j) \\ &= \omega_i^+(v^*) - \sum_{1 \leq j \leq p} \min_{1 \leq j' \leq h} \omega_{j'}^+(v_j) \\ &\geq \omega_i^+(v^*) - \sum_{1 \leq j \leq p} \omega_i^+(v_j) \geq 1 \end{aligned}$$

$\diamond$

Claim 10 proves that the guards in the configuration  $\omega_v^{min}$  control a spy located at  $v$ .

**Claim 11.** For every  $v \in V$  and  $v' \in N_s(v)$ , there is a feasible flow from  $\omega_v^{min}$  to  $\omega_{v'}^{min}$ .

*Proof of the claim.* Let  $\mathcal{C}_v = \{\omega_1, \dots, \omega_h\} \in \sigma$  (the configurations of  $\sigma$  when the spy is at  $v$ ) and  $\mathcal{C}_{v'} = \{\omega'_1, \dots, \omega'_{h'}\} \in \sigma$  (the configurations of  $\sigma$  when the spy is at  $v'$ ). Since  $\sigma$  is a winning strategy, it means that, for every  $1 \leq i \leq h$ , there is  $1 \leq \delta(i) \leq h'$ , such that there is a feasible flow from  $\omega_i \in \mathcal{C}_v$  to  $\omega'_{\delta(i)} \in \mathcal{C}_{v'}$ . That is, there is a function  $f^i : V \times V \rightarrow \mathbb{R}^+$  such that, for

every  $u \in V$ ,  $\omega'_{\delta(i)}(u) = \omega_i(u) + \sum_{w \in N(u)} (f^i(w, u) - f^i(u, w))$  and  $\sum_{w \in N(u)} f^i(u, w) \leq \omega_i(u)$ . Note that, such a function  $f^i$  can be defined as, for every  $u \in V$  and  $p \in V$ , the parent of  $u$  in  $T$  rooted in  $r$  (if  $u \neq r$ ),  $f^i(u, p) = \max\{\omega_i(u) - \omega'_{\delta(i)}(u), 0\}$  and  $f^i(p, u) = \max\{\omega'_{\delta(i)}(u) - \omega_i(u), 0\}$ .

Let  $u \in V$ ,  $X \subseteq \text{Children}(u)$  be any subset of the children of  $u$ , and  $1 \leq i \leq h$ . Because of the existence of the flow  $f^i$ ,  $\sum_{w \in X} (\omega'_{\delta(i)})^+(w) \leq \omega_i(u) + \sum_{w \in X} \omega_i^+(w)$ , hence:

$$\omega_i^+(u) = \omega_i(u) + \sum_{w \in X} \omega_i^+(w) + \sum_{w \in \text{Children}(u) \setminus X} \omega_i^+(w) \geq \sum_{w \in X} (\omega'_{\delta(i)})^+(w) + \sum_{w \in \text{Children}(u) \setminus X} \omega_i^+(w)$$

and so, since for every  $w \in V$ ,  $\alpha_{v'}(w) = \min_{1 \leq j \leq h'} \omega'_j(w)$  and  $\alpha_v(w) = \min_{1 \leq j \leq h} \omega_j(w)$ :

$$\omega_i^+(u) \geq \sum_{w \in X} \alpha_{v'}(w) + \sum_{w \in \text{Children}(u) \setminus X} \alpha_v(w)$$

The above inequality holds for every  $1 \leq i \leq h$ . Since  $\alpha_v(u) = \min_{1 \leq i \leq h} \omega_i^+(u)$ , it follows that:

$$\alpha_v(u) \geq \sum_{w \in X} \alpha_{v'}(w) + \sum_{w \in \text{Children}(u) \setminus X} \alpha_v(w)$$

By similar arguments (because, by symmetry, there is a flow from  $\omega'_j$  to some  $\omega_{j'}$  for every  $1 \leq j \leq h'$ ), we get

$$\alpha_{v'}(u) \geq \sum_{w \in X} \alpha_{v'}(w) + \sum_{w \in \text{Children}(u) \setminus X} \alpha_v(w)$$

We need to prove that there exists a function  $f : V \times V \rightarrow \mathbb{R}^+$  such that, for every  $u \in V$ ,  $\omega_{v'}^{min}(u) = \omega_v^{min}(u) + \sum_{w \in N(u)} (f(w, u) - f(u, w))$  and  $\sum_{w \in N(u)} f(u, w) \leq \omega_v^{min}(u)$ .

For every  $u \in V$ , let  $p \in V$  be the parent of  $u$  in  $T$  rooted in  $r$  (if  $u \neq r$ ). Let  $f^{min}(u, p) = \max\{\alpha_v(u) - \alpha_{v'}(u), 0\}$  and let  $f^{min}(p, u) = \max\{\alpha_{v'}(u) - \alpha_v(u), 0\}$ .

It is clear that, for every  $u \in V$ ,  $\omega_{v'}^{min}(u) = \omega_v^{min}(u) + \sum_{w \in N(u)} (f^{min}(w, u) - f^{min}(u, w))$ . Hence, we only need to prove that  $\sum_{w \in N(u)} f^{min}(u, w) \leq \omega_v^{min}(u)$ .

Let  $u \in V$ ,  $p$  its parent (if  $u \neq r$ ) and let  $X \subseteq \text{Children}(u)$  be the set of vertices such that, for every  $w \in X$ ,  $f^{min}(u, w) = \alpha_{v'}(w) - \alpha_v(w) > 0$ . There are two cases to be considered.

- First, let us assume that  $f^{min}(u, p) = 0$ .

$$\begin{aligned} \omega_{v'}^{min}(u) &= \alpha_v(u) - \sum_{w \in \text{Children}(u)} \alpha_v(w) \\ &= (\alpha_v(u) - \sum_{w \in \text{Children}(u) \setminus X} \alpha_v(w)) - \sum_{w \in X} \alpha_v(w) \\ &\geq \sum_{w \in X} (\alpha_{v'}(w) - \alpha_v(w)) = \sum_{w \in N(u)} f^{min}(u, w) \end{aligned}$$

- Second, assume that  $f^{min}(u, p) = \delta > 0$ .

$$\begin{aligned}
\omega_v^{min}(u) &= \alpha_v(u) - \sum_{w \in \text{Children}(u)} \alpha_v(w) \\
&= \alpha_{v'}(u) + \delta - \sum_{w \in \text{Children}(u)} \alpha_v(w) \\
&= \delta + (\alpha_{v'}(u) - \sum_{w \in \text{Children}(u) \setminus X} \alpha_v(w)) - \sum_{w \in X} \alpha_v(w) \\
&\geq \delta + \sum_{w \in X} (\alpha_{v'}(w) - \alpha_v(w)) = \sum_{w \in N(u)} f^{min}(u, w)
\end{aligned}$$

◇

Claim 11 shows that the moves that were valid in  $\sigma$  still hold for  $\sigma^{min}$ . This concludes the proof of the theorem. □

We can now prove the main theorem of this section.

**Theorem 12.** *Let  $T$  be a tree and let  $s \geq 2$  and  $d \geq 0$  be two integers. There is a polynomial-time algorithm that computes an integral winning strategy using  $gn_{s,d}(T)$  guards to control a spy with speed  $s$  at distance  $d$ .*

*Proof.* By Theorem 7, there exists an optimal (fractional) winning strategy that is spy-positional. By Theorem 2, such a strategy can be computed in polynomial time. By Theorem 3, an optimal integral winning strategy can be computed in polynomial time from any optimal fractional winning strategy. □

## 5 Spy game in Grid and Torus

It is clear that, for any  $n \times n$  grid  $G$ ,  $gn_{s,d}(G) \leq |V(G)| = O(n^2)$ . However, the exact order of magnitude of  $gn_{s,d}(G)$  is not known. In this section, we prove that there exists  $\beta > 0$ , such that  $\Omega(n^{1+\beta})$  guards are necessary to win against one spy in an  $n \times n$ -grid. Our lower bound actually holds for the fractional relaxation of the game. Precisely, we prove that  $fgn_{s,d}(G)$  is super-linear and sub-quadratic (in the side  $n$ ).

Let  $n, m \geq 2$  be two integers. We consider the  $n \times m$  toroidal grid  $TG_{n \times m} = (V, E)$ , i.e., the graph with vertices  $v_{i,j} = (i, j)$  and edges  $\{(i, j), (i+1 \bmod n, j)\}$  and  $\{(i, j), (i, j+1 \bmod m)\}$ , for all  $0 \leq i < n$  and  $0 \leq j < m$ . The  $n \times m$  grid  $G_{n \times m}$  is obtained from  $TG_{n \times m}$  by removing the edges  $\{(m-1, i), (0, i)\}; \{(j, n-1), (j, 0)\} \mid \forall 0 \leq i < n, 0 \leq j < m\}$ .

First, we show that the number of fractional (resp., integral) guards required in the grid and in the torus have the same order of magnitude. Precisely:

**Lemma 13.** *For every  $n, m \geq 2$ ,  $s \geq 2$ ,  $d \geq 0$ , and for every  $f \in \{gn_{s,d}, fgn_{s,d}, fgn_{s,d}^*\}$ :*

$$f(TG_{n \times m})/4 \leq f(G_{n \times m}) \leq 4 \cdot f(TG_{n \times m}).$$

*Proof.* Let us present the proof in the integral case, i.e., when  $f = gn_{s,d}$ , the other two cases are similar.

Let  $\sigma$  be a winning strategy using  $k$  guards in  $TG_{n \times m}$ . We define a winning strategy using  $4k$  guards in  $G_{n \times m}$ . For this purpose, let us label the guards used by  $\sigma$  as  $G_1, \dots, G_k$ . In  $G_{n \times m}$ , the behavior of Guard  $G_i$  ( $1 \leq i \leq k$ ) is “simulated” by four guards as follows. The guard  $G_i$  being at  $(x, y) \in V(TG_{n \times m})$  is simulated by one guard at each of the four vertices:  $(x, y)$ ,  $(n-1-x, y)$ ,  $(x, m-1-y)$  and  $(n-1-x, m-1-y)$ . Hence,  $gn_{s,d}(G_{n \times m}) \leq 4 \cdot gn_{s,d}(TG_{n \times m})$ .

Let  $\sigma$  be a winning strategy using  $k$  guards in  $G_{n \times m}$ . We define a winning strategy using  $4k$  guards in  $TG_{n \times m}$ . Our strategy actually allows to control four spies whose moves are correlated. Precisely, assume that when one spy occupies vertex  $(x, y)$ , the three other spies occupy respectively  $(n - 1 - x, y)$ ,  $(x, m - 1 - y)$  and  $(n - 1 - x, m - 1 - y)$ . We divide the  $4k$  guards into four teams, each of which uses the strategy  $\sigma$  (i.e., they all act as if they were in the grid) to control one of the four spies. When some spies cross an edge of  $E(TG_{n \times m}) \setminus E(G_{n \times m})$ , some teams will exchange their target. Hence,  $gn_{s,d}(TG_{n \times m}) \leq 4 \cdot gn_{s,d}(G_{n \times m})$ .  $\square$

This main result of this section is:

**Theorem 14.** *There exist  $\beta > 0$  and  $0 < \alpha \leq \log(3/2)$  such that, for every  $s \geq 2$ ,  $d \geq 0$ ,*

$$\Omega(n^{1+\beta}) = fgn_{s,d}(G_{n \times n}) \quad \text{and} \quad fgn_{s,d}^*(TG_{n \times n}) = O(n^{2-\alpha}).$$

**Corollary 15.** *There exists  $\beta > 0$  such that, for every  $s \geq 2$ ,  $d \geq 0$ ,  $\Omega(n^{1+\beta}) = gn_{s,d}(G_{n \times n})$ .*

Section 5.1 is devoted to prove the first part of Theorem 14, and Section 5.2 is devoted to its second part.

## 5.1 Lower bound in Grids

The goal of this section is to prove that there exists  $\beta > 0$  such that  $\Omega(n^{1+\beta}) = fgn_{s,d}(G_{n \times n})$ , i.e., the number of guards required in any  $n \times n$ -grid is super-linear in the side  $n$  of the grid.

For this purpose, let us define (yet) another parameter. For any  $s \geq 2$ ,  $d \geq 0$ ,  $t \geq 0$ ,  $q \geq 1$ , and any graph  $G$  (note that  $t$  may be a function of  $|V(G)|$ ), let  $gn_{s,d}^{q,t}(G)$  be the minimum number  $k$  of guards such that there is an integral strategy using  $k$  guards that ensures that at least  $q$  guards are at distance at most  $d$  from a spy with speed  $s$  during at least  $t$  steps. Note that, by definition,  $\sup_t gn_{s,d}^{1,t}(G) \leq gn_{s,d}(G)$ .

The first step of the proof is that  $gn_{s,d}^{q,2n}(G_{n \times n}) = \Omega(q \cdot n \log n)$  in any  $n \times n$ -grid and then to extend this result to the fractional strategies. This result will be used as a ‘‘bootstrap’’ in the induction proof for the main result. Let  $H : \mathbb{R}^+ \rightarrow \mathbb{R}^+$ ,  $H(x) = \sum_{1 \leq i \leq x} 1/i$  for every  $x \in \mathbb{R}^+$ .

**Lemma 16.**  $\exists \beta \geq 1/16$  such that for any  $s \geq 2, d \geq 0, q > 0$ ,  $g_{s,d}^{q,2n}(G_{n \times n}) \geq \beta \cdot q \cdot \frac{n}{d} H(\frac{n}{d})$ .

*Proof.* The proof is for  $s = 2$  since  $g_{s,d}^{q,2n}(G_{n \times n}) \geq g_{2,d}^{q,2n}(G_{n \times n})$ .

In order to prove the result, we will consider a *family* of strategies for the spy. For every  $0 \leq r < n$ , the spy starts at position  $(0, 0)$  and runs at full speed toward  $(r, 0)$ . Once there, it continues at full speed toward  $(r, n - 1)$ . We name  $P_r$  the path it follows during this strategy, which is completed in  $\lceil \frac{1}{2}(r + n - 1) \rceil$  steps. Note that the guards may be aware of the family of strategies played by the spy but do not know  $r$  in advance.

Let us assume that there exists a strategy using an amount  $k$  of guards that maintains at least  $q$  guards at distance at most  $d$  from the spy during at least  $2n$  turns. Moreover, the spy only plays the strategies described above.

Assuming that the guards are labeled with integers in  $\{1, \dots, k\}$ , we can name at any time of strategy  $P_r$  the labels of  $q$  guards that are at distance at most  $d$  from the spy. In this way, let  $c(i, j)$  denote this set of  $q$  guards that are at distance at most  $d$  from the spy, when the spy is at position  $(i, j)$ .

**Claim 17.** *If  $|j_2 - j_1| > 2d$ , then  $c(r, 2j_1)$  and  $c(r, 2j_2)$  are disjoint.*

*Proof of the claim.* Assuming  $j_1 < j_2$ , it takes  $j_2 - j_1$  steps for the spy in strategy  $P_r$  to go from  $(r, 2j_1)$  to  $c(r, 2j_2)$ . A guard cannot be at distance at most  $d$  from  $(r, 2j_1)$  and,  $j_2 - j_1$  steps later, at distance at most  $d$  from  $(r, 2j_2)$ . Indeed, to do so its speed must be at least  $2(j_2 - j_1 - d)/(j_2 - j_1) > 1$ , a contradiction.  $\diamond$

**Claim 18.** *If  $|r_2 - r_1| > 2d + 2 \min(j_1, j_2)$ , then  $c(2r_1, 2j_1)$  and  $c(2r_2, 2j_2)$  are disjoint.*

*Proof of the claim.* Assuming  $r_1 < r_2$ , note that strategies  $P_{2r_1}$  and  $P_{2r_2}$  are identical for the first  $r_1$  steps. By that time, the spy is at position  $(2r_1, 0)$ . If  $c(2r_1, 2j_1)$  intersects  $c(2r_2, 2j_2)$ , it means that at this instant some guard is simultaneously at distance at most  $d + j_1$  from  $(2r_1, 2j_1)$  (strategy  $P_{2r_1}$ ) and at distance at most  $d + |r_2 - r_1| + j_2$  from  $(2r_2, 2j_2)$  (strategy  $P_{2r_2}$ ). As those two points are at distance  $2|r_2 - r_1| + 2|j_2 - j_1|$  from each other, we have:

$$\begin{aligned} 2|r_2 - r_1| + 2|j_2 - j_1| &\leq (d + j_1) + (d + |r_2 - r_1| + j_2) \\ |r_2 - r_1| + 2|j_2 - j_1| &\leq 2d + j_1 + j_2 \\ |r_2 - r_1| &\leq 2d + 2 \min(j_1, j_2) \end{aligned} \quad \diamond$$

We can now proceed to prove that the number of guards is sufficiently large. To do so, we define a graph  $\mathcal{H}$  on a subset of  $V(G_{n \times n})$  and relate the distribution of the guards (as described by  $c$ ) with the independent sets of  $\mathcal{H}$ . Intuitively, an independent set  $I$  in  $\mathcal{H}$  will consist of a set of sets  $c(i, j)$  of guards that must be pairwise disjoint. It is defined over  $V(\mathcal{H}) = \{(2r, 4dj) : 0 \leq 2r < n, 0 \leq 4dj < n\}$ , where:

- $(2r, 4dj_1)$  is adjacent with  $(2r, 4dj_2)$  for  $j_1 \neq j_2$  (see Claim 17).
- $(2r_1, 4dj_1)$  is adjacent with  $(2r_2, 4dj_2)$  if  $|r_2 - r_1| > 4d(1 + \min(j_1, j_2))$  (see Claim 18).

By definition,  $c$  gives  $q$  colors to each vertex of  $\mathcal{H}$  and any set of vertices of  $\mathcal{H}$  receiving a common color is an independent set of  $\mathcal{H}$ . If we denote by  $\#c^{-1}(x)$  the number of vertices which received color  $x$ , and by  $\alpha_{(2r, 4dj)}(\mathcal{H})$  the maximum size of an independent set of  $\mathcal{H}$  containing  $(2r, 4dj)$ , we have:

$$\begin{aligned} k &\geq \sum_{(2r, 4dj) \in V(\mathcal{H})} \sum_{x \in c(2r, 4dj)} \frac{1}{\#c^{-1}(x)} \\ &\geq \sum_{(2r, 4dj) \in V(\mathcal{H})} \frac{q}{\alpha_{(2r, 4dj)}(\mathcal{H})} \end{aligned}$$

It is easy, however, to approximate this lower bound.

**Claim 19.**  $\alpha_{(2r, 4dj)}(\mathcal{H}) \leq 4d(j + 1) + 1$

*Proof of the claim.* An independent set  $S \subseteq V(\mathcal{H})$  containing  $(2r, 4dj)$  cannot contain two vertices with the same first coordinate. Furthermore,  $(2r, 4dj)$  is adjacent with any vertex  $(2r', 4dj')$  if  $|r' - r| > 4d(1 + j)$ .  $\diamond$

We can now finish the proof:

$$\begin{aligned} k &\geq \sum_{(2r, 4dj) \in V(\mathcal{H})} \frac{q}{\alpha_{(2r, 4dj)}(\mathcal{H})} \\ &\geq \sum_{(2r, 4dj) \in V(\mathcal{H})} \frac{q}{4d(j + 1) + 1} \\ &\geq \frac{n}{2} \sum_{j_1 \in \{0, \dots, n/4d\}} \frac{q}{4d(j + 1) + 1} \\ &\geq \frac{qn}{16d} \sum_{j \in \{1, \dots, n/4d+1\}} \frac{1}{j} \geq \frac{qn}{16d} H(n/4d) \end{aligned}$$

where  $H$  is the harmonic function.  $\square$

Next, we aim at transposing Lemma 16 in the case of fractional strategies.

**Lemma 20.** *Let  $n, a \in \mathbb{N}^*$  such that  $d = 2n/a \in \mathbb{N}$ . There exists  $\gamma > 0$  such that  $fgn_{s,d}(G_{n \times n}) \geq \gamma a H(a)$ , where  $H$  is the harmonic function. Moreover, against a smaller amount of guards, the spy wins after at most  $2n$  steps starting from a corner of  $G_{n \times n}$ .*

*Proof.* Let us start by the following claim.

**Claim 21.** *Let  $G$  be any graph with  $n$  vertices and  $d, s, t, q \in \mathbb{N}$ . Then,*

$$gn_{s,d}^{q,t}(G) \leq q \cdot fgn_{s,d}(G) + (t+1)n^2$$

*Proof of the claim.* From a fractional strategy using a total amount  $c$  of guards, let us define an integral strategy keeping at least  $k$  guards at distance at most  $d$  from the spy during at least  $t$  steps.

Initially, each vertex which has an amount  $x$  of guards receives  $\lfloor xq \rfloor + (t+1)n$  guards. That is, our integral strategy uses at most  $\sum_{v \in V(G)} (\lfloor x_v q \rfloor + (t+1)n) \leq (t+1)n^2 + \sum_{v \in V(G)} x_v q \leq (t+1)n^2 + cq$  guards.

We then ensure that, at every step  $t' \in \{1, \dots, t\}$ , a vertex occupied by an amount of  $x$  guards in the fractional strategy is occupied by at least  $\lfloor xq \rfloor + (t-t')n$  guards in the integral strategy. To this aim, whenever an amount  $x_{uv}$  of guards is to be moved from  $u$  to  $v$  in the fractional strategy, we move  $\lfloor x_{uv}q \rfloor + 1$  in the integral strategy.

Precisely, let  $x_v$  (resp.,  $x'_v$ ) be the amount of guards at  $v$  at step  $t'$  (resp., at  $t'+1$ ). Let  $A \subseteq N(v)$  be the set of neighbors of  $v$  sending it a positive amount of flow and let  $B \subseteq N(v)$  be the set of neighbors of  $v$  that receive a positive amount of flow from  $x$ . We have  $x_v + \sum_{u \in A} x_{uv} - \sum_{u \in B} x_{vu} = x'_v$ .

In the integral strategy, by induction on  $t'$ , we get that, after step  $t'+1$ , the number of guards at  $v$  is at least  $\lfloor x_v q \rfloor + (t-t')n + \sum_{u \in A} (\lfloor x_{uv}q \rfloor + 1) - \sum_{u \in B} (\lfloor x_{vu}q \rfloor + 1) \geq x_v q - 1 + (t-t')n + \sum_{u \in A} (x_{uv}q + 1 - 1) - \sum_{u \in B} (x_{vu}q + 1) \geq q(x_v + \sum_{u \in A} x_{uv} - \sum_{u \in B} x_{vu}) + (t-t')n - 1 - B = qx'_v + (t-t')n - 1 - B$ . Since  $B \subseteq N(v)$ ,  $|B| < n$  and so, the number of guards at  $v$  at step  $t'+1$  is at least  $qx'_v + (t-t'-1)n$ .

As our invariant is preserved throughout the  $t$  steps, the spy which had an amount of at least 1 guard within distance  $d$  in the fractional strategy now has at least  $q$  guards around it, which proves the result. Indeed, the number of guards at distance at most  $d$  from the spy (occupying vertex  $y$  at step  $t' \leq t$ ) is  $\sum_{v \in N_d(y)} (\lfloor x_v q \rfloor + (t-t'+1)n) \geq \sum_{v \in N_d(y)} (x_v q - 1 + (t-t'+1)n) \geq q \sum_{v \in N_d(y)} x_v \geq q$   $\diamond$

Previous claim holds for every  $q \in \mathbb{N}$ . Therefore,  $\limsup_{k \rightarrow \infty} \frac{g_{s,d}^{q,t}(G)}{q} \leq fgn_{s,d}(G)$ .

Finally, by Lemma 16, there exists  $\beta > 0$  such that  $g_{s,d}^{q,2n}(G_{n \times n}) \geq \beta \cdot q \cdot \frac{n}{d} H(\frac{n}{d})$ .

Altogether, for  $d = 2n/a \in \mathbb{N}$ , there exists  $\gamma > 0$  such that  $fgn_{s,d}(G_{n \times n}) \geq \gamma a H(a)$ .

Moreover, Lemma 16 shows that against strictly less than  $\beta \cdot q \cdot \frac{n}{d} H(\frac{n}{d})$ , the spy will win in  $2n$  steps, starting from the corner. By the claim, this result implies that the spy will win in  $2n$  steps, starting from the corner, against less than  $\gamma a H(a)$  fractional guards.  $\square$

The next lemma is a key argument for this purpose. While it holds for any graph and its proof is very simple, we have not been able to prove a similar lemma in the classical (i.e., non-fractional) case. Note that this is the only part in this section where we really need to consider the fractional variant of the spy game.

**Lemma 22.** *Let  $G = (V, E)$  be any graph and  $s \geq 2$ ,  $d \geq 0$  be two integers with  $fgn_{s,d}(G) > c \in \mathbb{Q}^*$  where the spy wins in at most  $t$  steps against  $c$  guards starting from  $v \in V(G)$ .*

*For any fractional strategy using a total amount  $k > 0$  of guards, there exists a strategy for the spy (with speed  $s$ ) starting from  $v \in V(G)$  such that after at most  $t$  steps, the amount of guards at distance at most  $d$  from the spy is less than  $k/c$ .*



*Proof.* For purpose of contradiction, assume that there is a strategy  $\mathcal{S}$  using  $k > 0$  guards that contradicts the lemma. Then consider the strategy  $\mathcal{S}'$  obtained from  $\mathcal{S}$  by multiplying the number of guards by  $c/k$ . That is, if  $w \in V$  is initially occupied by  $q > 0$  guards in  $\mathcal{S}$ , then  $\mathcal{S}'$  places  $qc/k$  guards at  $w$  initially (note that  $\mathcal{S}'$  uses a total amount of  $kc/k=c$  guards). Then, when  $\mathcal{S}$  moves an amount  $q$  of guards along an edge  $e \in E$ ,  $\mathcal{S}'$  moves  $qc/k$  guards along  $e$ . Since  $\mathcal{S}$  contradicts the lemma, at any step  $\leq t$ , at least an amount  $k/c$  of guards is at distance at most  $d$  from the spy, whatever be the strategy of the spy. Therefore,  $\mathcal{S}'$  ensures that an amount of at least 1 cop is at distance at most  $d$  from the spy during at least  $t$  steps. This contradicts that the spy wins after at most  $t$  steps against a total amount of  $c$  guards.  $\square$

From Lemmas 20 and 22, we get

**Corollary 23.** *Let  $a \in \mathbb{N}^*$ . For any strategy using a total amount of  $k > 0$  guards, there exists a strategy for the spy (with speed  $s$ ) starting from a corner of  $G_{n \times n}$  such that after at most  $2n$  steps, the amount of guards at distance at most  $2n/a$  from the spy is less than  $k \cdot (aH(a))^{-1}$ .*

**Theorem 24.**  $\exists \beta, \gamma > 0$  such that, for any  $n \times n$ -grid  $G_{n \times n}$  and  $s, d \in \mathbb{N}$  ( $s \geq 2$ ), the spy (with speed  $s$ ) can win (for distance  $d$ ) in at most  $2n$  steps against  $< \gamma n^{1+\beta}$  fractional guards.

*Proof.* Let  $a_0 \in \mathbb{N}$  be such that  $H(a_0)^{-1} \leq 1/2$ . Since  $fg_{n,s,d}(G_{n \times n})$  is non-decreasing as a function of  $n$ , it is sufficient to prove the lemma for  $n = (a_0)^i$  for any  $i \in \mathbb{N}^*$ .

We prove the result by induction on  $i$ . It is clearly true for  $i = 1$  since  $a_0$  is a constant. Assume by induction that there exist  $\gamma, \beta > 0$ , such that, for  $i \geq 1$  with  $n = (a_0)^i$ , the spy (with speed  $s$ ) can win (for distance  $d$ ) in at most  $2n$  steps against  $\gamma a_0^{i(1+\beta)}$  guards in any  $n \times n$  grid.

Let  $G$  be an  $n \times n$ -grid with  $n = (a_0)^{i+1}$ . Let  $k \leq \gamma n^{1+\beta}$ . By Corollary 23, there exists a strategy for the spy (with speed  $s$ ) starting from a corner of  $G$  such that after  $t \leq 2n$  steps, the amount of guards at distance at most  $2n/a_0$  from the spy is less than  $k * (a_0 H(a_0))^{-1} \leq k/(2a_0) \leq \gamma n^{1+\beta}/(2a_0)$ .

Let  $v$  be the vertex reached by the spy at step  $t$  of strategy  $\mathcal{S}$ . Let  $G'$  be any subgrid of  $G$  with side  $n/a_0$  and corner  $v$ . By the previous paragraph, at most  $\gamma n^{1+\beta}/(2a_0)$  guards can occupy the nodes at distance at most  $d$  from any node of  $G'$  during the next  $2n/a_0$  steps of the strategy. So, by the induction hypothesis, the spy playing an optimal strategy in  $G'$  against at most  $\gamma n^{1+\beta}/(2a_0)$  guards will win.  $\square$

The above theorem proves the first part of Theorem 14 and the Corollary 15.

## 5.2 Upper bound in Torus

This section is devoted to present a spy-positional fractional strategy in the  $n \times n$ -torus,  $TG_{n \times n}$ , that uses a sub-quadratic (in  $n$ ) total amount of guards to control a spy with speed  $s \geq 2$ , at distance  $d \geq 0$ .

For this purpose, we do several simplifications.

**Restricted moves for the spy.** First, let us weaken the spy by allowing it to move only “horizontally” or “vertically” at full speed. Precisely, when the spy occupies the vertex  $(i, j)$ , it is only allowed to move to one of the four vertices  $(i \pm s \bmod n, j)$  or  $(i, j \pm s \bmod n)$ . We say that such a spy has *restricted* moves. The next lemma shows that it does not change the order of magnitude of the number of guards required to control it.

**Lemma 25.** *Let  $n, s \geq 2$ , and  $d \geq 0$  be integers. Assume that there exists a (fractional or integral) winning strategy using  $k$  guards to control a spy, with speed  $s$  and restricted moves, at distance  $d$  in the  $n \times n$ -torus. Then, there exists a (fractional or integral) winning strategy using  $O(s^2 k)$  guards to control a spy, with speed  $s$ , at distance  $d$  in the  $n \times n$ -torus.*

*Proof.* The proof is written in the integral case. The fractional case is similar.

For any strategy of a (non-restricted) spy, we will define a strategy for a restricted spy, called the *spy's shadow*, that ensures that the shadow is always at distance at most  $2s$  from the non-restricted spy. To control the non-restricted spy, the strategy consists of applying the strategy  $\sigma$  against its shadow (i.e., using  $k$  guards) and replacing each guard  $\gamma$  of  $\sigma$  by  $O(s^2)$  guards, one at every vertex at distance at most  $2s$  from the position of  $\gamma$ .

The shadow starts at the same vertex as the spy and “follows” it but only using restricted moves. The shadow can easily stay at distance  $< 2s$  from the spy if the spy moves from a vertex at distance  $< 2s$  from the shadow to a vertex at distance at least  $2s$  (but  $< 3s$  since the spy has speed  $s$ ) from the shadow. This means, then, that the shadow is at a position such that one of its coordinates differs by at least  $s$  from one of the spy’s coordinates. So it can decrease its distance to the spy by exactly  $s$  using a restricted move. This means that after the shadow moves, the distance is still  $< 2s$ .  $\square$

Since we are only interested by the order of magnitude of the number of guards, in what follows, we will only consider a spy with restricted moves.

**Linear Program for spy-positional strategy in a torus.** Recall that, in a spy-positional strategy, the positions of the guards (configuration) only depends on the position of the spy. In the  $n \times n$  torus, as in any vertex-transitive graph, this implies that there is actually a unique configuration.

Precisely, we are looking for a function  $\omega : \{0, \dots, n-1\} \times \{0, \dots, n-1\} \rightarrow \mathbb{R}^+$  such that,  $\omega(i-x \bmod n, j-y \bmod n)$  is the amount of guards occupying the vertex  $(i, j)$  when the spy is occupying the vertex  $(x, y)$ . Moreover,  $\omega$  must satisfy the usual constraints that we recall together with a Linear Program (LP) that computes such an optimal strategy.

The LP aims at minimizing the total amount of guards.

$$\text{Minimize } \sum_{0 \leq i, j < n} \omega(i, j) \tag{9}$$

The first constraint states that, for every position  $v \in V$  of the spy (by vertex-transitivity, it is sufficient to check it when  $v = (0, 0)$ ), the amount of guards at distance at most  $d$  from the spy is at least 1, i.e., the guards always control the spy at distance  $d$ . Let  $B_d$  be the set of vertices at distance at most  $d$  from  $(0, 0)$ .

$$\sum_{(i, j) \in B_d} \omega(i, j) \geq 1 \tag{10}$$

The second family of constraints states that, for any move of the spy (from  $(0, 0)$  to  $(x, y) \in N_s[(0, 0)]$ ), there is a feasible flow from the configuration  $(\omega(i, j))_{(i, j) \in V(TG_{n \times n})}$  to  $(\omega(i-x, j-y))_{(i, j) \in V(TG_{n \times n})}$ .

The above LP, restricted to vertex-transitive graphs, is more efficient than the one presented in Section 3 since there is only one configuration to be considered (and so, much less variables and constraints). In particular, it gives interesting experimental results as presented in the conclusion. However, we were still not able to provide a formal analysis of it and that is why we provide several simplifications of it below. The first one is done by considering only the spy with restricted moves which allows to simplify the flow constraints, as shown in the next paragraph.

**Flow constraints when the spy has restricted moves.** By symmetry of the torus, when the spy has restricted moves, the flow constraints can be expressed by a unique set of constraints stating that, when the spy goes from  $(0, 0)$  to  $(s, 0)$ , there must be a feasible flow from the configuration  $(\omega(i, j))_{(i, j) \in V(TG_{n \times n})}$  to  $(\omega(i-s, j))_{(i, j) \in V(TG_{n \times n})}$ .

Equivalently, these constraints can be defined as a flow problem in a *transportation bipartite auxiliary network*  $H$  defined as follows (i.e., the constraints are satisfied if and only if there is feasible flow in  $H$ )  $\square$ . Let  $H = (V_1 \cup V_2, E(H))$  be the graph such that  $V_1$  and  $V_2$  are two copies of  $V(TG_{n \times n})$ . There is an arc from  $u \in V_1$  to  $v \in V_2$  if  $\{u, v\} \in E(TG_{n \times n})$ . Each vertex  $(i, j) \in V_1$  has a supply  $\omega(i, j)$  and every vertex  $(i', j') \in V_2$  has a demand  $\omega(i - s, j)$ .

By duality  $\square$ , there is a feasible flow in  $H$  if and only if, for every  $A \subseteq V_1$ , the total supply in  $N[A]$  is at least the demand in  $A \subseteq V_2$ , i.e., at least  $\sum_{(i,j) \in A} \omega(i - s, j)$ .

In other words, in the case of a spy with restricted moves, the flow constraints can be stated as:

$$\forall A \subseteq V(TG_{n \times n}), \quad \sum_{(i,j) \in N[A]} \omega(i, j) \geq \sum_{(i,j) \in A} \omega(i - s, j). \quad (11)$$

Overall, any function  $\omega : \{0, \dots, n-1\} \times \{0, \dots, n-1\} \rightarrow \mathbb{R}^+$  satisfying the constraints 10 and 11 corresponds to a fractional spy-positional strategy against a spy with restricted moves in an  $n \times n$  torus. The goal of the remaining part this section is to exhibit such a function that uses a total amount of guards that is sub-quadratic in  $n$ . For this purpose, we do our second simplification (considering only particular strategies for the guards) by restricting ourselves to particular kinds of strategies that are "distance-invariant".

**Distance-invariant spy-positional strategies.** From now on, let us consider some spy-positional strategies  $\omega : \{0, \dots, n-1\} \times \{0, \dots, n-1\} \rightarrow \mathbb{R}^+$  (in the torus, against a spy with restricted moves and speed  $s$ ) that are *distance-invariant*. Let  $d(u, v)$  denote the distance between two vertices  $u$  and  $v$ . Precisely, we consider only spy-positional strategies  $\omega^\alpha$  such that  $\omega^\alpha(i, j) = \frac{1}{(d((i,j), (0,0)) + 1)^\alpha}$  for every  $(i, j) \in V(TG_{n \times n})$ , for some  $0 < \alpha < 1$ . This implies that, for any  $(i, j), (i', j') \in V(TG_{n \times n})$  at the same distance from  $(0, 0)$ ,  $\omega^\alpha(i, j) = \omega^\alpha(i', j')$ .

We aim at deciding the range of  $\alpha$  such that the function  $\omega^\alpha$  satisfies constraint 11. For this purpose, we first aim at finding a set  $\mathcal{H}_s \subseteq V(TG_{n \times n})$  such that  $\kappa^\alpha(\mathcal{H}_s) = \sum_{(i,j) \in N[\mathcal{H}_s]} \omega^\alpha(i, j) - \sum_{(i,j) \in \mathcal{H}_s} \omega^\alpha(i - s, j)$  is minimum. For such a set  $\mathcal{H}_s$ , if  $\kappa^\alpha(\mathcal{H}_s) \geq 0$ , it implies that  $\omega^\alpha$  satisfies constraint 11.

Let  $\mathcal{H}_s$  be the set of vertices  $(i, j) \in V(TG_{n \times n})$  defined by:

$$\mathcal{H}_s = \{(i, j) \mid s/2 \leq i \leq (n+s)/2 \pmod n, 0 \leq j < n\}.$$

**Lemma 26.** *Let  $\alpha > 0$  and  $s \leq n/2$ . For every  $A \subseteq V(TG_{n \times n})$ ,  $\kappa^\alpha(A) \geq \kappa^\alpha(\mathcal{H}_s)$ .*

*Proof.* For simplicity of calculations, let us assume that both  $s$  and  $n$  are even. For any  $0 \leq i < n$ , the *row*  $L_i$  equals  $\{(i, j) \mid 0 \leq j < n\}$  and, for any  $0 \leq j < n$ , the *column*  $C_j$  equals  $\{(i, j) \mid 0 \leq i < n\}$ .

For any integer  $\ell$ , let  $f_\ell : V \rightarrow \mathbb{R}^+$  be the function such that, for any  $v_{i,j} \in V$ ,

$$f_\ell(i, j) = \frac{1}{(d((i, j), (\ell, 0)) + 1)^\alpha}$$

where  $d(x, y)$  denotes the distance between  $x$  and  $y$  in  $G$ .

Note that

**Claim 27.** *For any  $i, j$ ,  $f_s(i, j) = f_0(i - s \pmod n, j)$ .*

For any  $A \subseteq V(TG_{n \times n})$ , let us define the *border*  $\delta(A)$  of  $A$  as  $\delta(A) = \{w \notin A \mid \exists v \in A, \{v, w\} \in E\}$ , i.e., the set of vertices not in  $A$  that have a neighbor in  $A$ .

Note that:

$$\kappa^\alpha(A) = \sum_{v \in N[A]} f_0(v) - \sum_{v \in A} f_s(v) = \sum_{v \in A} (f_0(v) - f_s(v)) + \sum_{v \in \delta(A)} f_0(v).$$

To find a vertex-set minimizing the above function, we actually define another function lower bounding the previous one. We identify a set  $A_{min}$  minimizing this second function such that both functions achieve the same value for  $A_{min}$ . Therefore,  $A_{min}$  also minimizes the first function.

The *vertical border*  $\mu(A)$  equals  $\{(i, j) \notin A \mid (i+1 \bmod n, j) \in A \text{ or } (i-1 \bmod n, j) \in A\}$ , i.e., the set of vertices not in  $A$  that have a neighbor in  $A$  and in the same column. Note that  $\mu(A) \subseteq \delta(A)$  for any  $A \subseteq V$ .

Let us set

$$\gamma(A) = \sum_{v \in A} (f_0(v) - f_s(v)) + \sum_{v \in \mu(A)} f_0(v).$$

Since  $f_0$  is positive and  $\mu(A) \subseteq \delta(A)$ ,

**Claim 28.**  $\kappa^\alpha(A) \geq \gamma(A)$  for any  $A \subseteq V$ .

A useful property of  $\gamma$  is that columns are somehow “independent”.

**Claim 29.**  $\gamma(A) = \sum_{0 \leq j < m} \gamma(A \cap C_j)$ .

Note that  $\mathcal{H}_s = \{v_{i,j} \mid s/2 \leq i \leq (n+s)/2 \bmod n, 0 \leq j < n\}$  is the set of vertices  $v$  such that  $f_0(v) - f_s(v) \leq 0$ . Moreover, note that  $\mu(\mathcal{H}_s) = \delta(\mathcal{H}_s)$  and so:

**Claim 30.**  $\gamma(\mathcal{H}_s) = \kappa^\alpha(\mathcal{H}_s)$ .

Another useful property is that, by the first claim (and telescopic sum),

**Claim 31.** For any  $0 \leq j < n$ ,

$$\gamma(\mathcal{H}_s \cap C_j) = \sum_{-s/2+1 \leq i \leq s/2+1} f_0(n/2 + i \bmod n, j) - \sum_{-s/2 \leq i \leq s/2-2} f_0(i, j).$$

*Proof of the claim.*

$$\begin{aligned} \gamma(\mathcal{H}_s \cap C_j) &= \left[ \sum_{i=s/2}^{(n+s)/2 \bmod n} f_0(i, j) - f_s(i, j) \right] + f_0(s/2 - 1, j) + f_0((n+s)/2 + 1 \bmod n, j) \\ &= \left[ \sum_{i=s/2}^{(n+s)/2 \bmod n} f_0(i, j) - f_0(i-s, j) \right] + f_0(s/2 - 1, j) + f_0((n+s)/2 + 1 \bmod n, j) \\ &= \sum_{i=(n-s)/2+1}^{(n+s)/2+1 \bmod n} f_0(i, j) - \sum_{i=-s/2}^{s/2-2} f_0(i, j) \end{aligned}$$

◇

The above proof actually extends to the following. Let

$$H(a, b) \cap C_j = \{(i, j) \mid a \bmod n \leq i \leq b \bmod n\}$$

**Claim 32.** For any  $|a - b| > 1$ ,

$$\gamma(H(a, b) \cap C_j) = \sum_{-s/2+1 \leq i \leq s/2+1} f_0(b - s/2 + i, j) - \sum_{-s/2 \leq i \leq s/2-2} f_0(a - s/2 + i, j).$$

The remaining part of this section is devoted to prove that  $\mathcal{H}_s$  minimizes  $\kappa^\alpha$ . Precisely, let us prove that  $\gamma(\mathcal{H}_s) = \min_{A \subseteq V} \gamma(A)$ . This follows from the two following claims and previous claims.

**Claim 33.** *Let  $X$  be such that  $\gamma(X) = \min_{A \subseteq V} \gamma(A)$ . Then, for any  $0 \leq j < n$ ,  $X \cap C_j$  is connected.*

*Proof of the claim.* First, assume that there exists a vertex  $v \in C_j \setminus X$  such that its two neighbors in  $C_j$  are in  $X$ . Then,  $\gamma(X \cup \{v\}) = \gamma(X) - f_s(v) < \gamma(X)$ . Therefore, by minimality of  $\gamma(X)$ , there are no such vertices.

If there is  $(n+s)/2 < i < n+s/2 \pmod n$  such that  $u = (i, j) \in X$ ,  $w = (i+1, j) \notin X$ , and  $(i-1, j) \in X$ . Note that, by the previous paragraph,  $(i+2, j) \notin X$ . Therefore,  $\gamma(X \setminus u) = \gamma(X) - f_0(w) + f_s(u) < \gamma(X)$ . The last inequality is because  $f_0(w) > f_s(u)$  because of the choice of  $i$ . This contradicts the minimality of  $\gamma(X)$ . If on the other hand,  $(i-1, j) \notin X$ , then  $\gamma(X \setminus u) = \gamma(X) - f_0(w) - f_0(u) + f_s(u) < \gamma(X)$  which contradicts the minimality of  $\gamma(X)$ .

"Symmetrically", if there is  $s/2 \leq i \leq (n+s)/2$  such that  $u = (i, j) \notin X$ ,  $w = (i-1, j) \in X$ , and  $(i+2, j) \notin X$ . Note that, by the first paragraph,  $(i+1, j) \notin X$ . Therefore,  $\gamma(X \cup \{u\}) = \gamma(X) - f_s(u) + f_0(i+1, j) < \gamma(X)$ . The last inequality is because  $f_0(i+1, j) < f_s(u)$  because of the choice of  $i$ . This contradicts the minimality of  $\gamma(X)$ . If on the other hand,  $(i+2, j) \in X$ , then  $\gamma(X \cup \{u\}) = \gamma(X) - f_s(u) < \gamma(X)$  which contradicts the minimality of  $\gamma(X)$ .

If  $X \cap C_j$  would not be connected, one of the cases of the two previous paragraphs should occur. Therefore,  $X \cap C_j$  is connected.  $\diamond$

**Claim 34.** *Let  $0 \leq j < m$ . For any  $X \subseteq V$  such that  $X \cap C_j$  is connected,  $\gamma(\mathcal{H}_s \cap C_j) \leq \gamma(X \cap C_j)$ .*

*Proof of the claim.* Since  $X \cap C_j$  is connected, it has the form  $H(a, b) \cap C_j$  for some  $a$  and  $b$ . We assume that  $|a - b| > 1$  (the other case can be done similarly). Therefore, by previous claims, it remains to prove that, for any  $a$  and  $b$ ,  $\gamma(\mathcal{H}_s \cap C_j) \leq \gamma(H(a, b) \cap C_j)$ .

$$\begin{aligned} & \gamma(H(a, b) \cap C_j) - \gamma(\mathcal{H}_s \cap C_j) = \\ & \sum_{-s/2+1 \leq i \leq s/2+1} (f_0(b - s/2 + i, j) - f_0(n/2 + i, j)) - \sum_{-s/2 \leq i \leq s/2-2} (f_0(a - s/2 + i, j) - f_0(i, j)). \end{aligned}$$

Since the function  $f_0$  is maximum around  $i = 0$  and minimum around  $i = n/2$ , it is easy to check that, for any  $a$  and  $b$ :

$$\sum_{-s/2+1 \leq i \leq s/2+1} (f_0(b - s/2 + i, j) - f_0(n/2 + i, j)) \geq 0$$

and

$$\sum_{-s/2 \leq i \leq s/2-2} (f_0(a - s/2 + i, j) - f_0(i, j)) \leq 0.$$

Hence,  $\gamma(H(a, b) \cap C_j) - \gamma(\mathcal{H}_s \cap C_j) \geq 0$ .  $\diamond$

By previous claims,  $\kappa^\alpha(\mathcal{H}_s) = \gamma(\mathcal{H}_s) = \min_{A \subseteq V} \gamma(A) \leq \min_{A \subseteq V} \kappa^\alpha(A)$ .

Hence,  $\kappa^\alpha(\mathcal{H}_s) = \min_{A \subseteq V} \kappa^\alpha(A)$ .  $\square$

Finally, we are ready to present a winning strategy in the  $n \times n$  torus. Let  $d(v)$  (resp.,  $d(i, j)$ ) denote the distance between vertex  $v$  (resp.,  $(i, j)$ ) and vertex  $(0, 0)$  in  $TG_{n \times n}$ .

**Lemma 35.** *Let  $n, s \geq 2$ ,  $s \leq n/2$ ,  $d \geq 0$  and  $0 < \alpha \leq \log(3/2) < 1$ . There exists a constant  $B > 0$  (independent of  $n$ ) such that the function  $\omega^\alpha(v) : V(TG_{n \times n}) \rightarrow \mathbb{R}^+$  where  $\omega^\alpha(v) = \frac{B}{(d(v)+1)^\alpha}$  for every  $v \in V(TG_{n \times n})$  is a spy-positional winning fractional strategy that uses  $O(n^{2-\alpha})$  guards to control a spy with speed  $s$  at distance  $d$  in  $TG_{n \times n}$ .*

*Proof.* To verify that  $\omega^\alpha$  is a winning strategy, we need to prove that it satisfies constraints 10 and 11. Let  $B_d$  be the set of vertices at distance at most  $d$  from  $(0, 0)$  and let  $B = 1 / \sum_{v \in B_d} \frac{1}{(d(v)+1)^\alpha}$ .

The total amount of guards used by the strategy is:

$$\sum_{v \in V(TG_{n \times n})} \frac{B}{(d(v)+1)^\alpha} = B \left( \sum_{0 \leq i \leq \lfloor n/2 \rfloor} \frac{4(i+1)}{(i+1)^\alpha} + \sum_{\lfloor n/2 \rfloor < i < n} \frac{4(n-i)}{(i+1)^\alpha} \right) = O(n^{2-\alpha})$$

Constraint 10 states that  $\sum_{v \in B_d} \omega^\alpha(v) \geq 1$  which is satisfied by the choice of  $B$ .

Constraint 11 states that,  $\forall A \subseteq V(TG_{n \times n})$ ,  $\sum_{(i,j) \in N[A]} \omega^\alpha(i,j) \geq \sum_{(i,j) \in A} \omega^\alpha(i-s,j)$ .

By Lemma 26, we know that  $\kappa^\alpha(A) = \sum_{(i,j) \in N[A]} \omega^\alpha(i,j) - \sum_{(i,j) \in A} \omega^\alpha(i-s,j)$  is minimum for  $A = \mathcal{H}_s$ , where  $\mathcal{H}_s = \{(i,j) \mid s/2 \leq i \leq (n+s)/2 \pmod n, 0 \leq j < n\}$ . Hence, it is sufficient to show that  $\kappa^\alpha(\mathcal{H}_s) \geq 0$ .

Again, for ease of presentation, let us assume that  $s$  and  $n$  are even.

$$\kappa^\alpha(\mathcal{H}_s) = \sum_{s/2-1 \leq i \leq (n+s)/2+1, 0 \leq j < n} \frac{B}{(d(i,j)+1)^\alpha} - \sum_{s/2 \leq i \leq (n+s)/2, 0 \leq j < n} \frac{B}{(d(i-s,j)+1)^\alpha}$$

Because  $s \leq n/2$ , this can be simplified as :

$$\kappa^\alpha(\mathcal{H}_s) = \sum_{(n-s)/2+1 \leq i \leq (n+s)/2+1, 0 \leq j < n} \frac{B}{(d(i,j)+1)^\alpha} - \sum_{-s/2 \leq i \leq s/2-2, 0 \leq j < n} \frac{B}{(d(i,j)+1)^\alpha}$$

And so (the detailed computation is postponed in the Appendix):

$$\kappa^\alpha(\mathcal{H}_s) \geq 2(s+1) \sum_{0 \leq j \leq n/2-1} \frac{B}{(n/2+j)^\alpha} - 2(s-1) \sum_{1 \leq j \leq n/2} \frac{B}{j^\alpha} + O(1/n^\alpha)$$

Since  $0 < \alpha$ , then  $p(x) = \frac{1}{x^\alpha}$  is decreasing, and  $\int_a^{b+1} p(t) dt \leq \sum_{x=a}^b p(x) \leq \int_{a-1}^b p(t) dt$ . Hence,

$$\begin{aligned} \kappa^\alpha(\mathcal{H}_s)/(2B) &\geq (s+1) \int_0^{n/2} \frac{1}{(n/2+t)^\alpha} dt - (s-1) \int_0^{n/2} \frac{1}{t^\alpha} dt + O(1/n^\alpha) \\ &= \frac{1}{1-\alpha} [(s+1)((n/2)^\alpha - (n/2)^\alpha) - (s-1)(n/2)^\alpha] + O(1/n^\alpha) \\ &= \frac{n^{1-\alpha}}{1-\alpha} [(s+1)(1 - (1/2)^{1-\alpha}) - (s-1)(1/2)^{1-\alpha}] + O(1/n^\alpha) \end{aligned}$$

Hence,  $\kappa^\alpha(\mathcal{H}_s) \geq 0$  if  $0 \leq (s+1)(1 - (1/2)^{1-\alpha}) - (s-1)(1/2)^{1-\alpha}$ . In other words,  $\kappa^\alpha(\mathcal{H}_s) \geq 0$  if  $2^\alpha \leq \frac{s+1}{s}$  and  $0 < \alpha \leq \log(1 + 1/s) \leq \log(3/2) < 1$ .  $\square$

## Conclusion

The main open question is to determine the exact value of  $gn_{s,d}(G_{n \times n})$  in any  $n \times n$  grid  $G_{n \times n}$  (or torus). A first step toward such a result would be to prove that  $gn_{s,d}(G_{n \times n}) = O(gn_{s',d'}(G_{n \times n}))$  for any  $s, s' \geq 2$  and  $d, d' \geq 0$ . In trees, it would be interesting to design a combinatorial algorithm (i.e., not relying on the solution of a Linear Programme) that computes optimal strategies for controlling a spy with speed  $s$  at distance  $d$ . Finally, using the fractional framework to obtain new results in two-player combinatorial games in graphs seems promising.

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## Appendix

Proof of the computational part of Lemma 35

$$\begin{aligned}
\kappa^\alpha(\mathcal{H}_s) &= \sum_{(n-s)/2+1 \leq i \leq (n+s)/2+1, 0 \leq j < n} \frac{B}{(d(i, j) + 1)^\alpha} - \sum_{-s/2 \leq i \leq s/2-2, 0 \leq j < n} \frac{B}{(d(i, j) + 1)^\alpha} \\
&\geq (s+1) \sum_{0 \leq j < n} \frac{B}{(d(n/2, j) + 1)^\alpha} - (s-1) \sum_{0 \leq j < n} \frac{B}{(d(0, j) + 1)^\alpha} \\
&= 2(s+1) \sum_{0 \leq j \leq n/2} \frac{B}{(d(n/2, j) + 1)^\alpha} - \frac{(s+1) \cdot B}{(d(n/2, 0) + 1)^\alpha} - \frac{(s+1) \cdot B}{(d(n/2, n/2) + 1)^\alpha} \\
&\quad - 2(s-1) \sum_{0 \leq j \leq n/2} \frac{B}{(d(0, j) + 1)^\alpha} + \frac{(s-1)B}{(d(0, 0) + 1)^\alpha} + \frac{(s-1)B}{(d(0, n/2) + 1)^\alpha} \\
&= 2(s+1) \sum_{1 \leq j \leq n/2+1} \frac{B}{(n/2 + j)^\alpha} - 2(s-1) \sum_{1 \leq j \leq n/2+1} \frac{B}{j^\alpha} \\
&\quad - \frac{2B}{(n/2 + 1)^\alpha} - \frac{(s+1) \cdot B}{(n+1)^\alpha} + (s-1)B \\
&= 2(s+1) \sum_{0 \leq j \leq n/2-1} \frac{B}{(n/2 + j)^\alpha} - 2(s-1) \sum_{1 \leq j \leq n/2} \frac{B}{j^\alpha} \\
&\quad - \frac{2(s+1) \cdot B}{(n/2)^\alpha} + \frac{2(s+1) \cdot B}{n^\alpha} + \frac{2(s+1) \cdot B}{(n+1)^\alpha} - \frac{2(s-1) \cdot B}{(n/2 + 1)^\alpha} \\
&\quad - \frac{2B}{(n/2 + 1)^\alpha} - \frac{(s+1) \cdot B}{(n+1)^\alpha} + (s-1)B \\
&\geq 2(s+1) \sum_{0 \leq j \leq n/2-1} \frac{B}{(n/2 + j)^\alpha} - 2(s-1) \sum_{1 \leq j \leq n/2} \frac{B}{j^\alpha} + O(1/n^\alpha)
\end{aligned}$$