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Implicit MAC scheme for compressible Navier-Stokes equations: Low Mach asymptotic error estimates

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Abstract

We investigate error between any discrete solution of the implicit Marker and Cell (MAC) numerical scheme for compressible Navier-Stokes equations in low Mach number regime and an exact strong solution of the incompressible Navier-Stokes equations. The main tool is the relative energy method suggested on the continuous level in [7], whose discrete numerical version has been developed in [19]. We get unconditional error estimate in terms of explicitly determined positive powers of the space-time discretization parameters and Mach number in the case of well prepared initial data, and the boundedness of the error if the initial data are ill prepared. The multiplicative constant in the error estimate depends on the suitable norm of the strong solution but is independent on the numerical solution itself (and of course, on the discretization parameters and the Mach number). This is the first proof ever that the MAC scheme is unconditionally and uniformly asymptotically stable at the low Mach number regime.

Key words: Navier-Stokes system, finite difference numerical method, finite volume numerical method, Marker and Cell scheme, error estimate

AMS classification 35Q30, 65N12, 65N30, 76N10, 76N15, 76M10, 76M12

1 Introduction

In [20], we have derived unconditional error estimates for the Marker and Cell (MAC) numerical scheme for the compressible Navier-Stokes equations. The goal of this paper is to investigate the low Mach number asymptotic for this discretization. The aim is to estimate the error of the MAC discrete numerical solution on a mesh of size h and time step δt in the MAC discrete function space with respect to a convenient projection to the discrete numerical space of the unique strong solution of the incompressible Navier-Stokes equations in terms of the (positive) powers of h , δt and Mach number ε . The multiplicative constant in this estimate must be independent of the numerical solution (and of course of h , δt and ε); it may however depend on the norm of the strong solution (Π, \mathbf{V}) of the target problem in a convenient functional space of sufficiently regular functions. In particular, we shall not require any additional information on the numerical solution than the information provided by the algebraic numerical scheme itself.

Such type of estimates are referred as (unconditional) error estimates in the numerical analysis of PDEs. The numerical schemes possessing this type of error estimates are referred as (uniformly) asymptotic preserving. In spite of the importance of this property for applications, the mathematical literature on this subject is in a short supply, mostly due to the complexity of the problem: the rigorous asymptotic preserving error estimates are known solely on the level of the numerical schemes, and, in this case the error estimate depends on the space-time discretization. This philosophy is pursued for example

in papers [1], [4], [16], [25], [33], [34], [35], [36]. This type of estimates does not provide any information on the convergence of the scheme, and this is a serious drawback. To the best of our knowledge, we present here the first *unconditional and uniform* result providing quantitatively *an uniform convergence rate* in terms of space-time discretization $(h, \delta t)$ and Mach number ε for the MAC scheme (compare with [8] establishing asymptotic preserving estimates for an academic FEM/DG scheme). Its importance and interest is underlined by the fact that the Marker and Cell scheme in its explicit or semi-implicit form constitutes the basis for many industrially ran codes in fluid mechanics.

The relative energy method introduced on the continuous level in [11], [7], [9] and its numerical counterpart developed in Gallouët et al. [19] seem to provide the convenient strategy to achieve this goal.

We consider the compressible Navier-Stokes equations in the low Mach number regime in a space-time cylinder $Q_T = (0, T) \times \Omega$, where $T > 0$ is arbitrarily large and $\Omega \subset \mathbb{R}^3$ is a bounded Lipschitz domain:

$$\partial_t \varrho + \operatorname{div}_x(\varrho \mathbf{u}) = 0, \quad (1.1) \quad \{\text{i1}\}$$

$$\partial_t(\varrho \mathbf{u}) + \operatorname{div}_x(\varrho \mathbf{u} \otimes \mathbf{u}) + \frac{1}{\varepsilon^2} \nabla_x p(\varrho) = \mu \Delta \mathbf{u} + (\mu + \lambda) \nabla_x \operatorname{div}_x \mathbf{u}, \quad (1.2) \quad \{\text{i2}\}$$

In equations (1.1–1.2) $\varrho = \varrho(t, x) \geq 0$ and $\mathbf{u} = \mathbf{u}(t, x) \in \mathbb{R}^3$, $t \in [0, T)$, $x \in \Omega$ are unknown density and velocity fields, μ, λ are viscosity coefficients

$$\mu > 0, \quad \lambda + \frac{2}{d} \mu \geq 0, \quad (1.3) \quad \{\text{i3}\}$$

p is a pressure characterizing the fluid via the constitutive relations

$$p \in C^2(0, \infty) \cap C[0, \infty), \quad p(0) = 0, \quad p'(\varrho) > 0 \text{ for all } \varrho > 0, \quad (1.4) \quad \{\text{i4}\}$$

$$\lim_{\varrho \rightarrow \infty} \frac{p'(\varrho)}{\varrho^{\gamma-1}} = p_\infty > 0, \quad \inf_{\varrho \in (0,1)} \frac{p'(\varrho)}{\varrho} = p_0 > 0$$

where $\gamma \geq 1$. The (small) number $\varepsilon > 0$ is the Mach number. We notice that assumptions (1.4) are compatible with the isentropic pressure law $p(\varrho) = \varrho^\gamma$ provided $1 \leq \gamma \leq 2$.

Equations (1.1–1.2) are completed with the no-slip boundary conditions

$$\mathbf{u}|_{\partial\Omega} = 0, \quad (1.5) \quad \{\text{i6}\}$$

and initial conditions

$$\varrho(0, \cdot) = \varrho_0, \quad \mathbf{u}(0, \cdot) = \mathbf{u}_0, \quad \varrho_0 > 0 \text{ in } \overline{\Omega}. \quad (1.6) \quad \{\text{i7}\}$$

In parallel, we consider a strong solution of the incompressible Navier-Stokes equation

$$\overline{\varrho} \left(\partial_t \mathbf{V} + \mathbf{V} \cdot \nabla_x \mathbf{V} \right) + \nabla_x \Pi = \mu \Delta \mathbf{V}, \quad \operatorname{div} \mathbf{V} = 0, \quad (1.7) \quad \{\text{ns1}\}$$

$$\mathbf{V}|_{\partial\Omega} = 0, \quad \overline{\varrho} = \text{const} > 0 \quad (1.8) \quad \{\text{ns1+}\}$$

endowed with initial data

$$\mathbf{V}(0) = \mathbf{V}_0, \quad (1.9) \quad \{\text{ns2}\}$$

The solution of the incompressible target problem (1.7–1.9) is supposed to belong to the regularity class

$$\Pi \in \mathcal{Y}_T^p(\Omega) \equiv \{\Pi \in C([0, T]; C^1(\overline{\Omega})), \quad \partial_t \Pi \in L^1(0, T; L^p(\Omega))\}, \quad 2 \leq p \leq \infty, \quad \mathbf{V} \in X_T(\Omega), \quad (1.10) \quad \{\text{ns3}\}$$

$$\mathcal{X}_T(\Omega) \equiv \{\mathbf{V} \in C^1([0, T] \times \overline{\Omega}; \mathbb{R}^3), \quad \nabla^2 \mathbf{V} \in C([0, T] \times \overline{\Omega}; \mathbb{R}^3), \quad (\partial_t^2 \mathbf{V}, \partial_t \nabla \mathbf{V}) \in L^2(0, T; L^{6/5}(\Omega; \mathbb{R}^{12})).$$

2 The numerical scheme

2.1 MAC space and time discretization

2.1.1 Space discretization

We assume that the closure of the domain Ω is a union of closed rectangles ($d = 2$) or closed orthogonal parallelepipeds ($d = 3$) with mutually disjoint interiors, and, without loss of generality, we assume that the edges (or faces) of these rectangles (or parallelepipeds) are orthogonal to the canonical basis vectors, denoted by $(e^{(1)}, \dots, e^{(d)})$,

Definition 2.1 (MAC grid - definition notations and basic properties). *A discretization of Ω with MAC grid, denoted by \mathcal{D} , is given by $\mathcal{D} = (\mathcal{T}, \mathcal{E})$, where:*

- *The primal (or density or pressure) grid of domain Ω denoted by \mathcal{T} consists of union of possibly non uniform (closed) rectangles ($d=2$) or (closed) parallelepipeds ($d = 3$), the edges (or faces) of these rectangles (or parallelepipeds) are orthogonal to the canonical basis vectors; a generic cell of this grid is denoted by K (a closed set), and its mass center \mathbf{x}_K . It is a conforming grid, meaning that*

$$\overline{\Omega} = \cup_{K \in \mathcal{M}} K, \text{ where } \text{int}(K) \cap \text{int}(L) = \emptyset \text{ whenever } (K, L) \in \mathcal{M}^2, K \neq L, \quad (2.1) \quad \{\text{primalg}\}$$

and if $K \cap L \neq \emptyset$ then $K \cap L$ is a common face or edge or vertex of K and L . A generic face (or edge in the two-dimensional case) of such a cell is denoted by σ (a closed set), its interior in the \mathbb{R}^{d-1} topology is denoted by $\text{int}_{d-1}(\sigma)$ and its mass center \mathbf{x}_σ . Symbol $\mathcal{E}(K)$ denotes the set of all faces of K . We denote by $\mathbf{n}_{\sigma,K}$ the unit normal vector to σ outward K . The set of all faces of the mesh is denoted by \mathcal{E} ; we have $\mathcal{E} = \mathcal{E}_{\text{int}} \cup \mathcal{E}_{\text{ext}}$, where \mathcal{E}_{int} (resp. \mathcal{E}_{ext}) are the edges of \mathcal{E} that lie in the interior (resp. on the boundary) of the domain. The set of faces that are orthogonal to the i^{th} unit vector $\mathbf{e}^{(i)}$ of the canonical basis of \mathbb{R}^d is denoted by $\mathcal{E}^{(i)}$, for $i = 1, \dots, d$. We then have $\mathcal{E}^{(i)} = \mathcal{E}_{\text{int}}^{(i)} \cup \mathcal{E}_{\text{ext}}^{(i)}$, where $\mathcal{E}_{\text{int}}^{(i)}$ (resp. $\mathcal{E}_{\text{ext}}^{(i)}$) are the edges of $\mathcal{E}^{(i)}$ that lie in the interior (resp. on the boundary) of the domain. Finally, for $i = 1, \dots, d$ and $K \in \mathcal{T}$, we denote $\mathcal{E}^{(i)}(K) = \mathcal{E}(K) \cap \mathcal{E}^{(i)}$ and $\mathcal{E}_{\text{int}}^{(i)}(K) = \mathcal{E}(K) \cap \mathcal{E}_{\text{int}}^{(i)}$, $\mathcal{E}_{\text{ext}}^{(i)}(K) = \mathcal{E}(K) \cap \mathcal{E}_{\text{ext}}^{(i)}$.

- *For each $\sigma \in \mathcal{E}$, we write that $\sigma = K|L$ if $\sigma = K \cap L$ and we write that $\sigma = \overrightarrow{K|L}$ if, furthermore, $\sigma \in \mathcal{E}^{(i)}$ and $(\mathbf{x}_L - \mathbf{x}_K) \cdot \mathbf{e}^{(i)} > 0$ for some $i \in [1, d] = \{1, \dots, d\}$. A primal cell K will be denoted $K = [\overrightarrow{\sigma\sigma'}]$ if $\sigma, \sigma' \in \mathcal{E}^{(i)}(K)$ for some $i = 1, \dots, d$ are such that $(\mathbf{x}_{\sigma'} - \mathbf{x}_\sigma) \cdot \mathbf{e}^{(i)} > 0$. For a face $\sigma \in \mathcal{E}$, the distance d_σ is defined by:*

$$d_\sigma = \begin{cases} d(\mathbf{x}_K, \mathbf{x}_L) & \text{if } \sigma = K|L \in \mathcal{E}_{\text{int}}, \\ d(\mathbf{x}_K, \mathbf{x}_\sigma) & \text{if } \sigma \in \mathcal{E}_{\text{ext}} \cap \mathcal{E}(K) \end{cases} \quad (2.2) \quad \{\text{dsigma}\}$$

where $d(\cdot, \cdot)$ denotes the Euclidean distance in \mathbb{R}^d .

- *A dual cell D_σ associated to a face $\sigma \in \mathcal{E}$ is defined as follows:*
 - * *if $\sigma = K|L \in \mathcal{E}_{\text{int}}$ then $D_\sigma = D_{\sigma,K} \cup D_{\sigma,L}$, where $D_{\sigma,K}$ - a closed set (resp. $D_{\sigma,L}$ - a closed set) is the half-part of K (resp. L) adjacent to σ (see Fig. 1 for the two-dimensional case) ;*
 - * *if $\sigma \in \mathcal{E}_{\text{ext}}$ is adjacent to the cell K , then $D_\sigma = D_{\sigma,K}$.*

The dual grid $\{D_\sigma\}_{\sigma \in \mathcal{E}^{(i)}}$ of Ω (sometimes called the i -th velocity component grid) verifies for each fixed $i \in \{1, \dots, d\}$

$$\overline{\Omega} = \cup_{\sigma \in \mathcal{E}^{(i)}} D_\sigma, \quad \text{int}(D_\sigma) \cap \text{int}(D_{\sigma'}) = \emptyset, \quad \sigma, \sigma' \in \mathcal{E}^{(i)}, \quad \sigma \neq \sigma'. \quad (2.3) \quad \{\text{dualgri}\}$$

- *A dual face separating two neighboring dual cells D_σ and $D_{\sigma'}$ is denoted by $\epsilon = \sigma|\sigma'$ or $\epsilon = D_\sigma|D_{\sigma'}$ (a closed set). Symbol $\tilde{\mathcal{E}}(D_\sigma)$ denotes the set of the faces of D_σ ; it is decomposed to the set of external faces $\tilde{\mathcal{E}}_{\text{ext}}(D_\sigma) = \{\epsilon \in \tilde{\mathcal{E}}(D_\sigma) | \epsilon \subset \partial\Omega\}$ and the set of internal faces $\tilde{\mathcal{E}}_{\text{int}}(D_\sigma) = \{\epsilon \in$*

$\tilde{\mathcal{E}}(D_\sigma)|_{\text{int}_{d-1}\varepsilon \subset \Omega}$. Symbol $\tilde{\mathcal{E}}^{(i)}$ denotes the set of the faces of the i -th dual grid (associated to the i -th velocity component). It is decomposed into the internal and boundary edges: $\tilde{\mathcal{E}}^{(i)} = \tilde{\mathcal{E}}_{\text{int}}^{(i)} \cup \tilde{\mathcal{E}}_{\text{ext}}^{(i)}$, where $\tilde{\mathcal{E}}_{\text{int}}^{(i)} = \{\varepsilon = \sigma|\sigma' | \sigma, \sigma' \in \mathcal{E}^{(i)}\}$ and $\tilde{\mathcal{E}}_{\text{ext}}^{(i)} = \{\varepsilon = \partial D_\sigma \cap \partial\Omega | \sigma \in \mathcal{E}^{(i)}, \partial D_\sigma \cap \partial\Omega \neq \emptyset\}$. Finally, for $\varepsilon \in \tilde{\mathcal{E}}^{(i)}$ we write $\varepsilon = \overrightarrow{\sigma|\sigma'}$ if $(\mathbf{x}_{\sigma'} - \mathbf{x}_\sigma) \cdot \mathbf{e}^{(i)} > 0$. We denote by $\mathbf{n}_{\varepsilon, D_\sigma}$ the unit normal vector to $\varepsilon \in D_\sigma$ outward D_σ .

We denote for further convenience \mathbf{n}_ε and \mathbf{n}_σ a normal unit vector to face ε and σ , respectively. We write $\varepsilon \perp \sigma$ resp. $\sigma \perp \sigma'$ iff $\mathbf{n}_\varepsilon \cdot \mathbf{n}_\sigma = 0$ resp. $\mathbf{n}_\sigma \cdot \mathbf{n}_{\sigma'} = 0$. Similarly we write $\varepsilon \perp \mathbf{e}^{(j)}$ resp. $\sigma \perp \mathbf{e}^{(j)}$ iff \mathbf{n}_ε and $\mathbf{e}^{(j)}$ resp. \mathbf{n}_σ and $\mathbf{e}^{(j)}$ are parallel. We also denote by \mathbf{ab} the segment $\{\mathbf{a} + t(\mathbf{b} - \mathbf{a}) | t \in [0, 1]\}$, where $(\mathbf{a}, \mathbf{b}) \in \mathbb{R}^{2d}$, and by \mathbf{x}_ε resp. $\mathbf{x}_{\sigma \cap \varepsilon}$ the mass centers of the face ε resp. of the set $\sigma \cap \varepsilon$ (provided it is not empty).

- In order to define bi-dual grid, we introduce the set $\tilde{\mathcal{E}}^{(i,j)} = \{\varepsilon \in \tilde{\mathcal{E}}^{(i)} | \varepsilon \perp \mathbf{e}^{(j)}\}$ of dual faces of the i -th component velocity grid that are orthogonal to $\mathbf{e}^{(j)}$. A bi-dual cell D_ε associated to a face $\varepsilon = \sigma|\sigma' \in \tilde{\mathcal{E}}$ is defined as follows:

- * If $\varepsilon = \overrightarrow{\sigma|\sigma'} \in \tilde{\mathcal{E}}^{(i,j)} \cap \tilde{\mathcal{E}}_{\text{int}}^{(i)}$ then $D_\varepsilon = \varepsilon \times \mathbf{x}_\sigma \mathbf{x}_{\sigma'}$ (see Figure 2). (We notice that, if $\sigma, \sigma' \in \mathcal{E}^{(i)}$ with $K = [\overrightarrow{\sigma\sigma'}] \in \mathcal{T}$ and $\varepsilon = \sigma|\sigma'$ then $D_\varepsilon = K$.)
- * If $\varepsilon \in \tilde{\mathcal{E}}^{(i,j)} \cap \tilde{\mathcal{E}}_{\text{ext}}^{(i)}$ with $\varepsilon \in \tilde{\mathcal{E}}(D_\sigma)$ and $i \neq j$ then $D_\varepsilon = \varepsilon \times \mathbf{x}_\sigma \mathbf{x}_{\sigma \cap \varepsilon}$.

In the list above we did not consider the situation $\varepsilon \in \tilde{\mathcal{E}}^{(i,i)} \cap \tilde{\mathcal{E}}_{\text{ext}}^{(i)}$ with $\varepsilon \in \tilde{\mathcal{E}}(D_\sigma)$. In this case $\varepsilon = \sigma \subset \partial\Omega$, and we set for completeness $D_\varepsilon = \emptyset$.

It is to be noticed that, for each fixed couple $(i, j) \in \{1, \dots, d\}^2$

$$\bigcup_{\varepsilon \in \tilde{\mathcal{E}}^{(i,j)}} D_\varepsilon = \overline{\Omega}, \quad \text{int}(D_\varepsilon) \cap \text{int}(D_{\varepsilon'}) = \emptyset, \quad \varepsilon \neq \varepsilon', \quad \varepsilon, \varepsilon' \in \tilde{\mathcal{E}}^{(i,j)}. \quad (2.4) \quad \{\text{bidualg}\}$$

To any dual face $\varepsilon \in \tilde{\mathcal{E}}^{(i)}$, we associate a distance d_ε

$$d_\varepsilon = \begin{cases} d(\mathbf{x}_\sigma, \mathbf{x}_{\sigma'}) & \text{if } \varepsilon \in \tilde{\mathcal{E}}^{(i,j)} \cap \tilde{\mathcal{E}}_{\text{int}}^{(i)}, \\ d(\mathbf{x}_\sigma, \mathbf{x}_{\sigma \cap \varepsilon}) & \text{if } \varepsilon \in \tilde{\mathcal{E}}^{(i,j)} \cap \tilde{\mathcal{E}}_{\text{ext}}^{(i)} \text{ with } \varepsilon \in \tilde{\mathcal{E}}(D_\sigma) \text{ and } i \neq j, \\ d_\sigma & \text{if } \varepsilon \in \tilde{\mathcal{E}}^{(i,i)} \cap \tilde{\mathcal{E}}_{\text{ext}}^{(i)} \text{ with } \varepsilon \in \tilde{\mathcal{E}}(D_\sigma). \end{cases} \quad (2.5) \quad \{\text{depsilo}\}$$

(We notice that the last line in the above definition is irrelevant and pure convention, since in that case $D_\varepsilon = \emptyset$.)

- We define the size of the mesh by

$$h = \max\{h_K, K \in \mathcal{T}\} \quad (2.6) \quad \{\text{stepsiz}\}$$

where h_K stands for the diameter of K . Moreover if $K = [\overrightarrow{\sigma\sigma'}]$ where $\sigma, \sigma' \in \mathcal{E}^{(i)} \cap \mathcal{E}(K)$ for some $i = 1, \dots, d$ we will denote

$$h_K^{(i)} = \frac{|K|}{|\sigma|} = \frac{|K|}{|\sigma'|}. \quad (2.7)$$

We measure the regularity of the mesh through the positive real number $\eta_{\mathcal{T}}$ defined by

$$\eta_{\mathcal{T}} = \max\left\{\frac{|\sigma|}{|\sigma'|}, \sigma \in \mathcal{E}^{(i)}, \sigma' \in \mathcal{E}^{(j)}, (i, j) \in \{1, \dots, d\}^2, i \neq j\right\}. \quad (2.8) \quad \{\text{reg}\}$$

Finally, we denote by h_σ the diameter of the face $\sigma \in \mathcal{E}$.

Some geometric notions presented in Definition 2.1 are sketched on Figures 1 and 2 below.

Definition 2.2 (Discrete spaces). Let $\mathcal{D} = (\mathcal{T}, \mathcal{E})$ be a MAC grid in the sense of Definition 2.1. The discrete density and pressure space $L_{\mathcal{T}}$ is defined as the set of piecewise constant functions over each of the grid cells K of \mathcal{T} , and the discrete i -th velocity space $H_{\mathcal{E}}^{(i)}$ as the set of piecewise constant functions over each of the grid cells D_{σ} , $\sigma \in \mathcal{E}^{(i)}$. As in the continuous case, the Dirichlet boundary conditions (1.5) are (partly) incorporated in the definition of the velocity spaces, and, to this purpose, we introduce $H_{\mathcal{E},0}^{(i)} \subset H_{\mathcal{E}}^{(i)}$, $i = 1, \dots, d$, defined as follows:

$$H_{\mathcal{E},0}^{(i)} = \left\{ v \in H_{\mathcal{E}}^{(i)}, v(\mathbf{x}) = 0 \ \forall \mathbf{x} \in D_{\sigma}, \ \sigma \in \tilde{\mathcal{E}}_{\text{ext}}^{(i)} \right\}.$$

We then set $\mathbf{H}_{\mathcal{E},0} = \prod_{i=1}^d H_{\mathcal{E},0}^{(i)}$. Since we are dealing with piecewise constant functions, it is useful to introduce the characteristic functions \mathcal{X}_K , $K \in \mathcal{T}$ and $\mathcal{X}_{D_{\sigma}}$, $\sigma \in \mathcal{E}$ of the density (or pressure) and velocity cells. We can then write a function $\mathbf{v} \in \mathbf{H}_{\mathcal{E},0}$ as $\mathbf{v} = (v_1, \dots, v_d)$ with $v_i = \sum_{\sigma \in \mathcal{E}_{\text{int}}^{(i)}} v_{i,\sigma} \mathcal{X}_{D_{\sigma}}$, $i \in [1, d]$ and

a function $q \in L_{\mathcal{T}}$ as $q = \sum_{K \in \mathcal{T}} q_K \mathcal{X}_K$. If there is no confusion possible we shall write v_{σ} instead of $v_{i,\sigma}$, where $\sigma \in \mathcal{E}^{(i)}$.

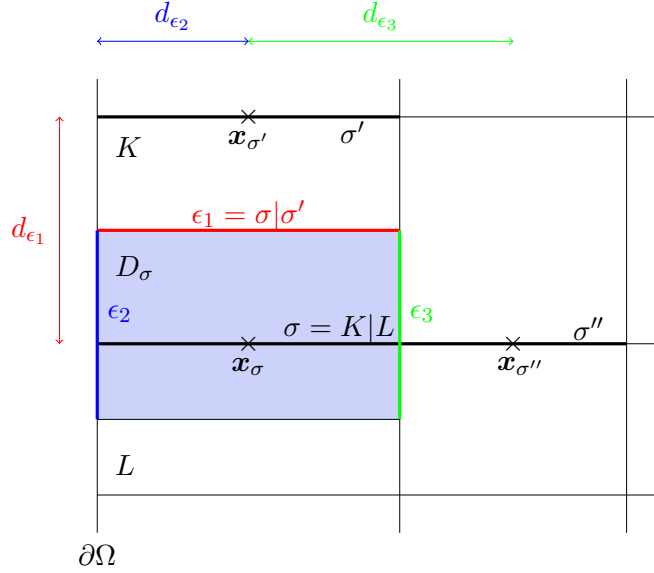


Figure 1: Notations for control volumes and dual cells

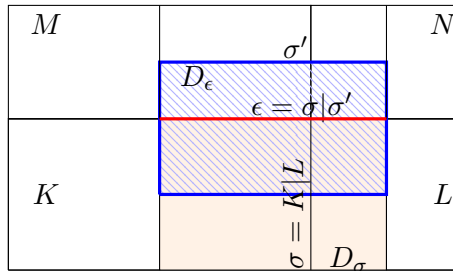


Figure 2: Notations for bi-dual cells

2.1.2 Time discretization

We consider a partition $0 = t^0 < t^1 < \dots < t^N = T$ of the time interval $(0, T)$, and, for the sake of simplicity, a constant time step $\delta t = t^n - t^{n-1}$; hence $t^n = n\delta t$ for $n \in \{0, \dots, N\}$. We denote respectively by $\{u_{i,\sigma}^n \equiv u_\sigma^n, \sigma \in \mathcal{E}_{\text{int}}^{(i)}, i \in \{1, \dots, d\}, n \in \{0, \dots, N\}\}$, and $\{\varrho_K^n, K \in \mathcal{T}, n \in \{1, \dots, N\}\}$ the sets of discrete velocity and density unknowns. For $\sigma \in \mathcal{E}_{\text{int}}^{(i)}$, $i \in \{1, \dots, d\}$ the value u_σ^n is an expected approximation of the mean value over $(t^{n-1}, t^n) \times D_\sigma$ of the i -th component of the velocity of a weak solution, while for $K \in \mathcal{T}$ the value ϱ_K^n is an expected approximation of the mean value over $(t^{n-1}, t^n) \times K$ of the density of a weak solution. To the discrete unknowns, we associate piecewise constant functions on time intervals and on primal or dual meshes, which are expected approximation of weak solutions, For the velocities, these constant functions are of the form:

$$u_i(t, \mathbf{x}) = \sum_{n=1}^N \sum_{\sigma \in \mathcal{E}_{\text{int}}^{(i)}} u_\sigma^n \mathcal{X}_{D_\sigma}(\mathbf{x}) \mathcal{X}_{(t^{n-1}, t^n)}(t), \quad (2.9) \quad \{\text{du}\}$$

where $\mathcal{X}_{(t^{n-1}, t^n)}$ is the characteristic function of the interval (t^{n-1}, t^n) . We denote by $X_{i,\mathcal{E},\delta t}$ the set of such piecewise constant functions on time intervals and dual cells, and we set $\mathbf{X}_{\mathcal{E},\delta t} = \prod_{i=1}^d X_{i,\mathcal{E},\delta t}$. For the density, the piecewise constant function is of the form:

$$\varrho(t, \mathbf{x}) = \sum_{K \in \mathcal{T}} \varrho_K^n(\mathbf{x}) \mathcal{X}_K(\mathbf{x}) \mathcal{X}_{(t^{n-1}, t^n)}(t) \quad (2.10) \quad \{\text{drho}\}$$

and we denote by $Y_{\mathcal{T},\delta t}$ the space of such piecewise constant functions.

For a given $\mathbf{u} \in \mathbf{X}_{\mathcal{E},\delta t}$ associated to the set of discrete velocity unknowns $\{u_\sigma^n, \sigma \in \mathcal{E}_{\text{int}}^{(i)}, i \in \{1, \dots, d\}, n \in \{1, \dots, N\}\}$, and for $n \in \{1, \dots, N\}$, we denote by $u_i^n \in H_{\mathcal{E},0}^{(i)}$ the piecewise constant function defined by $u_i^n(\mathbf{x}) = u_\sigma^n \equiv u_{i,\sigma}^n$ for $\mathbf{x} \in D_\sigma, \sigma \in \mathcal{E}_{\text{int}}^{(i)}$, and set $\mathbf{u}^n = (u_1^n, \dots, u_d^n)^t \in \mathbf{H}_{\mathcal{E},0}$. In a same way, given $\varrho \in Y_{\mathcal{T},\delta t}$ associated to the discrete density unknowns $\{\varrho_K^n, K \in \mathcal{T}, n \in \{1, \dots, N\}\}$ we denote by $\varrho^n \in L_{\mathcal{T}}$ the piecewise constant function defined by $\varrho^n(\mathbf{x}) = \varrho_K^n$ for $\mathbf{x} \in K, K \in \mathcal{T}$.

2.2 MAC discretization of differential operators

2.2.1 Upwind divergence and primal fluxes

The discrete "upwind" divergence is defined by

$$\text{div}_{\mathcal{T}}^{\text{up}} : \left\{ \begin{array}{l} L_{\mathcal{T}} \times \mathbf{H}_{\mathcal{E},0} \longrightarrow L_{\mathcal{T}} \\ (\varrho, \mathbf{u}) \longmapsto \text{div}_{\mathcal{T}}^{\text{up}}(\varrho \mathbf{u}) = \sum_{K \in \mathcal{T}} \frac{1}{|K|} \sum_{\sigma \in \mathcal{E}(K)} F_{\sigma,K}(\varrho, \mathbf{u}) \mathcal{X}_K, \end{array} \right. \quad (2.11) \quad \{\text{eq:divu}\}$$

where $F_{\sigma,K}(\varrho, \mathbf{u})$ stands for the mass flux across σ outward K , which, because of the Dirichlet boundary conditions, vanishes on external faces and is given on the internal faces by:

$$\forall \sigma = K|L \in \mathcal{E}_{\text{int}}, \quad F_{\sigma,K}(\varrho, \mathbf{u}) = |\sigma| \varrho_\sigma^{\text{up}} u_{\sigma,K}, \quad (2.12) \quad \{\text{eq:mass}\}$$

where $u_{\sigma,K}$ is an approximation of the normal velocity to the face σ outward K , defined by:

$$u_{\sigma,K} = u_\sigma \mathbf{e}^{(i)} \cdot \mathbf{n}_{\sigma,K} \text{ for } \sigma \in \mathcal{E}^{(i)} \cap \mathcal{E}(K). \quad (2.13) \quad \{\text{eq:edge}\}$$

Thanks to the boundary conditions, $u_{\sigma,K}$ vanishes for any external face σ . The density at the internal face $\sigma = K|L$ is obtained by an upwind technique:

$$\varrho_\sigma^{\text{up}} = \left\{ \begin{array}{ll} \varrho_K & \text{if } u_{\sigma,K} \geq 0, \\ \varrho_L & \text{otherwise.} \end{array} \right. \quad (2.14) \quad \{\text{eq:rho}\}$$

Note that any solution $(\varrho^n, \mathbf{u}^n) \in L_{\mathcal{T}} \times \mathbf{H}_{\mathcal{E},0}$ to (2.40a) satisfy $\varrho_K^n > 0$, $\forall K \in \mathcal{T}$ provided $\varrho_K^{n-1} > 0$, $\forall K \in \mathcal{T}$ and in particular $p(\varrho^n)$ makes sense. The positivity of the density ϱ^n in (2.40a) is not enforced in the scheme but results from the above upwind choice, see Proposition 2.1.

Note also that, with this definition, we have the usual finite volume property of local conservation of the flux through a primal face

$$F_{\sigma,K}(\varrho, \mathbf{u}) = -F_{\sigma,L}(\varrho, \mathbf{u}), \text{ where } \sigma = K|L. \quad (2.15) \quad \{\text{dod1}\}$$

2.2.2 Discrete convective operator and dual fluxes

The discrete divergence of the convective term $\varrho \mathbf{u} \otimes \mathbf{u}$ is defined by

$$\text{div}_{\mathcal{E}}^{\text{up}} : \left\{ \begin{array}{l} L_{\mathcal{T}} \times \mathbf{H}_{\mathcal{E},0} \longrightarrow \mathbf{H}_{\mathcal{E},0} \\ (\varrho, \mathbf{u}) \longmapsto \text{div}_{\mathcal{E}}^{\text{up}}(\varrho \mathbf{u} \otimes \mathbf{u}) = (\text{div}_{\mathcal{E}}^{(1)}(\varrho \mathbf{u} u_1), \dots, \text{div}_{\mathcal{E}}^{(d)}(\varrho \mathbf{u} u_d)), \end{array} \right. \quad (2.16)$$

where for any $1 \leq i \leq d$, the i^{th} component of the above operator reads:

$$\text{div}_{\mathcal{E}}^{(i)} : \left\{ \begin{array}{l} L_{\mathcal{T}} \times \mathbf{H}_{\mathcal{E},0} \longrightarrow H_{\mathcal{E},0}^{(i)} \\ (\varrho, \mathbf{u}) \longmapsto \text{div}_{\mathcal{E}}^{(i)}(\varrho \mathbf{u} u_i) = \sum_{\sigma \in \mathcal{E}_{\text{int}}^{(i)}} \frac{1}{|D_{\sigma}|} \sum_{\epsilon \in \tilde{\mathcal{E}}(D_{\sigma})} F_{\epsilon,\sigma}(\varrho, \mathbf{u}) u_{\epsilon} \mathcal{X}_{D_{\sigma}}. \end{array} \right. \quad (2.17) \quad \{\text{FLU}\}$$

Here for $\sigma \in \mathcal{E}_{\text{int}}^{(i)}$ and $\epsilon \in \mathcal{E}(D_{\sigma})$ the quantity $F_{\epsilon,\sigma} = F_{\epsilon,\sigma}(\varrho, \mathbf{u})$ stands for a mass flux through the dual faces of the mesh and are defined hereafter while u_{ϵ} stands for the centered approximation of i^{th} component of the velocity over the face ϵ : For internal dual face $\epsilon = D_{\sigma}|D_{\sigma'} \in \tilde{\mathcal{E}}_{\text{int}}^{(i)}$,

$$u_{\epsilon} \equiv u_{i,\epsilon} = \frac{u_{i,\sigma} + u_{i,\sigma'}}{2} \equiv \frac{u_{\sigma} + u_{\sigma'}}{2}. \quad (2.18) \quad \{\text{centered}\}$$

The dual fluxes $F_{\epsilon,\sigma}$ are defined as follows:

Since we consider homogenous Dirichlet boundary condition, the flux through a dual face ϵ included in the boundary is taken equal to zero. (For this reason $\tilde{\mathcal{E}}(D_{\sigma})$ in the sum (2.17) can be replaced by $\tilde{\mathcal{E}}_{\text{int}}(D_{\sigma})$, and it is not necessary to define the value u_{ϵ} at the external dual faces ϵ .)

Otherwise, we have to distinguish two cases (see Figure 2.2.2):

- First case – The vector $\mathbf{e}^{(i)}$ is normal to ϵ , so ϵ is included in a primal cell K , and we denote by σ' the second face of K which, in addition to σ , is normal to $\mathbf{e}^{(i)}$. We thus have $\epsilon = D_{\sigma}|D_{\sigma'}$. Then the mass flux through ϵ is given by:

$$F_{\epsilon,\sigma}(\varrho, \mathbf{u}) = \frac{1}{2} [F_{\sigma,K}(\varrho, \mathbf{u}) \mathbf{n}_{\epsilon,D_{\sigma}} \cdot \mathbf{n}_{\sigma,K} + F_{\sigma',K}(\varrho, \mathbf{u}) \mathbf{n}_{\epsilon,D_{\sigma}} \cdot \mathbf{n}_{\sigma',K}]. \quad (2.19) \quad \{\text{eq:flux}\}$$

- Second case – The vector $\mathbf{e}^{(i)}$ is tangent to ϵ , and ϵ is the union of the halves of two primal faces τ and τ' such that $\tau \in \mathcal{E}(K)$ and $\tau' \in \mathcal{E}(L)$. The mass flux through ϵ is then given by:

$$F_{\epsilon,\sigma}(\varrho, \mathbf{u}) = \frac{1}{2} [F_{\tau,K}(\varrho, \mathbf{u}) + F_{\tau',L}(\varrho, \mathbf{u})]. \quad (2.20) \quad \{\text{eq:flux}\}$$

- Third case – The vector $\mathbf{e}^{(i)}$ is tangent to ϵ , and ϵ is the halve of a primal face τ such that $\tau \in \mathcal{E}(K)$. In particular $\sigma \in \mathcal{E}_{\text{ext}}^{(i)}$. The mass flux through ϵ is then given by:

$$F_{\epsilon,\sigma}(\varrho, \mathbf{u}) = \frac{1}{2} F_{\tau,K}(\varrho, \mathbf{u}). \quad (2.21)$$

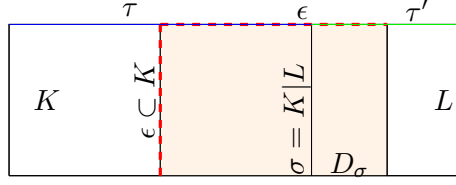


Figure 3: Notations for the dual fluxes of the first component of the velocity.

Note that, with this definition, we have the usual finite volume property of local conservativity of the flux through a dual face, $D_\sigma|D_{\sigma'}$,

$$F_{\epsilon,\sigma}(\varrho, \mathbf{u}) = -F_{\epsilon,\sigma'}(\varrho, \mathbf{u}).$$

The density on a dual cell is given by:

$$\begin{aligned} \text{for } \sigma \in \mathcal{E}_{\text{int}}, \sigma = K|L & \quad |D_\sigma| \varrho_{D_\sigma} = |D_{\sigma,K}| \varrho_K + |D_{\sigma,L}| \varrho_L, \\ \text{for } \sigma \in \mathcal{E}_{\text{ext}}, \sigma \in \mathcal{E}(K), & \quad \varrho_{D_\sigma} = \varrho_K. \end{aligned} \quad (2.22) \quad \{\text{eq:rho}\}$$

and we denote

$$\text{for } 1 \leq i \leq d, \quad \widehat{\varrho}^{(i)} = \sum_{\sigma \in \mathcal{E}^{(i)}} \varrho_{D_\sigma} \mathcal{X}_{D_\sigma}. \quad (2.23) \quad \{\text{rho}\}$$

The definition of the dual mass fluxes and the dual density ensures the validity of the mass balance equation over the diamond cells:

$$\forall 1 \leq i \leq d, \forall \sigma \in \mathcal{E}^{(i)}, \quad \frac{1}{\delta t} (\varrho_{D_\sigma}^n - \varrho_{D_\sigma}^{n-1}) + \frac{1}{|D_\sigma|} \sum_{\epsilon \in \widetilde{\mathcal{E}}(D_\sigma)} F_{\epsilon,\sigma}^n = 0. \quad (2.24) \quad \{\text{eq:mass}\}$$

This equation is necessary later for the derivation of the discrete energy balance.

2.2.3 Discrete divergence and gradient

The discrete divergence operator $\text{div}_{\mathcal{T}}$ is defined by:

$$\text{div}_{\mathcal{T}} : \left\{ \begin{array}{l} \mathbf{H}_{\mathcal{E}} \longrightarrow L_{\mathcal{T}} \\ \mathbf{u} \longmapsto \text{div}_{\mathcal{T}} \mathbf{u} = \sum_{K \in \mathcal{T}} \frac{1}{|K|} \sum_{\sigma \in \mathcal{E}(K)} |\sigma| u_{\sigma,K} \mathcal{X}_K, \end{array} \right. \quad (2.25) \quad \{\text{eq:div}\}$$

where $u_{\sigma,K}$ is defined in (2.13).

The discrete divergence of $\mathbf{u} = (u_1, \dots, u_d) \in \mathbf{H}_{\mathcal{E},0}$ may also be written as

$$\text{div}_{\mathcal{T}} \mathbf{u} = \sum_{i=1}^d \sum_{K \in \mathcal{T}} (\partial_i u_i)_K \mathcal{X}_K, \quad (2.26)$$

where the discrete derivative $(\partial_i u_i)_K$ of u_i on K is defined by

$$(\partial_i u_i)_K = \frac{|\sigma|}{|K|} (u_{\sigma'} - u_\sigma) \text{ with } K = [\overrightarrow{\sigma\sigma'}], \sigma, \sigma' \in \mathcal{E}^{(i)}. \quad (2.27) \quad \{\text{discret}\}$$

The gradient in the discrete momentum balance equation is defined as follows:

$$\nabla_{\mathcal{E}} : \left\{ \begin{array}{l} L_{\mathcal{T}} \longrightarrow \mathbf{H}_{\mathcal{E},0} \\ p \longmapsto \nabla_{\mathcal{E}} p \\ \nabla_{\mathcal{E}} p(\mathbf{x}) = (\partial_1 p(\mathbf{x}), \dots, \partial_d p(\mathbf{x}))^t, \end{array} \right. \quad (2.28) \quad \{\text{eq:grad}\}$$

where $\tilde{\partial}_i p \in H_{\mathcal{E},0}^{(i)}$ is the discrete derivative of p in the i -th direction, defined by:

$$\tilde{\partial}_i p = \sum_{\sigma=\overrightarrow{K}|L \in \mathcal{E}_{\text{int}}^{(i)}} \frac{|\sigma|}{|D_\sigma|} (p_L - p_K) \chi_{D_\sigma}, \quad i = 1, \dots, d. \quad (2.29) \quad \{\text{discder}\}$$

Note that in fact, the discrete gradient of a function of $L_{\mathcal{T}}$ should only be defined on the internal faces, and does not need to be defined on the external faces; we set it here in $\mathbf{H}_{\mathcal{E},0}$ (that is zero on the external faces) for the sake of simplicity.

The gradient in the discrete momentum balance equation is built as the dual operator of the discrete divergence. Indeed, we have the following lemma:

Lemma 2.1. [Discrete div – ∇ duality]

Let $q \in L_{\mathcal{T}}$ and $\mathbf{v} \in \mathbf{H}_{\mathcal{E},0}$ then we have:

$$\int_{\Omega} q \operatorname{div}_{\mathcal{T}} \mathbf{v} \, dx + \int_{\Omega} \nabla_{\mathcal{E}} q \cdot \mathbf{v} \, dx = 0. \quad (2.30) \quad \{\text{Ndiscre}\}$$

2.2.4 Discrete Laplace operator

For $i = 1, \dots, d$, we classically define the discrete Laplace operator on the i -th velocity grid by:

$$\begin{aligned} -\Delta_{\mathcal{E}}^{(i)} : \quad & \left| \begin{array}{l} H_{\mathcal{E},0}^{(i)} \longrightarrow H_{\mathcal{E},0}^{(i)} \\ u_i \longmapsto -\Delta_{\mathcal{E}}^{(i)} u_i \end{array} \right. \\ -\Delta_{\mathcal{E}}^{(i)} u_i = \quad & \sum_{\sigma \in \mathcal{E}_{\text{int}}^{(i)}} \left[\frac{1}{|D_\sigma|} \sum_{\epsilon \in \tilde{\mathcal{E}}(D_\sigma)} \phi_{\epsilon,\sigma}(u_i) \right] \chi_{D_\sigma}, \end{aligned} \quad (2.31) \quad \{\text{eq:lapi}\}$$

where $\tilde{\mathcal{E}}(D_\sigma)$, d_ϵ is defined in Definition 2.1, and

$$\phi_{\epsilon,\sigma}(u_i) = \begin{cases} \frac{|\epsilon|}{d_\epsilon} (u_{i,\sigma} - u_{i,\sigma'}) & \text{if } \epsilon = \sigma|\sigma' \in \tilde{\mathcal{E}}_{\text{int}}^{(i)}, \\ \frac{|\epsilon|}{d_\epsilon} u_{i,\sigma} & \text{if } \epsilon \in \tilde{\mathcal{E}}_{\text{ext}}^{(i)} \cap \tilde{\mathcal{E}}(D_\sigma). \end{cases} \quad (2.32)$$

Note that we have the usual finite volume property of local conservativity of the flux through an interface $\epsilon = \sigma|\sigma'$:

$$\phi_{\epsilon,\sigma}(u_i) = -\phi_{\epsilon,\sigma'}(u_i), \quad \forall \epsilon = \sigma|\sigma' \in \tilde{\mathcal{E}}_{\text{int}}^{(i)}. \quad (2.33) \quad \{\text{conserv}\}$$

Then the discrete Laplace operator of the full velocity vector is defined by

$$\begin{aligned} -\Delta_{\mathcal{E}} : \quad & \mathbf{H}_{\mathcal{E},0} \longrightarrow \mathbf{H}_{\mathcal{E},0} \\ \mathbf{u} \mapsto -\Delta_{\mathcal{E}} \mathbf{u} = \quad & (-\Delta_{\mathcal{E}}^{(1)} u_1, \dots, -\Delta_{\mathcal{E}}^{(d)} u_d)^t. \end{aligned} \quad (2.34)$$

Let us now recall the definition of the discrete H_0^1 inner product [5]; it is obtained by multiplying the discrete Laplace operator scalarly by a test function $\mathbf{v} \in \mathbf{H}_{\mathcal{E},0}$ and integrating over the computational domain. A simple reordering of the sums (which may be seen as a discrete integration by parts) yields, thanks to the conservativity of the diffusion flux (2.33):

$$\begin{aligned} \forall (\mathbf{u}, \mathbf{v}) \in \mathbf{H}_{\mathcal{E},0}^2, \quad & \int_{\Omega} -\Delta_{\mathcal{E}} \mathbf{u} \cdot \mathbf{v} \, dx = [\mathbf{u}, \mathbf{v}]_{1,\mathcal{E},0} = \sum_{i=1}^d [u_i, v_i]_{1,\mathcal{E}^{(i)},0}, \\ \text{with } [u_i, v_i]_{1,\mathcal{E}^{(i)},0} = \quad & \sum_{\substack{\epsilon \in \tilde{\mathcal{E}}_{\text{int}}^{(i)} \\ \epsilon = \sigma|\sigma'}} \frac{|\epsilon|}{d_\epsilon} (u_{i,\sigma} - u_{i,\sigma'}) (v_{i,\sigma} - v_{i,\sigma'}) + \sum_{\substack{\epsilon \in \tilde{\mathcal{E}}_{\text{ext}}^{(i)} \\ \epsilon \in \tilde{\mathcal{E}}(D_\sigma)}} \frac{|\epsilon|}{d_\epsilon} u_{i,\sigma} v_{i,\sigma} \end{aligned} \quad (2.35) \quad \{\text{ps}\}$$

The bilinear forms $\left| \begin{array}{l} H_{\mathcal{E},0}^{(i)} \times H_{\mathcal{E},0}^{(i)} \rightarrow \mathbb{R} \\ (u, v) \mapsto [u_i, v_i]_{1,\mathcal{E}^{(i)},0} \end{array} \right|$ and $\left| \begin{array}{l} \mathbf{H}_{\mathcal{E},0} \times \mathbf{H}_{\mathcal{E},0} \rightarrow \mathbb{R} \\ (\mathbf{u}, \mathbf{v}) \mapsto [\mathbf{u}, \mathbf{v}]_{1,\mathcal{E},0} \end{array} \right|$ are inner products on $H_{\mathcal{E},0}^{(i)}$ and $\mathbf{H}_{\mathcal{E},0}$ respectively, which induce the following discrete H_0^1 norms:

$$\|u_i\|_{1,\mathcal{E}^{(i)},0}^2 = [u_i, u_i]_{1,\mathcal{E}^{(i)},0} = \sum_{\substack{\epsilon \in \tilde{\mathcal{E}}_{\text{int}}^{(i)} \\ \epsilon = \sigma|\sigma'}} \frac{|\epsilon|}{d_\epsilon} (u_{i,\sigma} - u_{i,\sigma'})^2 + \sum_{\substack{\epsilon \in \tilde{\mathcal{E}}_{\text{ext}}^{(i)} \\ \epsilon \in \tilde{\mathcal{E}}(D_\sigma)}} \frac{|\epsilon|}{d_\epsilon} u_{i,\sigma}^2 \quad \text{for} \quad (2.36a) \quad \{\text{normi}\}$$

$$\|\mathbf{u}\|_{1,\mathcal{E},0}^2 = [\mathbf{u}, \mathbf{u}]_{1,\mathcal{E},0} = \sum_{i=1}^d \|u_i\|_{1,\mathcal{E}^{(i)},0}^2. \quad (2.36b) \quad \{\text{normful}\}$$

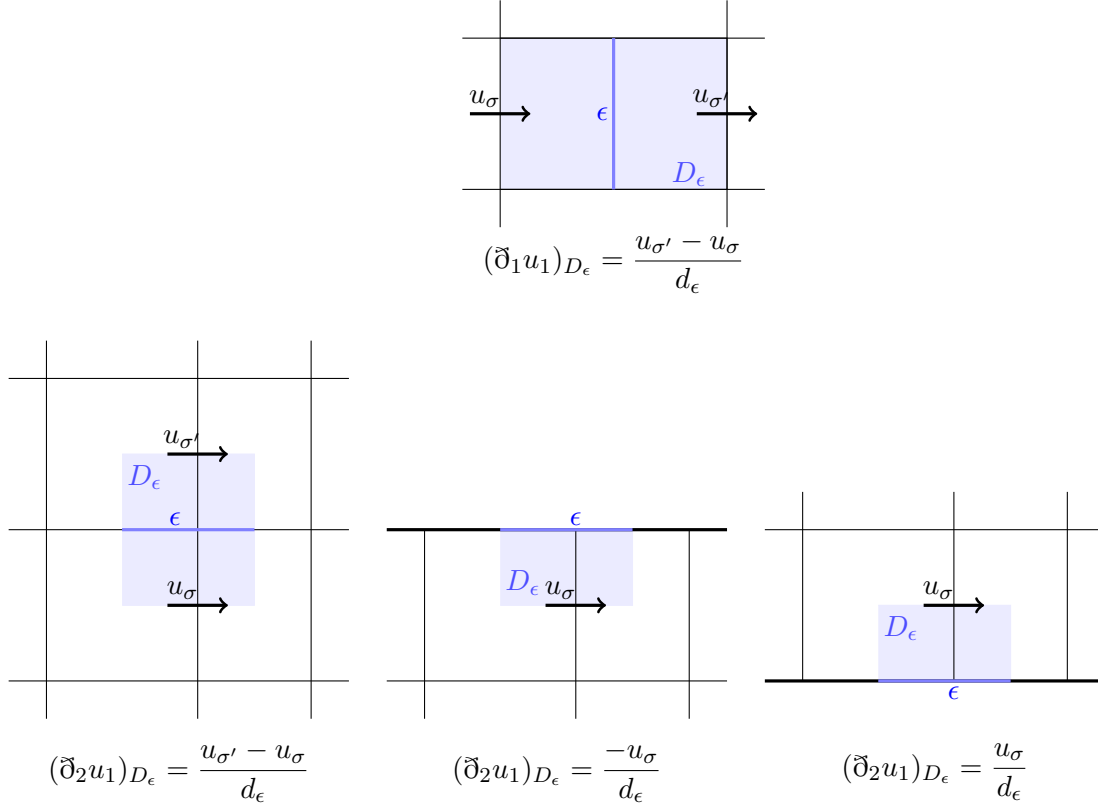


Figure 4: Notations for the definition of the partial space derivatives of the first component of the velocity, in two space dimensions.

We introduce the discrete gradient of velocity component u_i as follows:

$$\nabla_{\mathcal{E}^{(i)}} u_i = (\delta_1 u_i, \dots, \delta_d u_i) \quad \text{with} \quad \delta_j u_i = \sum_{\epsilon \in \tilde{\mathcal{E}}^{(i,j)}} (\delta_j u_i)_{D_\epsilon} \mathcal{X}_{D_\epsilon} \quad (2.37) \quad \{\text{partial}\}$$

where

$$(\delta_j u_i)_{D_\epsilon} = \begin{cases} \frac{u_{i,\sigma'} - u_{i,\sigma}}{d_\epsilon} & \text{if } \epsilon = \overrightarrow{\sigma|\sigma'} \in \tilde{\mathcal{E}}^{(i,j)} \cap \tilde{\mathcal{E}}_{\text{int}}^{(i)}, \\ -\frac{u_{i,\sigma}}{d_\epsilon} \mathbf{e}^{(j)} \cdot \mathbf{n}_{\epsilon,D_\sigma} & \text{if } \epsilon \in \tilde{\mathcal{E}}^{(i,j)} \cap \tilde{\mathcal{E}}_{\text{ext}}^{(i)} \cap \tilde{\mathcal{E}}(D_\sigma) \end{cases} \quad (2.38) \quad \{\text{Ddiscre}\}$$

(see Figure 4). Recall that all notations used above are introduced in Definition 2.1. Note, that this definition is compatible with the definition of the discrete derivative $(\delta_i u_i)_K$ given by (2.27). Finally

notice that the second line in (2.38) is equal to zero whenever $i = j$ (since $\mathbf{u} \in \mathbf{H}_{\mathcal{E},0}$). With this definition, it is easily seen that

$$\int_{\Omega} \nabla_{\mathcal{E}^{(i)}} u \cdot \nabla_{\mathcal{E}^{(i)}} v \, dx = [u, v]_{1,\mathcal{E}^{(i)},0}, \forall u, v \in H_{\mathcal{E},0}^{(i)}, \forall i = 1, \dots, d. \quad (2.39) \quad \{\text{gradient}\}$$

where $[u, v]_{1,\mathcal{E}^{(i)},0}$ is the discrete H_0^1 inner product defined by (2.35). Now we define the discrete gradient of the velocity field \mathbf{u} ,

$$\nabla_{\mathcal{E}} \mathbf{u} = (\nabla_{\mathcal{E}^{(1)}} u_1, \dots, \nabla_{\mathcal{E}^{(d)}} u_d)$$

and verify easily that

$$\int_{\Omega} \nabla_{\mathcal{E}} \mathbf{u} : \nabla_{\mathcal{E}} \mathbf{v} \, dx = [\mathbf{u}, \mathbf{v}]_{1,\mathcal{E},0}.$$

We will need discrete Sobolev inequalities for the discrete approximations. The following Theorem is proved in [5, Lemma 9.5].

Lemma 2.2. *[Discrete Sobolev inequalities]*

Let Ω be a bounded domain of \mathbb{R}^d , $d = 2$ or $d = 3$, compatible with the MAC grid and let $\mathcal{D} = (\mathcal{T}, \mathcal{E})$ be a MAC grid of Ω . Let $1 \leq q < +\infty$ if $d = 2$ and $q = 6$ if $d = 3$, $i = 1, \dots, d$. Then there exists $c = c(q, |\Omega|, \eta_{\mathcal{T}})$ (independent of h) depending on $\eta_{\mathcal{T}}$ in a nondecreasing way such that, for all $u \in H_{\mathcal{E},0}^{(i)}$,

$$\|u\|_{L^q(\Omega)} \leq c \|u\|_{1,\mathcal{E}^{(i)},0}.$$

2.3 The numerical scheme

Given $\varrho^0 \in L_{\mathcal{T}}$, $\varrho^0 > 0$ and $\mathbf{u}^0 \in \mathbf{H}_{\mathcal{E},0}$, we consider an implicit-in-time scheme for unknown $\varrho^n \in L_{\mathcal{T}}$, $\mathbf{u}^n \in \mathbf{H}_{\mathcal{E},0}$, $1 \leq n \leq N$, which reads

$$\frac{1}{\delta t} (\varrho^n - \varrho^{n-1}) + \operatorname{div}_{\mathcal{T}}^{\text{up}} (\varrho^n \mathbf{u}^n) = 0, \quad (2.40a) \quad \{\text{dcont}\}$$

$$\begin{aligned} \frac{1}{\delta t} (\widehat{\varrho^n^{(i)}} u_i^n - \widehat{\varrho^{n-1}^{(i)}} u_i^{n-1}) + \operatorname{div}_{\mathcal{E}}^{(i)} (\varrho^n \mathbf{u}^n u_i^n) - \mu \Delta_{\mathcal{E}}^{(i)} u_i^n \\ - (\mu + \lambda) \partial_i \operatorname{div}_{\mathcal{T}} \mathbf{u}^n + \frac{1}{\varepsilon^2} \partial_i p(\varrho^n) = 0, \quad i = 1, \dots, d. \end{aligned} \quad (2.40b) \quad \{\text{dmom}\}$$

Equation (2.40a) is a finite volume discretization of the mass balance (1.1) over the primal mesh. Equation (2.40b) is the discretization of the momentum balance equation (1.2) on the dual cells associated to the faces of the mesh. The discrete spaces $\mathcal{L}_{\mathcal{T}}$, $\mathbf{H}_{\mathcal{E},0}$ are defined in Section 2.1.1. We recall that the quantities $\widehat{\varrho^n^{(i)}}$ are defined in (2.23), while the discrete differential operators appearing (2.40) are defined in Sections 2.2.1–2.2.4.

Of course, the quantities $(\varrho, \mathbf{u}) \in \mathbf{Y}_{\mathcal{M},\delta t} \times \mathbf{X}_{\mathcal{E},\delta t}$ (see Sections 2.1.1–2.1.2) depend tacitly on h , δt and ε , meaning that $(\varrho, \mathbf{u}) \equiv (\varrho_{h,\delta t,\varepsilon}, \mathbf{u}_{h,\delta t,\varepsilon})$. However, in order to avoid a cumbersome notation, we shall omit the subscripts $h, \delta t, \varepsilon$ in most formulas. We shall keep some of them only when a confusion could arise.

It is well known that the (2.40) admits at least one solution. Indeed, the following existence theorem is proved in [20, Appendix A].

Proposition 2.1. *Let $(\varrho^0, \mathbf{u}^0) \in L_{\mathcal{T}} \times \mathbf{H}_{\mathcal{E},0}$ such that $\varrho^0 > 0$ (meaning that $\varrho_K^0 > 0$ for any $K \in \mathcal{T}$). There exists a solution $(\mathbf{u}, \varrho) \in \mathbf{H}_{\mathcal{E},0} \times L_{\mathcal{T}}$ of Problem (2.40). Moreover any solution is such that $\varrho > 0$ a.e in Ω (meaning that $\varrho_K^n > 0$ for any $n = 1, \dots, N$ and for any $K \in \mathcal{T}$).*

Uniqueness remains an open problem.

2.4 Projection operators

In this section we introduce several projection operators. We first define the mean-value interpolator over $L_{\mathcal{T}}$:

$$\mathcal{P}_{\mathcal{T}} : \begin{cases} L_{\text{loc}}^1(\Omega) & \longrightarrow & L_{\mathcal{T}} \\ \varphi & \mapsto & \mathcal{P}_{\mathcal{T}}\varphi = \sum_{K \in \mathcal{T}} \varphi_K \mathcal{X}_K, \end{cases} \quad (2.41) \quad \{\text{projpri}\}$$

with

$$\varphi_K = \frac{1}{|K|} \int_K \varphi(\mathbf{x}) d\mathbf{x}, \quad \forall K \in \mathcal{T}. \quad (2.42) \quad \{\text{meanval}\}$$

We also define over $H_{\mathcal{E},0}^{(i)}$ the following interpolation operator $\mathcal{P}_{\mathcal{E}}^{(i)}$:

$$\mathcal{P}_{\mathcal{E}}^{(i)} : \begin{cases} H_0^1(\Omega) & \longrightarrow & H_{\mathcal{E},0}^{(i)} \\ \varphi & \mapsto & \mathcal{P}_{\mathcal{E}}^{(i)}\varphi = \sum_{\sigma \in \mathcal{E}_{\text{int}}^{(i)}} \varphi_{\sigma} \mathcal{X}_{D_{\sigma}}, \end{cases} \quad (2.43) \quad \{\text{projdua}\}$$

with

$$\varphi_{\sigma} = \frac{1}{|\sigma|} \int_{\sigma} \varphi(\mathbf{x}) d\gamma(\mathbf{x}), \quad \forall \sigma \in \mathcal{E}_{\text{int}}^{(i)}, \quad (2.44) \quad \{\text{meanval}\}$$

where $d\gamma$ is the $(d-1)$ -Lebesgue measure on σ , and we denote

$$\mathcal{P}_{\mathcal{E}} = (\mathcal{P}_{\mathcal{E}}^{(1)}, \dots, \mathcal{P}_{\mathcal{E}}^{(d)}) \in \mathcal{L}(H_0^1(\Omega)^d, \mathbf{H}_{\mathcal{E},0}) \quad (2.45) \quad \{\text{projdua}\}$$

the vector valued extension. This operator preserves the divergence in the following sense, see [21].

Lemma 2.3.

$$\forall \mathbf{v} \in H_0^1(\Omega)^d, \forall q \in L_{\mathcal{T}}, \int_{\Omega} q \operatorname{div}_{\mathcal{T}} \mathcal{P}_{\mathcal{E}} \mathbf{v} d\mathbf{x} = \int_{\Omega} q \operatorname{div} \mathbf{v} d\mathbf{x}. \quad (2.46) \quad \{\text{L1-1}\}$$

In particular, if $\operatorname{div} \mathbf{v} = 0$ then $\operatorname{div}_{\mathcal{T}} \mathcal{P}_{\mathcal{E}}(\mathbf{v}) = 0$.

The next lemma deals with the properties of the projections defined by (2.41) and (2.43). It can be obtained by rescaling from the standard inequalities on the reference cell $[0, 1]^d$, see e.g [5] or [20, Lemma 3.2].

Lemma 2.4. [Mean value inequalities]

Let $K = \prod_{i=1}^d (a_i, b_i)$ be a bounded open square of \mathbb{R}^d , $d \geq 1$. Let $\sigma \subset \partial K$ be a face of K . Let $1 \leq p \leq \infty$. There exists c only depending on d and p such that $\forall v \in W^{1,p}(K)$,

$$\|v - v_{\sigma}\|_{L^p(K)} \leq c \operatorname{diam}(K) \|\nabla v\|_{L^p(K; \mathbb{R}^d)}, \quad (2.47) \quad \{\text{L1-2}\}$$

$$\|v - v_K\|_{L^p(K)} \leq c \operatorname{diam}(K) \|\nabla v\|_{L^p(K; \mathbb{R}^d)}, \quad (2.48) \quad \{\text{L1-3}\}$$

where v_K and v_{σ} are defined in (2.42), (2.44).

From Lemma 2.4 one deduces in almost straightforward way the following "global" properties of projections $\mathcal{P}_{\mathcal{T}}$, $\mathcal{P}_{\mathcal{E}}$ (see [20, Lemma 3.2]):

Lemma 2.5. Let $\mathcal{D} = (\mathcal{T}, \mathcal{E})$ be a MAC grid of the computational domain Ω . Let $1 \leq p \leq \infty$. There exists $c > 0$ only depending on d and p and $|\Omega|$ such that for any $i = 1, \dots, d$ one has

$$\forall v \in L^p(\Omega), \quad \|\mathcal{P}_{\mathcal{T}} v\|_{L^p(\Omega)} \leq c \|v\|_{L^p(\Omega)}, \quad (2.49) \quad \{\text{dod11}\}$$

$$\forall v \in W^{1,p}(\Omega), \quad \|\mathcal{P}_{\mathcal{T}} v - v\|_{L^p(\Omega)} \leq ch \|\nabla v\|_{L^p(\Omega; \mathbb{R}^d)},$$

$$\forall v \in W_0^{1,p}(\Omega), \quad \|\mathcal{P}_{\mathcal{E}}^{(i)} v\|_{L^p(\Omega)} \leq c \|\nabla v\|_{L^p(\Omega; \mathbb{R}^d)}, \quad (2.50) \quad \{\text{dod12}\}$$

$$\forall v \in W_0^{1,p}(\Omega), \quad \|\mathcal{P}_{\mathcal{E}}^{(i)} v - v\|_{L^p(\Omega)} \leq ch \|\nabla v\|_{L^p(\Omega; \mathbb{R}^d)}.$$

Lemma 2.6. *Let $\mathcal{D} = (\mathcal{T}, \mathcal{E})$ be a MAC grid of the computational domain Ω . Let $1 \leq p \leq \infty$. There exists $c > 0$ only depending on d and p and $|\Omega|$ and on $\eta_{\mathcal{T}}$ in a nondecreasing way such that for any $i = 1, \dots, d$ one has*

$$\forall v \in W^{1,p}(\Omega) \cap H_0^1(\Omega), \quad \|\nabla_{\mathcal{E}^{(i)}} \mathcal{P}_{\mathcal{E}}^{(i)} v\|_{L^p(\Omega; \mathbb{R}^d)} \leq c \|\nabla v\|_{L^p(\Omega; \mathbb{R}^d)}, \quad (2.51) \quad \{\text{dod13}\}$$

$$\forall v \in W^{2,\infty}(\Omega) \cap H_0^1(\Omega), \quad \|\tilde{\partial}_j \mathcal{P}_{\mathcal{E}}^{(i)} v - \partial_j v\|_{L^\infty(\Omega)} \leq ch \|\nabla^2 v\|_{L^\infty(\Omega; \mathbb{R}^{d^2})}.$$

Next we introduce and recall some properties of different velocity interpolators.

Definition 2.3. [Velocity interpolators]

1. *Velocity reconstruction operator with respect to (i, j)*

For a given MAC grid $\mathcal{D} = (\mathcal{T}, \mathcal{E})$, we define, for $i, j = 1, \dots, d$, the full grid velocity reconstruction operator with respect to (i, j) by

$$\mathcal{R}_{\mathcal{E}}^{(i,j)} : H_{\mathcal{E},0}^{(i)} \rightarrow H_{\mathcal{E},0}^{(j)}, \quad v \mapsto \mathcal{R}_{\mathcal{E}}^{(i,j)} v = \sum_{\sigma \in \mathcal{E}_{\text{int}}^{(j)}} \hat{v}_{\sigma}^{(i,j)} \chi_{D_{\sigma}}, \quad (2.52) \quad \{\text{def:ufu}\}$$

where

$$\hat{v}_{\sigma}^{(i,j)} = v_{\sigma} \text{ if } \sigma \in \mathcal{E}_{\text{int}}^{(i)}, \quad \hat{v}_{\sigma}^{(i,j)} = \frac{1}{\text{card}(\mathcal{N}_{\sigma})} \sum_{\sigma' \in \mathcal{N}_{\sigma}} v_{\sigma'} \text{ otherwise,} \quad \mathcal{N}_{\sigma} = \{\sigma' \in \mathcal{E}^{(i)}, D_{\sigma} \cap \sigma' \neq \emptyset\}. \quad (2.53)$$

2. *Velocity reconstruction to $L_{\mathcal{T}}$*

For any $i = 1, \dots, d$, we also define a projector from $H_{\mathcal{E}}^{(i)}$ into $L_{\mathcal{T}}$ by

$$\mathcal{R}_{\mathcal{T}}^{(i)} : H_{\mathcal{E}}^{(i)} \rightarrow L_{\mathcal{T}}, \quad v \mapsto \mathcal{R}_{\mathcal{T}}^{(i)} v = \sum_{K \in \mathcal{T}} v_K \chi_K, \quad (2.54) \quad \{\text{def:ufu}\}$$

where

$$v_K = \frac{1}{|K|} \int_K v(x) dx = \frac{1}{2} \sum_{\sigma \in \mathcal{E}^{(i)}(K)} v_{\sigma}. \quad (2.55) \quad \{\text{eq:inte}\}$$

We then set

$$\mathcal{R}_{\mathcal{T}} : \mathbf{H}_{\mathcal{E}} \rightarrow L_{\mathcal{T}}^d, \quad \mathbf{v} = (v_1, \dots, v_d) \mapsto \mathcal{R}_{\mathcal{T}} \mathbf{v} = (\mathcal{R}_{\mathcal{T}}^{(1)} v_1, \dots, \mathcal{R}_{\mathcal{T}}^{(d)} v_d). \quad (2.56) \quad \{\text{def:ufu}\}$$

3. *Upwind velocity reconstruction operator with respect to (i, j)*

Let $\sigma = K|L \in \mathcal{E}^{(j)}$ and let $u \in H_{\mathcal{E},0}^{(j)}$. We define

$$\sigma_u^{\text{up}} = \left\{ \begin{array}{l} K \text{ if } u_{\sigma,K} \equiv u_{j,\sigma} \mathbf{e}^{(j)} \cdot \mathbf{n}_{\sigma,K} > 0, \\ L \text{ if } u_{\sigma,K} \equiv u_{j,\sigma} \mathbf{e}^{(j)} \cdot \mathbf{n}_{\sigma,K} \leq 0 \end{array} \right\} \in \mathcal{T}.$$

For any $v \in H_{\mathcal{E},0}^{(i)}$ we define

$$\mathcal{R}_{\mathcal{E}}^{(i,j,u)}(v) = \sum_{\sigma \in \mathcal{E}^{(j)}} v_{\sigma_u^{\text{up}}} \chi(D_{\sigma}) \in H_{\mathcal{E},0}^{(j)}.$$

The following Lemmas 2.7-2.9 are straightforward consequence of Definition 2.3, see [20, Lemma 4] for the proofs.

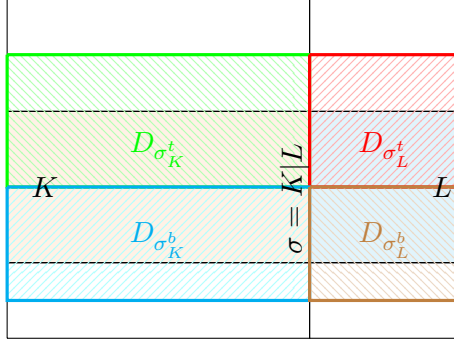


Figure 5: Full grid velocity interpolate.

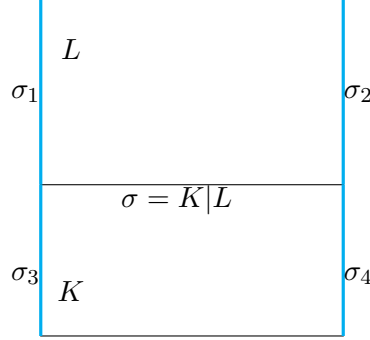


Figure 6: Set $\mathcal{N}_\sigma = \{\sigma_1, \sigma_2, \sigma_3, \sigma_4\}$ with $\sigma \in \mathcal{E}_{\text{int}}^{(j)}(K)$, $j \neq i$ in two dimensions ($i = 1, j = 2$)

Lemma 2.7. *There exists $c > 0$ depending on d, p such that for any $i = 1, \dots, d$, for any $v \in H_{\mathcal{E},0}^{(i)}$ and for any $1 \leq p \leq \infty$ one has*

$$\|\mathcal{R}_{\mathcal{T}}^{(i)} v\|_{L^p(\Omega)} \leq c\|v\|_{L^p(\Omega)}, \quad \|\mathcal{R}_{\mathcal{T}}^{(i)} v - v\|_{L^p(\Omega)} \leq ch\|\tilde{\partial}_i v\|_{L^p(\Omega)}. \quad (2.57) \quad \{\text{ufullpr}\}$$

Lemma 2.8. *There exists $c > 0$, depending only on d, p and on the regularity of the mesh (defined by (2.8)) in a nondecreasing way, such that, for any $v \in L^p(\Omega)$, for any $1 \leq p \leq \infty$ and for any $i, j = 1, \dots, d$,*

$$\|\mathcal{R}_{\mathcal{E}}^{(i,j)} v\|_{L^p(\Omega)} \leq c\|v\|_{L^p(\Omega)}, \quad \|\mathcal{R}_{\mathcal{E}}^{(i,j)} v - v\|_{L^p(\Omega)} \leq ch\|\nabla_{\mathcal{E}}^{(i)} v\|_{L^p(\Omega; \mathbb{R}^d)}. \quad (2.58) \quad \{\text{def:ufu}\}$$

Lemma 2.9. *There exists $c > 0$, depending only on d, p and on the regularity of the mesh (defined by (2.8)) in a nondecreasing way, such that, for any $v \in L^p(\Omega)$, for any $1 \leq p \leq \infty$ and for any $i, j = 1, \dots, d$,*

$$\|\mathcal{R}_{\mathcal{E}}^{(i,j,u)} v\|_{L^p(\Omega)} \leq c\|v\|_{L^p(\Omega)}, \quad \|\mathcal{R}_{\mathcal{E}}^{(i,j,u)} v - v\|_{L^p(\Omega)} \leq ch\|\nabla_{\mathcal{E}}^{(i)} v\|_{L^p(\Omega; \mathbb{R}^d)}. \quad (2.59) \quad \{\text{R2up}\}$$

The following algebraic identity derived in [20, Lemma 5] in the spirit of [27] is useful to transform terms involving the dual fluxes into terms involving primal fluxes.

Lemma 2.1. *Let $\varrho \in L_{\mathcal{T}}$ and $\mathbf{u} \in \mathbf{H}_{\mathcal{E},0}$. Let $i \in \{1, \dots, d\}$. Let $\varphi = \sum_{\sigma \in \mathcal{E}_{\text{int}}^{(i)}} \varphi_{\sigma} \chi_{D_{\sigma}} \in H_{\mathcal{E},0}^{(i)}$ be a discrete scalar function. Let the primal fluxes be given by (2.12) and let the dual fluxes $F_{\epsilon,\sigma}$ be given by (2.19) or (2.20) (depending on the direction of \mathbf{n}_{ϵ} with respect to $\mathbf{e}^{(i)}$). Then we have:*

$$\sum_{\sigma \in \mathcal{E}_{\text{int}}^{(i)}} \sum_{\epsilon \in \mathcal{E}(D_{\sigma})} F_{\epsilon,\sigma} u_{\epsilon} \varphi_{\sigma} = \sum_{K \in \mathcal{T}} (\mathcal{R}_{\mathcal{T}}^{(i)} \varphi) \Big|_K \sum_{j=1}^d \sum_{\sigma \in \mathcal{E}_{\text{int}}^{(j)}(K)} F_{\sigma,K} (\mathcal{R}_{\mathcal{E}}^{(i,j)} u_i) \Big|_{\sigma} + R^i(u_i, \varphi)$$

where

$$R^i(u_i, \varphi) = \sum_{K \in \mathcal{T}} \sum_{\sigma \in \mathcal{E}_{\text{int}}^{(i)}(K)} (\varphi_\sigma - (\mathcal{R}_{\mathcal{T}}^{(i)} \varphi)|_K) F_{\sigma, K}(u_\sigma - (\mathcal{R}_{\mathcal{T}}^{(i)} u_i)_K) \\ + \sum_{K \in \mathcal{T}} \sum_{\sigma \in \mathcal{E}_{\text{int}}^{(i)}(K)} (\varphi_\sigma - (\mathcal{R}_{\mathcal{T}}^{(i)} \varphi)|_K) \sum_{j=1, j \neq i}^d \sum_{\tau \in \mathcal{E}_{\text{int}}^{(j)}(K)} \sum_{\sigma' \in \mathcal{N}_{\tau, \sigma}} \frac{F_{\tau, K}}{2} \left(\frac{u_{i, \sigma} + u_{i, \sigma'}}{2} - (\mathcal{R}_{\mathcal{T}}^{(i)} u_i)|_K \right).$$

In the last sum we have denoted

$$\mathcal{N}_{\tau, \sigma} = \{\sigma' \in \mathcal{E}^{(i)} \mid \text{int}_{d-1} \tau \cap \text{int}_{d-1}(D_\sigma | D_{\sigma'}) \neq \emptyset\},$$

where $\sigma \in \mathcal{E}_{\text{int}}^{(i)}(K)$, $\tau \in \mathcal{E}_{\text{int}}^{(j)}(K)$, $j \neq i$.

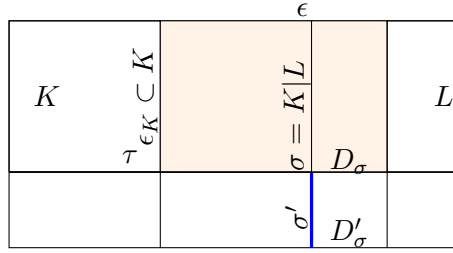


Figure 7: Set $\mathcal{N}_{\tau, \sigma} = \{\sigma'\}$ with $\tau \in \mathcal{E}_{\text{int}}^{(j)}(K)$, $\sigma \in \mathcal{E}^{(i)}(K)$, $j \neq i$ in two dimensions ($i = 1$, $j = 2$)

2.5 Main result: asymptotic preserving error estimates

Now, we are ready to state the main results of this paper. For the sake of clarity, we shall state the theorem and perform the proofs only in the most interesting three dimensional case. The modifications to be done for the two dimensional case, which is in fact more simple, are mostly due to the different Sobolev embeddings and are left to the interested reader.

2.5.1 Relative energy and relative energy functional

Before the announcement of the main theorems, we introduce relative energy function

$$E : [0, \infty) \times (0, \infty) \rightarrow [0, \infty), \quad E(\varrho|z) = \mathcal{H}(\varrho) - \mathcal{H}'(z)(\varrho - z) - \mathcal{H}(z), \quad \text{where } \mathcal{H}(\varrho) = \varrho \int_1^\varrho \frac{p(s)}{s^2} ds. \quad (2.60) \quad \{\text{E}\}$$

We notice that under assumption $p'(\varrho) > 0$, function $\varrho \mapsto H(\varrho)$ is strictly convex on $(0, \infty)$; whence

$$E(\varrho|z) \geq 0 \quad \text{and} \quad E(\varrho|z) = 0 \Leftrightarrow \varrho = z.$$

In fact E obeys stronger coercivity property.

Lemma 2.10. *Let p satisfies assumptions (1.4). Let $\bar{\varrho} > 0$. Then there exists $c = c(\bar{\varrho}) > 0$ such that for all $\varrho \in [0, \infty)$ there holds*

$$E(\varrho|\bar{\varrho}) \geq c \left(1_{R_+ \setminus [\bar{\varrho}/2, 2\bar{\varrho}]}(\varrho) + \varrho^\gamma 1_{R_+ \setminus [\bar{\varrho}/2, 2\bar{\varrho}]}(\varrho) + (\varrho - \bar{\varrho})^2 1_{[\bar{\varrho}/2, 2\bar{\varrho}]}(\varrho) \right). \quad (2.61) \quad \{\text{added}\}$$

Finally we introduce the corresponding relative energy functional,

$$\mathcal{E}_\varepsilon(\varrho, \mathbf{u}|z, \mathbf{v}) = \int_\Omega \left(\varrho |\mathbf{u} - \mathbf{v}|^2 + \frac{1}{\varepsilon^2} E(\varrho|z) \right) dx, \quad (2.62) \quad \{\text{calE}\}$$

where $\varrho \geq 0$, $z > 0$, \mathbf{u} , \mathbf{v} are measurable functions on Ω .

2.5.2 Error estimates

We are at the point to announce the main theorem of the paper.

Theorem 2.1. [Error estimate in the low Mach number regime]

Let $\Omega \subset \mathbb{R}^3$ be a bounded domain compatible with the MAC grid and let $\mathcal{D} = (\mathcal{T}, \mathcal{E})$ be a MAC grid of Ω (see Definition 2.1) with step size h (see (2.6)) and regularity $\eta_{\mathcal{T}}$ where $\eta_{\mathcal{T}}$ is defined in (2.8). Let us consider a partition $0 = t^0 < t^1 < \dots < t^N = T$ of the time interval $[0, T]$, which, for the sake of simplicity, we suppose uniform where δt stands for the constant time step.

Let $(\varrho, \mathbf{u}) \in Y_{\mathcal{T}, \delta t} \times \mathbf{X}_{\mathcal{E}, \delta t}$ (see Section 2.1.2) be a solution of the discrete problem (2.40) emanating from the initial data $(\varrho_\varepsilon^0, \mathbf{u}_\varepsilon^0) \in L_{\mathcal{T}} \times \mathbf{H}_{\mathcal{E}, 0}$ such that $\varrho_\varepsilon^0 > 0$ and

$$M_{0, \mathcal{T}, \varepsilon} = \int_{\Omega} \varrho_\varepsilon^0 dx, \quad \mathcal{E}_{0, \mathcal{T}, \varepsilon} = \int_{\Omega} \varrho_\varepsilon^0 |\mathbf{u}_\varepsilon^0|^2 dx + \frac{1}{\varepsilon^2} \int_{\Omega} E(\varrho_\varepsilon^0 | \bar{\varrho}) dx, \quad (2.63) \quad \{\text{energies}\}$$

where

$$M_0/2 \leq M_{0, \mathcal{T}, \varepsilon} \leq 2M_0, \quad \bar{\varrho}|\Omega| = M_0, \quad \mathcal{E}_{0, \mathcal{T}, \varepsilon} \leq E_0, \quad E_0 > 0 \quad (2.64) \quad \{\text{dod2}\}$$

(existence of which is guaranteed by Proposition 2.1).

Suppose that $[\Pi, \mathbf{V}]$ is a classical solution to the initial-boundary value problem (1.7–1.9) in $(0, T) \times \Omega$ in the regularity class (1.10) with $p = \max(2, \gamma')$, emanating from the initial data $\mathbf{V}(0) \equiv \mathbf{V}^0 \in L^2(\Omega)$.

Then there exists a positive number

$$C = C\left(M_0, E_0, \bar{\varrho}, \|\mathbf{V}\|_{\mathcal{X}_T(\Omega)}, \|\Pi\|_{\mathcal{Y}_T^p(\Omega)}\right)$$

depending on these parameters in a nondecreasing way, on $\eta_{\mathcal{T}}$ in a nondecreasing way and dependent tacitly also on $|\Omega|$, T , γ , p_0 , p_∞ (and independent in particular on h , δt , ε) such that

$$\begin{aligned} \sup_{1 \leq n \leq N} \mathcal{E}_\varepsilon(\varrho^n, \mathbf{u}^n | \bar{\varrho}, \mathbf{V}_\varepsilon^n) + \mu \delta t \sum_{n=1}^N \|\mathbf{u}^n - \mathbf{V}_\varepsilon^n\|_{1, \mathcal{E}, 0}^2 + (\mu + \lambda) \delta t \sum_{n=1}^N \|\operatorname{div}_{\mathcal{T}}(\mathbf{u}^n - \mathbf{V}_\varepsilon^n)\|_{L^2(\Omega)}^2 \\ \leq C \left(\sqrt{\delta t} + h^A + \varepsilon + \mathcal{E}_\varepsilon(\varrho_\varepsilon^0, \mathbf{u}_\varepsilon^0 | \bar{\varrho}, \mathbf{V}_\varepsilon^0) \right), \end{aligned} \quad (2.65) \quad \{\text{M1}\}$$

where

$$A = \min\left(\frac{2\gamma - 3}{\gamma}, 1\right). \quad (2.66) \quad \{\text{A1}\}$$

Here we have denoted $\mathbf{V}_\varepsilon^n = \mathcal{P}_\mathcal{E}(\mathbf{V}(t_n))$, where $\mathcal{P}_\mathcal{E}$ is the projection to the discrete space $\mathbf{H}_{\mathcal{E}, 0}$ defined in Section 2.4. Operator $\operatorname{div}_{\mathcal{T}}$ is defined in (2.25) and the norm $\|\cdot\|_{1, \mathcal{E}, 0}$ is given in (2.36a–2.36b). Finally, in the above

$$\begin{aligned} \|\mathbf{V}\|_{\mathcal{X}_T(\Omega)} &= \|\mathbf{V}\|_{C^1([0, T] \times \bar{\Omega}; \mathbb{R}^3)} + \|\nabla^2 \mathbf{V}\|_{C([0, T] \times \bar{\Omega}; \mathbb{R}^{27})} \\ &+ \|\partial_t^2 \mathbf{V}\|_{L^2(0, T; L^{6/5}(\Omega; \mathbb{R}^3))} + \|\partial_t \nabla \mathbf{V}\|_{L^2(0, T; L^{6/5}(\Omega; \mathbb{R}^9))}, \\ \|\Pi\|_{\mathcal{Y}_T(\Omega)} &= \|\Pi\|_{C^1([0, T] \times \bar{\Omega})} + \|\partial_t \Pi\|_{L^1(0, T; L^p(\Omega))}. \end{aligned} \quad (2.67) \quad \{\text{normspa}\}$$

Remark 2.1.

1. Due to (1.7) and (1.9–1.10), \mathbf{V}_0 belongs necessarily to $C^1(\bar{\Omega}; \mathbb{R}^3)$ and it is divergence free. If the initial data are ill prepared, meaning that

$$\int_{\Omega} E(\varrho_\varepsilon^0 | \bar{\varrho}) \lesssim \varepsilon^2, \quad \int_{\Omega} \varrho_\varepsilon^0 |\mathbf{u}_\varepsilon^0 - \mathbf{V}_\varepsilon^0|^2 \lesssim 1,$$

we obtain in Theorem 2.1 for the error solely a bound independent of ε . On the other hand, if the initial data are well prepared, with a convergence rate, ε^ξ , $\xi > 0$, meaning

$$\int_{\Omega} E(\varrho_\varepsilon^0 | \bar{\varrho}) \lesssim \varepsilon^{2+\xi}, \quad \int_{\Omega} \varrho_\varepsilon^0 |\mathbf{u}_\varepsilon^0 - \mathbf{V}_\varepsilon^0|^2 \lesssim \varepsilon^\xi,$$

Theorem 2.1 gives uniform convergence as $(h, \delta t, \varepsilon) \rightarrow 0$ of the numerical solution to the strong solution of the incompressible Navier-Stokes equations, provided the strong solution exists, including the rates of convergence. These results are in agreement with the theory of low Mach number limits in the continuous case.

2. In view of Lemma 2.10, formula (2.65) provides the bound for the "essential part" of the solution (where the numerical density remains bounded from above and from below outside zero):

$$\|\varrho^m - \bar{\varrho}\|_{L^2(\Omega \cap \{\underline{\varrho}/2 \leq \varrho^m \leq 2\bar{\varrho}\})}^2 + \|\mathbf{u}^m - \mathbf{V}_{\mathcal{E}}^m\|_{L^2(\Omega \cap \{\underline{\varrho}/2 \leq \varrho^m \leq 2\bar{\varrho}\})}^2,$$

and for the "residual part" of the solution, where the numerical density can be "close" to zero or infinity:

$$|\{\varrho^m \leq \underline{\varrho}/2\}| + |\{\varrho^m \geq 2\bar{\varrho}\}| + \|\varrho^m\|_{L^\gamma(\Omega \cap \{\varrho^m \geq 2\bar{\varrho}\})}^\gamma + \|\varrho^m |\mathbf{u}^m - \mathbf{V}_{\mathcal{E}}^m|^2\|_{L^1(\Omega \cap \{\varrho^m \geq 2\bar{\varrho}\})}.$$

In the above, for $B \subset \Omega$, $|B|$ denotes the Lebesgue measure of B .

Moreover, in the particular case of $p(\varrho) = \varrho^2$, we have $E(\varrho|r) = (\varrho - r)^2$ and the error estimate (2.65) provides a bound for the Lebesgue norms

$$\|\varrho^m - \bar{\varrho}\|_{L^2(\Omega)}^2 + \|\varrho^m |\mathbf{u}^m - \mathbf{V}_{\mathcal{E}}^m|^2\|_{L^1(\Omega)}.$$

3. Theorem 2.1 remains valid also for two dimensional bounded domains compatible with the MAC discretization described in Section 2.2.1 with any $0 < A < \frac{2\gamma-2}{\gamma}$ if $\gamma \in (1, 2]$, and $A = 1$ if $\gamma > 2$.

3 Mesh independent estimates

3.1 Conservation of mass

Due to (2.15), summing (2.40a) over $K \in \mathcal{T}$, we obtain immediately the total conservation of mass,

$$\forall n = 1, \dots, N, \quad \int_{\Omega} \varrho^n dx = \int_{\Omega} \varrho^0 dx. \quad (3.1) \quad \{\text{masscon}\}$$

3.2 Energy Identity

In the next theorem we report the energy identity for any solution of the numerical scheme (2.40). This theorem shows that the scheme (2.40) is unconditionally stable meaning that the discrete energy inequality holds without any extra assumptions on the discrete solution. The theorem whose detailed proof can be find in [20, Theorem 4] reads

Lemma 3.1. [Energy identity]

Let $(\varrho, \mathbf{u}) \in Y_{\mathcal{T}, \delta t} \times \mathbf{X}_{\mathcal{E}, \delta t}$ be a solution of with pressure obeying hypotheses (1.4)₁. Then there exists $\varrho^{n-1, n} \in L_{\mathcal{T}}$, $\min(\varrho_K^{n-1}, \varrho_K^n) \leq \varrho_K^{n-1, n} \leq \max(\varrho_K^{n-1}, \varrho_K^n)$ and $\varrho_\sigma^n \in [\min(\varrho_K^n, \varrho_L^n), \max(\varrho_K^n, \varrho_L^n)]$, $\sigma = K|L \in \mathcal{E}_{\text{int}}$, $n = 1, \dots, N$ such that for all $n = 1, \dots, N$ we have,

$$\begin{aligned} & \frac{1}{\delta t} \frac{1}{\varepsilon^2} \int_{\Omega} \mathcal{H}(\varrho^n) - \mathcal{H}(\varrho^{n-1}) dx + \frac{1}{2\delta t} \int_{\Omega} \varrho^n |\mathbf{u}^n|^2 - \varrho^{n-1} |\mathbf{u}^{n-1}|^2 dx \\ & \quad + \mu \|\mathbf{u}^n\|_{1, \mathcal{E}, 0}^2 + (\mu + \lambda) \|\operatorname{div}_{\mathcal{T}} \mathbf{u}^n\|_{L^2(\Omega)}^2 + \frac{1}{2\delta t} \int_{\Omega} \varrho^{n-1} |\mathbf{u}^n - \mathbf{u}^{n-1}|^2 dx \\ & \quad + \frac{1}{2\delta t} \frac{1}{\varepsilon^2} \int_{\Omega} \mathcal{H}''(\varrho^{n-1, n}) (\varrho^n - \varrho^{n-1})^2 dx + \frac{1}{2} \frac{1}{\varepsilon^2} \sum_{\substack{\sigma \in \mathcal{E}_{\text{int}} \\ \sigma = K|L}} |\sigma| \mathcal{H}''(\varrho_\sigma^n) (\varrho_K^n - \varrho_L^n)^2 |\mathbf{u}_{\sigma, K}^n| = 0. \end{aligned} \quad (3.2) \quad \{\text{ied1g}\}$$

From now on, the letter c denotes a positive number that may tacitly depend on T , $|\Omega|$, γ , p_0 , p_∞ , μ , λ and on $\eta_{\mathcal{T}}$ in a nondecreasing way. This dependence on the above parameters will not be indicated in the argument of c . The number c may be dependent further in a nondecreasing way on M_0 , E_0 , $\bar{\varrho}$ (see (2.64)), and on given functions denoted $(\Pi, \mathbf{U}) \in \mathcal{X}_T(\Omega) \times \mathcal{Y}_T^p(\Omega)$, see (1.10). The dependence on these quantities (if any) is always explicitly indicated in the argument of c .

The numbers c can take different values even in the same formula. They are always independent of the size of the discretisation δt and h and on the Mach number ε .

Now, for fixed number $\bar{\varrho} > 0$ and fixed functions ϱ^n , $n = 0, \dots, N$, we introduce the residual and essential subsets of Ω (relative to ϱ^n) as follows:

$$\Omega_{\text{ess}}^n = \{x \in \Omega \mid \frac{1}{2}\bar{\varrho} \leq \varrho^n(x) \leq 2\bar{\varrho}\}, \quad \Omega_{\text{res}}^n = \Omega \setminus \Omega_{\text{ess}}^n, \quad (3.3) \quad \{\text{essres}\}$$

and we set

$$[g]_{\text{ess}}(x) = g(x)1_{\Omega_{\text{ess}}^n}(x), \quad [g]_{\text{res}}(x) = g(x)1_{\Omega_{\text{res}}^n}(x), \quad x \in \Omega, \quad g \in L^1(\Omega).$$

Corollary 3.1. *Let $(\varrho, \mathbf{u}) \in Y_{\mathcal{T}, \delta t} \times \mathbf{X}_{\mathcal{E}, \delta t}$ be a solution of (2.40) with pressure p obeying (1.4) emanating from initial data (2.63-2.64). Then we have*

1. *Induced standard energy estimates:*

$$\|\mathbf{u}\|_{L^2(0,T;\mathbf{H}_{\mathcal{E},0}(\Omega))} \leq c, \quad (3.4) \quad \{\text{est0}\}$$

$$\|\mathbf{u}\|_{L^2(0,T;L^6(\Omega;\mathbb{R}^3))} \leq c, \quad (3.5) \quad \{\text{est1}\}$$

$$\|\varrho|\mathbf{u}|^2\|_{L^\infty(0,T;L^1(\Omega))} \leq c, \quad (3.6) \quad \{\text{est2}\}$$

$$\max_{0 \leq n \leq N} \int_{\Omega} E(\varrho^n | \bar{\varrho}) dx \leq c\varepsilon^2 \quad (3.7) \quad \{\text{est3-}\}$$

$$\max_{0 \leq n \leq N} \left(\|\varrho^n\|_{L^q(\Omega_{\text{res}}^n)}^q \leq c\varepsilon^2, \quad 1 \leq q \leq \gamma, \quad \max_{0 \leq n \leq N} |\Omega_{\text{res}}^n| \leq c\varepsilon^2 \right) \quad (3.8) \quad \{\text{est3}\}$$

$$\max_{0 \leq n \leq N} \|\varrho^n - \bar{\varrho}\|_{L^q(\Omega_{\text{ess}}^n)} \leq c(\bar{\varrho})\varepsilon^2, \quad 2 \leq q < \infty.$$

2. *Estimates of numerical dissipation*

$$\begin{aligned} \delta t \sum_{n=1}^N \sum_{\sigma=K|L \in \mathcal{E}_{\text{int}}} |\sigma| \frac{(\varrho_K^n - \varrho_L^n)^2}{[\max(\varrho_K^n, \varrho_L^n)]^{(2-\gamma)^+}} 1_{\{\varrho_\sigma^n \geq 1\}} |u_{\sigma,K}^n| \\ + \delta t \sum_{\sigma=K|L \in \mathcal{E}_{\text{int}}} |\sigma| (\varrho_K^n - \varrho_L^n)^2 1_{\{\varrho_\sigma^n < 1\}} |u_{\sigma,K}^n| \leq c(M_0, E_0)\varepsilon^2, \end{aligned} \quad (3.9) \quad \{\text{dissipa}\}$$

$$\begin{aligned} \sum_{n=1}^N \sum_{K \in \mathcal{M}} |K| \frac{(\varrho_K^n - \varrho_K^{n-1})^2}{[\max(\varrho_K^{n-1}, \varrho_K^n)]^{(2-\gamma)^+}} 1_{\{\varrho_K^{n-1,n} \geq 1\}} \\ + \sum_{n=1}^N \sum_{K \in \mathcal{M}} |K| (\varrho_K^n - \varrho_K^{n-1})^2 1_{\{\varrho_K^{n-1,n} < 1\}} \leq c(M_0, E_0)\varepsilon^2. \end{aligned} \quad (3.10) \quad \{\text{dissipa}\}$$

$$\sum_{n=1}^m \sum_{i=1}^3 \sum_{\sigma \in \mathcal{E}_{\text{int}}^{(i)}} |D_\sigma| \varrho_\sigma^{n-1} |u_\sigma^n - u_\sigma^{n-1}|^2 \leq c. \quad (3.11) \quad \{\text{dis}\}$$

The quantities ϱ_σ^n and $\varrho^{n-1,n}$ are defined in Lemma 3.1.

Proof of Corollary 3.1

Lemma 3.1 in combination with the conservation of mass (3.1) and the definition (2.60) of $E(\cdot|\cdot)$, yields

$$\begin{aligned} & \frac{1}{\delta t} \frac{1}{\varepsilon^2} \int_{\Omega} E(\varrho^n|\bar{\varrho}) - E(\varrho^{n-1}|\bar{\varrho}) \, dx + \frac{1}{2\delta t} \int_{\Omega} \varrho^n |\mathbf{u}^n|^2 - \varrho^{n-1} |\mathbf{u}^{n-1}|^2 \, dx \\ & \quad + \mu \|\mathbf{u}^n\|_{1,\mathcal{E},0}^2 + (\mu + \lambda) \|\operatorname{div}_{\mathcal{T}} \mathbf{u}^n\|_{L^2(\Omega)}^2 + \frac{1}{2\delta t} \int_{\Omega} \varrho^{n-1} |\mathbf{u}^n - \mathbf{u}^{n-1}|^2 \, dx \\ & \quad + \frac{1}{2\delta t} \frac{1}{\varepsilon^2} \int_{\Omega} \mathcal{H}''(\varrho^{n-1,n})(\varrho^n - \varrho^{n-1})^2 \, dx + \frac{1}{2} \frac{1}{\varepsilon^2} \sum_{\substack{\sigma \in \mathcal{E}_{\text{int}} \\ \sigma = K|L}} |\sigma| \mathcal{H}''(\varrho_{\sigma}^n)(\varrho_K^n - \varrho_L^n)^2 |\mathbf{u}_{\sigma,K}^n| = 0. \end{aligned} \quad (3.12) \quad \{\text{dodano4}$$

This yields immediately (after multiplication by δt and summation over $n = 1, \dots, N$) estimates (3.4), (3.6), (3.7) and (3.11). We obtain (3.5) from (3.4) and the discrete Sobolev inequality reported in Lemma 2.2. We obtain (3.9) and (3.10) from the corresponding terms in Lemma 3.1 after employing (1.4)

Integrating inequality (2.61) we deduce

$$\int_{\Omega} \left([1]_{\text{res}} + [(\varrho^n)^{\gamma}]_{\text{res}} + [\varrho^n - \bar{\varrho}]_{\text{ess}}^2 \right) \, dx \leq c(\bar{\varrho}) \int_{\Omega} E(\varrho^n, \mathbf{u}^n | \bar{\varrho}, 0) \, dx; \quad (3.13) \quad \{\text{rentrop}$$

whence estimates (3.8) follow from (3.7). This completes the proof of Corollary 3.1.

4 Relative energy inequality for the discrete problem

4.1 Exact relative energy inequality for the discrete problem

In this Section, we report the exact relative energy inequality for the numerical scheme (2.40). The proof of this inequality is available in [20, Proposition 2].

Theorem 4.1. [Exact discrete relative energy inequality]

Any solution $(\varrho, \mathbf{u}) \in Y_{\mathcal{T},\delta t} \times \mathbf{X}_{\mathcal{E},\delta t}$ of the discrete problem (2.40) with pressure p obeying hypotheses (1.4)₁ satisfies relative energy inequality that reads:

$$\frac{1}{\delta t} \left(\mathcal{E}_{\varepsilon}(\varrho^n, \mathbf{u}^n | r^n, \mathbf{U}^n) - \mathcal{E}_{\varepsilon}(\varrho^{n-1}, \mathbf{u}^{n-1} | r^{n-1}, \mathbf{U}^{n-1}) \right) \quad (4.1) \quad \{\text{erelati}$$

$$+ \mu \|\mathbf{u}^n - \mathbf{U}^n\|_{1,\mathcal{E},0}^2 + (\mu + \lambda) \|\operatorname{div}_{\mathcal{T}}(\mathbf{u}^n - \mathbf{U}^n)\|_{L^2(\Omega)}^2 \leq \sum_{k=1}^5 T_k^n$$

for any couple of discrete test functions (r, \mathbf{U}) , $0 < r \in Y_{\mathcal{T},\delta t}$, $\mathbf{U} \in \mathbf{X}_{\mathcal{E},\delta t}$, where

$$T_1^n = \int_{\Omega} \varrho^{n-1} \frac{\mathbf{U}^{n-1} - \mathbf{U}^n}{\delta t} \cdot \left(\mathbf{u}^{n-1} - \frac{1}{2}(\mathbf{U}^{n-1} + \mathbf{U}^n) \right) \, dx,$$

$$T_2^n = \sum_{i=1}^3 \sum_{\sigma \in \mathcal{E}_{\text{int}}^{(i)}} \sum_{\epsilon \in \tilde{\mathcal{E}}(D_{\sigma}), \epsilon = D_{\sigma}|D_{\sigma'}} F_{\epsilon,\sigma}(\varrho^n, \mathbf{u}^n) \mathbf{U}_{\sigma}^n \cdot (\mathbf{u}_{\epsilon}^n - \mathbf{U}_{\epsilon}^n),$$

$$T_3^n = \mu [\mathbf{U}^n - \mathbf{u}^n, \mathbf{U}^n]_{1,\mathcal{E},0} + (\mu + \lambda) \int_{\Omega} \operatorname{div}_{\mathcal{T}}(\mathbf{U}^n - \mathbf{u}^n) \operatorname{div}_{\mathcal{T}} \mathbf{U}^n \, dx,$$

$$T_4^n = -\frac{1}{\varepsilon^2} \int_{\Omega} p(\varrho^n) \operatorname{div}_{\mathcal{T}} \mathbf{U}^n \, dx,$$

$$T_5^n = \frac{1}{\varepsilon^2} \int_{\Omega} (r^n - \varrho^n) \frac{\mathcal{H}'(r^n) - \mathcal{H}'(r^{n-1})}{\delta t} \, dx + \frac{1}{\varepsilon^2} \int_{\Omega} \operatorname{div}_{\mathcal{T}}^{\text{up}}(\varrho^n \mathbf{u}^n) \mathcal{H}'(r^{n-1}) \, dx.$$

In the above formulas, flux $F_{\epsilon,\sigma}$ is defined in (2.19–2.20), $\mathbf{U}_{\sigma} = (U_{i,\sigma})_{i=1,2,3}$, see last alinea in Section 2.1.2, $\operatorname{div}_{\mathcal{T}}$ is defined in (2.25), the bilinear form $[\cdot, \cdot]_{1,\mathcal{E},0}$ and corresponding norm $\|\cdot\|_{1,\mathcal{E},0}$ are given in (2.35), (2.36a–2.36b). Finally, the operation denoted by $_{\varepsilon}$ is defined in (2.18), i.e. $u_{i,\varepsilon}^n = \frac{u_{i,\sigma} + u_{i,\sigma'}}{2}$ if $\sigma, \sigma' \in \mathcal{E}_{\text{int}}^{(i)}$, $\varepsilon = D_{\sigma}|D_{\sigma'}$, $\mathbf{u}_{\varepsilon}^n = (u_{i,\varepsilon}^n)_{i=1,2,3}$, and similarly for $\mathbf{U}_{\varepsilon}^n$.

4.2 Approximate relative energy inequality for the discrete problem

The exact relative energy inequality in Theorem 4.1 is an intrinsic inequality for the given MAC scheme. In what follows, we shall write this inequality with particular discrete test functions $r = \bar{\varrho} \in Y_{\mathcal{T},\delta t}$, $\mathbf{U} = \mathcal{P}_{\mathcal{E}}(\mathbf{V}^n)$, where \mathbf{V} is divergence free function with zero traces in the regularity class $X_T(\Omega; \mathbb{R}^3)$. At the same time we shall transform some of the terms in the resulting inequality to the form convenient for comparison with an integral identity satisfied by any strong solution to problem (1.7–1.9). This identity will be derived later. The modified relative energy inequality and the latter mentioned inequality will give the wanted error estimate announced in Theorem 2.1. The lemma reads:

Lemma 4.1. [Approximate discrete relative energy]

Let $(\varrho, \mathbf{u}) \in Y_{\mathcal{T},\delta t} \times \mathbf{X}_{\mathcal{E},\delta t}$ be a solution of the discrete problem (2.40) with pressure p satisfying relations (1.4) _{$\gamma \geq 3/2$} emanating from initial data obeying (2.63–2.64). Let $\mathbf{V} \in C^1([0, T] \times \bar{\Omega}; \mathbb{R}^3)$ be such that

$$\mathbf{V}|_{\partial\Omega} = 0, \quad \operatorname{div} \mathbf{V} = 0.$$

Then there exists

$$c = c(M_0, E_0, \|\mathbf{V}, \nabla \mathbf{V}, \partial_t \mathbf{V}\|_{L^\infty(Q_T; \mathbb{R}^{15})})$$

such that for all $m = 1, \dots, N$ we have:

$$\begin{aligned} & \mathcal{E}_\varepsilon(\varrho^m, \mathbf{u}^m | \bar{\varrho}, \mathbf{V}_{\mathcal{E}}^m) - \mathcal{E}_\varepsilon(\varrho^0, \mathbf{u}^0 | \bar{\varrho}, \mathbf{V}_{\mathcal{E}}^0) \\ & + \delta t \sum_{n=1}^m \left(\mu \|\mathbf{u}^n - \mathbf{V}_{\mathcal{E}}^n\|_{1,\mathcal{E},0}^2 + (\mu + \lambda) \|\operatorname{div}_{\mathcal{T}}(\mathbf{u}^n - \mathbf{V}_{\mathcal{E}}^n)\|_{L^2(\Omega)}^2 \right) \leq \sum_{k=1}^3 S_k + \mathcal{R}_{\mathcal{T},h,\delta t}^m + \mathcal{G}_{\mathcal{T},\delta t}^m, \end{aligned} \quad (4.2) \quad \{\text{relative}\}$$

where

$$\begin{aligned} S_1 &= \delta t \sum_{n=1}^m \int_{\Omega} \varrho^{n-1} \left(\frac{\mathbf{V}_{\mathcal{E}}^n - \mathbf{V}_{\mathcal{E}}^{n-1}}{\delta t} \right) \cdot (\mathbf{V}_{\mathcal{E}}^n - \mathbf{u}^n), \\ S_2 &= \sum_{i=1}^3 \sum_{j=1}^3 \sum_{K \in \mathcal{T}} \sum_{\sigma \in \mathcal{E}^{(j)}(K) \cap \mathcal{E}_{\text{int}}^{(j)}} |\sigma| \left(V_{i,\mathcal{E},K}^n - V_{i,\sigma}^n \right) \varrho_{\sigma}^{n,\text{up}} V_{j,\sigma}^n \mathbf{e}^{(j)} \cdot \mathbf{n}_{\sigma,K} \left(u_{i,K}^n - V_{i,K}^n \right) \\ S_3 &= \delta t \sum_{n=1}^m \mu [\mathbf{V}_{\mathcal{E}}^n - \mathbf{u}^n, \mathbf{V}_{\mathcal{E}}^n]_{1,\mathcal{E},0}, \end{aligned}$$

for any divergence free vector field $\mathbf{V} \in \mathcal{X}_T(\Omega; \mathbb{R}^3)$ (see (1.10)) vanishing at the boundary of Ω . In the above inequality,

$$|\mathcal{G}_{\mathcal{T},\delta t}^m| \leq c \delta t \sum_{n=1}^m \mathcal{E}_\varepsilon(\varrho^m, \mathbf{u}^m | \bar{\varrho}, \mathbf{V}_{\mathcal{E}}^m) \quad (4.3)$$

$$|\mathcal{R}_{\mathcal{T},\delta t}^m| \leq c(\sqrt{\delta t} + h^A), \quad (4.4)$$

and where A is by formula (2.66). Here, we have used the abbreviated notation $V_{i,\mathcal{E}} = \mathcal{P}_{\mathcal{E}}^{(i)}(V_i)$, $\mathbf{V}_{\mathcal{E}} = \mathcal{P}_{\mathcal{E}} \mathbf{V} = (V_{i,\mathcal{E}})_{i=1,2,3}$, where projections $\mathcal{P}_{\mathcal{E}}^{(i)}$, $\mathcal{P}_{\mathcal{E}}$ are defined in Section 2.4. Further, $V_{i,\mathcal{E},K} = [V_{i,\mathcal{E}}]_K = [\mathcal{R}_{\mathcal{T}}^{(i)} V_{i,\mathcal{E}}]_K$, where the interpolator $\mathcal{R}_{\mathcal{T}} = (\mathcal{R}_{\mathcal{T}}^{(i)})_{i=1,2,3}$ is defined in Definition 2.3. Operator $\operatorname{div}_{\mathcal{T}}$ is defined in (2.25), the bilinear form $[\cdot, \cdot]_{1,\mathcal{E},0}$ and corresponding norm $\|\cdot\|_{1,\mathcal{E},0}$ are given in (2.35), (2.36a–2.36b).

Proof of Lemma 4.1

We shall use in the relative energy inequality (4.1) test functions $r = \bar{\varrho}$ and $\mathbf{U} = \mathbf{V}_{\mathcal{E}}$. Since $\bar{\varrho}$ is constant, term $T_5^n = 0$. According to Lemma 2.3, $\operatorname{div}_{\mathcal{T}} \mathbf{V}_{\mathcal{E}} = 0$; whence $T_4^n = 0$. Term T_3^n will be kept as it stays. It remains to transform terms T_1^n and T_2^n . This will be done in several steps.

Step 1: Term T_1^n .

We have

$$T_1^n = T_{1,1}^n + R_{1,1}^n + R_{1,2}^n, \quad (4.5) \quad \{\text{S1}\}$$

$$T_{1,1}^n = \int_{\Omega} \varrho^{n-1} \left(\frac{\mathbf{V}_{\mathcal{E}}^n - \mathbf{V}_{\mathcal{E}}^{n-1}}{\delta t} \right) \cdot (\mathbf{V}_{\mathcal{E}}^n - \mathbf{u}^n) \, dx,$$

$$R_{1,1}^n = - \int_{\Omega} \frac{1}{2} \varrho^{n-1} \frac{\mathbf{V}_{\mathcal{E}}^n - \mathbf{V}_{\mathcal{E}}^{n-1}}{\delta t} \cdot (\mathbf{V}_{\mathcal{E}}^n - \mathbf{V}_{\mathcal{E}}^{n-1}) \, dx, \quad R_{1,2}^n = \int_{\Omega} \varrho^{n-1} \frac{\mathbf{V}_{\mathcal{E}}^n - \mathbf{V}_{\mathcal{E}}^{n-1}}{\delta t} \cdot (\mathbf{u}^n - \mathbf{u}^{n-1}) \, dx.$$

By virtue of the first order integral Taylor formula applied to \mathbf{V} in the interval (t_{n-1}, t_n) , definition (2.43) of projection $\mathcal{P}_{\mathcal{E}}$, Cauchy-Schwartz inequality and numerical dissipation (3.11), we easily get

$$|\delta t \sum_{n=1}^m R_{1,1}^n| \leq \delta t c(M_0, \|\partial_t \mathbf{V}\|_{L^\infty(Q_T; \mathbb{R}^3)}), \quad |\delta t \sum_{n=1}^m R_{1,2}^n| \leq \sqrt{\delta t} c(M_0, E_0, \|\partial_t \mathbf{V}\|_{L^\infty(Q_T; \mathbb{R}^3)}). \quad (4.6) \quad \{\text{S1r}\}$$

Step 2: Term T_2^n

This step will consist of several successive transformations performed in four movements.

Step 2a:

Employing Lemma 2.1 we get

$$T_2^n = T_{2,1}^n + R_{2,1}^n \quad (4.7) \quad \{\text{S2}\}$$

$$T_{2,1}^n = \sum_{i=1}^3 \sum_{j=1}^3 \sum_{K \in \mathcal{T}} \sum_{\sigma \in \mathcal{E}^{(i)}(K) \cap \mathcal{E}_{\text{int}}^{(j)}} \left(V_{i,\mathcal{E},K}^n - V_{i,\sigma}^n \right) F_{\sigma,K}(\varrho^n, \mathbf{u}^n) (\mathcal{R}_{\mathcal{E}}^{(i,j)}(u_i^n - V_{i,\mathcal{E}}^n)) \Big|_{D_\sigma},$$

$$R_{2,1}^n = \sum_{i=1}^3 R^i(u_i^n - V_{i,\mathcal{E}}^n, V_{i,\mathcal{E}}^n), \text{ where } R^i \text{ is defined in Lemma 2.1.}$$

We have used the local conservation of primal fluxes (2.15) in order to replace in $T_{2,1}^n$ expression $V_{i,\mathcal{E},K}^n$ by the difference $V_{i,\mathcal{E},K}^n - V_{i,\sigma}^n$.

We easily see from definitions of the projections and interpolates in Section 2.4, and first order Taylor formula that

$$|V_{i,\mathcal{E},K}^n - V_{i,\sigma}^n| \leq ch \|\nabla V_i\|_{L^\infty(Q_T; \mathbb{R}^3)} \quad (4.8) \quad \{\text{F1}\}$$

Recalling definition of $F_{\sigma,K}(\varrho^n, \mathbf{u}^n)$ we obtain by Hölder's inequality

$$\left| \sum_{i=1}^3 R^i(u_i^n - V_{i,\mathcal{E}}^n, V_{i,\mathcal{E}}^n) \right| \leq c \|\varrho^n\|_{L^{\gamma_0}(\Omega)} \|\mathbf{u}^n\|_{L^6(\Omega; \mathbb{R}^3)} \|\mathcal{R}_{\mathcal{T}}(\mathbf{u}^n - \mathbf{V}_{\mathcal{E}}^n) - (\mathbf{u}^n - \mathbf{V}_{\mathcal{E}}^n)\|_{L^q(\Omega; \mathbb{R}^3)}, \quad (4.9) \quad \{\text{F1+}\}$$

where $\frac{1}{\gamma_0} + \frac{1}{q} = \frac{5}{6}, \gamma_0 = \min(\gamma, 3)$. Due to Lemmas 2.5 2.7, and Lemma 2.2

$$\|\mathcal{R}_{\mathcal{T}} \mathbf{V}^n - \mathbf{V}^n\|_{L^q(\Omega)} \leq ch \|\nabla \mathbf{V}^n\|_{L^q(\Omega)},$$

$$\|\mathcal{R}_{\mathcal{T}} \mathbf{u}^n - \mathbf{u}^n\|_{L^2(\Omega)} \leq ch \|\mathbf{u}^n\|_{1,\mathcal{E},0}, \quad \|\mathcal{R}_{\mathcal{T}} \mathbf{u}^n - \mathbf{u}^n\|_{L^6(\Omega)} \leq c \|\mathbf{u}^n\|_{1,\mathcal{E},0};$$

whence interpolation of L^q between L^2 and L^6 yields

$$\|\mathcal{R}_{\mathcal{T}} \mathbf{u}^n - \mathbf{u}^n\|_{L^q(\Omega)} \leq ch^{\frac{2\gamma_0-3}{\gamma_0}} \|\mathbf{u}^n\|_{1,\mathcal{E},0}. \quad (4.10) \quad \{\text{F2}\}$$

Consequently, coming back to formula (4.9), we arrive to the estimate

$$|\delta t \sum_{n=1}^m R_{2,1}^n| \leq h^A c(M_0, E_0, \bar{\varrho}, \|\nabla_x \mathbf{V}\|_{L^\infty(Q_T; \mathbb{R}^9)}) \quad (4.11) \quad \{\text{S2r}\}$$

after employing the known bounds (3.4–3.8) derived in Corollary 3.1. Here A is defined in (2.66).

Step 2b:

We rewrite $T_{2,1}^n$ by using the definition (2.12) of $F_{\sigma,K}$ as follows

$$T_{2,1}^n = T_{2,2}^n + R_{2,2}^n, \quad (4.12) \quad \{\text{S2+}\}$$

where

$$\begin{aligned} T_{2,2}^n &= \sum_{i=1}^3 \sum_{j=1}^3 \sum_{K \in \mathcal{T}} \sum_{\sigma \in \mathcal{E}^{(j)}(K) \cap \mathcal{E}_{\text{int}}^{(j)}} |\sigma| \left(V_{i,\mathcal{E},K}^n - V_{i,\sigma}^n \right) \varrho_{\sigma}^{n,\text{up}} u_{j,\sigma}^n \mathbf{e}^{(j)} \cdot \mathbf{n}_{\sigma,K} \left(u_{i,\sigma_{u_j^n}^{\text{up}}} - V_{i,\sigma_{u_j^n}^{\text{up}}}^n \right) \\ R_{2,2}^n &= \sum_{i=1}^3 \sum_{j=1}^3 \sum_{K \in \mathcal{T}} \sum_{\sigma \in \mathcal{E}^{(j)}(K) \cap \mathcal{E}_{\text{int}}^{(j)}} |\sigma| \left((\mathcal{R}_{\mathcal{T}}^{(i)}(V_{i,\mathcal{E}}^n))_{\underline{K}} - V_{i,\sigma}^n \right) \varrho_{\sigma}^{n,\text{up}} u_{j,\sigma}^n \mathbf{e}^{(j)} \cdot \mathbf{n}_{\sigma,K} \\ &\quad \times \left(\widehat{[u_i^n]_{\sigma}}^{(i,j)} - \widehat{[V_{i,\mathcal{E}}^n]_{\sigma}}^{(i,j)} - (u_{i,\sigma_{u_j^n}^{\text{up}}} - V_{i,\sigma_{u_j^n}^{\text{up}}}^n) \right). \end{aligned}$$

Here, the number $\widehat{[\cdot]_{\sigma}}^{i,j}$, the primal cell $\sigma_{u_j^n}^{\text{up}}$, and the related operators $\mathcal{R}_{\mathcal{E}}^{i,j}$, $\mathcal{R}_{\mathcal{E}}^{i,j,u_j^n}$ used in the next formulas are defined in items 1. and 3. of Definition 2.3. Using (4.8) and the Hölder's inequality, we get

$$|R_{2,2}^n| \leq c \|\varrho^n\|_{L^{\gamma_0}(\Omega)} \|\mathbf{u}^n\|_{L^6(\Omega)} \sum_{i=1}^3 \sum_{j=1}^3 \|\mathcal{R}_{\mathcal{E}}^{i,j}(u_i^n - V_{i,\mathcal{E}}^n) - \mathcal{R}_{\mathcal{E}}^{i,j,u_j^n}(u_i^n - V_{i,\mathcal{E}}^n)\|_{L^q(\Omega)}, \quad (4.13) \quad \{\text{F3}\}$$

$\frac{1}{\gamma_0} + \frac{1}{q} = \frac{5}{6}$, $\gamma_0 = \min(\gamma, 3)$ as in (4.9). Due to Lemmas 2.8 and 2.9

$$\begin{aligned} \|\mathcal{R}_{\mathcal{E}}^{i,j}(u_i^n - V_{i,\mathcal{E}}^n) - \mathcal{R}_{\mathcal{E}}^{i,j,u_j^n}(u_i^n - V_{i,\mathcal{E}}^n)\|_{L^2(\Omega)} &\leq ch \|\nabla_{\mathcal{E}^{(i)}}(u_i^n - V_{i,\mathcal{E}}^n)\|_{L^2(\Omega; \mathbb{R}^3)}, \\ \|\mathcal{R}_{\mathcal{E}}^{i,j}(u_i^n - V_{i,\mathcal{E}}^n) - V_{i,\mathcal{E}}^n - \mathcal{R}_{\mathcal{E}}^{i,j,u_j^n}(u_i^n - V_{i,\mathcal{E}}^n)\|_{L^6(\Omega)} &\leq c \|u_i^n - V_{i,\mathcal{E}}^n\|_{L^6(\Omega)}, \end{aligned}$$

where by the discrete Sobolev inequality evoked in Lemma 2.2

$$\|\mathbf{u}^n - \mathbf{V}_{\mathcal{E}}^n\|_{L^6(\Omega; \mathbb{R}^3)} \leq \|\mathbf{u}^n - \mathbf{V}^n\|_{1,\mathcal{E},0}.$$

Now, by interpolation of L^q between L^2 and L^6 ,

$$\|\mathcal{R}_{\mathcal{E}}^{i,j}(u_i^n - V_{i,\mathcal{E}}^n) - \mathcal{R}_{\mathcal{E}}^{i,j,u_j^n}(u_i^n - V_{i,\mathcal{E}}^n)\|_{L^q(\Omega)} \leq ch^{\frac{2\gamma_0-3}{\gamma_0}} (\|\mathbf{u}^n\|_{1,\mathcal{E},0} + \|\mathbf{V}_{\mathcal{E}}^n\|_{1,\mathcal{E},0}),$$

similarly as in (4.10). Consequently, employing formula (4.13), the above estimates and estimates (3.4), (3.8) from Corollary 3.1, we get

$$|\delta t \sum_{n=1}^m R_{2,2}^n| \leq h^A c(M_0, E_0, \bar{\varrho}, \|\nabla_x \mathbf{V}\|_{L^\infty(Q_T; \mathbb{R}^9)}). \quad (4.14) \quad \{\text{S2r+}\}$$

Step 2c:

In the next step, we write

$$T_{2,2}^n = T_{2,3}^n + R_{2,3}^n \quad (4.15) \quad \{\text{S2++}\}$$

with

$$T_{2,3}^n = \sum_{i=1}^3 \sum_{j=1}^3 \sum_{K \in \mathcal{T}} \sum_{\sigma \in \mathcal{E}^{(j)}(K) \cap \mathcal{E}_{\text{int}}^{(j)}} |\sigma| \left(V_{i,\mathcal{E},K}^n - V_{i,\sigma}^n \right) \varrho_{\sigma}^{n,\text{up}} V_{j,\sigma}^n \mathbf{e}^{(j)} \cdot \mathbf{n}_{\sigma,K} \left(u_{i,\sigma_{u_j^n}^{\text{up}}} - V_{i,\sigma_{u_j^n}^{\text{up}}}^n \right)$$

and

$$R_{2,3}^n = \sum_{i=1}^3 \sum_{j=1}^3 \sum_{K \in \mathcal{T}} \sum_{\sigma \in \mathcal{E}^{(j)}(K) \cap \mathcal{E}_{\text{int}}^{(j)}} |\sigma| \left(V_{i,\mathcal{E},K}^n - V_{i,\sigma}^n \right) \varrho_{\sigma}^{n,\text{up}} \left(u_{j,\sigma}^n - V_{j,\sigma}^n \right) \mathbf{e}^{(j)} \cdot \mathbf{n}_{\sigma,K} \left(u_{i,\sigma_{u_j^n}^{\text{up}}} - V_{i,\mathcal{E},\sigma_{u_j^n}^{\text{up}}}^n \right);$$

Noticing that

$$\int_{\Omega} \varrho^n |\mathbf{u}^n|^2 dx = \sum_{i=1}^3 \sum_{K=\overrightarrow{[\sigma,\sigma']}, \sigma \in \mathcal{E}^{(i)}} \left(|D_{\sigma,K}| \varrho_K^n |u_{i,\sigma}^n|^2 + |D_{\sigma',K}| \varrho_K^n |u_{i,\sigma'}^n|^2 \right)$$

and recalling the definition of the primal cell $[\cdot]_{\sigma}^{\text{up}}$ in Definition 2.3, formula (4.8) and definition of relative energy $\mathcal{E}_{\varepsilon}(\cdot|\cdot)$ (see (2.62)), we conclude that

$$\delta t \left| \sum_{n=1}^m R_{2,3}^n \right| \leq c(\|\nabla \mathbf{V}\|_{L^{\infty}(Q_T; \mathbb{R}^3)}) \delta t \sum_{n=1}^m \mathcal{E}_{\varepsilon}(\varrho^n, \mathbf{u}^n | \bar{\varrho}, \mathbf{V}_{\varepsilon}^n). \quad (4.16) \quad \{\text{S2r++}\}$$

Step 2d:

Finally,

$$T_{2,3}^n = T_{2,4}^n + R_{2,4}^n, \quad (4.17) \quad \{\text{S2+++}\}$$

where

$$T_{2,4}^n = \sum_{i=1}^3 \sum_{j=1}^3 \sum_{K \in \mathcal{T}} \sum_{\sigma \in \mathcal{E}^{(j)}(K) \cap \mathcal{E}_{\text{int}}^{(j)}} |\sigma| \left(V_{i,\mathcal{E},K}^n - V_{i,\sigma}^n \right) \varrho_{\sigma}^{n,\text{up}} V_{j,\sigma}^n \mathbf{e}^{(j)} \cdot \mathbf{n}_{\sigma,K} \left(u_{i,K}^n - V_{i,K}^n \right)$$

and

$$\begin{aligned} R_{2,4}^n &= \sum_{i=1}^3 \sum_{j=1}^3 \sum_{K \in \mathcal{T}} \sum_{\sigma \in \mathcal{E}^{(j)}(K) \cap \mathcal{E}_{\text{int}}^{(j)}} |\sigma| \left(V_{i,\mathcal{E},K}^n - V_{i,\sigma}^n \right) \varrho_{\sigma}^{n,\text{up}} V_{j,\sigma}^n \mathbf{e}^{(j)} \cdot \mathbf{n}_{\sigma,K} \\ &\quad \times \left(u_{i,\sigma_{u_j^n}^{\text{up}}}^n - V_{i,\sigma_{u_j^n}^{\text{up}}}^n - (u_{i,K}^n - V_{i,K}^n) \right). \end{aligned}$$

Next, by the Hölder and Minkowski inequalities and (4.8),

$$|R_{2,4}^n| \leq c(\|\mathbf{V}\|_{L^{\infty}(0,T;W^{1,\infty}(\Omega;\mathbb{R}^3))}) \|\varrho^n\|_{L^{\gamma_0}(\Omega)} \sum_{i=1}^3 \sum_{j=1}^3 \left(\|\mathcal{R}_{\mathcal{E}}^{(i,j,u_j^n)} u_i^n - \mathcal{R}_T^{(i)} u_i^n\|_{L^q(\Omega)} + \|\mathcal{R}_{\mathcal{E}}^{(i,j,u_j^n)} V_{i,\mathcal{E}}^n - \mathcal{R}_T^{(i)} V_{i,\mathcal{E}}^n\|_{L^q(\Omega)} \right)$$

where $\gamma_0 = \min(\gamma_0, 2)$, $\frac{1}{\gamma_0} + \frac{1}{q} = 1$. Now we estimate

$$\|\mathcal{R}_{\mathcal{E}}^{(i,j,u_j^n)} u_i^n - \mathcal{R}_T^{(i)} u_i^n\|_{L^2(\Omega)} \leq ch \|\nabla_{\mathcal{E}^{(i)}} u_i^n\|_{L^2(\Omega)}$$

and

$$\|\mathcal{R}_{\mathcal{E}}^{(i,j,u_j^n)} u_i^n - \mathcal{R}_T^{(i)} u_i^n\|_{L^6(\Omega)} \leq c \|u_i^n\|_{L^6(\Omega)} \leq c \|\nabla_{\mathcal{E}^{(i)}} u_i^n\|_{L^2(\Omega)}$$

by virtue of Lemmas 2.7, 2.9 and 2.2. Similar estimates are true if we replace u_i^n by $V_{i,\mathcal{E}}^n$ in the argument of $\mathcal{R}_{\mathcal{E}}^{(i,j,u_j^n)}$ and of $\mathcal{R}_T^{(i)}$. Consequently, by interpolation of L^q between L^2 and L^6 ,

$$\|\mathcal{R}_{\mathcal{E}}^{(i,j,u_j^n)} u_i^n - \mathcal{R}_T^{(i)} u_i^n\|_{L^q(\Omega)} \leq ch^{\frac{5\gamma_0-6}{2\gamma_0}} \|\nabla_{\mathcal{E}^{(i)}} u_i^n\|_{L^2(\Omega)}$$

Putting together these estimates, and employing in addition estimates for the numerical solution deduced in Corollary 3.1, we get

$$\delta t \left| \sum_{n=1}^m R_{2,4}^n \right| \leq h^{\frac{5\gamma_0-6}{2\gamma_0}} c(M_0, E_0, \|\nabla \mathbf{V}\|_{L^{\infty}(Q_T; \mathbb{R}^9)}). \quad (4.18) \quad \{\text{S2r+++}\}$$

Now we put together formulas (4.5) and (4.17) together with estimates of remainders (4.6), (4.11), (4.14), (4.16), (4.18) in order to get the required result. Lemma 4.1 is proved.

5 An identity for the strong solution. Consistency error.

The goal of this section is to prove the following lemma.

Lemma 5.1. [Consistency error]

Let $(\varrho, \mathbf{u}) \in Y_{\mathcal{T}, \delta t} \times \mathbf{X}_{\mathcal{E}, \delta t}$ be a solution of the discrete problem (2.40) with pressure p satisfying relations (1.4) $_{\gamma \geq 3/2}$ emanating from initial data obeying (2.63–2.64). Let the couple (Π, \mathbf{V}) belonging to the regularity class (1.10) $_{p=\max(2, \gamma')}$ be a strong solution to the incompressible Navier-Stokes equations (1.7–1.9).

Then there exists

$$c = c(M_0, E_0, \bar{\varrho}, \|\mathbf{V}, \nabla \mathbf{V}, \nabla^2 \mathbf{V}, \nabla \Pi\|_{L^\infty(Q_T; \mathbb{R}^{42})}, \|\partial_t \nabla \mathbf{V}\|_{L^2(0, T; L^{6/5}(\Omega; \mathbb{R}^9))}, \|\partial_t \Pi\|_{L^1(0, T; L^p(\Omega))}),$$

$p = \max(2, \gamma')$ such that for all $m = 1, \dots, N$ we have

$$\sum_{k=1}^3 \mathcal{S}_k + \mathcal{R}_{h, \delta t}^m = 0, \quad (5.1) \quad \{\text{consist}\}$$

where

$$\begin{aligned} \mathcal{S}_1 &= \delta t \sum_{n=1}^m \int_{\Omega} \bar{\varrho} \left(\frac{\mathbf{V}_{\mathcal{E}}^n - \mathbf{V}_{\mathcal{E}}^{n-1}}{\delta t} \right) \cdot (\mathbf{V}_{\mathcal{E}}^n - \mathbf{u}^n), \\ \mathcal{S}_2 &= \delta t \sum_{n=1}^m \sum_{i=1}^3 \sum_{j=1}^3 \sum_{K \in \mathcal{T}} \sum_{\sigma \in \mathcal{E}^{(j)}(K)} \left(V_{i, \mathcal{E}, K}^n - V_{i, \sigma}^n \right) \bar{\varrho} V_{j, \sigma}^n \mathbf{e}^{(j)} \cdot \mathbf{n}_{\sigma, K} (u_{i, K}^n - V_{i, \mathcal{E}, K}^n) \\ \mathcal{S}_3 &= \delta t \sum_{n=1}^m \mu [\mathbf{V}_{\mathcal{E}}^n - \mathbf{u}^n, \mathbf{V}_{\mathcal{E}}^n]_{1, \mathcal{E}, 0}, \end{aligned}$$

and

$$|\mathcal{R}_{h, \delta t}^m| \leq c(h + \delta t + \varepsilon).$$

Here we use the same notation as in Lemma 4.1.

The rest of this section is devoted to the proof of Lemma 4.1.

5.1 Getting started

Since (Π, \mathbf{V}) satisfies (1.7–1.9) on $(0, T) \times \Omega$ and belongs to the class (1.10), equation (1.7) can be rewritten in the form

$$\bar{\varrho} \partial_t \mathbf{V} + \bar{\varrho} \mathbf{V} \cdot \nabla \mathbf{V} + \nabla \Pi - \mu \Delta \mathbf{V} = 0 \quad \text{in } (0, T) \times \Omega.$$

From this fact, we deduce the identity

$$\sum_{s=1}^4 \delta t \sum_{n=1}^m \mathcal{T}_s^n = 0, \quad m = 1, \dots, N, \quad (5.2) \quad \{\text{strongC}\}$$

where

$$\begin{aligned} \mathcal{T}_1^n &= \int_{\Omega} \bar{\varrho} [\partial_t \mathbf{V}]^n \cdot (\mathbf{V}^n - \mathbf{u}^n) \, dx, \quad \mathcal{T}_2^n = \int_{\Omega} \bar{\varrho} \mathbf{V}^n \cdot \nabla \mathbf{V}^n \cdot (\mathbf{V}^n - \mathbf{u}^n) \, dx, \\ \mathcal{T}_3^n &= - \int_{\Omega} (\mu \Delta \mathbf{V}^n) \cdot (\mathbf{V}^n - \mathbf{u}^n) \, dx, \quad \mathcal{T}_4^n = - \int_{\Omega} \nabla \Pi^n \cdot \mathbf{u}^n \, dx. \end{aligned}$$

In the steps below, we deal with each of the terms \mathcal{T}_s .

5.2 Term with the time derivative

We proceed in two steps.

Step 1:

$$\mathcal{T}_1^n = \mathcal{T}_{1,1}^n + \mathcal{R}_{1,1}^n, \quad (5.3) \quad \{\text{calS1}\}$$

where

$$\begin{aligned} \mathcal{T}_{1,1}^n &= \int_{\Omega} \bar{\varrho} \frac{\mathbf{V}^n - \mathbf{V}^{n-1}}{\delta t} \cdot (\mathbf{V}_{\mathcal{E}}^n - \mathbf{u}^n) dx, \\ \mathcal{R}_{2,1}^n &= \int_{\Omega} \bar{\varrho} [\partial_t \mathbf{V}]^n \cdot (\mathbf{V}^n - \mathbf{V}_{\mathcal{E}}^n) dx + \int_{\Omega} \bar{\varrho} \left([\partial_t \mathbf{V}]^n - \frac{\mathbf{V}^n - \mathbf{V}^{n-1}}{\Delta t} \right) \cdot (\mathbf{V}_{\mathcal{E}}^n - \mathbf{u}^n) dx. \end{aligned}$$

Realizing that

$$[\partial_t \mathbf{V}]^n - \frac{\mathbf{V}^n - \mathbf{V}^{n-1}}{\Delta t} = \frac{1}{\delta t} \int_{t_{n-1}}^{t_n} \int_s^{t_n} \partial_t^2 \mathbf{V}(z, \cdot) dz ds,$$

we get by using Hölder's inequality (in particular with $\mathbf{V}_{\mathcal{E}}^n - \mathbf{u}^n$ in $L^6(\Omega; \mathbb{R}^3)$), Lemma 2.5, Lemma 2.2

$$\delta t \left| \sum_{n=1}^m \mathcal{R}_{2,1}^n \right| \leq (\delta t + h) c \left(M_0, E_0, \bar{\varrho}, \|\mathbf{V}, \partial_t \mathbf{V}, \nabla \mathbf{V}\|_{L^\infty(Q_T; \mathbb{R}^{15})}, \|\partial_t^2 \mathbf{V}\|_{L^2(0, T; L^{6/5}(\Omega; \mathbb{R}^3))} \right), \quad (5.4) \quad \{\text{calS1r}\}$$

where we have used Lemma 2.2 and the energy bound (3.4) from Corollary 3.1 for \mathbf{u}^n .

Step 2:

Finally, we write

$$\mathcal{T}_{1,1}^n = \mathcal{T}_{1,2}^n + \mathcal{R}_{1,2}^n, \quad (5.5) \quad \{\text{calS1+}\}$$

where

$$\begin{aligned} \mathcal{T}_{1,2}^n &= \int_{\Omega} \bar{\varrho} \frac{\mathbf{V}_{\mathcal{E}}^n - \mathbf{V}_{\mathcal{E}}^{n-1}}{\delta t} \cdot (\mathbf{V}_{\mathcal{E}}^n - \mathbf{u}^n) dx, \\ \mathcal{R}_{1,2}^n &= \int_{\Omega} \bar{\varrho} \left(\frac{\mathbf{V}^n - \mathbf{V}^{n-1}}{\delta t} - \frac{\mathbf{V}_{\mathcal{E}}^n - \mathbf{V}_{\mathcal{E}}^{n-1}}{\delta t} \right) \cdot (\mathbf{V}_{\mathcal{E}}^n - \mathbf{u}^n) dx. \end{aligned}$$

We have by Hölder's inequality and Lemma 2.5

$$|\mathcal{R}_{1,2}^n| \leq h c \left\| \nabla \frac{\mathbf{V}^n - \mathbf{V}^{n-1}}{\delta t} \right\|_{L^{6/5}(\Omega; \mathbb{R}^9)} \|\mathbf{V}_{\mathcal{E}}^n - \mathbf{u}^n\|_{L^6(\Omega; \mathbb{R}^3)},$$

where $\nabla \frac{\mathbf{V}^n - \mathbf{V}^{n-1}}{\delta t} = \frac{1}{\delta t} \int_{t_{n-1}}^{t_n} \partial_t \nabla \mathbf{V}(z, \cdot) dz$; whence after taking into account Corollary 3.1, we deduce

$$|\mathcal{R}_{1,2}^n| \leq h c (M_0, E_0, \bar{\varrho}, \|\mathbf{V}\|_{L^\infty(Q_T; \mathbb{R}^3)}, \|\partial_t \nabla \mathbf{V}\|_{L^2(0, T; L^{6/5}(\Omega; \mathbb{R}^9))}). \quad (5.6) \quad \{\text{calS1r+}\}$$

5.3 Convective term

Step 1:

We decompose term \mathcal{T}_2^n as follows:

$$\mathcal{T}_2^n = \mathcal{T}_{2,1}^n + \mathcal{R}_{2,1}^n, \quad (5.7) \quad \{\text{calS2}\}$$

with

$$\begin{aligned} \mathcal{T}_{2,1}^n &= \sum_{K \in \mathcal{T}} \int_K \bar{\varrho} \mathbf{V}_{\mathcal{E}, K}^n \cdot \nabla \mathbf{V}^n \cdot (\mathbf{V}_{\mathcal{E}, K}^n - \mathbf{u}_K^n) dx, \\ \mathcal{R}_{2,1}^n &= \sum_{K \in \mathcal{T}} \left(\int_K \bar{\varrho} \mathbf{V}^n \cdot \nabla \mathbf{V}^n \cdot (\mathbf{V}^n - \mathbf{V}_{\mathcal{E}, K}^n - (\mathbf{u}^n - \mathbf{u}_K^n)) dx + \int_K \bar{\varrho} (\mathbf{V}^n - \mathbf{V}_{\mathcal{E}, K}^n) \cdot \nabla \mathbf{V}^n \cdot (\mathbf{V}_{\mathcal{E}, K}^n - \mathbf{u}_K^n) dx \right). \end{aligned}$$

Consequently, by Lemmas 2.7 2.5, 2.6 and estimate (3.4) in Corollary 3.1,

$$\delta t \left| \sum_{n=1}^m \mathcal{R}_{2,1}^n \right| \leq h \, c(M_0, E_0, \bar{\varrho}, \|\mathbf{V}, \nabla \mathbf{V}\|_{L^\infty(Q_T; \mathbb{R}^{12})}). \quad (5.8) \quad \{\text{calS2r}\}$$

Step 2:

Integrating by parts in $\mathcal{T}_{2,1}^n$ while using the fact that $\sum_{\sigma \in \mathcal{E}(K)} \int_{\sigma} \mathbf{V}_{\mathcal{E},K}^n \cdot \mathbf{n}_{\sigma,K} = 0$, we get

$$\mathcal{T}_{2,1}^n = \sum_{K \in \mathcal{T}} \sum_{\sigma \in \mathcal{E}(K)} |\sigma| \bar{\varrho} \mathbf{V}_{\mathcal{E},K}^n \cdot \mathbf{n}_{\sigma,K} (\mathbf{V}_{\sigma}^n - \mathbf{V}_{\mathcal{E},K}) \cdot (\mathbf{V}_{\mathcal{E},K}^n - \mathbf{u}_K^n).$$

Now, we rewrite the last expression as follows

$$\mathcal{T}_{2,1}^n = \mathcal{T}_{2,2}^n + \mathcal{R}_{2,2}^n, \quad (5.9) \quad \{\text{calS2+}\}$$

where

$$\mathcal{T}_{2,2}^n = \sum_{K \in \mathcal{T}} \sum_{\sigma \in \mathcal{E}(K)} |\sigma| \bar{\varrho} \mathbf{V}_{\mathcal{E}}^n \cdot \mathbf{n}_{\sigma,K} (\mathbf{V}_{\sigma}^n - \mathbf{V}_{\mathcal{E},K}) \cdot (\mathbf{V}_{\mathcal{E},K}^n - \mathbf{u}_K^n)$$

and

$$\mathcal{R}_{2,2}^n = \sum_{K \in \mathcal{T}} \sum_{\sigma \in \mathcal{E}(K)} |\sigma| \bar{\varrho} (\mathbf{V}_{\mathcal{E},K}^n - \mathbf{V}_{\mathcal{E}}^n) \cdot \mathbf{n}_{\sigma,K} (\mathbf{V}_{\sigma}^n - \mathbf{V}_{\mathcal{E},K}) \cdot (\mathbf{V}_{\mathcal{E},K}^n - \mathbf{u}_K^n).$$

By Hölder's inequality, after application of Lemmas 2.5, 2.7 and 2.6, we get

$$\delta t \left| \sum_{n=1}^m \mathcal{R}_{2,2}^n \right| \leq h \, c(M_0, E_0, \bar{\varrho}, \|\mathbf{V}, \nabla \mathbf{V}\|_{L^\infty(Q_T; \mathbb{R}^{12})}). \quad (5.10) \quad \{\text{calS2r+}\}$$

Expression $\mathcal{T}_{2,2}^n$ written explicitly in coordinates is exactly term \mathcal{S}_2 in formula (5.1)

5.4 Viscous term

Step 1:

$$\mathcal{T}_3^n = \mathcal{T}_{3,1}^n + \mathcal{R}_{3,1}^n, \quad (5.11) \quad \{\text{calS3}\}$$

$$\mathcal{T}_{3,1}^n = \int_{\Omega} \mu \Delta \mathbf{V}^n \cdot (\mathbf{V}_{\mathcal{E}}^n - \mathbf{u}^n) dx,$$

$$\mathcal{R}_{3,1}^n = \int_{\Omega} \mu \Delta \mathbf{V}^n \cdot (\mathbf{V} - \mathbf{V}_{\mathcal{E}}^n) dx,$$

where by virtue of the Cauchy-Schwartz inequality and Lemma 2.5

$$\delta t \left| \sum_{n=1}^m \mathcal{R}_{3,1}^n \right| \leq h \, c(\|\mathbf{V}, \nabla^2 \mathbf{V}\|_{L^\infty(Q_T; \mathbb{R}^{12})}). \quad (5.12) \quad \{\text{calS3r1}\}$$

Step 2:

In this step we decompose $\mathcal{T}_{3,1}^n$ as follows

$$\begin{aligned} \mathcal{T}_{3,1}^n &= \sum_{i=1}^3 \sum_{\sigma \in \mathcal{E}^{(i)}} \int_{D_{\sigma}} \mu \Delta V_i^n (V_{i,\sigma}^n - u_{i,\sigma}^n) dx \\ &= \sum_{i=1}^3 \sum_{\sigma \in \mathcal{E}^{(i)}} \sum_{\varepsilon \in \tilde{\mathcal{E}}(D_{\sigma})} \int_{\varepsilon} \mu \mathbf{n}_{\varepsilon, D_{\sigma}} \cdot \nabla V_i^n \cdot (V_{i,\sigma}^n - u_{i,\sigma}^n) d\gamma \end{aligned}$$

$$\begin{aligned}
&= \sum_{i=1}^3 \sum_{j=1}^3 \sum_{\varepsilon=\vec{\sigma} | \vec{\sigma}' \in \tilde{\mathcal{E}}^{(i)}, \varepsilon \perp \mathbf{e}^{(j)}} \int_{\varepsilon} \partial_j V_i (V_{i,\sigma} - u_{i,\sigma} - (V_{i,\sigma'} - u_{i,\sigma'})) d\gamma. \\
&= \sum_{i=1}^3 \sum_{j=1}^3 \sum_{\varepsilon=\vec{\sigma} | \vec{\sigma}' \in \tilde{\mathcal{E}}^{(i)}, \varepsilon \perp \mathbf{e}^{(j)}} |\varepsilon| d_{\varepsilon} \left[\frac{1}{|\varepsilon|} \int_{\varepsilon} \partial_j V_i d\gamma \right] \bar{\partial}_j (u_{i,\varepsilon} - V_{i,\varepsilon}) \Big|_{D_{\varepsilon}},
\end{aligned}$$

where we have used integration by parts and definition (2.37) of $\bar{\partial}_j$. Here $d\gamma$ is $d - 1$ dimensional Hausdorff measure on σ . Consequently, we may write

$$\mathcal{T}_{3,1}^n = \mathcal{T}_{3,2} + \mathcal{R}_{3,2}^n, \quad (5.13) \quad \{\text{calS3+}\}$$

$$\begin{aligned}
\mathcal{T}_{3,2} &= \sum_{i=1}^3 \sum_{j=1}^3 \int_{\Omega} \bar{\partial}_j V_{i,\varepsilon} \bar{\partial}_j (u_{i,\varepsilon} - V_{i,\varepsilon}) dx, \\
\mathcal{R}_{3,2}^n &= \sum_{i=1}^3 \sum_{j=1}^3 \sum_{\varepsilon=\vec{\sigma} | \vec{\sigma}' \in \tilde{\mathcal{E}}^{(i)}, \varepsilon \perp \mathbf{e}^{(j)}} |\varepsilon| d_{\varepsilon} \left(\left[\frac{1}{|\varepsilon|} \int_{\varepsilon} \partial_j V_i d\gamma \right] - \bar{\partial}_j V_{i,\varepsilon} \Big|_{D_{\varepsilon}} \right) \bar{\partial}_j (u_{i,\varepsilon} - V_{i,\varepsilon}) \Big|_{D_{\varepsilon}},
\end{aligned}$$

where, due to the Cauchy-Schwartz inequality, Lemma 2.6 combined with the first order Taylor formula applied to $\left[\frac{1}{|\varepsilon|} \int_{\varepsilon} \partial_j V_i \right] - \bar{\partial}_j V_{i,\varepsilon} \Big|_{D_{\varepsilon}}$ and Corollary 3.1, we get

$$\delta t \left| \sum_{n=1}^m \mathcal{R}_{3,2}^n \right| \leq h \, c(M_0, E_0 \| \mathbf{V}, \nabla \mathbf{V}, \nabla^2 \mathbf{V} \|_{L^\infty(Q_T; \mathbb{R}^{39})}). \quad (5.14) \quad \{\text{calS3r2}\}$$

5.5 Pressure term

Step 1: The following lemma about the consistency of the upwind discretization will be crucial.

Lemma 5.2. *For any $r, G \in \mathcal{L}_{\mathcal{T}}$, any $\mathbf{u} \in \mathbf{H}_{\mathcal{E},0}$ and any $\phi \in C^1(\bar{\Omega})$ there holds*

$$\begin{aligned}
&\int_{\Omega} r \mathbf{u} \cdot \nabla \phi dx + \sum_{K \in \mathcal{T}} \sum_{\sigma \in \mathcal{E}^{(i)}} F_{\sigma,K}(r, \mathbf{u}) G_K \\
&= \sum_{i=1}^3 \sum_{\sigma=\vec{K} | \vec{L} \in \mathcal{E}^{(i)}(K)} |\sigma| (r_K - r_L) (\phi_{\sigma} - G_K) u_{i,\sigma} - \sum_{i=1}^3 \sum_{\sigma=\vec{K} | \vec{L} \in \mathcal{E}^{(i)}(K)} |\sigma| (r_L - r_{\sigma}^{\text{up}}) (G_K - G_L) u_{i,\sigma},
\end{aligned}$$

where the primal fluxes $F_{\sigma,K}$ are defined in (2.12).

Proof of Lemma 5.2

Using integration by parts,

$$\begin{aligned}
\int_{\Omega} r \mathbf{u} \cdot \nabla \phi dx &= \sum_{K \in \mathcal{T}} \int_K r \mathbf{u} \cdot \nabla (\phi - G_K) dx \\
&= \sum_{i=1}^3 \sum_{K \in \mathcal{T}} \sum_{\sigma \in \mathcal{E}^{(i)}(K)} |\sigma| r_K \phi_{\sigma} u_{i,\sigma} \mathbf{n}^{(i)} \cdot \mathbf{n}_{\sigma,K} - \sum_{i=1}^3 \sum_{K \in \mathcal{T}} \sum_{\sigma \in \mathcal{E}^{(i)}(K)} |\sigma| r_K G_K u_{i,\sigma} \mathbf{n}^{(i)} \cdot \mathbf{n}_{\sigma,K} \\
&= \sum_{i=1}^3 \sum_{\sigma=\vec{K} | \vec{L} \in \mathcal{E}^{(i)}} |\sigma| (r_K - r_L) (\phi_{\sigma} - G_K) u_{i,\sigma} \\
&\quad - \sum_{i=1}^3 \sum_{\sigma=\vec{K} | \vec{L} \in \mathcal{E}^{(i)}} |\sigma| r_L (G_K - G_L) u_{i,\sigma}
\end{aligned}$$

$$\begin{aligned}
&= \sum_{i=1}^3 \sum_{\sigma=\overrightarrow{K}|\overrightarrow{L} \in \mathcal{E}^{(i)}} |\sigma|(r_K - r_L)(\phi_\sigma - G_K)u_{i,\sigma} \\
&- \sum_{i=1}^3 \sum_{\sigma=\overrightarrow{K}|\overrightarrow{L} \in \mathcal{E}^{(i)}} |\sigma|(r_L - r_\sigma^{\text{up}})(G_K - G_L)u_{i,\sigma} - \sum_{i=1}^3 \sum_{\sigma=\overrightarrow{K}|\overrightarrow{L} \in \mathcal{E}^{(i)}} |\sigma|r_\sigma^{\text{up}}(G_K - G_L)u_{i,\sigma},
\end{aligned}$$

where for the latter term, we have

$$\sum_{i=1}^3 \sum_{\sigma=\overrightarrow{K}|\overrightarrow{L} \in \mathcal{E}^{(i)}} |\sigma|r_\sigma^{\text{up}}(G_K - G_L)u_{i,\sigma} = \sum_{i=1}^3 \sum_{K \in \mathcal{T}} \sum_{\sigma \in \mathcal{E}^{(i)}(K)} |\sigma|r_\sigma^{\text{up}}G_K u_{i,\sigma} \mathbf{n}^{(i)} \cdot \mathbf{n}_{\sigma,K}$$

Lemma 5.2 is proved.

Step 2:

We shall now evaluate the error in the upwind discretization. We have

$$\mathcal{T}_4^n = -\frac{1}{\bar{\varrho}} \int_{\Omega} \varrho^n \mathbf{u}^n \cdot \nabla \Pi^n dx + \frac{1}{\bar{\varrho}} \int_{\Omega} (\varrho^n - \bar{\varrho}) \mathbf{u}^n \cdot \nabla \Pi^n dx = \mathcal{T}_{4,1}^n + \mathcal{R}_{4,1}^n, \quad (5.15) \quad \{\text{calS41}\}$$

where

$$\delta t \left| \sum_{n=1}^N \mathcal{R}_{4,1}^n \right| \leq \varepsilon c(M_0, E_0, \bar{\varrho}, \|\nabla \Pi\|_{L^\infty((0,T) \times \Omega)}), \quad (5.16) \quad \{\text{calS4r}\}$$

by virtue of Hölder's inequality and estimates (3.5), (3.8) from Corollary 3.1.

Next we deduce from the discrete continuity equation (2.40a) and Lemma 5.2

$$\mathcal{T}_{4,1}^n = \mathcal{J}_1^n + \mathcal{J}_2^n + \mathcal{J}_3^n, \quad (5.17) \quad \{\text{calS4+}\}$$

where

$$\begin{aligned}
\mathcal{J}_1^n &= \frac{1}{\bar{\varrho}} \int_{\Omega} \frac{\varrho^n - \varrho^{n-1}}{\delta t} \Pi_{\mathcal{T}}^n dx, \\
\mathcal{J}_2^n &= \frac{1}{\bar{\varrho}} \sum_{i=1}^3 \sum_{\sigma=\overrightarrow{K}|\overrightarrow{L} \in \mathcal{E}^{(i)}(K)} |\sigma|(\varrho_L^n - \varrho_K^n)(\Pi_\sigma^n - \Pi_K^n)u_{i,\sigma}^n, \\
\mathcal{J}_3^n &= \frac{1}{\bar{\varrho}} \sum_{i=1}^3 \sum_{\sigma=\overrightarrow{K}|\overrightarrow{L} \in \mathcal{E}^{(i)}(K)} |\sigma|(\varrho_L^n - \varrho_\sigma^{n,\text{up}})(\Pi_K^n - \Pi_L^n)u_{i,\sigma}^n,
\end{aligned}$$

where $\Pi_{\mathcal{T}}^n = \mathcal{P}_{\mathcal{T}} \Pi^n$ is defined in (2.41).

Now we estimate each of terms \mathcal{J}_1^n , \mathcal{J}_2^n , \mathcal{J}_3^n separately.

Step 2a:

We get by direct calculation

$$\begin{aligned}
\delta t \sum_{n=1}^N \mathcal{J}_1^n &= \frac{\delta t}{\bar{\varrho}} \sum_{n=1}^N \int_{\Omega} \frac{\varrho^n - \varrho^{n-1}}{\delta t} \Pi_{\mathcal{T}}^n dx = \frac{1}{\bar{\varrho}} \sum_{n=1}^N \int_{\Omega} ((\varrho^n - \bar{\varrho}) - (\varrho^{n-1} - \bar{\varrho})) \Pi_{\mathcal{T}}^n dx \\
&= \frac{1}{\bar{\varrho}} \sum_{n=1}^N \int_{\Omega} ((\varrho^n - \bar{\varrho}) \Pi_{\mathcal{T}}^n - (\varrho^{n-1} - \bar{\varrho}) \Pi_{\mathcal{T}}^{n-1}) dx + \frac{1}{\bar{\varrho}} \sum_{n=1}^N \int_{\Omega} (\varrho^{n-1} - \bar{\varrho}) (\Pi_{\mathcal{T}}^{n-1} - \Pi_{\mathcal{T}}^n) dx \\
&= \frac{1}{\bar{\varrho}} \int_{\Omega} (\varrho^n - \bar{\varrho}) \Pi_{\mathcal{T}}^N - \int_{\Omega} (\varrho^0 - \bar{\varrho}) \Pi_{\mathcal{T}}^0 + \frac{\delta t}{\bar{\varrho}} \sum_{n=1}^N \int_{\Omega} (\varrho^{n-1} - \bar{\varrho}) \frac{\Pi_{\mathcal{T}}^{n-1} - \Pi_{\mathcal{T}}^n}{\delta t} dx.
\end{aligned}$$

Therefore, by virtue of Hölders inequality, Lemma 2.5, the first order Taylor formula

$$\delta t \left| \sum_{n=1}^N \mathcal{J}_1^n \right| \leq \varepsilon(1 + \delta t) c(M_0, E_0, \bar{\varrho}, \|\Pi\|_{L^\infty(Q_T)}, \|\partial_t \Pi\|_{L^1(0,T;L^p(\Omega))}), \quad p = \max(2, \gamma'),$$

where we have used estimates (3.5) and (3.8) in Corollary 3.1.

Step 2b:

First, we have by using Hölder's inequality,

$$\begin{aligned} & \left| \sum_{i=1}^3 \sum_{\sigma=\overrightarrow{K} | \vec{L} \in \mathcal{E}^{(i)}(K)} |\sigma| (\varrho_L^n - \varrho_K^n) (\Pi_\sigma^n - \Pi_K^n) u_{i,\sigma}^n \right| \\ & \leq \sqrt{h} \|\nabla_x \Pi\|_{L^\infty((0,T) \times \Omega)} \left(\sum_{i=1}^3 \sum_{\sigma=\overrightarrow{K} | \vec{L} \in \mathcal{E}^{(i)}(K)} |\sigma| 1_{E_i}(\sigma) \frac{[\varrho_K^n - \varrho_L^n]^2}{\max(\varrho_K^n, \varrho_L^n)^\delta} |u_i^n| \right)^{1/2} \\ & \times \left(\sum_{i=1}^3 \sum_{\sigma=\overrightarrow{K} | \vec{L} \in \mathcal{E}^{(i)}(K)} |\sigma| h \max(\varrho_K^n, \varrho_L^n)^\gamma \right)^{\frac{\delta}{2\gamma}} \left(\sum_{i=1}^3 \sum_{\sigma=\overrightarrow{K} | \vec{L} \in \mathcal{E}^{(i)}(K)} |\sigma| h |u_i^n|^{\frac{\gamma}{\gamma-\delta}} \right)^{\frac{\gamma-\delta}{2\gamma}} \\ & + \sqrt{h} \|\nabla_x \Pi\|_{L^\infty((0,T) \times \Omega)} \left(\sum_{i=1}^3 \sum_{\sigma=\overrightarrow{K} | \vec{L} \in \mathcal{E}^{(i)}(K)} |\sigma| 1_{\mathcal{E}^{(i)} \setminus E_i}(\sigma) [\varrho_K^n - \varrho_L^n]^2 |u_i^n| \right)^{1/2} \\ & \times \left(\sum_{i=1}^3 \sum_{\sigma=\overrightarrow{K} | \vec{L} \in \mathcal{E}^{(i)}(K)} |\sigma| h |u_i^n| \right)^{1/2} \end{aligned}$$

with any $0 \leq \delta < \gamma$ and any $E_i \subset \mathcal{E}^{(i)}$, where we have used estimate Lemma 2.5 to evaluate the difference $\Pi_\sigma^n - \Pi_K^n$. Now employing estimates (3.5), (3.8), (3.9) in Corollary 3.1 we obtain

$$\delta t \sum_{n=1}^N |\mathcal{J}_2^n| \lesssim \varepsilon h^{1/2} c(M_0, E_0, \bar{\varrho}, \|\nabla \Pi\|_{L^\infty((0,T) \times \Omega)}),$$

where $\varepsilon \sqrt{h} \leq \frac{1}{2}(\varepsilon^2 + h)$. The same estimate as above holds also for \mathcal{J}_3^n by the same argument.

Resuming calculations in step 2, we get

$$\delta t \sum_{n=1}^N |\mathcal{T}_{4,1}^n| \leq (\varepsilon + h + \delta t) c(M_0, E_0, \bar{\varrho}, \|\nabla \Pi\|_{L^\infty((0,T) \times \Omega)}, \|\partial_t \Pi\|_{L^1(0,T;L^p(\Omega))}), \quad p = \max(2, \gamma'). \quad (5.18) \quad \{\text{calS4}++\}$$

The statement of Lemma 4.1 follows when we put together principal terms (5.3), (5.5), (5.7), (5.9), (5.11), (5.13) and residual terms (5.4), (5.6), (5.8), (5.10), (5.12), (5.14), (5.16), (5.16), (5.17), (5.17).

6 A Gronwall inequality

In this Section we put together the relative energy inequality (4.2) and the identity (5.1) derived in the previous section. The final inequality resulting from this manipulation is formulated in the following lemma.

Lemma 6.1. *Let $(\varrho^n, \mathbf{u}^n)$ be a solution of the discrete problem (2.40a–2.40b) with the pressure satisfying (1.4), where $\gamma \geq 3/2$, emanating from initial data (2.63), (2.64). Then there exists a positive number*

$$c = c(M_0, E_0, \bar{\varrho}, \|\mathbf{V}\|_{X_T(\Omega)}, \|\Pi\|_{Y_T^p(\Omega)}), \quad p = \max(2, \gamma')$$

such that for all $m = 1, \dots, N$, there holds:

$$\begin{aligned} & \mathcal{E}_\varepsilon(\varrho^m, \mathbf{u}^m | \bar{\varrho}, \mathbf{V}_\mathcal{E}^m) + \delta t \sum_{n=1}^m \left(\mu \|\mathbf{u}^n - \mathbf{V}_\mathcal{E}^n\|_{1,\mathcal{E},0}^2 + (\mu + \lambda) \|\operatorname{div}_\mathcal{T}(\mathbf{u}^n - \mathbf{V}_\mathcal{E}^n)\|_{L^2(\Omega)}^2 \right) \\ & \leq c \left[h^A + \sqrt{\delta t} + \varepsilon + \mathcal{E}_\varepsilon(\varrho^0, \mathbf{u}^0 | \bar{\varrho}, \mathbf{V}_\mathcal{E}(0)) \right] + c \delta t \sum_{n=1}^m \mathcal{E}_\varepsilon(\varrho^n, \mathbf{u}^n | \bar{\varrho}, \mathbf{V}_\mathcal{E}^n), \end{aligned}$$

with any couple (Π, \mathbf{V}) belonging to (1.10) satisfying (1.7–1.9) on $[0, T) \times \Omega$, where A is defined in (2.66) and \mathcal{E}_ε is given in (2.62).

Proof of Lemma 6.1

Gathering the formulae (4.2) and (5.1), one gets

$$\mathcal{E}_\varepsilon(\varrho^m, \mathbf{u}^m | \bar{\varrho}, \mathbf{V}_\mathcal{E}^m) - \mathcal{E}_\varepsilon(\varrho^0, \mathbf{u}^0 | \bar{\varrho}, \mathbf{V}_\mathcal{E}(0)) \tag{6.1} \quad \{\text{relative}\}$$

$$+ \delta t \sum_{n=1}^m \left(\mu \|\mathbf{u}^n - \mathbf{V}_\mathcal{E}^n\|_{1,\mathcal{E},0}^2 + (\mu + \lambda) \|\operatorname{div}_\mathcal{T}(\mathbf{u}^n - \mathbf{V}_\mathcal{E}^n)\|_{L^2(\Omega)}^2 \right) \leq \mathcal{P}_1 + \mathcal{P}_2 + \mathcal{Q},$$

where

$$\begin{aligned} \mathcal{P}_1 &= \delta t \sum_{n=1}^m \int_\Omega \left(\varrho^{n-1} - \bar{\varrho} \right) \left(\frac{\mathbf{V}_\mathcal{E}^n - \mathbf{V}_\mathcal{E}^{n-1}}{\delta t} \right) \cdot (\mathbf{V}_\mathcal{E}^n - \mathbf{u}^n), \\ \mathcal{P}_2 &= \delta t \sum_{n=1}^m \sum_{i=1}^3 \sum_{j=1}^3 \sum_{K \in \mathcal{T}} \sum_{\sigma \in \mathcal{E}^{(j)}(K)} \left(\varrho_\sigma^{n,\text{up}} - \bar{\varrho} \right) \left(V_{i,\mathcal{E},K}^n - V_{i,\sigma}^n \right) V_{j,\sigma}^n \mathbf{e}^{(j)} \cdot \mathbf{n}_{\sigma,K} (u_{i,K}^n - V_{i,\mathcal{E},K}^n), \\ \mathcal{Q} &= \mathcal{R}_{\mathcal{T},h,\delta t}^m + \mathcal{G}_{\mathcal{T},\delta t}^m - \mathcal{R}_{h,\delta t}^m. \end{aligned}$$

We use Hölder's inequality, together with the Taylor type formula (4.8) in order to get

$$\begin{aligned} |\mathcal{P}_1| &\leq \delta t \sum_{n=1}^m \left(\|[\varrho^{n-1}]_{\text{res}}\|_{L^q(\Omega)} |\Omega_{\text{res}}|^{1/r} + \|[\varrho^{n-1} - \bar{\varrho}]_{\text{ess}}\|_{L^2(\Omega)} \right) \left\| \frac{\mathbf{V}_\mathcal{E}^n - \mathbf{V}_\mathcal{E}^{n-1}}{\delta t} \right\|_{L^\infty(\Omega; \mathbb{R}^3)} \|\mathbf{V}_\mathcal{E}^n - \mathbf{u}^n\|_{L^6(\Omega; \mathbb{R}^3)}, \\ |\mathcal{P}_2| &\leq c \|\nabla V\|_{L^\infty(Q_T; \mathbb{R}^3)} \delta t \sum_{n=1}^m \left(\|[\varrho^n]_{\text{res}}\|_{L^q(\Omega)} |\Omega_{\text{res}}^n|^{1/r} + \|[\varrho^n - \bar{\varrho}]_{\text{ess}}\|_{L^2(\Omega)} \right) \|\mathbf{V}_\mathcal{E}^n\|_{L^\infty(\Omega; \mathbb{R}^3)} \|\mathbf{V}_\mathcal{E}^n - \mathbf{u}^n\|_{L^6(\Omega; \mathbb{R}^3)}, \end{aligned}$$

where $q = \min(\gamma, 2)$, $\frac{1}{r} + \frac{1}{q} + \frac{1}{6} = 1$, and symbols $[\cdot]_{\text{res}}$, $[\cdot]_{\text{ess}}$ and the sets Ω_{res}^n are defined in (3.3). Evoking estimates (3.5) and (3.8) from Corollary 3.1, one gets

$$|\mathcal{P}_1| + |\mathcal{P}_2| \leq \varepsilon c(M_0, E_0, \bar{\varrho}, \|\mathbf{V}, \nabla \mathbf{V}, \partial_t \mathbf{V}\|_{L^\infty(Q_T; \mathbb{R}^{15})}).$$

This formula, and the bounds of expressions $\mathcal{R}_{\mathcal{T},h,\delta t}^m$, $\mathcal{G}_{\mathcal{T},\delta t}^m$, $\mathcal{R}_{h,\delta t}^m$ evoked in (4.2), (5.1) yield the statement of Lemma 6.1.

Lemma 6.1 implies immediately error estimate (2.65) by the standard discrete Gronwall inequality. Theorem 2.1 is proved.

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