

# On the strength of time-indexed formulations for the resource-constrained project scheduling problem

Christian Artigues

LAAS-CNRS, Université de Toulouse, CNRS, Toulouse, France  
artigues@laas.fr

## Abstract

For non-preemptive scheduling, time-indexed zero-one linear programming formulations have been deeply analyzed. This note clarifies the current knowledge about the strength of these formulations and shows that some formulations that have been proposed “new” in the literature are in fact weaker or equivalent to those already known. Much of the arguments used follow from a PhD thesis by Sousa, which has been largely overlooked in the literature.

**Keywords** resource-constrained project scheduling; the linear programming; time-indexed variables; on/off variables; relaxation strength; non-singular transformations

## 1 Introduction

For non-preemptive scheduling problems, time-indexed zero-one linear programming formulations, based on the discretization of the scheduling horizon into unit-time intervals, have been deeply analyzed and widely used for various contexts (single machine, parallel machine and resource- and precedence-constrained problems) [17, 18, 5, 8, 21, 22, 19, 20, 6].

In particular, Sousa’s PhD thesis [21] provides a polyhedral analysis of three time-indexed formulations in a single machine setting and shows their equivalence. The first formulation is based on the pulse variable, equal to one solely at the start time period of a job. The second formulation is based on the on/off variable, equal to one if the job is in process during the time period. The third formulation is based on the step variable, equal to one if the job has been started at the time period of before.

For the resource-constrained project scheduling problem (RCPSP),  $V = \{1, \dots, n\}$  denotes a set of activities. Each activity  $i \in V$  has a duration  $p_i \geq 1$ . There is a set  $E$  of precedence constraints and a set  $\mathcal{R}$  of resources. Each resource  $k$  has a constant availability  $B_k$ . Each activity  $i$  has an earliest start time  $ES_i \geq 0$ , a latest start time  $LS_i \geq ES_i$  and requires a number  $b_{ik} \geq 0$  of units on each resource  $k \in \mathcal{R}$  during all its processing time. For convenience, we will also use notation  $LC_i = LS_i + p_i$  for the latest completion time of activity

$i$  and  $EC_i = ES_i + p_i$  for the earliest completion time. The RCPSP consists in assigning a start time  $S_i \geq 0$  to each activity  $i \in V$  such that a total cost  $f(S) = \sum_{i \in V} w_i S_i$  is minimized while time-window, precedence and resource constraints are satisfied. We assume in addition that all the data are integer. In this case it is dominant to start an activity at any integer value in  $\{ES_i, \dots, LS_i\}$ . Note that if the precedence constraints are such that an activity is successor of all other ones, then assign a unit weight to this activity and a zero weight to all other activities amounts to minimize the makespan, i.e. the total project duration, which is actually the most encountered RCPSP objective.

There are two common ways of modeling the precedence constraints with time-indexed variables, a weak one and a strong one (in terms of LP relaxation strength) that can in turn be applied to each of the three above-defined variable types, yielding six formulations. Due to the results by Sousa [21], the three weak formulations are equivalent and the three strong formulations are equivalent. To summarize there is a family of weak formulations and a family of strong formulations. Inside a family there is a potentially infinite number of formulations (among which the pulse, step, and on/off ones but also many possible others) that can be obtained from each other by non-singular linear transformations.

However a significant number of claimed “new” formulation have emerged since the 90’s and are still regularly appearing, most often without comparing them to the two standard (weak and strong) formulations.

In Section 2 we recall the previously established results on time-indexed formulations and we give the six weak and strong formulations based on pulse, step and on/off variables. In Section 3 we review a number of formulations presented in more or less recent literature and we show that they are all weaker than or equivalent to previously proposed ones. Concluding remarks are drawn in section 4.

## 2 Known results on time-indexed formulations

As a preamble we formally recall the equivalence relation between two integer programming formulations  $F_1$  and  $F_2$  in terms of linear programming relaxation strength, where  $F_1 \equiv^{LP} F_2$  means that  $F_1$  is equivalent to  $F_2$ ,  $F_1 \preceq^{LP} F_2$  means that  $F_1$  is not stronger than  $F_2$  and  $F_1 \prec^{LP} F_2$  means that  $F_1$  is strictly weaker than  $F_2$ .

**Definition 1.**  $F_1 \equiv^{LP} F_2$  if there exists an affine non-singular (i.e. bijective) transformation that allows to obtain one formulation from the other.

Hence if  $F_1 \equiv^{LP} F_2$ , then  $F_1$  and  $F_2$  have the same relaxation strength and consequently give the same lower bound of the integer program for a minimization objective. Definitions 2 and 3 below were also recently recalled in similar terms in [4].

**Definition 2.**  $F_1 \preceq^{LP} F_2$  if there exists an affine transformation of  $F_2$  on the solution space  $P_1$  of  $F_1$  that gives formulation  $F'_2$  and solution space  $P'_2$  such that  $P'_2 \subseteq P_1$ .

Hence, if  $F_1 \preceq^{LP} F_2$ , then the lower bound given by  $F_1$  is not better than the lower bound given by  $F_2$ .

**Definition 3.**  $F_1 \prec^{LP} F_2$  if  $F_1 \preceq^{LP} F_2$  and if, in addition, we can find a point  $x \in P_1$  such that  $x \notin P'_2$ , which yields  $P'_2 \subset P_1$ .

Note that if  $F_1 \prec^{LP} F_2$ , then there exist an objective coefficient vector such that the lower bound obtained by  $F_1$  is strictly lower than the lower bound obtained by  $F_2$ . Finally, note that  $F_1 \equiv^{LP} F_2$  if and only if  $F_1 \preceq^{LP} F_2$  and  $F_2 \preceq^{LP} F_1$ .

We first define the scheduling horizon  $H$  as the set of integers from 0 to  $T = \max_{i \in V} LC_i$ . We also assume that any variable indexed by a negative time stands for 0.

## 2.1 Pulse formulations

We start by giving a member of the strong time-indexed family of formulations, involving a pseudo-polynomial number of variables and constraints and based on pulse variables. The pulse binary variables  $x_{it}$ ,  $i \in V$ ,  $t \in H$  is such that  $x_{it} = 1$  iff activity  $i$  starts at period  $t$ . An activity that starts at  $t$  must be interpreted as the fact that the activity is in process during interval  $[t, t + 1]$  while it was not in process at time interval  $[t - 1, t]$  if  $t > 0$ . The strong formulation based on pulse variables is based on a disaggregated way of modeling the precedence constraints. Hence we call this formulation the pulse disaggregated discrete time formulation (PDDT). It was proposed by Christofides *et al.* [5] and is given below.

$$\min \sum_{i \in V} \sum_{t \in H} w_i t x_{it} \quad (1)$$

$$\sum_{\tau=0}^{t-p_i} x_{i\tau} - \sum_{\tau=0}^t x_{j\tau} \geq 0 \quad (i, j) \in E, t \in H \quad (2)$$

$$\sum_{i \in V} \sum_{\tau=t-p_i+1}^t b_{ik} x_{i\tau} \leq B_k \quad t \in H, k \in \mathcal{R} \quad (3)$$

$$\sum_{t \in H} x_{it} = 1 \quad i \in V \quad (4)$$

$$x_{it} = 0 \quad i \in V, t \in H \setminus [ES_i, LS_i] \quad (5)$$

$$x_{it} \in \{0, 1\} \quad i \in V, t \in H \quad (6)$$

Objective (1) minimizes the total cost, observing that, according to the above-definition of the pulse variables,  $S_i = \sum_{t \in H} t x_{it}$ . Constraints (2) are the precedence constraints. Constraints (3) are the resource constraints. Constraints (4) state that each activity has to be started exactly once in the scheduling horizon. Constraints (5) set to 0 all variables outside  $[ES_i, LS_i]$ . (6) defines the pulse decision variables.

A formulation having a weaker LP relaxation was proposed by Pritsker *et al* [18], called the aggregated discrete time formulation based on pulse start variables (PDT). It consists in replacing constraints (2) by the so-called aggregated precedence constraints (7) in formulation (1–6).

$$\sum_{t \in H} t x_{jt} - \sum_{t \in H} t x_{it} \geq p_i \quad i \in V, j \in V \setminus \{i\} \quad (7)$$

They are a direct translation of the precedence constraints  $S_j \geq S_i + p_i$ . The formulation is weaker than (PDDT) since constraints (7) are implied by constraints (4) and (2) for  $0 \leq x_{it} \leq 1$ ,  $i \in V, t \in H$ .

## 2.2 Step formulations

We consider another formulation, based now on binary step variables  $z_{it}$  such that  $z_{it} = 1$  iff activity  $i$  starts at time  $t$  or before. For a given activity, variable  $z_{it}$  with  $t < S_i$  are all equal to 0 while the variables with  $t \geq S_i$  are all equal to one. With these definitions, the start time can be expressed as:

$$S_i = \sum_{t \in H} t(z_{it} - z_{i,t-1}) \quad (8)$$

We present only the disaggregated variant of the discrete time formulation based on step variables (SDDT), which can be written:

$$\min \sum_{i \in V} \sum_{t \in H} w_i t (z_{i,t} - z_{i,t-1}) \quad (9)$$

$$z_{i,t-p_i} - z_{jt} \geq 0 \quad (i, j) \in E, t \in H \quad (10)$$

$$\sum_{i \in V} b_{ik} (z_{it} - z_{i,t-p_i}) \leq B_k \quad t \in H, k \in \mathcal{R} \quad (11)$$

$$z_{i,LS_i} = 1 \quad i \in V \quad (12)$$

$$z_{i,t} - z_{i,t-1} \geq 0 \quad i \in V, t \in H \quad (13)$$

$$z_{it} = 0 \quad i \in V, t \in H, t < ES_i \quad (14)$$

$$z_{it} \in \{0, 1\} \quad i \in V, t \in H \quad (15)$$

Objective (9) is directly obtained by replacing the start time variable by its expression in function of  $z$  (8). Disaggregated precedence constraints (10) state that if an activity  $j$  is started at time  $t$  or before (i.e.  $z_{jt} = 1$ ), then activity  $i$  has also to be started at time  $t - p_i$  or before. Resource constraints (11) follow from the fact that an activity is in process at time  $t$  iff  $z_{it} - z_{i,t-p_i} = 1$ . Otherwise,  $i$  is in process at time  $t$  iff it has been started at time  $t$  but not at time  $t - p_i$ . Constraints (12) state that each activity has to be started at or before its latest start time  $LS_i$ . Constraints (13) define the step function, together with constraints (12). Note that these constraints also set to 1 all variables  $x_{it}$  with  $t \geq LS_i$ . Constraints (14) set to 0 all variables  $x_{it}$  with  $t < LS_i$ . Finally constraints (15) defines the binary step variables.

Although it is presented as new in [11], the weak step formulation was already presented by Pritsker and Watters [17]. The strong step formulation has been theoretically studied and compared to the pulse formulation by de Souza and Wolsey [6] and Sankaran *et al* [20]. This has been also underlined in [15]. If we omit resource constraints in the (SDDT) formulation and if we relax integrality constraints, i.e. considering only constraints (10, 12–14), and  $0 \leq z_{it} \leq 1$ ,  $i \in V, t \in H$ , de Souza and Wolsey [6] and Sankaran *et al* [20] observed that the constraint matrix satisfies a sufficient unimodularity condition. It follows from this observation that, without resource-constraints the solution of the LP relaxation of (SDDT) is 0-1. So is the solution of the LP relaxation of (PDDT),

according to the non-singular transformation proposed by Sousa [21]. For all  $t \in H$ , we have  $x_{it} = z_{it} - z_{it-1}$ . Conversely, the inverse transformation defines  $z_{it} = \sum_{\tau=0}^t x_{i\tau}$  and gives the (PDDT) formulation from the (SDDT) formulation. Note that, in both cases, this transformation does not change the value of the LP-relaxation. An aggregated discrete-time formulation based on step variables (SDT) could also be defined this way. (PDT) and (SDT) formulations are also equivalent, for the same reason, but yield weaker relaxations as fractional solutions can be obtained by solving the LP relaxations without resource constraints [15, 20].

### 2.3 On/off formulations

We now consider on/off binary variables  $y_{it}$  where  $y_{it} = 1$  iff activity  $i$  is in process at time  $t$ . Such variables were presumably considered for the first time for preemptive jobs by Lawler in [13] and a non-preemptive formulation as well as a polyhedral study was proposed by Sousa for the single machine scheduling problem [21]. The model is based on the following non-singular transformations between binary variables  $x_{it}$ ,  $z_{it}$  and  $y_{it}$ , also proposed by Sousa [21].

An activity  $i \in V$  is in process at time  $t \in H$  iff  $z_{it} - z_{i,t-p_i} = 1$  and, equivalently, if  $\sum_{\tau=t-p_i+1}^t x_{i\tau} = 1$ . So, for any activity  $i \in V$  and for any time  $t \in H$ , we obtain the non-singular transformations  $y_{it} = z_{it} - z_{i,t-p_i}$  and  $y_{i\tau} = \sum_{\tau=t-p_i+1}^t x_{i\tau}$ . To obtain the inverse transformation for  $z_{it}$  we sum all  $y_{i\tau}$  for  $\tau = t - kp_i$  and  $k = 0, \dots, \lfloor t/p_i \rfloor$ , which gives

$$z_{it} = \sum_{k=0}^{\lfloor t/p_i \rfloor} y_{i,t-kp_i}$$

which means that  $i$  is started at  $t$  or before iff it is in process at time  $t - kp_i$  for some  $k \in \mathbb{N}$ . Furthermore, as  $x_{it} = z_{it} - z_{it-1}$  we obtain the inverse transformation for  $x_{it}$ ,

$$x_{it} = \sum_{k=0}^{\lfloor t/p_i \rfloor} y_{i,t-kp_i} - \sum_{k=0}^{\lfloor (t-1)/p_i \rfloor} y_{i,t-kp_i-1}$$

The start time  $S_i$  is then equal to

$$S_i = \sum_{t \in H} t \left( \sum_{k=0}^{\lfloor t/p_i \rfloor} y_{i,t-kp_i} - \sum_{k=0}^{\lfloor (t-1)/p_i \rfloor} y_{i,t-kp_i-1} \right)$$

Substituting variables  $x_{it}$  by variables  $y_{it}$  in formulation (PDDT), we obtain the formulation (OODDT) below.

$$\min \sum_{i \in V} w_i \sum_{t \in H} t \left( \sum_{k=0}^{\lfloor t/p_i \rfloor} y_{i,t-kp_i} - \sum_{k=0}^{\lfloor (t-1)/p_i \rfloor} y_{i,t-kp_i-1} \right) \quad (16)$$

$$\sum_{k=0}^{\lfloor (t-p_i)/p_i \rfloor} y_{i,t-(k+1)p_i} \geq \sum_{k=0}^{\lfloor t/p_j \rfloor} y_{j,t-kp_j} \quad (i, j) \in E, t \in H \quad (17)$$

$$\sum_{i \in V, p_i > 0} b_{ik} y_{it} \leq B_k \quad t \in H, k \in \mathcal{R} \quad (18)$$

$$\sum_{k=0}^{\lfloor (LC_i-1)/p_i \rfloor} y_{i, LC_i-1-kp_i} = 1 \quad i \in V \quad (19)$$

$$\sum_{k=0}^{\lfloor t/p_i \rfloor} y_{i,t-kp_i} \geq \sum_{k=0}^{\lfloor (t-1)/p_i \rfloor} y_{i,t-kp_i-1} \quad i \in V, t \in H \setminus \{0\} \quad (20)$$

$$y_{it} = 0 \quad i \in V, t \in H \setminus [ES_i, LC_i[ \quad (21)$$

$$y_{it} \in \{0, 1\} \quad i \in V, t \in H \quad (22)$$

Constraints (17) are the disaggregated precedence constraints, given the expression of start time variables  $S_i$  in function of on/off variables  $y_{it}$ . Constraints (18) are the resource constraints. Constraints (19) state that each activity has to be in process in exactly one time period among time periods  $t = LC_i - 1$ ,  $t = LC_i - 1 - p_i$ ,  $t = LC_i - 1 - 2p_i \dots$ . Constraints (20) are obtained by substitution of constraints (13) in (SDDT), or, equivalently, of constraints  $x_{it} \geq 0$  on (PDDT). They ensure, together with constraints (19) that exactly  $p_i$  consecutive variables will be switched-on, i.e. in a non-preemptive fashion [21]. Constraints (21) set to 0 variables that are outside the time window. Constraints (22) define the binary variables.

In the model presented by Sousa [21], the precedence and resource constraints (17,18) were not present but are immediately obtained by the transformation.

## 2.4 Synthesis

Finally, concerning the 6 weak and strong formulations, giving for each of them the presumably original publication, we can make the following synthesis concerning the LP relaxation strengths:

$$\begin{aligned} \text{(PDT)}[18] &\equiv^{LP} \text{(OODT)} [21] \equiv^{LP} \text{(SDT)} [17] \prec^{LP} \\ \text{(PDDT)}[5] &\equiv^{LP} \text{(OODDT)} [21] \equiv^{LP} \text{(SDDT)} [6]. \end{aligned}$$

Variable substitutions can be performed by the non-singular linear transformations to obtain any weak formulation from any other one, and also any strong formulation from any other one. Hence as shown in [21], all the weak formulations are equivalent in terms of LP relaxation/polyhedral structure and all the strong formulations are equivalent in the same sense.

## 3 Relaxation strength of alternative formulations

In this section, we review time-indexed formulations from the literature that were proposed alternatively without comparing them to the standard formulations in terms of LP relaxation quality.

### 3.1 The KF, DH and DH' on/off formulations

Klein [11] presents a variant of the on/off formulation, called Formulation 2 in [11] and denoted (KF) in this paper, based on the formulation of Kaplan [10] for the preemptive RCPSP. This formulation looks like (*OODDT*) with the following differences. Precedence constraints are replaced by constraints (23). Non-preemption/duration constraints (19,20) are replaced by duration constraints (24) and non-preemption constraints (25).

$$p_i y_{jt} - \sum_{q=ES_i}^{t-1} y_{iq} \leq 0 \quad (i, j) \in E, t \in H \cap [ES_j, LC_i[ \quad (23)$$

$$\sum_{t=ES_i}^{LC_i-1} y_{it} = p_i \quad i \in V \quad (24)$$

$$p_i (y_{it} - y_{i,t+1}) - \sum_{q=t-p_i+1}^{t-1} y_{iq} \leq 1 \quad i \in V, t \in [ES_i, LC_i-2] \quad (25)$$

Precedence constraints (23) state that for an activity  $j$  to be in process at time  $t$ , its predecessor  $i$  must have been entirely processed during interval  $[ES_i, t]$ . Duration constraints (24) are straightforward. Non-preemption constraints (25) model the fact that if an activity completes at time  $t+1$ , in which case the term in factor of  $p_i$  is equal to one, the  $p_i-1$  units that precede  $t$  must be switched-on. We remark that constraints (25) match the first formulation of non-preemption constraints presented by Sousa in Section 3.3 of his thesis [21].

Demeulemeester and Herroelen [7] (Section 2.1.3) present a variant of (KF), denoted (DH), with the same duration and non-preemption constraints but replacing precedence constraints (23) by exclusive constraints (26) and (27)

$$y_{jt} \leq y_{i,t-p_i}, \quad (i, j) \in E, t \in H, t \leq ES_j - 1 + p_i \quad (26)$$

$$y_{jt} \leq \sum_{q=ES_i+p_i-1}^{t-p_i} y_{iq}, \quad (i, j) \in E, t \in H, t \geq ES_j + p_i \quad (27)$$

These constraints aim at expressing the fact that if the successor  $j$  of an activity  $i$  is in process at a time  $t$ , then  $i$  has to be in process at time  $t-p_i$  or before, considering the non-preemption constraints. However the formulation has a slight mistake as explained below through a counter-example that also helps understanding the logic of the constraints.

**Proposition 1.** *Formulation (DH) is incorrect.*

*Proof.* Suppose that two activities  $i$  and  $j$  with  $(i, j) \in E$  are such that  $p_i = 4$ ,  $ES_i = 0$  and  $ES_j = 5$ . A necessary and sufficient condition to have  $y_{j8} = 0$  w.r.t. the precedence relation  $(i, j)$  is to have  $S_i + p_i \geq 9$ . Considering non-preemption and the fact that  $ES_i = 0$  and  $p_i = 4$ , we have  $S_i + p_i \geq 9$  if and only if  $y_{i3} + y_{i4} = 0$ , which yields constraint  $y_{j8} \leq y_{i3} + y_{i4}$ . However for  $t = 8$ , as we have  $ES_j - 1 + p_i = 8$ , the time period falls in the range of constraints (26). Consequently, we obtain  $y_{j8} \leq y_{i4}$ , which over-constrains the problem, preventing task  $j$  from being in process at time 8 when  $S_i + p_i = 4$ , while this should be allowed.  $\square$

In fact the counterexample situation happens as soon as  $ES_i + p_i < ES_j$ . To correct the formulation it suffices to replace  $ES_j$  by  $ES_i + p_i = EC_i$  in the range of constraints (26,27), which gives constraints (28,29) and formulation (DH').

$$y_{jt} \leq y_{i,t-p_i}, \quad (i, j) \in E, t \in H, t \leq EC_i - 1 + p_i \quad (28)$$

$$y_{j\bar{t}} \leq \sum_{q=ES_i+p_i-1}^{t-p_i} y_{iq} \quad (i, j) \in E, t \in H, t \geq EC_i + p_i \quad (29)$$

Constraints (28) state that, to be in process at time  $t$ , an activity  $j$  must have its predecessor in process at time  $t - p_i$  for any  $t$  such that  $t - p_i$  falls strictly before the earliest end time of  $i$ ,  $EC_i = ES_i + p_i$ . Indeed, if  $i$  starts at  $t - p_i$  (which allows  $j$  to be in process at  $t$ ) or before,  $i$  is necessarily in process at time  $t - p_i$  (i.e.  $S_i \leq t - p_i \Leftrightarrow i$  is in process at time  $t - p_i$ ). If  $t - p_i$  exceeds the earliest completion time of  $i$ , constraints (29) state that activity  $j$  can only be in process at time  $t$  if its predecessor starts at  $t - p_i$  or before, which means that it has to be in process on at least one time period between  $ES_i + p_i - 1$  and  $t - p_i$ .

We now focus on the relative strengths of the proposed (OODDT) formulation and the (KF) and (DH') formulations.

**Proposition 2.**  $(KF), (DH') \prec^{LP} (OODDT)$

*Proof.* Let us first compare the non-preemption constraints (19,20) in (OODDT) and their equivalent in (KF) and (DH'), i.e. duration constraints (24) and non-preemption constraints (25). The comparison is immediate from the results of Sousa [21]. A  $p$ -connected vector of a chain of length  $n$  is defined as an ordered set of binary elements containing exactly  $p$  consecutive non zero elements. Obviously, there is a one-to-one relation between the set of  $p_i$ -connected vectors of length  $n$  and the set of non-preemptive assignments of variables  $y_{it}$ . Sousa showed that the extreme points of the polytope defined by constraints (19,20) and  $y_i \geq 0$  for a given activity  $i$  are precisely the  $p_i$ -connected vectors of a chain of length  $n$ . Consequently, the description of the set of connected vectors given by (24,25) cannot be better.

We next show that disaggregated precedence constraints (17) are not weaker than (DH') precedence constraints (28) and (29). Consider a precedence constraint  $(i, j)$ . Writing the disaggregated precedence constraints (17) we get  $\sum_{k=0}^{K_{i,t-p_i}} y_{i,t-(k+1)p_i} - \sum_{k=0}^{K_{j,t}} y_{j,t-kp_j} \geq 0 \implies y_{jt} \leq \sum_{k=0}^{K_{i,t-p_i}} y_{i,t-(k+1)p_i}$ . We distinguish two cases. Consider the case where  $t \leq EC_i - 1 + p_i$ . Since  $t - p_i \leq EC_i - 1$ , any time  $t - (k+1)p_i$  with  $k \geq 1$  is strictly before  $ES_i$ . So we have  $\sum_{k=0}^{K_{i,t-p_i}} y_{i,t-(k+1)p_i} = y_{i,t-p_i}$ . This gives constraints (28). Consider now a time period  $t \geq EC_i - 1 + p_i$ . All non-zero terms of expression  $\sum_{k=0}^{K_{i,t-p_i}} y_{i,t-(k+1)p_i}$  such that  $t \geq EC_i - 1$  are also included in expression  $\sum_{q=EC_i-1}^{t-p_i} y_{iq}$ . Furthermore, for the unique  $k \geq 1$  such that  $t - (k+1)p_i < EC_i - 1$ , we have  $y_{i,EC_i-1} \geq y_{i,t-(k+1)p_i}$  by constraints (20). Hence we obtain:

$$y_{jt} \leq \sum_{k=0}^{K_{i,t-p_i}} y_{i,t-(k+1)p_i} \leq \sum_{q=EC_i}^{t-p_i} y_{iq}$$

which yields constraints (29).



Note that an alternative proof consists in using the total-unimodularity of the constraint matrix, established by de Souza and Wolsey [6] and Sankaran *et al* [20] for the  $z_{it}$  formulation. Using the non-singular transformation of the  $z_{it}$  variables in  $y_{it}$  variables, it comes that the polytope defined by constraints (17,19,20) and  $0 \leq y_{it} \leq 1$ ,  $i \in V, t \in H$  is 0–1. Hence formulations (KF) and (DH') cannot be stronger. It remains to show that they are strictly weaker.

Consider now a simple instance with a single activity, no precedence constraints and no resource constraints such that  $p_i = 2$ ,  $ES_i = 0$  and  $LC_i = T = 3$ . Writing the LP relaxation of (ODDT), we obtain:

$$\begin{aligned}
y_{i0} + y_{i2} &= 1 & (19) \\
y_{i0} &\geq 0 & (20), t = 0 \\
y_{i1} - y_{i0} &\geq 0 & (20), t = 1 \\
y_{i2} + y_{i0} - y_{i1} &\geq 0 & (20), t = 2 \\
y_{i1} - y_{i2} - y_{i0} &\geq 0 & (20), t = 3 \\
0 \leq y_{it} &\leq 1 & t = 0, 1, 2
\end{aligned}$$

Writing the LP relaxation of (KF) and (DH'), we obtain

$$\begin{aligned}
y_{i0} + y_{i1} + y_{i2} &= 2 & (24) \\
2y_{i0} - 2y_{i1} &\leq 1 & (25), t = 0 \\
2y_{i1} - 2y_{i2} - y_{i0} &\leq 1 & (25), t = 1 \\
0 \leq y_{it} &\leq 1 & t = 0, 1, 2
\end{aligned}$$

Consider now the fractional solution  $y_{i0} = y_{i1} = y_{i2} = \frac{2}{3}$ . We observe that this solution satisfies constraints (24) and (25). However, constraints (20) for  $t = 2$  and  $t = 3$  imply that  $y_{i1} = y_{i2} + y_{i0}$ , which is violated by the considered solution.  $\square$

### 3.2 The KF+ formulation

Klein [11] introduced two step formulations. The first one (Formulation 3, Section 3.2.1.4 in [11]) is precisely the (SDT) formulation. The second one (Formulation 4 Section 3.2.1.5 in [11]) is a variant of the (SDDT) formulation obtained by introducing another step binary variable  $\gamma_{it} = 1$  if  $i$  completes at time  $t$  or after. This formulation has an advantage when durations are decision variables. Indeed, in all formulations we presented so far, durations  $p_i$  have to be fixed parameters because they are present in the variables indices. Mixing  $z_{it}$  and  $\gamma_{it}$  allows to get rid of this drawback. Observe that we have  $z_{it} + \gamma_{it} - 1 = y_{it}$ . We obtain formulation (KF+) by adding to (SDDT) the following constraints, defining the  $\gamma$  variables and establishing the link with the  $z$  variables,

$$\sum_{t=ES_i}^{LC_{i-1}} z_{it} + \gamma_{it} - 1 = p_i \quad i \in V \quad (30)$$

$$\gamma_{i,EC_i-1} = 1 \quad i \in V \quad (31)$$

$$\gamma_{i,t-1} - \gamma_{i,t} \geq 0 \quad i \in V, t \in H \quad (32)$$

$$\gamma_{it} = 0 \quad i \in V, t \in H, t \geq LC_i \quad (33)$$

$$\gamma_{it} \in \{0, 1\} \quad i \in V, t \in H \cap [EC_i, LC_i[ \quad (34)$$

replacing precedence constraints (10) by

$$\gamma_{i,t} + z_{jt} \leq 1 \quad (i, j) \in E, t \in H \quad (35)$$

and resource constraints (11) by

$$\sum_{i \in V} b_{ik}(z_{it} + \gamma_{it} - 1) \leq B_k \quad t \in H, k \in \mathcal{R} \quad (36)$$

**Proposition 3.**  $(KF+) \preceq^{LP} (SDDT)$ .

*Proof.* Observe that we have non-singular transformations by setting  $z_{it} = \sum_{\tau=0}^t x_{i\tau}$  and  $\gamma_{it} = 1 - \sum_{\tau=0}^{t-p_i-1} x_{i\tau}$ . Using these non-singular transformations we could obtain aggregated or disaggregated formulations based on  $x_{it}$  variables and equivalent to the ones already presented and consequently not weaker than  $(KF+)$ .  $\square$

### 3.3 The BC step formulation

Bianco and Caramia [3, 4] propose a variant (BC) of the step formulation. Note that the formulation in [3] contained mistakes that were corrected in [16]. It involves the 0-1 step variable  $z_{it}$  and another 0-1 variable  $z'_{it}$  which equals 1 iff activity  $i$  is completed at  $t$  or before. Another variable  $\pi_{it}$  is introduced, giving the fraction of activity  $i$  that has been performed up to time  $t$ .

Formulation (BC) is obtained from (SDDT) by the following modifications. Constraints (10) are replaced by

$$z'_{i,t-1} - z_{jt} \geq 0 \quad (i, j) \in E, t \in H \quad (37)$$

The following constraints are added to model the step behavior of the  $z'_{it}$  variables and to set  $\pi_{it}$  variables to the correct value.

$$z'_{i, LS_i + p_i} = 1 \quad i \in V \quad (38)$$

$$z'_{i,t} - z'_{i,t-1} \geq 0 \quad i \in V, t \in H \quad (39)$$

$$z'_{it} = 0 \quad i \in V, t < ES_i + p_i \quad (40)$$

$$z'_{it} \in \{0, 1\} \quad i \in V, t \in H \quad (41)$$

$$\pi_{it} - \pi_{i,t-1} = \frac{1}{p_i}(z_{it} - z'_{i,t-1}) \quad i \in V, t \in H \quad (42)$$

$$z'_{it} \leq \pi_{it} \leq z_{it} \quad i \in V, t \in H \quad (43)$$

$$\pi_{it} \geq 0 \quad i \in V, t \in H \quad (44)$$

The resource constraints (11) are finally replaced by

$$\sum_{i \in V} b_{ik} p_i (\pi_{i,t+1} - \pi_{i,t}) \leq B_k \quad i \in V, t \in H \quad (45)$$

Note that in [4] the author proved that (BC) is strictly stronger than (DT) and (SDT). We can remark that comes from the fact that (BC) is actually a disaggregated model and that it is not stronger than the standard ones. Even if it is not mentioned in [3], we can also add in (BC) the following valid constraints:

$$z'_{it} = z_{t-p_i+1} \quad i \in V, t \in H \quad (46)$$

Then, we can show the following.

**Proposition 4.**  $(BC) \preceq^{LP} (SDDT)$

*Proof.* If we introduce variables  $z'$  and constraints (37–44) plus constraint (46) in the (SDDT) model, we obtain a formulation equivalent to (SDDT). Indeed, it can be shown that for any feasible value of variable  $z_{it}$  in the LP relaxation of SDDT, we can obtain values for variables  $z'_{it}$  immediately through constraints (46) and also values for variables  $\pi_{it}$  that satisfy (42–44). Remarking in addition that  $\pi_{it} - \pi_{i,t-1} = \frac{1}{p_i}(z_{it} - z'_{i,t-1}) = \frac{1}{p_i}(z_{it} - z_{it-d_i})$ , we precisely obtain resource constraints (11). As (BC) does not incorporate constraints (46), it comes that  $(BC) \preceq^{LP} (SDDT)$ .  $\square$

### 3.4 The OOPDT and OOPDDT formulations

Kopanos *et al.* [12] present two formulations incorporating on/off variables  $y_{it}$  into pulse formulations (PDT) and (PDDT). To that purpose, (PDT) and (PDDT) are modified in the sense that pulse resource constraints (3) are replaced by on/off resource constraints (18), all other constraints remaining the same. On/off duration constraints (24) and linking constraints  $y_{it} = \sum_{\tau=t-p_i+1}^t x_{i\tau}$ ,  $i \in V, t \in H$  are added. We name these formulations (OOPDT) and (OOPDDT).

**Proposition 5.**  $(OOPDT) \preceq^{LP} (PDT)$  and  $(OOPDDT) \preceq^{LP} (PDDT)$ .

*Proof.* Only variable substitutions are performed via the non-singular transformation so the “new” formulations cannot be stronger than their counterpart.  $\square$

## 4 Concluding remarks

We have discussed the well known pulse (PDDT) and step (SDDT) time-indexed formulations and the less well-known on/off time-indexed formulation (OODDT) for the RCPSp, that are all equivalent in terms of LP-relaxation and belong to the family of strong time-indexed formulations. Weak counterparts (PDT), (SDT) and (OODT), based on an aggregated form of the precedence constraints, define a second family of “weak” formulations and are also all equivalent in terms of LP relaxation. Other time-indexed formulations were proposed in the literature without any mention of the relative strengths of their LP relaxations. We remarked that for the ones presented in this paper, the LP relaxations are either weaker than or equivalent to the three mentioned ones. Note that the present paper extends and corrects the part on time-indexed formulations of the book chapter [2] as well as the technical report [1], in which the transformation was rediscovered as the authors were unfortunately not aware of Sousa [21] results.

We have to acknowledge that the practical performance of a formulation, in terms of integer solving, is not necessarily related to the LP relaxation strength. It is well known for instance that the weak formulation (PDT) may outperform the strong formulation (PDDT) on some instances, due to memory problems or to the CPU time / relaxation strength well-known trade-off of the branch-and-bound scheme. Bianco and Caramia [3] showed through extensive experiments that their formulation generally outperformed other formulations in terms of solution time and quality. The way constraints and/or additional redundant variables are introduced and formulated influences the solver performance in terms of memory usage, preprocessing, cutting plane generation and branching.

However, this should not however hide the fact that, in any “new” formulation, constraints that are equivalent, via non-singular transformations, to previously proposed ones should be identified and distinguished from actual stronger formulations, such as the one of Mingozzi *et al* [14] and strong cutting plane inequalities, such as the ones proposed in [9].

## Acknowledgement

The author would like to thank referee #1 for her/his constructive comments and, in particular, for pointing out the results of Sousa’s thesis, as well as referee #2 for her/his help in a clearer structuration of the paper.

## References

- [1] C. Artigues. A note on time-indexed formulations for the resource-constrained project scheduling problem. LAAS report 13206, LAAS-CNRS, 2013.
- [2] C. Artigues, O. Koné, P. Mopez, and M. Mongeau. Mixed-integer linear programming formulations. In C. Schwindt and J. Zimmermann, editors, *Handbook on Project Management and Scheduling*, volume 1 of *International Handbooks on Information Systems*. Springer, 2015.
- [3] L. Bianco and M. Caramia. A new formulation for the project scheduling problem under limited resources. *Flexible Services and Manufacturing Journal*, 25:6–24, 2013.
- [4] L. Bianco and M. Caramia. The resource constrained project scheduling problem: a theoretical comparison between a recent formulation and the main time indexed linear programming based approaches. *RAIRO-Operations Research*, <http://dx.doi.org/10.1051/ro/2016036>, 2016.
- [5] N. Christofides, R. Alvarez-Valdés, and J. Tamarit. Project scheduling with resource constraints: a branch and bound approach. *European Journal of Operational Research*, 29:262–273, 1987.
- [6] C. C. de Souza and L. A. Wolsey. Scheduling projects with labour constraints. Technical report, Instituto de Computação, Universidade Estadual de Campinas, 1997.
- [7] E. Demeulemeester and W. Herroelen. *Project scheduling: A research handbook*. Kluwer Academic Publishers, Dordrecht, 2002.
- [8] M. E. Dyer and L. A. Wolsey. Formulating the single machine sequencing problem with release dates as a mixed integer program. *Discrete Applied Mathematics*, 26:255–270, 1990.
- [9] J. R. Hardin, G. L. Nemhauser, and M. W. Savelsbergh. Strong valid inequalities for the resource-constrained scheduling problem with uniform resource requirements. *Discrete Optimization*, 5(1):19–35, 2008.

- [10] L. A. Kaplan. *Resource-constrained project scheduling with preemption of jobs*. PhD thesis, University of Michigan, 1988.
- [11] R. Klein. *Scheduling of resource-constrained projects*. Kluwer Academic Publishers, Dordrecht, 2000.
- [12] G. M. Kopanos, T. S. Kyriakidis, and M. C. Georgiadis. New continuous-time and discrete-time mathematical formulations for resource-constrained project scheduling problems. *Computers and Chemical Engineering*, 68:9–106, 2014.
- [13] E. L. Lawler. On scheduling problems with deferral costs. *Management Science*, 11:280–288, 1964.
- [14] A. Mingozzi, V. Maniezzo, S. Ricciardelli, and L. Bianco. An exact algorithm for the resource-constrained project scheduling problem based on a new mathematical formulation. *Management Science*, 44:714–729, 1998.
- [15] R. H. Möhring, A. S. Schulz, F. Stork, and M. Uetz. On project scheduling with irregular starting time costs. *Operations Research Letters*, 28:149–154, 2001.
- [16] A. Naber and R. Kolisch. Comments on the rcpsp-model of bianco and caramia. *Available at SSRN*, <http://dx.doi.org/10.2139/ssrn.2286718>, 2013.
- [17] A. Pritsker and L. Watters. A zero-one programming approach to scheduling with limited resources. Technical Report RM-5561-PR, The RAND Corporation, 1968.
- [18] A. Pritsker, L. Watters, and P. Wolfe. Multi-project scheduling with limited resources: a zero-one programming approach. *Management Science*, 16:93–108, 1969.
- [19] M. Queyranne and A. Schulz. Polyhedral approaches to machine scheduling. Technical Report 408/1994, Technische Universität Berlin, 1994.
- [20] J. K. Sankaran, D. L. Bricker, and S.H. Juang. A strong fractional cutting plane algorithm for resource-constrained project scheduling. *International Journal of Industrial Engineering - Applications and Practice*, 6:99–111, 1999.
- [21] J. P. Sousa. *Time indexed formulations of non-preemptive single-machine scheduling problems*. PhD thesis, Université Catholique de Louvain, Louvain-la-Neuve, Belgium, 1989.
- [22] J. P. Sousa and L. A. Wolsey. A time indexed formulation of non-preemptive single machine scheduling problems. *Mathematical Programming*, 54:353–367, 1992.