W 2,2 INTERIOR CONVERGENCE FOR SOME CLASS OF ELLIPTIC ANISOTROPIC SINGULAR PERTURBATIONS PROBLEMS.

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$W^{2,2}$ INTERIOR CONVERGENCE FOR SOME CLASS OF ELLIPTIC ANISOTROPIC SINGULAR PERTUBATIONS PROBLEMS

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Abstract. In this paper, we deal with anisotropic singular perturbations of some class of elliptic problem. We study the asymptotic behavior of the solution in certain second order pseudo Sobolev space.

1. Description of the problem

In this paper, we study diffusion problems when the diffusion coefficients in certain directions are going toward zero. More precisely we are interested in studying the asymptotic behavior of the solution in certain second order pseudo Sobolev space. We consider the following elliptic problem

$$\begin{cases} -\operatorname{div}(A_\epsilon \nabla u_\epsilon) = f \\ u_\epsilon \in W_0^{1,2}(\Omega) \end{cases} \tag{1}$$

where $0 < \epsilon \leq 1$ and Ω is a bounded domain (i.e. open bounded connected subset) of $\mathbb{R}^N$ and $f \in L^2(\Omega)$. We denote by $x = (x_1, ..., x_N) = (X_1, X_2)$ the points in $\mathbb{R}^N$ where

$$X_1 = (x_1, ..., x_q) \text{ and } X_1 = (x_{q+1}, ..., x_N),$$

with this notation we set

$$\nabla = (\partial_{x_1}, ..., \partial_{x_N})^T = \begin{pmatrix} \nabla_{X_1} \\ \nabla_{X_2} \end{pmatrix},$$

where

$$\nabla_{X_1} = (\partial_{x_1}, ..., \partial_{x_q})^T \text{ and } \nabla_{X_2} = (\partial_{x_{q+1}}, ..., \partial_{x_N})^T$$

The diffusion matrix A_ϵ is given by

$$A_\epsilon = (a_{ij}^\epsilon) = \begin{pmatrix} \epsilon^2 A_{11} & \epsilon A_{12} \\ \epsilon A_{21} & A_{22} \end{pmatrix} \text{ with } A = (a_{ij}) = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix},$$

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where A_{11} and A_{22} are $q\times q$ and $(N-q)\times(N-q)$ matrices. The coefficients a_{ij}^{ϵ} are given by

$$a_{ij}^{\epsilon}=\begin{cases}\epsilon^2 a_{ij} \text{ for } i,j\in\{1,..,q\}\\ a_{ij} \text{ for } i,j\in\{q+1,..,N\}\\ \epsilon a_{ij} \text{ for } i\in\{1,..,q\},\ j\in\{q+1,..,N\}\\ \epsilon a_{ij}^{\epsilon} \text{ for } i\in\{q+1,..,N\},\ j\in\{1,..,q\}\end{cases}$$

We assume that $A\in L^{\infty}(\Omega)$ and for some $\lambda>0$ we have

$$A(x)\zeta\cdot\zeta\geq\lambda|\zeta|^2,\forall\zeta\in\mathbb{R}^N,\ \text{a.e } x\in\Omega. \tag{2}$$

Recall the Hilbert space introduced in [2]

$$V^{1,2}=\left\{u\in L^2(\Omega)\ \mid \nabla_{X_2}u\in L^2(\Omega) \text{ and } u(X_1,\cdot)\in W_0^{1,2}(\Omega_{X_1}) \text{ a.e } X_1\in\Omega^1\right\},.$$

equipped with the norm

$$\|u\|_{1,2}=\left(\|u\|^2_{L^2(\Omega)}+\|\nabla_{X_2}u\|^2_{L^2(\Omega)}\right)^{\frac{1}{2}}.$$

Here $\Omega_{X_1}=\left\{X_2\in\mathbb{R}^{N-q}:(X_1,X_2)\in\Omega\right\}$ and $\Omega^1=P_1(\Omega)$ where P_1 is the natural projector $\mathbb{R}^N\to\mathbb{R}^q$.

We introduce the second order local pseudo Sobolev space

$$V^{2,2}_{loc}=\left\{u\in V^{1,2}\mid\nabla^2_{X_2}u\in L^2_{loc}(\Omega)\right\},$$

equipped with the family of norms $(\|\cdot\|^{\omega}_{2,2})_{\omega}$ given by

$$\|u\|^{\omega}_{2,2}=\left(\|u\|^2_{L^2(\Omega)}+\|\nabla_{X_2}u\|^2_{L^2(\Omega)}+\left\|\nabla^2_{X_2}u\right\|^2_{L^2(\omega)}\right)^{\frac{1}{2}},\ \omega\subset\subset\Omega \text{ open}$$

where $\nabla^2_{X_2}u$ is the Hessian matrix of u taken in the X_2 direction, the term $\left\|\nabla^2_{X_2}u\right\|^2_{L^2(\omega)}$ is given by

$$\left\|\nabla^2_{X_2}u\right\|^2_{L^2(\omega)}=\sum_{i,j=q+1}^{N}\left\|\partial^2_{ij}u\right\|^2_{L^2(\omega)}.$$

We can show that $V^{2,2}_{loc}$ is a Fréchet space (i.e. locally convex, metrizable and complete). We also define the following

$$\left\|\nabla^2_{X_1}u\right\|^2_{L^2(\omega)}=\sum_{i,j=1}^{q}\left\|\partial^2_{ij}u\right\|^2_{L^2(\omega)},$$

and

$$\left\|\nabla^2_{X_1X_2}u\right\|^2_{L^2(\omega)}=\sum_{i=1}^{q}\sum_{j=q+1}^{N}\left\|\partial^2_{ij}u\right\|^2_{L^2(\omega)}.$$

As $\epsilon\to0$, the Limit problem is given by

$$\begin{cases}-\operatorname{div}(A_{22}\nabla u_0(X_1,\cdot)=f(X_1,\cdot)\\ u_0(X_1,\cdot)\in W_0^{1,2}(\Omega_{X_1}) \qquad\qquad \text{a.e } X_1\in\Omega^1\end{cases} \tag{3}$$

The existence and the uniqueness of the $W_0^{1,2}$ weak solutions to (1) and (3) follow from the Lax-Milgram theorem. In [1] the authors studied the relationship

between u_ϵ and u_0 and they proved that $u_0 \in V^{1,2}$ and the following convergences (see Theorem 2.1 in the above reference)

$$u_\epsilon \to u_0 \text{ in } V^{1,2} \text{ and } \epsilon \nabla_{X_1} u_\epsilon \to 0 \text{ in } L^2(\Omega). \tag{4}$$

For the L^p case we refer the reader to [6], and [2],[4], [5] for other related problems. In this paper, we deal with the asymptotic behavior of the second derivatives of u_ϵ, in other words we show the convergence of u_ϵ in the space $V^{2,2}_{loc}$ introduced previously. The arguments are based on the Riesz-Fréchet-Kolmogorov compacity theorem in L^p spaces. Let us give the main result

Theorem 1. *Assume that $A \in L^\infty(\Omega) \cap C^1(\Omega)$ with (2), suppose that $f \in L^2(\Omega)$ then $u_0 \in V^{2,2}_{loc}$ and $u_\epsilon \to u_0$ in $V^{2,2}_{loc}$, where $u_\epsilon \in W^{1,2}_0(\Omega) \cap W^{2,2}_{loc}(\Omega)$ and u_0 are the unique weak solutions to (1) and (3) respectively. In addition, the convergences $\epsilon^2 \nabla^2_{X_1} u_\epsilon \to 0$, $\epsilon \nabla^2_{X_1 X_2} u_\epsilon \to 0$ hold in $L^2_{loc}(\Omega)$.*

2. Some useful tools

Proposition 1. *The vector space $V^{2,2}_{loc}$ equipped with the family of norms $(\|\cdot\|^\omega_{2,2})_\omega$ is a Fréchet space.*

Proof. Let $(\omega_n)_{n\in\mathbb{N}}$ be a countable open covering of Ω with $\omega_n \subset\subset \Omega$, $\omega_n \subset \omega_{n+1}$ for every $n \in \mathbb{N}$. The countable family $(\|\cdot\|^{\omega_n}_{2,2})_{n\in\mathbb{N}}$ define a base of norms for the $V^{2,2}$ topology. The general theory of locally convex topological vector spaces shows that this topology is metrizable, explicitly a distance d which define this topology is given by (see for instance [8])

$$d(u,v) = \sum_{n=0}^{\infty} 2^{-n} \frac{\|u-v\|^{\omega_n}_{2,2}}{1+\|u-v\|^{\omega_n}_{2,2}}, \quad u, v \in V^{2,2}_{loc}. \tag{5}$$

Let (u_m) be a Cauchy sequence in $V^{2,2}_{loc}$ then (u_m) is a Cauchy sequence for each norm $\|\cdot\|^{\omega_n}_{2,2}$, $n \in \mathbb{N}$. Whence, there exist $u, v \in L^2(\Omega)$ such that

$$u_m \to u,\ \nabla_{X_2} u_m \to v \text{ in } L^2(\Omega),$$

and for every $n \in \mathbb{N}$ fixed there exists $w_n \in L^2(\omega_n)$ such that

$$\nabla^2_{X_2} u_m \to w_n \text{ in } L^2(\omega_n).$$

The continuity of ∇_{X_2} and $\nabla^2_{X_2}$ on $D'(\Omega)$ and $D'(\omega_n)$ shows that $v = \nabla_{X_2} u$ and $\nabla^2_{X_2} u = w_n$ for every $n \in \mathbb{N}$. Hence $u \in V^{2,2}_{loc}$ and

$$\forall n \in \mathbb{N} : \|u_m - u\|^{\omega_n}_{2,2} \to 0 \text{ as } m \to \infty.$$

Finally the normal convergence of the series (5) implies

$$d(u_m, u) \to 0 \text{ as } m \to \infty,$$

and therefore the completion of $V^{2,2}_{loc}$ follows. □

Remark 1. *Notice that a sequence (u_m) in $V^{2,2}_{loc}$ converges to u with respect to d if and only if $\|u_m - u\|^\omega_{2,2} \to 0$ as $m \to \infty$, for every $\omega \subset\subset \Omega$ open.*

Now, let us give two useful lemmas

Lemma 1. *Let $f \in L^2(\mathbb{R}^N)$, for every $\epsilon \in (0,1]$ let $u_\epsilon \in W^{2,2}(\mathbb{R}^N)$ such that*

$$-\epsilon^2 \Delta_{X_1} u_\epsilon(x) - \Delta_{X_2} u_\epsilon(x) = f(x) \text{ a.e } x \in \mathbb{R}^N \tag{6}$$

then for every $\epsilon \in (0,1]$ we have the bounds

$$\begin{aligned} \left\|\nabla^2_{X_2} u_\epsilon\right\|_{L^2(\mathbb{R}^N)} &\leq \|f\|_{L^2(\mathbb{R}^N)}, \\ \epsilon^2 \left\|\nabla^2_{X_1} u_\epsilon\right\|_{L^2(\mathbb{R}^N)} &\leq \|f\|_{L^2(\mathbb{R}^N)}, \\ \sqrt{2}\epsilon \left\|\nabla^2_{X_1 X_2} u_\epsilon\right\|_{L^2(\mathbb{R}^N)} &\leq \|f\|_{L^2(\mathbb{R}^N)}. \end{aligned}$$

Proof. Let $\mathcal{F}$ be the Fourier transform defined on $L^2(\mathbb{R}^N)$ as the extension, by density, of the Fourier transform defined on the Schwartz space $\mathcal{S}(\mathbb{R}^N)$ by

$$\mathcal{F}(u)(\xi) = (2\pi)^{-\frac{N}{2}} \int_{\mathbb{R}^N} u(x) e^{-ix\cdot\xi} dx, \quad u \in \mathcal{S}(\mathbb{R}^N)$$

where $\cdot$ is the standard scalar product of $\mathbb{R}^N$. Applying $\mathcal{F}$ on (6) we obtain

$$\left(\epsilon^2 \sum_{i=1}^{q} \xi_i^2 + \sum_{i=q+1}^{N} \xi_i^2\right) \mathcal{F}(u_\epsilon)(\xi) = \mathcal{F}(f)(\xi),$$

then

$$\left(\epsilon^4 \sum_{i,j=1}^{q} \xi_i^2 \xi_j^2 + \sum_{i,j=q+1}^{N} \xi_i^2 \xi_j^2 + 2\epsilon^2 \sum_{j=q+1}^{N} \sum_{i=1}^{q} \xi_i^2 \xi_j^2\right) \left|\mathcal{F}(u_\epsilon)(\xi)\right|^2 = \left|\mathcal{F}(f)(\xi)\right|^2, \tag{7}$$

thus

$$\sum_{i,j=q+1}^{N} \xi_i^2 \xi_j^2 \left|\mathcal{F}(u_\epsilon)(\xi)\right|^2 \leq \left|\mathcal{F}(f)(\xi)\right|^2,$$

hence

$$\sum_{i,j=q+1}^{N} \left|\mathcal{F}(\partial^2_{ij} u_\epsilon)(\xi)\right|^2 \leq \left|\mathcal{F}(f)(\xi)\right|^2,$$

then

$$\sum_{i,j=q+1}^{N} \left\|\mathcal{F}(\partial^2_{ij} u_\epsilon)\right\|^2_{L^2(\mathbb{R}^N)} \leq \|\mathcal{F}(f)\|^2_{L^2(\mathbb{R}^N)},$$

and the Parseval identity gives

$$\sum_{i,j=q+1}^{N} \left\|\partial^2_{ij} u_\epsilon\right\|^2_{L^2(\mathbb{R}^N)} \leq \|f\|^2_{L^2(\mathbb{R}^N)}.$$

Hence

$$\left\|\nabla^2_{X_2} u_\epsilon\right\|_{L^2(\mathbb{R}^N)} \leq \|f\|_{L^2(\mathbb{R}^N)}.$$

Similarly we obtain from (7) the bounds

$$\epsilon^2 \left\|\nabla^2_{X_1} u_\epsilon\right\|_{L^2(\mathbb{R}^N)} \leq \|f\|_{L^2(\mathbb{R}^N)},$$

$$\sqrt{2}\epsilon \left\|\nabla^2_{X_1 X_2} u_\epsilon\right\|_{L^2(\mathbb{R}^N)} \leq \|f\|_{L^2(\mathbb{R}^N)}.$$

□

Notation 1. *For a function $u \in L^p(\mathbb{R}^N)$ and $h \in \mathbb{R}^N$ we denote $\tau_h u(x) = u(x+h)$, $x \in \mathbb{R}^N$.*

Lemma 2. *Let Ω be an open bounded subset of $\mathbb{R}^N$ and let $(u_k)_{k\in\mathbb{N}}$ be a converging sequence in $L^p(\Omega)$,$1 \leq p < \infty$ and let $\omega \subset\subset \Omega$ open, then for every $\sigma > 0$ there exists $0 < \delta < dist(\partial\Omega, \omega)$ such that*

$$\forall h \in \mathbb{R}^N, |h| \leq \delta, \forall k \in \mathbb{N} : \|\tau_h u_k - u_k\|_{L^p(\omega)} \leq \sigma$$

in other words we have $\lim\limits_{h\to 0} \sup\limits_{k\in\mathbb{N}} \|\tau_h u_k - u_k\|_{L^p(\omega)} = 0$.

Proof. Let $\omega \subset\subset \Omega$ open. For a function $v \in L^p(\Omega)$, extend v by 0 outside of Ω, since the translation $h \to \tau_h v$ is continuous from $\mathbb{R}^N$ to $L^p(\mathbb{R}^N)$ (see for instance [8]) then for every $\sigma > 0$ there exists $0 < \delta < dist(\partial\Omega, \omega)$ such that

$$\forall h \in \mathbb{R}^N, |h| \leq \delta : \|\tau_h v - v\|_{L^p(\omega)} \leq \sigma. \tag{8}$$

We denote $\lim u_k = u \in L^p(\Omega)$, and let $\sigma > 0$ then (8) shows that there exists $0 < \delta < dist(\partial\Omega, \omega)$ such that

$$\forall h \in \mathbb{R}^N, |h| \leq \delta : \|\tau_h u - u\|_{L^p(\omega)} \leq \frac{\sigma}{2}.$$

By the triangular inequality and the invariance of the Lebesgue measure under translations we have for every $k \in \mathbb{N}$ and $|h| \leq \delta$

$$\|\tau_h u_k - u_k\|_{L^p(\omega)} \leq 2\|u_k - u\|_{L^p(\Omega)} + \|\tau_h u - u\|_{L^p(\omega)} \tag{9}$$

Since $u_k \to u$ in $L^p(\Omega)$ then there exists $k_0 \in \mathbb{N}$, such that

$$\forall k \geq k_0 : \|u_k - u\|_{L^p(\Omega)} \leq \frac{\sigma}{4}.$$

Then from (9) we obtain

$$\forall h \in \mathbb{R}^N, |h| \leq \delta, \forall k \geq k_0 : \|\tau_h u_k - u_k\|_{L^p(\omega)} \leq \sigma \tag{10}$$

Similarly (8) shows that for every $k \in \{0, 1, 2, ..., k_0 - 1\}$ there exists $0 < \delta_k < dist(\partial\Omega, \omega)$ such that

$$\forall h \in \mathbb{R}^N, |h| \leq \delta_k : \|\tau_h u_k - u_k\|_{L^p(\omega)} \leq \sigma,\ k \in \{0, 1, 2, ..., k_0 - 1\} \tag{11}$$

Taking $\delta' = \min\limits_{k\in\{0,..,k_0-1\}} (\delta_k, \delta)$ and combining (10) and (11) we obtain

$$\forall h \in \mathbb{R}^N, |h| \leq \delta', \forall k \in \mathbb{N} : \|\tau_h u_k - u_k\|_{L^p(\omega)} \leq \sigma.$$

□

3. The perturbed Laplace equation

In this section we will prove **Theorem 1** for the perturbed Laplace equation. We suppose that $A = Id$, and let $u_\epsilon \in W_0^{1,2}(\Omega)$ be the unique solution to

$$\begin{cases} -\epsilon^2 \Delta_{X_1} u_\epsilon - \Delta_{X_2} u_\epsilon = f \\ u_\epsilon \in W_0^{1,2}(\Omega). \end{cases} \tag{12}$$

Notice that the elliptic regularity [7] shows that $u_\epsilon \in W^{2,2}_{loc}(\Omega)$. Now, let $(\epsilon_k)_{k\in\mathbb{N}}$ be a sequence in $(0, 1]$ with $\lim \epsilon_k = 0$, and let $u_k = u_{\epsilon_k}$ be the solution of (12) with ϵ replaced by ϵ_k. then one can prove the following

Proposition 2. *1) Let $\omega \subset\subset \Omega$ open then*

$$\begin{aligned}
&\lim_{h\to 0}\sup_{k\in\mathbb{N}} \left\|\tau_h\nabla^2_{X_2}u_k - \nabla^2_{X_2}u_k\right\|_{L^2(\omega)} = 0,\\
&\lim_{h\to 0}\sup_{k\in\mathbb{N}} \left\|\epsilon_k^2(\tau_h\nabla^2_{X_1}u_k - \nabla^2_{X_1}u_k)\right\|_{L^2(\omega)} = 0,\\
&\lim_{h\to 0}\sup_{k\in\mathbb{N}} \left\|\epsilon_k(\tau_h\nabla^2_{X_1X_2}u_k - \nabla^2_{X_1X_2}u_k)\right\|_{L^2(\omega)} = 0.
\end{aligned}$$

2) The sequences $\left(\nabla^2_{X_2}u_k\right)$, $(\epsilon_k^2\nabla^2_{X_1}u_k)$, $(\epsilon_k\nabla^2_{X_1X_2}u_k)$ are bounded in $L^2_{loc}(\Omega)$ i.e. for every $\omega \subset\subset \Omega$ open there exists $M \geq 0$ such that

$$\sup_{k\in\mathbb{N}}\left\|\epsilon_k^2\nabla^2_{X_1}u_k\right\|_{L^2(\omega)}, \sup_{k\in\mathbb{N}}\left\|\nabla^2_{X_2}u_k\right\|_{L^2(\omega)}, \sup_{k\in\mathbb{N}}\left\|\epsilon_k\nabla^2_{X_1X_2}u_k\right\|_{L^2(\omega)} \leq M.$$

Proof. 1) Let $\omega \subset\subset \Omega$ open, then one can choose ω' open such that $\omega \subset\subset \omega' \subset\subset \Omega$, let $\rho \in \mathcal{D}(\mathbb{R}^N)$ with $\rho = 1$ on ω, $0 \leq \rho \leq 1$ and $Supp(\rho) \subset \omega'$. Let $0 < h < dist(\omega', \partial\Omega)$, to make the notations less heavy we set $U_k^h = \tau_h u_k - u_k$, then $U_k^h \in W^{2,2}(\omega')$. Notice that translation and derivation commute then we have

$$-\epsilon_k^2\Delta_{X_1}U_k^h(x) - \Delta_{X_2}U_k^h(x) = F^h(x), \text{ a.e } x \in \omega',$$

with $F^h = \tau_h f - f$.

We set $\mathcal{W}_k^h = \rho U_k^h$ then we get

$$\begin{aligned}
-\epsilon_k^2\Delta_{X_1}\mathcal{W}_k^h(x) - \Delta_{X_2}\mathcal{W}_k^h(x) &= \rho(x)F^h(x) - 2\epsilon_k^2\nabla_{X_1}\rho(x)\cdot\nabla_{X_1}U_k^h(x)\\
&\quad - 2\nabla_{X_2}\rho(x)\cdot\nabla_{X_2}U_k^h(x) - U_k^h(x)(\epsilon_k^2\Delta_{X_1}\rho(x) - \Delta_{X_2}\rho(x)),
\end{aligned}$$

for a.e $x \in \omega'$.

Since $U_k^h \in W^{2,2}(\omega')$ then $\mathcal{W}_k^h \in W_0^{2,2}(\omega')$, so we can extend $\mathcal{W}_k^h$ by 0 outside of ω' then $\mathcal{W}_k^h \in W^2(\mathbb{R}^N)$. The right hand side of the above equality is extended by 0 outside of ω', hence the equation is satisfied in the whole space, and thus by **Lemma 1** we get

$$\begin{aligned}
\left\|\nabla^2_{X_2}\mathcal{W}_k^h\right\|_{L^2(\mathbb{R}^N)} &\leq \left\|\rho F^h\right\|_{L^2(\mathbb{R}^N)} + 2\epsilon_k^2\left\|\nabla_{X_1}\rho\cdot\nabla_{X_1}U_k^h\right\|_{L^2(\mathbb{R}^N)}\\
&\quad + 2\left\|\nabla_{X_2}\rho\cdot\nabla_{X_2}U_k^h\right\|_{L^2(\mathbb{R}^N)} + \left\|U_k^h(\epsilon_k^2\Delta_{X_1}\rho - \Delta_{X_2}\rho)\right\|_{L^2(\mathbb{R}^N)}.
\end{aligned}$$

Then

$$\begin{aligned}
\left\|\nabla^2_{X_2}U_k^h\right\|_{L^2(\omega)} &\leq \left\|F^h\right\|_{L^2(\omega')} + 2\epsilon_k\left\|\nabla_{X_1}\rho\right\|_\infty\left\|\epsilon_k\nabla_{X_1}U_k^h\right\|_{L^2(\omega')}\\
&\quad + 2\left\|\nabla_{X_2}\rho\right\|_\infty\left\|\nabla_{X_2}U_k^h\right\|_{L^2(\omega')} + \left\|(\epsilon_k^2\Delta_{X_1}\rho - \Delta_{X_2}\rho)\right\|_\infty\left\|U_k^h\right\|_{L^2(\omega')}.
\end{aligned}$$

Notice that by (4) we have $u_k \to u$ in $V^{1,2}$ and $\epsilon_k\nabla_{X_1}u_k \to 0$ in $L^2(\Omega)$, then by **Lemma 2** we deduce

$$\lim_{h\to 0}\sup_{k\in\mathbb{N}}\left\|\epsilon_k\nabla_{X_1}U_k^h\right\|_{L^2(\omega')} = \lim_{h\to 0}\sup_{k\in\mathbb{N}}\left\|\epsilon_k(\tau_h\nabla_{X_1}u_k - \nabla_{X_1}u_k)\right\|_{L^2(\omega')} = 0,$$

and similarly we obtain

$$\begin{aligned}
\lim_{h\to 0}\sup_{k\in\mathbb{N}}\left\|\nabla_{X_2}U_k^h\right\|_{L^2(\omega')} &= 0, \lim_{h\to 0}\sup_{k\in\mathbb{N}}\left\|F^h\right\|_{L^2(\omega')} = 0,\\
\lim_{h\to 0}\sup_{k\in\mathbb{N}}\left\|U_k^h\right\|_{L^2(\omega')} &= 0.
\end{aligned}$$

and hence

$$\lim_{h\to 0}\sup_{k\in\mathbb{N}}\left\|\tau_h\nabla^2_{X_2}u_k - \nabla^2_{X_2}u_k\right\|_{L^2(\omega)} = \lim_{h\to 0}\sup_{k\in\mathbb{N}}\left\|\nabla^2_{X_2}U_k^h\right\|_{L^2(\omega)} = 0.$$

Similarly we obtain

$$\lim_{h\to 0}\sup_{k\in\mathbb{N}}\left\|\epsilon_k^2(\tau_h\nabla^2_{X_1}u_k-\nabla^2_{X_1}u_k)\right\|_{L^2(\omega)}=0,$$

and

$$\lim_{h\to 0}\sup_{k\in\mathbb{N}}\left\|\epsilon_k(\tau_h\nabla^2_{X_1X_2}u_k-\nabla^2_{X_1X_2}u_k)\right\|_{L^2(\omega)}=0.$$

2) Following the same arguments, we get the estimation

$$\begin{aligned}&\epsilon_k^2\left\|\nabla^2_{X_1}u_k\right\|_{L^2(\omega)}+\left\|\nabla^2_{X_2}u_k\right\|_{L^2(\omega)}+\sqrt{2}\epsilon_k\left\|\nabla^2_{X_1X_2}u_k\right\|_{L^2(\omega)}\leq\\&3\left\|f\right\|_{L^2(\omega')}+6\epsilon_k\left\|\nabla_{X_1}\rho\right\|_\infty\left\|\epsilon_k\nabla_{X_1}u_k\right\|_{L^2(\omega')}\\&\qquad+6\left\|\nabla_{X_2}\rho\right\|_\infty\left\|\nabla_{X_2}u_k\right\|_{L^2(\omega')}+3\left\|(\epsilon_k^2\Delta_{X_1}\rho-\Delta_{X_2}\rho)\right\|_\infty\left\|u_k\right\|_{L^2(\omega')}.\end{aligned}$$

The convergences $u_k\to u$ in $V^{1,2}$, $\epsilon_k\nabla_{X_1}u_k\to 0$ in $L^2(\Omega)$ and boundedness of ρ and its derivatives show that the right hand side of the above inequality is uniformly bounded in k, i.e. for some $M\geq 0$ independent of k we have

$$\epsilon_k^2\left\|\nabla^2_{X_1}u_k\right\|_{L^2(\omega)}+\left\|\nabla^2_{X_2}u_k\right\|_{L^2(\omega)}+\sqrt{2}\epsilon_k\left\|\nabla^2_{X_1X_2}u_k\right\|_{L^2(\omega)}\leq M,\ \forall k\in\mathbb{N},$$

and therefore, the sequences $\left(\nabla^2_{X_2}u_k\right)$, $(\epsilon_k^2\nabla^2_{X_1}u_k)$, $(\epsilon_k\nabla^2_{X_1X_2}u_k)$ are bounded in $L^2_{loc}(\Omega)$. □

Now, we are ready to prove the following

Theorem 2. *Let $u_\epsilon\in W^{1,2}_0(\Omega)\cap W^{2,2}_{loc}(\Omega)$ be the solution of (12) then $u_\epsilon\to u_0$ strongly in $V^{2,2}_{loc}$ where $u_0\in V^{2,2}_{loc}$ is the solution of the limit problem. In addition, we have*

$$\epsilon^2\nabla^2_{X_1}u_\epsilon\to 0\ \textit{and}\ \epsilon\nabla^2_{X_1X_2}u_\epsilon\to 0,\ \textit{strongly in}\ L^2_{loc}(\Omega).$$

Proof. Let $u_0\in V^{1,2}$ be the solution of the limit problem and let $(u_k)_{k\in\mathbb{N}}$, $u_k=u_{\epsilon_k}\in W^{1,2}_0(\Omega)\cap W^{2,2}_{loc}(\Omega)$ be a sequence of solutions to (12) with ϵ replaced by ϵ_k. Then **Proposition 2** shows that the hypothesis of the Riesz-Fréchet-Kolmogorov theorem are fulfilled (For the statement of the theorem, see for instance [3]). Whence, it follows that $\left\{\nabla^2_{X_2}u_k\right\}_{k\in\mathbb{N}}$ is relatively compact in $L^2(\omega)$ for every $\omega\subset\subset\Omega$ open. Now, for $\omega\subset\subset\Omega$ fixed there exists $u_0^\omega\in L^2(\omega)$ and a subsequence still labeled $(\nabla^2_{X_2}u_k)_{k\in\mathbb{N}}$ such that $\nabla^2_{X_2}u_k\to u_0^\omega$ in $L^2(\omega)$ strongly. Since $u_k\to u_0$ in $L^2(\omega)$ and the second order differential operators ∂^2_{ij} are continuous on $\mathcal{D}'(\omega)$ then $u_0^\omega=\nabla_{X_2}u_0$ on ω. Whence, since ω is arbitrary we get $\nabla^2_{X_2}u_0\in L^2_{loc}(\Omega)$, i.e. $u_0\in V^{2,2}_{loc}$.

Now, Let (ω_n) be a countable covering of Ω with $\omega_n\subset\subset\Omega$, $\omega_n\subset\omega_{n+1}$,$\forall n\in\mathbb{N}$. Then by the diagonal process one can construct a subsequence still labeled (u_k) such that

$$\nabla^2_{X_2}u_k\to\nabla^2_{X_2}u_0\text{ in }L^2_{loc}(\Omega)\text{ strongly.}$$

Combining this with the convergence $u_k\to u_0$ of (4) we get

$$u_k\to u_0\text{ strongly in }V^{2,2}_{loc},\ \text{i.e. }d(u_k,u_0)\to 0\text{ as }k\to\infty,$$

where d is the distance of the Fréchet space $V^{2,2}_{loc}$.

To prove the convergence of the whole sequence $(u_\epsilon)_{0<\epsilon\leq 1}$ we can reason by contradiction. Suppose that there exists $\delta>0$ and a subsequence (u_k) such that

$d(u_k, u_0) > \delta$. It follows by the first part of this proof that there exists a subsequence still labeled (u_k) such that $d(u_k, u_0) \to 0$, which is a contradiction..

By using the same arguments we can show easily (see the end of subsection 4.1) that

$$\epsilon^2 \nabla^2_{X_1} u_\epsilon \to 0 \text{ and } \epsilon \nabla^2_{X_1 X_2} u_\epsilon \to 0 \text{ strongly in } L^2_{loc}(\Omega).$$

□

4. General elliptic problems

4.1. **Proof of the main theorem.** In this subsection we shall prove **Theorem 1**. Firstly, we suppose that the coefficients of A are constants then we have the following

Proposition 3. *Suppose that the coefficients of A are constants and assume (2), let $(u_\epsilon)_{0<\epsilon\leq 1}$ be a sequence in $W^{2,2}(\mathbb{R}^N)$ such that $-\sum_{i,j} a^\epsilon_{ij} \partial^2_{ij} u_\epsilon = f$, with $f \in L^2(\mathbb{R}^N)$ then we have for every $\epsilon \in (0,1]$:*

$$\begin{aligned} \lambda \left\| \nabla^2_{X_2} u_\epsilon \right\|_{L^2(\mathbb{R}^N)} &\leq \|f\|_{L^2(\mathbb{R}^N)}, \\ \lambda \epsilon^2 \left\| \nabla^2_{X_1} u_\epsilon \right\|_{L^2(\mathbb{R}^N)} &\leq \|f\|_{L^2(\mathbb{R}^N)}, \\ \sqrt{2}\lambda \epsilon \left\| \nabla^2_{X_1 X_2} u \right\|_{L^2(\mathbb{R}^N)} &\leq \|f\|_{L^2(\mathbb{R}^N)}. \end{aligned}$$

Proof. As in proof of **Lemma 1,** we use the Fourier transform and we obtain

$$\left(\sum_{i,j} a^\epsilon_{ij} \xi_i \xi_j \right) \mathcal{F}(u_\epsilon)(\xi) = \mathcal{F}(f)(\xi), \ \xi \in \mathbb{R}^N.$$

From the ellipticity assumption (2) we deduce

$$\lambda^2 \left(\epsilon^2 \sum_{i=}^{q} \xi_i^2 + \sum_{i=q+1}^{N} \xi_i^2 \right)^2 |\mathcal{F}(u_\epsilon)(\xi)|^2 \leq |\mathcal{F}(f)(\xi)|^2.$$

Thus, similarly we obtain the desired bounds. □

Now, suppose that $A \in L^\infty(\Omega) \cap C^1(\Omega)$ and assume (2), and let $u_\epsilon \in W^{1,2}_0(\Omega)$ be the unique weak solution to (1), then it follows by the elliptic regularity that $u_\epsilon \in W^{2,2}_{loc}(\Omega)$. We denote $u_k = u_{\epsilon_k}$the solution to (1) where (ϵ_k) is a sequence in $(0,1]$ such that, $\epsilon_k \to 0$ as $k \to \infty$.

Under the above assumption we can prove the following

Proposition 4. *Let $z_0 \in \Omega$ fixed then there exists $\omega_0 \subset\subset \Omega$ open with $z_0 \in \omega_0$ such that the sequences $\left(\nabla^2_{X_2} u_k\right)$, $\left(\nabla^2_{X_1} u_k\right)$ and $\left(\nabla^2_{X_1 X_2} u_k\right)$ are bounded in $L^2(\omega_0)$.*

Proof. Since $u_k \in W^{1,2}_0(\Omega) \cap W^{2,2}_{loc}(\Omega)$ and $A \in C^1(\Omega)$ then u_k satisfies

$$-\sum_{i,j} a^k_{ij}(x) \partial^2_{ij} u_k(x) - \sum_{i,j} \partial_i a^k_{ij}(x) \partial_j u_k(x) = f(x), \text{ for a.e } x \in \Omega \tag{13}$$

where we have set $a^k_{ij} = a^{\epsilon_k}_{ij}$.

Let $z_0 \in \Omega$ fixed, and let $\theta > 0$ such that

$$\min\left\{\left[\lambda - 3\theta(N-q)\right], \left[\lambda - 3\theta q\right], \left[\sqrt{2}\lambda - 6(N-q)q\theta\right]\right\} \geq \frac{\lambda}{2}. \tag{14}$$

By using the continuity of the a_{ij} one can choose $\omega_1 \subset\subset \Omega$, $z_0 \in \omega_1$ such that

$$\max_{i,j} \sup_{x\in\omega_1} |a_{ij}(x) - a_{ij}(z_0)| \leq \theta \tag{15}$$

Let $\omega_0 \subset\subset \omega_1$ open with $z_0 \in \omega_0$ and let $\rho \in \mathcal{D}(\mathbb{R}^N)$ such $\rho = 1$ on ω_0, $0 \leq \rho \leq 1$ and $Supp(\rho) \subset \omega_1$. We set $U_k = \rho u_k$, and we extend it by 0 on the outside of ω_1 then $U_k \in W^{2,2}(\mathbb{R}^N)$. Therefore from (13) we obtain

$$-\sum_{i,j} a_{ij}^k(z_0)\partial_{ij}^2 U_k(x) = \sum_{i,j}(a_{ij}^k(x) - a_{ij}^k(z_0))\partial_{ij}^2 U_k(x) + g_k(x), \text{ for a.e } x \in \mathbb{R}^N,$$

where g_k is given by

$$\begin{aligned} g_k(x) &= \rho(x)f(x) + \rho(x)\sum_{i,j}\partial_i a_{ij}^k(x)\partial_j u_k(x) \\ &\quad -u_k(x)\sum_{i,j} a_{ij}^k(x)\partial_{ij}^2\rho(x) - \sum_{i,j} a_{ij}^k(x)\partial_i\rho(x)\partial_j u_k(x) - \sum_{i,j} a_{ij}^k(x)\partial_j\rho(x)\partial_i u_k(x), \end{aligned} \tag{16}$$

and we have extended g_k by 0 outside of ω_1.

Now, applying **Proposition 3** to the above differential equality we get

$$\begin{aligned} &\lambda\left\|\nabla_{X_2}^2 U_k\right\|_{L^2(\omega_1)} + \lambda\epsilon_k^2\left\|\nabla_{X_1}^2 U_k\right\|_{L^2(\omega_1)} + \sqrt{2}\lambda\epsilon_k\left\|\nabla_{X_1X_2}^2 U_k\right\|_{L^2(\omega_1)} \\ &\qquad \leq 3\left\|\sum_{i,j}(a_{ij}^k - a_{ij}^k(z_0))\partial_{ij}^2 U_k\right\|_{L^2(\omega_1)} + 3\|g\|_{L^2(\omega_1)} \end{aligned}$$

Whence, by using (15) we get

$$\begin{aligned} &\lambda\left\|\nabla_{X_2}^2 U_k\right\|_{L^2(\omega_1)} + \lambda\epsilon_k^2\left\|\nabla_{X_1}^2 U_k\right\|_{L^2(\omega_1)} + \sqrt{2}\lambda\epsilon_k\left\|\nabla_{X_1X_2}^2 U_k\right\|_{L^2(\omega_1)} \\ &\leq 3\theta\epsilon_k^2\sum_{i,j=1}^{q}\left\|\partial_{ij}^2 U_k\right\|_{L^2(\omega_1)} + 3\theta\sum_{i,j=q+1}^{N}\left\|\partial_{ij}^2 U_k\right\|_{L^2(\omega_1)} \\ &\qquad + 6\theta\epsilon_k\sum_{i=1}^{q}\sum_{j=q+1}^{N}\left\|\partial_{ij}^2 U_k\right\|_{L^2(\omega_1)} + 3\|g\|_{L^2(\omega_1)}, \end{aligned}$$

and thus by the discrete Cauchy-Schwarz inequality we deduce

$$\begin{aligned} &\lambda\left\|\nabla_{X_2}^2 U_k\right\|_{L^2(\omega_1)} + \lambda\epsilon_k^2\left\|\nabla_{X_1}^2 U_k\right\|_{L^2(\omega_1)} + \sqrt{2}\lambda\epsilon_k\left\|\nabla_{X_1X_2}^2 U_k\right\|_{L^2(\omega_1)} \\ &\leq 3\theta(N-q)\left\|\nabla_{X_2}^2 U_k\right\|_{L^2(\omega_1)} + \epsilon_k^2 3\theta q\left\|\nabla_{X_1}^2 U_k\right\|_{L^2(\omega_1)} \\ &\qquad + \epsilon_k 6(N-q)q\theta\left\|\nabla_{X_1X_2}^2 U_k\right\|_{L^2(\omega_1)}^2 + 3\|g\|_{L^2(\omega_1)}, \end{aligned}$$

and thus

$$\begin{aligned} &\left[\lambda - 3\theta(N-q)\right]\left\|\nabla_{X_2}^2 U_k\right\|_{L^2(\omega_1)} + \epsilon_k^2\left[\lambda - 3\theta q\right]\left\|\nabla_{X_1}^2 U_k\right\|_{L^2(\omega_1)} + \\ &\qquad \epsilon_k\left[\sqrt{2}\lambda - 6(N-q)q\theta\right]\left\|\nabla_{X_1X_2}^2 U_k\right\|_{L^2(\omega_1)} \leq 3\|g_k\|_{L^2(\omega_1)}. \end{aligned}$$

Hence, by (14) we get

$$\left\|\nabla^2_{X_2}u_k\right\|_{L^2(\omega_0)}+\epsilon_k^2\left\|\nabla^2_{X_1}u_k\right\|_{L^2(\omega_0)}+\epsilon_k\left\|\nabla^2_{X_1X_2}u_k\right\|_{L^2(\omega_0)}\leq\frac{6}{\lambda}\left\|g_k\right\|_{L^2(\omega_1)}.$$

To complete the proof, we will show the boundedness of (g_k) in $L^2(\omega_1)$. Indeed, ρ and its derivatives, a_{ij} and their first derivatives are bounded on ω_1, moreover (4) shows that the sequences $(\epsilon_k\nabla_{X_1}u_k)$, $(\nabla_{X_2}u_k)$ and (u_k) are bounded in $L^2(\Omega)$, and therefore from (16) the boundedness of (g_k) in $L^2(\omega_1)$ follows. □

Corollary 1. *The sequences* $\left(\nabla^2_{X_2}u_k\right)$, $\left(\epsilon_k^2\nabla^2_{X_1}u_k\right)$, $\left(\epsilon_k\nabla^2_{X_1X_2}u_k\right)$ *are bounded in* $L^2_{loc}(\Omega)$.

Proof. Let $\omega\subset\subset\Omega$ open, for every $z\in\bar{\omega}$ there exists $\omega_z\subset\subset\Omega$, $z\in\omega_z$ which satisfies the affirmations of **Proposition 4** in $L^2(\omega_z)$. By using the compacity of $\bar{\omega}$, one can extract a finite cover (ω_{z_i}), and hence the sequences $\left(\nabla^2_{X_2}u_k\right)$, $\left(\epsilon_k^2\nabla^2_{X_1}u_k\right)$, $\left(\epsilon_k\nabla^2_{X_1X_2}u_k\right)$ are bounded in $L^2(\omega)$. □

Proposition 5. *Let* $z_0\in\Omega$ *then there exists* $\omega_0\subset\subset\Omega$, $z_0\in\omega_0$ *such that*

$$\begin{aligned}
&\lim_{h\to0}\sup_{k\in\mathbb{N}}\left\|\tau_h\nabla^2_{X_2}u_k-\nabla^2_{X_2}u_k\right\|_{L^p(\omega_0)}=0,\\
&\lim_{h\to0}\sup_{k\in\mathbb{N}}\left\|\epsilon_k^2(\tau_h\nabla^2_{X_1}u_k-\nabla^2_{X_1}u_k)\right\|_{L^p(\omega_0)}=0,\\
&\lim_{h\to0}\sup_{k\in\mathbb{N}}\left\|\epsilon_k(\tau_h\nabla^2_{X_1X_2}u_k-\nabla^2_{X_1X_2}u_k)\right\|_{L^p(\omega_0)}=0.
\end{aligned}$$

Proof. Let $z_0\in\Omega$ fixed and let $\theta>0$ then using the continuity of the a_{ij} one can choose $\omega_1\subset\subset\Omega$, $z_0\in\omega_1$ such that we have (15) with θ is chosen as in (14). Let $\omega_0\subset\subset\omega_1$, with $z_0\in\omega_0$, and let $\rho\in\mathcal{D}(\mathbb{R}^N)$ with $\rho=1$ on ω_0, $0\leq\rho\leq1$, and $Supp(\rho)\subset\omega_1$. Let $0<h<dist(\omega_1,\partial\Omega)$, we set $\mathcal{W}_k^h=\rho U_k^h$, with $U_k^h=(\tau_hu_k-u_k)$ and extend it by 0 on the outside of ω_1 then $\mathcal{W}_k^h\in W^{2,2}(\mathbb{R}^N)$, therefore using (13) we have:

$$-\sum_{i,j}a_{ij}^k(z_0)\partial^2_{ij}\mathcal{W}_k^h(x)=\sum_{i,j}(a_{ij}^k(x)-a_{ij}^k(z_0))\partial^2_{ij}\mathcal{W}_k^h(x)+G_k^h(x),\ \text{a.e}\ x\in\mathbb{R}^N$$

where

$$\begin{aligned}
-G_k^h(x) &= U_k^h\sum_{i,j}a_{ij}^k(x)\partial^2_{ij}\rho+\sum_{i,j}a_{ij}^k(x)\partial_i\rho\partial_jU_k^h+\sum_{i,j}a_{ij}^k(x)\partial_j\rho\partial_iU_k^h \qquad (17)\\
&\quad+\rho\sum_{i,j}\left(a_{ij}^k(x)-\tau_ha_{ij}^k(x)\right)\tau_h\partial^2_{ij}u_k(x)+\rho(x)\left(f(x)-\tau_hf(x)\right)\\
&\quad+\rho\sum_{i,j}\left[\partial_ia_{ij}^k(x)\partial_ju_k(x)-\partial_i\tau_ha_{ij}^k(x)\partial_j\tau_hu_k(x)\right],
\end{aligned}$$

and G_k^h is extended by 0 outside of ω_1.

Then, as in proof of **Proposition 4**, we obtain

$$\begin{aligned}
&\left\|\tau_h\nabla^2_{X_2}u_k-\nabla^2_{X_2}u_k\right\|_{L^2(\omega_0)}+\epsilon_k^2\left\|\tau_h\nabla^2_{X_1}u_k-\nabla^2_{X_1}u_k\right\|_{L^2(\omega_0)}\\
&\qquad+\epsilon_k\left\|\tau_h\nabla^2_{X_1X_2}u_k-\nabla^2_{X_1X_2}u_k\right\|_{L^2(\omega_0)}\leq\frac{6}{\lambda}\left\|G_k^h\right\|_{L^2(\omega_1)}.
\end{aligned}$$

To complete the proof, we have to show that $\lim_{h\to0}\sup_{k\in\mathbb{N}}\left\|G_k^h\right\|_{L^2(\omega_1)}=0$.

Using the boundedness of the a_{ij} and the boundedness of ρ and its derivatives on ω_1 we get from (17)

$$\begin{aligned}\left\|G_k^h\right\|_{L^2(\omega_1)} &\leq M\left\|U_k^h\right\|_{L^2(\omega_1)} + M\epsilon_k\left\|\nabla_{X_1}U_k^h\right\|_{L^2(\omega_1)} \\ &+M\left\|\nabla_{X_2}U_k^h\right\|_{L^2(\omega_1)} + \|\tau_h f - f\|_{L^2(\omega_1)} \\ &+\sum_{i,j}\left\|\left(a_{ij}^k - \tau_h a_{ij}^k\right)\tau_h\partial_{ij}^2 u_k\right\|_{L^2(\omega_1)} \\ &+\sum_{i,j}\left\|\partial_i a_{ij}^k\partial_j u_k - \tau_h\partial_i a_{ij}^k\tau_h\partial_j u_k\right\|_{L^2(\omega_1)},\end{aligned} \tag{18}$$

where $M \geq 0$ is independent of h and k. Now, estimating the fifth term of the right hand side of the above inequality

$$\sum_{i,j}\left\|\left(a_{ij}^k - \tau_h a_{ij}^k\right)\tau_h\partial_{ij}^2 u_k\right\|_{L^2(\omega_1)} \leq C_{q,N}\max_{i,j}\sup_{x\in\omega_1}|a_{ij}(x) - \tau_h a_{ij}(x)| \times$$
$$\left(\left\|\nabla_{X_2}^2 u_k\right\|_{L^2(\omega_1+h)} + \epsilon_k^2\left\|\nabla_{X_1}^2 u_k\right\|_{L^2(\omega_1+h)} + \epsilon_k\left\|\nabla_{X_1X_2}^2 u_k\right\|_{L^2(\omega_1+h)}\right),$$

where $C_{q,N} > 0$ is only depends in q and N.

Let $\delta > 0$ small enough such that for every $|h| \leq \delta$ we have $\omega_1 + h \subset\subset \Omega$. Then it follows by **Corollary 1,** applied on $\omega_1 + h$, that the quantity

$$\left\|\nabla_{X_2}^2 u_k\right\|_{L^2(\omega_1+h)} + \epsilon_k^2\left\|\nabla_{X_1}^2 u_k\right\|_{L^2(\omega_1+h)} + \epsilon_k\left\|\nabla_{X_1X_2}^2 u_k\right\|_{L^2(\omega_1+h)}$$

is uniformly bounded in k and h (for $|h| \leq \delta$). Since the a_{ij} are uniformly continuous on every $\omega \subset\subset \Omega$ open then

$$\lim_{h\to 0}\max_{i,j}\sup_{x\in\omega_1}|a_{ij}(x) - \tau_h a_{ij}(x)| = 0,$$

and hence

$$\lim_{h\to 0}\sup_{k\in\mathbb{N}}\sum_{i,j}\left\|\left(a_{ij}^k - \tau_h a_{ij}^k\right)\tau_h\partial_{ij}^2 u_k\right\|_{L^2(\omega_1)} = 0. \tag{19}$$

Now, estimating the last term of (18). By the triangular inequality we obtain

$$\begin{aligned}\sum_{i,j}\left\|\partial_i a_{ij}^k\partial_j u_k - \tau_h\partial_i a_{ij}^k\tau_h\partial_j u_k\right\|_{L^2(\omega_1)} &\leq \sum_{i,j}\left\|\partial_i a_{ij}^k\partial_j u_k - \tau_h\partial_i a_{ij}^k\partial_j u_k\right\|_{L^2(\omega_1)} \\ &+\sum_{i,j}\left\|\tau_h\partial_i a_{ij}^k\partial_j u_k - \partial_i\tau_h a_{ij}^k\tau_h\partial_j u_k\right\|_{L^2(\omega_1)},\end{aligned}$$

and thus, by using the boundedness of the first derivatives of the a_{ij} on ω_1 we get

$$\begin{aligned}&\sum_{i,j}\left\|\partial_i a_{ij}^k\partial_j u_k - \partial_i\tau_h a_{ij}^k\partial_j\tau_h u_k\right\|_{L^2(\omega_1)} \\ &\leq C'_{q,N}\max_{i,j}\sup_{x\in\omega_1}|\partial_i a_{ij}(x) - \partial_i\tau_h a_{ij}(x)|\left(\epsilon_k\left\|\nabla_{X_1}u_k\right\|_{L^2(\omega_1)} + \left\|\nabla_{X_2}u_k\right\|_{L^2(\omega_1)}\right) \\ &+M'\left(\epsilon_k\left\|\nabla_{X_1}U_k^h\right\|_{L^2(\omega_1)} + \left\|\nabla_{X_2}U_k^h\right\|_{L^2(\omega_1)}\right),\end{aligned}$$

where $M' \geq 0$ and $C'_{q,N} > 0$ are independent of h and k. Now, since the $\partial_i a_{ij}$ are uniformly continuous (recall that $A \in C^1(\Omega)$) on every $\omega \subset\subset \Omega$ then

$$\lim_{h\to 0}\max_{i,j}\sup_{x\in\omega_1}|\partial_i a_{ij}(x) - \tau_h\partial_i a_{ij}(x)| = 0,$$

and therefore, from the above inequality we get

$$\lim_{h\to 0}\sup_{k\in\mathbb{N}}\sum_{i,j}\left\|\partial_i a_{ij}^k\partial_j u_k-\partial_i\tau_h a_{ij}^k\partial_j\tau_h u_k\right\|_{L^2(\omega_1)}=0, \tag{20}$$

where we have used (4) and **Lemma 2.**

Passing to the limit in (18) by using (19), (20) and (4) with **Lemma 2** we deduce

$$\lim_{h\to 0}\sup_{k\in\mathbb{N}}\left\|G_k^h\right\|_{L^2(\omega_1)}=0.$$

and the proposition follows. □

Corollary 2. *For every $\omega\subset\subset\Omega$ open we have*

$$\begin{aligned}
&\lim_{h\to 0}\sup_{k\in\mathbb{N}}\left\|\tau_h\nabla_{X_2}^2u_k-\nabla_{X_2}^2u_k\right\|_{L^p(\omega)}=0,\\
&\lim_{h\to 0}\sup_{k\in\mathbb{N}}\left\|\epsilon_k^2(\tau_h\nabla_{X_1}^2u_k-\nabla_{X_1}^2u_k)\right\|_{L^p(\omega)}=0,\\
&\lim_{h\to 0}\sup_{k\in\mathbb{N}}\left\|\epsilon_k(\tau_h\nabla_{X_1X_2}^2u_k-\nabla_{X_1X_2}^2u_k)\right\|_{L^p(\omega)}=0.
\end{aligned}$$

Proof. Similar to proof of **Corollary 1**,where we use the compacity of $\bar{\omega}$ and **Proposition 5**. □

Now, we are able to give the proof of the main theorem. Indeed it is similar to proof of **Theorem 2,** where we will use **Corollary 1** and **Corollary 2.** Let us prove the convergence

$$\epsilon^2\nabla_{X_1}^2u_\epsilon\to 0 \text{ in } L^2_{loc}(\Omega).$$

Fix $\omega\subset\subset\Omega$ open, and let $u_k\in W_0^{1,2}(\Omega)\cap W_{loc}^{2,2}(\Omega)$ be a sequence of solutions of (1), then it follows from **Corollary 1** and **2** that the subset $\left\{\epsilon_k^2\nabla_{X_1}^2u_k\right\}_{k\in\mathbb{N}}$ is relatively compact in $L^2(\omega)$ then there exists $v^\omega\in L^2(\omega)$ and a subsequence still labeled $(\epsilon_k^2\nabla_{X_1}^2u_k)$ such that

$$\epsilon_k^2\nabla_{X_1}^2u_k\to v^\omega \text{ in } L^2(\omega),$$

and since $\epsilon_k^2u_k\to 0$ in $L^2(\omega)$ then $v^\omega=0$ (we used the continuity of $\nabla_{X_1}^2$ on $\mathcal{D}'(\omega)$). Hence by the diagonal process one can construct a sequence still labeled $(\epsilon_k^2\nabla_{X_1}^2u_k)$ such that

$$\epsilon_k^2\nabla_{X_1}^2u_k\to 0 \text{ in } L^2_{loc}(\Omega).$$

To prove the convergence for the whole sequence $(\epsilon^2\nabla_{X_1}^2u_\epsilon)_{0<\epsilon\leq 1}$, we can reason by contradiction (recall that $L^2_{loc}(\Omega)$ equipped with the family of semi norms $(\|\cdot\|_{L^2(\omega)})_{\omega\subset\subset\Omega}$ is a Fréchet space), and the proof of the main theorem is finished.

4.2. **A convergence result for some class of semilinear problem.** In this section we deal with the following semilinear elliptic problem

$$\begin{cases} -\operatorname{div}(A_\epsilon\nabla u_\epsilon)=a(u_\epsilon)+f\\ \qquad u_\epsilon\in W_0^{1,2}(\Omega)\end{cases}, \tag{21}$$

where $a:\mathbb{R}\to\mathbb{R}$ a continuous nonincreasing real valued function which satisfies the growth condition

$$\forall x\in\mathbb{R}:|a(x)|\leq c\left(1+|x|\right), \tag{22}$$

for some $c \geq 0$. This problem has been treated in [6] for $f \in L^p(\Omega)$, $1 < p \leq 2$, and the author have proved the convergences

$$\epsilon \nabla_{X_1} u_\epsilon \to 0,\ u_\epsilon \to u_0,\ \nabla_{X_2} u_\epsilon \to \nabla_{X_2} u_0 \text{ in } L^p(\Omega), \tag{23}$$

where u_0 is the solution of the limit problem.

Let $f \in L^2(\Omega)$ and assume A as in **Theorem 1** then the unique $W_0^{1,2}(\Omega)$ weak solution u_ϵ to (21) belongs to $W^{2,2}_{loc}(\Omega)$. Following the same arguments exposed in the above subsection one can prove the theorem

Theorem 3. *Under the above assumptions we have $u_\epsilon \to u_0$ in $V^{2,2}_{loc}$, $\epsilon^2 \nabla^2_{X_1} u_\epsilon \to 0$ and $\epsilon \nabla^2_{X_1 X_2} u_\epsilon \to 0$ strongly in $L^2_{loc}(\Omega)$.*

Proof. The arguments are similar, we only give the proof for the Laplacian case, so assume that $A = Id$.

Let $\omega \subset\subset \Omega$ open, then one can choose ω' open such that $\omega \subset\subset \omega' \subset\subset \Omega$, let $\rho \in \mathcal{D}(\mathbb{R}^N)$ with $\rho = 1$ on ω, $0 \leq \rho \leq 1$ and $Supp(\rho) \subset \omega'$. Let $0 < h < dist(\partial\omega', \Omega)$, we use the same notations of the above subsection, we set $U_k^h = \tau_h u_k - u_k$, then $U_k^h \in W^{1,2}(\omega')$ and we have

$$-\epsilon_k^2 \Delta_{X_1} U_k^h(x) - \Delta_{X_2} U_k^h(x) = F^h(x) + \tau_h a(u)(x) - a(u)(x), \ \text{ a.e } x \in \omega',$$

with $F^h = \tau_h f - f$. We set $\mathcal{W}_k^h = \rho U_k^h$ then we get as in **Proposition 2**

$$\begin{aligned} \left\|\tau_h \nabla^2_{X_2} u_k - \nabla^2_{X_2} u_k\right\|_{L^2(\omega)} &\leq \left\|F^h\right\|_{L^2(\omega')} + M \left\|\epsilon_k \nabla_{X_1} U_k^h\right\|_{L^2(\omega')} \\ &+ \left\|\tau_h a(u_k) - a(u_k)\right\|_{L^2(\omega')} \\ &+ M \left\|\nabla_{X_2} U_k^h\right\|_{L^2(\omega')} + M \left\|U_k^h\right\|_{L^2(\omega')}. \end{aligned}$$

We can prove easily, by using the continuity of the function a and (22), that the Nemytskii operator a maps continuously $L^2(\Omega)$ to $L^2(\Omega)$. Therefore, the convergence $u_k \to u_0$ in $L^2(\Omega)$ gives $a(u_k) \to a(u_0)$ in $L^2(\Omega)$, and hence **Lemma 2** gives

$$\lim_{h \to 0} \sup_{k \in \mathbb{N}} \left\|\tau_h a(u_k) - a(u_k)\right\|_{L^2(\omega)} = 0,$$

and finally the convergences (23) give

$$\lim_{h \to 0} \sup_{k \in \mathbb{N}} \left\|\tau_h \nabla^2_{X_2} u_k - \nabla^2_{X_2} u_k\right\|_{L^2(\omega)} = 0.$$

Similarly, using boundedness of the sequences (u_k), $(\epsilon_k \nabla_{X_1} u_k)$, $(\nabla_{X_2} u_k)$and $a(u_k)$ in $L^2(\Omega)$, and boundedness of ρ and its derivatives we get

$$\left\|\nabla^2_{X_2} u_k\right\|_{L^2(\omega)} \leq M',$$

and we conclude as in proof of **Theorem 2.** □

We complete this paper by giving an open question

Problem 1. *Let $f \in L^p(\Omega)$ with $1 < p < 2$, and consider problem (1). In* [6] *the author have proved the convergence $u_\epsilon \to u_0$ in the Banach space $V^{1,p}$ defined by*

$$V^{1,p} = \left\{ u \in L^p(\Omega) \ \mid \nabla_{X_2} u \in L^p(\Omega) \text{ and } u(X_1, \cdot) \in W_0^{1,p}(\Omega_{X_1}) \text{ a.e } X_1 \in \Omega^1 \right\},$$

equipped with the norm

$$\|u\|_{1,p} = \left(\|u\|^p_{L^p(\Omega)} + \|\nabla_{X_2} u\|^p_{L^p(\Omega)} \right)^{\frac{1}{p}}.$$

Similarly we introduce the Fréchet space

$$V_{loc}^{2,p} = \left\{ u \in V^{1,p} \ \mid \nabla_{X_2}^2 u \in L^p(\Omega) \right\},$$

equipped with family of norms

$$\|u\|_{2,p}^{\omega} = \left(\|u\|_{L^p(\Omega)}^p + \|\nabla_{X_2} u\|_{L^p(\Omega)}^p + \left\|\nabla_{X_2}^2 u\right\|_{L^p(\omega)}^p \right)^{\frac{1}{p}}, \ \omega \subset\subset \Omega \text{ open.}$$

Can one prove that $u_\epsilon \to u_0$ *in* $V_{loc}^{2,p}$ *?*

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