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$W^{2,2}$ INTERIOR REGULARITY FOR SOME CLASS OF ELLIPTIC ANISOTROPIC SINGULAR PERTUBATIONS PROBLEMS

CHOKRI OGABI

ABSTRACT. In this note we study the interior regularity of the asymptotic behavior of a linear elliptic anisotropic singular pertubations problem. We show the convergence of the seconde derivatives.

1. INTRODUCTION

In this note we study the regularity of the asymptotic behavior of the following elliptic problem

$$\begin{cases} -\operatorname{div}(A_\epsilon \nabla u_\epsilon) = f \\ u_\epsilon = 0 \end{cases} \quad \text{on } \partial\Omega, \quad (1)$$

where $0 < \epsilon \leq 1$, Ω is bounded Lipschitz domain and $f \in L^2(\Omega)$. we denote by $x = (x_1, \dots, x_N) = (X_1, X_2)$ the points in \mathbb{R}^N where

$$X_1 = (x_1, \dots, x_q) \text{ and } X_2 = (x_{q+1}, \dots, x_N),$$

with this notation we set

$$\nabla = (\partial_{x_1}, \dots, \partial_{x_N})^T = \begin{pmatrix} \nabla_{X_1} \\ \nabla_{X_2} \end{pmatrix},$$

where

$$\nabla_{X_1} = (\partial_{x_1}, \dots, \partial_{x_q})^T \text{ and } \nabla_{X_2} = (\partial_{x_{q+1}}, \dots, \partial_{x_N})^T$$

The diffusion matrix A_ϵ is given by

$$A_\epsilon = (a_{ij}^\epsilon) = \begin{pmatrix} \epsilon^2 A_{11} & \epsilon A_{12} \\ \epsilon A_{21} & A_{22} \end{pmatrix} \text{ with } A = (a_{ij}) = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix},$$

where A_{11} and A_{22} are $q \times q$ and $(N - q) \times (N - q)$ matrices. The coefficients a_{ij}^ϵ are given by

$$a_{ij}^\epsilon = \begin{cases} \epsilon^2 a_{ij} & \text{for } i, j \in \{1, \dots, q\} \\ a_{ij} & \text{for } i, j \in \{q + 1, \dots, N\} \\ \epsilon a_{ij} & \text{for } i \in \{1, \dots, q\}, j \in \{q + 1, \dots, N\} \\ \epsilon a_{ij}^\epsilon & \text{for } i \in \{q + 1, \dots, N\}, j \in \{1, \dots, q\} \end{cases}$$

We assume $A \in C^1(\Omega) \cap L^\infty(\Omega)$ and satisfies the ellipticity assumption

$$\exists \lambda > 0, \forall x \in \Omega, \forall \zeta \in \mathbb{R}^N : A(x)\zeta \cdot \zeta \geq \lambda |\zeta|^2. \quad (2)$$

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Recall the first order pseudo Sobolev space introduced in [2]

$$V^{1,2} = \left\{ u \in L^2(\Omega) \mid \nabla_{X_2} u \in L^2(\Omega) \text{ and } u(X_1, \cdot) \in W_0^{1,2}(\Omega_{X_1}) \text{ a.e } X_1 \in \Omega^1 \right\},$$

equipped with the norm

$$\|u\|_{1,2} = \left(\|u\|_{L^2(\Omega)}^2 + \|\nabla_{X_2} u\|_{L^2(\Omega)}^2 \right)^{\frac{1}{2}}.$$

Here $\Omega_{X_1} = \{X_2 \in \mathbb{R}^{N-q} : (X_1, X_2) \in \Omega\}$ and $\Omega^1 = P_1(\Omega)$ where P_1 is the natural projector $\mathbb{R}^N \rightarrow \mathbb{R}^q$.

We introduce the second order local pseudo Sobolev space

$$V_{loc}^{2,2} = \left\{ u \in V^{1,2} \mid \nabla_{X_2}^2 u \in L_{loc}^2(\Omega) \right\},$$

equipped with the family of norms $\|\cdot\|_{2,2}^\omega$ given by

$$\|u\|_{2,2}^\omega = \left(\|u\|_{L^2(\Omega)}^2 + \|\nabla_{X_2} u\|_{L^2(\Omega)}^2 + \|\nabla_{X_2}^2 u\|_{L^2(\omega)}^2 \right)^{\frac{1}{2}}, \quad \omega \subset\subset \Omega$$

where $\nabla_{X_2}^2 u$ is the Hessian matrix of u taken in the X_2 direction, the term $\|\nabla_{X_2}^2 u\|_{L^2(\omega)}^2$ is given by

$$\|\nabla_{X_2}^2 u\|_{L^2(\omega)}^2 = \sum_{i,j=q+1}^N \|\partial_{ij}^2 u\|_{L^2(\omega)}^2.$$

We can show easily that $V_{loc}^{2,2}$ is a Fréchet space (locally convex, metrizable and complete). We also define the following

$$\|\nabla_{X_1}^2 u\|_{L^2(\omega)}^2 = \sum_{i,j=1}^q \|\partial_{ij}^2 u\|_{L^2(\omega)}^2,$$

and

$$\|\nabla_{X_1 X_2}^2 u\|_{L^2(\omega)}^2 = \sum_{i=1}^q \sum_{j=q+1}^N \|\partial_{ij}^2 u\|_{L^2(\omega)}^2.$$

The formal passage to the limit gives the unperturbed problem

$$\begin{cases} -\operatorname{div}(A_{22} \nabla u_0(X_1, \cdot)) = f(X_1, \cdot) \\ u_0(X_1, \cdot) = 0 \end{cases} \quad \text{on } \partial\Omega_{X_1}, \text{ a.e } X_1 \in \Omega^1 \quad (3)$$

Since $A \in C^1(\Omega)$ which satisfies (2) then it is well known by the elliptic regularity [5] that the unique $W_0^{1,2}(\Omega)$ weak solution of (1) belongs to $W_{loc}^{2,2}(\Omega)$. Similarly the unique $W_0^{1,2}(\Omega_{X_1})$ weak solution $u_0(X_1, \cdot)$ of (3) belongs to $W_{loc}^{2,2}(\Omega_{X_1})$.

In [1] the authors have proved the convergences $u_\epsilon \rightarrow u_0$ in $V^{1,2}$ and $\epsilon \nabla_{X_1} u_\epsilon \rightarrow 0$ in $L^2(\Omega)$ (see [4] for the L^p case). In this paper we deal with the convergences in the space $V_{loc}^{2,2}$ introduced above. Let us give our main result.

Theorem 1. *Assume that $A \in L^\infty(\Omega) \cap C^1(\Omega)$ which satisfies (2), suppose that $f \in L^2(\Omega)$ then $u_0 \in V_{loc}^{2,2}$ and $u_\epsilon \rightarrow u_0$ in $V_{loc}^{2,2}$ where $u_\epsilon \in W_0^{1,2}(\Omega) \cap W_{loc}^{2,2}(\Omega)$ and u_0 are the unique weak solutions to (1) and (3) respectively, in addition the convergences $\epsilon^2 \nabla_{X_1}^2 u_\epsilon \rightarrow 0$, $\epsilon \nabla_{X_1 X_2}^2 u_\epsilon \rightarrow 0$ hold in $L_{loc}^2(\Omega)$.*

The arguments are based on the Riesz-Fréchet-Kolmogorov compactness theorem in L^p spaces. We begin by a basic case, the Laplace equation, and we give the proof in the general case using standard elliptic equations techniques.

2. THE PERTURBED LAPLACE EQUATION

Let us begin with some useful lemmas

Lemma 1. *Let $f \in L^2(\mathbb{R}^N)$, for every $\epsilon \in (0, 1]$ let $u_\epsilon \in W^{2,2}(\mathbb{R}^N)$ such that*

$$-\epsilon^2 \Delta_{X_1} u_\epsilon(x) - \Delta_{X_2} u_\epsilon(x) = f(x) \text{ a.e } x \in \mathbb{R}^N \quad (4)$$

then for every $\epsilon \in (0, 1]$ we have the bounds

$$\begin{aligned} \|\nabla_{X_2}^2 u_\epsilon\|_{L^2(\mathbb{R}^N)} &\leq \|f\|_{L^2(\mathbb{R}^N)}, \\ \epsilon^2 \|\nabla_{X_1}^2 u_\epsilon\|_{L^2(\mathbb{R}^N)} &\leq \|f\|_{L^2(\mathbb{R}^N)}, \\ \sqrt{2}\epsilon \|\nabla_{X_1 X_2}^2 u_\epsilon\|_{L^2(\mathbb{R}^N)} &\leq \|f\|_{L^2(\mathbb{R}^N)}. \end{aligned}$$

Proof. Let \mathcal{F} be the Fourier transform on $L^2(\mathbb{R}^N)$ defined as the extension, by density, of the Fourier transform defined on the Schwartz space $\mathcal{S}(\mathbb{R}^N)$ by

$$\mathcal{F}(u)(\xi) = (2\pi)^{-\frac{N}{2}} \int_{\mathbb{R}^N} u(x) e^{-ix \cdot \xi} dx, \quad u \in \mathcal{S}(\mathbb{R}^N)$$

where \cdot is the standard scalare product of \mathbb{R}^N . Applying \mathcal{F} on (4) we obtain

$$\left(\epsilon^2 \sum_{i=1}^q \xi_i^2 + \sum_{i=q+1}^N \xi_i^2 \right) \mathcal{F}(u_\epsilon)(\xi) = \mathcal{F}(f)(\xi),$$

then

$$\left(\epsilon^4 \sum_{i,j=1}^q \xi_i^2 \xi_j^2 + \sum_{i,j=q+1}^N \xi_i^2 \xi_j^2 + 2\epsilon^2 \sum_{j=q+1}^N \sum_{i=1}^q \xi_i^2 \xi_j^2 \right) |\mathcal{F}(u_\epsilon)(\xi)|^2 = |\mathcal{F}(f)(\xi)|^2, \quad (5)$$

thus

$$\sum_{i,j=q+1}^N \xi_i^2 \xi_j^2 |\mathcal{F}(u_\epsilon)(\xi)|^2 \leq |\mathcal{F}(f)(\xi)|^2,$$

hence

$$\sum_{i,j=q+1}^N |\mathcal{F}(\partial_{ij}^2 u_\epsilon)(\xi)|^2 \leq |\mathcal{F}(f)(\xi)|^2,$$

then

$$\sum_{i,j=q+1}^N \|\mathcal{F}(\partial_{ij}^2 u_\epsilon)\|_{L^2(\mathbb{R}^N)}^2 \leq \|\mathcal{F}(f)\|_{L^2(\mathbb{R}^N)}^2,$$

and the Parseval identity gives

$$\sum_{i,j=q+1}^N \|\partial_{ij}^2 u_\epsilon\|_{L^2(\mathbb{R}^N)}^2 \leq \|f\|_{L^2(\mathbb{R}^N)}^2.$$

Hence

$$\|\nabla_{X_2}^2 u_\epsilon\|_{L^2(\mathbb{R}^N)} \leq \|f\|_{L^2(\mathbb{R}^N)}.$$

Similarly we obtain from (5) the bounds

$$\epsilon^2 \|\nabla_{X_1}^2 u_\epsilon\|_{L^2(\mathbb{R}^N)} \leq \|f\|_{L^2(\mathbb{R}^N)},$$

$$\sqrt{2}\epsilon \|\nabla_{X_1 X_2}^2 u_\epsilon\|_{L^2(\mathbb{R}^N)} \leq \|f\|_{L^2(\mathbb{R}^N)}.$$

□

Notation 1. For any function $u \in L^p(\mathbb{R}^N)$ and $h \in \mathbb{R}^N$ we denote $\tau_h u(x) = u(x+h)$, $x \in \mathbb{R}^N$.

Lemma 2. Let Ω be an open bounded subset of \mathbb{R}^N and let $(u_k)_{k \in \mathbb{N}}$ be a converging sequence in $L^p(\Omega)$, $1 \leq p < \infty$ and let $\omega \subset\subset \Omega$ open, then for every $\sigma > 0$ there exists $0 < \delta < \text{dist}(\partial\Omega, \omega)$ such that

$$\forall h \in \mathbb{R}^N, |h| \leq \delta, \forall k \in \mathbb{N} : \|\tau_h u_k - u_k\|_{L^p(\omega)} \leq \sigma$$

in other words we have $\lim_{h \rightarrow 0} \sup_{k \in \mathbb{N}} \|\tau_h u_k - u_k\|_{L^p(\omega)} = 0$.

Proof. Let $\omega \subset\subset \Omega$ open. For any function $v \in L^p(\Omega)$, extend v by 0 outside of Ω , since the translation $h \rightarrow \tau_h v$ is continuous from \mathbb{R}^N to $L^p(\mathbb{R}^N)$ (see for instance [6]) then for every $\sigma > 0$ there exists $0 < \delta < \text{dist}(\partial\Omega, \omega)$ such that

$$\forall h \in \mathbb{R}^N, |h| \leq \delta : \|\tau_h v - v\|_{L^p(\omega)} \leq \sigma. \quad (6)$$

We denote $\lim u_k = u \in L^p(\Omega)$, and let $\sigma > 0$ then (6) shows that there exists $0 < \delta < \text{dist}(\partial\Omega, \omega)$ such that

$$\forall h \in \mathbb{R}^N, |h| \leq \delta : \|\tau_h u - u\|_{L^p(\omega)} \leq \frac{\sigma}{2}.$$

By the triangular inequality and the invariance of the Lebesgue measure under translations we have for every $k \in \mathbb{N}$ and $|h| \leq \delta$

$$\|\tau_h u_k - u_k\|_{L^p(\omega)} \leq 2 \|u_k - u\|_{L^p(\Omega)} + \|\tau_h u - u\|_{L^p(\omega)} \quad (7)$$

Since $u_k \rightarrow u$ in $L^p(\Omega)$ then there exists $k_0 \in \mathbb{N}$, such that

$$\forall k \geq k_0 : \|u_k - u\|_{L^p(\Omega)} \leq \frac{\sigma}{4}.$$

Then from (7) we obtain

$$\forall h \in \mathbb{R}^N, |h| \leq \delta, \forall k \geq k_0 : \|\tau_h u_k - u_k\|_{L^p(\omega)} \leq \sigma \quad (8)$$

Similarly (6) shows that for every $k \in \{0, 1, 2, \dots, k_0 - 1\}$ there exists $0 < \delta_k < \text{dist}(\partial\Omega, \omega)$ such that

$$\forall h \in \mathbb{R}^N, |h| \leq \delta_k : \|\tau_h u_k - u_k\|_{L^p(\omega)} \leq \sigma, \quad k \in \{0, 1, 2, \dots, k_0 - 1\} \quad (9)$$

Taking $\delta' = \min_{k \in \{0, \dots, k_0 - 1\}} (\delta_k, \delta)$ and combining (8) and (9) we obtain

$$\forall h \in \mathbb{R}^N, |h| \leq \delta', \forall k \in \mathbb{N} : \|\tau_h u_k - u_k\|_{L^p(\omega)} \leq \sigma.$$

□

Now, suppose that $A = Id$ be the identity matrix then (1) becomes

$$\begin{cases} -\epsilon^2 \Delta_{X_1} u_\epsilon - \Delta_{X_2} u_\epsilon = f \\ u_\epsilon = 0 \end{cases} \quad \text{on } \partial\Omega, \quad (10)$$

Let $(\epsilon_k)_{k \in \mathbb{N}}$ be sequence in $(0, 1]$ with $\lim \epsilon_k = 0$ and let $u_k = u_{\epsilon_k} \in W_0^{1,2}(\Omega) \cap W_{loc}^{2,2}(\Omega)$ be the solution of (10) then we have the following

Proposition 1. 1) Let $\omega \subset\subset \Omega$ open then

$$\begin{aligned} \limsup_{h \rightarrow 0} \sup_{k \in \mathbb{N}} \|\tau_h \nabla_{X_2}^2 u_k - \nabla_{X_2}^2 u_k\|_{L^2(\omega)} &= 0, \\ \limsup_{h \rightarrow 0} \sup_{k \in \mathbb{N}} \|\epsilon_k^2 (\tau_h \nabla_{X_1}^2 u_k - \nabla_{X_1}^2 u_k)\|_{L^2(\omega)} &= 0, \\ \limsup_{h \rightarrow 0} \sup_{k \in \mathbb{N}} \|\epsilon_k (\tau_h \nabla_{X_1 X_2}^2 u_k - \nabla_{X_1 X_2}^2 u_k)\|_{L^2(\omega)} &= 0. \end{aligned}$$

2) The sequences $(\nabla_{X_2}^2 u_k)$, $(\epsilon_k^2 \nabla_{X_1}^2 u_k)$, $(\epsilon_k \nabla_{X_1 X_2}^2 u_k)$ are bounded in $L_{loc}^2(\Omega)$ i.e. for every $\omega \subset\subset \Omega$ there exists $M \geq 0$ such that

$$\sup_{k \in \mathbb{N}} \|\epsilon_k^2 \nabla_{X_1}^2 u_k\|_{L^2(\omega)}, \sup_{k \in \mathbb{N}} \|\nabla_{X_2}^2 u_k\|_{L^2(\omega)}, \sup_{k \in \mathbb{N}} \|\epsilon_k \nabla_{X_1 X_2}^2 u_k\|_{L^2(\omega)} \leq M.$$

Proof. 1) Let $\omega \subset\subset \Omega$ open, then one can choose ω' open such that $\omega \subset\subset \omega' \subset\subset \Omega$, let $\rho \in \mathcal{D}(\mathbb{R}^N)$ with $\rho = 1$ on ω , $0 \leq \rho \leq 1$ and $Supp(\rho) \subset \omega'$. Let $0 < h < dist(\omega', \partial\Omega)$, to make the notations less heavy we set $U_k^h = \tau_h u_k - u_k$, then $U_k^h \in W^{2,2}(\omega')$. Notice that translation and derivation commute then we have

$$-\epsilon_k^2 \Delta_{X_1} U_k^h(x) - \Delta_{X_2} U_k^h(x) = F^h(x), \quad \text{a.e } x \in \omega',$$

with $F^h = \tau_h f - f$. We set $\mathcal{W}_k^h = \rho U_k^h$ then we get for a.e $x \in \omega'$

$$\begin{aligned} -\epsilon_k^2 \Delta_{X_1} \mathcal{W}_k^h(x) - \Delta_{X_2} \mathcal{W}_k^h(x) &= \rho(x) F^h(x) - 2\epsilon_k^2 \nabla_{X_1} \rho(x) \cdot \nabla_{X_1} U_k^h(x) \\ &\quad - 2\nabla_{X_2} \rho(x) \cdot \nabla_{X_2} U_k^h(x) - U_k^h(x) (\epsilon_k^2 \Delta_{X_1} \rho(x) - \Delta_{X_2} \rho(x)). \end{aligned}$$

Since $U_k^h \in W^{2,2}(\omega')$ then $\mathcal{W}_k^h \in W_0^{2,2}(\omega')$, so we can extend \mathcal{W}_k^h by 0 outside of ω' then $\mathcal{W}_k^h \in W^2(\mathbb{R}^N)$. The right hand side of the above equality is extended by 0 outside of ω' , hence the equation is satisfied in the whole space, so by **Lemma 1** we get

$$\begin{aligned} \|\nabla_{X_2}^2 \mathcal{W}_k^h\|_{L^2(\mathbb{R}^N)} &\leq \|\rho F^h\|_{L^2(\mathbb{R}^N)} + 2\epsilon_k^2 \|\nabla_{X_1} \rho \cdot \nabla_{X_1} U_k^h\|_{L^2(\mathbb{R}^N)} \\ &\quad + 2 \|\nabla_{X_2} \rho \cdot \nabla_{X_2} U_k^h\|_{L^2(\mathbb{R}^N)} + \|U_k^h (\epsilon_k^2 \Delta_{X_1} \rho - \Delta_{X_2} \rho)\|_{L^2(\mathbb{R}^N)}. \end{aligned}$$

Then

$$\begin{aligned} \|\nabla_{X_2}^2 U_k^h\|_{L^2(\omega)} &\leq \|F^h\|_{L^2(\omega')} + 2\epsilon_k \|\nabla_{X_1} \rho\|_\infty \|\epsilon_k \nabla_{X_1} U_k^h\|_{L^2(\omega')} \\ &\quad + 2 \|\nabla_{X_2} \rho\|_\infty \|\nabla_{X_2} U_k^h\|_{L^2(\omega')} + \|(\epsilon_k^2 \Delta_{X_1} \rho - \Delta_{X_2} \rho)\|_\infty \|U_k^h\|_{L^2(\omega')}. \end{aligned}$$

Recall that $u_k \rightarrow u$ in $V^{1,2}$ and $\epsilon_k \nabla_{X_1} u_k \rightarrow 0$ in $L^2(\Omega)$, then by **Lemma 2** we have

$$\limsup_{h \rightarrow 0} \sup_{k \in \mathbb{N}} \|\epsilon_k \nabla_{X_1} U_k^h\|_{L^2(\omega')} = \limsup_{h \rightarrow 0} \sup_{k \in \mathbb{N}} \|\epsilon_k (\tau_h \nabla_{X_1} u_k - \nabla_{X_1} u_k)\|_{L^2(\omega')} = 0,$$

and similarly we have

$$\begin{aligned} \limsup_{h \rightarrow 0} \sup_{k \in \mathbb{N}} \|\nabla_{X_2} U_k^h\|_{L^2(\omega')} &= 0, \quad \limsup_{h \rightarrow 0} \sup_{k \in \mathbb{N}} \|F^h\|_{L^2(\omega')} = 0, \\ \limsup_{h \rightarrow 0} \sup_{k \in \mathbb{N}} \|U_k^h\|_{L^2(\omega')} &= 0. \end{aligned}$$

and hence

$$\limsup_{h \rightarrow 0} \sup_{k \in \mathbb{N}} \|\tau_h \nabla_{X_2}^2 u_k - \nabla_{X_2}^2 u_k\|_{L^2(\omega)} = \limsup_{h \rightarrow 0} \sup_{k \in \mathbb{N}} \|\nabla_{X_2}^2 U_k^h\|_{L^2(\omega)} = 0.$$

Similarly we obtain

$$\limsup_{h \rightarrow 0} \sup_{k \in \mathbb{N}} \|\epsilon_k^2 (\tau_h \nabla_{X_1}^2 u_k - \nabla_{X_1}^2 u_k)\|_{L^2(\omega)} = 0,$$

and

$$\limsup_{h \rightarrow 0} \sup_{k \in \mathbb{N}} \|\epsilon_k (\tau_h \nabla_{X_1 X_2}^2 u_k - \nabla_{X_1 X_2}^2 u_k)\|_{L^2(\omega)} = 0.$$

2) Following the same arguments, we obtain

$$\begin{aligned} &\epsilon_k^2 \|\nabla_{X_1}^2 u_k\|_{L^2(\omega)} + \|\nabla_{X_2}^2 u_k\|_{L^2(\omega)} + \sqrt{2} \epsilon_k \|\nabla_{X_1 X_2}^2 u_k\|_{L^2(\omega)} \leq \\ &3 \|f\|_{L^2(\omega')} + 6 \epsilon_k \|\nabla_{X_1} \rho\|_{\infty} \|\epsilon_k \nabla_{X_1} u_k\|_{L^2(\omega')} \\ &\quad + 6 \|\nabla_{X_2} \rho\|_{\infty} \|\nabla_{X_2} u_k\|_{L^2(\omega')} + 3 \|(\epsilon_k^2 \Delta_{X_1} \rho - \Delta_{X_2} \rho)\|_{\infty} \|u_k\|_{L^2(\omega')}. \end{aligned}$$

The convergences $u_k \rightarrow u$ in $V^{1,2}$, $\epsilon_k \nabla_{X_1} u_k \rightarrow 0$ in $L^2(\Omega)$ and boundedness of ρ and its derivatives show that the right hand side of the above inequality is uniformly bounded in k , i.e for some $M > 0$ independent of k

$$\epsilon_k^2 \|\nabla_{X_1}^2 u_k\|_{L^2(\omega)} + \|\nabla_{X_2}^2 u_k\|_{L^2(\omega)} + \sqrt{2} \epsilon_k \|\nabla_{X_1 X_2}^2 u_k\|_{L^2(\omega)} \leq M.$$

□

Now, we are ready to prove the following

Theorem 2. *Let $u_\epsilon \in W_0^{1,2}(\Omega) \cap W_{loc}^{2,2}(\Omega)$ be the solution of (10) then $u_\epsilon \rightarrow u_0$ strongly in $V_{loc}^{2,2}$ where $u_0 \in V_{loc}^{2,2}$ is the solution of the limit problem. In addition, we have*

$$\epsilon^2 \nabla_{X_1}^2 u_\epsilon \rightarrow 0 \text{ and } \epsilon \nabla_{X_1 X_2}^2 u_\epsilon \rightarrow 0, \text{ strongly in } L_{loc}^2(\Omega).$$

Proof. Let $u_0 \in V^{1,2}$ be the solution of the limit problem and let $(u_k)_{k \in \mathbb{N}}$, $u_k \in W_0^{1,2}(\Omega) \cap W_{loc}^{2,2}(\Omega)$ be a subsequence of solutions to (10), then **Proposition 1** shows by the Riesz-Fréchet-Kolmogorov theorem ([3]) that the subset $\{\nabla_{X_2}^2 u_k\}_{k \in \mathbb{N}}$ is relatively compact in $L^2(\omega)$ for every $\omega \subset\subset \Omega$ open. Now, fix $\omega \subset\subset \Omega$ then there exists $u_0^\omega \in L^2(\omega)$ and a subsequence still labeled $(\nabla_{X_2}^2 u_k)_{k \in \mathbb{N}}$ such that $\nabla_{X_2}^2 u_k \rightarrow u_0^\omega$ in $L^2(\omega)$ strongly. Since $u_k \rightarrow u_0$ in $L^2(\omega)$ and the second order differential operators ∂_{ij}^2 are continuous on $\mathcal{D}'(\omega)$ then $u_0^\omega = \nabla_{X_2}^2 u_0$ on ω . Whence, since ω is arbitrary we get $\nabla_{X_2}^2 u_0 \in L_{loc}^2(\Omega)$. Now, taking a countable covering (ω_n) of Ω with $\omega_n \subset\subset \Omega$ then by the diagonal process one can construct a subsequence still labeled (u_k) such that $\nabla_{X_2}^2 u_k \rightarrow \nabla_{X_2}^2 u_0$ in $L_{loc}^2(\Omega)$ strongly, combining this with $u_k \rightarrow u_0$ in $V^{1,2}$, we get $u_k \rightarrow u_0$ strongly in $V_{loc}^{2,2}$.

To prove the convergence of the whole sequence $(u_\epsilon)_{0 < \epsilon \leq 1}$ we can reasoning by contradiction, so suppose that there exists $\delta > 0$ and a subsequence (u_k) such that $d_{V_{loc}^{2,2}}(u_k, u_0) > \delta$, here $d_{V_{loc}^{2,2}}$ is the distance of the Fréchet space $V_{loc}^{2,2}$. It follows by

the first part of this proof that there exists a subsequence still labeled (u_k) such that $d_{V^{2,2}}(u_k, u_0) \rightarrow 0$, and the proof of the theorem is finished. Similarly we show that $\epsilon^2 \nabla_{X_1}^2 u_\epsilon \rightarrow 0$ and $\epsilon \nabla_{X_1 X_2}^2 u_\epsilon \rightarrow 0$ strongly in $L^2_{loc}(\Omega)$. \square

3. GENERAL ELLIPTIC PROBLEMS

3.1. Proof of the main theorem. In this subsection we shall prove **Theorem 1**. Firstly, we suppose that the coefficients of A are constants then we have the following

Proposition 2. *Suppose that the coefficients of A are constants and assume (2), let (u_ϵ) be a sequence of $W^{2,2}(\mathbb{R}^N)$ such that $-\sum_{i,j} a_{ij}^\epsilon \partial_{ij}^2 u_\epsilon = f$, with $f \in L^2(\mathbb{R}^N)$*

then we have the bounds

$$\begin{aligned} \lambda \|\nabla_{X_2}^2 u_\epsilon\|_{L^2(\mathbb{R}^N)} &\leq \|f\|_{L^2(\mathbb{R}^N)}, \\ \lambda \epsilon^2 \|\nabla_{X_1}^2 u_\epsilon\|_{L^2(\mathbb{R}^N)} &\leq \|f\|_{L^2(\mathbb{R}^N)}, \\ \sqrt{2} \lambda \epsilon \|\nabla_{X_1 X_2}^2 u_\epsilon\|_{L^2(\mathbb{R}^N)} &\leq \|f\|_{L^2(\mathbb{R}^N)}. \end{aligned}$$

Proof. As in the proof of **Proposition 1** we use the Fourier transform and we obtain

$$\left(\sum_{i,j} a_{ij}^\epsilon \xi_i \xi_j \right) \mathcal{F}(u_\epsilon)(\xi) = \mathcal{F}(f)(\xi)$$

From the ellipticity assumption (2) we deduce

$$\lambda^2 \left(\epsilon^2 \sum_{i=1}^q \xi_i^2 + \sum_{i=q+1}^N \xi_i^2 \right)^2 |\mathcal{F}(u_\epsilon)(\xi)|^2 \leq |\mathcal{F}(f)(\xi)|^2$$

Thus, as in proof of **Proposition 1**, we obtain the desired bounds. \square

Now, suppose that $A \in L^\infty(\Omega) \cap C^1(\Omega)$ and assume (2), We denote $u_k = u_{\epsilon_k} \in W_0^{1,2}(\Omega) \cap W_{loc}^{2,2}(\Omega)$ the solution to (1) with $\epsilon_k \rightarrow 0$ as $k \rightarrow \infty$, then we have the following.

Proposition 3. *Let $z_0 \in \Omega$ fixed then there exists $\omega_0 \subset \subset \Omega$ open with $z_0 \in \omega_0$ such that the sequences $(\nabla_{X_2}^2 u_k)$, $(\nabla_{X_1}^2 u_k)$ and $(\nabla_{X_1 X_2}^2 u_k)$ are bounded in $L^2(\omega_0)$.*

Proof. Since $u_k \in W_0^{1,2}(\Omega) \cap W_{loc}^{2,2}(\Omega)$ and $A \in C^1(\Omega)$ then u_k satisfies for a.e $x \in \Omega$

$$-\sum_{i,j} a_{ij}^k(x) \partial_{ij}^2 u_k(x) - \sum_{i,j} \partial_i a_{ij}^k(x) \partial_j u_k(x) = f(x), \quad (11)$$

where $a_{ij}^k = a_{ij}^{\epsilon_k}$.

Let $z_0 \in \Omega$ fixed, and let $\theta > 0$ then using the continuity of the a_{ij} one can choose $\omega_1 \subset \subset \Omega$, $z_0 \in \omega_1$ such that

$$\sup_{x \in \omega_1} |a_{ij}(x) - a_{ij}(z_0)| \leq \theta \text{ for every } i, j \in \{1, \dots, N\} \quad (12)$$

Let $\omega_0 \subset\subset \omega_1$ open with $z_0 \in \omega_0$ and let $\rho \in \mathcal{D}(\mathbb{R}^N)$ with $\rho = 1$ on ω_0 , $0 \leq \rho \leq 1$ and $Supp(\rho) \subset \omega_1$. We set $U_k = \rho u_k$, and we extend it by 0 on the outside of ω_1 then $U_k \in W^{2,2}(\mathbb{R}^N)$. therefore we obtain

$$-\sum_{i,j} a_{ij}^k(z_0) \partial_{ij}^2 U_k(x) = \sum_{i,j} (a_{ij}^k(x) - a_{ij}^k(z_0)) \partial_{ij}^2 U_k(x) + g_k(x), \text{ for a.e } x \in \mathbb{R}^N,$$

where g_k is given by

$$\begin{aligned} g_k(x) &= \rho(x) f(x) + \rho(x) \sum_{i,j} \partial_i a_{ij}^k(x) \partial_j u_k(x) \\ &\quad - u_k(x) \sum_{i,j} a_{ij}^k(x) \partial_{ij}^2 \rho(x) - \sum_{i,j} a_{ij}^k(x) \partial_i \rho(x) \partial_j u_k(x) - \sum_{i,j} a_{ij}^k(x) \partial_j \rho(x) \partial_i u_k(x) \end{aligned} \quad (13)$$

Now applying **Proposition 2** to the above differential equality we get

$$\begin{aligned} &\lambda \|\nabla_{X_2}^2 U_k\|_{L^2(\omega_1)} + \lambda \epsilon_k^2 \|\nabla_{X_1}^2 U_k\|_{L^2(\omega_1)} + \sqrt{2} \lambda \epsilon_k \|\nabla_{X_1 X_2}^2 U_k\|_{L^2(\omega_1)} \\ &\leq 3 \left\| \sum_{i,j} (a_{ij}^k(x) - a_{ij}^k(z_0)) \partial_{ij}^2 U_k \right\|_{L^2(\omega_1)} + 3 \|g\|_{L^2(\omega_1)} \end{aligned}$$

Whence we use (12) we get

$$\begin{aligned} &\lambda \|\nabla_{X_2}^2 U_k\|_{L^2(\omega_1)} + \lambda \epsilon_k^2 \|\nabla_{X_1}^2 U_k\|_{L^2(\omega_1)} + \sqrt{2} \lambda \epsilon_k \|\nabla_{X_1 X_2}^2 U_k\|_{L^2(\omega_1)} \\ &\leq 3\theta \epsilon_k^2 \sum_{i,j=1}^q \|\partial_{ij}^2 U_k\|_{L^2(\omega_1)} + 3\theta \sum_{i,j=q+1}^N \|\partial_{ij}^2 U_k\|_{L^2(\omega_1)} \\ &\quad + 6\theta \epsilon_k \sum_{i=1}^q \sum_{j=q+1}^N \|\partial_{ij}^2 U_k\|_{L^2(\omega_1)} + 3 \|g\|_{L^2(\omega_1)}, \end{aligned}$$

and thus we deduce by Cauchy-Schwarz

$$\begin{aligned} &\lambda \|\nabla_{X_2}^2 U_k\|_{L^2(\omega_1)} + \lambda \epsilon_k^2 \|\nabla_{X_1}^2 U_k\|_{L^2(\omega_1)} + \sqrt{2} \lambda \epsilon_k \|\nabla_{X_1 X_2}^2 U_k\|_{L^2(\omega_1)} \\ &\leq 3\theta(N-q) \|\nabla_{X_2}^2 U_k\|_{L^2(\omega_1)} + \epsilon_k^2 3\theta q \|\nabla_{X_1}^2 U_k\|_{L^2(\omega_1)} \\ &\quad + \epsilon_k 6(N-q)q\theta \|\nabla_{X_1 X_2}^2 U_k\|_{L^2(\omega_1)}^2 + 3 \|g\|_{L^2(\omega_1)} \\ &[\lambda - 3\theta(N-q)] \|\nabla_{X_2}^2 U_k\|_{L^2(\omega_1)} + \epsilon_k^2 [\lambda - 3\theta q] \|\nabla_{X_1}^2 U_k\|_{L^2(\omega_1)} + \\ &\quad \epsilon_k [\sqrt{2} \lambda - 6(N-q)q\theta] \|\nabla_{X_1 X_2}^2 U_k\|_{L^2(\omega_1)} \leq 3 \|g_k\|_{L^2(\omega_1)}, \end{aligned}$$

we can choose a priori θ small enough such that

$$\min \left\{ [\lambda - 3\theta(N-q)], [\lambda - 3\theta q], [\sqrt{2} \lambda - 6(N-q)q\theta] \right\} \geq \frac{\lambda}{2}, \quad (14)$$

and then we deduce

$$\begin{aligned} &\|\nabla_{X_2}^2 u_k\|_{L^2(\omega_0)} + \epsilon_k^2 \|\nabla_{X_1}^2 u_k\|_{L^2(\omega_0)} + \epsilon_k \|\nabla_{X_1 X_2}^2 u_k\|_{L^2(\omega_0)} \\ &\leq \frac{6}{\lambda} \|g_k\|_{L^2(\omega_1)}. \end{aligned}$$

To finish the proof it suffices to show that the sequence (g_k) is bounded in $L^2(\omega_1)$. In fact ρ and its derivatives, a_{ij} and their first derivatives are bounded on ω_1 ,

moreover the sequences $(\epsilon_k \nabla_{X_1} u_k)$, $(\nabla_{X_2} u_k)$ and (u_k) are bounded in $L^2(\Omega)$, then from (13) the boundedness of (g_k) in $L^2(\omega_1)$ follows. \square

Corollary 1. *The sequences $(\nabla_{X_2}^2 u_k)$, $(\epsilon_k^2 \nabla_{X_1}^2 u_k)$, $(\epsilon_k \nabla_{X_1 X_2}^2 u_k)$ are bounded in $L^2_{loc}(\Omega)$.*

Proof. Let $\omega \subset\subset \Omega$ open, for every $z_\mu \in \bar{\omega}$ there exists $\omega_\mu \subset\subset \Omega$, $z_\mu \in \omega_\mu$ which satisfies the affirmations of **Proposition 3** in $L^2(\omega_\mu)$. Since $\bar{\omega}$ is compact then one can extract a finit cover (ω_{μ_m}) , then $(\nabla_{X_2}^2 u_k)$, $(\epsilon_k^2 \nabla_{X_1}^2 u_k)$, $(\epsilon_k \nabla_{X_1 X_2}^2 u_k)$ are bounded in $L^2(\omega)$ and the corollary follows. \square

Proposition 4. *Let $z_0 \in \Omega$ then there exists $\omega_0 \subset\subset \Omega$, $z_0 \in \omega_0$ such that*

$$\begin{aligned} \lim_{h \rightarrow 0} \sup_{k \in \mathbb{N}} \|\tau_h \nabla_{X_2}^2 u_k - \nabla_{X_2}^2 u_k\|_{L^p(\omega_0)} &= 0, \\ \lim_{h \rightarrow 0} \sup_{k \in \mathbb{N}} \|\epsilon_k^2 (\tau_h \nabla_{X_1}^2 u_k - \nabla_{X_1}^2 u_k)\|_{L^p(\omega_0)} &= 0, \\ \lim_{h \rightarrow 0} \sup_{k \in \mathbb{N}} \|\epsilon_k (\tau_h \nabla_{X_1 X_2}^2 u_k - \nabla_{X_1 X_2}^2 u_k)\|_{L^p(\omega_0)} &= 0. \end{aligned}$$

Proof. Let $z_0 \in \Omega$ fixed and let $\theta > 0$ then using the continuity of the a_{ij} one can choose $\omega_1 \subset\subset \Omega$, $z_0 \in \omega_1$ such that we have (12) with θ chosen such that (14). Let $\omega_0 \subset\subset \omega_1$, with $z_0 \in \omega_0$, and let $\rho \in \mathcal{D}(\mathbb{R}^N)$ with $\rho = 1$ on ω_0 , $0 \leq \rho \leq 1$, and $Supp(\rho) \subset \omega_1$. Let $0 < h < dist(\omega_1, \partial\Omega)$, we set $\mathcal{W}_k^h = \rho U_k^h$, with $U_k^h = (\tau_h u_k - u_k)$ and extend it by 0 on the outside of ω_1 then $\mathcal{W}_k^h \in W^{2,2}(\mathbb{R}^N)$, therefore using (11) we have for a.e $x \in \mathbb{R}^N$:

$$-\sum_{i,j} a_{ij}^k(z_0) \partial_{ij}^2 \mathcal{W}_k^h(x) = \sum_{i,j} (a_{ij}^k(x) - a_{ij}^k(z_0)) \partial_{ij}^2 \mathcal{W}_k^h(x) + G_k^h(x),$$

where

$$\begin{aligned} -G_k^h(x) &= U_k^h \sum_{i,j} a_{ij}^k(x) \partial_{ij}^2 \rho + \sum_{i,j} a_{ij}^k(x) \partial_i \rho \partial_j U_k^h + \sum_{i,j} a_{ij}^k(x) \partial_j \rho \partial_i U_k^h \quad (15) \\ &+ \rho \sum_{i,j} (a_{ij}^k(x) - \tau_h a_{ij}^k(x)) \tau_h \partial_{ij}^2 u_k(x) + \rho(x) (f(x) - \tau_h f(x)) \\ &+ \rho \sum_{i,j} [\partial_i a_{ij}^k(x) \partial_j u_k(x) - \partial_i \tau_h a_{ij}^k(x) \partial_j \tau_h u_k(x)]. \end{aligned}$$

Then, as in proof of **Proposition 3**, we obtain

$$\begin{aligned} \|\tau_h \nabla_{X_2}^2 u_k - \nabla_{X_2}^2 u_k\|_{L^2(\omega_0)} + \epsilon_k^2 \|\tau_h \nabla_{X_1}^2 u_k - \nabla_{X_1}^2 u_k\|_{L^2(\omega_0)} \\ + \epsilon_k \|\tau_h \nabla_{X_1 X_2}^2 u_k - \nabla_{X_1 X_2}^2 u_k\|_{L^2(\omega_0)} \leq \frac{6}{\lambda} \|G_k^h\|_{L^2(\omega_1)}. \end{aligned}$$

Now, to finish the proof we have to show that $\lim_{h \rightarrow 0} \sup_{k \in \mathbb{N}} \|G_k^h\|_{L^2(\omega_1)} = 0$. Using the boundedness of the a_{ij} and the boundedness of ρ and its derivatives on ω_1 we

get from (15)

$$\begin{aligned}
\|G_k^h\|_{L^2(\omega_1)} &\leq M \|U_k^h\|_{L^2(\omega_1)} + M \epsilon_k \|\nabla_{X_1} U_k^h\|_{L^2(\omega_1)} \\
&+ M \|\nabla_{X_2} U_k^h\|_{L^2(\omega_1)} + \|\tau_h f - f\|_{L^2(\omega_1)} \\
&+ \sum_{i,j} \|(a_{ij}^k - \tau_h a_{ij}^k) \tau_h \partial_{ij}^2 u_k\|_{L^2(\omega_1)} \\
&+ \sum_{i,j} \|\partial_i a_{ij}^k \partial_j u_k - \tau_h \partial_i a_{ij}^k \tau_h \partial_j u_k\|_{L^2(\omega_1)},
\end{aligned} \tag{16}$$

where $M > 0$ independent of h and k . Now, estimating the fifth term of the right hand side of the above inequality

$$\begin{aligned}
\sum_{i,j} \|(a_{ij}^k - \tau_h a_{ij}^k) \tau_h \partial_{ij}^2 u_k\|_{L^2(\omega_1)} &\leq C_{q,N} \sup_{x \in \omega_1} |a_{ij}(x) - \tau_h a_{ij}(x)| \times \\
&\left(\|\nabla_{X_2}^2 u_k\|_{L^2(\omega_{1+h})} + \epsilon_k^2 \|\nabla_{X_1}^2 u_k\|_{L^2(\omega_{1+h})} + \epsilon_k \|\nabla_{X_1 X_2}^2 u_k\|_{L^2(\omega_{1+h})} \right),
\end{aligned}$$

where $C_{q,N} > 0$ is only depends in q and N .

For h small enough then it follows from **Corollary 1** that the quantity

$$\|\nabla_{X_2}^2 u_k\|_{L^2(\omega_{1+h})} + \epsilon_k^2 \|\nabla_{X_1}^2 u_k\|_{L^2(\omega_{1+h})} + \epsilon_k \|\nabla_{X_1 X_2}^2 u_k\|_{L^2(\omega_{1+h})},$$

is uniformly bounded. Since the a_{ij} are uniformly continuous on every $\omega \subset\subset \Omega$ then $\lim_{h \rightarrow 0} \sup_{x \in \omega_1} |a_{ij}(x) - \tau_h a_{ij}(x)| = 0$ and hence

$$\lim_{h \rightarrow 0} \sup_{k \in \mathbb{N}} \sum_{i,j} \|(a_{ij}^k - \tau_h a_{ij}^k) \tau_h \partial_{ij}^2 u_k\|_{L^2(\omega_1)} = 0. \tag{17}$$

Now, estimating the last term of (16)

$$\begin{aligned}
\sum_{i,j} \|\partial_i a_{ij}^k \partial_j u_k - \tau_h \partial_i a_{ij}^k \tau_h \partial_j u_k\|_{L^2(\omega_1)} &\leq \sum_{i,j} \|\partial_i a_{ij}^k \partial_j u_k - \tau_h \partial_i a_{ij}^k \partial_j u_k\|_{L^2(\omega_1)} \\
&+ \sum_{i,j} \|\tau_h \partial_i a_{ij}^k \partial_j u_k - \partial_i \tau_h a_{ij}^k \tau_h \partial_j u_k\|_{L^2(\omega_1)},
\end{aligned}$$

and thus using boundedness of the first derivatives of the a_{ij} on ω_1 we get

$$\begin{aligned}
&\sum_{i,j} \|\partial_i a_{ij}^k \partial_j u_k - \partial_i \tau_h a_{ij}^k \partial_j \tau_h u_k\|_{L^2(\omega_1)} \\
&\leq C'_{q,N} \sup |\partial_i a_{ij}(x) - \partial_i \tau_h a_{ij}(x)| \left(\epsilon_k \|\nabla_{X_1} u_k\|_{L^2(\omega_1)} + \|\nabla_{X_2} u_k\|_{L^2(\omega_1)} \right) \\
&\quad + M' \left(\epsilon_k \|\nabla_{X_1} U_k^h\|_{L^2(\omega_1)} + \|\nabla_{X_2} U_k^h\|_{L^2(\omega_1)} \right),
\end{aligned}$$

where $M' > 0$ is independent of h and k . Now, since the $\partial_i a_{ij}$ are uniformly continuous (recall that $A \in C^1(\Omega)$) on every $\omega \subset\subset \Omega$ then

$$\lim_{h \rightarrow 0} \sup_{x \in \omega_1} |\partial_i a_{ij}(x) - \tau_h \partial_i a_{ij}(x)| = 0,$$

and hence from the above inequality we get

$$\lim_{h \rightarrow 0} \sup_{k \in \mathbb{N}} \sum_{i,j} \|\partial_i a_{ij}^k \partial_j u_k - \partial_i \tau_h a_{ij}^k \partial_j \tau_h u_k\|_{L^2(\omega_1)} = 0, \tag{18}$$

where we have used the convergence of $(\nabla_{X_2} u_k)$, $(\epsilon_k \nabla_{X_1} u_k)$ and **Lemma 2**.

Passing to the limit in (16) by using (17), (18) and **Lemma 2** we deduce

$$\lim_{h \rightarrow 0} \sup_{k \in \mathbb{N}} \|G_k^h\|_{L^2(\omega_1)} = 0.$$

and the proposition follows. \square

Corollary 2. *For every $\omega \subset\subset \Omega$ open we have*

$$\begin{aligned} \lim_{h \rightarrow 0} \sup_{k \in \mathbb{N}} \|\tau_h \nabla_{X_2}^2 u_k - \nabla_{X_2}^2 u_k\|_{L^p(\omega)} &= 0, \\ \lim_{h \rightarrow 0} \sup_{k \in \mathbb{N}} \|\epsilon_k^2 (\tau_h \nabla_{X_1}^2 u_k - \nabla_{X_1}^2 u_k)\|_{L^p(\omega)} &= 0, \\ \lim_{h \rightarrow 0} \sup_{k \in \mathbb{N}} \|\epsilon_k (\tau_h \nabla_{X_1 X_2}^2 u_k - \nabla_{X_1 X_2}^2 u_k)\|_{L^p(\omega)} &= 0. \end{aligned}$$

Proof. Similar to proof of **Corollary 1** where we use compactness of $\bar{\omega}$ and **Proposition 4**. \square

Now, we are able to give the proof of the main theorem. Indeed the proof is similar to proof of **Theorem 2** where we use **Corollary 1** and **Corollary 2**. Let us give the proof of the convergence $\epsilon^2 \nabla_{X_1}^2 u_\epsilon \rightarrow 0$ in $L_{loc}^2(\Omega)$. Fix $\omega \subset\subset \Omega$ open, and let $u_k \in W_0^{1,2}(\Omega) \cap W_{loc}^{2,2}(\Omega)$ be a sequence of solutions of (1) (ϵ replaced by $\epsilon_k \rightarrow 0$), then it follows from **Corollary 1** and **2** that the subset $\{\epsilon_k^2 \nabla_{X_1}^2 u_k\}_{k \in \mathbb{N}}$ is relatively compact in $L^2(\omega)$ then there exists $v^\omega \in L^2(\omega)$ and a subsequence still labeled $(\epsilon_k^2 \nabla_{X_1}^2 u_k)$ such that $\epsilon_k^2 \nabla_{X_1}^2 u_k \rightarrow v^\omega$ in $L^2(\omega)$. Since $\epsilon_k^2 u_k \rightarrow 0$ in $L^2(\omega)$ then $v^\omega = 0$, since $\nabla_{X_1}^2$ is continuous on $\mathcal{D}'(\omega)$. Hence by the diagonal process one can construct a sequence still labeled $(\epsilon_k^2 \nabla_{X_1}^2 u_k)$ such that $\epsilon_k^2 \nabla_{X_1}^2 u_k \rightarrow 0$ in $L_{loc}^2(\Omega)$. To prove the convergence of the whole sequence we can reasoning by contradiction as in proof of **Theorem 2** (recall that $L_{loc}^2(\Omega)$ equipped with the family of semi norms $\|\cdot\|_{L^2(\omega)}$ is a Fréchet space), and the proof of the main theorem is finished.

3.2. $W_{loc}^{2,2}$ regularity for some class of semilinear problem. In this section we deal with the following semilinear elliptic problem

$$\begin{cases} -\operatorname{div}(A_\epsilon \nabla u_\epsilon) = a(u_\epsilon) + f \\ u_\epsilon = 0 \end{cases} \quad \text{on } \partial\Omega, \quad ,$$

where $a : \mathbb{R} \rightarrow \mathbb{R}$ a continuous nonincreasing real valued function which satisfies the growth condition

$$\forall x \in \mathbb{R} : |a(x)| \leq c(1 + |x|), \quad (19)$$

for some $c \geq 0$. This problem has been treated in [4] for $f \in L^p(\Omega)$, $1 < p \leq 2$, and the author have proved the convergences

$$\epsilon \nabla_{X_1} u_\epsilon \rightarrow 0, u_\epsilon \rightarrow u_0, \nabla_{X_2} u_\epsilon \rightarrow \nabla_{X_2} u_0 \text{ in } L^p(\Omega), \quad (20)$$

where u_0 is the solution of the unperturbed problem.

Let $f \in L^2(\Omega)$ and assume A as in **Theorem 1** then the unique $W_0^{1,2}(\Omega)$ weak solution u_ϵ belongs to $W_{loc}^{2,2}(\Omega)$ by the elliptic regularity. Following the same arguments exposed in the above section one can prove the theorem

Theorem 3. *Under the above assumptions we have $u_\epsilon \rightarrow u_0$ in $V_{loc}^{2,2}$, $\epsilon^2 \nabla_{X_1}^2 u_\epsilon \rightarrow 0$ and $\epsilon \nabla_{X_1 X_2}^2 u_\epsilon \rightarrow 0$ strongly in $L_{loc}^2(\Omega)$.*

Proof. The arguments are similar, we only give the proof for the Laplacian case, so assume that $A = Id$. Let $\omega \subset\subset \Omega$ open, then one can choose ω' open such that $\omega \subset\subset \omega' \subset\subset \Omega$, let $\rho \in \mathcal{D}(\mathbb{R}^N)$ with $\rho = 1$ on ω , $0 \leq \rho \leq 1$ and $Supp(\rho) \subset \omega'$. Let $0 < h < dist(\partial\omega', \Omega)$, we use the same notations so we set $U_k^h = \tau_h u_k - u_k$, then $U_k^h \in W^{1,2}(\omega')$ and we have

$$-\epsilon_k^2 \Delta_{X_1} U_k^h(x) - \Delta_{X_2} U_k^h(x) = F^h(x) + \tau_h a(u)(x) - a(u)(x), \text{ a.e } x \in \omega',$$

with $F^h = \tau_h f - f$. We set $\mathcal{W}_k^h = \rho U_k^h$ then we get as in **Proposition 1**

$$\begin{aligned} \|\tau_h \nabla_{X_2}^2 u_k - \nabla_{X_2}^2 u_k\|_{L^2(\omega)} &\leq \|F^h\|_{L^2(\omega')} + M \|\epsilon_k \nabla_{X_1} U_k^h\|_{L^2(\omega')} \\ &+ \|\tau_h a(u_k) - a(u_k)\|_{L^2(\omega')} \\ &+ M \|\nabla_{X_2} U_k^h\|_{L^2(\omega')} + M \|U_k^h\|_{L^2(\omega')}. \end{aligned}$$

We show easily using continuity of the function a and (19) that the Nemytskii operator a maps continuously L^2 to L^2 , therefore the convergence $u_k \rightarrow u_0$ in $L^2(\Omega)$ gives $a(u_k) \rightarrow a(u_0)$ in $L^2(\Omega)$, and hence

$$\limsup_{h \rightarrow 0} \sup_{k \in \mathbb{N}} \|\tau_h a(u_k) - a(u_k)\|_{L^2(\omega)} = 0,$$

and finally the convergences (20) give

$$\limsup_{h \rightarrow 0} \sup_{k \in \mathbb{N}} \|\tau_h \nabla_{X_2}^2 u_k - \nabla_{X_2}^2 u_k\|_{L^2(\omega)} = 0.$$

Similarly, using boundedness (u_k) , $(\epsilon_k \nabla_{X_1} u_k)$, $(\nabla_{X_2} u_k)$ and $a(u_k)$ in $L^2(\Omega)$, and boundedness of ρ and its derivatives we obtain

$$\|\nabla_{X_2}^2 u_k\|_{L^2(\omega)} \leq M',$$

and we conclude as in proof of **Theorem 2**. \square

REFERENCES

- [1] M. Chipot and S. Guesmia, On the asymptotic behaviour of elliptic, anisotropic singular perturbations problems, Com. Pur. App. Ana, 8 (2009), 179-193
- [2] M. Chipot and S. Guesmia, On a class of integro-differential problems, Commun. Pure Appl. Anal., 9 2010, 1249-1262.
- [3] M. Chipot, Elliptic Equations, An Introductory Cours, Birkhauser, ISBN: 978-3764399818, 2009
- [4] C. Ogabi, On the L^p theory of anisotropic singular perturbations elliptic problems. Com. Pur. App. Ana, Volume 15, 1157 - 1178, July 2016
- [5] Trudinger & Gilbarg, Elliptic Partial Differential Equations of Second Order..
- [6] Vo khac Khoan, Distributions, analyse de Fourier, opérateurs aux dérivées partielles Tome 1.

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