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Article

Is Turbulence a State of Maximum Energy Dissipation?

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Abstract: Turbulent flows are known to enhance turbulent transport. It has then even been suggested that turbulence is a state of *maximum energy dissipation*. In this paper, we re-examine critically this suggestion at the light of several recent works around the Maximum Entropy Production principle (MEP) that has been used in several out-of-equilibrium systems. We provide a set of 4 different optimization principles, based on maximization of energy dissipation, entropy production, Kolmogorov-Sinai entropy and minimization of mixing time, and study the connection between these principles using simple out-of-equilibrium models describing mixing of a scalar quantity. We find that there is a chained-relationship between most probable stationary states of the system, and their ability to obey one of the 4 principle. This provides an empirical justification of the Maximum Entropy Production principle in this class of systems, including some turbulent flows, for special boundary conditions. Otherwise, we claim that the minimization of the mixing time would be a more appropriate principle. We stress that this principle might actually be limited to flows where symmetry or dynamics impose pure mixing of a quantity (like angular momentum, momentum or temperature). The claim that turbulence is a state of maximum energy dissipation, a quantity intimately related to entropy production, is therefore limited to special situations that nevertheless include classical systems such as Shear flows, Rayleigh Benard convection and von Karman flows, forced with constant velocity or temperature conditions.

Keywords: Maximum Entropy Production; Turbulence; Kolmogorov-Sinai entropy

1. Introduction: Turbulence as a maximum Energy Dissipation State?

A well-known feature of any turbulent flow is the Kolmogorov-Richardson cascade by which energy is transferred from scale to scale until scales at which it can be dissipated. This cascade is a non-linear, non-equilibrium process. It is believed to be the origin of the significant enhancement of dissipation observed in turbulent flow, often characterized via the introduction of a *turbulent viscosity*.

It has then sometimes been suggested that turbulence is a state of *maximum energy dissipation*. This principle inspired early works by Malkus [1,2] or Spiegel [3] to compute analytically the heat or momentum profiles in thermal boundary layers or linear shear flows. While there were many

27 criticisms about this principle, there are a few experimental situation where this principle seems to
28 work. A good example is provided by the von Karman flow. This flow is generated by two-counter
29 rotating impellers inside a cylindrical vessel filled e.g. with water (see Figure 1). The impellers
30 produce a source of angular momentum at the top and bottom of the vessel, angular momentum
31 that is then transferred and mixed within the flow throughout the turbulent motions [4], in analogy
32 with heat transferred through a Rayleigh-Benard cell. For most impellers, the resulting mean large
33 scale stationary motion is the superposition of a two-cell azimuthal motion, and a two cell poloidal
34 motion bringing the flow from the top and bottom end of the experiment towards its center plane
35 $z = 0$ (see Figure 1). This mean flow is thus symmetrical with respect to the plane $z = 0$. For some
36 types of impellers, however, this symmetrical state is unstable, and bifurcates after a certain time
37 towards another state that breaks the system symmetry [5,6]-see Figure 2. This state corresponds to
38 a global rotation in the direction of either the top or the bottom impeller. The energy dissipation
39 corresponding to either one of these 3 states can be measured through monitoring of the torque
40 applied to the impellers by the flow. When monitored during a bifurcation (see Figure 3), this energy
41 dissipation displays a jump (by a factor 4) at the moment of the bifurcation from the symmetrical state
42 towards either one of the non-symmetrical states. Once the system is in the bifurcated state, it never
43 bifurcates back towards the symmetrical state, indicating that the most stable state is the state with
44 larger dissipation.

45 This observation is in agreement with a general principle inspired from Malkus principle, that could
46 be formulated as follow:

47 *Principle A: In certain non-equilibrium systems with coexistence of several stationary state, the most*
48 *stable one is that of Maximum Energy Dissipation.*

49
50 This principle is of course very appealing. There are however no derivation of it from any first
51 principles, and we are not aware of any theories that could lead to its proof (while there are probably
52 many immediate counter-example that can be provided). If it is true or approximately true for
53 some types of flows (like the von Karman flow, or the Rayleigh-Benard flow or the plane Couette
54 flow), it may then lead to interesting applications allowing the computation of mean velocity or
55 temperature profile without the need to integrate the whole Navier-Stokes equations. A way to
56 proceed with its justification is to transform it into an equivalent principle, that uses notions more
57 rooted in non-equilibrium physics. Indeed, energy dissipation is not a handful quantity to work with
58 in general, because of its dependence on the small scale processes that produce it. In general, energy
59 dissipation is a signature of entropy production. The connection between energy dissipation and
60 entropy production was heuristically made by Lorenz [7] and theoretically discussed in nonlinear
61 chemical thermodynamics by Dewar [8] and Moroz [9,10]. This last notion seems more appealing to
62 work with and a first natural step is to modify slightly the principle A into a more appealing version
63 as:

64 *Principle B: In certain non-equilibrium systems with coexistence of several stationary state, the most*
65 *stable one is that of Maximum Entropy Production (MEP).*

66
67 From the point of view of non-equilibrium physics, this principle appears as a counterpart of the well
68 known principle of Maximum Entropy that governs stability of equilibria in statistical physics, the
69 analog of equilibria here being the stationary states. This principle was discovered by Ziegler [11,12]
70 and it is sometimes referred to as the Ziegler's principle. It has found several applications in climate
71 dynamics: first it was used by Paltridge [13] to derive a good approximation of the mean temperature
72 distribution in the mean atmosphere of the Earth. This approach has been extend to more exhaustive
73 climate models by Herbert [14]. Kleidon et al. [15] used an atmospheric general circulation model to
74 show that MEP states can be used as a criterion to determine the boundary layer friction coefficients.

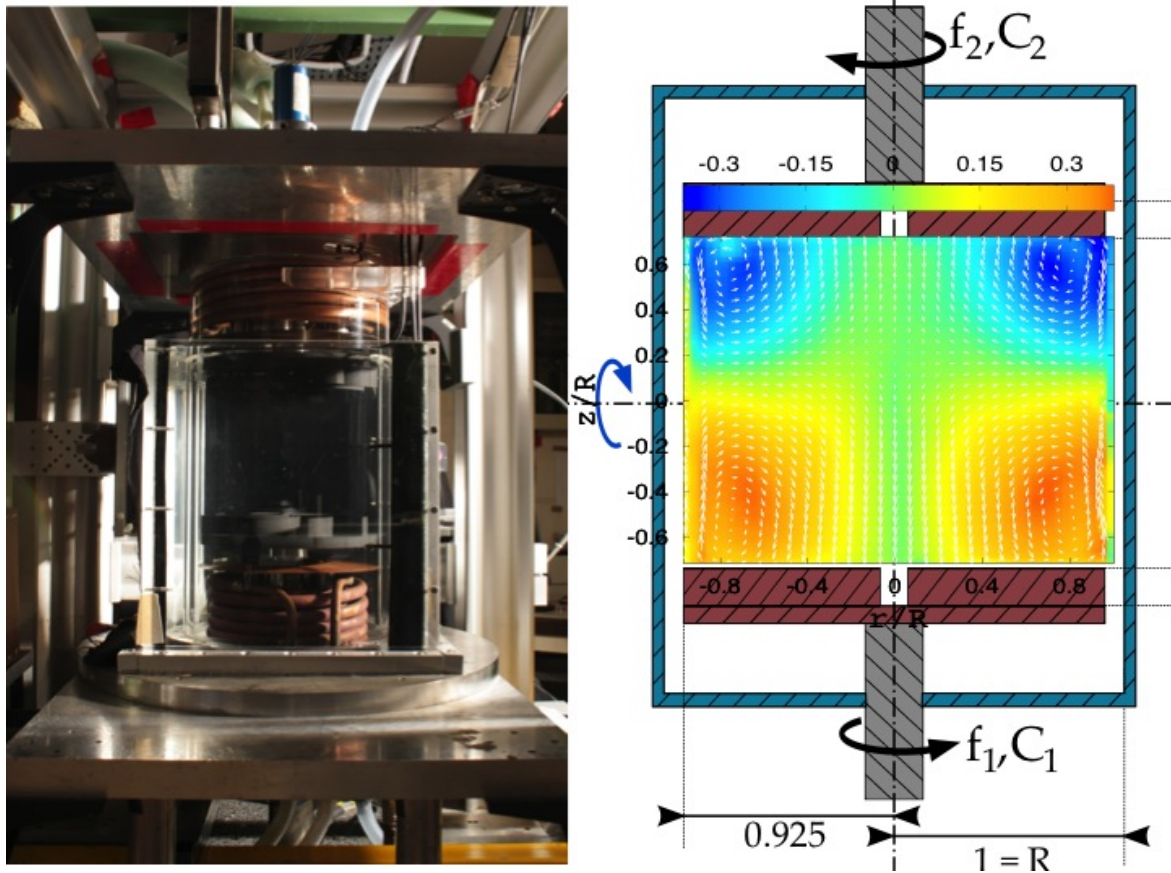


Figure 1. Von Karman experiment. The flow is generated inside a cylindrical vessel through counter-rotation of two impellers. The impellers inject angular-momentum at the top and the bottom, inducing a large scale circulation inside the flow. At low Reynolds numbers, the circulation is symmetrical with respect to a π -rotation around a horizontal axis through the origin (blue arrow). One can impose the torque C_i or the rotation frequency f_i to the flow, generating different turbulent regimes. The right picture shows a representation of the mean velocity fields in a plane passing for the axis of the cylinder (arrows) and orthogonal to this plane (colorscale in m/s), obtained by averaging several thousands instantaneous fields. (Pictures courtesy Brice Saint-Michel).

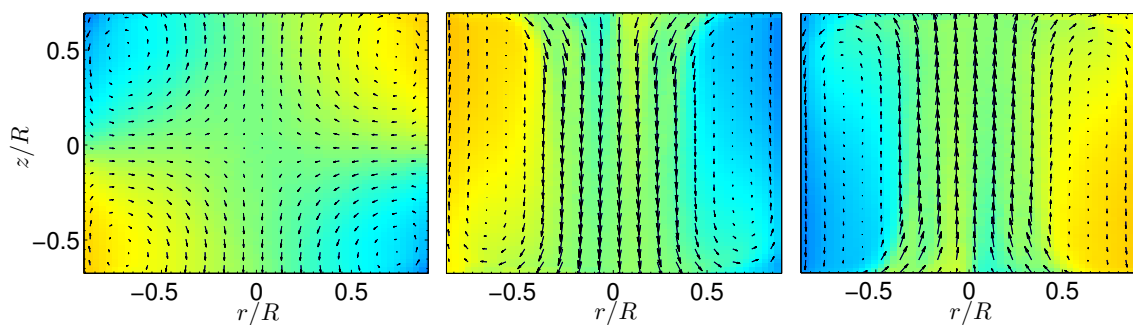


Figure 2. The 3 stationary states of the von Karman flow: left: The symmetric state. Middle and right: the two bifurcated states, that are symmetric to each other with respect to a π -rotation along an axis going through the rotation axis [5,6]. Color coding as in Figure 1.

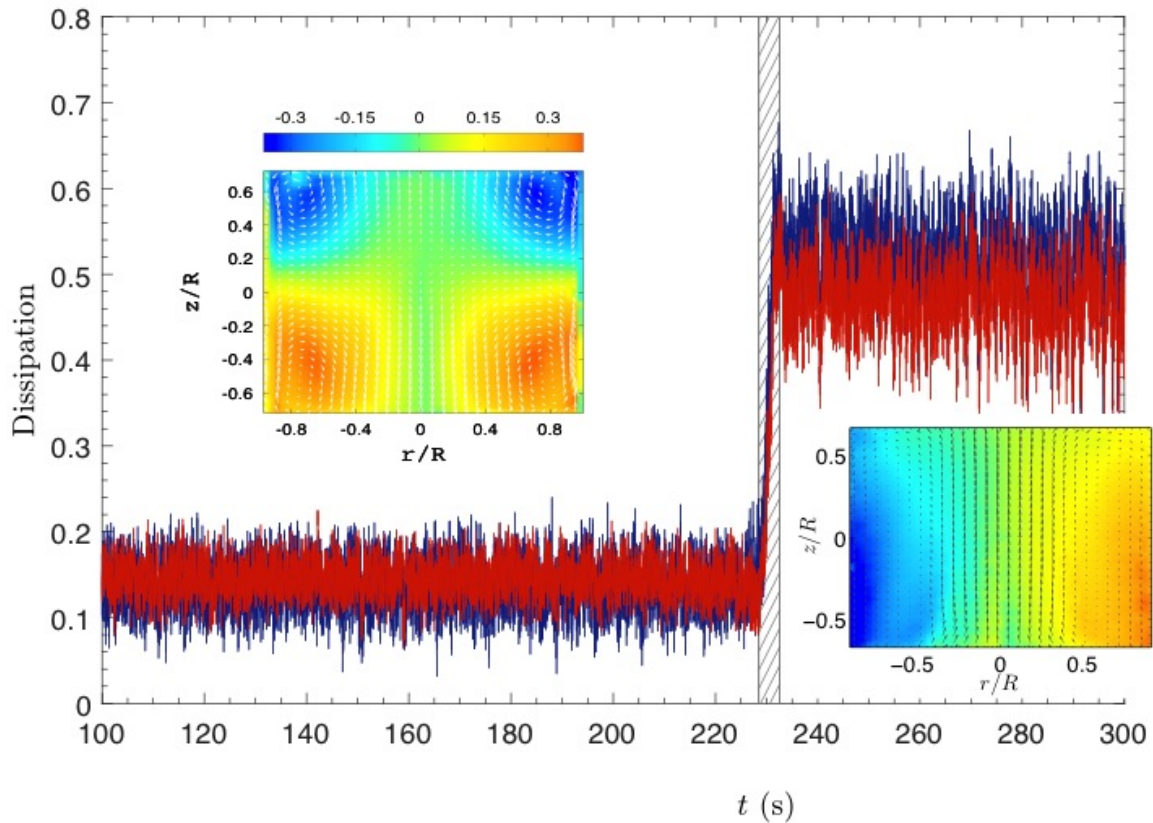


Figure 3. Spontaneous bifurcation in the von Karman flow: the flow, initially started in a symmetrical state, bifurcates after a certain time toward a bifurcated state, that produces a 4 times larger dissipation [5]. The dissipation is measured through monitoring of the torque, applied on each propeller, by the turbulent flow. This flow being turbulent, the resulting torque is widely fluctuating around a mean value, characterizing the dissipation. Colour coding of the flow as in Figure 1. Colour coding of the dissipation is blue for the dissipation measured on the lower propeller, and red for the dissipation measured on the upper-propeller. The total dissipation is the some of the two contributions.

MEP seems also a valuable principle to describe planetary atmospheres, as those of Mars and Titan where it has been used to determine latitudinal temperature gradients [16]. MEP have been also applied to several geophysical problems: to describe thermally driven winds [17] and convection [18], and to oceanic circulation [19]. A detailed overview on the usefulness of MEP in climate science can be found in [20,21] and references therein. It is therefore interesting to evaluate the soundness of this principle and understand its limitation and its possible improvements, to extend as possible the scope of its applications. The usual path to prove the validity of a principle is to provide some rigorous demonstration of the principle itself. This task has been attempted without convincing results in the past years [8,22,23]. In the absence of any theory of out-of-equilibrium systems, we may turn to equilibrium theory as a guide to find a path for justification of the selection of stationary states. In equilibrium systems or conceptual models, this selection can be studied using the dynamical systems theory, where other quantities than thermodynamics entropy are relevant. One of this quantity is the Kolmogorov Sinai Entropy (KSE) [24], which is indeed different from the thermodynamic one. The KSE appears a good candidate for the selection of preferred metastable states because it is related to the concept of mixing time [25]. The goal of this paper is therefore to show that MEP principle could find a justification in a linked relationship which involves studying the connections among MEP, KSE and mixing times. The paper follow this structure: after discussing the relation between MEP and the Prigogine minimization principle (section II), we connect MEP and maximum KSE in conceptual models of turbulence (section III). In section IV we establish the link between maximum KSE and mixing times for Markov chains. Then, we summarize the results and discuss the implications of our findings.

2. Maximization or Minimization of Entropy production?

At first sight, Principle B appears in conflict with an established result of Prigogine, according to which the stationary states of a system close to equilibrium are states with *minimum* entropy production. In fact, both principles can be reconciled if Principle B is viewed as a MaxMin, principle: Martyushev et al. [26] and Kleidon [21] suggest that Prigogine's principle select the steady state of minimum entropy production compared to transient states. For a steady-state condition, the minimum entropy production principle does not give any further information if many steady state conditions are possible given the imposed boundary conditions. To provide a simple example, let us consider a system characterized by two parameters, T and U , where T controls the departure from equilibrium and U labels an additional constraint of the system, allowing the existence of several stationary states at a given T (see Figure 4). In our von Karman system, T could for example label the velocity fluctuations, and U the angular momentum transport. In the T direction, application of the Prigogine principle selects the value of T corresponding to the stationary state. When there are several possible stationary state, the Principle B then selects the most stable state as the one with the largest entropy production, thereby fixing the corresponding value of U . This was precisely the procedure followed by Paltridge.

On the other hand, it appears that the stability of the stationary state may depend on boundary conditions. For example, Niven [27] or Kawazura and Yoshida [28] provide explicit examples of out-of-equilibrium systems, in which the entropy production is *sl* maximized for fixed temperature boundary conditions, while it is *minimized for fixed heat flux* boundary conditions. In the experimental von Karman system discussed in Section 1, we observe a similar phenomenon: for fixed propeller velocities, the selected stationary state is the one of maximum dissipation. However, when one changes the boundary conditions into fixed applied torque at each propeller, the stationary state regime disappears and is replaced by a dynamical regime, in which the system switches between different meta-stable states of low and high energy dissipation [6]. This case is discussed in more detail in Section 6. It shows however that we cannot trust the Maximum Entropy Production principle blindly, and must find ways to understand why and when it works, using tools borrowed from

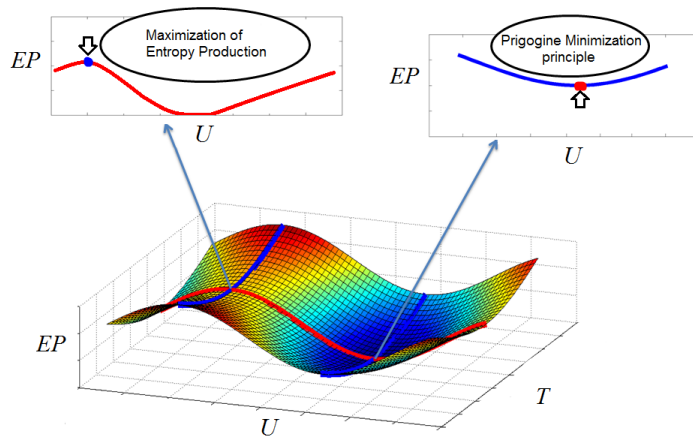


Figure 4. Illustration of the relation between maximization of entropy production EP and Prigogine principle: the system is set out of equilibrium by the parameters T and U . T controls the departure from equilibrium and U labels an additional constraint of the system. In the T direction, application of the Prigogine principle selects the value of T corresponding to the stationary state, where MEP selects the one with largest EP in the U direction.

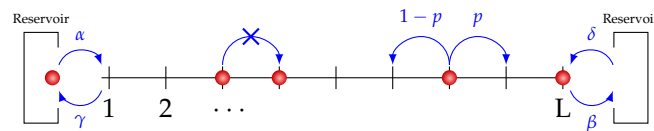


Figure 5. The dynamical rules of the toy model of scalar transport: the particle can jump to the left or to the right with probabilities denoted by p . At both end, two reservoirs sets the flux of incoming or outgoing particles. The particle can be a boson (several particles per box are allowed), in which case the process is called zero-range process (ZRP). When the particle is a fermion, jump towards a box that is already occupied are forbidden. The corresponding process is called asymmetric exclusion process (ASEP).

123 non-equilibrium theories. A justification has been attempted [8], and dismissed [22,23], following the
 124 ideas of Jaynes that non-equilibrium systems should be characterized by a probability distribution
 125 on the trajectories in phase space, instead of just the points in phase space at equilibrium. A more
 126 pragmatic way to evaluate the validity of Principle B is to consider its application to toy models of
 127 non-equilibrium statistics, that mimics the main processes at work in the von Karman flow, and that
 128 can guide us on a way to a justification (or dismissal). This is the topic of the next section.

129 3. From Maximum entropy production to Maximum Kolmogorov-Sinai entropy in toy models of 130 turbulence

131 3.1. From passive scalar equation to Markovian box models

In the von Karman flow, angular momentum is transported from the vessel ends towards the center. In Rayleigh-Benard, the temperature is transported from the bottom to the top plates. In shear flows, the linear momentum is transported from one side to the other. On Earth, the heat is transported from the equator towards the pole. All this system in which Principle B seems to provide a non-trivial answer have then in common that they deal with the transport of a scalar quantity T by a given velocity field $u(x, t)$, and that may be sketched as:

$$\partial_t T + u \nabla T = \kappa (\nabla)^2 T, \quad (1)$$

132 with appropriate boundary conditions. Here κ is the diffusivity. To transform this process into a
 133 tractable toy model, we stick to a one dimensional case and divide the accessible space ℓ into L
 134 boxes. We impose the boundary conditions through two reservoirs located at each end of the chain
 135 (mimicking e.g. the top and bottom propeller or solar heat flux at pole and equator). The boxes
 136 contains bosonic or fermionic particles that can jump in between two adjacent boxes via decorrelated
 137 jumps (to the right or to the left) following a 1D Markov dynamics governed by a coupling with
 138 the two reservoirs imposing a difference of chemical potential at the ends. The different jumps are
 139 described as follow. At each time step a particle can jump right with probability pw_n or jump left with
 140 probability $(1-p)w_n$. w_n is a parameter depending on the number of particles inside the box and
 141 on the nature of particles. Choices of different w_n give radically different behaviors. For fermionic
 142 particles, it prevents a jump on to a site, if this site is already occupied by a particle. The corresponding
 143 process is called Asymmetric Exclusion Process (ASEP). For boson, $w_n = 1$ and the process is called
 144 Zero Range Process (ZRP). At the edges of the lattice the probability rules are different: at the left
 145 edge a particle can enter with probability α and exit with probability γw_n whereas at the right edge a
 146 particle can exit with probability βw_n and enter with probability δ .

147 Without loss of generality, we may consider only $p \geq 1/2$ which corresponds to a particle flow
 148 from the left to the right and note $U = 2p - 1$. After a sufficiently long time the system reaches a
 149 non-equilibrium steady state, with a well defined density profile (or fugacity profile) across the boxes
 150 ranging between ρ_a , the density of the left reservoir and ρ_b , the density of the right reservoir, given by
 151 $\rho_a = \alpha(\gamma + \epsilon\alpha)$ and $\rho_b = \delta(\beta + \epsilon\delta)$, where $\epsilon = 1$ for ASEP (fermion) and $\epsilon = -1$ for ZRP (boson). In
 152 the sequel, we fix $\gamma + \alpha = 1$ and $\beta + \delta = 1$, and denote $\Delta T = \rho_a - \rho_b$ the parameter that measures the
 153 balance between the input rate of the left reservoir (the equivalent of the heat or momentum injection),
 154 and the removal rate of the right reservoir (the equivalent of the heat or momentum dissipation). Once
 155 β (say), ΔT are fixed, we can compute all the other parameter α , γ and δ of the model. In the sequel,
 156 we fix $\beta = 0.75$, and vary ΔT and/or U .

Taking the continuous limit of this process, it may be checked that the fugacity $Z = \rho/(1 + \rho)$
 of stationary solutions of a system consisting of boxes of size $\frac{1}{L}$ follow the continuous equation
 [29]:

$$U \frac{\partial Z}{\partial x} - \frac{1}{2L} \frac{\partial^2 Z}{\partial x^2} = 0, \quad (2)$$

corresponding to stationary solution of a passive scalar equation with velocity U and diffusivity $\frac{1}{2L}$.
 Therefore, the fugacity Z is a passive scalar obeying a convective-diffusion equation. We thus see
 that $U = 0$ corresponds to a purely conductive regime whereas the larger U the more convective the
 regime. This toy model therefore mimics in the continuous limit the behavior of scalar transport in the
 von Karman, Rayleigh-Bénard, Couette or Earth system we are trying to understand. The toy model
 is a discrete Markov process with 2^L states. It is characterized by its transition matrix $P = (p_{ij})$ which
 is irreducible. Thus, the probability measure on the states converges to the stationary probability
 measure $\mu = (\mu_{1_{stat}}, \dots, \mu_{2^L_{stat}})$ which satisfies:

$$\mu_{i_{stat}} = \sum_{j=1}^{2^L} \mu_{j_{stat}} \cdot p_{ji} \quad \forall i \in \llbracket 1, 2^L \rrbracket. \quad (3)$$

157 This Markov property makes our model simple enough so that exact computations are analytically
 158 tractable and numerical simulations are possible up to $L = 10$ (ASEP model) to $L = 1000$ (ZRP
 159 model) on a laptop computer. The idea now is to apply the Principle B in these toy models, and see
 160 what useful information we can derive from it.

161

162 3.2. Maximum Entropy production in Zero Range and ASymmetric Exclusion Process

163 We turn to the definition of entropy production in our toy model system. For a macroscopic system
 164 subject to thermodynamic forces X_i and fluxes J_i , the thermodynamic entropy production is given by:
 165 [30,31]:

$$\sigma_{thermo} = \sum_i J_i X_i \quad (4)$$

166 The fluxes to consider for a diffusive particles model are fluxes of particles and the thermodynamic
 167 forces can be written $X = \Delta(-\frac{\nu}{T})$ where T is the temperature and ν the chemical potential
 168 proportional to $\log(\rho)$ for an ideal gas [30]. So, as the temperature is here fixed, the thermodynamic
 169 Entropy production of a given stationary state takes the form:

$$\sigma_{thermo} \propto \sum_{i=1}^L J_i (\log(\rho_i) - \log(\rho_{i+1})) = J \cdot (\log(\rho_1) - \log(\rho_L)) \quad (5)$$

where ρ is the stationary density distribution and $J = J_i$ the particle fluxes in the stationary state,
 where all fluxes become equal and independent on the site. ρ and J are both (nonlinear) function of
 U . It is easy to show [32] that this definition is just the continuous limit of the classical thermodynamic
 entropy production in an ideal gas, that reads:

$$\sigma_{thermo} = - \int_{A_+}^{B_-} J(x,t) \frac{\partial \log(\rho(x,t))}{\partial x} dx \quad (6)$$

In the case of bosonic particles (ZRP model), this entropy production takes an compact analytical
 shape in the (thermodynamic) limit $L \rightarrow 0$ [33]:

$$\sigma(U) = \frac{\alpha U}{U + \gamma} \left(\log\left(\frac{\alpha}{U + \gamma - \alpha}\right) - \log\left(\frac{(\alpha + \delta)U + \gamma\delta}{U(\beta - \alpha - \delta) + \beta\gamma - \gamma\delta}\right) \right) \quad (7)$$

170 Because $U = 2p - 1 \geq 0$ the entropy production is positive if and only if $\rho_a \leq \rho_b$. This means that
 171 fluxes are in the opposite direction of the gradient. We remark than if $U = 0$ then $\sigma(U) = 0$. Indeed
 172 J is proportional to U in this model. Moreover, $\sigma(U)$ is zero also when $\rho_1 = \rho_L$. This happens when
 173 U increases, $\rho_a(U)$ decreases and $\rho_b(U)$ increases till they take the same value. Thus it exists U , large
 174 enough, for which $\sigma(U) = 0$. Between these two values of U the entropy production has at least one
 175 maximum. By computing numerically the entropy production, we observe in fact that it is also true
 176 for the fermionic particles, even though we cannot prove it analytically. This is illustrated in Figure 6
 177 for $L = 100$ (ZRP) and $L = 10$ (ASEP).

The value of $U_{maxEP}(T)$ at which this maximum occurs depends on the distance to equilibrium of the
 system, characterized by the parameter $\Delta T = \alpha - \delta$. In the case of the ZRP model, it can be computed
 as [33]:

$$U_{maxEP,ZRP} = \frac{\Delta T}{4M\alpha} + 3 \frac{\Delta T^2(\alpha + 1)}{8M^2\alpha^2(\alpha - 1)} + o((\Delta T)^2), \quad (8)$$

178 where $M = (1 + 2\rho_a)(1 + 2\rho_b)$. This means that at equilibrium ($\Delta T = 0$, $\rho_a = \rho_b$), the maximum
 179 is attained for $U = 0$, i.e. the symmetric case. Numerical simulations of the ASEP system suggest
 180 that this behavior is qualitatively valid also for fermionic particles: the entropy production displays
 181 a maximum, that varies linearly in ΔT . Such behavior therefore appears quite generic of this class of
 182 toy model. When the system is close to equilibrium ($\Delta T \ll 1$), the maximum is very near zero, and,
 183 the density profile is linear, corresponding to a *conductive* case. When the system is out-of-equilibrium
 184 ($\Delta T \geq 0$) the maximum is shifted towards larger values of $U > 0$, corresponding to a *convective state*,
 185 with flattened profile. An example is provided in Figure 7 for the ZRP and ASEP model.

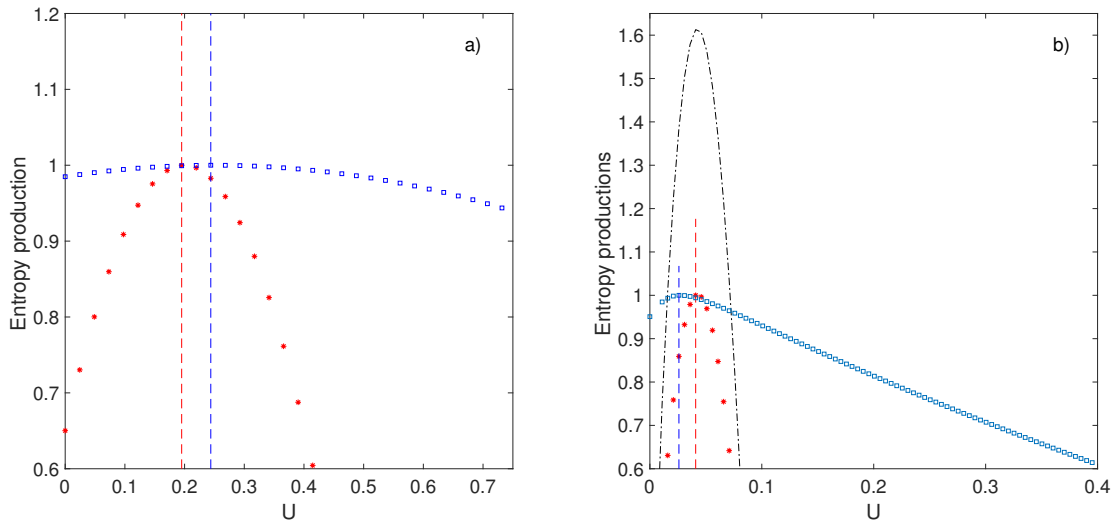


Figure 6. Entropy productions as a function of U for $\beta = 0.75$ and $\Delta T = 0.25$ for two toys models. Red stars: Thermodynamic entropy production $\sigma(U)$; blue squares: Kolmogorov-Sinai entropy $h_{KS}(U)$. The location of the maxima are denoted by vertical dashed line (red for $\sigma(U)$; and blue for $h_{KS}(U)$). a) Case $L=10$ ASEP (fermion) ; b) Case $L=100$ ZRP (boson) . The dot-dashed line is the asymptotic law for $\sigma(U)$ given by Eq. (7).

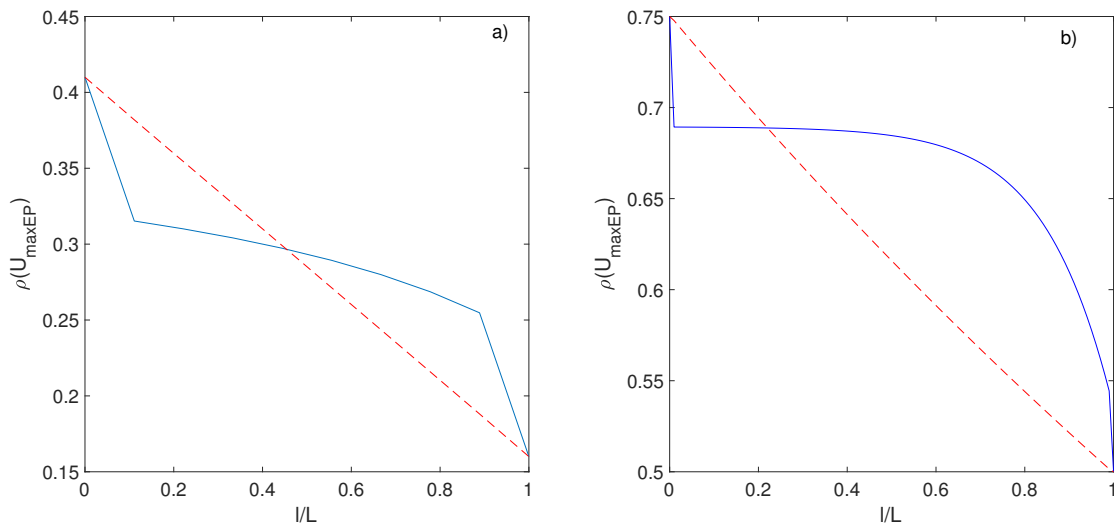


Figure 7. Profiles of density profiles (blue line) corresponding to models with $U = U_{maxEP}$ for $\beta = 0.75$ and $\Delta T = 0.25$ for two toys models. The red dashed line is the density profile obtained at $U = 0$, i.e. in the *conductive case*. a) Case $L=10$ ASEP (fermion) ; b) Case $L=100$ ZRP (boson)

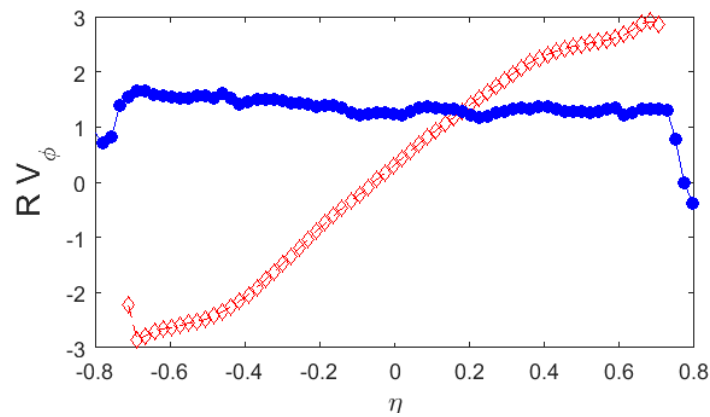


Figure 8. Profiles of angular momentum RV_ϕ as a function of the height from the central plane η in the von Karman laboratory experiment. Blue symbols with line: in the *bifurcated* state with higher energy dissipation. Red line with symbols: in the symmetric state.

186 Our toy models are examples of systems with deviation from equilibrium (labelled by ΔT), admitting
 187 several stationary states (labelled by U). So if we were to apply our MinMax/Principle B to these
 188 toy models, we would select the model corresponding to $U_{max_{EP}}$ as the most stable one, i.e. the
 189 conductive state with linear profile at equilibrium, and the convective state with flattened profile at
 190 non-equilibrium. Interestingly enough, this selection corresponds qualitatively to the type of profiles
 191 that are selected by the non-linear dynamics in the von Karman, Rayleigh-Benard or Couette system,
 192 as illustrated in Figure 8 for the VK flow: for low levels of fluctuations (low Reynolds or impeller
 193 with moderately bent blades) corresponding to close to equilibrium state, the most stable state is the
 194 symmetric state, with linear angular momentum profile. At larger fluctuation rates, the most stable
 195 state is the bifurcated state, with flat angular momentum at the center.

196 The ability of Principle B, based only on entropy i.e. *equilibrium* notions, to predict at least
 197 qualitatively the correct behavior of scalar transport in several non-equilibrium turbulent system
 198 is puzzling. It would be more satisfying to connect this Principle to other notions that seem more
 199 appropriate in the case of non-equilibrium system. This is the topic of the next section.

200 3.3. From Maximum Entropy Production to Kolmogorov-Sinai Entropy

The physical meaning of the thermodynamic entropy production is the measure of irreversibility: the larger σ the more irreversible the system [34]. It is however only a *static* quantity, being unconnected to the behavior of trajectories in the phase space. In that respect, it is not in agreement with the ideas of Jaynes that non-equilibrium systems should be characterized by a probability distribution on the trajectories in phase space, instead of just the points in phase space at equilibrium. In the context of Markov chains, Jaynes' idea provides a natural generalization of equilibrium statistical mechanics [35], by considering the Kolmogorov-Sinai entropy (KSE). There are many ways to estimate the Kolmogorov-Sinai entropy associated with a Markov chain [35,36]. The most useful one in our context is the one defined as the time derivative of the Jaynes entropy. To characterize the dynamics of the system during the time interval $[0, t]$, one considers the possible dynamical trajectories $\Gamma_{[0,t]}$ and the associated probabilities $p_{\Gamma_{[0,t]}}$. The dynamical trajectories entropy- the Jaynes entropy- reads:

$$S_{Jaynes}(t) = - \sum_{\Gamma_{[0,t]}} p_{\Gamma_{[0,t]}} \cdot \log(p_{\Gamma_{[0,t]}}) \quad (9)$$

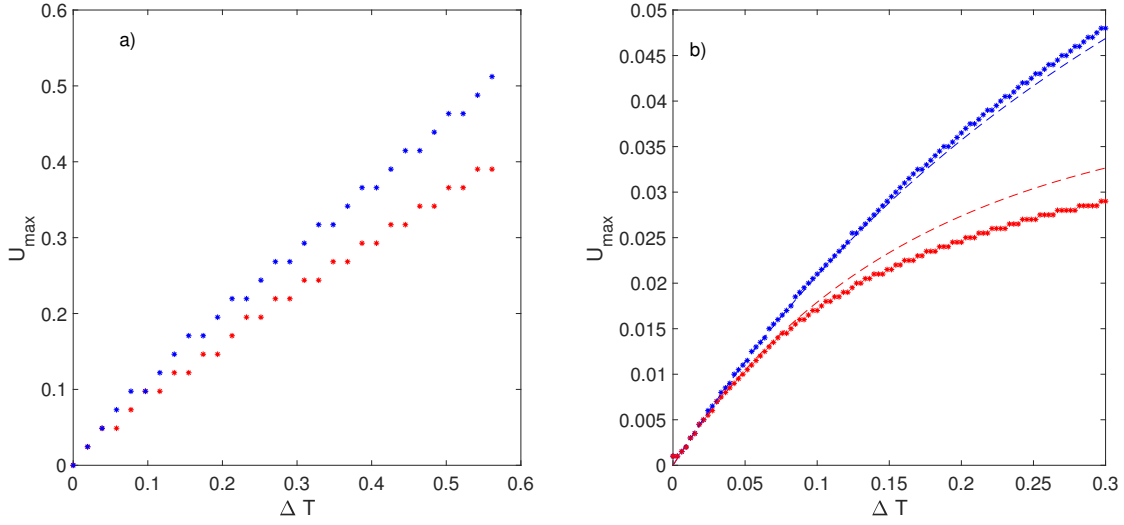


Figure 9. Location of the maximum of thermodynamic entropy production U_{maxEP} (red stars) and maxim of Kolmogorov-Sinai entropy (blue stars) as a function of ΔT . a) Case $L=10$ ASEP (fermion). b) Case $L=100$ ZRP (boson). The red dashed line is the second order approximation given by Eq. (8). The blue-dashed line is the first order approximation to the location of the maxima (Eq. (12)).

For a Markov chain we find that:

$$S_{Jaynes}(t) - S_{Jaynes}(t-1) = - \sum_{(i,j)} \mu_{i_{stat}} p_{ij} \log(p_{ij}) \quad (10)$$

Thus, the Kolmogorov-Sinai Entropy for the Markov chain is:

$$h_{KS} = - \sum_{(i,j)} \mu_{i_{stat}} p_{ij} \log(p_{ij}), \quad (11)$$

201 where μ_i is the stationary measure and p_{ij} the transition matrix.

In the case of bosonic particles (ZRP model), the KSE can be computed analytically and it admits a maximum as a function of U [33]. The value of U corresponding to this maximum can also be computed analytically, and leads to:

$$U_{max_{KS}} = \frac{\Delta T}{4M\alpha} + o(\Delta T), \quad (12)$$

202 Such first order approximation appear to be valid even far from equilibrium (see Figure 9).
 203 Comparing with the value for the maximum of entropy production, we see that the two maxima
 204 coincide, to first order in ΔT and $1/L$: in the continuous limit, Principle B provides the same kind of
 205 information than a third principle that we may state as:

206 *Principle C: In certain non-equilibrium systems with coexistence of several stationary state, the most*
 207 *stable one is that of Maximum Kolmogorov-Sinai Entropy.*

208 Is this third principle any better that the principle B? In our toy models, it seems to give the same
 209 information than the Principle B: for ZRP, we have shown analytically that the maxima of each
 210 principle coincide to first order in ΔT and $1/L$. Numerical simulation of the ASEP system suggests
 211 that it is also true for fermionic particles: for a given value of ΔT the difference between the two
 212 maxima $\Delta U_{max} = U_{maxEP} - U_{max_{KS}}$ decreases with increasing L [32]. For a fixed L , the difference

213 between the two maxima increases with ΔT , but never exceeds a few percent at $L = 10$ [32], see
 214 Figure 9).

215 In turbulent system, the test of this principle is more elaborate, because the computation of h_{KS} is not
 216 straightforward. It would however be interesting to test it in numerical simulations.

217 4. From Maximum Kolmogorov-Sinai entropy to Minimum Mixing Time in Markov 218 Chains

219 Principle C is appealing because it involves a quantity clearly connected with dynamics in the phase
 220 space, but it is still unclear why the maximization of the entropy associated with paths in the phase
 221 space should select the most stable stationary state, if any. To make such a link, we need to somehow
 222 connect to the relaxation towards a stationary state, i.e. the time a system takes to reach its stationary
 223 state. In Markov chains, such a time is well defined, and is called the mixing time. Intuitively, one
 224 may think that the smaller the mixing time, the most probable it is to observe a given stationary state.
 225 So there should be a link between the maximum of Kolmogorov-Sinai entropy, and the Minimum
 226 mixing time. This link has been derived in [25], for general Markov chains defined by their adjacency
 227 matrix A and transition matrix P , defined as follows: $A(i, j) = 1$ if and only if there is a link between
 228 the nodes i and j and 0 otherwise. $P = (p_{ij})$ where p_{ij} is the probability for a particle in i to hop on
 229 the j node. Specifically, it has been shown that Kolmogorov-Sinai entropy increases with decreasing
 230 mixing time (see Figure 1 in [25]). More generally, for a given degree of sparseness of the matrix
 231 A (number of 0), the Markov process maximizing the Kolmogorov-Sinai entropy is close (using an
 232 appropriate distance) to the Markov process minimizing the mixing time. The degree of closeness
 233 depends on the sparseness, and becomes very large with decreasing sparseness, i.e. for unconstrained
 234 dynamics (Figure 3 in [25]).

235 This result provides us with a fourth principle in Markov chains:

236 *Principle D: In certain non-equilibrium systems with coexistence of several stationary states, the*
 237 *most stable one is that of Minimum Mixing Time.*

238

239 Given what we have seen before, there are 4 general principle that select the same stationary state,
 240 in the limit of large size and small deviations from out-of-equilibrium (see Figure 10). Among all 4,
 241 the Principle that provides the better understanding of its application is the Principle D, because the
 242 smaller the mixing time, the less time it is required to reach a given state and so the most probable the
 243 corresponding state. This phenomenological understanding can actually be given a deeper meaning
 244 when switching from Markov chains to Langevin systems.

245 5. From Minimum Mixing Time in Markov Chains to Mean Escape rate in Langevin 246 systems

The notions we have derived in Markov chains have actually a natural counter-part in general Langevin systems. Consider indeed the general Langevin model:

$$\dot{x} = f(x) + \zeta, \quad (13)$$

where $f(x)$ is the force and ζ is a delta-correlated Gaussian noise:

$$\langle \zeta \rangle = 0, \quad \langle \zeta(t)\zeta(t') \rangle = 2D\delta(t - t'). \quad (14)$$

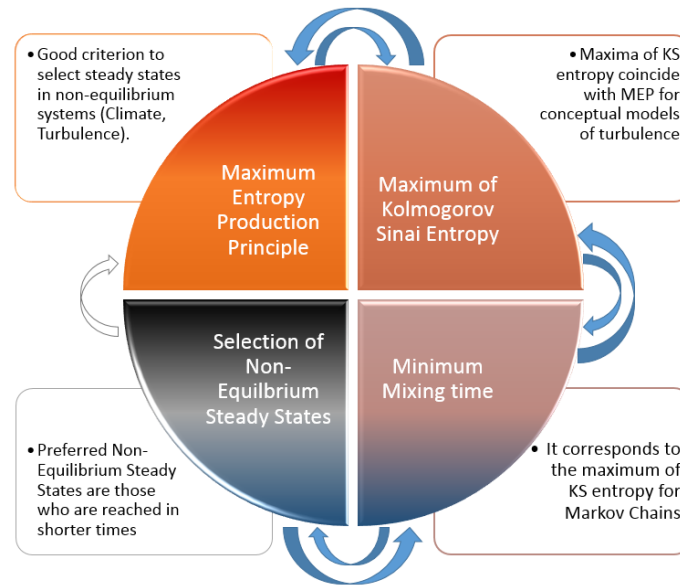


Figure 10. Conceptual path to demonstrate the validity of the Maximum Entropy Principle followed in this paper.

247 The probability distribution of x , $P(x, t)$ then obeys a Fokker-Planck equation:

$$\begin{aligned}\partial_t P(x, t) &= -\partial_x(f(x)P(x, t)) + D\partial_x\partial_x P(x, t), \\ &= \partial_x(J(x, t)P(x, t)),\end{aligned}\quad (15)$$

where $J = f(x)P(x, t) - D\partial_x P(x, t)$ is the current. It is well known (see e.g. [37]) that the discretization of the Langevin model on a lattice of grid spacing a (so that $x = na$) is a Markov chain, governed by the master equation:

$$\partial_t P_n = w_{n-1}^+ P_{n-1} - w_n^+ P_n + w_{n+1}^- - w_n^- P_n, \quad (16)$$

248 where P_n is the probability of having the particle at node n and w_n^\pm are the probability of forward and
249 backward jump at node n , given by:

$$\begin{aligned}w_n^+ &= \frac{D}{a^2}, \\ w_n^- &= \frac{D}{a^2} - \frac{f_n}{a}.\end{aligned}\quad (17)$$

From this, we can compute the Kolomogorov-Sinai entropy as [38]

$$h_{KSL} = \langle w_n^+ \ln \frac{w_n^+}{w_{n+1}^-} + w_n^- \ln \frac{w_n^-}{w_{n-1}^+} \rangle, \quad (18)$$

which in the continuous limit $a \rightarrow 0$ becomes:

$$h_{KSLc} = \langle \frac{1}{D} |f(x)|^2 + f'(x) \rangle, \quad (19)$$

250 which is the well know entropy production.

On the other hand, the dissipated power in the Langevin process can easily be computed as :

$$\mathcal{P} = \langle f\dot{x} \rangle = \int J(x, t)f(x, t)dx = \langle |f|^2 + Df'(x) \rangle = Dh_{KSLc}. \quad (20)$$

251 The Kolmogorov-Sinai entropy and the dissipated power are thus proportional to each other. In such
 252 example, it is thus clear that maxima of h_{KS} and maxima of entropy production coincide.

253 On the other hand, when f derives from a potential, $f = -\partial_x(U)$, there may be several meta-stable
 254 positions at U local minima. In such a case, it is known from diffusion maps theory and spectral
 255 clustering [39] that the exit times from the meta-stable states are connected to the smallest eigenvalues
 256 of the operator \mathcal{H} , such that $\mathcal{H}P = D\Delta P - \nabla P \nabla U$, which is the equivalent of the Liouville operator
 257 in Markov chain. More specifically, if U has N local minima, then the spectra of \mathcal{H} has a cluster
 258 of N eigenvalues $\mu_1 < \mu_2 < \dots < \mu_N$ located near 0, each of which being associated with the exit
 259 time it takes to get out from the local minima S_i to the state corresponding to the deepest minimum
 260 (the equilibrium state). For example, $\mu_1 = \frac{1}{\tau_1}$, where τ_1 is the mean exit time to jump the highest
 261 barrier of energy onto the deepest well. We see from that that the smaller the eigenvalue (equivalent
 262 to the mixing time), the longer time it takes to jump from this metastable state, and so the *more*
 263 *stable* is this state. This provides a quantitative justification of the notion that *the most stable stationary*
 264 *states* are the one with the minimum mixing time. It is worth mentioning that Langevin systems
 265 are now incorporated in numerical weather prediction to provide some flexibility to the sub-grid
 266 scales parameterizations [40]. Models based on these so-called stochastic parameterizations have
 267 usually better prediction skills than models based on deterministic parameterizations. Stochastic
 268 parameterization is therefore increasingly used in different aspects of weather and climate modeling
 269 [41]. We might speculate that, in these large simulations, the stochasticity helps models to reach more
 270 realistic results by favoring jumps to *more stable* states as outlined above. Another way to select these
 271 more realistic states could possibly be to search for the ones that maximize dissipation or entropy
 272 production, as was done empirically in a simple way in Paltridge's model [13,14].

273 6. Summary: Turbulence as a minimum Mixing Time State?

274 Considering a *mixing time* to characterize turbulence is natural, given the well known enhancement
 275 of mixing properties observed in turbulent flows. The mixing time defined for Markov chains is
 276 also comparable to the mixing time one would naturally define for turbulence, namely the smallest
 277 time after which a given partition of a scale quantity is uniformly spread over the volume. This time
 278 corresponds to the one defined by Arnold for dynamical systems [42] when introducing the concept
 279 of strong mixing. Here, it is the time associated with eigenvalues of the Liouville operator of the
 280 processes describing the turbulence action. In the specific example we consider here, namely von
 281 Karman flow, the turbulence is characterized by a symmetry along the rotation axis, which favors
 282 stationary states in which angular momentum is mixed along meridional planes [43,44]. In this case,
 283 there is a clear connection between the equation obeying the angular momentum and the classical
 284 Fokker-Planck equation Eq. (15). One can thus hope in such a case to find a Langevin process
 285 describing the angular momentum mixing. This was actually done in [43] and shown to reproduce
 286 very well the power statistics in both regimes of constant torque and constant speed forcing, in case
 287 where there is only one stationary state. For multiple stationary states, obtained in the regime with
 288 fixed applied torque, the corresponding Langevin process has been derived in [45] and it turns out
 289 to be a non-linear stochastic oscillator, featuring multiple metastable states and limit cycles. Such
 290 oscillator is found to describe the dynamics of the reduced frequency $\theta = (f_1 - f_2)/(f_1 + f_2)$ which
 291 is a global observable respecting the symmetries of the flow. The challenge is then to compute the
 292 mixing times of the different metastable states arising in the non-linear stochastic oscillator.

293 In [46–48] we have shown that the mixing time τ of turbulent flows can be easily obtained by fitting
 294 the Langevin process (or auto-regressive process) $x_t = \phi x_{t-1} + \zeta(t)$ to data. Here $x(t)$ is a global
 295 quantity tracing the symmetry of the flows, $\zeta(t)$ is a random variable normally distributed and $-1 <$
 296 $\phi < 1$ is the so called auto-regressive coefficient. The link with the mixing time is made through
 297 the parameter $|\phi|$ which is indeed proportional to τ : the larger this quantity, the slower the mixing

298 in the system, because the dynamics weight more the present observation x_t when updating x_{t+1} .
299 In [46], only flow configurations with a single stationary state have been analysed and ϕ computed
300 using the complete time series. To extend the results to the flow regimes featuring multistability, we
301 use the strategy outlined in [49]. First of all, we reconstruct the dynamics by using the embedding
302 methodology on the series of partial maxima of θ , denoted as θ_i . A 2D section of the attractor is
303 shown in the upper panel of Figure 11 and it is obtained by plotting θ_i as a function of the subsequent
304 maxima θ_{i+1} . The histogram of θ_i is reported in the lower panel and show the correspondence to three
305 metastable states s_1, s_2, s_3 . Since we are not dealing with a stationary process, we cannot compute a
306 single ϕ for the full time series of θ . The method introduced in [49] consists in computing a value of
307 ϕ_i for each θ_i , taking the 50 previous observations of the complete time series. The distribution of $|\phi|$
308 is shown in colorscale in Figure 11. It is evident that the most represented states (s_1 and s_2) are those
309 with the minimum mixing time whereas the most unstable one (s_3) is the one with the largest mixing
310 time. This example shows that the results outlined in Section 5 can be extended to higher dimensions
311 and that there is a simple strategy to compute the mixing times in complex systems.

312 It is quite plausible that such study can be generalized to turbulent systems with other symmetry,
313 such as symmetry by translation along an axis. The turbulent shear flow enter in that category.
314 Finally, we note that in Rayleigh-Bénard systems or in stratified turbulence, the temperature is also a
315 quantity that is mixed within the flow, and that should also be liable to a Langevin description. It is
316 therefore not a coincidence that shear flow, Rayleigh-Bénard convection and von Karman flows are
317 so far the only systems in which the principle of Maximum Energy dissipation has been applied with
318 some success. They are systems where a Langevin description is possible, and where the Maximum
319 Energy dissipation principle in fact coincides with the Minimum Mixing time principle, connected
320 to the longest exit time from meta-stable states. As observed by [50], these flows tends to be in
321 a steady state with a distribution of eddies that produce the maximum rate of entropy increase in
322 the nonequilibrium surroundings. In more general turbulence, it is not clear that such a Langevin
323 description is possible, so that the statement of *Turbulence as a minimum Mixing Time State* might
324 actually be limited to quite special situations, where symmetry or dynamics impose pure mixing
325 of a quantity (like angular momentum, momentum or temperature). Shear flow, Rayleigh Benard
326 convection and von Karman flows belong to this category.

327 Other conceptual pathways allow to link MEP to the underlying dynamics of the system: Moroz
328 [51] suggests that the dissipation time minimization is linked to the least action principle, used
329 in chemistry, biology and physics to derive the equations of motion. Although this theoretical
330 formulation goes in the same direction of the results provided in this paper, our approach provides a
331 rather practical way to connect dynamics and thermodynamics through statistical quantities directly
332 computable from experimental time series.

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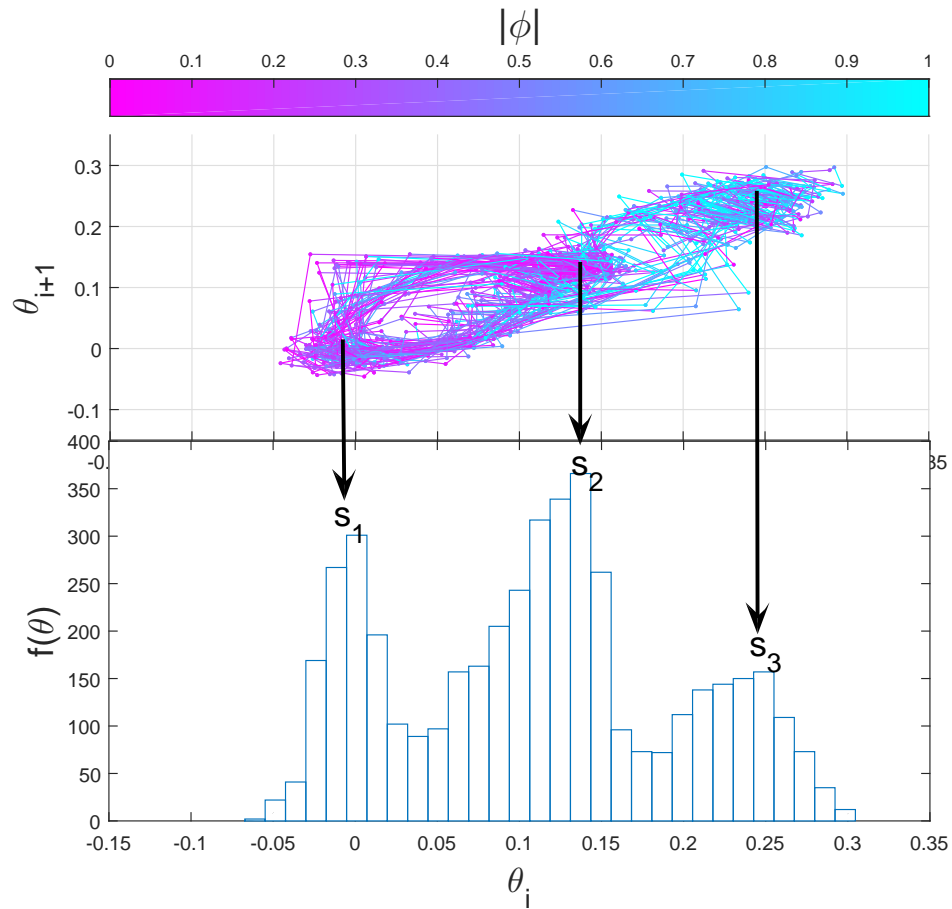


Figure 11. Upper panel: 2D Poincaré section of the Von Karman attractor obtained embedding the partial maxima of the reduced frequency θ_i . The colors represent $|\phi|$, the autoregressive coefficient computed for each of the θ_i , using the 50 previous observations of the full time series of θ . Lower panel: histogram of the partial maxima θ_i . The metastable state s_3 is visited less than s_1 and s_2 and corresponds to higher mixing time (values of $|\phi|$ close to 1)

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