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# On nonconservativeness of Eringen's nonlocal elasticity in beam mechanics: correction from a discrete-based approach

**Abstract** In this paper, the self-adjointness of Eringen's nonlocal elasticity is investigated based on simple one-dimensional beam models. It is shown that Eringen's model may be nonself-adjoint and that it can result in an unexpected stiffening effect for a cantilever's fundamental vibration frequency with respect to increasing Eringen's small length scale coefficient. This is clearly inconsistent with the softening results of all other boundary conditions as well as the higher vibration modes of a cantilever beam. By using a (discrete) microstructured beam model, we demonstrate that the vibration frequencies obtained decrease with respect to an increase in the small length scale parameter. Furthermore, the microstructured beam model is consistently approximated by Eringen's nonlocal model for an equivalent set of beam equations in conjunction with variationally based boundary conditions (conservative elastic model). An equivalence principle is shown between the Hamiltonian of the microstructured system and the one of the nonlocal continuous beam system. We then offer a remedy for the special case of the cantilever beam by tweaking the boundary condition for the bending moment of a free end based on the microstructured model.

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## 1 Introduction

Nonlocal elasticity theories provide efficient reversible mechanisms for capturing small scale effects induced by the microstructure or the discrete nature of the material at a smaller scale. In order to allow for these small length scale effects in microstructured materials (such as nanostructures for instance), Eringen's nonlocal elasticity based on an implicit differential equation between the stress and the strain quantities may be used [1,2]. Eringen's nonlocal elasticity can be presented in its simplest form, expressed for uniaxial case, as

$$\sigma(x) - l_c^2 \sigma''(x) = E \varepsilon(x) \quad (1)$$

where  $\sigma(x)$  is the uniaxial stress,  $\varepsilon(x)$  the uniaxial strain,  $x$  the spatial coordinate,  $l_c$  the length scale parameter responsible for the nonlocal effect,  $E$  the modulus of elasticity, and the double prime denotes the second derivative with respect to  $x$ . The first nonlocal solutions in structural mechanics are rather recent and date from the beginning of this century, with specific applications to nanotechnology. Surprisingly, we have only one decade of active research in the field of nonlocal structural mechanics, since the appearance of two pioneering papers by Peddieson et al. [3] and Sudak [4] in 2003. New solutions have then been found and developed following these two papers, which include nonlocal Euler-Bernoulli, nonlocal Timoshenko, and nonlocal higher-order beam theories (see for instance [5–8]).

Eringen's nonlocal elasticity appears to be not only an engineering phenomenological model that is able to introduce, rather arbitrarily, some scale effects, but it has also been shown to capture Born-Karman lattice dispersion curves related to the discreteness of the lattice structure (see [1] for the calibration of the small length scale on wave properties of lattice structures). Hence, such nonlocal models have found numerous applications in the field of wave propagation in heterogeneous materials (see [9] or [10]). The calibration and the micromechanics justification of Eringen's nonlocal elasticity in beam mechanics are more recent. Challamel et al. [11, 12] or Wang et al. [13] showed, with a so-called continualization method, that microstructured bending systems globally behave as nonlocal beams of Eringen's type. It has been shown also that the spatial nonlocality is different for static analysis (buckling problems) as compared to dynamics analysis. The same kind of results has been also obtained for discrete shear systems continualized as nonlocal Timoshenko beam elements (see [14] or [15]).

## 2 Self-adjointness of Eringen's elasticity applied to beam mechanics

One question that is still debated about the applicability and the limitations of Eringen's model is whether the model is self-adjoint and if it can be justified from energy considerations. In a seminal paper dated from 2007, Reddy [5] mentioned the possible nonself-adjointness of Eringen's model and the impossibility to build the underlying quadratic functional for such nonlocal beam models (see, also [6]). Later, following the works of Adali [16,17], it has been shown that the governing equations of Eringen's type nonlocal elasticity applied to several beam mechanics models may be derived from a single-energy functional, with some variationally based boundary conditions (see also [7] or [8]). In the present paper, we attempt to reconcile both approaches: strictly speaking, it can be shown for specific boundary conditions that Eringen's nonlocal elasticity is not self-adjoint and one cannot derive the governing equations (including the consistent boundary conditions) of the nonlocal models from an energy functional. This confirms Reddy's anticipated result [5]. However, we also find that Eringen's nonlocal elasticity may be self-adjoint for some other boundary conditions. We also find that the nonself-adjointness of Eringen's nonlocal model is not physically justified from the microstructural point of view. In other words, a correction of the boundary conditions is needed when considering the nonlocal model of Eringen for accurately describing microstructure effects.

The focus of this paper on the possible nonself-adjointness of the differential-based Eringen's model does not cover some other integral-based nonlocal models available in the seminal book of Eringen [2], where for instance the kernel of the nonlocal model is assumed to have symmetrical spatial properties, a property that may be lost with the differential-based nonlocal model investigated in this paper herein.

As recently shown in [8], it is possible to build a functional for the bending, vibration, and buckling problems using nonlocal Euler-Bernoulli beams, as well as higher-order beam models, to obtain the correct

differential equations of Eringen's beam models. The works of Adali [16,17] and Elishakoff et al. [7] for nonlocal Timoshenko or higher-order beam models contain energy functionals. Apparently, it is possible, by an inverse procedure that uses the governing equations of the nonlocal model to construct the functional, to have a self-adjoint formulation of Eringen's beam models. However, there is a need for the clarification of the meaning of variationally consistent boundary conditions (i.e., boundary conditions derived from the functional constructed using the inverse procedure) that may differ from the boundary conditions obtained by the direct approach.

Considering both buckling and vibration problems, we will show in this paper that:

- the buckling problem of the Eringen nonlocal beams (Euler-Bernoulli, Timoshenko or higher-order shear beam models) is self-adjoint, i.e., it is possible to derive both the differential equations and the associated boundary conditions from the variation of a single functional; and
- the vibration problem is different in nature. The hinged-hinged beam problem is self-adjoint, but not self-adjoint for the clamped-free nonlocal beam (or for any boundary condition that involves the specification of a generalized force including a concentrated force and a concentrated torque).

The meaning of this possible nonself-adjointness for the free-edge condition is discussed at the end of this paper.

### 3 Dependency of Eringen's kernel on boundary conditions

Let us consider the moment-curvature relation in the nonlocal Euler-Bernoulli beam theory based on Eringen's differential model:

$$M(x) - l_c^2 M''(x) = EI \chi(x) \text{ with } \chi(x) = w''(x) \quad (2)$$

where  $M$  is the bending moment,  $I$  the second moment of area,  $\chi$  the curvature, and  $w$  the deflection of the beam. Eringen's nonlocal elasticity appears to be efficient in capturing the scale effects of microstructured media such as microstructured beams (see for instance [11,12]). The length scale parameter  $l_c$  can then be computed from the microstructure cell size.

As pointed out by Eringen [1], this differential equation shows that the bending moment is a spatially weighted average of the curvature variable with the weighting function being the Green's function  $G(x, y)$  of the differential system associated with the relevant boundary conditions:

$$M(x) = EI \int_0^L G(x, y) \chi(y) dy \quad (3)$$

where  $L$  is the length of the beam. If for instance, the beam is rotationally free at the boundary (hinge-hinge supported beams for instance), the Green's function is expressed as [7]:

$$M(0) = M(L) = 0 \Rightarrow \begin{cases} G(x, y) = \frac{1}{l_c} \frac{\sinh\left(\frac{L-y}{l_c}\right)}{\sinh\left(\frac{L}{l_c}\right)} \sinh\left(\frac{x}{l_c}\right) & \text{if } x \leq y \\ G(x, y) = \frac{1}{l_c} \frac{\sinh\left(\frac{L-x}{l_c}\right)}{\sinh\left(\frac{L}{l_c}\right)} \sinh\left(\frac{y}{l_c}\right) & \text{if } x \geq y \end{cases} \quad (4)$$

Note that the Green's function is symmetrical in this case, i.e.,  $G(x, y) = G(y, x)$ . The Green's functions strongly depend on the boundary conditions and are obtained from the differential equations (see for instance [18]):

$$G(x, y) - l_c^2 \frac{\partial^2 G(x, y)}{\partial x^2} = \delta(x - y) \quad (5)$$

with the solution expressed by

$$\begin{cases} G(x, y) = A(y) \cosh \frac{x}{l_c} + B(y) \sinh \frac{x}{l_c} & \text{if } x < y \\ G(x, y) = C(y) \cosh \frac{x}{l_c} + D(y) \sinh \frac{x}{l_c} & \text{if } x > y \end{cases} \quad (6)$$

The introduction of the four following boundary conditions leads to the Green's function of Eq. (4):

$$G(0, y) = 0, \quad G(L, y) = 0, \quad G(y^-, y^-) = G(y^+, y^+) \quad \text{and} \quad \frac{\partial G}{\partial x}(y^+, y^+) - \frac{\partial G}{\partial x}(y^-, y^-) = \frac{-1}{l_c^2} \quad (7)$$

Following the same procedure, it is also possible to express other Green's function. For instance, in the case of translationally free boundaries (see also [7]):

$$V(0) = M'(0) = V(L) = M'(L) = 0 \quad \Rightarrow \quad \begin{cases} G(x, y) = \frac{1}{l_c} \frac{\cosh\left(\frac{L-y}{l_c}\right)}{\sinh\left(\frac{L}{l_c}\right)} \cosh\left(\frac{x}{l_c}\right) & \text{if } x \leq y \\ G(x, y) = \frac{1}{l_c} \frac{\cosh\left(\frac{L-x}{l_c}\right)}{\sinh\left(\frac{L}{l_c}\right)} \cosh\left(\frac{y}{l_c}\right) & \text{if } x \geq y \end{cases} \quad (8)$$

where  $V$  is the shear force variable. For such boundary conditions, the nonlocal operator is self-adjoint and the nonlocal energy functional is given, for instance by Challamel and Wang [19] (see also [20] or [2] for more general energy functionals applied to three-dimensional nonlocal elastic media):

$$\pi = \int_0^L \int_0^L \frac{1}{2} EIG(x, y) \chi(y) \chi(x) dx dy = \int_0^L \frac{1}{2} M(x) \chi(x) dx \quad (9)$$

As detailed in [19] and citing Roach [18] for instance, following the Green-type identity, the nonlocal operator is self-adjoint for unmixed or periodic boundary conditions. This is typically the case of the moment-free boundary conditions (see for instance Eq. (4) for the hinge-hinge supported beam, where  $M$  is prescribed and equal to 0), or shear-free boundary conditions (see Eq. (8) for translational-free boundary conditions).

This is no longer the case for the clamped-free boundary condition. Therefore, Eq. (9) is not valid for all the boundary conditions, and this explains a part of the apparent paradox (the self-adjointness of Eringen's nonlocal model depends on the considered boundary conditions).

The nonlocal system is self-adjoint if:

$$\int_0^L M(x) \delta \chi(x) dx = \int_0^L \delta M(x) \chi(x) dx \quad (10)$$

It can be shown by integration by parts that [19]:

$$\int_0^L \chi(x) \delta M(x) dx = \int_0^L M(x) \delta \chi(x) dx - \frac{l_c^2}{EI} [M' \delta M]_0^L + \frac{l_c^2}{EI} [M \delta M']_0^L \quad (11)$$

Hence, the self-adjointness is controlled by the following equality:

$$[M' \delta M]_0^L - [M \delta M']_0^L = 0 \quad (12)$$

As a consequence, in order to fulfil the conditions Eq. (12), the following boundary conditions are associated with a self-adjoint formulation of Eringen's nonlocal Euler-Bernoulli beam model:

$$M(0) = M(L) = 0, \quad M'(0) = M'(L) = 0, \quad M(0) = M'(L) = 0 \text{ or } M'(0) = M(L) = 0 \quad (13)$$

, whereas the following boundary conditions (including the clamped-free boundary conditions) lead to a nonself-adjoint formulation of Eringen's-based nonlocal Euler-Bernoulli beam model:

$$M(0) = M'(0) = 0 \quad \text{or} \quad M(L) = M'(L) = 0 \quad (14)$$

In the case of Eq. (14), the self-adjointness of the problem cannot be established (and the condition Eq. (12) is not necessarily valid). For instance, for the clamped-free boundary conditions

$$V(0) = M(0) = 0 \quad \Rightarrow \quad \begin{cases} G(x, y) = 0 & \text{if } x \leq y \\ G(x, y) = \frac{1}{l_c} \sinh\left(\frac{y-x}{l_c}\right) & \text{if } x \geq y \end{cases} \quad (15)$$

It appears in this case that  $G(x, y)$  is no longer symmetrical in this case, and the energy formulation Eq. (9) cannot be used in such a case.

#### 4 Buckling problem

The buckling equations for a beam loaded by an axial load  $P$  are obtained from the equilibrium equations

$$M'' + Pw'' = 0 \quad (16)$$

and the Eringen's-based nonlocal moment-curvature law:

$$M - l_c^2 M'' = EI w'' \quad (17)$$

leading to the differential equation for the deflection:

$$(EI - Pl_c^2) w^{(4)} + Pw'' = 0 \quad (18)$$

In this case, it is possible to show that the problem is self-adjoint whatever the boundary conditions, and the energy functional  $U$  is shown to be expressed simply by (see [7, 16] or [8]):

$$U[w] = \pi[w] - W_e[w] = \int_0^L \frac{1}{2} M w'' - \frac{1}{2} P w'^2 dx = \int_0^L \frac{1}{2} (EI - Pl_c^2) w''^2 - \frac{1}{2} P w'^2 dx \quad (19)$$

where  $\pi$  is the internal bending energy, whereas  $W_e$  is the work done by the axial force. The differential equation for the buckling of a nonlocal Euler-Bernoulli column is obtained from the application of the variational principle  $\delta U = 0$ , leading to the differential equations Eq. (18), with the natural and essential boundary conditions:

$$[(EI - Pl_c^2) w'' \delta w']_0^L = 0 \quad \text{and} \quad [-(EI - Pl_c^2) w''' + Pw']_0^L \delta w = 0 \quad (20)$$

which can also be expressed with the static variables as

$$[M \delta w']_0^L = 0 \quad \text{and} \quad [-(M' + Pw')]_0^L \delta w = 0 \quad (21)$$

In this case, the problem is self-adjoint whatever the boundary conditions, as the bending moment is directly proportional to the curvature:

$$M = (EI - Pl_c^2) \chi \quad (22)$$

which is a kind of modified "locality", without any Green's functions. Therefore, the discussion on the self-adjoint property is simplified in this case, due to the fact that the nonlocal law degenerates into a kind of local law in Eq. (22).

#### 5 Vibration problem

Now, for the vibrations analysis, we have

$$\begin{cases} M - l_c^2 M'' = EI w'' \\ M'' = -\rho A \ddot{w} \end{cases} \quad (23)$$

where  $\rho$  is the volumetric mass density and  $A$  is the cross-sectional area. We finally obtain the nonlocal bending wave equation:

$$EI \frac{\partial^4 w}{\partial x^4} - \mu l_c^2 \frac{\partial^4 w}{\partial x^2 \partial t^2} + \mu \frac{\partial^2 w}{\partial t^2} = 0 \quad (24)$$

with  $\mu = \rho A$ . As mentioned in [8], this wave equation corresponds to the vibration problem of a Rayleigh's beam, when the radius of gyration of the cross section  $r$  is assumed to be equal to the nonlocal length scale

$l_c$ . Adali [17] suggested that the nonlocal wave equation can be derived from the following strain energy and kinetic energy functionals:

$$U[w] = \int_0^L \frac{1}{2} EI w''^2 dx \quad \text{and} \quad T[w] = \frac{1}{2} \int_0^L (\mu \dot{w}^2 + \mu l_c^2 \dot{w}'^2) dx \quad (25)$$

where  $\mu$  is the mass per unit length of the beam material. The dynamic equations are obtained via the Hamilton principle, leading to Eq. (24), with the variationally based boundary conditions:

$$[(-EI w''' + \mu l_c^2 \ddot{w}') \delta w]_0^L = 0 \quad \text{and} \quad [(EI w'') \delta w']_0^L = 0 \quad (26)$$

or equivalently:

$$[-M' \delta w]_0^L = 0 \quad \text{and} \quad [(M + \mu l_c^2 \ddot{w}) \delta w']_0^L = 0 \quad (27)$$

It means that the bending-type free boundary condition  $(M + \mu l_c^2 \ddot{w}) = 0$  is different in this case from Eringen's model which should be simply  $M = 0$  and then the variationally based model based on Eq. (27) is not strictly equivalent to Eringen's model:

$$[-M' \delta w]_0^L = 0 \quad \text{and} \quad [M \delta w']_0^L = 0 \quad (28)$$

Note that only the boundary conditions differ from the variationally based model and Eringen's nonlocal model. As a second consequence, Eringen's model can be considered from the following variational statement:

$$\int_{t_1}^{t_2} (\delta U - \delta T - [\mu l_c^2 \ddot{w} \delta w']_0^L) dt = 0 \quad (29)$$

which is clearly nonself-adjointed, as  $[\mu l_c^2 \ddot{w} \delta w']_0^L$  cannot be derived from a potential. This additional term  $[\mu l_c^2 \ddot{w} \delta w']_0^L$  responsible of the nonconservativeness of the nonlocal vibrations problem can be attributed to a nonconservative inertia moment applied at the end of a Rayleigh's beam. Note that for the hinged-hinged beam problem, this last term is vanishing as  $w = 0$  along the boundaries, and the nonlocal elasticity problem is then self-adjointed (equivalent to the considering of the energy functional). This proof confirms the nonself-adjointness of Eringen's model without boundary corrections. This result is in agreement with Reddy's analysis [5] who has already mentioned the possible nonself-adjointness of Eringen's model, and the impossibility to build the underlying quadratic functional for such nonlocal beam models (see, also [6]). This nonself-adjointness explains why nonsymmetrical mass matrix can be derived for Eringen's nonlocal beam analyses in finite element beam analyses [21].

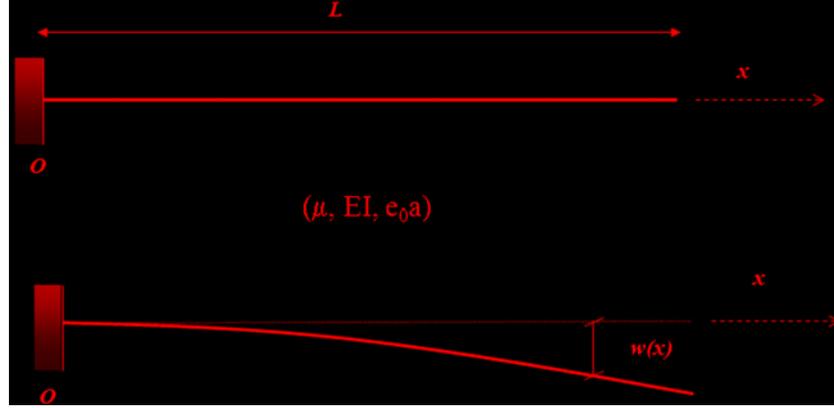
The particular boundary conditions for having the vibration problem nonself-adjoint is the clamped-free boundary condition (see Fig. 1). In Fig. 2, the two boundary conditions Eq. (27) called the proposed nonlocal model (exact self-adjoint nonlocal model) and Eq. (28) (exact nonself-adjoint Eringen's nonlocal model) are compared for the frequency sensitivity with respect to the small length scale. It is shown in Fig. 2 that the small length scale effect has a softening influence in case of the self-adjoint nonlocal elasticity model, whereas surprisingly, a stiffening effect is noticed with the nonself-adjoint nonlocal elasticity mode ([22,23] or [24]).

By assuming free harmonic motion, the differential equation is obtained:

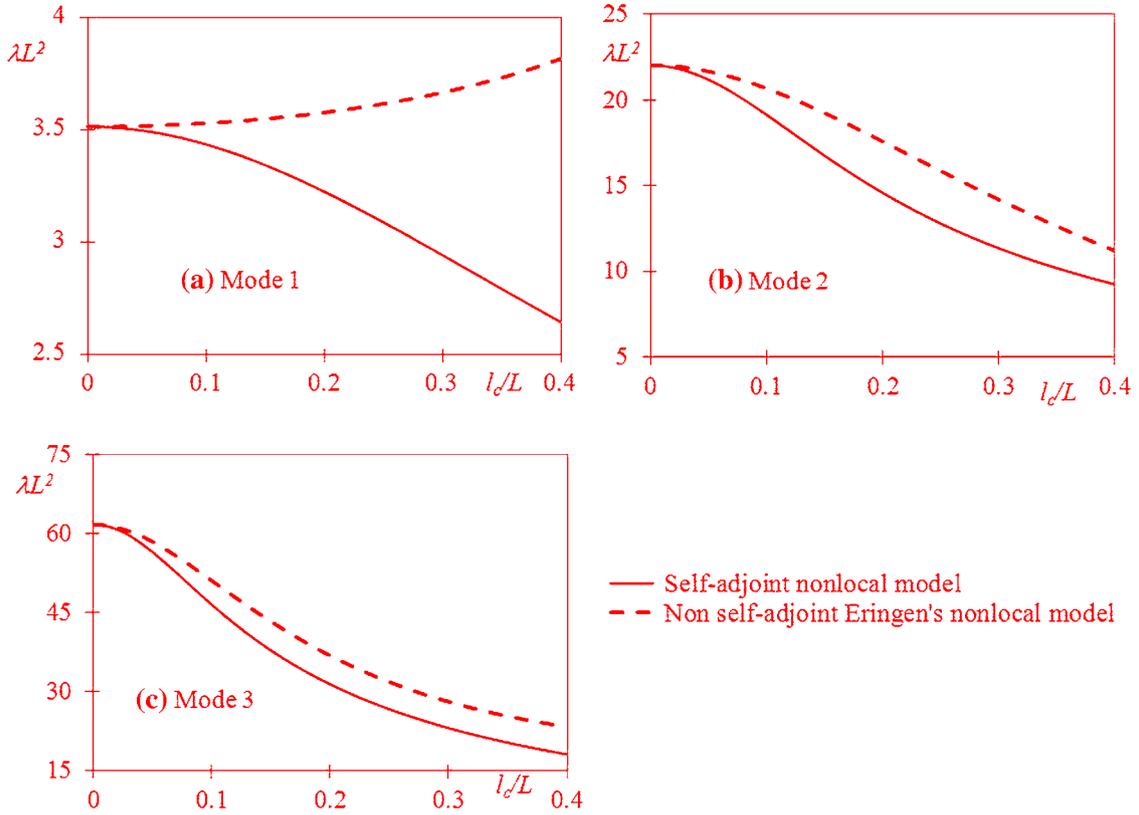
$$EI w^{(4)} + \mu l_c^2 \omega^2 w'' - \mu \omega^2 w = 0 \quad (30)$$

where  $\omega$  is the circular frequency of vibration. The frequency equation of the variationally based corrected nonlocal Euler-Bernoulli beam model is obtained from the fourth-order differential equation:

$$\frac{d^4 w}{dx^4} + \lambda^2 (e_0 a)^2 \frac{d^2 w}{dx^2} - \lambda^2 w = 0 \quad (31)$$



**Fig. 1** Clamped-free continuous nonlocal beam model



**Fig. 2** The first three vibration frequencies of nonlocal Euler beams with clamped-free boundaries

where  $\lambda^2 = \frac{\mu}{EI} \omega^2$ ,  $e_0 a = l_c$  and  $\mu = \rho A$ .  $e_0$  is a dimensionless number to be calibrated, and  $a$  is a length scale related to the size of the microstructured cell, responsible for possible nonlocal effects. The general solution for Eq. (31) is

$$w(x) = C_1 \cos \alpha x + C_2 \sin \alpha x + C_3 \cosh \beta x + C_4 \sinh \beta x \quad (32)$$

where

$$\alpha = \left( \frac{\sqrt{\lambda^4 (e_0 a)^4 + 4\lambda^2} + (e_0 a)^2 \lambda^2}{2} \right)^{\frac{1}{2}}, \quad \beta = \left( \frac{\sqrt{\lambda^4 (e_0 a)^4 + 4\lambda^2} - (e_0 a)^2 \lambda^2}{2} \right)^{\frac{1}{2}}$$

The variationally based boundary conditions Eq. (27) in the case of vibrating, clamped-free, nonlocal Euler-Bernoulli beam are written as:

$$w(0) = 0, \quad w'(0) = 0, \quad w''(L) = 0 \quad \text{and} \quad EIw'''(L) + \mu l_c^2 \omega^2 w'(L) = 0 \quad (33)$$

We note that these boundary conditions are different from the ones of the local model, especially for the last one dealing with the shear force. Considering the last boundary condition in a local format such as  $EIw'''(L) = 0$ , it would lead to significantly different numerical results for this case. The variationally based boundary conditions are expressed for the nonlocal Euler-Bernoulli beam problem as:

$$\left[ (-EIw''' - \mu l_c^2 \omega^2 w') \delta w \right]_0^L = 0 \quad \text{and} \quad \left[ (EIw'') \delta w' \right]_0^L = 0 \quad (34)$$

By using the correct boundary conditions of the variationally based clamped-free problem listed in Eq. (33), one gets the following determinant of coefficient matrix

$$\begin{vmatrix} 1 & 0 & 1 & 0 \\ 0 & \alpha & 0 & \beta \\ \gamma \sin(\alpha L) & -\gamma \cos(\alpha L) & \eta \sinh(\beta L) & \eta \cosh(\beta L) \\ -\alpha^2 \cos(\alpha L) & -\alpha^2 \sin(\alpha L) & \beta^2 \cosh(\beta L) & \beta^2 \sinh(\beta L) \end{vmatrix} = 0 \quad (35)$$

where  $\gamma = \alpha^3 - \lambda^2 l_c^2 \alpha = \alpha \beta^2$  and  $\eta = \beta^3 + \lambda^2 l_c^2 \beta = \beta \alpha^2$ . Therefore, the characteristic equation is given by

$$(\gamma \beta \alpha^2 + \alpha \eta \beta^2) + (\gamma \beta^3 + \eta \alpha^3) \cos(\alpha L) \cosh(\beta L) + (-\eta \beta \alpha^2 + \gamma \alpha \beta^2) \sin(\alpha L) \sinh(\beta L) = 0 \quad (36)$$

This is the frequency equation of the self-adjoint, nonlocal clamped-free Euler-Bernoulli beam. This frequency equation can be also presented in the form of

$$2 + \left( \frac{\beta^2}{\alpha^2} + \frac{\alpha^2}{\beta^2} \right) \cos(\alpha L) \cosh(\beta L) + \left( \frac{\beta}{\alpha} + \frac{-\alpha}{\beta} \right) \sin(\alpha L) \sinh(\beta L) = 0 \quad (37)$$

or in a more condensed form of

$$2 + (2 + l_c^4 \alpha^2 \beta^2) \cos(\alpha L) \cosh(\beta L) - l_c^2 \alpha \beta \sin(\alpha L) \sinh(\beta L) = 0 \quad (38)$$

The frequency equation of the exact nonself-adjoint Eringen's nonlocal model is calculated from the following boundary conditions

$$w(0) = 0, \quad w'(0) = 0, \quad EIw''(L) + \mu l_c^2 \omega^2 w(L) = 0 \quad \text{and} \quad EIw'''(L) + \mu l_c^2 \omega^2 w'(L) = 0 \quad (39)$$

By using these last boundary conditions of the nonself-adjoint clamped-free problem, one gets the following determinant of coefficient matrix

$$\begin{vmatrix} 1 & 0 & 1 & 0 \\ 0 & \alpha & 0 & \beta \\ \gamma \sin(\alpha L) & -\gamma \cos(\alpha L) & \eta \sinh(\beta L) & \eta \cosh(\beta L) \\ -\frac{\gamma}{\alpha} \cos(\alpha L) & -\frac{\gamma}{\alpha} \sin(\alpha L) & \frac{\eta}{\beta} \cosh(\beta L) & \frac{\eta}{\beta} \sinh(\beta L) \end{vmatrix} = 0 \quad (40)$$

Therefore, the characteristic equation is given by

$$\gamma \beta^3 + \alpha^3 \eta + 2\gamma \eta \cos(\alpha L) \cosh(\beta L) + (-\eta \beta^3 + \gamma \alpha^3) \sin(\alpha L) \sinh(\beta L) = 0 \quad (41)$$

which can be also presented in the form:

$$\frac{\beta^2}{\alpha^2} + \frac{\alpha^2}{\beta^2} + 2 \cos(\alpha L) \cosh(\beta L) + \left( \frac{-\beta}{\alpha} + \frac{\alpha}{\beta} \right) \sin(\alpha L) \sinh(\beta L) = 0 \quad (42)$$

or in a more condensed form as shown by [22]:

$$2 + l_c^4 \alpha^2 \beta^2 + 2 \cos(\alpha L) \cosh(\beta L) + l_c^2 \alpha \beta \sin(\alpha L) \sinh(\beta L) = 0 \quad (43)$$

This is the frequency equation of the nonself-adjoint, nonlocal clamped-free Euler-Bernoulli beam. These frequency sensitivity is shown in Fig. 2, based on these two frequency equations Eq. (38) (exact self-adjoint nonlocal model) and Eq. (43) (exact nonself-adjoint Eringen's nonlocal model).

## 6 Microstructured beam model: discrete versus nonlocal equivalent model

### 6.1 Microstructured model and continualization method

The self-adjointness of Eringen's nonlocal model is now investigated through a microstructured model, as the micromechanics-based Eringen's model. The microstructured cantilever that comprises  $n$ -cells is shown in Fig. 3. The buckling, or vibration of this discretized beam of length  $L$  and  $n$  rigid segments connected by elastic rotational springs of stiffness  $C$ , is studied in this part. The beam is loaded by an axial force denoted by  $P$ . The beam is composed of  $n$  repetitive cells of length denoted by  $a$  (the total length of the structure  $L$  is equal to  $L = n \times a$ ).

The elastic potential energy of the deformed rotational springs of this discrete clamped system comprising  $n$ -cells is given by

$$V = \frac{C}{2} \sum_{i=2}^{n-1} \left( \frac{w_{i+1} - 2w_i + w_{i-1}}{a} \right)^2 + \frac{C_1}{2} \left( \frac{w_1}{a} \right)^2 \quad \text{where } w_i = w(x = x_i = ia) \quad (44)$$

and  $C = EI/a$  is the stiffness of each repetitive rotational spring.  $C_1$  is the stiffness of the connection attached to the clamped part.

The work done  $W_e$  by the axial load is given by

$$W_e = \frac{P}{2} a \sum_{i=1}^n \left( \frac{w_{i+1} - w_i}{a} \right)^2 \quad (45)$$

The kinetic energy  $T$  of the discrete system with equal concentrated inertia masses denoted by  $m$  is given by

$$T = \frac{m}{2} \sum_{i=1}^{n-1} \dot{w}_i^2 + \frac{m}{4} \dot{w}_n^2 \quad (46)$$

where the superdot denotes differentiation with respect to time  $t$ ,  $\mu = m/a$  for the correspondence between the discrete and the continuous system, and  $\mu$  is the mass per unit length of the beam material. Only half of the characteristic mass  $m$  is attributed to the end nodes, as the end nodes have only one contributing rigid segment.

The application of Hamilton's principle leads to the equations of motion for the discrete system:

$$\frac{P}{a} (w_{i+1} - 2w_i + w_{i-1}) + \frac{EI}{a^3} (w_{i+2} - 4w_{i+1} + 6w_i - 4w_{i-1} + w_{i-2}) + \mu a \ddot{w}_i = 0 \quad (47)$$

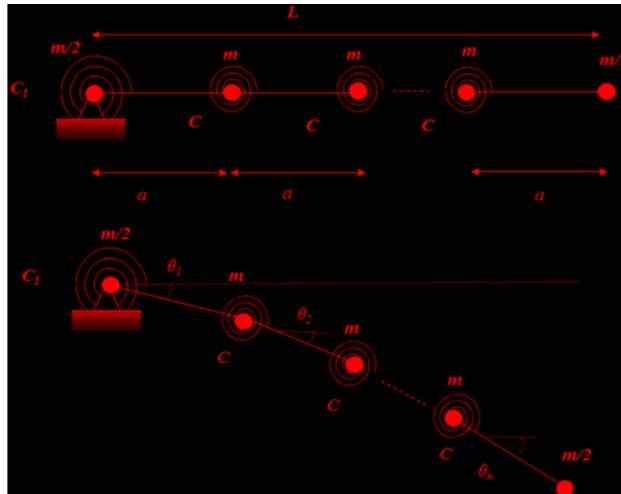


Fig. 3 Clamped-free microstructured model

This discrete equation is also the finite difference equation for a “local” continuous Euler-Bernoulli beam problem. As already outlined by Silverman (1951), the discrete microstructured system is mathematically equivalent to the finite difference format of the so-called “local” continuum (see [25,26], and the use of Hencky’s chain as a physical support of numerical-based finite difference method).

The fourth-order finite difference equation can be also introduced from the constitutive law and the governing equations as

$$M_i = EI \frac{w_{i-1} - 2w_i + w_{i+1}}{a^2} \quad (48)$$

and

$$\frac{M_{i-1} - 2M_i + M_{i+1}}{a^2} + P \frac{w_{i-1} - 2w_i + w_{i+1}}{a^2} + \mu \ddot{w}_i = 0 \quad (49)$$

The discrete equations can be extended to an equivalent continuum via a continualization method. The following relation between the discrete and the equivalent continuous system  $w_i = w(x = ia)$  holds for a sufficiently smooth deflection function as:

$$w(x+a) = \sum_{k=0}^{\infty} \frac{a^k \partial_x^k}{k!} w(x) = e^{a\partial_x} w(x) \quad \text{with } \partial_x = \frac{\partial}{\partial x} \quad (50)$$

The generalized bending problem is then governed by the following pseudo-differential equation (see recently [27])

$$\mu \frac{\partial^2 w}{\partial t^2} + 16 \frac{EI}{a^4} \sinh^4\left(\frac{a}{2}\partial_x\right) w + \frac{4P}{a^2} \sinh^2\left(\frac{a}{2}\partial_x\right) w = 0 \quad (51)$$

The pseudo-differential operator can be efficiently approximated by the Padé’s approximant (see for instance [28,29] or [30] for axial wave applications):

$$\frac{4}{a^2} \sinh^2\left(\frac{a}{2}\partial_x\right) = \frac{\partial_x^2}{1 - l_c^2 \partial_x^2} + \dots \quad \text{with } l_c^2 = \frac{a^2}{12} \quad (52)$$

The generalized Laplacian can be defined from this pseudo-operator as:

$$\Delta_2 = \frac{4}{a^2} \sinh^2\left(\frac{a}{2}\partial_x\right) = \frac{\partial_x^2}{1 - l_c^2 \partial_x^2} + \dots \quad (53)$$

Such a generalized Laplacian has been recently used and extended by Michelitsch et al. [31] for discrete systems with long-range interactions. It is also possible to introduce the first central difference as the square root of this generalized Laplacian,  $\Delta_2 = \delta\delta = \delta^2$ :

$$\delta = \frac{2}{a} \sinh\left(\frac{a}{2}\partial_x\right) \quad (54)$$

where the first central difference is introduced by Sheppard (1899) (see [32,33]). The bending equation can be reformulated using this generalized Laplacian:

$$\mu \frac{\partial^2 w}{\partial t^2} + EI \delta^4 w + P \delta^2 w = 0 \quad (55)$$

which is very analogous to the continuous Euler-Bernoulli beam model, except that the spatial differentiation is replaced by the central difference  $\delta$ .

By using the Padé’s approximants of the pseudo-differential operators, the pseudo-differential equation can be efficiently approximated by the linear differential equation given by

$$\mu (1 - 2l_c^2 \partial_x^2) \partial_t^2 w + EI \partial_x^4 w + P (1 - l_c^2 \partial_x^2) \partial_x^2 w = 0 \quad (56)$$

which can be considered as the nonlocal equivalent medium of the microstructured model. As already discussed by Challamel et al. [12] or Wang et al. [13], this nonlocal Euler-Bernoulli equation shows that the length scale of the dynamics problem is different from the one of the buckling problem.

## 6.2 Energy equivalence between continuum and discrete media

It can be shown that the nonlocal equations Eq. (56) obtained from expanding the difference system in a partial differential system using Padé's approximant can be derived from Hamilton's principle by considering the Lagrangian  $L = T - U$  of the nonlocal continuous system:

$$\int_{t_1}^{t_2} \delta L dt = \int_{t_1}^{t_2} (\delta T - \delta U) dt = 0 \text{ with}$$

$$T[w] = \frac{1}{2} \int_0^L (\mu \dot{w}^2 + 2\mu l_c^2 \dot{w}'^2) dx, U[w] = \int_0^L \frac{1}{2} (EI - Pl_c^2) w''^2 - \frac{1}{2} P w'^2 dx \quad \text{and} \quad l_c^2 = \frac{a^2}{12} \quad (57)$$

One question is now to obtain this Lagrangian directly from the foregoing Lagrangian of the discrete system

$$L = \sum_{i=1}^{n-1} \frac{\mu a}{2} \dot{w}_i^2 - \frac{EI}{2a} \left( \frac{w_{i+1} - 2w_i + w_{i-1}}{a} \right)^2 + \frac{1}{2} Pa \left( \frac{w_{i+1} - w_i}{a} \right)^2 \quad (58)$$

where the specific boundary effects have been omitted. This question has been already investigated by Rosenau [34,35] for axial vibrating systems, where the discrete energy of the discrete axial system is continualized for obtaining a nonlocal equivalent energy.

With regard to the bending problem, one may use a change of variable. Consider the regular deflection variable  $w$  defined by

$$\partial_x^2 w = \frac{w_{i+1} - 2w_i + w_{i-1}}{a^2} \quad (59)$$

This condition can be inverted, i.e.,

$$w_i = Q(a\partial_x) w \quad \text{with} \quad Q(a\partial_x) = \frac{\partial_x^2}{\frac{4}{a^2} \sinh^2\left(\frac{a}{2}\partial_x\right)} \quad (60)$$

Rosenau [34,35] has shown the equivalence between the Hamiltonian of the discrete axial system and the one of the nonlocal continuous system. In this paper, we follow the same reasoning and show the equivalence between the Hamiltonian of the discrete bending system and the one of the nonlocal continuous system.

It is possible to expand this differential operator

$$Q(a\partial_x) w = \left[ 1 - \frac{(a\partial_x)^2}{12} + o(a^4) \right] w \quad (61)$$

The kinetic energy of the discrete system can be continualized thanks to this differential operator:

$$\int_0^L \dot{w}_i^2 dx = \int_0^L Q(a\partial_x) \dot{w} Q(a\partial_x) \dot{w} dx = \int_0^L \dot{w} Q^*(a\partial_x) Q(a\partial_x) \dot{w} dx \quad (62)$$

where  $Q^*(a\partial_x) = Q(-a\partial_x)$  is the self-adjoint operator (see for instance [36]). The summation is here approximated with the integral operator, omitting the quadrature approximation. In this case, for symmetry reasons, the self-adjoint operator is equal to the operator itself  $Q^* = Q$ , and the pseudo-differential operator involved in the kinetic energy can be expanded as:

$$Q^* Q = Q^2 = \left[ 1 - \frac{(a\partial_x)^2}{6} + o(a^4) \right] \quad (63)$$

The kinetic term can then be rewritten as:

$$\int_0^L \dot{w}_i^2 dx = \int_0^L \dot{w} Q^* Q \dot{w} dx = \int_0^L \dot{w} Q^2 \dot{w} dx = \int_0^L \dot{w} \left( \dot{w} - \frac{a^2}{6} \dot{w}'' + o(a^4) \right) dx \quad (64)$$

An integration by parts finally gives

$$\int_0^L \dot{w}_i^2 dx = \int_0^L \dot{w}^2 + \frac{a^2}{6} \dot{w}'^2 + o(a^4) dx - \frac{a^2}{6} [\dot{w} \dot{w}']_0^L \quad (65)$$

By omitting the boundary terms, the kinetic energy of the nonlocal system is then corrected by:

$$T[w] = \frac{1}{2} \int_0^L (\mu \dot{w}^2 + 2\mu l_c^2 \dot{w}'^2) dx \quad \text{and} \quad l_c^2 = \frac{a^2}{12} \quad (66)$$

which is similar to the reasoning of Rosenau [34,35] for axial vibrating systems.

The reasoning for the expansion of the potential energy of the discrete system is similar. The potential energy depends on the difference:

$$\frac{w_{i+1} - w_i}{a} = \frac{a \partial_x [\exp(a \partial_x) - 1]}{4 \sinh^2 \frac{a \partial_x}{2}} \partial_x w \quad \text{with} \quad w_i = \frac{a^2 \partial_x^2}{4 \sinh^2 \frac{a \partial_x}{2}} w \quad (67)$$

A new differential operator can be introduced as:

$$\frac{w_{i+1} - w_i}{a} = R(a \partial_x) \partial_x w \quad \text{with} \quad R(a \partial_x) = \frac{a \partial_x [\exp(a \partial_x) - 1]}{4 \sinh^2 \frac{a \partial_x}{2}} \quad (68)$$

The potential energy can then be obtained from:

$$\int_0^L \left( \frac{w_{i+1} - w_i}{a} \right)^2 dx = \int_0^L R(a \partial_x) (\partial_x w) R(a \partial_x) (\partial_x w) dx = \int_0^L (\partial_x w) R^*(a \partial_x) R(a \partial_x) (\partial_x w) dx \quad (69)$$

It is possible to expand this differential operator as:

$$R^*(a \partial_x) R(a \partial_x) = \frac{(a \partial_x)^2}{4 \sinh^2 \left( \frac{a \partial_x}{2} \right)} = \left[ 1 - \frac{(a \partial_x)^2}{12} + o(a^4) \right] \quad (70)$$

The potential energy term can then be rewritten as:

$$\int_0^L \left( \frac{w_{i+1} - w_i}{a} \right)^2 dx = \int_0^L w' \left( w' - \frac{a^2}{12} w''' + o(a^4) \right) dx \quad (71)$$

An integration by parts finally gives

$$\int_0^L \left( \frac{w_{i+1} - w_i}{a} \right)^2 dx = \int_0^L w'^2 + \frac{a^2}{12} w''^2 + o(a^4) dx - \frac{a^2}{12} [w' w'']_0^L \quad (72)$$

By omitting the boundary terms, the potential energy of the nonlocal system is then corrected by:

$$U[w] = \int_0^L \frac{1}{2} (EI - Pl_c^2) w''^2 - \frac{1}{2} P w'^2 dx \quad \text{and} \quad l_c^2 = \frac{a^2}{12} \quad (73)$$

This closes the proof of the energy equivalence between the discrete bending system and the nonlocal continuous bending system.

### 6.3 Clamped condition defined from continuum system

There is some debate about the calibration of the end stiffness of the discrete system for the equivalent clamped section. One can consider that the continuum kinematics at the clamped section is the reference, and one looks to the clamped equivalent behaviour of the discrete system. This approach has been followed by Challamel et al. [12] and is presented here again for the sake of completeness. Another approach followed thereafter is based on the calibration of the stiffness equivalence with respect to the discrete medium.

In this first approach where the continuum is chosen as the reference medium, the end stiffness is calibrated from the deflection and moment equivalence principle at the end connection

$$w_1 = w(a) \text{ and } M_0 = M(0) \quad (74)$$

where  $w_1$  is the deflection at the end of the first cell ( $i = 1$ ) and  $M_0$  is the moment at the clamped boundary. In the first element connected to the clamped extremity, these quantities are related by

$$w_1 = w(a) = \theta_1 \times a = \frac{M_0}{C_1} \times a \quad (75)$$

For simplicity, the calibration is proposed from using the local Euler-Bernoulli theory, even if we have in mind that using the nonlocal model would have been more accurate. For the local Euler-Bernoulli beam, the natural frequencies expression of the clamped-free cantilever is obtained from (see for instance [37]):

$$\cos\left(\sqrt[4]{\frac{\mu\omega^2}{EI}}L\right) \times \cosh\left(\sqrt[4]{\frac{\mu\omega^2}{EI}}L\right) = -1 \quad (76)$$

and the vibrations mode is given by:

$$\begin{aligned} \frac{w}{w_0} &= \cos\left(\sqrt[4]{\frac{\mu\omega^2}{EI}}x\right) - \cosh\left(\sqrt[4]{\frac{\mu\omega^2}{EI}}x\right) \\ &- \frac{\cos\sqrt[4]{\frac{\mu\omega^2}{EI}}L + \cosh\sqrt[4]{\frac{\mu\omega^2}{EI}}L}{\sin\sqrt[4]{\frac{\mu\omega^2}{EI}}L + \sinh\sqrt[4]{\frac{\mu\omega^2}{EI}}L} \left[ \sin\left(\sqrt[4]{\frac{\mu\omega^2}{EI}}x\right) - \sinh\left(\sqrt[4]{\frac{\mu\omega^2}{EI}}x\right) \right] \end{aligned} \quad (77)$$

Now using the correspondence principle of Eq. (74), the equivalent stiffness in the clamped section of the discrete system is computed from:

$$M(0) = EIw''(0) = -2EIw_0\omega\sqrt{\frac{\mu}{EI}} = \frac{C_1}{a}w(a) \quad \text{and } a = \frac{L}{n} \quad (78)$$

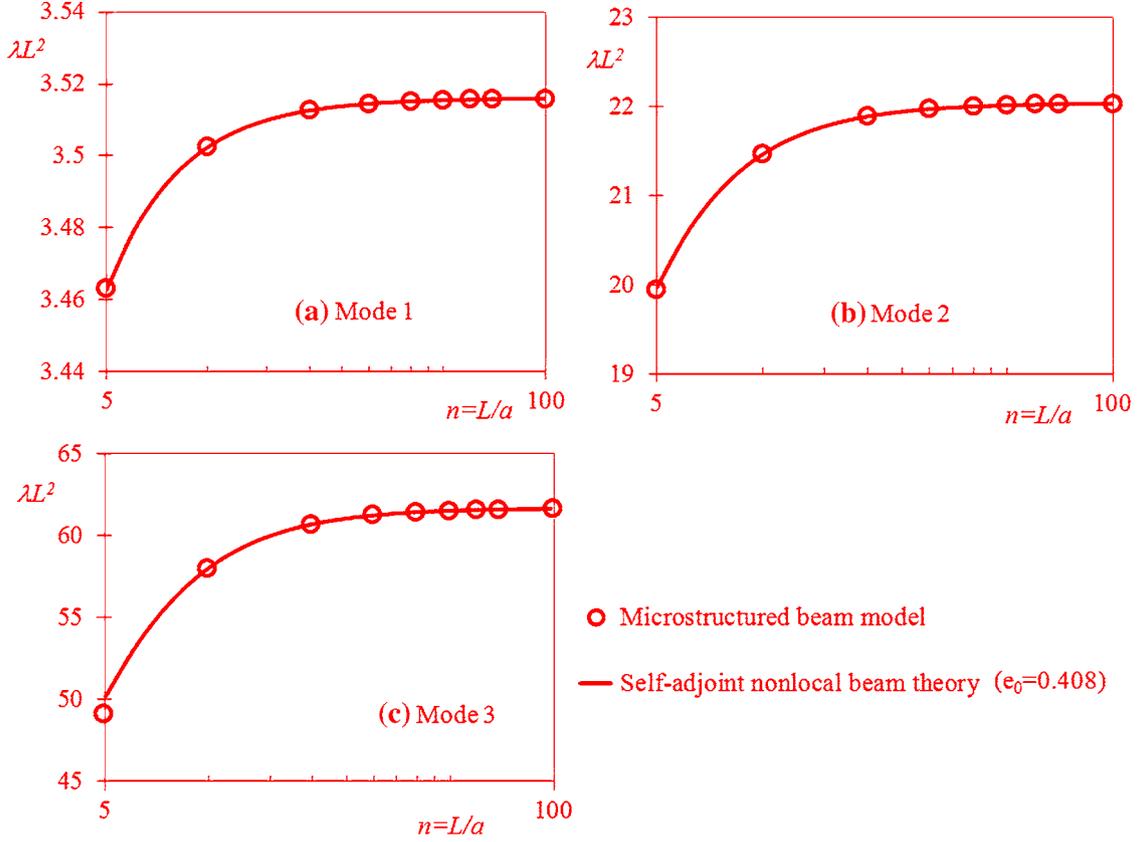
One finally obtains a complex frequency-dependent stiffness law:

$$\frac{C_1}{C} = \frac{-2\omega\sqrt{\frac{\mu}{EI}}\frac{L^2}{n^2}}{\cos\left(\sqrt[4]{\frac{\mu\omega^2}{EI}}\frac{L}{n}\right) - \cosh\left(\sqrt[4]{\frac{\mu\omega^2}{EI}}\frac{L}{n}\right) - \frac{\cos\sqrt[4]{\frac{\mu\omega^2}{EI}}L + \cosh\sqrt[4]{\frac{\mu\omega^2}{EI}}L}{\sin\sqrt[4]{\frac{\mu\omega^2}{EI}}L + \sinh\sqrt[4]{\frac{\mu\omega^2}{EI}}L} \left[ \sin\left(\sqrt[4]{\frac{\mu\omega^2}{EI}}\frac{L}{n}\right) - \sinh\left(\sqrt[4]{\frac{\mu\omega^2}{EI}}\frac{L}{n}\right) \right]} \quad (79)$$

A first-order expansion leads to:

$$\frac{C_1}{C} = 2 + \frac{A}{n} + \dots \quad \text{with } A = \frac{2}{3}\sqrt[4]{\frac{\mu\omega^2}{EI}}L \frac{\cos\sqrt[4]{\frac{\mu\omega^2}{EI}}L + \cosh\sqrt[4]{\frac{\mu\omega^2}{EI}}L}{\sin\sqrt[4]{\frac{\mu\omega^2}{EI}}L + \sinh\sqrt[4]{\frac{\mu\omega^2}{EI}}L} \quad (80)$$

Figure 4 shows excellent agreement between the frequencies obtained from microstructured beam model and the self-adjoint nonlocal beam theory (adopting  $e_0 = 0.408$ ) for a clamped-free beam. On the other hand, matching of frequencies is not possible if the nonself-adjoint nonlocal Eringen beam theory was used. This demonstrates that the microstructured beam model can point to the ‘‘correct’’ boundary conditions for Eringen’s beam theory when confronted with a nonself-adjoint formulation such as the vibration problem of a clamped-free beam.



**Fig. 4** Comparison between frequencies obtained from microstructured beam model (with the clamped stiffness calibration) and self-adjoint nonlocal Eringen beam theory with  $e_0 = 0.408$  for a clamped-free beam; calibration of the clamped stiffness of the discrete model with respect to the continuous beam model

#### 6.4 Clamped condition defined from discrete system

Another point of view for the calibration of the end stiffness is to consider that the reference medium is the discrete system, and one seeks the best match for the equivalent continuum.

The finite difference system is exactly solved in this part, as already investigated in [27, 38, 39] for discrete bending systems. Analytical solutions for the vibrations of pure discrete shear system are available in [40], whereas discrete shear/bending systems have been recently analysed in [14, 15] within nonlocal mechanics. The fourth-order linear finite difference Eq. (47) restricted to the vibration terms can be written as

$$w_{i+2} - 4w_{i+1} + 6w_i - 4w_{i-1} + w_{i-2} - \frac{\Omega^2}{n^4}w_i = 0 \quad \text{with } \Omega^2 = \omega^2 \frac{\mu L^4}{EI} \quad (81)$$

The vibrations mode can be obtained from  $w_i = A\lambda^i$ , which by substitution in Eq. (81) leads to the characteristic equation:

$$\left(\frac{1}{\lambda} + \lambda\right)^2 - 4\left(\frac{1}{\lambda} + \lambda\right) + 4 - \frac{\Omega^2}{n^4} = 0 \quad (82)$$

Equation (82) admits four solutions which may be expressed as (see [27, 39]):

$$\begin{aligned} \lambda_1 &= \cos \theta - j \sin \theta, \quad \lambda_2 = \cos \theta + j \sin \theta, \quad \lambda_3 = 2 - \cos \theta - \sqrt{(2 - \cos \theta)^2 - 1} \text{ and} \\ \lambda_4 &= 2 - \cos \theta + \sqrt{(2 - \cos \theta)^2 - 1} \quad \text{with } \theta = \arccos\left(1 - \frac{\Omega}{2n^2}\right) \text{ and } j^2 = -1 \end{aligned} \quad (83)$$

The solution can be expressed with a real functional basis, in the following format:

$$w_i = A_1 \cos(\theta i) + A_2 \sin(\theta i) + A_3 \lambda_3^i + A_4 \lambda_4^i \quad (84)$$

For the clamped-free problem investigated herein, the boundary conditions are detailed below. The deflection of the discrete system vanishes at the boundary, i.e.,

$$w_0 = 0 \Rightarrow A_1 + A_3 + A_4 = 0 \quad (85)$$

The second condition at the clamped section is the continuity of the deflection in both parts of the clamped section:

$$w_{-1} = w_1 \Rightarrow 2A_2 \sin \theta + A_3 \left( \lambda_3 - \frac{1}{\lambda_3} \right) + A_4 \left( \lambda_4 - \frac{1}{\lambda_4} \right) = 0 \quad (86)$$

By writing the moment equivalence between the finite difference system and the physically based microstructured system, one obtains the following equivalent clamped stiffness  $C_1 = 2C$ , as shown by:

$$M_0 = C_1 \theta_1 = \frac{w_{-1} - 2w_0 + w_1}{a} C = 2C \theta_1 \quad \text{if } w_0 = 0 \quad \text{and} \quad w_{-1} = w_1 \quad (87)$$

The third condition is the vanishing of the bending moment at the free end, i.e.,  $M_n = 0$ :

$$w_{n+1} - 2w_n + w_{n-1} = 0 \Rightarrow -A_1 \cos(n\theta) - A_2 \sin(n\theta) + A_3 \lambda_3^n + A_4 \lambda_4^n = 0 \quad (88)$$

Finally, the last boundary condition deals with the free shear force condition which is written here for the discrete system as:

$$\begin{aligned} w_{n+1} - 3w_n + 3w_{n-1} - w_{n-2} &= \frac{-\Omega^2}{2n^4} w_n \Rightarrow \\ A_1 [\sin \theta \sin(n\theta) + \cos(n\theta)] + A_2 [-\sin \theta \cos(n\theta) + \sin(n\theta)] & \\ -A_3 \lambda_3^{n-1} \left( 1 - \frac{\Omega}{2n^2} \lambda_3 \right) - A_4 \lambda_4^{n-1} \left( 1 - \frac{\Omega}{2n^2} \lambda_4 \right) &= 0 \end{aligned} \quad (89)$$

The half factor in the last shear boundary condition is due to the half term of the inertia load in the last cell (see Fig. 3). The crucial role of the free end of a discrete system has already been pointed out by Wallis [41] or Kivshar et al [42] for the axial chain, in relation with surface modes. In the present study, the free end has a lumped mass  $m/2$  (see Fig. 3), as also considered for instance by Leckie and Lindberg [38] for the discrete bending system.

This shear force boundary condition is also equivalent to:

$$M_n - M_{n-1} = -\omega^2 a^2 \frac{\mu}{2} w_n \quad (90)$$

As  $M_n = 0$ , this last boundary condition can be also written as:

$$M_{n-1} = \omega^2 a^2 \frac{\mu}{2} w_n \quad (91)$$

as detailed by Leckie and Lindberg [38] for instance for free-free discrete finite difference beams. Indeed, Leckie and Lindberg [38] studied the free vibrations of discrete finite difference beams with various boundary conditions including the clamped-clamped case, the simply supported case of the free-free beams, but did not report results for the clamped-free case. Interestingly, Leckie and Lindberg [38] also mentioned the analogy between the microstructured model and the finite difference beam equations in the dynamics case.

The natural frequencies of the discrete clamped-free system are then obtained from the determinant calculation:

$$\begin{vmatrix} 1 & 0 & 1 & 1 \\ 0 & 2 \sin \theta & \lambda_3 - \frac{1}{\lambda_3} & \lambda_4 - \frac{1}{\lambda_4} \\ -\cos(n\theta) & -\sin(n\theta) & \lambda_3^n & \lambda_4^n \\ \cos(n\theta) + \sin \theta \sin(n\theta) & \sin(n\theta) - \sin \theta \cos(n\theta) & -\lambda_3^{n-1} \left( 1 - \frac{\Omega}{2n^2} \lambda_3 \right) & -\lambda_4^{n-1} \left( 1 - \frac{\Omega}{2n^2} \lambda_4 \right) \end{vmatrix} = 0 \quad (92)$$

where  $\theta$ ,  $\lambda_3$ , and  $\lambda_4$  are given by Eq. (83). These results coincide with the numerical discrete results based on the equivalent clamped stiffness  $C_1 = 2C$ . These results can be also compared to the nonlocal Eringen's model with the modified clamped boundary conditions. Equations (31) and (32) are still valid in this case (nonlocal Eringen's type Euler-Bernoulli beam), but the new boundary conditions are

$$w(0) = 0, \quad w(a) = w(-a), \quad w''(L) = 0 \quad \text{and} \quad EIw'''(L) + \mu l_c^2 \omega^2 w'(L) = 0 \quad \text{with} \quad l_c^2 = (e_0 a)^2 = \frac{a^2}{6} \quad (93)$$

By using these last boundary conditions, one gets the following determinant of coefficient matrix

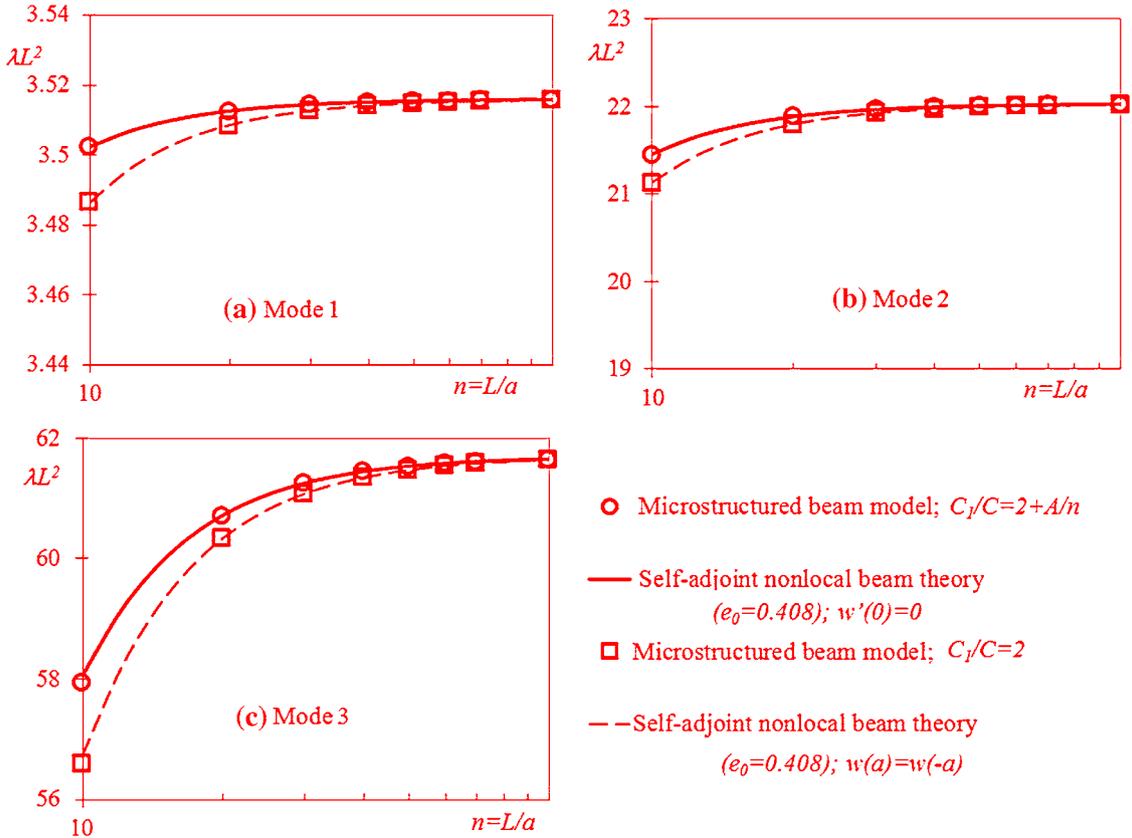
$$\begin{vmatrix} 1 & 0 & 1 & 0 \\ 0 & \sin(\alpha a) & 0 & \sinh(\beta a) \\ \gamma \sin(\alpha L) & -\gamma \cos(\alpha L) & \eta \sinh(\beta L) & \eta \cosh(\beta L) \\ -\alpha^2 \cos(\alpha L) & -\alpha^2 \sin(\alpha L) & \beta^2 \cosh(\beta L) & \beta^2 \sinh(\beta L) \end{vmatrix} = 0 \quad (94)$$

where  $\gamma = \alpha^3 - \lambda^2 l_c^2 \alpha = \alpha \beta^2$  and  $\eta = \beta^3 + \lambda^2 l_c^2 \beta = \beta \alpha^2$ . Therefore, the characteristic equation is given by

$$\begin{aligned} & [\gamma \alpha^2 \sinh(\beta a) + \eta \beta^2 \sin(\alpha a)] + [\gamma \beta^2 \sinh(\beta a) + \eta \alpha^2 \sin(\alpha a)] \cos(\alpha L) \cosh(\beta L) \\ & + [-\eta \alpha^2 \sinh(\beta a) + \gamma \beta^2 \sin(\alpha a)] \sin(\alpha L) \sinh(\beta L) = 0 \end{aligned} \quad (95)$$

This is the frequency equation of the self-adjoint, nonlocal clamped-free Euler-Bernoulli beam with modified clamped boundary conditions. This frequency equation can be also presented in the form of

$$\begin{aligned} & [\beta \alpha^2 \sinh(\beta a) + \alpha \beta^2 \sin(\alpha a)] + [\beta^3 \sinh(\beta a) + \alpha^3 \sin(\alpha a)] \cos(\alpha L) \cosh(\beta L) \\ & + [-\alpha^3 \sinh(\beta a) + \beta^3 \sin(\alpha a)] \sin(\alpha L) \sinh(\beta L) = 0 \end{aligned} \quad (96)$$



**Fig. 5** Comparison between frequencies obtained from microstructured beam model and self-adjoint nonlocal Eringen beam theory with  $e_0 = 0.408$  for a clamped-free beam; parameter  $A$  defined in Eq. (80); effect of the boundary conditions at the clamped section

In Fig. 5, the discrepancies are clearly highlighted for a small number of cells, between the two ways for modelling the clamped section. However, in both cases, the softening effect of the small length scale effect is noticed. Furthermore, as expected, a perfect agreement is also observed between the nonlocal Eringen's model with the clamped boundary conditions  $w(0) = 0$  and  $w(a) = w(-a)$ , and the discrete model based on  $w_{-1} = w_1$ .

## 7 Conclusions

The following conclusions may be drawn from this study:

1. The buckling problem of Eringen's beam is shown to be self-adjoint for all boundary conditions. A modified equivalent energy functional of the nonlocal elasticity beam can be given in term of the local deflection. This nonlocal elasticity problem is clearly a conservative problem (in the sense of deriving from a potential).
2. Eringen's Euler-Bernoulli beam mechanics is also shown to be self-adjoint for hinged-hinged boundary conditions, whatever the analysis considered (buckling or vibration analysis).
3. For vibrations analyses, the clamped-free problem of the Eringen's-based nonlocal beam leads to a nonself-adjoint problem. In this case, it is not possible to build an associated functional energy, and the problem is clearly nonconservative. In this last case, a surprising and paradoxical result has been observed, where the small length scale effect tends to stiffen the local system, in contrast to the softening result obtained for all the other boundary conditions.
4. Eringen's nonlocal elasticity has been recently shown to be an accurate description of microstructured (discrete) media. As the discrete media is considered as a conservative system at the local scale, it is rational to have a conservative macro-scale (conservative nonlocal elasticity equivalent continuum, or self-adjoint macroscopic system). From the microstructure equivalent correspondence, Eringen's corrected nonlocal elasticity with self-adjoint boundary conditions is recommended to describe the softening phenomenon induced by the small length scale effect, a conclusion that cannot be reached within nonself-adjoint Eringen elasticity. We therefore recommend the use of the differential equations with the proposed associated boundary conditions within the micromechanics-based energy arguments.
5. It has been shown in this paper how some discrete systems may be accurately approximated by some Eringen's-based nonlocal continuum models. The discrete system considered in this study is based on some unidimensional repetitive bending cells, which can be viewed as a structural paradigm in structural mechanics. However, alternative nonlocal or gradient theories may also be supported by some other physical or theoretical supports (see the historical perspective [43]). For instance, higher gradient elasticity theories may be also deduced starting from discrete repetitive truss cells [44].

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