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► **To cite this version:**

Hugo Gimbert, Rasmus Ibsen-Jensen. A short proof of correctness of the quasi-polynomial time algorithm for parity games. 2017. hal-01457996v2

HAL Id: hal-01457996

<https://hal.science/hal-01457996v2>

Preprint submitted on 14 Feb 2017 (v2), last revised 24 Apr 2017 (v4)

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A short proof of correctness of the quasi-polynomial time algorithm for parity games

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Abstract

Recently Cristian S. Calude, Sanjay Jain, Bakhadyr Khoussainov, Wei Li and Frank Stephan proposed a quasi-polynomial time algorithm for parity games [1]. These notes provide a short proof of correctness of their algorithm.

Parity games

A parity game is given by a directed graph (V, E) , a starting node $s \in V$, a function which attaches to each $v \in V$ a priority $\text{pty}(v)$ from a set $\{1, 2, \dots, m\}$; the main parameter of the game is n , the number of nodes, and the second parameter is m . Two players Anke and Boris move alternately in the graph with Anke moving first. A move from a node v to another node w is valid if (v, w) is an edge in the graph; furthermore, it is required that from every node one can make at least one valid move. The alternate moves by Anke and Boris define an infinite sequence of nodes which is called a play. Anke wins a play through nodes v_0, v_1, \dots iff $\limsup_t \text{pty}(v_t)$ is even, otherwise Boris wins the play.

We say that a player *wins the parity game* if she has a strategy which guarantees the play to be winning for her. Parity games are determined [3] thus either Anke or Boris wins the parity game.

Statistics

The core of the algorithm of Calude et al. is to keep track of statistics about the game, in the form of *partial* functions

$$f : 0 \dots k \rightarrow 1 \dots m .$$

The integer k is chosen such that 2^k is strictly larger than the number of vertices. The domain of f is denoted $\text{dom}(f)$ and its image $\text{im}(f)$. Statistics are assumed to be *increasing*, i.e. $\forall i, j \in \text{dom}(f), (i \leq j \implies f(i) \leq f(j))$. A statistic f can be modified by *inserting* a priority c at an index ℓ , which results in removing all pairs of index $\leq \ell$ from f and adding the pair (ℓ, c) .

The initial statistic is the empty statistic $f_0 = \emptyset$, which is updated successively by all the priorities visited during the play, thus producing a sequence of statistics. The update of a statistic f by a priority c is performed by applying successively the following two rules.

- **Type I update:** If c is even then it is inserted at the highest index $j \in 0 \dots k$ such that f is defined and even on $0 \dots j - 1$.
- **Type II update:** If $\text{im}(f)$ contains at least one value $< c$ then c is inserted at the highest index $j \in \text{dom}(f)$ such that $f(j) < c$.

Applying both rules in succession ensures that the update of an increasing statistic is increasing. If rule II triggers an insertion then we say the update is a type II update. Notice that in this case, applying or not rule I in the first place does not change the result. If rule I triggers an insertion but rule II does not then we say the update is a type I update.

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Anke (resp. Boris) *wins the statistics game* if she (resp. he) has a strategy to enforce (resp. to avoid) a visit to a statistic whose domain contains k . Similarly to the game of chess, statistics games are determined: either Anke or Boris has a winning strategy [2].

1 Correctness of the algorithm

► **Theorem 1** (Calude et al). *Anke wins the parity game iff she wins the statistics games.*

Since statistics games are determined, the direct implication follows from:

► **Lemma 2.** *If Boris wins the statistics games, he wins the parity game.*

Proof. Every play won by Boris in the statistics game is won by Boris in the parity game because $c = \limsup_t c_t$ is odd in every sequence of statistics updates $f_0 \rightarrow_{c_0} f_1 \rightarrow_{c_1} \dots$ such that $\forall t \geq 0, k \notin \text{dom}(f_t)$, the proof of which follows.

An easy case is when the sequence of statistics is ultimately constant equal to some f then $f \rightarrow_c f$ thus c is odd because an update by an even priority always performs an insertion. In the opposite case define (ℓ, d) the maximal pair (for the dichotomic order) inserted infinitely often. Since d is inserted infinitely often then $d \leq \limsup_t c_t = c$. And $d \geq c$ otherwise c would be inserted infinitely often at an index $\geq \ell$ which would contradict the maximality of (ℓ, d) . Since (ℓ, c) is inserted infinitely often then it is removed infinitely often, let (ℓ', c') be a pair used infinitely often to remove (ℓ, c) . Since (ℓ, c) is removed by the insertion of (ℓ', c') then $\ell \leq \ell'$ hence $\ell = \ell'$ by maximality of ℓ . Since $(\ell', c') = (\ell, c) \neq (\ell, c)$ then $c' < c$ by maximality of c . Since (ℓ, c) is removed by the insertion of (ℓ', c') and $c' < c$ then this insertion is necessarily performed by a type I update and since $\ell = \ell'$ then c is odd otherwise the insertion would occur at a higher index. ◀

The converse implication (Corollary 7) relies on several crucial properties of statistics.

► **Definition 3** (Counter value). With every statistic f is associated its *counter value*

$$\text{bin}(f) = \sum_{j \in \text{dom}(f)} 2^j .$$

► **Lemma 4** (Increments). *Let $f \rightarrow_c f'$ be a statistic update inserting c at index $\ell \in 1 \dots k$. The following conditions are equivalent:*

- $\text{bin}(f') > \text{bin}(f)$,
- $\text{bin}(f') = \text{bin}(f) + 1$,
- $\ell \notin \text{dom}(f)$.

If any of these conditions hold then $f \rightarrow_c f'$ is a type I update called an increment.

Proof. If $\ell \in \text{dom}(f)$ then $\text{dom}(f') \subseteq \text{dom}(f)$ thus $\text{bin}(f') \leq \text{bin}(f)$ and none of the conditions hold. If $\ell \notin \text{dom}(f)$ then by definition of updates, $f \rightarrow_c f'$ is a type I update and f is defined and even on $1 \dots \ell - 1$. Thus $\text{bin}(f') - \text{bin}(f) = 2^\ell - (2^0 + 2^1 \dots + 2^{\ell-1}) = 1$ and the three conditions hold. ◀

In the sequel we fix a sequence $f_0 \rightarrow_{c_0} f_1 \dots \rightarrow_{c_N} f_{N+1}$ of statistics updates.

► **Definition 5** (Even statistics and even factorizations). A statistic f is *even* if all values in $\text{im}(f)$ are even. An *even factorization* of length j is a sequence $0 \leq t_0 < \dots < t_j = N + 1$ such that for every $i \in 0 \dots j - 1$, the maximum of $c_{t_i}, c_{t_i+1}, \dots, c_{t_{i+1}-1}$ is even.

Long even factorizations exist whenever the last statistic is even.

► **Lemma 6.** *Assume $f_0 = \emptyset$ and f_{N+1} is even. Then there is an even factorization of length at least $\text{bin}(f_{N+1})$.*

► **Corollary 7.** *If Anke wins the statistics game then she wins the parity game.*

Proof. By definition of the statistics game, Anke can enforce the play to reach a statistic f_{N+1} such that $k \in \text{dom}(f_{N+1})$. If N is chosen minimal then $f_N \rightarrow_{c_N} f_{N+1}$ is an increment by Lemma 4, thus $\text{bin}(f_{N+1}) = 2^k$. Moreover f_{N+1} is even, because $f_{N+1} = \{(k, c_N)\}$ by minimality of N , and c_N is even since $f_N \rightarrow_{c_N} f_{N+1}$ is an increment. According to Lemma 6, such a play has an even factorization $t_0 < t_1 < \dots < t_j = N + 1$ of length $j \geq 2^k$. Since 2^k is $>$ than the number of vertices, the play loops on the same vertex at some dates t_i and $t_{i'}$ with $0 \leq i < i' \leq j$. By definition of even factorizations, the maximal priority on this loop is even. Thus Boris has no positional winning strategy in the parity game, and since parity games are positional [3], Boris has no winning strategy at all in the parity game. ◀

Proof of Lemma 6. The proof is by induction on N . We first show two easy cases: if $\text{bin}(f_{N+1}) = 0$ then $t_0 = N + 1$ is an even factorization of length 0; and if $N = 0$ and $0 < \text{bin}(f_1)$ then according to Lemma 4, $f_0 \rightarrow_{c_0} f_1$ is an increment and c_0 is even thus $t_0 = 0, t_1 = 1$ is an even factorization of length 1. Assume now $N > 0$ and $0 < \text{bin}(f_{N+1})$. Then f_{N+1} is not empty, let $\ell = \min \text{dom}(f_{N+1})$ and

$$K_\ell = \{t' \in 0 \dots N \mid f_{t'} \rightarrow_{c_{t'}} f_{t'+1} \text{ is a type I update triggering an insertion at index } \ell\}.$$

We make use of property

(A) for every $t' \in 0 \dots N$, if $\ell \notin \text{dom}(f_{t'})$ then there is $t'' \in K_\ell$ such that $t'' \geq t'$.

To prove property (A), choose for t' the maximal date such that $\ell \notin \text{dom}(f_{t'})$. According to (A) and since $f_0 = \emptyset$ then K_ℓ is not empty, and $t = \max K_\ell$ is well defined. Moreover,

(B) for every $t' \in t \dots N$, $\ell \in \text{dom}(f_{t'})$ and $f_{t'} \rightarrow_{c_{t'}} f_{t'+1}$ is *not* an insertion at index $> \ell$.

(A) implies (B) because an insertion at index $> \ell$ removes ℓ from the domain. We show:

i) $\text{bin}(f_{t+1}) \leq \text{bin}(f_t) + 1$,

ii) $\text{bin}(f_{t+1}) = \text{bin}(f_{N+1})$,

iii) f_t is even,

iv) the largest priority c in c_t, \dots, c_N is $f_{N+1}(\ell)$ and thus even.

Property i) is a straightforward application of Lemma 4: in case $\ell \notin \text{dom}(f_t)$ then $f_t \rightarrow_{c_t} f_{t+1}$ is an increment and in case $\ell \in \text{dom}(f_t)$ then $\text{bin}(f_{t+1}) \leq \text{bin}(f_t)$. According to (B), f_{t+1} and f_{N+1} coincide on $\ell + 1 \dots N$. Since moreover $\ell = \min \text{dom}(f_{t+1}) = \min \text{dom}(f_{N+1})$ it implies ii). Since f_{N+1} is even and $f_{t+1}(\ell) = c_t$ is even (because $t \in K_\ell$) then f_{t+1} is even as well. And the update $f_t \rightarrow_{c_t} f_{t+1}$ has type I thus f_t is even as well hence iii).

Now we show iv). Observe first that if $f_{t'}(\ell) = c$ for some $t' \in t + 1 \dots N + 1$, then $f_{N+1}(\ell) = c$: after date t' , by maximality of c no type II update can remove c and by (B) and the maximality of t , type I updates modify only the indices $< \ell$. If $c = c_t$, then $f_{t+1}(\ell) = c$ because $t \in K_\ell$ thus $f_{N+1}(\ell) = c$. Otherwise, if $c \neq c_t$ let $t^* \in t + 1 \dots N$ be the minimal date such that $c_{t^*} = c$. The update at t^* is a type II update on index ℓ : $\ell \in \text{dom}(f_{t^*})$ by property (B), $f_{t^*}(\ell) < c$ by maximality of c and minimality of t^* and the insertion of c occurs at index $\leq \ell$ by property (B) again. Hence, $f_{t^*+1}(\ell) = c$ and thus $f_{N+1}(\ell) = c$.

Since $t \leq N$ and according to iii) we can apply the induction hypothesis to $f_0 \rightarrow_{c_0} f_1 \dots \rightarrow_{c_{t-2}} f_t$ and get an even factorization $0 \leq t_0 < t_1 < \dots < t_n = t$ of length $\geq \text{bin}(f_t)$. According to iv), $0 \leq t_0 < t_1 < \dots < t_n = t < N + 1$ is an even factorization as well, of length $\geq 1 + \text{bin}(f_t)$. According to i) and ii), $1 + \text{bin}(f_t) \geq \text{bin}(f_{t+1}) = \text{bin}(f_{N+1})$ which terminates the proof of the induction step. ◀

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