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A short proof of correctness of the quasi-polynomial time algorithm for parity games

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Abstract

Recently Cristian S. Calude, Sanjay Jain, Bakhadyr Khousainov, Wei Li and Frank Stephan proposed a quasi-polynomial time algorithm for parity games [1]. This paper proposes a short proof of correctness of their algorithm.

Parity games

A parity game is given by a directed graph (V, E) , a starting node $s \in V$, a function which attaches to each $v \in V$ a priority $\text{pty}(v)$ from a set $\{1, 2, \dots, m\}$; the main parameter of the game is n , the number of nodes, and the second parameter is m . Two players Anke and Boris move alternately in the graph with Anke moving first. A move from a node v to another node w is valid if (v, w) is an edge in the graph; furthermore, it is required that from every node one can make at least one valid move. The alternate moves by Anke and Boris define an infinite sequence of nodes which is called a play. Anke wins a play through nodes v_0, v_1, \dots iff $\limsup_t \text{pty}(v_t)$ is even, otherwise Boris wins the play.

We say that a player *wins the parity game* if she has a strategy which guarantees the play to be winning for her. Parity games are determined [3] thus either Anke or Boris wins the parity game.

Statistics

The core of the algorithm is to keep track of statistics about the game, in the form of *partial* functions

$$f : 0 \dots k \rightarrow 1 \dots m .$$

The integer k is chosen such that 2^k is strictly larger than the number of vertices. The domain of f is denoted $\text{dom}(f)$ and its image $\text{im}(f)$. Statistics are assumed to be *increasing*, i.e. $\forall i, j \in \text{dom}(f), (i \leq j \implies f(i) \leq f(j))$. A statistic f can be modified by *inserting* a priority c at an index ℓ , which results in removing all pairs of index $\leq \ell$ from f and adding the pair (ℓ, c) .

The initial statistic is the empty statistic $f_0 = \emptyset$, which is updated successively by all the priorities visited during the play, thus producing a sequence of statistics. The update of a statistic f by a priority c is performed by applying successively the following two rules.

- **Type I update:** If c is even then it is inserted at the highest index $j \in 0 \dots k$ such that f is defined and even on $0 \dots j - 1$.
- **Type II update:** If $\text{im}(f)$ contains at least one value $< c$ then c is inserted at the highest index $j \in \text{dom}(f)$ such that $f(j) < c$.

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Applying both rules I and II successively, in this order, gives the same result than applying only rule II. Both rules ensures that the update of an increasing statistic is increasing.

Anke (resp. Boris) *wins the statistics game* if she (resp. he) has a strategy to enforce (resp. to avoid) a visit to a statistic whose domain contains k . Similarly to the game of chess, statistics games are determined: either Anke or Boris has a winning strategy [2].

The main result of [1] is:

► **Theorem 1** (Calude et al.). *Anke wins the parity game iff she wins the statistics games.*

Since statistics games are determined, the direct implication follows from:

► **Lemma 2.** *If Boris wins the statistics games, he wins the parity game.*

Proof. We prove that any play won by Boris in the statistics game is won by Boris in the parity game, for that we show that $c = \limsup_t c_t$ is odd in every sequence of statistics update $f_0 \rightarrow_{c_0} f_1 \rightarrow_{c_1} \dots$ such that $\forall t \geq 0, k \notin \text{dom}(f_t)$.

An easy case is when the sequence of statistics is ultimately constant equal to some f then $f \rightarrow_c f$ thus c is odd because an update by an even priority always performs an insertion. In the opposite case define (ℓ, d) the maximal pair (for the dichotomic order) inserted infinitely often. Since d is inserted infinitely often then $d \leq \limsup_t c_t = c$. And $d \geq c$ otherwise c would be inserted infinitely often at a index $\geq \ell$ which would contradict the maximality of (ℓ, d) . Since (ℓ, c) is inserted infinitely often then it is removed infinitely often, let (ℓ', c') be a pair used infinitely often to remove (ℓ, c) . Since (ℓ, c) is removed by the insertion of (ℓ', c') then $\ell \leq \ell'$ hence $\ell = \ell'$ by maximality of ℓ . Since $(\ell', c') = (\ell, c) \neq (\ell, c)$ then $c' < c$ by maximality of c . Since (ℓ, c) is removed by the insertion of (ℓ', c') and $c' < c$ then this insertion is necessarily performed by a type I update and since $\ell = \ell'$ then c is odd otherwise the insertion would occur at a higher index. ◀

The converse implication (Corollary 7) relies on several crucial properties of statistics.

► **Definition 3** (Counter value). With every statistic f is associated its *counter value*

$$\text{bin}(f) = \sum_{j \in \text{dom}(f)} 2^j .$$

► **Lemma 4** (Increments). *Let $f \rightarrow_c f'$ be a statistic update inserting c at index $\ell \in 1 \dots k$. The following conditions are equivalent:*

- $\text{bin}(f') > \text{bin}(f)$,
- $\text{bin}(f') = \text{bin}(f) + 1$,
- $\ell \notin \text{dom}(f)$.

If any of these conditions hold then $f \rightarrow_c f'$ is a type I update called an increment.

Proof. If $\ell \in \text{dom}(f)$ then $\text{dom}(f') \subseteq \text{dom}(f)$ thus $\text{bin}(f') \leq \text{bin}(f)$ and none of the conditions hold. If $\ell \notin \text{dom}(f)$ then by definition of updates, $f \rightarrow_c f'$ is a type I update and f is defined and even on $1 \dots \ell - 1$. Thus $\text{bin}(f') - \text{bin}(f) = 2^\ell - (2^0 + 2^1 \dots + 2^{\ell-1}) = 1$ and the three conditions hold. ◀

In the sequel we fix a sequence $f_0 \rightarrow_{c_0} f_1 \dots \rightarrow_{c_N} f_{N+1}$ of statistics updates.

► **Definition 5** (Even statistics and even factorizations). A statistic f is *even* if all values in $\text{im}(f)$ are even. An *even factorization* of length j is a sequence $0 \leq t_0 < \dots < t_j = N + 1$ such that for every $i \in 0 \dots j - 1$, the maximum of $c_{t_i}, c_{t_i+1}, \dots, c_{t_{i+1}-1}$ is even.

Long even factorizations exist whenever the last statistic is even.

► **Lemma 6.** *Assume $f_0 = \emptyset$ and f_{N+1} is even. Then there is an even factorization of length at least $\text{bin}(f_{N+1})$.*

► **Corollary 7.** *If Anke wins the statistics game then she wins the parity game.*

Proof. By definition of the statistics game, Anke can enforce the play to reach a statistic f_{N+1} such that $k \in \text{dom}(f_{N+1})$. If N is chosen minimal then $f_N \rightarrow_{c_N} f_{N+1}$ is an increment by Lemma 4, thus $\text{bin}(f_{N+1}) = 2^k$. Moreover f_{N+1} is even, because f_{N+1} only contains the just inserted pair (k, c_N) , by definition of bin , and c_N is even by Lemma 4. According to Lemma 6, such a play has an even factorization $t_0 < t_1 < \dots < t_j = N + 1$ of length $j \geq 2^k$. Since 2^k is $>$ than the number of vertices, the play loops on the same vertex at some dates t_i and $t_{i'}$ with $0 \leq i < i' \leq j$. By definition of even factorizations, the maximal priority on this loop is even. Thus Boris has no positional winning strategy in the parity game, and since parity games are positional [3], Boris has no winning strategy at all in the parity game. ◀

Proof of Lemma 6. The proof is by induction on N . We first rule out two easy cases: if $\text{bin}(f_{N+1}) = 0$ then $t_0 = N + 1$ is an even factorization of length 0; and if $N = 0$ and $0 < \text{bin}(f_1)$ then according to Lemma 4, $f_0 \rightarrow_{c_0} f_1$ is an increment and c_0 is even thus $t_0 = 0, t_1 = 1$ is an even factorization of length 1.

From now on assume $N > 0$ and $0 < \text{bin}(f_{N+1})$. Then f_{N+1} is not empty, let $\ell = \min \text{dom}(f_{N+1})$. We will use the following *insertion property* several times: assume that $\ell \notin \text{dom}(f_t)$ for some t , then there exists $t' \in t + 1 \dots N$ such that $f_{t'} \rightarrow_{c_{t'}} f_{t'+1}$ is a type I insertion at index ℓ (take t' the maximal date such that $\ell \notin \text{dom}(f_{t'})$).

Since $f_0 = \emptyset$ and according to the insertion property there exists a maximal date $t \in 0 \dots N$ such that $f_t \rightarrow_{c_t} f_{t+1}$ is a type I insertion at index ℓ . We show that:

- i) $\text{bin}(f_t) \geq \text{bin}(f_{t+1}) - 1$,
- ii) $\text{bin}(f_{t+1}) = \text{bin}(f_{N+1})$,
- iii) f_t is even,
- iv) the largest priority c in c_t, \dots, c_N is $f_{N+1}(\ell)$ and thus even.

Property i) is a straightforward application of Lemma 4: in case $\ell \notin \text{dom}(f_t)$ then $f_t \rightarrow_{c_t} f_{t+1}$ is an increment and in case $\ell \in \text{dom}(f_t)$ then $\text{bin}(f_{t+1}) \leq \text{bin}(f_t)$.

An insertion of index $> \ell$ removes ℓ from the domain. According to the insertion property and by maximality of t , there is not such insertion after date t thus $\text{dom}(f_{t+1}) \cap \ell + 1 \dots N = \text{dom}(f_{N+1}) \cap \ell + 1 \dots N$. Moreover $\ell = \min \text{dom}(f_{t+1}) = \min \text{dom}(f_{N+1})$ thus $\text{dom}(f_{t+1}) = \text{dom}(f_{N+1})$, hence ii). Another consequence is that f_{t+1} and f_{N+1} coincide on $\ell + 1 \dots k$. Since moreover f_{N+1} is even and $f_{t+1}(\ell) = c_t$ is even then f_{t+1} is even as well. Since the update $f_t \rightarrow_{c_t} f_{t+1}$ has type I then f_t is even as well hence iii).

Now we show iv). Observe first that if $f_{t'}(\ell) = c$ for some $t' \in t + 1 \dots N + 1$, then $f_{N+1}(\ell) = c$ because by maximality of t , all updates at index ℓ after date t have type II thus they cannot remove the maximal priority c . If $c = c_t$, then $f_{t+1}(\ell) = c$ by definition of t and thus $f_{N+1}(\ell) = c$. Otherwise, if $c \neq c_t$ let $t^* \in t + 1, \dots, N$ be the minimal date such that $c_{t^*} = c$. The update t^* is a type II update of index ℓ (according to the insertion property and the maximality of t , $\ell \in \text{dom}(f_{t^*})$ and there is no insertion at index $> \ell$ after date t , and $f_{t^*}(\ell) < c$ by definition of c and t^*). Hence, $f_{t^*+1}(\ell) = c$ and thus $f_{N+1}(\ell) = c$.

Since $t \leq N$ and according to iii) we can apply the induction hypothesis to $f_0 \rightarrow_{c_0} f_1 \dots \rightarrow_{c_{t-2}} f_t$ and get an even factorization $0 \leq t_0 < t_1 < \dots < t_n = t$ of length $\geq \text{bin}(f_t)$. According to iv), $0 \leq t_0 < t_1 < \dots < t_n = t < N + 1$ is an even factorization as well, of length $\geq 1 + \text{bin}(f_t)$. According to i) and ii), $1 + \text{bin}(f_t) \geq \text{bin}(f_{t+1}) = \text{bin}(f_{N+1})$ thus the proof of the induction step is over. ◀

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