HAL
open science

Outerplanar graph drawings with few slopes

Kolja Knauer, Piotr Micek, Bartosz Walczak

To cite this version:

Kolja Knauer, Piotr Micek, Bartosz Walczak. Outerplanar graph drawings with few slopes . Computational Geometry: Theory and Applications Computational Geometry @ ScienceDirect, 2014. hal-01457974

HAL Id: hal-01457974

https://hal.science/hal-01457974

Submitted on 6 Feb 2017

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

Outerplanar graph drawings with few slopes☆

Kolja Knauer[a,1], Piotr Micek[b,2], Bartosz Walczak[b,c,3]

[a] *Université Montpellier 2, France*
[b] *Theoretical Computer Science Department, Faculty of Mathematics and Computer Science, Jagiellonian University, Kraków, Poland*
[c] *École Polytechnique Fédérale de Lausanne, Switzerland*

Abstract

We consider straight-line outerplanar drawings of outerplanar graphs in which a small number of distinct edge slopes are used, that is, the segments representing edges are parallel to a small number of directions. We prove that $\Delta - 1$ edge slopes suffice for every outerplanar graph with maximum degree $\Delta \geqslant 4$. This improves on the previous bound of $O(\Delta^5)$, which was shown for planar partial 3-trees, a superclass of outerplanar graphs. Our bound is tight: for every $\Delta \geqslant 4$ there is an outerplanar graph with maximum degree Δ that requires at least $\Delta - 1$ distinct edge slopes in an outerplanar straight-line drawing.

1. Introduction

A *straight-line drawing* of a graph G is a mapping of the vertices of G into distinct points of the plane and of the edges of G into straight-line segments connecting the points representing their end-vertices and passing through no

☆Preliminary version of this paper appeared in: J. Gudmundsson, J. Mestre, T. Viglas (Eds.), Computing and Combinatorics, Vol. 7434 of Lecture Notes Comput. Sci., Springer, 2012, pp. 323–334.

Email addresses: `kolja.knauer@math.univ-montp2.fr` (Kolja Knauer), `micek@tcs.uj.edu.pl` (Piotr Micek), `walczak@tcs.uj.edu.pl` (Bartosz Walczak)

[1]Supported by DFG grant FE-340/8-1 as part of ESF EuroGIGA project GraDR and by ANR TEOMATRO grant ANR-10-BLAN 0207.

[2]Supported by MNiSW grant 884/N-ESF-EuroGIGA/10/2011/0 as part of ESF EuroGIGA project GraDR.

[3]Supported by MNiSW grant 884/N-ESF-EuroGIGA/10/2011/0 as part of ESF EuroGIGA project GraDR and by Swiss NSF grant 200020-144531.

other points representing vertices. If it leads to no confusion, in notation and terminology, we make no distinction between a vertex and the corresponding point, and between an edge and the corresponding segment. The *slope* of an edge in a straight-line drawing is the family of all straight lines parallel to this edge. The *slope number* of a graph G, a parameter introduced by Wade and Chu [1], is the smallest number s such that there is a straight-line drawing of G using s slopes.

Since at most two edges at each vertex can use the same slope, $\lceil \frac{\Delta}{2} \rceil$ is a lower bound on the slope number of a graph with maximum degree Δ. Dujmović and Wood [2] asked whether the slope number can be bounded from above by a function of the maximum degree. This has been answered independently by Barát, Matoušek, and Wood [3], Pach and Pálvölgyi [4], and Dujmović, Suderman, and Wood [5] in the negative: graphs with maximum degree 5 can have arbitrarily large slope number. On the other hand, Mukkamala and Pálvölgyi [6] proved that graphs with maximum degree 3 have slope number at most 4, improving earlier results of Keszegh, Pach, Pálvölgyi, and Tóth [7] and of Mukkamala and Szegedy [8]. The question whether the slope number of graphs with maximum degree 4 is bounded by a constant remains open.

The situation looks different for *planar* straight-line drawings, that is, straight-line drawings in which no two edges intersect in a point other than a common endpoint. It is well known that every planar graph admits a planar straight-line drawing [9, 10, 11]. The *planar slope number* of a planar graph G is the smallest number s such that there is a planar straight-line drawing of G using s slopes. This parameter was first studied by Dujmović, Eppstein, Suderman, and Wood [12] in relation to the number of vertices. They also asked whether the planar slope number of planar graphs is bounded in terms of the maximum degree. Jelínek, Jelínková, Kratochvíl, Lidický, Tesař, and Vyskočil [13] gave an upper bound of $O(\Delta^5)$ for planar graphs of treewidth at most 3. The problem has been solved in full generality by Keszegh, Pach, and Pálvölgyi [14], who showed (with a non-constructive proof) that the planar slope number is bounded from above by an exponential function of the maximum degree. It is still an open problem whether this can be improved to a polynomial upper bound.

In the present paper, we consider drawings of outerplanar graphs. The above-mentioned result of Jelínek et al. implies that outerplanar graphs admit planar drawings with $O(\Delta^5)$ slopes, as they have treewidth at most 2. A straight-line drawing of a graph G is *outerplanar* if it is planar and all vertices

of G lie on the outer face. The *outerplanar slope number* of an outerplanar graph G is the smallest number s such that there is an outerplanar straight-line drawing of G using s slopes. It is proved in [12] that the outerplanar slope number of any outerplanar graph is at most the number of its vertices. We provide a tight bound on the outerplanar slope number in terms of the maximum degree.

Main Theorem. *For $\Delta \geqslant 4$, every outerplanar graph with maximum degree at most Δ has outerplanar slope number at most $\Delta - 1$.*

That the bound of $\Delta - 1$ is tight is witnessed by a graph consisting of a cycle C with $2\Delta - 3$ vertices $v_1, \ldots, v_{2\Delta-3}$ each of which has $\Delta - 2$ additional private neighbors. In any outerplanar straight-line drawing of this graph with $\Delta - 2$ edge slopes, C must be the boundary of an inner face. It cannot be strictly convex, as in a strictly convex polygon each slope can be used by at most two edges. Therefore, some angle of this face, say at v_i, is not strictly convex. Each of the private neighbors of v_i needs to be connected with v_i by an edge lying outside the cycle. This is a contradiction, because at most $\Delta - 3$ slopes are available for such edges. Moreover, for $\Delta \in \{2, 3\}$, the lower bound is 3 as witnessed by the triangle.

Note that the tight bounds for the outerplanar slope number with respect to the maximum degree Δ are 1 for $\Delta = 1$ and 3 for $\Delta \in \{2, 3\}$. For the latter, the upper bound follows from the Main Theorem for $\Delta = 4$, while the tightness is witnessed by a triangle.

The proof of our theorem is constructive and yields an algorithm to produce a claimed drawing that performs a linear number of arithmetic operations on rationals.

2. Basic definitions

For the remainder of the paper, we assume that an outerplanar drawing of a graph G with maximum degree at most Δ is given, where $\Delta \geqslant 4$. This drawing determines the cyclic ordering of edges around every vertex. We produce an outerplanar straight-line drawing of G with few edge slopes which preserves this ordering at every vertex. The set of slopes that we use depends only on Δ, so we can draw each connected component of G separately. Therefore, for the remainder of the paper, we assume that G is connected. Our construction is inductive—it composes the entire drawing of G from drawings of subgraphs of G that we call bubbles.

We distinguish the *outer face* of G (the one that is unbounded in the given drawing of G and contains all vertices on the boundary) from the *inner faces*. The edges on the boundary of the former are *outer edges*, while all remaining ones are *inner edges*. A *snip* is a simple closed counterclockwise-oriented curve γ which

- passes through some pair of vertices u and v of G (possibly being the same vertex) and through no other vertex of G,
- on the way from v to u goes entirely through the outer face of G crossing no edges on the way,
- on the way from u to v (considered only if $u \neq v$) goes through inner faces of G possibly crossing some inner edges of G, each at most once.

Every snip γ defines a *bubble* H in G as the subgraph of G induced on the vertices lying on or inside γ. Note that H is a connected subgraph of G as γ crosses no outer edges. The *roots* of H are the vertices u and v together with all vertices of H adjacent to $G - H$. The snip γ breaks the cyclic clockwise order of the edges of H around u or v, making it a linear order, which we envision as going from left to right. In particular, we call the first edge in this order *leftmost* and the last one *rightmost*. Similar left-to-right orderings of edges are defined at the remaining roots of H, except that in their case the cyclic order is broken by the edges connecting H to $G - H$. The *root-path* of H is the simple oriented path P in H that starts at u with the rightmost edge, continues counterclockwise along the boundary of the outer face of H, and ends at v with the leftmost edge. If $u = v$, then the root-path consists of that single vertex only. All roots of H lie on the root-path—their sequence in the order along the root-path is the *root-sequence* of H. A bubble with k roots is called a *k-bubble*. See Fig. 1 for an illustration.

Except at the very end of the proof where we regard the entire G as a bubble, we deal with bubbles H whose first root u and last root v are adjacent to $G - H$. For such bubbles H, all the roots, the root-path, the root-sequence and the left-to-right order of edges at every root do not depend on the particular snip γ used to define H.

Bubbles admit a natural decomposition, which is the base of our recursive drawing.

Lemma 1. *Let H be a bubble with root-path $v_1 \dots v_k$. Every component of $H - \{v_1, \dots, v_k\}$ is adjacent to either one vertex among $v_1, \dots, v_k$ or*

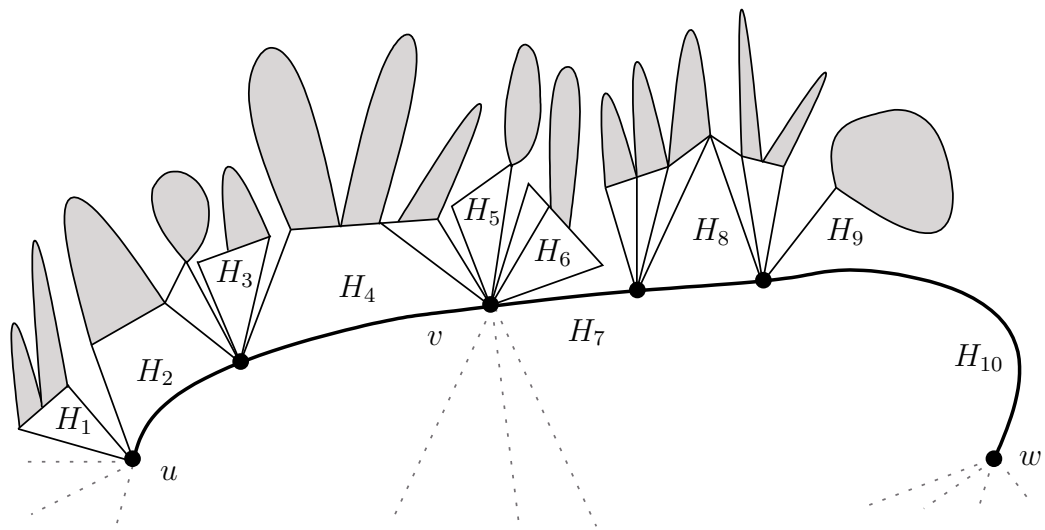

Figure 1: A 3-bubble H with root-path drawn thick, root-sequence u, v, w (connected to the remaining graph by dotted edges), and splitting sequence $(H_1, \ldots, H_{10})$, in which H_1, H_3, H_5, H_6, and H_9 are v-bubbles, while H_2, H_4, H_7, H_8, and H_{10} are e-bubbles. There are six 2-bubbles with roots u and v contained in H: (H_1, H_2, H_3, H_4), $(H_1, H_2, H_3, H_4, H_5)$, $(H_1, H_2, H_3, H_4, H_5, H_6)$, (H_2, H_3, H_4), (H_2, H_3, H_4, H_5), and $(H_2, H_3, H_4, H_5, H_6)$.

two consecutive vertices from $v_1, \ldots, v_k$. Moreover, there is at most one component adjacent to v_i and v_{i+1} for $1 \leqslant i < k$.

Proof. Let C be a connected component of $H - \{v_1, \ldots, v_k\}$. As H itself is connected, C must be adjacent to a vertex from $v_1, \ldots, v_k$. In order to get a contradiction, suppose that C is connected to two non-consecutive vertices v_i and v_j. Let P be a simple v_i, v_j-path having all internal vertices in C. Let $P' = v_i \ldots v_j$ be the subpath of the root-path of H connecting v_i and v_j. Since $v_1 \ldots v_k$ is the root-path of H, all edges connecting the internal vertices of P' to $G - H$ are inner edges. Hence, also the edges of P' lie on inner faces which are not faces of H. The symmetric difference of all these inner faces considered as sets of edges is a simple cycle containing P' as a subpath. Let P'' denote the other v_i, v_j-subpath of that cycle. It is internally disjoint from P and P'. Moreover, P'' and P together enclose P' and thus the internal vertices of P' do not lie on the outer face—contradiction.

Now, to prove the second statement, suppose that for some i two components C and C' of $H - \{v_1, \ldots, v_k\}$ are adjacent to both v_i and v_{i+1}. We

find two internally disjoint v_i, v_{i+1}-paths P and P' through C and C', respectively. As in the paragraph above, we use the fact that v_iv_{i+1} is contained in an inner face, which is not a face of H. The third path P'' is obtained from that face by deleting the edge v_iv_{i+1}. It follows that P, P', and P'' form a subdivision of $K_{2,3}$, which contradicts the outerplanarity of G. $\square$

Lemma 1 allows us to assign each component of $H - \{v_1, \ldots, v_k\}$ to a vertex of P or an edge of P so that every edge is assigned at most one component. For a component C assigned to a vertex v_i, the graph induced on $C \cup \{v_i\}$ is called a *v-bubble*. For a component C assigned to an edge v_iv_{i+1}, the graph induced on $C \cup \{v_i, v_{i+1}\}$ is called an *e-bubble*. If no component is assigned to an edge of P, then we let that edge alone be a *trivial e-bubble*. Note that v- and e-bubbles are special kinds of 1- and 2-bubbles, respectively. All v-bubbles of v_i in H are naturally ordered by their clockwise arrangement around v_i in the drawing. All this leads to a decomposition of the bubble H into a sequence $(H_1, \ldots, H_b)$ of v- and e-bubbles such that the naturally ordered v-bubbles of v_1 precede the e-bubble of v_1v_2, which precedes the naturally ordered v-bubbles of v_2, and so on. We call this sequence the *splitting sequence* of H and write $H = (H_1, \ldots, H_b)$. The splitting sequence of a single-vertex 1-bubble is empty. Every 1-bubble with more than one vertex is a v-bubble or a bouquet of several v-bubbles. The splitting sequence of a 2-bubble may consist of several v- and e-bubbles. For an illustration, see Fig. 1.

The general structure of the induction in our proof is covered by the following lemma (see Fig. 2):

Lemma 2.

2.1. *Let H be a v-bubble rooted at v. Let $v^1, \ldots, v^k$ be the neighbors of v in H from left to right. It follows that $H - v$ is a k-bubble with root-sequence $v^1, \ldots, v^k$.*

2.2. *Let H be a v-bubble rooted at v_0. Consider an induced path $v_0 \ldots v_n$ in H that starts with the rightmost edge at v_0 and continues counterclockwise along the outer face of H so that $v_1, \ldots, v_{n-1}$ are not cut-vertices in H. It follows that $H - \{v_0, \ldots, v_n\}$ has a unique component H' adjacent to both v_0 and v_n. Moreover, let X be the subgraph of H induced on $v_0, \ldots, v_n$ and the vertices of H'. Let $v_0^1, \ldots, v_0^{k_0}, v_1$ be the neighbors of v_0 in X in clockwise order. For $1 \leqslant i \leqslant n-1$, let $v_{i-1}, v_i^1, \ldots, v_i^{k_i}, v_{i+1}$ be the neighbors of v_i in X in clockwise order. Let $v_{n-1}, v_n^1, \ldots, v_n^{k_n}$*

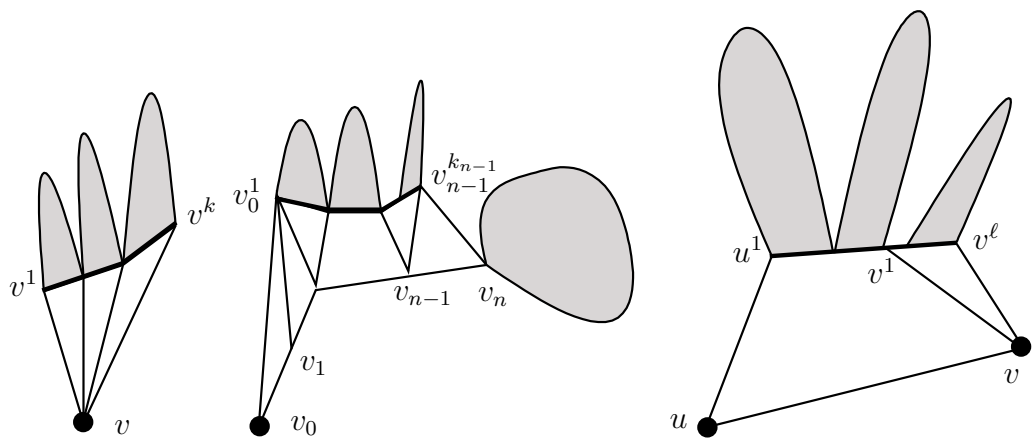

Figure 2: Three ways of obtaining smaller bubbles from v- and e-bubbles as described in Lemma 2. The new bubbles are marked gray, and the new root-paths are drawn thick.

be the neighbors of v_n in X in clockwise order. It follows that H' is a bubble with root-sequence $v_0^1, \ldots, v_0^{k_0}, v_1^1, \ldots, v_1^{k_1}, \ldots, v_n^1, \ldots, v_n^{k_n}$ in which $v_i^{k_i}$ and v_{i+1}^1 coincide whenever the inner face of H containing $v_i v_{i+1}$ is a triangle.

2.3. *Let H be an e-bubble with roots u and v. Let $u^1, \ldots, u^k, v$ be the neighbors of u in H from left to right and $u, v^1, \ldots, v^\ell$ be the neighbors of v in H from left to right. It follows that $H - \{u, v\}$ is a bubble with root-sequence $u^1, \ldots, u^k, v^1, \ldots, v^\ell$ in which u^k and v^1 coincide if the inner face of H containing uv is a triangle.*

Proof. First we prove 2.1. Since H is a v-bubble, $H - v$ is connected. The symmetric difference of the inner faces of H incident to v, considered as sets of edges, gives a simple clockwise cycle in H passing through v and $v^1, \ldots, v^k$ in this order. Let γ be a closed curve going counterclockwise from v^k through the outer face of G to v^1, and then through the inner faces of H at v, crossing the edges $vv^2, \ldots, vv^{k-1}$ in this order, back to v_k. Clearly, γ is a snip defining the bubble $H - v$ with root-sequence $v^1, \ldots, v^k$.

Next we prove 2.2. Since none of $v_1, \ldots, v_{n-1}$ is a cut-vertex in H, the graph $H - \{v_0, \ldots, v_n\}$ has a component adjacent to both v_0 and v_n. Moreover, since the path $v_0 \ldots v_n$ consists only of outer edges, such a component is unique. Thus H' is well defined. Now, the symmetric difference of the inner faces of X incident to any of $v_0, \ldots, v_n$, considered as sets of edges, gives

a simple clockwise cycle in H passing through $v_n, \ldots, v_0$ and then through $v_0^1, \ldots, v_0^{k_0}, v_1^1, \ldots, v_1^{k_1}, \ldots, v_n^1, \ldots, v_n^{k_n}$ in this order. Let γ be a closed curve going counterclockwise from $v_n^{k_n}$ through the outer face of G to v_0^1, and then through the inner faces of H at $v_0, \ldots, v_n$, crossing the edges $v_0v_0^2, \ldots, v_0v_0^{k_0}$, $v_1v_1^1, \ldots, v_1v_1^{k_1}, \ldots, v_nv_n^1, \ldots, v_nv_n^{k_n-1}$ in this order, back to $v_n^{k_n}$. Clearly, γ is a snip defining the bubble H' with root-sequence $v_0^1, \ldots, v_0^{k_0}, v_1^1, \ldots, v_1^{k_1}, \ldots, v_n^1, \ldots, v_n^{k_n}$.

Finally we show 2.3. Since H is an e-bubble, $H - \{u, v\}$ is connected. Again, the symmetric difference of the inner faces of H incident to u or v, considered as sets of edges, gives a simple clockwise cycle in H passing through v, u, and $v^1, \ldots, v^k$ in this order. Let γ be a closed curve going counterclockwise from v^ℓ through the outer face of G to u^1, and then through the inner faces of H at u and v, crossing the edges $uu^2, \ldots, uu^k, vv^1, \ldots, vv^{\ell-1}$ in this order, back to v_ℓ. Clearly, γ is a snip defining the bubble $H - \{u, v\}$ with root-sequence $u^1, \ldots, u^k, v^1, \ldots, v^\ell$. $\square$

3. Bounding regions

Depending on the maximum degree Δ of G, define the set S of $\Delta - 1$ slopes to consist of the horizontal slope and the slopes of vectors $\mathbf{f}_1, \ldots, \mathbf{f}_{\Delta-2}$, where

$$\mathbf{f}_i = (-\tfrac{1}{2} + \tfrac{i-1}{\Delta-3}, 1) \quad \text{for } i = 1, \ldots, \Delta - 2.$$

An important property of S is that it cuts the horizontal segment L from $(-\frac{1}{2}, 1)$ to $(\frac{1}{2}, 1)$ into $\Delta - 3$ segments of equal length $\frac{1}{\Delta-3}$. We construct an outerplanar straight-line drawing of G using only slopes from S and preserving the given cyclic ordering of edges at each vertex of G.

The essential tool in proving that our construction does not make bubbles overlap are bounding regions. Their role is to bound the area of the plane occupied by bubbles. The bounding region of a bubble is parametrized by ℓ and r which depend on the degrees of the roots in the bubble. Let v be a point in the plane. For a vector x, let $R(v; x) = \{v + \alpha x : \alpha \geqslant 0\}$. For $0 \leqslant \ell \leqslant \Delta - 1$, we define $LB(v; \ell)$ to be the cone consisting of v and all points p such that

- $p_y \geqslant v_y$,
- p lies on $R(v; \mathbf{f}_1)$ or to the right of it if $\ell = 1$,

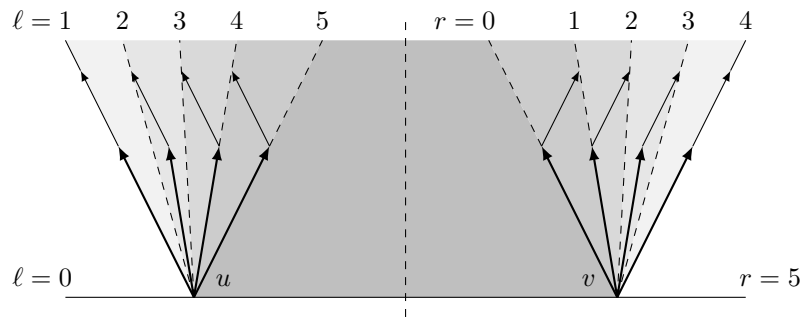

Figure 3: Boundaries of $LB(u;\ell)$ (left) and $RB(v;r)$ (right) for $\Delta = 6$. Vectors $\mathbf{f}_i$ at u and v are indicated by thick arrows. Vectors $\frac{1}{2}\mathbf{f}_1$ at $u + \mathbf{f}_i$ and $\frac{1}{2}\mathbf{f}_4$ at $v + \mathbf{f}_i$ are indicated by thin arrows. Note that $u + \mathbf{f}_3$ lies on the boundary of $LB(u;4)$ and $v + \mathbf{f}_2$ lies on the boundary of $RB(v;1)$.

- $p_x > v_x$ if $\Delta = 4$ and $\ell = 2$,
- p lies to the right of $R(v; \mathbf{f}_\ell + \frac{1}{\Delta-4}\mathbf{f}_1)$ if $\Delta \geqslant 5$ and $2 \leqslant \ell \leqslant \Delta - 2$,
- p lies to the right of $R(v; \mathbf{f}_{\Delta-2})$ if $\ell = \Delta - 1$.

Similarly, for $0 \leqslant r \leqslant \Delta - 1$, we define $RB(v;r)$ to be the cone consisting of v and all points p such that

- $p_y \geqslant v_y$,
- p lies to the left of $R(v; \mathbf{f}_1)$ if $r = 0$,
- $p_x < v_x$ if $\Delta = 4$ and $r = 1$,
- p lies to the left of $R(v; \mathbf{f}_r + \frac{1}{\Delta-4}\mathbf{f}_{\Delta-2})$ if $\Delta \geqslant 5$ and $1 \leqslant r \leqslant \Delta - 3$,
- p lies on $R(v; \mathbf{f}_{\Delta-2})$ or to the left of it if $r = \Delta - 2$.

See Fig. 3 for an illustration. Now, for points u, v in the plane such that $u_y = v_y$ and $u_x \leqslant v_x$, we define bounding regions as follows:

$$B(uv;\ell,r) = LB(u;\ell) \cap RB(v;r) \qquad \text{for } 0 \leqslant \ell, r \leqslant \Delta - 1,$$
$$\bar{B}(uv;\ell,r;h) = B(uv;\ell,r) \cap \{p : p_y < u_y + h\} \qquad \text{for } 0 \leqslant \ell, r \leqslant \Delta - 1 \text{ and } h > 0.$$

We denote $B(vv;\ell,r)$ simply by $B(v;\ell,r)$ and $\bar{B}(vv;\ell,r;h)$ simply by $\bar{B}(v;\ell,r;h)$. Note that the bottom border of a bounding region is always included, the left

border (if exists) is included if $\ell = 1$, the right border (if exists) is included if $r = \Delta - 2$, and the top border (if exists) is never included. If $\ell > r$, then $B(v; \ell, r) = \{v\}$.

We use $B(v; \ell, r)$ and $\bar{B}(v; \ell, r; h)$ to bound drawings of 1-bubbles H with root v such that $r - \ell + 1 = d_H(v)$. Note that every 1-bubble drawn inside $B(v; \ell, r)$ can be scaled to fit inside $\bar{B}(v; \ell, r; h)$ for any $h > 0$ without changing slopes. We use $B(uv; \ell, r)$ and $\bar{B}(uv; \ell, r; h)$ with $u \neq v$ to bound drawings of 2-bubbles H whose root-path starts at u and ends at v, such that $\ell = \Delta - d_H(u)$ and $r = d_H(v) - 1$. Here H cannot be scaled if the positions of both u and v are fixed, so the precise value of h matters. However, every 2-bubble drawn inside $B(uv; \ell, r)$ can be scaled to fit inside $\bar{B}(uw; \ell, r; h)$ for any $h > 0$ without changing slopes, where w is some point of the segment uv.

Lemma 3. *Bounding regions have the following geometric properties ($*$ stands for any value which if generally feasible is irrelevant to the statement).*

3.1. *If $u'v' \subseteq uv$, $\ell' \geqslant \ell$, and $r' \leqslant r$, then $B(u'v'; \ell', r') \subseteq B(uv; \ell, r)$.*

3.2. *If $i < \ell$, then a vector at u in direction $\mathbf{f}_i$ points outside $B(uv; \ell, *)$ to the left of it. If $i > r$, then a vector at v in direction $\mathbf{f}_i$ points outside $B(uv; *, r)$ to the right of it.*

3.3. *If u, v, w are points on a horizontal line in this order from left to right and $\ell - 1 \geqslant r + 1$, then $B(uv; *, r) \cap B(vw; \ell, *) = \{v\}$.*

Moreover, the following holds for $\Delta \geqslant 5$.

3.4. *For $\ell < r$, $h > 0$, $u' = u + h\mathbf{f}_\ell$, and $v' = v + h\mathbf{f}_r$, we have $\bar{B}(u'v'; 1, \Delta - 2; \frac{h}{\Delta-4}) \subseteq \bar{B}(uv; \ell, r; \frac{\Delta-3}{\Delta-4}h)$.*

3.5. *If u, v, w are points on a horizontal line in this order from left to right, $r' \leqslant \Delta - 3$, and $r \geqslant 1$, then $\bar{B}(uv; \ell, r'; (\frac{\Delta-3}{\Delta-4})^2|vw|) \subseteq B(uw; \ell, r)$.*

3.6. *If u, v, w, x are points on a horizontal line in this order from left to right, $|uv| = |wx| \leqslant |vw|$, $\ell \geqslant 2$, and $r \leqslant \Delta - 3$, then $\bar{B}(uv; *, r; \frac{\Delta-3}{\Delta-4}|uv|) \cap \bar{B}(wx; \ell, *; \frac{\Delta-3}{\Delta-4}|wx|) = \emptyset$.*

Proof. Statement 3.1 clearly follows from the definition. Statement 3.2 is implied by the definition and for $\Delta \geqslant 5$ by the fact that

$$\begin{aligned} \mathbf{f}_\ell + \tfrac{1}{\Delta-4}\mathbf{f}_1 &= \mathbf{f}_{\ell-1} + \tfrac{1}{\Delta-4}\mathbf{f}_{\Delta-3} && \text{for } \ell = 2, \ldots, \Delta - 2, \\ \mathbf{f}_r + \tfrac{1}{\Delta-4}\mathbf{f}_{\Delta-2} &= \mathbf{f}_{r+1} + \tfrac{1}{\Delta-4}\mathbf{f}_2 && \text{for } r = 1, \ldots, \Delta - 3. \end{aligned}$$

Statement 3.2 directly yields 3.3: the vector at v in direction $\mathbf{f}_{r+1}$ points outside both $B(uv;*,r)$ and $B(vw;\ell,*)$. To see 3.4, note that the point $u' + \frac{h}{\Delta-4}\mathbf{f}_1$, which is the top-left corner of $\bar{B}(u'v';1,\Delta-2;\frac{h}{\Delta-4})$, equals $u+h(\mathbf{f}_\ell + \frac{1}{\Delta-4}\mathbf{f}_1)$. Hence, it lies at the top-left corner of $\bar{B}(uv;\ell,r;\frac{\Delta-3}{\Delta-4}h)$. Similarly, the top-right corner of $\bar{B}(u'v';1,\Delta-2;\frac{h}{\Delta-4})$ lies at the top-right corner of $\bar{B}(uv;\ell,r;\frac{\Delta-3}{\Delta-4}h)$. To prove 3.5, it suffices to consider the case $r=1$ and $r'=\Delta-3$. The top-right corner of $\bar{B}(uv;\ell,r';(\frac{\Delta-3}{\Delta-4})^2|vw|)$ is

$$\begin{aligned} v + \tfrac{\Delta-3}{\Delta-4}|vw|(\mathbf{f}_{\Delta-3} + \tfrac{1}{\Delta-4}\mathbf{f}_{\Delta-2}) &= v + |vw|(\mathbf{f}_{\Delta-2} + \tfrac{1}{\Delta-4}\mathbf{f}_1 + \tfrac{\Delta-3}{\Delta-4}\cdot\tfrac{1}{\Delta-4}\mathbf{f}_{\Delta-2}) \\ &= w + \tfrac{\Delta-3}{\Delta-4}|vw|(\mathbf{f}_1 + \tfrac{1}{\Delta-4}\mathbf{f}_{\Delta-2}). \end{aligned}$$

Therefore, it lies on the right side of $B(uw;\ell,1)$, and the conclusion of 3.5 follows. Finally, for the proof of 3.6, it suffices to consider the case $\ell=2$, $r=\Delta-3$, and $|uv|=|vw|=|wx|=\lambda$. The top-right corner of $\bar{B}(uv;*,\Delta-3;\frac{\Delta-3}{\Delta-4}\lambda)$ and the top-left corner of $\bar{B}(wx;2,*;\frac{\Delta-3}{\Delta-4}\lambda)$ are respectively

$$\begin{aligned} v + \lambda(\mathbf{f}_{\Delta-3} + \tfrac{1}{\Delta-4}\mathbf{f}_{\Delta-2}) &= v + \lambda(\mathbf{f}_{\Delta-2} + \tfrac{1}{\Delta-4}\mathbf{f}_2), \\ w + \lambda(\mathbf{f}_2 \quad + \tfrac{1}{\Delta-4}\mathbf{f}_1) \quad &= w + \lambda(\mathbf{f}_1 \quad + \tfrac{1}{\Delta-4}\mathbf{f}_{\Delta-3}). \end{aligned}$$

They coincide if $\Delta=5$, otherwise the former lies to the left of the latter. □

4. The drawing

We present the construction of a drawing first for $\Delta \geqslant 5$ and then for $\Delta=4$. Both constructions follow the same idea but differ in technical details.

Lemma 4. *Suppose $\Delta \geqslant 5$.*

4.1. *Let H be a 1-bubble with root v such that $d_H(v) \leqslant \Delta-1$. Suppose that the position of v is fixed. Let ℓ and r be such that $0 \leqslant \ell, r \leqslant \Delta-1$ and $r-\ell+1 = d_H(v)$. It follows that there is a straight-line drawing of H inside $B(v;\ell,r)$.*

4.2. *Let H be a 2-bubble with first root u and second root v. Suppose that the positions of u and v are fixed on a horizontal line in this order from left to right. Let $\ell = \Delta - d_H(u)$ and $r = d_H(v)-1$. It follows that there is a straight-line drawing of H inside $\bar{B}(uv;\ell,r;\frac{\Delta-3}{\Delta-4}|uv|)$ such that the root-path of H is drawn as the segment uv.*

4.3. *Let H be a k-bubble with root-sequence $v_1, \ldots, v_k$. If $k = 1$, then suppose $d_H(v_1) \leqslant \Delta - 2$, otherwise suppose $d_H(v_1), d_H(v_k) \leqslant \Delta - 1$. Suppose that the positions of $v_1, \ldots, v_k$ are fixed in this order from left to right on a horizontal line so that $|v_1v_2| = \ldots = |v_{k-1}v_k| = \lambda$, for some $\lambda > 0$. It follows that there is a straight-line drawing of H inside $\bar{B}(v_1v_k; 1, \Delta - 2; \frac{\Delta-3}{\Delta-4}\lambda)$ such that the root-path of H is drawn as the segment v_1v_k.*

The drawings claimed above use only slopes from S and preserve the order of edges around each vertex w of H under the assumption that all edges connecting w to $G - H$ (if exist) are drawn in the correct order outside the considered bounding region.

Proof. The proof constructs the required drawing by induction on the size of H. That is, to prove any of 4.1–4.3 for a bubble H, we assume that the entire lemma holds for any bubble with fewer vertices than H has. The construction we are going to describe clearly preserves the order of edges at every vertex of H and uses only slopes from S, and we do not explicitly state this observation anywhere further in the proof.

Proof of 4.1. We consider several cases depending on the values of ℓ and r and on whether H is a single v-bubble or a bouquet of several v-bubbles.

Case 1. $\ell > r$.

In this case v is the only vertex of H and the statement is trivial.

Case 2. *H is a v-bubble and* $1 \leqslant \ell \leqslant r \leqslant \Delta - 2$.

Define $H' = H - v$, and let $v^\ell, \ldots, v^r$ be the neighbors of v in H from left to right. By 2.1, the graph H' is an $(r - \ell + 1)$-bubble with root-sequence $v^\ell, \ldots, v^r$. Put each vertex v^i at point $v + \mathbf{f}_i$. Consider two subcases.

Subcase 2.1. $1 \leqslant \ell = r \leqslant \Delta - 2$.

By the induction hypothesis 4.1, the 1-bubble H' can be drawn inside $B(v^r; 0, d_{H'}(v^r) - 1)$ as well as inside $B(v^r; \Delta - d_{H'}(v^r), \Delta - 1)$. Choose the former drawing if $\ell = r = \Delta - 2$, the latter if $\ell = r = 1$, or any of the two otherwise. After appropriate scaling, the chosen drawing fits within $B(v; r, r)$.

Subcase 2.2. $1 \leqslant \ell < r \leqslant \Delta - 2$.

It follows that $v^\ell, \ldots, v^r$ lie on a common horizontal line L and partition L into segments of length $\frac{1}{\Delta-3}$. Apply the induction hypothesis 4.3 to draw H' inside $\bar{B}(v^\ell v^r; 1, \Delta - 2; \frac{1}{\Delta-4})$ (see Fig. 4, left). It follows from 3.4 that this bounding region is contained in $B(v; \ell, r)$.

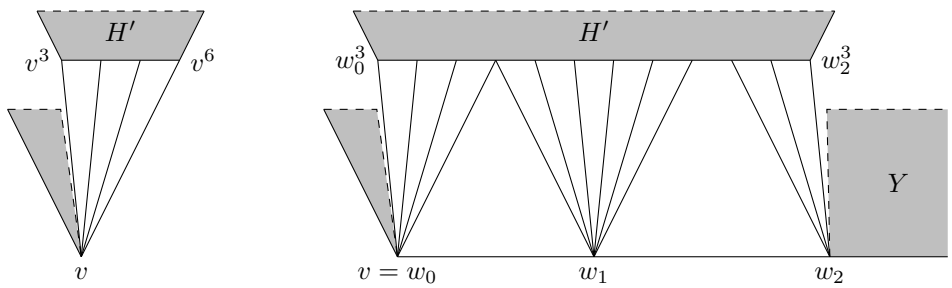

Figure 4: Sample drawings of 1-bubbles for $\Delta = 8$, illustrating Subcase 2.2 (left) and Subcase 3.2 (right) of the proof of 4.1. The gray areas labeled H' and Y denote bounding regions of the bubbles H' and Y. The unlabeled gray areas are where other parts of a 1-bubble may be added by repeating the procedure in Case 4 of the proof of 4.1.

Case 3. *H is a v-bubble, $0 \leqslant \ell \leqslant r \leqslant \Delta - 1$, and $\ell = 0$ or $r = \Delta - 1$.*

As $d_H(v) \leqslant \Delta - 1$, the cases $\ell = 0$ and $r = \Delta - 1$ cannot hold simultaneously. Therefore, by symmetry, it is enough to consider only the case that $1 \leqslant \ell \leqslant r = \Delta - 1$. Consider two subcases of the latter.

Subcase 3.1. $\ell = r = \Delta - 1$.

It follows that v has only one neighbor in H, say w, and $H' = H - v$ is a 1-bubble rooted at w. Put w horizontally to the right of v. Draw H' inside $B(w; \Delta - d_{H'}(w), \Delta - 1)$ by the induction hypothesis 4.1, scaling the drawing appropriately to fit it within $B(v; \Delta - 1, \Delta - 1)$.

Subcase 3.2. $1 \leqslant \ell < r = \Delta - 1$.

It follows that v has at least two neighbors in H. Let $P = w_0 \dots w_n$ be the simple path of length $n \geqslant 1$ that stars at $w_0 = v$ with the rightmost edge and continues counterclockwise along the outer face of H so that

- the vertices $w_1, \dots, w_{n-1}$ have degree Δ and are not cut-vertices in H,
- the vertex w_n has degree at most $\Delta - 1$ or is a cut-vertex in H.

Note that the first condition is satisfied vacuously if $n = 1$. Since the degrees of $w_1, \dots, w_{n-1}$ are at least 3 and by outerplanarity, P is an induced path. Therefore, by 2.2, the graph $H - P$ has exactly one component H' adjacent to both w_0 and w_n. All other components of $H - P$ are adjacent to w_n. Together with w_n they form a (possibly trivial) 1-bubble Y rooted at w_n. Let X denote the subgraph of H induced on $w_0, \dots, w_n$ and the vertices of H'. Define $r_X = d_X(w_n) - 1$ and $\ell_Y = \Delta - d_Y(w_n)$. Let $w_0^\ell, \dots, w_0^{\Delta-2}, w_1$ be the neighbors

of w_0 in X ordered clockwise. Let $w_{i-1}, w_i^1, \ldots, w_i^{\Delta-2}, w_{i+1}$ be the neighbors of w_i in X ordered clockwise, for $1 \leqslant i \leqslant n-1$. Let $w_{n-1}, w_n^1, \ldots, w_n^{r_X}$ be the neighbors of w_n in X ordered clockwise. It follows from 2.2 that H' is a bubble with root-sequence $w_0^\ell, \ldots, w_0^{\Delta-2}, w_1^1, \ldots, w_1^{\Delta-2}, \ldots, w_n^1, \ldots, w_n^{r_X}$ in which $w_i^{\Delta-2}$ and w_{i+1}^1 coincide whenever the inner face of H containing $w_i w_{i+1}$ is a triangle. For $i = 0, \ldots, n-1$, define

$$\lambda_i = \begin{cases} 1 & \text{if } w_i^{\Delta-2} = w_{i+1}^1, \\ \frac{\Delta-2}{\Delta-3} & \text{if } w_i^{\Delta-2} \neq w_{i+1}^1. \end{cases}$$

Put the vertices $w_1, \ldots, w_n$ in this order from left to right on the horizontal line going through w_0 in such a way that $|w_i w_{i+1}| = \lambda_i$ for $0 \leqslant i \leqslant n-1$. Put each vertex w_i^j at point $w_i + \mathbf{f}_j$. Note that if $w_i^{\Delta-2}$ and w_{i+1}^1 are the same vertex, then they correctly end up at the same point. All w_i^j lie on a common horizontal line L at distance 1 above the segment $w_0 w_n$ and partition L into segments of length $\frac{1}{\Delta-3}$. Define

$$B_X = \bar{B}(w_0 w_n; \ell, r_X; \tfrac{\Delta-3}{\Delta-4}),$$
$$B_Y = \bar{B}(w_n; \ell_Y, \Delta-1; 1).$$

Draw H' inside $\bar{B}(w_0^\ell w_n^{r_X}; 1, \Delta-2; \frac{1}{\Delta-4})$ using the induction hypothesis 4.3 (see Fig. 4, right). Note that if H' is a 1-bubble, then $n = 1$ and the root of H' has at least two edges outside H' (the ones going to w_0 and w_1), so 4.3 can indeed be applied. By 3.4, this bounding region is contained in B_X. Draw Y inside B_Y using the induction hypothesis and scaling. This way it lies entirely below the line L. By 3.2, the drawing of Y lies to the right of the edge $w_n w_n^{r_X}$ and thus does not overlap with the drawing of X. Clearly, B_X and B_Y are contained in $B(w_0; \ell, \Delta-1)$.

Case 4. *H consists of at least two v-bubbles.*

Let $(H_1, \ldots, H_b)$ be the splitting sequence of H. Thus all $H_1, \ldots, H_b$ are v-bubbles and $b \geqslant 2$. Define $X = (H_2, \ldots, H_b)$, $r_1 = \ell + d_{H_1}(v) - 1$, and $\ell_X = \Delta - d_X(v)$. By the induction hypothesis 4.1, we can draw H_1 inside $B(v; \ell, r_1)$ and X inside $B(v; \ell_X, r)$. We scale the drawing of H_1 to make it so small that it lies entirely below the horizontal lines determined by all the vertices of X other than v and those lying on the horizontal line passing through v (see Fig. 4). Since $r_1 + 1 = \ell_X$ and by 3.2, our scaled drawing of H_1 lies to the left of the leftmost edge at the root v of X. Thus the drawings of H_1 and X do not overlap. By 3.1, they both fit within $B(uv; \ell, r)$.

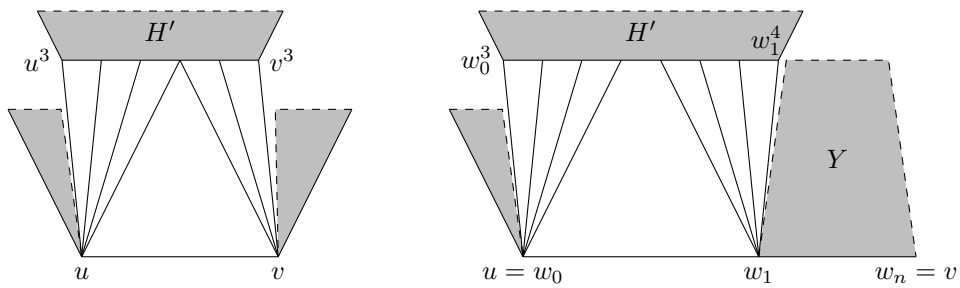

Figure 5: Sample drawings of 2-bubbles for $\Delta = 8$, illustrating Case 2 (left, with $u^6 = v^1$) and Case 3 (right, with $w_0^6 \neq w_1^1$) of the proof of 4.2. The gray areas labeled H' and Y denote bounding regions of the bubbles H' and Y. The unlabeled gray areas are where initial and final v-bubbles may be added by repeating the procedure in Case 4 of the proof of 4.2.

Proof of 4.2. We consider several cases depending on the bubbles forming the splitting sequence of H. The cases are not pairwise disjoint, but they cover all possible situations.

Case 1. *The splitting sequence of H contains a trivial single-edge e-bubble.*

Let $(H_1, \ldots, H_b)$ be the splitting sequence of H, and let H_i be a trivial e-bubble. Let u_i and v_i be the first and the second roots of H_i, respectively. That is, H_i consists only of the edge u_iv_i. Define $X = (H_1, \ldots, H_{i-1})$ and $Y = (H_{i+1}, \ldots, H_b)$. If $i = 1$, then X is a trivial one-vertex bubble, while if $i = b$, then Y is a trivial one-vertex bubble. Choose a small $\lambda > 0$. If X is a 1-bubble, then $u_i = u$, and otherwise put u_i on the segment uv so that $|uu_i| = \lambda$. Similarly, if Y is a 1-bubble, then $v_i = v$, and otherwise put v_i on uv so that $|v_iv| = \lambda$. Draw X inside $\bar{B}(uu_i; \ell, d_X(u_i) - 1; \frac{\Delta-3}{\Delta-4}\lambda)$ and Y inside $\bar{B}(v_iv; \Delta - d_Y(v_i), r; \frac{\Delta-3}{\Delta-4}\lambda)$ using the induction hypothesis 4.1 or 4.2. Clearly, both bounding regions are contained in $\bar{B}(uv; \ell, r; \frac{\Delta-3}{\Delta-4}|uv|)$. Moreover, if λ has been chosen small enough, then the two bounding regions are disjoint.

Case 2. *H is a non-trivial e-bubble.*

It follows that $d_H(u), d_H(v) \geqslant 2$ and therefore $\ell \leqslant \Delta - 2$ and $r \geqslant 1$. Let $u^\ell, \ldots, u^{\Delta-2}, v$ be the neighbors of u in H ordered clockwise, and let $u, v^1, \ldots, v^r$ be the neighbors of v in H ordered clockwise. Let $H' = H - \{u, v\}$. By 2.3, the graph H' is a bubble with root-sequence $u^\ell, \ldots, u^{\Delta-2}$, $v^1, \ldots, v^r$ in which $u^{\Delta-2}$ and v^1 coincide if the inner face of H containing uv

is a triangle. Define

$$h = \begin{cases} |uv| & \text{if } u^{\Delta-2} = v^1, \\ \frac{\Delta-3}{\Delta-2}|uv| & \text{if } u^{\Delta-2} \neq v^1. \end{cases}$$

Put each vertex u^i at point $u + h\mathbf{f}_i$ and each vertex v^i at point $v + h\mathbf{f}_i$. Note that if $u^{\Delta-2}$ and v^1 are the same vertex, then they correctly end up at the same point. All u^i and v^i lie on a common horizontal line and partition it into segments of length $\frac{h}{\Delta-3}$. Draw H' inside $\bar{B}(u^\ell v^r; 1, \Delta-2; \frac{h}{\Delta-4})$ using the induction hypothesis 4.3 (see Fig. 5, left). This bounding region is contained in $\bar{B}(uv; \ell, r; \frac{\Delta-3}{\Delta-4}h)$ by 3.4. Since $h \leqslant |uv|$, we have $\bar{B}(uv; \ell, r; \frac{\Delta-3}{\Delta-4}h) \subseteq \bar{B}(uv; \ell, r; \frac{\Delta-3}{\Delta-4}|uv|)$.

Case 3. *The splitting sequence of H starts with a non-trivial e-bubble and contains some other e-bubbles but no trivial e-bubbles.*

Let $(H_1, \ldots, H_b)$ be the splitting sequence of H and $w_0 \ldots w_n$ be the root-path of H. Thus $b \geqslant 2$, $w_0 = u$, $w_n = v$, and $n \geqslant 2$. We split H into the e-bubble H_1 with roots w_0 and w_1 and the rest $Y = (H_2, \ldots, H_b)$ being a 2-bubble with roots w_1 and w_n. Define $r_1 = d_{H_1}(w_1) - 1$ and $\ell_Y = \Delta - d_Y(w_1)$. Since $H_1, \ldots, H_b$ are non-trivial, we have $d_{H_1}(w_0), d_{H_1}(w_1), d_Y(w_1), d_Y(w_n) \geqslant 2$ and therefore $\ell \leqslant \Delta - 2$, $1 \leqslant r_1 \leqslant \Delta - 3$, $2 \leqslant \ell_Y \leqslant \Delta - 2$, and $r \geqslant 1$. Let $w_0^\ell, \ldots, w_0^{\Delta-2}, w_1$ be the neighbors of w_0 in H_1 ordered clockwise. Similarly, let $w_0, w_1^1, \ldots, w_1^{r_1}$ be the neighbors of w_1 in H_1 ordered clockwise. Note that $w_0^{\Delta-2}$ and w_1^1 are the same vertex if the inner face of H containing w_0w_1 is a triangle. Define

$$\alpha = \begin{cases} 1 & \text{if } w_0^{\Delta-2} = w_1^1, \\ \frac{\Delta-3}{\Delta-2} & \text{if } w_0^{\Delta-2} \neq w_1^1. \end{cases}$$

Fix the position of w_1 on the segment w_0w_n so that $\alpha|w_0w_1| = \frac{\Delta-3}{\Delta-4}|w_1w_n| = h$. Put each vertex w_0^i at point $w_0 + h\mathbf{f}_i$ and each vertex w_1^i at point $w_1 + h\mathbf{f}_i$. Note that if $w_0^{\Delta-2}$ and w_1^1 are the same vertex, then they correctly end up at the same point. All w_0^i and w_1^i lie on a common horizontal line L at distance h to the segment w_0w_n and partition L into segments of length $\frac{h}{\Delta-3}$. Define

$$B_1 = \bar{B}(w_0w_1; \ell, r_1; \tfrac{\Delta-3}{\Delta-4}h),$$
$$B_Y = \bar{B}(w_1w_n; \ell_Y, r; h).$$

Let $H' = H_1 - \{w_0, w_1\}$. By 2.3, the graph H' is a bubble with root-sequence $w_0^\ell, \ldots, w_0^{\Delta-2}, w_1^1, \ldots, w_1^{r_1}$ in which $w_0^{\Delta-2}$ and w_1^1 may coincide. Draw H'

inside $\bar{B}(w_0^\ell w_1^{r_1}; 1, \Delta-2; \frac{h}{\Delta-4})$ using the induction hypothesis 4.2 (see Fig. 5, right). By 3.4, this bounding region is contained in B_1. Since $r_1 \leqslant \Delta - 3$ and by 3.5, we have $B_1 \subseteq B(w_0w_n; \ell, r)$. This and the fact that $h \leqslant |w_0w_n|$ imply $B_1 \subseteq \bar{B}(w_0w_n; \ell, r; \frac{\Delta-3}{\Delta-4}|w_0w_n|)$. To complete the drawing of H, apply the induction hypothesis 4.2 to draw Y inside B_Y. This way it lies entirely below L and therefore does not overlap with the drawing of H'. By 3.2, it also lies to the right of the edge $w_1w_1^{r_1}$. Clearly, $B_Y \subseteq \bar{B}(w_0w_n; \ell, r; \frac{\Delta-3}{\Delta-4}|w_0w_n|)$.

Case 4. *The splitting sequence of H starts or ends with a v-bubble.*

The two cases are symmetric, so it is enough to consider only the case that the splitting sequence of H starts with a v-bubble. Hence, let $(H_1, \ldots, H_b)$ be the splitting sequence of H, where H_1 is a v-bubble. Define $X = (H_2, \ldots, H_b)$, $r_1 = \ell + d_{H_1}(u) - 1$, and $\ell_X = \Delta - d_Y(u)$. By the induction hypothesis 4.1, we can draw H_1 inside $B(u; \ell, r_1)$, and by the induction hypothesis 4.2, we can draw X inside $\bar{B}(uv; \ell_X, r; \frac{\Delta-3}{\Delta-4}|uv|)$. We scale the drawing of H_1 to make it so small that it lies entirely below the horizontal lines determined by all the vertices of X not lying on the root-path as well as below the horizontal line bounding from above the requested bounding region of H (see Fig. 5). Since $r_1 + 1 = \ell_X$ and by 3.2, our scaled drawing of H_1 lies to the left of the leftmost edge at the root u of X. Thus the drawings of H_1 and X do not overlap. By 3.1, they both fit within $\bar{B}(uv; \ell, r; \frac{\Delta-3}{\Delta-4}|uv|)$.

Proof of 4.3. If $k = 1$, then the claim follows directly from 4.1 and 3.1 by scaling. Thus assume $k \geqslant 2$. There is a splitting of H into 2-bubbles $X_1, \ldots, X_{k-1}$ so that the splitting sequences of $X_1, \ldots, X_{k-1}$ together form the splitting sequence of H. In particular,

- the roots of X_i are v_i and v_{i+1} for $i = 1, \ldots, k-1$,
- $X_{i-1} \cap X_i = \{v_i\}$ for $i = 2, \ldots, k-1$.

Apply 4.2 to draw each X_i inside $\bar{B}(v_iv_{i+1}; \Delta - d_{X_i}(v_i), d_{X_i}(v_{i+1}) - 1; \frac{\Delta-3}{\Delta-4}\lambda)$. Consecutive bounding regions do not overlap by 3.3, while non-consecutive ones are disjoint by 3.6. By 3.1, they are all contained in $\bar{B}(v_1v_k; 1, \Delta - 2; \frac{\Delta-3}{\Delta-4}\lambda)$. □

Lemma 5. *Suppose* $\Delta = 4$.

5.1. *Let H be a 1-bubble with root v. Suppose that the position of v is fixed. Let ℓ and r be such that $0 \leqslant \ell, r \leqslant 3$ and $r - \ell + 1 = d_H(v) \leqslant 3$. It follows that there is a straight-line drawing of H inside $B(v; \ell, r)$.*

5.2. *Let H be a 2-bubble with first root u and second root v. Suppose that the positions of u and v are fixed on a horizontal line in this order from left to right. Let $\ell = 4 - d_H(u)$ and $r = d_H(v) - 1$. It follows that there is a straight-line drawing of H inside $B(uv; \ell, r)$ such that the root-path of H is drawn as the segment uv.*

The drawings claimed above use only slopes from S and preserve the order of edges around each vertex w of H under the assumption that all edges connecting w to $G - H$ (if exist) are drawn in the correct order outside the considered bounding region.

Note that 5.2 differs from 4.2 in that the bounding region of H is unbounded from above, and in fact no such bound independent of the size of H is possible. This is why the case of $\Delta = 4$ needs to be dealt with separately.

Proof. The proof, like for Lemma 4, constructs the required drawing by induction on the size of H. That is, to prove either of 5.1 and 5.2 for a bubble H, we assume that the entire lemma holds for any bubble with fewer vertices than H has. We proceed along the same lines as in the proof of Lemma 4, focusing only on those details in which the two proofs differ.

Proof of 5.1. We consider the same cases as in the proof of 4.1.

Case 1. $\ell > r$.

As in the proof of 4.1, the statement is trivial in this case.

Case 2. *H is a v-bubble and* $1 \leqslant \ell \leqslant r \leqslant 2$.

Subcase 2.1. $1 \leqslant \ell = r \leqslant 2$.

This is handled the same way as in Subcase 2.1 of the proof of 4.1.

Subcase 2.2. $\ell = 1$ *and* $r = 2$.

Define $H' = H - v$, and let v^1 and v^2 be the left and right neighbors of v in H, respectively. As in Subcase 2.2 of the proof of 4.1, H' is a 2-bubble with roots v^1 and v^2. Draw H' inside $B(v^1v^2; 1, 2)$ using the induction hypothesis 5.2. This bounding region is clearly contained in $B(v; 1, 2)$.

Case 3. *H is a v-bubble,* $0 \leqslant \ell \leqslant r \leqslant 3$, *and* $\ell = 0$ *or* $r = 3$.

As $d_H(v) \leqslant 3$, the cases $\ell = 0$ and $r = 3$ cannot hold simultaneously. Therefore, by symmetry, it is enough to consider only the case that $1 \leqslant \ell \leqslant r = 3$. Consider two subcases of the latter.

Subcase 3.1. $\ell = r = 3$.

This is handled the same way as in Subcase 3.1 of the proof of 4.1.

Subcase 3.2. $1 \leqslant \ell < r = 3$.

Define path $w_0 \dots w_n$, bubble H', index r_X, and vertices w_i^j like in Subcase 3.2 of the proof of 4.1. For $i = 0, \dots, n-1$, define

$$\lambda_i = \begin{cases} 1 & \text{if } w_i^2 = w_{i+1}^1, \\ 1+\varepsilon & \text{if } w_i^2 \neq w_{i+1}^1, \end{cases}$$

for a small $\varepsilon > 0$. Draw vertices w_i and w_i^j like in Subcase 3.2 of the proof of 4.1 but with the new definition of λ_i. All w_i^j lie on a common horizontal line, and moreover the segments $w_i^2 w_{i+1}^1$ (for $w_i^2 \neq w_{i+1}^1$) have length ε. Define $B_X = B(w_0 w_n; \ell, r_X)$. To obtain the required drawing of H, it suffices to draw H' inside B_X, and then the remaining part of H can be drawn like in Subcase 3.2 of the proof of 4.1. But here the drawing of H' inside B_X is more tricky.

If H' is a 1-bubble, then $n = 1$, $w_0^2 = w_1^1$, and thus we can draw H' inside $B(w_0^2; 1, 2)$ by the induction hypothesis 5.1, scaling the drawing appropriately to fit it within B_X. Thus we assume that H' is a k-bubble with $k \geqslant 2$. We split H' into 2-bubbles $X_1, \dots, X_{k-1}$ so that the splitting sequences of $X_1, \dots, X_{k-1}$ together form the splitting sequence of H'. The roots of $X_1, \dots, X_{k-1}$ are pairs of vertices consecutive in the sequence of all w_i^j. Define $\ell_i = 4 - d_{X_i}(u_i)$ and $r_i = d_{X_i}(v_i)$, where u_i and v_i denote respectively the first and the last root of X_i. We draw each X_i inside $B(u_i v_i; \ell_i, r_i)$ using the induction hypothesis 5.2. Since each root of H' has degree at most 3 in H', we have $\ell_i \geqslant 2$ for $2 \leqslant i \leqslant k-1$ and $r_i \leqslant 1$ for $1 \leqslant i \leqslant k-2$. Thus the bounding regions for X_i do not overlap. Moreover, for $2 \leqslant i \leqslant k-2$, we clearly have $B(u_i v_i; \ell_i, r_i) \subseteq B_X$. Thus to complete the proof for the case of H being a v-bubble, it remains to show that the drawings of X_1 and X_{k-1} are contained in B_X. We do not necessarily have $B(u_1 v_1; \ell_1, r_1) \subseteq B_X$. However, this inclusion may not hold only if $\ell = 2$ and $w_0^2 \neq w_1^1$. In this case we have $|u_1 v_1| = \varepsilon$ and thus the drawing of X_1 indeed lies within B_X provided that ε is small enough. Similarly, the drawing of X_{k-1} in contained in B_X for ε small enough.

Case 4. *H consists of at least two v-bubbles.*

This is handled the same way as in Case 4 of the proof of 4.1.

Proof of 5.2. Again, we consider the same cases as in the proof of 4.2.

Case 1. *The splitting sequence of H contains a trivial single-edge e-bubble.*

This is handled the same way as in Case 1 of the proof of 4.2. The only difference is that here the bounding regions of X and Y (defined as in Case 1 of the proof of 4.2) obtained from the application of the induction hypothesis 5.2 are not bounded from above, so we need to choose λ small enough for the drawings of X and Y not to overlap and to fit within $B(uv; \ell, r)$.

Case 2. *H is a non-trivial e-bubble.*

It follows that $d_H(u), d_H(v) \geqslant 2$ and therefore $\ell \leqslant 2$ and $r \geqslant 1$. Define vertices u^j, v^j and bubble H' like in Case 2 of the proof of 4.2. Define

$$h = \begin{cases} |uv| & \text{if } u^2 = v^1, \\ |uv| - \varepsilon & \text{if } u^2 \neq v^1, \end{cases}$$

for a small $\varepsilon > 0$. Put each vertex u^i at point $u + h\mathbf{f}_i$ and each vertex v^i at point $v + h\mathbf{f}_i$, so that if u^2 and v^1 are the same vertex, then they correctly end up at the same point. All u^i and v^i lie on a common horizontal line, and moreover the segment u^2v^1 (if exists) has length ε. The same argument as in Subcase 3.2 above shows that H' can be drawn inside $B(u^\ell v^r; 1, 2)$.

Case 3. *The splitting sequence of H contains at least two e-bubbles but no trivial e-bubbles.*

Let $(H_1, \ldots, H_b)$ be the splitting sequence of H and $w_0 \ldots w_n$ be the root-path of H. Thus $b \geqslant 2$, $w_0 = u$, $w_n = v$, and $n \geqslant 2$. Since none of the edges $w_0w_1, \ldots, w_{n-1}w_n$ is a bridge in H. We split H into 2-bubbles $X_1, \ldots, X_n$ so that the roots of X_i are w_{i-1} and w_i and the splitting sequences of $X_1, \ldots, X_n$ together form the splitting sequence $(H_1, \ldots, H_b)$ of H. Define $\ell_i = 4 - d_{X_i}(w_{i-1})$ and $r_i = d_{X_i}(w_i)$. We draw each X_i inside $B(w_{i-1}w_i; \ell_i, r_i)$ using the induction hypothesis 5.2. Since none of $X_1, \ldots, X_n$ is a trivial e-bubble, we have $\ell_1 = \ell \leqslant 2$, $\ell_i = 2$ for $2 \leqslant i \leqslant n$, $r_i = 1$ for $1 \leqslant i \leqslant n-1$, and $r_n = r \geqslant 1$. Thus the bounding regions for X_i do not overlap and are contained in $B(w_0w_n; \ell, r)$.

Case 4. *The splitting sequence of H starts or ends with a v-bubble.*

This is handled the same way as in Case 4 of the proof of 4.2. □

Now, to prove the Main Theorem, pick any vertex v of G of degree less than Δ (such a vertex always exists in an outerplanar graph), fix its position in the plane, and apply 4.1 or 5.1 to the graph G considered as a 1-bubble with root v.

Acknowledgments

We thank Vít Jelínek and Dömötör Pálvölgyi for introducing us to the problem at the meeting in Prague in the summer of 2011. We also thank an anonymous referee whose comments improved the quality of the paper.

References

[1] G. A. Wade, J.-H. Chu, Drawability of complete graphs using a minimal slope set, Comput. J. 37 (2) (1994) 139–142.

[2] V. Dujmović, D. R. Wood, On linear layouts of graphs, Discrete Math. Theor. Comput. Sci. 6 (2) (2004) 339–358.

[3] J. Barát, J. Matoušek, D. R. Wood, Bounded-degree graphs have arbitrarily large geometric thickness, Electron. J. Combin. 13 (1) (2006) #R3, 14 pp.

[4] J. Pach, D. Pálvölgyi, Bounded-degree graphs can have arbitrarily large slope numbers, Electron. J. Combin. 13 (1) (2006) #N1, 4 pp.

[5] V. Dujmović, M. Suderman, D. R. Wood, Graph drawings with few slopes, Comput. Geom. 38 (3) (2007) 181–193.

[6] P. Mukkamala, D. Pálvölgyi, Drawing cubic graphs with the four basic slopes, in: M. van Kreveld, B. Speckmann (Eds.), Graph Drawing, Vol. 7034 of Lecture Notes Comput. Sci., Springer, 2012, pp. 254–265.

[7] B. Keszegh, J. Pach, D. Pálvölgyi, G. Tóth, Drawing cubic graphs with at most five slopes, Comput. Geom. 40 (2) (2008) 138–147.

[8] P. Mukkamala, M. Szegedy, Geometric representation of cubic graphs with four directions, Comput. Geom. 42 (9) (2009) 842–851.

[9] I. Fáry, On straight line representation of planar graphs, Acta Univ. Szeged. Sect. Sci. Math. 11 (1948) 229–233.

[10] P. Koebe, Kontaktprobleme der konformen Abbildung, Berichte Verhande. Sächs. Akad. Wiss. Leipzig, Math.-Phys. Klasse 88 (1936) 141–164.

[11] K. Wagner, Bemerkungen zum Vierfarbenproblem, Jahresber. Deutsch. Math. Verein. 46 (1936) 26–32.

[12] V. Dujmović, D. Eppstein, M. Suderman, D. R. Wood, Drawings of planar graphs with few slopes and segments, Comput. Geom. 38 (3) (2007) 194–212.

[13] V. Jelínek, E. Jelínková, J. Kratochvíl, B. Lidický, M. Tesař, T. Vyskočil, The planar slope number of planar partial 3-trees of bounded degree, Graphs Combin. 29 (4) (2013) 981–1005.

[14] B. Keszegh, J. Pach, D. Pálvölgyi, Drawing planar graphs of bounded degree with few slopes, SIAM J. Discrete Math. 27 (2) (2013) 1171–1183.