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Massimiliano Berti, Jean-Marc Delort

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Massimiliano Berti
SISSA, Via Bonomea 265
34136, Trieste, Italy[1]
&
Jean-Marc Delort[2]
Université Paris 13,
Sorbonne Paris Cité, LAGA, CNRS (UMR 7539),
99, Avenue J.-B. Clément,
F-93430 Villetaneuse

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Abstract

The goal of this monograph is to prove that any solution of the Cauchy problem for the capillarity-gravity water waves equations, in one space dimension, with periodic, even in space, initial data of small size ϵ, is almost globally defined in time on Sobolev spaces, i.e. it exists on a time interval of length of magnitude ϵ^{-N} for any N, as soon as the initial data are smooth enough, and the gravity-capillarity parameters are taken outside an exceptional subset of zero measure. In contrast to the many results known for these equations on the real line, with decaying Cauchy data, one cannot make use of dispersive properties of the linear flow. Instead, our method is based on a normal forms procedure, in order to eliminate those contributions to the Sobolev energy that are of lower degree of homogeneity in the solution.

Since the water waves equations are a quasi-linear system, usual normal forms approaches would face the well known problem of losses of derivatives in the unbounded transformations. In this monograph, to overcome such a difficulty, after a paralinearization of the capillarity-gravity water waves equations, necessary to obtain energy estimates, and thus local existence of the solutions, we first perform several paradifferential reductions of the equations to obtain a diagonal system with constant coefficients symbols, up to smoothing remainders. Then we may start with a normal form procedure where the small divisors are compensated by the previous paradifferential regularization. The reversible structure of the water waves equations, and the fact that we look for solutions even in x, guarantees a key cancellation which prevents the growth of the Sobolev norms of the solutions.

Keywords: Capillarity-gravity water waves equations, Long-time existence, Paradifferential calculus, Normal forms. MSC 76B15, 76B45, 35Q35, 35S50, 37J40, 70K45.

Contents

Chapter 0

Introduction

0.1 Main Theorem

The capillarity-gravity water waves equations describe the motion of the interface between an incompressible irrotational fluid in a gravity field and air, in the presence of surface tension. In the case of the one dimensional problem with finite depth, corresponding to a two-dimensional fluid, the velocity of the fluid is given by the gradient of an harmonic potential Φ, called velocity potential, defined on the time dependent domain

$$\Omega_t = \left\{(x,y) \in \mathbb{R}^2; -h < y < \eta(t,x)\right\}.$$

As soon as the profile $\eta(t,x)$ is known, Φ is determined by the knowledge of its restriction to $y = \eta(t,x)$, say $\psi(t,x) = \Phi(t,x,\eta(t,x))$, and by the Neumann boundary condition at the bottom $\partial_y \Phi|_{y=-h} = 0$. The resulting system on (η,ψ) is the Craig-Sulem-Zakharov formulation of the capillarity-gravity water waves equations

$$\begin{aligned} &\partial_t \eta = G(\eta)\psi \\ &\partial_t \psi = -g\eta + \kappa H(\eta) - \frac{1}{2}(\partial_x \psi)^2 + \frac{1}{2}\frac{(\eta' \partial_x \psi + G(\eta)\psi)^2}{1+\eta'^2} \end{aligned} \tag{0.1.1}$$

where $g > 0$ is the acceleration of gravity, $\kappa > 0$ the surface tension, $\eta' = \partial_x \eta$,

$$H(\eta) = \partial_x[(\partial_x \eta)(1 + (\partial_x \eta)^2)^{-1/2}]$$

is the mean curvature of the wave profile, and $G(\eta)$ is the Dirichlet-Neumann operator, defined in terms of the potential Φ by

$$G(\eta)\psi = (\partial_y \Phi - \partial_x \eta \partial_x \Phi)(t,x,\eta(t,x))$$

(see [25] and [66] for the derivation of (0.1.1) when the surface tension vanishes, and the book of Lannes [53] for a presentation of different models of

water waves). Local and global existence of solutions for these equations has been the object of intensive studies during the last years.

In the case $\kappa = 0$, the problem of local existence of solutions with smooth and *small* Cauchy data with Sobolev regularity, defined on $\mathbb{R}$, has been solved by Nalimov [58], for the infinite depth problem, and by Yosihara [65], for finite depth (see also Craig [23]). For large Cauchy data, local existence in infinite depth has been proved by S. Wu [61] (see [63] for the case of two space dimensions, i.e. a 3D fluid). The similar question in finite depth (for a variable bottom in any dimension) has been solved by Lannes [52]. The case of local existence for the free surface incompressible Euler equation has been settled by Lindblad [55].

Concerning the case of positive κ, local existence of solutions with data in Sobolev spaces is due to Beyer and Gunther [17] and to Coutand and Shkroller [22] in the case of solutions of the incompressible free boundary Euler equation. Ifrim and Tataru [43] studied recently local existence when the fluid has constant vorticity. The case of finite depth has been settled by Ming and Zhang [56] and arbitrary bottoms have been considered by Alazard, Burq and Zuily [2] for rough initial data. Finally, the problem of local existence with Cauchy data that are periodic in space, instead of lying in a Sobolev space on $\mathbb{R}^d$, has been established by Ambrose [10] and Ambrose-Masmoudi [11] for $\kappa \geq 0$ in the case of infinite depth, and by Schweizer [60] for finite depth (even with a non zero vorticity). The case of non-localized Cauchy data lying in uniformly local spaces has been treated by Alazard, Burq and Zuily, in the case of arbitrary rough bottoms [4].

Regarding to long time existence of solutions of the water waves equations, most results have been obtained when Cauchy data are small, smooth and decaying at infinity, which allows to exploit the dispersive properties of the flow of the linear part of the equations. The first contribution has been the one of Sijue Wu [62], who showed that in one space dimension, i.e. for a two dimensional fluid of infinite depth, solutions of the water waves equations with $\kappa = 0$ exist over a time interval of exponential length $e^{c/\epsilon}$ when the size ϵ of the initial data goes to zero. In two space dimensions, i.e. for three dimensional fluids, global existence with small decaying data has been obtained independently by Germain, Masmoudi and Shatah [36] and by Wu [64]. Global existence for small data in one space dimension has been proved independently by Ionescu and Pusateri [45], Alazard and Delort [5] and by Ifrim and Tataru [44], for infinite depth fluids.

For the capillarity-gravity water waves equations, i.e. when $\kappa > 0$, global existence is known in two space dimensions (three dimensional fluids) in infinite depth by Deng, Ionescu, Pausader and Pusateri [31]. When the surface tension is positive, but the gravity g vanishes, global solutions in infinite depth fluids have been proved to exist by Germain, Masmoudi and Shatah [37] in dimension 2 and by Ionescu and Pusateri [46] in dimension 1.

Finally, long time existence results have been obtained independently

by Ifrim and Tataru [42] and Ionescu and Pusateri [46] for small data, that do not necessarily decay at infinity. These authors proved that, in the case $\kappa > 0$, $g = 0$ and for infinite depth fluids, data of size ϵ in some Sobolev space (periodic or on the line), give rise to solutions defined on interval times of length at least c/ϵ^2 (instead of the usual lifetime in c/ϵ that holds true in general for a nonlinear equation with quadratic non-linearity). In [43], Ifrim and Tataru obtain a similar result when $\kappa = 0$, $g > 0$ and constant vorticity. The case of a zero vorticity had been previously treated by Hunter, Ifrim and Tataru in [41]. It is implicitly contained in the energy estimates of Wu [61]. Very recently, Harrop-Griffiths, Ifrim and Tataru [40] obtained a $c\epsilon^{-2}$ lower bound for the time of existence in the case $\kappa = 0$, $g > 0$, for irrotational incompressible fluids with finite depth.

The goal of this monograph is to get, for one dimensional space periodic Cauchy data of size ϵ, solutions of the water waves equations (0.1.1) with $g > 0, \kappa > 0$ defined over a time interval of length $c_N\epsilon^{-N}$, for arbitrary N. We shall be able to achieve this almost global existence result if the initial datum (η_0, ψ_0) is smooth enough, even in x and η_0 has zero average.

Solutions $(\eta(t,x), \psi(t,x))$ of (0.1.1) that are even in x are called "standing waves". This property is preserved during the evolution of (0.1.1). In this case also the velocity potential $\Phi(t,x,y)$ is even and 2π-periodic in x and so the x-component of the velocity field (Φ_x, Φ_y) vanishes at $x = k\pi$, $\forall k \in \mathbb{Z}$. Hence there is no flux of fluid through the lines $x = k\pi$, $k \in \mathbb{Z}$, and a solution of the system (0.1.1) physically describes the motion of a liquid confined between two walls.

The assumption that η_0 has zero average is preserved during the evolutions because the "mass" $\int_{\mathbb{T}^1} \eta(x)dx$ is a prime integral of (0.1.1). The component $\eta(t,\cdot)$ of the solution will thus lie in a Sobolev space $H_0^{s+\frac{1}{4}}(\mathbb{T}^1)$ of periodic functions with zero mean.

Concerning ψ, notice that the right hand side of (0.1.1) is well defined when ψ is in a space of functions modulo constants (which is natural, as only the gradient of the velocity potential Φ has a physical meaning). Thus, projecting the second equation on functions modulo constants, we see that we may look for ψ in an homogeneous Sobolev space $\dot{H}^{s-\frac{1}{4}}(\mathbb{T}^1)$ of periodic functions modulo constants.

The main result proved in this monograph is the following:

Theorem 1 (Almost global existence of periodic capillarity-gravity waves) *There is a zero measure subset $\mathcal{N}$ in $]0,+\infty[^2$ such that, for any (g,κ) in $]0,+\infty[^2 - \mathcal{N}$, for any N in $\mathbb{N}$, there is $s_0 > 0$ and, for any $s \geq s_0$, there are $\epsilon_0 > 0, c > 0, C > 0$ such that, for any $\epsilon \in]0,\epsilon_0[$, any even function (η_0, ψ_0) in $H_0^{s+\frac{1}{4}}(\mathbb{T}^1,\mathbb{R}) \times \dot{H}^{s-\frac{1}{4}}(\mathbb{T}^1,\mathbb{R})$ with*

$$\|\eta_0\|_{H_0^{s+\frac{1}{4}}} + \|\psi_0\|_{\dot{H}^{s-\frac{1}{4}}} < \epsilon\,,$$

system (0.1.1) has a unique classical solution (η, ψ) *defined on* $]-T_\epsilon, T_\epsilon[\times\mathbb{T}^1$ *with* $T_\epsilon \geq c\epsilon^{-N}$*, belonging to the space*

$$C^0\Big(]-T_\epsilon, T_\epsilon[, H_0^{s+\frac{1}{4}}(\mathbb{T}^1,\mathbb{R})\times \dot{H}^{s-\frac{1}{4}}(\mathbb{T}^1,\mathbb{R})\Big)$$

satisfying the initial condition $\eta|_{t=0} = \eta_0, \psi|_{t=0} = \psi_0$*. Moreover, this solution is even in space and it stays at any time in the ball of center 0 and radius* $C\epsilon$ *of* $H_0^{s+\frac{1}{4}}(\mathbb{T}^1,\mathbb{R})\times \dot{H}^{s-\frac{1}{4}}(\mathbb{T}^1,\mathbb{R})$*.*

The above theorem establishes almost global existence of solutions to the capillarity-gravity water waves equations on a compact manifold, the circle, when the parameters (g, κ) avoid a zero measure subset. It is of different nature than the results we mentioned above, for decaying Cauchy data on the line, as there is no longer any dispersive property which makes decay the solutions, providing long time existence.

Notice also that the assumption that the parameters (g, κ) avoid a subset of zero measure is essential for the normal forms reduction that will play a seminal role in the proof. Without any restriction on the parameters, it is known that resonances may occur at the level of the quadratic part of the non-linearity (the so called "Wilton ripples") or for higher order terms in the Taylor expansion of the non-linearity. We refer for example to Craig and Sulem [26] that discuss the possible dynamical instabilities that might be generated by this phenomenon.

We do not know if the almost global solutions of the Cauchy problem proved in Theorem 1 are global in time or not. Let us mention nevertheless that one may construct classes of global solutions, namely time periodic and also quasi-periodic ones.

In the case of zero surface tension, existence of small amplitude time periodic standing wave solutions has been proved first by Plotnikov-Toland in [59] for a fluid in finite depth, and by Iooss, Plotnikov, Toland in [51], see also [47], [48], in the case of infinite depth. For the problem with surface tension existence of time periodic standing wave solutions has been proved by Alazard-Baldi [1]. Recently Berti-Montalto [16] have extended this result proving the existence of also time quasi-periodic capillarity-gravity standing wave solutions, as well as their linear stability. All the above results prove the existence of solutions when the parameters (g, κ, h) satisfy suitable non-resonance conditions. Actually the problem of the existence of periodic/quasi-periodic solutions is faced with small divisors difficulties which are particularly hard due to the quasi-linear nature of the water waves system (0.1.1).

It is well known that the existence of quasi-periodic motions is possible just for systems with some algebraic structure which excludes "secular motions" (using the language of Celestial mechanics) and growth of the Sobolev norms. The most common ones are the Hamiltonian and the reversible structure. The water-waves system (0.1.1) exhibits both of them.

In the proof of Theorem 1, it is the reversible structure of (0.1.1), and the fact that we restrict to solutions even in x, that ensure that terms that could generate a growth of the Sobolev norms of the standing waves actually vanish. Let us describe the meaning of that notion of reversibility. We say that the capillarity-gravity water waves system (0.1.1) is reversible with respect to the involution $S = \left[\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right]$ since it may be written as a dynamical system $\left[\begin{smallmatrix} \dot\eta \\ \dot\psi \end{smallmatrix}\right] = F\left[\begin{smallmatrix} \eta \\ \psi \end{smallmatrix}\right]$, where the vector field F satisfies

$$SF\left[\begin{smallmatrix} \eta \\ \psi \end{smallmatrix}\right] = -F\Big(S\left[\begin{smallmatrix} \eta \\ \psi \end{smallmatrix}\right]\Big)\,. \tag{0.1.2}$$

Equivalently, denoting by Φ^t the flow of (0.1.1), the reversibility property (0.1.2) is equivalent to

$$S \circ \Phi^{-t} = \Phi^t \circ S\,.$$

We shall discuss in the next sections some intuitive reasons why the presence of this symmetry may produce stability results.

To conclude we also mention some existence results of other global solutions, as the traveling wave periodic solutions (standing waves are not traveling because they are even in space). In dimension 2 small amplitude traveling gravity water waves were proved by Levi-Civita [54]. In dimension 3 the existence of traveling capillarity-gravity periodic water waves has been proved by Craig-Nicholls [24] (it is not a small divisor problem) and by Iooss-Plotinikov [50]-[49] in the case of zero surface tension (in such a case it is a small divisor problem).

In the following sections, we shall describe the main steps in the proof of Theorem 1 performed in Chapter 3 and Chapter 4. The other chapters are devoted to the construction of the tools needed in those two chapters, and to the reformulation of the water waves system (0.1.1) as a paradifferential equation. For simplicity we take the depth $h = 1$ and the gravity $g = 1$.

0.2 Introduction to the proof

In this section we shall present on a very simplified model some ideas of the normal forms method for semilinear PDEs, and we shall point out the deep modifications that this procedure requires for proving the long time existence Theorem 1 for the *quasi-linear* water waves system (0.1.1). In the following sections of this introduction, we shall explain in detail our normal form approach, including several technical aspects of the proof.

Consider a semilinear equation of the form

$$\begin{aligned} (D_t - m_\kappa(D_x))u &= P(u,\bar u) \\ u|_{t=0} &= \epsilon u_0, \end{aligned} \tag{0.2.1}$$

where $D_t = \frac{1}{i}\frac{\partial}{\partial t}$ and $m_\kappa(D_x) = \mathcal{F}^{-1} \circ m_\kappa(\xi) \circ \mathcal{F}$ is an even Fourier multiplier with real valued symbol that depends on some parameter $\kappa \in]0, +\infty[$, acting on complex valued functions $u(x)$ of $L^2(\mathbb{T}^1)$. In the present case

$$m_\kappa(\xi) = (\xi \tanh \xi)^{\frac{1}{2}}(1 + \kappa\xi^2)^{1/2} \tag{0.2.2}$$

comes from the linearization of the water waves equation at the flat surface and zero velocity potential. Notice that since $m_\kappa(\xi)$ is even in ξ, the operator $m_\kappa(D_x)$ leaves invariant the subspace of functions even in x. Assume that the nonlinearity P is a polynomial, homogeneous of degree $p \geq 2$, depending on $(u, \bar{u})$. The first serious difficulty of (0.1.1), compared to (0.2.1), is, of course, the quasi-linear nature of the nonlinear vector field in (0.1.1). We shall discuss later how to take it into account.

The initial datum u_0 in (0.2.1) is in the Sobolev space $H^s(\mathbb{T}^1, \mathbb{C})$, for some large s. Writing the Sobolev energy inequality associated to (0.2.1), one gets an a priori estimate for the solutions like

$$\|u(t,\cdot)\|_{H^s}^2 \leq \|u(0,\cdot)\|_{H^s}^2 + C\left| \int_0^t \|u(\tau,\cdot)\|_{H^s}^{p+1}\, d\tau \right|. \tag{0.2.3}$$

Since the initial datum $\|u(0,\cdot)\|_{H^s} \sim \epsilon$, one deduces by a bootstrap argument the a priori bound $\|u(t,\cdot)\|_{H^s} = O(\epsilon)$ as long as $|t|$ is smaller than $c\epsilon^{-p+1}$, for some c small. As a consequence, using that (0.2.1) is locally well posed in Sobolev spaces, the solution exists up to times of magnitude $c\epsilon^{-p+1}$. Nevertheless, one may get a longer existence time combining such energy bounds to Shatah's style normal form methods, in the framework of periodic functions of space (instead of functions defined on $\mathbb{R}$). In the Hamiltonian setting this procedure is often called Birkhoff normal form, that for semilinear equations is by now classical, see for example [14], [30], [13]. The strategy is to look for some polynomial correction $Q(u, \bar{u})$ to u, homogeneous of degree p, such that

$$\begin{aligned}(D_t - m_\kappa(D_x))[u + Q(u,\bar{u})] &= (\text{terms of order } q > p) \\ &+ (\text{terms of order } p \text{ that do not contribute to the energy}).\end{aligned} \tag{0.2.4}$$

Then energy inequalities for this new equation imply that solutions with initial datum of size ϵ exist up to times of magnitude $\epsilon^{-q} \gg \epsilon^{-p}$. If one may repeat the process as many times as desired, one will get finally an existence time of order ϵ^{-N} for any N. The fact that the terms of order p that are left in (0.2.4) do not contribute to the energy inequality follows from a suitable algebraic properties of the nonlinearity P. In many instances, the Hamiltonian nature of the vector field P provides that property. Here, we shall instead exploit another classical structure, namely *reversibility*, that for semilinear PDEs has been used in [34, 35]. For the model equation (0.2.1), considering the involution $S : u \mapsto \bar{u}$ acting on the space of functions even in

x, this amounts to require that the nonlinearity satisfies $P(u,\bar u)=\overline{P(\bar u,u)}$. To construct Q from P, let us limit ourselves to the case when for instance $P(u,\bar u)=a_{p,\ell}u^\ell\bar u^{p-\ell}$ for a scalar coefficient $a_{p,\ell}$. Notice that it satisfies the above reversibility condition if and only if $a_{p,\ell}$ is *real*. We look for $Q(u,\bar u)$ as

$$Q(u,\bar u)=M(\underbrace{u,\dots,u}_{\ell},\underbrace{\bar u,\dots,\bar u}_{p-\ell})$$

for some p-linear map M to be determined in such a way that

$$(D_t-m_\kappa(D_x))M(u,\dots,u,\bar u,\dots,\bar u)=-a_{p,\ell}u^\ell\bar u^{p-\ell}+\text{ higher order terms}\,.$$

Distributing the time derivative on each argument of M and replacing $D_tu, D_t\bar u$ from its expression coming from (0.2.1), we obtain

$$\begin{aligned}(0.2.5)\quad (D_t-m_\kappa(D_x))&M(u,\dots,u,\bar u,\dots,\bar u)\\ &=\sum_{j=1}^{\ell}M(u,\dots,m_\kappa(D_x)u,\dots,u,\bar u,\dots,\bar u)\\ &-\sum_{j=\ell+1}^{p}M(u,\dots,u,\bar u,\dots,m_\kappa(D_x)\bar u,\dots,\bar u)\\ &\qquad -m_\kappa(D_x)M(u,\dots,\bar u)+\text{ higher order terms}\end{aligned}$$

where in the sums $m_\kappa(D_x)$ acts on the j-th component and where we used that $m_\kappa(\xi)$ is even. Denote by Π_n, n in $\mathbb{N}$, the spectral projector associated to the nth mode of $-\Delta$ on the circle. Then as $m_\kappa(\xi)$ is even, $m_\kappa(D_x)\Pi_n=m_\kappa(n)\Pi_n$, so that if we replace in the right hand side of (0.2.5)

$$\begin{aligned}&M(u,\dots,m_\kappa(D_x)u,\dots,u,\bar u,\dots,\bar u)\\ &M(u,\dots,u,\bar u,\dots,m_\kappa(D_x)\bar u,\dots,\bar u)\\ &m_\kappa(D_x)M(u,\dots,\bar u)\end{aligned}$$

by respectively

$$\begin{aligned}&\Pi_{n_{p+1}}M(\Pi_{n_1}u_1,\dots,m_\kappa(n_j)\Pi_{n_j}u_j,\dots,\Pi_{n_\ell}u_\ell,\Pi_{n_{\ell+1}}u_{\ell+1},\dots,\Pi_{n_p}u_p)\\ &\Pi_{n_{p+1}}M(\Pi_{n_1}u_1,\dots,\Pi_{n_\ell}u_\ell,\Pi_{n_{\ell+1}}u_{\ell+1},\dots,m_\kappa(n_j)\Pi_{n_j}u_j,\dots,\Pi_{n_p}u_p)\\ &m_\kappa(n_{p+1})\Pi_{n_{p+1}}M(\Pi_{n_1}u_1,\dots,\Pi_{n_\ell}u_\ell,\Pi_{n_{\ell+1}}u_{\ell+1},\dots,\Pi_{n_p}u_p)\end{aligned}$$

we get for the terms homogeneous of degree p in the right hand side of (0.2.5) the expression

(0.2.6)

$$\mathcal{D}_{\kappa,\ell}(n_1,\dots,n_{p+1})\Pi_{n_{p+1}}M(\Pi_{n_1}u_1,\dots,\Pi_{n_\ell}u_\ell,\Pi_{n_{\ell+1}}u_{\ell+1},\dots,\Pi_{n_p}u_p),$$

with the "small divisors"

$$\mathcal{D}_{\kappa,\ell}(n_1,\dots,n_{p+1})=\sum_{j=1}^{\ell}m_\kappa(n_j)-\sum_{j=\ell+1}^{p+1}m_\kappa(n_j)\,.$$

In order to obtain (0.2.4), one would have thus to choose M so that
(0.2.7)

$$\mathcal{D}_{\kappa,\ell}(n_1,\dots,n_{p+1})\Pi_{n_{p+1}}M(\Pi_{n_1}u_1,\dots,\Pi_{n_p}u_p) = -\Pi_{n_{p+1}}\big(a_{p,\ell}\prod_{j=1}^{p}\Pi_{n_j}u_j\big)$$

for any $n_1,\dots,n_{p+1}$ such that $\sum_{j=1}^{p+1}\epsilon_j n_j = 0$ for some choice of the signs $\epsilon_j \in \{-1,+1\}$. Suppose that the following non-resonance condition holds: there is N_0 such that

$$|\mathcal{D}_{\kappa,\ell}(n_1,\dots,n_{p+1})| \geq c(\text{third largest among } n_1,\dots,n_{p+1})^{-N_0} \tag{0.2.8}$$

for any $n_1,\dots,n_{p+1}$ except in the case when

$$p \text{ is odd, } \ell = \frac{p+1}{2}, \text{ and } \{n_1,\dots,n_\ell\} = \{n_{\ell+1},\dots,n_{p+1}\}, \tag{0.2.9}$$

for which $\mathcal{D}_{\kappa,\ell}(n_1,\dots,n_{p+1})$ vanishes for any κ. In several instances, it turns out that, when the symbol $\xi \mapsto m_\kappa(\xi)$ is analytic/sub-analytic and satisfies suitable "non degeneracy" conditions, then (0.2.8) can be verified for any κ taken outside a subset of zero measure. This is the case, for example, for the symbol $m_\kappa(\xi)$ given in (0.2.2), see Proposition 6 below. Notice also that, for $m_\kappa(\xi)$ in (0.2.2), the divisor $\mathcal{D}_{\kappa,\ell}(n_1,\dots,n_{p+1})$ vanishes as well when all frequencies n_k are zero, but we ignore this point in this introduction, as we shall dispose of it by the fact that in the water waves system (0.1.1) the function η has zero average and ψ is defined modulo constants.

Except in case (0.2.9), one may define M by division of (0.2.7). The small divisor condition (0.2.8) and the fact that P is semilinear imply that M is a multilinear map that is *bounded* on H^s for any large enough s, and thus the associated transformation $u \mapsto u+Q(u,\bar u)$ is bounded and invertible in a neighborhood of the origin of H^s. In the "resonant" case (0.2.9), the corresponding term homogeneous of degree p, that is of the form

$$-\Pi_{n_\ell}\Big[a_{2\ell-1,\ell}\prod_{j=1}^{\ell-1}|\Pi_{n_j}u|^2\Pi_{n_\ell}u\Big],$$

cannot be eliminated in the right hand side of (0.2.5), but it does not contribute to the energy because the coefficient $a_{2\ell-1,\ell}$ is real. In dynamical systems language the "actions" $\|\Pi_n u\|_{L^2}^2$ are prime integrals of the resulting "resonant" system. The above procedure can be iterated at higher orders. The fact that the terms of order p that are left at each step do not contribute to the growth of the Sobolev norms is a consequence of the preservation of the reversible structure and the fact that we restrict to a subspace of functions even in x.

In trying to implement the above procedure for the complete water waves system (0.1.1) many considerable problems arise.

1. The first difficulty, that is already present in the local theory of water waves type equations, is the fact that (0.1.1) is a quasi-linear system for which a direct application of energy inequalities makes apparently lose derivatives. This problem is now well understood (see references in section 0.3 below) and, following the now classical approach of Alazard-Métivier [7], we settle it writing (0.1.1) as a paradifferential system in terms of the so called "good unknown". However we cannot directly use the results in [7, 2, 6] since we have to ensure that the new paradifferential system admits a polynomial expansion as the solution tends to zero. We explain this procedure in detail in section 0.3. An important fact is that the "good unknown" change of variable preserves the reversible structure.

2. Before performing a normal form analysis as described above, we reduce the terms of the water waves para-differential system that are of positive order to constant coefficients. This step is essential, otherwise the transformations performed above to decrease the size of the nonlinearity, like $u + Q(u, \bar{u})$, would be unbounded. We perform such a reduction along the lines of the recent works of Alazard-Baldi [1] and Berti-Montalto [16] concerning respectively periodic and quasi-periodic solutions of (0.1.1), ending up with a system as

$$\Big(D_t - m_\kappa(D_x)(1 + \underline{\zeta}(U;t))\left[\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right] - H(U;t,D_x)\Big)V = R(U;t)V$$

where $H(U;t,\xi)$ is a diagonal matrix of Fourier multipliers of order one (independent of x), with imaginary part of order 0, $\underline{\zeta}(U;t)$ is a real valued function of t, independent of x, and $R(U;t)$ is a smoothing operator (we neglect here another smoothing remainder $R_2(U;t)U$, see (0.4.7)). This reduction to constant coefficients of the unbounded terms is possible because the dispersion relation $\xi \mapsto m_\kappa(\xi)$ is super-linear. We explain this procedure in detail in section 0.4.

3. An energy inequality for the above system implies an estimate like (0.2.3) (with $p = 2$) proving existence of the solution up to times $c\epsilon^{-1}$ (this time corresponds essentially to local existence theory). In order to prove almost global existence we have thus to eliminate first those contributions to $\mathrm{Im}\, H$ and R that have a low degree of homogeneity in U, by a normal form method similar to the one described above. As in the model case (0.2.1), not all such terms may be eliminated, and one has to check that those which remain do not make grow the energy. This is a consequence of reversibility of the system, and of the fact that our initial data are even real valued function. We explain this in more detail in section 0.5 of this introduction.

0.3 Paradifferential formulation and good unknown

In order to start the proof, it will be useful to rewrite the water waves equations (0.1.1) as a paradifferential system. The classes of paradifferential operators we shall need will be introduced and studied in detail in Chapters 2 and 5. In this introduction, let us just consider symbols given by functions $(x,\xi)\to a(x,\xi)$, with limited smoothness in x, satisfying for some real m estimates

$$|\partial_\xi^\beta a(x,\xi)|\leq C_\beta\langle\xi\rangle^{m-\beta},\quad\forall\beta\in\mathbb{N}\,, \tag{0.3.1}$$

where $\langle\xi\rangle=\sqrt{1+\xi^2}$ (the further information we shall need is to track the dependence of these symbols with respect to the unknown dynamical variables). One defines for u a test function or a tempered distribution, the action of the Bony-Weyl quantization of a on u as

$$Op^{\rm BW}(a)u=\frac{1}{2\pi}\int e^{i(x-y)\xi}a_\chi\Big(\frac{x+y}{2},\xi\Big)u(y)\,dyd\xi,$$

where a_χ is the cut-off symbol whose x-Fourier transform $\hat a_\chi(\eta,\xi)$ is given by $\hat a_\chi(\eta,\xi)=\chi(\eta/\langle\xi\rangle)\hat a(\eta,\xi)$, χ being a $C_0^\infty(\mathbb{R})$ function equal to one close to zero, with small enough support. If moreover the symbol a is 2π periodic in x, then $Op^{\rm BW}(a)$ acts from $\dot H^s(\mathbb{T}^1)$ to $\dot H^{s-m}(\mathbb{T}^1)$ for any s, and if (0.3.1) is satisfied also by all x-derivatives of a, then the difference $Op^{\rm BW}(a)-a(x,D)$ sends H^s to $H^{s'}$ for any s,s'. Using the paralinearization formula of Bony, that asserts that

$$R(u,v)=uv-Op^{\rm BW}(u)v-Op^{\rm BW}(v)u$$

has a degree of smoothness equal to the sum of the degrees of smoothness of u and v (minus some universal constant), one may write, following essentially Alazard and Métivier [7] the capillarity-gravity wave equations as a paradifferential system

$$\Big(D_t-Op^{\rm BW}(A(\eta,\psi;t,x,\xi))\Big)\left[\begin{smallmatrix}\eta\\ \psi\end{smallmatrix}\right]=R(\eta,\psi)\left[\begin{smallmatrix}\eta\\ \psi\end{smallmatrix}\right]$$

where $D_t=\frac{1}{i}\partial_t$, A is a matrix of symbols (that depend on t,x through the functions η,ψ) and R is a smoothing operator. One cannot obtain immediately energy estimates for the above paradifferential system because the eigenvalues of the matrix A have unbounded imaginary part when the frequency ξ goes to infinity. This apparent loss of derivatives is not the indication of a genuine instability of the system, but comes from the fact that one did not write the problem using the right unknowns. This difficulty has been overcome in several different ways: Sijue Wu [61] uses a Lagrangian formulation of the water waves system; Alazard-Métivier [7], Alazard-Burq-Zuily [2] and Lannes [52] use the "good unknown" of Alinhac [9], together

with paradifferential calculus for the first two groups of authors and Nash-Moser methods for the last one; Hunter, Ifrim and Tataru [41], as well as the last two authors in their subsequent works [42, 44], rely on a blending of the preceding ideas, reformulating the problem in convenient complex coordinates.

We adopt here the point of view of the good unknown of Alinhac, rewriting in Chapter 6 the water waves equations (0.1.1) as a paradifferential system in (η, ω) where ω, the good unknown, is defined by

$$\omega = \psi - Op^{\mathrm{BW}}(B)\eta\,, \tag{0.3.2}$$

the function B being given by

$$B = B(\eta, \psi) = \frac{G(\eta)\psi + \eta' \partial_x \psi}{1 + \eta'^2} = \partial_z \tilde{\Phi}|_{z=0}, \tag{0.3.3}$$

where z is a new vertical coordinate in which the free surface of the fluid is given by $z = 0$, and $\tilde{\Phi}(x, z)$ is the harmonic velocity potential $\Phi(x, y)$ expressed in the new coordinate system (x, z). An important feature of the new system satisfied by the new unknown is that its right hand side still satisfies the reversibility condition (0.1.2) as (0.1.1), while the Hamiltonian structure is lost. The fact that the reversibility property is preserved through the different reductions made in the proof of Theorem 1 will play an essential role in section 0.5 (and section 4.3), when dealing with normal forms. We shall discuss this issue more in detail at the end of that section.

The paradifferential water waves system satisfied by the new unknown (η, ω) may be rewritten conveniently in a new complex variable (see section 6.3), as presented in detail in section 3.2. Introduce the Fourier multiplier operator of order $-\frac{1}{4}$ given by

$$\Lambda_\kappa(D) = \Big(\frac{D \tanh D}{1 + \kappa D^2}\Big)^{1/4}.$$

If η is in $H_0^{s+\frac{1}{4}}(\mathbb{T}^1, \mathbb{R})$ and ψ is in $\dot{H}^{s-\frac{1}{4}}(\mathbb{T}^1, \mathbb{R})$, then $\omega = \psi - Op^{\mathrm{BW}}(B)\eta$ belongs to the same space $\dot{H}^{s-\frac{1}{4}}(\mathbb{T}^1, \mathbb{R})$ as well, and we introduce

$$u = \Lambda_\kappa(D)\omega + i\Lambda_\kappa(D)^{-1}\eta. \tag{0.3.4}$$

We obtain an element of $\{u \in H^s(\mathbb{T}^1; \mathbb{C}); \int_{\mathbb{T}^1} \mathrm{Im}\, u\, dx = 0\}/\mathbb{R}$ that we may identify to the space $\dot{H}^s(\mathbb{T}^1, \mathbb{C})$ of complex valued H^s functions modulo complex constants. Actually, we are interested only on the subspace of even functions, that is endowed with the norm $\sum_1^{+\infty} \|\Pi_n u\|_{L^2}^2$, where Π_n is the spectral projector associated to the n-th mode, acting on even periodic functions, i.e.

$$\Pi_n u = \langle u, \varphi_n \rangle \varphi_n\,, \quad \langle u, \varphi_n \rangle = \frac{1}{\sqrt{\pi}} \int_{\mathbb{T}^1} u(x) \cos(nx)\, dx\,, \quad \varphi_n = \frac{\cos(nx)}{\sqrt{\pi}}\,.$$

The reduction of the water waves equations (0.1.1) to a complex system in the good unknown will lead in section 6.3 to the following proposition (see Proposition 6.3.1).

Proposition 2 (Water waves equations in complex coordinates) *Let $s \gg K \gg \rho \gg 1$ be integers. Let (η, ψ) be a continuous function of time t, defined on some interval $I \subset \mathbb{R}$, symmetric with respect to $t = 0$, with values in the space $H_0^{s+\frac{1}{4}}(\mathbb{T}^1, \mathbb{R}) \times \dot{H}^{s-\frac{1}{4}}(\mathbb{T}^1, \mathbb{R})$, even in x, that solves system (0.1.1). Define ω from (η, ψ) by (0.3.2). Assume moreover that for $k \leq K$, the ∂_t^k derivative of (η, ω) belongs to $H_0^{s+\frac{1}{4}-\frac{3}{2}k}(\mathbb{T}^1, \mathbb{R}) \times \dot{H}^{s-\frac{1}{4}-\frac{3}{2}k}(\mathbb{T}^1, \mathbb{R})$, and is small enough in that space. Then the function $U = \left[\begin{smallmatrix} u \\ \bar{u} \end{smallmatrix}\right]$, with u given by (0.3.4), solves a paradifferential system of the form*

$$D_t U = Op^{\mathrm{BW}}(A(U; t, x, \xi))U + R(U; t)U \tag{0.3.5}$$

where the 2×2 matrix of symbols $A(U; t, x, \xi)$ has the form

$$\begin{aligned} A(U; t, x, \xi) = &\Big(m_\kappa(\xi)(1 + \zeta(U; t, x)) + \lambda_{1/2}(U; t, x, \xi)\Big)\left[\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right] \\ &+ \Big(m_\kappa(\xi)\zeta(U; t, x) + \lambda_{-1/2}(U; t, x, \xi)\Big)\left[\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right] \\ &+ \lambda_1(U; t, x, \xi)\left[\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right] + \lambda_0(U; t, x, \xi)\left[\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right] \end{aligned}$$

and

$$m_\kappa(\xi) = (\xi \tanh \xi)^{1/2}(1 + \kappa\xi^2)^{1/2} \tag{0.3.6}$$

is a constant coefficients symbol of order $3/2$,

$$\zeta(U; t, x) = [(1 + \eta'^2)^{-3/2} - 1]/2$$

is a real valued function of (t, x), $\lambda_j(U; t, x, \xi)$ is a symbol of order j, for $j = 1, 1/2, 0, -1/2$, with $\operatorname{Im} \lambda_j$ of order $j - 1$ when $j = 1$ or $1/2$, and $R(U; t)$ is a 2×2 matrix of smoothing operators that gain ρ derivatives.

Remarks: • The symbol m_κ in (0.3.6) describes the dispersion relation of the linearized system (0.1.1) at $\eta = 0$, $\psi = 0$. Notice that $\xi \mapsto m_\kappa(\xi)$ is even but, in the subspace of functions even in x, the linear frequencies of oscillations $m_\kappa(n)$, $n \in \mathbb{N}$, are simple.
• One may check that the eigenvalues of the matrix $A(U; t, x, \xi)$ are symbols whose imaginary part is of order zero (actually of order $-1/2$). If A where a diagonal matrix, this would imply that $Op^{\mathrm{BW}}(A(U; t, x, \xi))$ is self-adjoint, modulo a bounded operator, allowing to derive energy estimates for the solutions of (0.3.5) for small times. Actually in Proposition 3 below we shall diagonalize the principal symbol of the system (0.3.5) obtaining in this way energy estimates. The property that the imaginary parts of the eigenvalues

have order zero is a consequence of the fact that we passed to the good unknown. Had we not done this preliminary reduction, we would have found eigenvalues with imaginary part of order 1/2, provoking the instability we mentioned above.

• The matrix A satisfies three algebraic properties that will turn out to be essential for the proof of the theorem, namely:

- The *Reality* condition $\overline{A(U;t,x,-\xi)} = -SA(U;t,x,\xi)S$, where S is the matrix $S = -\left[\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right]$, which is the translation, in the present complex formulation, of the involution map S introduced in section 0.1 in the definition (0.1.2) of reversibility. This property of A is equivalent, at the operator level, to

$$\overline{Op^{\mathrm{BW}}(A(U;t,x,\xi))V} = -SOp^{\mathrm{BW}}(A(U;t,x,\xi))S\overline{V}$$

  and reflects the fact that in system (0.3.5), the second equation is obtained from the first one by complex conjugation (i.e. that (0.1.1) is a real system).

- The *Parity preserving* condition $A(U;t,-x,-\xi) = A(U;t,x,\xi)$, that implies that the operator $Op^{\mathrm{BW}}(A)$ preserves the space of even functions of x.

- The *Reversibility* condition $A(U;-t,x,\xi) = -SA(U_S;t,x,\xi)S$, where $U_S(t) = SU(-t)$. At the level of operators, this condition reads

$$Op^{\mathrm{BW}}(A(U;-t,\cdot)) = -SOp^{\mathrm{BW}}(A(U_S;t,\cdot))S\,.$$

  We shall see below, in Lemma 3.1.3, that, for the homogeneous components A_p of the symbol A introduced in (0.3.7), the above condition amounts to the autonomous reversibility property

$$Op^{\mathrm{BW}}(A_p(SU;\cdot)) = -SOp^{\mathrm{BW}}(A_p(U;\cdot))S,$$

  so that the non-linearity $F(U) = Op^{\mathrm{BW}}(A_p(U;\cdot))U$ satisfies the reversibility condition $SF(U) = -F(SU)$.

The smoothing operator $R(U;t)$ in the right hand side of (0.3.5) satisfies similar properties.

From a dynamical point of view we can heuristically understand why these algebraic properties play a key role. The action of the involution S on a vector U of complex functions even in x, of the form

$$U = \left[\begin{smallmatrix} u \\ \bar{u} \end{smallmatrix}\right], \quad u(x) = \sum_{n\geq 1} u_n \cos(nx)\,,$$

reads $u_n \mapsto -\bar{u}_n$, for any $n \geq 1$. Introducing action-angle variables by the relation $u_n := i\sqrt{I_n}e^{i\theta_n}$, this involution reads $(\theta_n, I_n) \mapsto (-\theta_n, I_n)$. An autonomous vector field written in action-angle variables

$$\dot{\theta} = g(\theta, I)\,, \quad \dot{I} = f(\theta, I)$$

is reversible if $f(\theta, I)$ is odd in θ and $g(\theta, I)$ is even in θ. Now, since the angles θ are expected to rotate faster than the actions I, at the first order we could expect the evolution of the actions to be approximated by the θ-averaged equation $\dot{I}(t) = 0$, in accordance with the naive "averaging principle". Thus reversibility appears as a natural algebraic property, independent of the Hamiltonian nature of a system, which may prevent a systematic drift of the action variables, i.e. growth of Sobolev norms in the PDE language. The concept of reversibility was introduced in KAM theory by Moser [57], see also Arnold [12] and [21] for further developments, and then it has also been used to prove normal form stability results, see for example the exponential estimates in [38], [39] near an elliptic equilibrium. Concerning PDEs we refer, for KAM results, to Zhang, Gao, Yuan [67] for reversible derivative Schrödinger equations and Berti, Biasco, Procesi [15] for reversible derivative wave equations, and to Faou-Grébert [35] and Fang-Han-Zhang [34] for normal form results for semi-linear reversible PDEs.

In view of the normal form procedure that will conclude the proof of Theorem 1, it is not sufficient to define our symbols of paradifferential operators using just estimates (0.3.1). It will be important to know that these symbols are polynomial in U up to the order $N-1$, plus a non-homogeneous symbol which vanishes as $O(\|U\|^N)$ as $U \to 0$. More precisely, the classes of symbols we shall use are given by finite sums

$$A(U; t, x, \xi) = \sum_{p=0}^{N-1} A_p(U, \dots, U; x, \xi) + A_N(U; t, x, \xi) \tag{0.3.7}$$

where A_0 is a constant coefficients symbol of order m, independent of U, and $A_p(U, \dots, U; x, \xi)$, $p = 1, \dots, N-1$, resp. $A_N(U; t, x, \xi)$, are symbols of order m that depend on U as monomials of degree p, resp. in a non-homogeneous way vanishing at order $O(\|U\|^N)$ as $U \to 0$. More precisely $A_p(U_1, \dots, U_p; x, \xi)$ are symmetric p-linear functions of $(U_1, \dots, U_p)$ satisfying bounds of the form

$$|\partial_x^\alpha \partial_\xi^\beta A_p(\Pi_{n_1} U_1, \dots, \Pi_{n_p} U_p; x, \xi)| \leq C|n|^{\mu+\alpha}\langle\xi\rangle^{m-\beta} \prod_1^p \|\Pi_{n_j} U_j\|_{L^2}$$

where as above Π_{n_j} is the spectral projector associated to the n_j-th mode, and μ is a fixed integer. The meaning of this inequality is that each time we make act one x-derivative on the symbol, we lose one power of n, i.e.

one derivative acting on $U_1, \dots, U_p$. We allow a fixed extra loss of μ derivatives. Assuming that $U_1, \dots, U_p$ are in some Sobolev space $\dot{H}^\sigma$, we see that estimates of the form (0.3.1) are satisfied by about $\partial_x^{\sigma-\mu}$ space derivatives of the symbol $A_p(U, \dots, U; x, \xi)$. Finally the non-homogeneous symbol $A_N(U; t, x, \xi)$ satisfies similar estimates with a constant in the right hand side which vanishes at order N when U goes to zero, see Definition 2.1.2. In the whole monograph, for the homogeneous symbols the dependence on time t will enter only through the function $U = U(t)$, while the non-homogeneous symbols may depend explicitly on time t.

Similar decompositions, in multilinear contributions plus a remainder vanishing at order N when U goes to zero, have to be assumed as well on the smoothing operators. Consequently, one has to establish a symbolic calculus for symbols of paradifferential operators that admit such a decomposition, with remainders of the same type. This is what is done in Chapter 2. Moreover, one has to check that the paradifferential symbols in (0.3.5) do belong to such classes. This obliges us to revisit the paralinearization of the water wave equations made in [7, 2, 6], in order to verify such a property. This is the object of Chapters 5 and 6. We show that the paralinearization of the Dirichlet-Neumann operator gives rise to a symbol having the wanted decomposition in multilinear contributions plus a symbol vanishing at large order when U goes to zero. We need also to get similar information on the smoothing remainders. Because of that, we do not make use of a variational method to study the Dirichlet-Neumann boundary value problem as in [7, 2, 3], but we construct a paradifferential parametrix à la Boutet de Monvel [18, 19, 20], introducing classes of para-Poisson operators whose symbols have a decomposition in multilinear terms. Next, we apply these results to the construction of the good unknown and the paralinearization of the water waves system. In particular, in Proposition 6.1.5 and section 7.2 we provide the paralinearization formula of the Dirichlet-Neumann operator, and in Proposition 6.2.1 the paralinearization of the equation for $\partial_t \omega$.

Removed reference to Wang.

Once the water waves system has been written under the paradifferential form (0.3.5), one may start the reasoning that will ultimately prove the existence of its solutions over a time interval of length of order ϵ^{-N} if the initial data are of size ϵ and smooth enough.

As already mentioned in section 0.2, our approach consists in two main steps. The first is a reduction of (0.3.5) to a nonlinear system with paradifferential operators with constant coefficients symbols (in x), up to smoothing remainders. This paradifferential reduction is presented in section 0.4. The second step is a normal form procedure that decreases the size in U of the non-linear terms, and it is presented in section 0.5. We underline that all the transformations used to reduce to constant coefficients the water waves system (0.3.5) up to smoothing remainders are defined by paradifferential operators, and they are bounded maps acting on Sobolev spaces H^s. Also all the transformations in the normal form procedure are bounded maps

on Sobolev spaces H^s, for s large enough. Actually the small divisors will be compensated either by the paradifferential regularization of the symbols defining such transformations, either by the smoothing character of the remainders, as we shall explain in detail in section 0.5.

Notice that it is essential for us to reduce the water waves system (0.3.5) to constant coefficients *before* starting a normal forms method which reduces the size in U of the nonlinear terms, as otherwise the quasi-linear character of the equations would generate losses of derivatives at each transformation. Such unbounded changes of variables would just provide a *formal* normal form, similarly to the works of Craig-Worfolk [27] and Dyachenko and Zakharov [33], for the pure gravity equations. We mention that, in presence of capillarity, Craig-Sulem [26] have recently proved the boundedness of the third order Birkhoff normal form transformation. Such a construction of a bounded normal form at order three is also related to the result in Hunter, Ifrim and Tataru which proves an ϵ^{-2} lifespan for the solution of the pure gravity water waves equations with non localized data of size ϵ in [41] (see also the recent result of the last two authors in the case of a constant non zero vorticity [43], and [42] for capillarity water waves equations).

In the present work, we overcome the issue of the boundedness of the normal form using that, after reducing the water waves system to constant coefficients in x, up to sufficiently regularizing operators, one falls into a normal form framework similar to the one applicable for semi-linear PDEs.

0.4 Reduction to constant coefficients

In order to prove Theorem 1 we are going to constuct for any integer N, a modified energy $E_s(U(t,\cdot)) \sim \|U(t,\cdot)\|^2_{\dot{H}^s}$, equivalent to the square of the H^s-Sobolev norm, that satisfies, along any small amplitude solutions of (0.3.5), the bound

$$\frac{d}{dt}E_s(U(t,\cdot)) = O(\|U(t,\cdot)\|^{N+2}_{\dot{H}^s}), \quad \text{as} \quad U \to 0\,. \tag{0.4.1}$$

This implies the energy inequality

$$E_s(U(t,\cdot)) \leq E_s(U(0,\cdot)) + C\left|\int_0^t \|U(\tau,\cdot)\|^{N+2}_{\dot{H}^s}\,d\tau\right|$$

so that, if $E_s(U(0,\cdot)) \sim \epsilon$, one may prove by a bootstrap argument that, if ϵ is small enough, and t satisfies $|t| \leq c\epsilon^{-N}$ for some small enough c, then $E_s(U(t,\cdot)) = O(\epsilon^2)$, thus $\|U(t,\cdot)\|_{\dot{H}^s} = O(\epsilon)$. This a priori estimate, combined with the fact that local existence with smooth Cauchy data holds true according to [60], implies that the solution may be extended up to times of magnitude $c\epsilon^{-N}$ (actually local existence could be deduced by system (0.4.7) below, but, to avoid further technicalities, we directly rely on the results in [60]).

The construction of a modified energy E_s which satisfies (0.4.1) will rely, as already said, on two main conceptually different procedures. First we shall perform a series of non-linear para-differential changes of variables, similar to the transformations used in Alazard-Baldi [1] and Berti-Montalto [16] to reduce the linearized equations (which arise in a Nash-Moser iteration to prove the existence of periodic and quasi-periodic solutions) to constant coefficients. Then we shall develop a normal form method parallel to those used by Bambusi, Delort, Grébert and Szeftel [13], [14], [30], and Faou-Grébert [35] for semi-linear PDEs. The modified energy E_s is explicitly constructed in (4.4.29) (with $q = N - 1$).

We describe below the reduction steps, and will explain the normal form method in the next section.

Step 1: Diagonalization of the system

We prove in section 3.3, using symbolic calculus, that one may replace the matrix of symbols A in the right hand side of (0.3.5) by a *diagonal* matrix, up to a modification of the smoothing operator $R(U;t)$. More precisely, we get (Proposition 3.3.2):

Proposition 3 (Diagonalization of the matrix symbol A) *There exist 2×2 matrices of symbols of order 0, $P(U;t,\cdot)$, $Q(U;t,\cdot)$, such that $Op^{\mathrm{BW}}(P) \circ Op^{\mathrm{BW}}(Q) - \mathrm{Id}$ is a ρ-smoothing operator (for a large given ρ) so that, if $W = Op^{\mathrm{BW}}(Q)U$, then W solves the system*

$$\big(D_t - Op^{\mathrm{BW}}(A^{(1)}(U;t,\cdot))\big)W = R'(U;t)W + R''(U;t)U \tag{0.4.2}$$

where $R'(U;t), R''(U;t)$ are ρ-smoothing operators and $A^{(1)}$ is a diagonal *matrix of symbols*

$$A^{(1)}(U;t,x,\xi) = \Big(m_\kappa(\xi)(1+\zeta^{(1)}(U;t,x)) + \lambda^{(1)}_{1/2}(U;t,x,\xi)\Big)\left[\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right] + \lambda^{(1)}_1(U;t,x,\xi)\left[\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right] \tag{0.4.3}$$

where $\zeta^{(1)}$ is a real valued function of (t,x), $\lambda^{(1)}_j$ is a symbol of order j, whose imaginary part is of order $j-1$. Moreover, $A^{(1)}$ has a homogeneous development of the form (0.3.7), and it satisfies the reality, parity preserving and reversibility conditions.

The proof of the above proposition is made through conjugation by paradifferential operators, in order to decrease successively the order of the non diagonal terms in the matrix of symbols in (0.3.5). We may easily explain the idea of that diagonalization at principal order on the form taken by equation (0.1.1) on the good unknown. In Chapter 6, we shall introduce some real valued function V, depending on η, ψ such that, if we set

$c(t,x) = (1+\eta'^2)^{-3/2}$, the couple (η,ω) satisfies an equation of the form $\mathcal{P}\left[\begin{smallmatrix}\eta\\ \omega\end{smallmatrix}\right] = R$, where

$$\mathcal{P} = \partial_t + \begin{bmatrix} V\partial_x & -D\tanh D \\ 1+\kappa c(t,x)D^2 & V\partial_x \end{bmatrix} \tag{0.4.4}$$

and R is given by the action of operators of non positive order on $\left[\begin{smallmatrix}\eta\\ \omega\end{smallmatrix}\right]$, and is at least quadratic in (η,ω). Actually, the first equation in (0.4.4) comes from the expression (6.1.55) of $G(\eta)\psi$ and the second one follows from (6.2.8). If we conjugate the operator $\mathcal{P}$ by $\left[\begin{smallmatrix}\Lambda_\kappa(D) & 0\\ 0 & -\Lambda_\kappa(D)\end{smallmatrix}\right]$, we obtain the operator

$$\partial_t + \begin{bmatrix} V\partial_x & -m_\kappa(D) \\ c(t,x)m_\kappa(D) & V\partial_x \end{bmatrix}$$

up to remainders whose action on the unknown gives terms of the same form as R. Conjugating again this system by the matrix valued multiplication operator $\left[\begin{smallmatrix}1 & 0\\ 0 & c(t,x)^{1/2}\end{smallmatrix}\right]$, we obtain

$$\partial_t + \begin{bmatrix} V\partial_x & -c(t,x)^{1/2}m_\kappa(D) \\ c(t,x)^{1/2}m_\kappa(D) & V\partial_x \end{bmatrix}$$

acting on some new unknown $\left[\begin{smallmatrix}\tilde\eta\\ \tilde\omega\end{smallmatrix}\right]$. If we set $u = \tilde\omega + i\tilde\eta$, we get the complex equation

$$\partial_t u = -ic(t,x)^{1/2}m_\kappa(D)u + V\partial_x u$$

modulo again remainders as above. This is, at principal order, the diagonalized equation we are seeking for. Actually in order to obtain, after having performed the above conjugation, another paradifferential equation we have to use above the paraproduct instead of the multiplication operator.

Step 2: Reduction to constant coefficients at principal order

The goal of the next steps is to reduce to constant coefficients the matrix symbol $A^{(1)}$ in (0.4.3). We shall exploit in an essential way that the dispersion relation $m_\kappa(\xi)$ given by (0.3.6) is *superlinear*. We start making this reduction for the principal part of the matrix symbol (0.4.3) given according to (0.3.6) by the product of $\left[\begin{smallmatrix}1 & 0\\ 0 & -1\end{smallmatrix}\right]$ and of $(1+\zeta^{(1)}(U;t,x))\sqrt{\kappa}|\xi|^{3/2}$. We would like to eliminate the x-dependence in that symbol. If we make a time dependent change of variables $y = \Phi_U(t,x)$, i.e. $x = \Phi_U^{-1}(t,y)$, the above principal symbol becomes

$$(1+\zeta^{(1)}(U;t,\Phi_U(t,x)))\sqrt{\kappa}\Big|\big(\partial_y\Phi_U^{-1}(t,y)\big)_{|y=\Phi_U(t,x)}\Big|^{3/2}|\xi|^{3/2}.$$

To get a constant coefficients symbol, we cannot just choose the diffeomorphism Φ_U so that $(1+\zeta^{(1)}(U;t,y))|\partial_y\Phi_U^{-1}(t,y)|^{3/2} = 1$, namely to take $\partial_y\Phi_U^{-1}(t,y) = (1+\zeta^{(1)}(U;t,y))^{-2/3}$, since Φ_U^{-1}, and thus Φ_U, must be a

diffeomorphisms of $\mathbb{T}^1$, i.e. we need $\Phi_U^{-1}(t,y)-y$ to be periodic. Instead, we choose $\Phi_U^{-1}(t,y) = y+\gamma(t,y)$ where $\gamma(t,y)$ is the unique periodic function of y, with zero mean, solving the equation

$$\partial_y\gamma(y) = (1+\underline{\zeta}(U;t))^{2/3}(1+\zeta^{(1)}(U;t,y))^{-2/3}-1$$

where $\underline{\zeta}(U;t)$ is defined in order to make zero the space average on $\mathbb{T}^1$ of the right handside. The difficulty that one encounters is due to the fact that this change of variables Φ_U depends on U, so has only limited smoothness. Because of that, instead of defining a new unknown function V by the composition $V = W\circ\Phi_U$, we use a paracomposition operator in the sense of Alinhac [8]. We set $V = \Phi_U^\star W$, where the paracomposition operator $\Phi_U^\star$ is defined and studied in section 2.5 (we provide an alternative definition using flows). In that way, when we compute the equation satisfied by the unknown V, we still get a paradifferential equation. More precisely, we prove (Proposition 4.1.1):

Proposition 4 (Reduction of the highest order) *If we define Φ_U as above and set $V = \Phi_U^\star W$, then V solves the system*

(0.4.5)
$$\Big(D_t - Op^{\rm BW}\Big(\big[(1+\underline{\zeta}(U;t))m_\kappa(\xi)+\tilde\lambda(U;t,x,\xi)\big]\left[\begin{smallmatrix}1&0\\0&-1\end{smallmatrix}\right]+\mu(U;t,x,\xi)\mathcal{I}_2\Big)\Big)V = R'(U;t)V+R''(U;t)U$$

where $\tilde\lambda$ is of order $1/2$ with $\operatorname{Im}\tilde\lambda$ of order $-1/2$, μ is of order one, with $\operatorname{Im}\mu$ of order zero, R', R'' are smoothing operators. Moreover, the reality, parity preserving and reversibility conditions are still satisfied by the matrix symbol in the left hand side of (0.4.5) and the smoothing operators in its right hand side.

The next step of the proof will be to eliminate the non constant coefficients parts of $\tilde\lambda$ and μ up to remainders of very negative order.

Step 3: Reduction to constant coefficients at arbitrary order

Denote by $F(U)$ a diagonal 2×2 matrix of symbols of order $1/2$ which is self-adjoint, up to contributions of order zero. We define in section 4.2, for any θ between -1 and 1 the operator $\Omega_{F(U)}(\theta) = \exp[i\theta Op^{\rm BW}(F(U))]$, which is the flow generated by the linear system (4.2.3). Symbolic calculus shows that the diagonal matrix of symbols of order $3/2$ given by

$$D_0(U;\cdot) = (1+\underline{\zeta}(U;t))m_\kappa(\xi)\left[\begin{smallmatrix}1&0\\0&-1\end{smallmatrix}\right]$$

transforms as

$$\Omega_{F(U)}(-1)Op^{\rm BW}\big(D_0(U;\cdot)\big)\Omega_{F(U)}(1) = Op^{\rm BW}\big(D_0(U;\cdot)\big) + (1+\underline{\zeta}(U;t))Op^{\rm BW}\big(\{F(U,\cdot),m_\kappa(\xi)\}\big)\left[\begin{smallmatrix}1&0\\0&-1\end{smallmatrix}\right] + \text{ lower order terms.}$$

If one conjugates system (0.4.5) by $\Omega_{F(U)}(1)$, one obtains a similar system with a new symbol given by

$$\big[(1+\underline{\zeta}(U;t))m_\kappa(\xi)+\tilde\lambda(U;t,x,\xi)+(1+\underline{\zeta}(U;t))\{F,m_\kappa\}\big]\left[\begin{smallmatrix}1&0\\0&-1\end{smallmatrix}\right] + \mu(U;t,x,\xi)\mathcal{I}_2 + \text{ lower order terms.}$$

We decompose $\tilde\lambda=\tilde\lambda^{\mathrm{D}}+\tilde\lambda^{\mathrm{ND}}$, where $\tilde\lambda^{\mathrm{D}}$ is the x-average of $\tilde\lambda$ and $\tilde\lambda^{\mathrm{ND}}$ has zero average. We write in the same way $\tilde\mu=\tilde\mu^{\mathrm{D}}+\tilde\mu^{\mathrm{ND}}$. Solving the equation

$$\Big((1+\underline{\zeta}(U;t))\{F,m_\kappa\}+\tilde\lambda^{\mathrm{ND}}\Big)\left[\begin{smallmatrix}1&0\\0&-1\end{smallmatrix}\right]+\tilde\mu^{\mathrm{ND}}\mathcal{I}_2=0 \tag{0.4.6}$$

at principal order, we reduce ourselves to the case when the contributions of order 1 and 1/2 in (0.4.5) have constant coefficients. Repeating the reasoning up to some very negative order, we get (Proposition 4.2.4):

Proposition 5 (Reduction to constant coefficients of (0.4.5)**)** *There is a diagonal matrix $F(U)$ of symbols of order 1/2 such that if we set $\tilde V=\Omega_{F(U)}(-1)V$, then $\tilde V$ solves the system*

$$\Big(D_t-Op^{\mathrm{BW}}\Big((1+\underline{\zeta}(U;t))m_\kappa(\xi)\left[\begin{smallmatrix}1&0\\0&-1\end{smallmatrix}\right]-H(U;t,\xi)\Big)\Big)\tilde V = R_1(U;t)\tilde V+R_2(U;t)U \tag{0.4.7}$$

where R_1,R_2 are ρ-smoothing operators and H is a diagonal matrix of constant coefficients *symbols (in x) of order one, such that* $\operatorname{Im}H$ *is of order zero. Moreover, the reality, parity preserving and reversibility conditions are satisfied.*

Actually the constant coefficient symbols of H are of order 1/2, see the remark after Proposition 4.2.4. This information is not necessary for the subsequent normal form arguments but it is in agreement with the asymptotic expansion of the Floquet exponents of the periodic and quasi-periodic solutions found in [1], [16].

In equation (0.4.6) it is essential that the symbol m_κ defined in (0.3.6) is of order strictly larger than 1, i.e. that the capillarity-gravity linear dispersion relation is superlinear. This is what allows to construct a symbol $F(U)$ that is of order strictly smaller than the one of $\tilde\mu$ in the right hand side, and to start the induction that eliminates all variable coefficient symbols up to some order as negative as we want. This is also what implies that the contributions coming from the conjugation of D_t by $\Omega_{F(U)}(1)$ enter into the remainders, as they are of order strictly smaller than the main part. If the linear part of the operator were just of order one, like for instance for Klein-Gordon equations, it would be no longer possible to reduce in this way the system to constant coefficients before performing the normal forms introduced in next section. We refer to [28, 29] and references therein for long

time existence results for quasi-linear Hamiltonian Klein-Gordon equations.

In (0.4.7), the operators $Op^{\mathrm{BW}}(1+\underline{\zeta}(U;t))m_\kappa(\xi)\left[\begin{smallmatrix}1&0\\0&-1\end{smallmatrix}\right]$ and $Op^{\mathrm{BW}}(\mathrm{Re}\, H)$ are self adjoint. Consequently, the L^2 energy inequality associated to that system reads, for example in the case $t \geq 0$,

$$\|\tilde{V}(t,\cdot)\|_{L^2}^2 \leq \|\tilde{V}(0,\cdot)\|_{L^2}^2 + \int_0^t \|Op^{\mathrm{BW}}(\mathrm{Im}\, H(\tau,\xi))\tilde{V}(\tau,\cdot)\|_{L^2}\|\tilde{V}(\tau,\cdot)\|_{L^2}\, d\tau$$
$$+ \int_0^t \|R_1(U;t)\tilde{V}(\tau,\cdot) + R_2(U;t)U(\tau,\cdot)\|_{L^2}\|\tilde{V}(\tau,\cdot)\|_{L^2}\, d\tau\,.$$

Moreover, since the coefficients in left hand side of (0.4.7) are constant in x, we may commute as many space derivatives as we wants with the equation, and deduce from the above L^2-estimate a similar inequality for Sobolev norms. As a consequence, if the symbol $\mathrm{Im}\, H$, which is of order zero, were vanishing as $O(\|U\|^N)$ when U goes to zero, as well as the smoothing terms R_1, R_2, we would deduce from that the estimate

$$\|\tilde{V}(t,\cdot)\|_{\dot{H}^s}^2 \leq \|\tilde{V}(0,\cdot)\|_{\dot{H}^s}^2 + C\int_0^t \|U(\tau,\cdot)\|_{\dot{H}^\sigma}^N(\|\tilde{V}(\tau,\cdot)\|_{\dot{H}^s}^2 + \|U(\tau,\cdot)\|_{\dot{H}^s}^2)\, d\tau.$$

Using that at $t = 0$, U is of size ϵ in $\dot{H}^s$, and that, as long as U remains small, U and $\tilde{V}$ are of the same magnitude, one would deduce from that, by bootstrap, an a priori estimate $\|U(t,\cdot)\|_{\dot{H}^s} = O(\epsilon)$ over a time interval of length $c\epsilon^{-N}$ for some small c. Together with local existence theory, this would imply that the solution may be extended up to such an interval of time. In conclusion, to prove our main theorem, we have to show that we may modify equation (0.4.7) in order to ensure that $\mathrm{Im}\, H(U;\cdot)$ and $R_1(U;t)$, $R_2(U;t)$ will vanish as $O(U^N)$ as U goes to zero. We shall achieve this goal through a normal forms procedure.

0.5 Normal forms

Let $B(U;t,\xi)$ be a diagonal matrix of constant coefficients symbols of order zero to be determined. We conjugate system (0.4.7) by the operator $\exp\big(Op^{\mathrm{BW}}(B(U;t,\xi))\big)$. Setting $\tilde{V}^1 = \exp\big(Op^{\mathrm{BW}}(B(U;t,\xi))\big)\tilde{V}$, we get

(0.5.1)
$$\Big(D_t - Op^{\mathrm{BW}}\big[D_tB(U;t,\xi) + (1+\underline{\zeta}(U;t))m_\kappa(\xi)\left[\begin{smallmatrix}1&0\\0&-1\end{smallmatrix}\right] - H(U;t,\xi)\big]\Big)\tilde{V}^1$$
$$= \text{ smoothing terms.}$$

We want to choose B in order to eliminate the contributions to $\mathrm{Im}\, H$ which are homogeneous of degree $p < N$. Thus we decompose

$$H(U;t,\xi) = \sum_{p=1}^{N-1} H_p(U,\dots,U;\xi) + H_N(U;t,\xi)$$

with $H_p(U_1,\dots,U_p;\xi)$ a symmetric p-linear map in $(U_1,\dots,U_p)$ with values in diagonal matrices of constant coefficient symbols, and H_N a symbol vanishing at least at order N when U goes to zero. We look for B as

$$B=\sum_{p=1}^{N-1} B_p(U,\dots,U;\xi)$$

with B_p a p-linear map. Rewriting equation (0.3.5) as

$$D_tU = m_\kappa(D)\mathcal{K} + \text{ terms vanishing at least at order 1 in } U,$$

(such terms are unbounded operators of order 3/2) where $\mathcal{K}=\left[\begin{smallmatrix}1&0\\0&-1\end{smallmatrix}\right]$, we may write

$$\begin{aligned} D_t(B_p(U,\dots,U)) &= \sum_{j=1}^{p} B_p(U,\dots,U,D_tU,U,\dots,U)\\ &= \sum_{j=1}^{p} B_p(U,\dots,U,m_\kappa(D)\mathcal{K}U,U,\dots,U) + \text{ terms of higher order in } U. \end{aligned}$$

To eliminate the lower order terms contributions in $\operatorname{Im} H$, we are thus reduced to find B_p so that

$$\sum_{j=1}^{p} B_p(U,\dots,U,m_\kappa(D)\mathcal{K}U,U,\dots,U) = i\operatorname{Im}\tilde{H}_p(U,\dots,U) \tag{0.5.2}$$

where $\tilde{H}_p$ is computed recursively from H_p and from $B_{p'}$, $p'<p$. Replacing $U=\left[\begin{smallmatrix}u\\ \bar{u}\end{smallmatrix}\right]$ by $\Pi^+_{n_j}U=\left[\begin{smallmatrix}\Pi_{n_j}u\\0\end{smallmatrix}\right]$ in the first ℓ components and by $\Pi^-_{n_j}U=\left[\begin{smallmatrix}0\\ \Pi_{n_j}\bar{u}\end{smallmatrix}\right]$ in the last $p-\ell$ ones, and using that

$$m_\kappa(D)\mathcal{K}\Pi^\pm_{n_j} = \pm m_\kappa(n_j)\Pi^\pm_{n_j}\,,$$

we see that it is enough to determine B_p so that

$$\begin{aligned} &\Big(\sum_{j=1}^{\ell} m_\kappa(n_j) - \sum_{j=\ell+1}^{p} m_\kappa(n_j)\Big)\\ &\qquad\times B_p(\Pi^+_{n_1}U_1,\dots,\Pi^+_{n_\ell}U_\ell,\Pi^-_{n_{\ell+1}}U_{\ell+1},\dots,\Pi^-_{n_p}U_p;\xi)\\ &\qquad\qquad = i\operatorname{Im}\tilde{H}_p(\Pi^+_{n_1}U_1,\dots,\Pi^+_{n_\ell}U_\ell,\Pi^-_{n_{\ell+1}}U_{\ell+1},\dots,\Pi^-_{n_p}U_p;\xi). \end{aligned} \tag{0.5.3}$$

Thus we just have to be able to divide the right hand side of (0.5.3) by the "small divisor"

$$\mathcal{D}_\ell(n_1,\dots,n_p) = \sum_{j=1}^{\ell} m_\kappa(n_j) - \sum_{j=\ell+1}^{p} m_\kappa(n_j)$$

when the parameter κ is taken outside a convenient subset of zero measure. Clearly, this will not be possible if p is even, $\ell = p/2$ and one has a two by two cancellation between terms of the first and second sum so that $\mathcal{D}_\ell(n_1, \dots, n_p) = 0$. The possibility to solve nevertheless equation (0.5.3) follows from the following proposition:

Proposition 6 *(i) There is a zero measure subset $\mathcal{N}$ of $]0, +\infty[$ such that, if κ is taken outside $\mathcal{N}$, there are $N_0 \in \mathbb{N}$, and $c > 0$ such that for any $n_1, \dots, n_p$ in $\mathbb{N}^*$,*

$$|\mathcal{D}_\ell(n_1, \dots, n_p)| \geq c(n_1 + \cdots + n_p)^{-N_0} \tag{0.5.4}$$

except if p is even, $\ell = p/2$ and $\{n_1, \dots, n_\ell\} = \{n_{\ell+1}, \dots, n_p\}$.
(ii) In the latter case

$$\operatorname{Im} \tilde{H}_p(\Pi^+_{n_1} U_1, \dots, \Pi^+_{n_\ell} U_\ell, \Pi^-_{n_{\ell+1}} U_{\ell+1}, \dots, \Pi^-_{n_p} U_p; \xi) \equiv 0. \tag{0.5.5}$$

Proposition 6 allows consequently to solve equation (0.5.3) in any case, and thus to eliminate the contributions to $\operatorname{Im} H$ of lower degree of homogeneity. By the estimate (0.5.4) the small divisor that appears dividing the right hand side of (0.5.3) by $\mathcal{D}_\ell$ produces a loss of $O(N_0)$ derivatives on the smoothness of B_p as a function of x, but the associated paradifferential operator $Op^{\mathrm{BW}}(B_p(U; t, \xi))$ remains bounded on any H^s if U is in H^σ with a large σ (depending on N_0). Indeed the H^s-boundedness of the paradifferential operator $Op^{\mathrm{BW}}(B_p(U; t, \xi))$ depends only on a finite number (*independent of s*) of derivatives of U.

One may proceed in a similar way to eliminate the smoothing contributions, which are homogeneous of low order in U, in the right hand side of (0.5.1). More precisely, in section 4.4 we shall construct iteratively, by an analogous normal form method, quasi-invariant modified energy Sobolev norms for the system (0.5.1), see (4.4.29), constructing ultimately a modified energy E_s satisfying (0.4.1). We require at each iterative step a small divisor estimate as

$$\Big| \sum_{j=0}^{\ell} m_\kappa(n_j) - \sum_{j=\ell+1}^{p+1} m_\kappa(n_j) \Big| \geq c \max(n_0, \dots, n_{p+1})^{-N_0} \tag{0.5.6}$$

for any integer $n_0, \dots, n_{p+1}$ except in the case when

$$p \text{ is even}, \quad \ell = \frac{p}{2}, \quad \text{and} \quad \{n_0, \dots, n_\ell\} = \{n_{\ell+1}, \dots, n_{p+1}\},$$

and we verify in Lemma 4.4.2-(ii) a cancellation similar to (0.5.5). Notice that the small divisors estimate (0.5.6) is very weak because the right hand side contains the maximum of all integers $n_0, \dots, n_{p+1}$ and not just the third largest $\max_3(n_0, \dots, n_{p+1})$, as in (0.2.8). The latter condition is

essential for the normal form approach developed in the papers [14], [13], [35], concerning semilinear PDEs. It could be imposed as well in the present context. The use of the weak non resonance condition (0.5.6) produces losses of $O(N_0)$ derivatives at each normal form transformation which, however, in the present approach, are compensated by the smoothing character of the remainders, when the Sobolev regularity of H^s is large enough.

In conclusion the preceding Proposition 6, which is proved in sections 4.3 and 7.1, is the last remaining step in the proof of the main Theorem 1.

Idea of the proof of Proposition 6: The small divisor estimate (0.5.4) of Part (i) follows from more general results of Delort-Szeftel [30] concerning measure estimates of sublevels of subanalytic functions. Part (ii) is a consequence of the fact that the matrices of symbols $\tilde{H}_p$ obtained by the successive reductions described above still satisfy the reality, parity preserving and reversibility properties. Combined together, these three properties may be shown to imply the vanishing of the left hand side of (0.5.5), see Lemma 4.3.2. □

As explained above, it is essential that the cancellation property (0.5.5) holds in the case when the left hand side of (0.5.3) vanishes identically. This cancellation follows ultimately from the reversible nature of the capillarity-gravity water waves equations (0.1.1), and it is essential for us that this property, that holds for system (0.1.1), be preserved by all the different reductions we make. In particular, a key point is that the definition $\omega = \psi - Op^{\mathrm{BW}}(B)\eta$ of the new unknown shows, according to (0.3.3), that ω is an odd function (actually a linear function) of ψ. Because of that, the water waves system satisfied by (η, ω) will still satisfy the reversibility property (0.1.2). Actually we prove in Corollary 6.2.2 that the new water waves system in the (η, ω) variables satisfies a time dependent reversibility property.

As already mentioned elsewhere, reversibility is not the only algebraic information that might be used to ensure the solvability of an equation of the form (0.5.3), and so proving that the terms that could generate a growth in modified Sobolev norms, actually vanish. An alternative property, that has been used extensively in other works, is the Hamiltonian character of the system. In this case, one controls a modified Sobolev energy of the solution, defined through composition of the usual Sobolev energy by canonical transformations. In that way, one may pursue all the reductions staying in the Hamiltonian framework. Equation (0.5.3) is then replaced by a similar equation at the level of Hamiltonians, often called "homological equation", that may be solved in all cases, except the one corresponding to (ii) of the preceding proposition. But the contributions to the Hamiltonian generated by the indices satisfying $\{n_1, \dots, n_\ell\} = \{n_{\ell+1}, \dots, n_p\}$ when p is even and $\ell = p/2$ depend actually only on action coordinates, and thus cannot generate growth of the energy. We refer to the works of Bambusi-Grébert [14] and Bambusi, Delort, Grébert and Szeftel [13] were such ideas are developed for

semi-linear equations, and to [28, 29] for the application of a similar method to quasi-linear Klein-Gordon type equations.

We have not followed this Hamiltonian approach for the capillarity-gravity water wave equations because the passage from the initial unknown to the good one does not seem to preserve the Hamiltonian character of the system. We exploit instead the reversible structure, whose preservation through the change of the good unknown is trivial.

Let us mention that an index of notation is provided at the end of the volume.

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Chapter 1

Main result

1.1 The periodic capillarity-gravity equations

Let $h > 0$ be a constant, $\mathbb{T}^1 = \mathbb{R}/(2\pi\mathbb{Z})$ the circle, and consider an incompressible and irrotational perfect fluid, occupying at time t a two dimensional domain, periodic in the horizontal variable, given by

$$\Omega_t = \big\{(x,y) \in \mathbb{T}^1 \times \mathbb{R};\ -h < y < \eta(t,x)\big\}, \tag{1.1.1}$$

where $\eta : \mathbb{R} \times \mathbb{T}^1 \to \mathbb{R}$ is a smooth enough function such that $\|\eta\|_{L^\infty} < h$. The velocity field in Ω_t is then the gradient of some harmonic function Φ, called the velocity potential. The evolution of the motion of the fluid is determined by the knowledge of the two functions

$$(t,x) \to \eta(t,x), \quad (t,x;y) \to \Phi(t,x,y)$$

defined for t in some interval, x in $\mathbb{T}^1$, (x,y) in Ω_t, subject to the following boundary conditions (see for instance the book of Lannes [53])

$$\begin{aligned} &\partial_t\eta = \partial_y\Phi - (\partial_x\eta)(\partial_x\Phi) \text{ on } y = \eta(t,x) \\ &\partial_t\Phi = -g\eta + \kappa H(\eta) - \frac{1}{2}\big[(\partial_x\Phi)^2 + (\partial_y\Phi)^2\big] \text{ on } y = \eta(t,x) \\ &\partial_y\Phi = 0 \text{ on } y = -h, \end{aligned} \tag{1.1.2}$$

where $g > 0$ is the acceleration of gravity, $\kappa > 0$ the surface tension, and

$$H(\eta) = \partial_x[(\partial_x\eta)(1 + (\partial_x\eta)^2)^{-1/2}]$$

is the mean curvature. From now on, we shall write η' for $\partial_x\eta$. We define

$$\psi(t,x) = \Phi(t,x,\eta(t,x))$$

the restriction of the velocity potential Φ to the free interface and, following Zakharov [66] and Craig-Sulem [25], we express (1.1.2) as a system in the variables (η,ψ).

Define the Dirichlet-Neumann operator $G(\eta)$ of the upper boundary of the set Ω_t as

$$G(\eta)\psi = \sqrt{1+\eta'^2}(\partial_n\Phi)|_{y=\eta(t,x)} = (\partial_y\Phi - \eta'\partial_x\Phi)(t,x,\eta(t,x)) \tag{1.1.3}$$

where ∂_n is the outward unit normal at the free interface $y = \eta(t,x)$. According to Craig-Sulem [25], (η,ψ) satisfies the capillarity-gravity water waves system

$$\begin{aligned} &\partial_t\eta = G(\eta)\psi \\ &\partial_t\psi = -g\eta + \kappa H(\eta) - \frac{1}{2}(\partial_x\psi)^2 + \frac{1}{2}\frac{(\eta'\partial_x\psi + G(\eta)\psi)^2}{1+\eta'^2}. \end{aligned} \tag{1.1.4}$$

Our goal is to prove that, for fixed h, and almost all values of the parameters $(g,\kappa) \in]0,+\infty[^2$, for any fixed integer N, Cauchy data that are smooth enough, of small enough size $\epsilon > 0$, even in x, generate a solution defined on a time interval of length at least $c\epsilon^{-N}$.

Notice that the right hand side of (1.1.4) remains invariant if we replace $(\eta(x),\psi(x))$ by $(\eta(-x),\psi(-x))$, so that initial data which are even in x give rise to a solution that remains even in x at any future time t.

1.2 Statement of the main theorem

Let us introduce some notation. For n in $\mathbb{N}^*$, we denote by Π_n the orthogonal projector from $L^2(\mathbb{T}^1,\mathbb{C})$ or $L^2(\mathbb{T}^1,\mathbb{C}^2)$ to the subspace spanned by $\{e^{inx}, e^{-inx}\}$. For any s in $\mathbb{R}$, we define

$$\begin{aligned} &H^s_0(\mathbb{T}^1,\mathbb{R}) \stackrel{\text{def}}{=} \Big\{ u \in H^s(\mathbb{T}^1,\mathbb{R})\,;\, \int_{\mathbb{T}^1} u(x)\,dx = 0 \Big\} \\ &\dot{H}^s(\mathbb{T}^1,\mathbb{R}) \stackrel{\text{def}}{=} H^s(\mathbb{T}^1,\mathbb{R})/\mathbb{R}\,. \end{aligned} \tag{1.2.1}$$

Both H^s_0 and the homogeneous Sobolev space $\dot{H}^s$ will be endowed with the same norm

$$\|u\|_{H^s_0} = \|u\|_{\dot{H}^s} = \Big(\sum_{n=1}^{+\infty} \|\Pi_n u\|^2_{L^2}\Big)^{1/2}. \tag{1.2.2}$$

Furthermore, we denote by $H^s_{0,\text{ev}} = H^s_{0,\text{ev}}(\mathbb{T}^1,\mathbb{R})$, $\dot{H}^s_{\text{ev}} = \dot{H}^s_{\text{ev}}(\mathbb{T}^1,\mathbb{R})$, the subspaces of $H^s_0(\mathbb{T}^1,\mathbb{R}), \dot{H}^s(\mathbb{T}^1,\mathbb{R})$ formed by the functions even in x. Our main result is the following:

Theorem 1.2.1 (Almost global existence of periodic capillarity-gravity waves) *There is a zero measure subset $\mathcal{N}$ in $]0,+\infty[^2$ such that, for any (g,κ) in $]0,+\infty[^2 - \mathcal{N}$, for any N in $\mathbb{N}$, there is $s_0 > 0$ and for any*

$s \geq s_0$, there are $\epsilon_0 > 0, c > 0, C > 0$ such that, for any $\epsilon \in]0, \epsilon_0[$, any function $(\eta_0, \psi_0) \in H^{s+\frac{1}{4}}_{0,\mathrm{ev}}(\mathbb{T}^1, \mathbb{R}) \times \dot{H}^{s-\frac{1}{4}}_{\mathrm{ev}}(\mathbb{T}^1, \mathbb{R})$ satisfying

$$\|\eta_0\|_{H^{s+\frac{1}{4}}_0} + \|\psi_0\|_{\dot{H}^{s-\frac{1}{4}}} < \epsilon\,,$$

system (1.1.4) has a unique classical solution (η, ψ) defined on $]-T_\epsilon, T_\epsilon[\times\mathbb{T}^1$ with $T_\epsilon \geq c\epsilon^{-N}$, belonging to the space

$$C^0(]-T_\epsilon, T_\epsilon[, H^{s+\frac{1}{4}}_{0,\mathrm{ev}}(\mathbb{T}^1, \mathbb{R}) \times \dot{H}^{s-\frac{1}{4}}_{\mathrm{ev}}(\mathbb{T}^1, \mathbb{R}))$$

satisfying the initial condition $\eta|_{t=0} = \eta_0, \psi|_{t=0} = \psi_0$. Moreover this solution stays at any time in the ball of center 0 and radius $C\epsilon$ of $H^{s+\frac{1}{4}}_{0,\mathrm{ev}} \times \dot{H}^{s-\frac{1}{4}}_{\mathrm{ev}}$.

Remarks: • The above theorem provides an "almost global" solution of system (1.1.4) with small smooth periodic even initial data, when the parameters (g, κ) stay outside a subset of zero measure. Notice that we assume that $g > 0$ and $\kappa > 0$, which will be essential for our proof.

• We assume that at $t = 0$, η has zero average. It is well known that the quantity $\int_{\mathbb{T}^1} \eta(t, x)\, dx$ is conserved during the evolution, so that $\eta(t, \cdot)$ has zero average at any time.

• The function ψ is taken in a Sobolev space defined modulo constants, i.e. the homogeneous Sobolev spaces in (1.2.1). This is related to the fact that only $\partial_x\psi$ has a physical meaning.

• An essential property is that the water waves system (1.1.4) is reversible in the following sense. Let S be the linear involution i.e. $S^2 = \mathrm{Id}$,

$$S : \mathbb{R}^2 \to \mathbb{R}^2\,, \quad \text{with matrix} \quad \left[\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right]. \tag{1.2.3}$$

Then, denoting by $F = F(\eta, \psi) = \left[\begin{smallmatrix} F_1(\eta,\psi) \\ F_2(\eta,\psi) \end{smallmatrix}\right]$ the right hand side in (1.1.4), we have

$$F \circ S = -S \circ F\,, \tag{1.2.4}$$

namely that $F_1(\eta, -\psi) = -F_1(\eta, \psi)$ is odd in ψ and $F_2(\eta, -\psi) = F_2(\eta, \psi)$ is even in ψ.

Denoting by Φ^t the flow of (1.1.4), the reversibility property (1.2.4) is equivalent to

$$S \circ \Phi^{-t} = \Phi^t \circ S\,.$$

In other words, if $\left[\begin{smallmatrix} \eta(t;\eta_0,\psi_0) \\ \psi(t;\eta_0,\psi_0) \end{smallmatrix}\right]$ is the solution of (1.1.4) with initial condition $\eta(0; \eta_0, \psi_0) = \eta_0$, $\psi(0; \eta_0, \psi_0) = \psi_0$, then

$$S\left[\begin{smallmatrix} \eta(-t;\eta_0,\psi_0) \\ \psi(-t;\eta_0,\psi_0) \end{smallmatrix}\right] = \left[\begin{smallmatrix} \eta(-t;\eta_0,\psi_0) \\ -\psi(-t;\eta_0,\psi_0) \end{smallmatrix}\right]$$

is solution as well of (1.1.4), and, at $t = 0$, it takes the value $\left[\begin{smallmatrix} \eta_0 \\ -\psi_0 \end{smallmatrix}\right]$. Therefore, by uniqueness of the solution,

$$\eta(-t;\eta_0,\psi_0) = \eta(t;\eta_0,-\psi_0)\,, \quad -\psi(-t;\eta_0,\psi_0) = \psi(t;\eta_0,-\psi_0)\,,$$

which may be written as

$$\left[\begin{smallmatrix} \eta \\ \psi \end{smallmatrix}\right]\Big(-t;\left[\begin{smallmatrix} \eta_0 \\ \psi_0 \end{smallmatrix}\right]\Big) = S\left[\begin{smallmatrix} \eta \\ \psi \end{smallmatrix}\right]\Big(t;S\left[\begin{smallmatrix} \eta_0 \\ \psi_0 \end{smallmatrix}\right]\Big). \tag{1.2.5}$$

The reversibility of the water waves system plays an essential role in the proof of our theorem. Together with evenness, this condition will ensure that some quantities, that could generate a growth in modified Sobolev norms, actually vanish, see Lemmas 4.3.2 and 4.4.2.

The first step in the proof of Theorem 1.2.1 will be to rewrite the water waves system (1.1.4) in terms of the "good unknown" of Alinhac. This will be essential in order to obtain energy estimates. To do so, we introduce tools of paradifferential calculus in the next chapter.

Let us fix some notation that will be used in the rest of the monograph. Since the quotient map induces an isometry from H^s_0 to $\dot{H}^s$, we shall consider (η,ψ) as an element of $\dot{H}^{s+\frac{1}{4}}_{\rm ev} \times \dot{H}^{s-\frac{1}{4}}_{\rm ev}$, identifying thus η to its inverse image through this isometry.

To simplify notation, we shall assume that the depth h is equal to 1. Moreover, we may reduce to the case $g = 1$: actually, considering instead of (η,ψ) the couple of functions $(\eta(\sqrt{g}t,x), \sqrt{g}\psi(\sqrt{g}t,x))$, we reduce (1.1.4) to the same system, with (g,κ) replaced by $(1,\kappa/g)$.

Let us mention that in the whole monograph, we shall denote indifferently $\|\cdot\|_{L^2}$ or $\|\cdot\|_{\dot{H}^0}$. It should be understood that in any case the zero frequency of functions at hand is discarded.

Chapter 2

Paradifferential calculus

2.1 Classes of symbols

We define in this chapter several classes of symbols of paradifferential operators that we shall use in the whole monograph. Our classes are essentially standard ones, except that our symbols are depending on some functions (that in the application will be the solution U of the equation itself), and we control the semi-norms of the symbols from these functions.

To fix ideas, let us mention that in the whole monograph we shall deal with parameters satisfying

$$s \gg \sigma \gg K \gg \rho \gg N \tag{2.1.1}$$

where N is the exponent in the lower bound of the existence time of Theorem 1.2.1 and ρ will be the smoothing degree at which we shall stop the symbolic calculus of paradifferential operators. As explained in the introduction N coincides with the number of normal form steps performed in sections 4.3 and 4.4. In order to implement these Birkhoff normal form procedures we have to choose the parameter κ such that the small divisor estimate (7.1.3) of Proposition 7.1.1 holds. The effect of the small divisors is to produce a loss of derivatives proportional to N_0 and N, that may be compensated by taking the regularizing index $\rho \gg N$ large enough. This requires also to develop the paradifferential calculus in Sobolev spaces $\dot{H}^s$ with a regularity s large enough, $s \geq \sigma$ with $\sigma \gg \rho$. Moreover we look for classical solutions $U(t)$ of the capillarity-gravity water waves equations which are K-times differentiable in time, with derivatives $\partial_t^k U$ in $\dot{H}^{\sigma-\frac{3}{2}k}$, $k = 0, \dots, K$, and thus we need $\sigma \gg K$. Finally, the reason why $K \gg \rho$ is that, in order to obtain system (4.2.9), which has constant coefficients up to ρ-smoothing remainders, see also (0.5.1), we shall perform a large number of conjugations, large with respect to ρ, of the water waves system (1.1.4), to obtain para-differential operators which are enough regularizing. Each conjugation consumes one time derivative, so that we shall need $K \gg \rho$.

We also notice that our solutions will remain in a small ball of the space of C^k functions of time with values in $\dot{H}^{\sigma-\frac{3}{2}k}$, and their time derivative of order k will be continuous in t with values in $\dot{H}^{s-\frac{3}{2}k}$, for $k \leq K$.

We define below several classes of symbols and operators. We shall stick to the following convention: for p in $\mathbb{N}$, we shall denote by $\tilde{A}_p$ (with A replaced by several letters, affected eventually with some indices) classes of symbols, i.e. functions of (x,ξ), (or operators), which are symmetric p-linear maps of some argument $\mathcal{U} = (U_1,\dots,U_p)$ belonging to a functional space, where each $U_j = U_j(t)$ are functions of time. The notation $A_N[r]$ with N in $\mathbb{N}^*$, $r > 0$ will be used for symbols (or operators) depending in a non-homogeneous way on a function U staying in the ball of center zero, radius r in a convenient space, and vanishing at order N when U goes to zero in that space. Finally, $\Sigma A_p[r,N]$ will denote sums of functions homogeneous in U of degree q, $p \leq q \leq N-1$, defined as restrictions at $U_1 = \cdots = U_q = U$ of a q-linear map of $\tilde{A}_q$, and of an element of $A_N[r]$. In the whole monograph, for the homogeneous symbols (or operators) the dependence on time t will enter only through the function $U = U(t)$, while the non-homogeneous symbols (or operators) in $A_N[r]$ may depend explicitly on time t.

Before giving the first instance where these conventions will be used, we introduce some notation. If K is in $\mathbb{N}$, I is an interval of $\mathbb{R}$, symmetric with respect to the origin $t = 0$, σ is in $\mathbb{R}$, we denote by $C^K_*(I,\dot{H}^\sigma(\mathbb{T}^1,\mathbb{C}^2))$ the space of continuous functions U of $t \in I$ with values in $\dot{H}^\sigma(\mathbb{T}^1,\mathbb{C}^2)$, which are differentiable in t with values in $\dot{H}^{\sigma-\frac{3}{2}k}(\mathbb{T}^1,\mathbb{C}^2)$, for any $0 \leq k \leq K$, with continuous derivatives $\partial_t^k U \in \dot{H}^{\sigma-\frac{3}{2}k}(\mathbb{T}^1,\mathbb{C}^2)$. The space $C^K_*(I,\dot{H}^\sigma(\mathbb{T}^1,\mathbb{C}^2))$ is endowed with the norm

$$\sup_{t\in I} |\!|\!| U(t,\cdot) |\!|\!|_{K,\sigma} \quad \text{where} \quad |\!|\!| U(t,\cdot) |\!|\!|_{K,\sigma} = \sum_{k=0}^{K} \|\partial_t^k U(t,\cdot)\|_{\dot{H}^{\sigma-\frac{3}{2}k}}. \tag{2.1.2}$$

We denote by $C^K_{*\mathbb{R}}(I,\dot{H}^\sigma(\mathbb{T}^1,\mathbb{C}^2))$ the subspace of $C^K_*(I,\dot{H}^\sigma(\mathbb{T}^1,\mathbb{C}^2))$ made of the functions of t with values in $\left\{U \in \dot{H}^\sigma(\mathbb{T}^1;\mathbb{C}^2); U = \left[\begin{smallmatrix} u \\ \bar{u} \end{smallmatrix}\right]\right\}$.

If $\mathcal{U} = (U_1,\dots,U_p)$ with U_j in $\dot{H}^\infty(\mathbb{T}^1,\mathbb{C}^2)$, respectively if U is in $C^K_*(I,\dot{H}^\sigma(\mathbb{T}^1,\mathbb{C}^2))$, we set

$$\begin{aligned} \mathcal{G}^\sigma_{0,p}(\mathcal{U}) &= \prod_{j=1}^{p} \|U_j\|_{\dot{H}^\sigma}\,, \\ \mathcal{G}^\sigma_{K,p}(U,t) &= |\!|\!| U(t,\cdot) |\!|\!|^p_{K,\sigma}\,. \end{aligned} \tag{2.1.3}$$

When $p = 0$, we set $\mathcal{G}^\sigma_{K,0} = 1$ by convention. If $\mathcal{U} = (U_1,\dots,U_p)$ is a p-tuple of functions, $n = (n_1,\dots,n_p)$ an element of $(\mathbb{N}^*)^p$, we set

$$\Pi_n \mathcal{U} = (\Pi_{n_1} U_1,\dots,\Pi_{n_p} U_p) \tag{2.1.4}$$

where Π_{n_j} is the spectral projector defined before (1.2.1). Finally, let us mention that in chapters 5 and 6 we shall use similar notations where U_j will be in $\dot{H}^\sigma(\mathbb{T}^1,\mathbb{R})$ or $\dot{H}^\sigma(\mathbb{T}^1,\mathbb{R}^2)$.

Definition 2.1.1 (p-homogeneous symbol) *Let m in $\mathbb{R}$, p in $\mathbb{N}^*$. We denote by $\tilde{\Gamma}^m_p$ the space of symmetric p-linear maps from $\dot{H}^\infty(\mathbb{T}^1,\mathbb{C}^2)^p$ to the space of C^∞ functions in $(x,\xi)\in\mathbb{T}^1\times\mathbb{R}$*

$$\mathcal{U}\to((x,\xi)\to a(\mathcal{U};x,\xi))$$

satisfying the following: There is $\mu\geq 0$ and for any $\alpha,\beta\in\mathbb{N}$, there is $C>0$ and for any $\mathcal{U}$ in $\dot{H}^\infty(\mathbb{T}^1,\mathbb{C}^2)^p$, any $n=(n_1,\dots,n_p)$ in $(\mathbb{N}^*)^p$,

$$|\partial_x^\alpha\partial_\xi^\beta a(\Pi_n\mathcal{U};x,\xi)|\leq C|n|^{\mu+\alpha}\langle\xi\rangle^{m-\beta}\mathcal{G}^0_{0,p}(\Pi_n\mathcal{U})\,. \tag{2.1.5}$$

Moreover, we assume that if, for some $(n_0,\dots,n_p)$ in $\mathbb{N}\times(\mathbb{N}^*)^p$,

$$\Pi_{n_0}a(\Pi_{n_1}U_1,\dots,\Pi_{n_p}U_p;\cdot)\not\equiv 0, \tag{2.1.6}$$

then there is a choice of signs $\epsilon_0,\dots,\epsilon_p\in\{-1,1\}$ such that $\sum_0^p\epsilon_jn_j=0$.

When $p=0$, we denote by $\tilde{\Gamma}^m_0$ the space of constant coefficient symbols $\xi\to a(\xi)$, that satisfy inequality (2.1.5) for $\alpha=0$, with in the right hand side the $|n|$ factor replaced by one, and with the convention $\mathcal{G}^0_{0,0}(\Pi_n\mathcal{U})=1$.

Remarks: • In the sequel, we shall consider functions $\mathcal{U}=(U_1,\dots,U_p)$ where U_j depends also on time t, so that the corresponding symbols in the above definition are functions of (t,x,ξ) that we denote also by $a(\mathcal{U};t,x,\xi)$.

• If the dependence of a on $(U_1,\dots,U_p)$ is polynomial with constant coefficients in x, then, as $\Pi_{n_j}U_j$ is a linear combination of e^{in_jx}, e^{-in_jx}, condition (2.1.6) holds automatically.

• If $a\in\tilde{\Gamma}^m_p$ and $b\in\tilde{\Gamma}^{m'}_q$ then $ab\in\tilde{\Gamma}^{m+m'}_{p+q}$. If $a\in\tilde{\Gamma}^m_p$ then $\partial_xa\in\tilde{\Gamma}^m_p$ and $\partial_\xi a\in\tilde{\Gamma}^{m-1}_p$.

• Let $K\in\mathbb{N}$ and $\sigma_0\geq\frac{3}{2}K+\mu+\frac{1}{2}$. For $\sigma\geq\sigma_0$ let consider a function

$$U\in C^K_*(I,\dot{H}^\sigma(\mathbb{T}^1,\mathbb{C}^2))\,.$$

Take a in $\tilde{\Gamma}^m_p$, $p\in\mathbb{N}^*$. We claim that, for all $0\leq k\leq K$, $\alpha\leq\sigma-\sigma_0$, $\beta\in\mathbb{N}$, the function

$$a(\underbrace{U;\dots,U}_{p};t,x,\xi)=a(U(t,\cdot),\dots,U(t,\cdot);x,\xi) \tag{2.1.7}$$

satisfies the estimate

$$|\partial_t^k\partial_x^\alpha\partial_\xi^\beta a(U,\dots,U;t,x,\xi)|\leq C\langle\xi\rangle^{m-\beta}|\!|\!|U(t,\cdot)|\!|\!|^{p-1}_{k,\sigma_0}|\!|\!|U(t,\cdot)|\!|\!|_{k,\sigma_0+\alpha}. \tag{2.1.8}$$

Indeed, if we make act k time derivatives on a, we have, by the p-linearity of a in each argument U,

$$\partial_t^k \partial_x^\alpha \partial_\xi^\beta a(U(t,\cdot),\dots,U(t,\cdot);x,\xi) = \sum_{n_1,\dots,n_p\in\mathbb{N}^*} \sum_{k_1+\dots+k_p=k} C_{k_1,\dots,k_p} \partial_x^\alpha \partial_\xi^\beta a(\partial_t^{k_1}\Pi_{n_1}U(t),\dots,\partial_t^{k_p}\Pi_{n_p}U(t);x,\xi)$$

for suitable binomial coefficients $C_{k_1,\dots,k_p}$. By (2.1.5) and recalling the definition of the norm (2.1.2), each term in the right hand side of the above sum, with for instance $n_1 \geq n_2,\dots,n_p$, is bounded by

$$C\langle\xi\rangle^{m-\beta}|n|^{\mu+\alpha}\prod_1^p \|\partial_t^{k_j}\Pi_{n_j}U(t,\cdot)\|_{L^2} \leq$$

$$C\langle\xi\rangle^{m-\beta} n_1^{\mu-\sigma_0+\frac{3}{2}k_1} c^1_{n_1} \|\partial_t^{k_1}U(t)\|_{\dot H^{\sigma_0+\alpha-\frac{3}{2}k_1}} \prod_2^p n_j^{-\sigma_0+\frac{3}{2}k_j} c^j_{n_j} \|\partial_t^{k_j}U(t)\|_{\dot H^{\sigma_0-\frac{3}{2}k_j}}$$

$$\leq C\langle\xi\rangle^{m-\beta} n_1^{\mu-\sigma_0+\frac{3}{2}k} \prod_2^p n_j^{-\sigma_0} c^1_{n_1}\dots c^p_{n_p} |\!|\!| U(t,\cdot)|\!|\!|_{k,\sigma_0+\alpha} |\!|\!| U(t,\cdot)|\!|\!|^{p-1}_{k,\sigma_0}$$

where $(c^j_{n_j})_{n_j}$ stands for some ℓ^2 sequence. Since $\sigma_0 - \frac{3}{2}K > \mu + \frac{1}{2}$, we obtain (2.1.8). In conclusion, the function $a(U,\dots,U;t,x,\xi)$ is a symbol of a pseudo-differential operator with limited smoothness in x, with a control of its semi-norms in terms of U, vanishing in U with degree p. Note that the bound in (2.1.8) is tame, namely it is linear in the highest norm $|\!|\!| U(t,\cdot)|\!|\!|_{k,\sigma_0+\alpha}$.

We define next symbols $a(U;t,x,\xi)$ for which the U-dependence is not homogeneous, using estimates of type (2.1.8), where moreover we allow also K' extra time derivatives in the right hand side. For $\sigma\in\mathbb{R}$, $k\in\mathbb{N}$, $r>0$, I an interval of $\mathbb{R}$, we define

$$B^k_\sigma(I,r) = \Big\{ U\in C^k_*(I,\dot H^\sigma(\mathbb{T}^1,\mathbb{C}^2));\ \sup_{t\in I} |\!|\!| U(t,\cdot)|\!|\!|_{k,\sigma} < r \Big\}. \tag{2.1.9}$$

Definition 2.1.2 (Non-homogeneous symbol) *Let m be in $\mathbb{R}$, p in $\mathbb{N}$, $K'\leq K$ in $\mathbb{N}$, $r>0$. One denotes by $\Gamma^m_{K,K',p}[r]$ the space of functions $(U,t,x,\xi)\to a(U;t,x,\xi)$, defined for U in $B^K_{\sigma_0}(I,r)$, for some large enough σ_0, t in I, x in $\mathbb{T}^1$, ξ in $\mathbb{R}$, with complex values, such that for any $0\leq k\leq K-K'$, any $\sigma\geq\sigma_0$, there are $C>0$, $r(\sigma)\in]0,r[$ and for any U in $B^K_{\sigma_0}(I,r(\sigma))\cap C^{k+K'}_*(I,\dot H^\sigma(\mathbb{T}^1,\mathbb{C}^2))$, any α,β in $\mathbb{N}$, with $\alpha\leq\sigma-\sigma_0$*

$$|\partial_t^k\partial_x^\alpha\partial_\xi^\beta a(U;t,x,\xi)| \leq C\langle\xi\rangle^{m-\beta}\mathcal{G}^{\sigma_0}_{k+K',p-1}(U,t)\mathcal{G}^{\sigma}_{k+K',1}(U,t), \tag{2.1.10}$$

(where, if $p=0$, the right hand side has to be replaced by $C\langle\xi\rangle^{m-\beta}$).

Notice that the above non-homogeneous symbols $a(U;t,x,\xi)$ depend explicitly on the time variable t (not only through the function $U(t)$ as for the homogeneous symbols). We introduce also a definition for the subclass of non-homogeneous symbols that depend on time only through U.

Definition 2.1.3 (Autonomous non-homogeneous symbol) *We denote by $\Gamma^m_{K,0,p}[r,\mathrm{aut}]$ the subspace of $\Gamma^m_{K,0,p}[r]$ made of the non-homogeneous symbols $(U,x,\xi)\to a(U;x,\xi)$ that satisfy estimates (2.1.10) with $K'=0$, the time dependence being only through $U=U(t)$.*

Remarks: • By (2.1.8), a symbol $a(\mathcal{U};\cdot)$ of $\tilde\Gamma^m_p$ defines, by restriction to the diagonal, the symbol $a(U,\dots,U;\cdot)$ of $\Gamma^m_{K,0,p}[r,\mathrm{aut}]$ for any $r>0$.

• If a is in $\Gamma^m_{K,K',p}[r]$ with $K'\le K-1$, then $\partial_t a$ is in $\Gamma^m_{K,K'+1,p}[r]$.

• If $a\in\Gamma^m_{K,K',p}[r]$ then $\partial_\xi a\in\Gamma^{m-1}_{K,K',p}[r]$ and $\partial_x a\in\Gamma^m_{K,K',p}[r]$, by the fact that we may increase the value of σ_0 in Definition 2.1.2.

• If $a\in\Gamma^m_{K,K',p}[r]$ and $b\in\Gamma^{m'}_{K,K',q}[r]$ then $ab\in\Gamma^{m+m'}_{K,K',q+p}[r]$. Let us prove this remark. Notice that if $\alpha=\alpha_1+\alpha_2$, we have

(2.1.11)
$$|\!|\!|U|\!|\!|_{k_1,\alpha_1+\sigma_0}|\!|\!|V|\!|\!|_{k_2,\alpha_2+\sigma_0}\le C\big(|\!|\!|U|\!|\!|_{k_1,\alpha+\sigma_0}|\!|\!|V|\!|\!|_{k_2,\sigma_0}+|\!|\!|U|\!|\!|_{k_1,\sigma_0}|\!|\!|V|\!|\!|_{k_2,\alpha+\sigma_0}\big).$$

Actually, recalling the definition (2.1.2) of the norm $|\!|\!|\cdot|\!|\!|_{K,\sigma}$, we have

$$\begin{aligned}|\!|\!|U|\!|\!|_{k_1,\alpha_1+\sigma_0}|\!|\!|V|\!|\!|_{k_2,\alpha_2+\sigma_0}&=\sum_{k_1'=0}^{k_1}\sum_{k_2'=0}^{k_2}\|\partial_t^{k_1'}U\|_{\dot H^{\alpha_1+\sigma_0-\frac32k_1'}}\|\partial_t^{k_2'}V\|_{\dot H^{\alpha_2+\sigma_0-\frac32k_2'}}\\&\le\sum_{k_1'=0}^{k_1}\sum_{k_2'=0}^{k_2}\|\partial_t^{k_1'}U\|_{\dot H^{\alpha+\sigma_0-\frac32k_1'}}\|\partial_t^{k_2'}V\|_{\dot H^{\sigma_0-\frac32k_2'}}+\\&\qquad\|\partial_t^{k_1'}U\|_{\dot H^{\sigma_0-\frac32k_1'}}\|\partial_t^{k_2'}V\|_{\dot H^{\alpha+\sigma_0-\frac32k_2'}}\end{aligned}$$

by interpolation. By the last inequality, (2.1.11) follows. Taking now $a\in\Gamma^m_{K,K',p}[r]$ and $b\in\Gamma^{m'}_{K,K',q}[r]$, write for $0\le k\le K-K'$, α,β in $\mathbb{N}$, with $\alpha\le\sigma-\sigma_0$

$$\begin{aligned}&\partial_t^k\partial_x^\alpha\partial_\xi^\beta\big(a(U;t,x,\xi)b(U;t,x,\xi)\big)\\&=\sum_{\substack{k_1+k_2=k,\alpha_1+\alpha_2=\alpha,\\ \beta_1+\beta_2=\beta}}C^{k_2,\alpha_2,\beta_2}_{k_1,\alpha_1,\beta_1}(\partial_t^{k_1}\partial_x^{\alpha_1}\partial_\xi^{\beta_1}a(U;t,x,\xi))(\partial_t^{k_2}\partial_x^{\alpha_2}\partial_\xi^{\beta_2}a(U;t,x,\xi))\end{aligned}$$

for suitable binomial coefficients $C^{k_2,\alpha_2,\beta_2}_{k_1,\alpha_1,\beta_1}$. Using (2.1.10), we get

$$\begin{aligned}
&\Big|\partial_t^k\partial_x^\alpha\partial_\xi^\beta\big(a(U;t,x,\xi)b(U;t,x,\xi)\big)\Big| \\
&\qquad\le C\sum_{\substack{k_1+k_2=k,\alpha_1+\alpha_2=\alpha,\\ \beta_1+\beta_2=\beta}}\langle\xi\rangle^{m-\beta_1}\mathcal{G}^{\sigma_0}_{k_1+K',p-1}(U,t)\mathcal{G}^{\alpha_1+\sigma_0}_{k_1+K',1}(U,t) \\
&\qquad\qquad\times\langle\xi\rangle^{m'-\beta_2}\mathcal{G}^{\sigma_0}_{k_2+K',q-1}(U,t)\mathcal{G}^{\alpha_2+\sigma_0}_{k_2+K',1}(U,t) \\
&\qquad\le C\sum_{\alpha_1+\alpha_2=\alpha}\langle\xi\rangle^{m+m'-\beta}|\!|\!|U|\!|\!|^{p+q-2}_{k+K',\sigma_0}|\!|\!|U|\!|\!|_{k+K',\alpha_1+\sigma_0}|\!|\!|U|\!|\!|_{k+K',\alpha_2+\sigma_0} \\
&\qquad\qquad\le C\langle\xi\rangle^{m+m'-\beta}|\!|\!|U|\!|\!|^{p+q-1}_{k+K',\sigma_0}|\!|\!|U|\!|\!|_{k+K',\alpha+\sigma_0} \\
&\qquad\qquad\qquad\le C\langle\xi\rangle^{m+m'-\beta}\mathcal{G}^{\sigma_0}_{k+K',p+q-1}(U,t)\mathcal{G}^{\sigma}_{k+K',1}(U,t)
\end{aligned}$$

where we used (2.1.11) and took $\sigma\ge\alpha+\sigma_0$. Thus ab satisfies the estimate (2.1.10) of elements of $\Gamma^{m+m'}_{K,K',p+q}[r]$.

Finally, we define symbols that may be expressed as the sum of homogeneous elements of $\tilde\Gamma^m_q$, $q=p-1,\dots,N-1$, plus a non-homogeneous symbol vanishing in U at order N.

Definition 2.1.4 (Symbols) *Let m be in $\mathbb{R}$, p in $\mathbb{N}$, $K'\le K$ in $\mathbb{N}$, $r>0$, N in $\mathbb{N}$ with $p\le N$. One denotes by $\Sigma\Gamma^m_{K,K',p}[r,N]$ the space of functions $(U,t,x,\xi)\to a(U;t,x,\xi)$ such that there are homogeneous symbols a_q in $\tilde\Gamma^m_q$ for $q=p,\dots,N-1$, and a non-homogeneous symbol a_N in $\Gamma^m_{K,K',N}[r]$, so that*

$$a(U;t,x,\xi)=\sum_{q=p}^{N-1}a_q(U,\dots,U;x,\xi)+a_N(U;t,x,\xi)\,. \tag{2.1.12}$$

We set $\Sigma\Gamma^{-\infty}_{K,K',p}[r,N]=\bigcap_m\Sigma\Gamma^m_{K,K',p}[r,N]$.

We define the subclasses of autonomous symbols $\Sigma\Gamma^m_{K,0,p}[r,N,\mathrm{aut}]$ by formula (2.1.12), where a_N is taken in the class $\Gamma^m_{K,0,N}[r,\mathrm{aut}]$ of Definition 2.1.3. We set finally $\Sigma\Gamma^{-\infty}_{K,0,p}[r,N,\mathrm{aut}]=\bigcap_m\Sigma\Gamma^m_{K,0,p}[r,N,\mathrm{aut}]$.

For $p=N$ we mean that the symbol $a=a_N$ is purely non-homogeneous.

Remarks: • We have the following simple inclusions

$$\begin{aligned}
&m_1\le m_2\,, && \Sigma\Gamma^{m_1}_{K,K',p}[r,N]\subseteq\Sigma\Gamma^{m_2}_{K,K',p}[r,N]\,,\\
&p_1\le p_2\,, && \Sigma\Gamma^{m}_{K,K',p_2}[r,N]\subseteq\Sigma\Gamma^{m}_{K,K',p_1}[r,N]\,,\\
&K_1'\le K_2'\,, && \Sigma\Gamma^{m}_{K,K_1',p}[r,N]\subseteq\Sigma\Gamma^{m}_{K,K_2',p}[r,N]\,,
\end{aligned}$$

as well as the similar ones for autonomous symbols.

• If $a\in\Sigma\Gamma^m_{K,K',p}[r,N]$ and $b\in\Sigma\Gamma^{m'}_{K,K',p'}[r,N]$ then $ab\in\Sigma\Gamma^{m+m'}_{K,K',p+p'}[r,N]$. A similar property holds for autonomous symbols.

2.2 Quantization of symbols

If $(x,\xi) \to a(x,\xi)$ is a smooth symbol, one defines, for every $\tau \in [0,1]$, a quantization of the symbol a as the operator acting on test functions u (or on tempered distributions) by

$$\mathrm{Op}_\tau(a)u = \frac{1}{2\pi}\int_{\mathbb{R}\times\mathbb{R}} e^{i(x-y)\xi} a(\tau x + (1-\tau)y, \xi)u(y)\, dyd\xi\,. \tag{2.2.1}$$

In the case $\tau = \frac{1}{2}$, one gets the Weyl quantization

$$Op^{\mathrm{W}}(a)u = \frac{1}{2\pi}\int_{\mathbb{R}\times\mathbb{R}} e^{i(x-y)\xi} a\Big(\frac{x+y}{2}, \xi\Big)u(y)\, dyd\xi \tag{2.2.2}$$

and for $\tau = 1$ the usual quantization

$$\mathrm{Op}(a)u = \frac{1}{2\pi}\int_{\mathbb{R}\times\mathbb{R}} e^{i(x-y)\xi} a(x,\xi)u(y)\, dyd\xi\,.$$

These formulas are meaningful, in particular, for u in $C^\infty(\mathbb{T}^1)$ (identifying u to a 2π-periodic function, and thus to a tempered distribution). If we decompose u in Fourier series as

$$u(x) = \sum_{n\in\mathbb{Z}} \hat{u}(n)\frac{e^{inx}}{\sqrt{2\pi}}\,,$$

we may compute the oscillatory integral (2.2.1), and we find out that the action of $\mathrm{Op}_\tau(a)$ on u is given by the periodic function

$$\mathrm{Op}_\tau(a)u = \frac{1}{\sqrt{2\pi}}\sum_{k\in\mathbb{Z}}\Big(\sum_{n\in\mathbb{Z}} \hat{a}(k-n, (1-\tau)k + \tau n)\hat{u}(n)\Big)\frac{e^{ikx}}{\sqrt{2\pi}}$$

where $\hat{a}(k,\xi)$ stands for the Fourier coefficients of the periodic function $x \to a(x,\xi)$. In particular, the Weyl quantization is given by

$$Op^{\mathrm{W}}(a)u = \frac{1}{\sqrt{2\pi}}\sum_{k\in\mathbb{Z}}\Big(\sum_{n\in\mathbb{Z}} \hat{a}\Big(k-n, \frac{k+n}{2}\Big)\hat{u}(n)\Big)\frac{e^{ikx}}{\sqrt{2\pi}}, \tag{2.2.3}$$

and the usual quantization, corresponding to $\tau = 1$, by

$$\begin{aligned}\mathrm{Op}(a)u &= \frac{1}{\sqrt{2\pi}}\sum_{k\in\mathbb{Z}}\Big(\sum_{n\in\mathbb{Z}} \hat{a}(k-n,n)\hat{u}(n)\Big)\frac{e^{ikx}}{\sqrt{2\pi}} \\ &= \sum_{n\in\mathbb{Z}} a(x,n)\hat{u}(n)\frac{e^{inx}}{\sqrt{2\pi}}\,.\end{aligned} \tag{2.2.4}$$

If the symbol $a = a(\xi)$ does not depend of x, then, for any $\tau \in [0,1]$, the associated operator $\mathrm{Op}_\tau(a) = \mathrm{Op}(a) = a(D)$ is the usual Fourier multiplier.

The above formulas allow to transform the symbols between different quantization, in particular we have

$$\text{Op}(a) = Op^{\text{W}}(b) \quad \text{where} \quad \hat{b}(n,\xi) = \hat{a}\big(n, \xi - \frac{n}{2}\big)\,,\ \forall n \in \mathbb{Z}\,. \tag{2.2.5}$$

We shall need in Chapter 6 an asymptotic expansion of the Weyl symbol b in terms of the symbol a. Since the distributional kernel of the operator $\text{Op}^W(b)$ defined in (2.2.2) is the oscillatory integral

$$K(x,y) = \frac{1}{2\pi}\int_{\mathbb{R}} e^{i(x-y)\xi} b\big(\frac{x+y}{2}, \xi\big) d\xi$$

we may recover the Weyl symbol b from the kernel by the inverse formula

$$b(x,\xi) = \int_{\mathbb{R}} K\big(x + \frac{t}{2}, x - \frac{t}{2}\big) e^{-it\xi} dt\,. \tag{2.2.6}$$

Now, identifying an operator $\text{Op}(a)$ with its distributional kernel, we derive by (2.2.6) that

$$\text{Op}(a) = \frac{1}{2\pi}\int_{\mathbb{R}} e^{i(x-y)\xi} a(x,\xi) d\xi = \text{Op}^W(b) \tag{2.2.7}$$

where b is the symbol

$$b(x,\xi) = \frac{1}{2\pi}\int_{\mathbb{R}^2} e^{-iz\zeta} a\big(x + \frac{z}{2}, \xi - \zeta\big)\, dz d\zeta\,.$$

Lemma 2.2.1 *Let a be in $\tilde{\Gamma}^m_p$, $\mathcal{U} = (U_1, \dots, U_p)$ in $\dot{H}^\infty(\mathbb{T}^1, \mathbb{C}^2)^p$, and consider the function*

$$b(\mathcal{U}; x, \xi) = \frac{1}{2\pi}\int_{\mathbb{R}^2} e^{-iz\zeta} a\big(\mathcal{U}; x + \frac{z}{2}, \xi - \zeta\big)\, dz d\zeta \tag{2.2.8}$$

(interpreted as an oscillatory integral). Then for any integer $A \geq 1$ we have the expansion

$$b(\mathcal{U}; x, \xi) = \sum_{\alpha=0}^{A-1} \frac{(-1)^\alpha}{2^\alpha \alpha!} \big(\partial_x^\alpha D_\xi^\alpha a\big)(\mathcal{U}; x, \xi) + \tilde{b}(\mathcal{U}; x, \xi) \tag{2.2.9}$$

where $D_\xi = \frac{1}{i}\partial_\xi$ and $\tilde{b}$ satisfies the bounds: for any $\alpha, \beta \in \mathbb{N}$, $n \in (\mathbb{N}^*)^p$,

$$|\partial_x^\alpha \partial_\xi^\beta \tilde{b}(\Pi_n \mathcal{U}; x, \xi)| \leq Q\bigg(\frac{|n|}{\langle \xi \rangle}\bigg) |n|^{\mu+\alpha+\beta} \langle \xi \rangle^{m-\beta} \mathcal{G}^0_{0,p}(\Pi_n \mathcal{U}) \tag{2.2.10}$$

where Q is a polynomial of valuation larger or equal to A and degree at most $2A-1$, and μ depends only on m.

Proof: Let χ be in $C_0^\infty(\mathbb{R})$, equal to one close to zero, with small enough support. We split the integral in (2.2.8) as the sum of

$$\frac{1}{2\pi}\int_{\mathbb{R}^2}\chi(\zeta\langle\xi\rangle^{-1})e^{-iz\zeta}a\big(\mathcal{U};x+\frac{z}{2},\xi-\zeta\big)\,dzd\zeta \tag{2.2.11}$$

and

$$\frac{1}{2\pi}\int_{\mathbb{R}^2}(1-\chi)(\zeta\langle\xi\rangle^{-1})e^{-iz\zeta}a\big(\mathcal{U};x+\frac{z}{2},\xi-\zeta\big)\,dzd\zeta\,. \tag{2.2.12}$$

We first consider the integral (2.2.12) with $\mathcal{U}$ replaced by $\Pi_n\mathcal{U}$. For any $N_0\in\mathbb{N}$, we perform N_0 ∂_z-integrations by parts in (2.2.12) and two ∂_ζ integrations by parts. Thus we gain a decaying factor $\langle z\rangle^{-2}\langle\zeta\rangle^{-N_0}$ and, using (2.1.5), the integral (2.2.12) is bounded by

$$\begin{aligned}C\int_{\mathbb{R}}\langle\xi-\zeta\rangle^m\langle\zeta\rangle^{-N_0}\mathbf{1}_{\{|\zeta|\geq\delta\langle\xi\rangle\}}\,d\zeta\;|n|^{\mu+N_0}\mathcal{G}^0_{0,p}(\Pi_n\mathcal{U})\\ \leq C'\langle\xi\rangle^{M(m)-N_0}|n|^{\mu+N_0}\mathcal{G}^0_{0,p}(\Pi_n\mathcal{U})\end{aligned}$$

for any $N_0>m+1$, where we set $M(m)=m+1$ if $m>-1$, $M(-1)=\epsilon$ for a small $\epsilon>0$ and $M(m)=0$ if $m<-1$. If we make act $\partial_x^\alpha\partial_\xi^\beta$ derivatives on (2.2.12), we get in the same way a bound in

$$\langle\xi\rangle^{M(m-\beta)-N_0}|n|^{\mu+N_0+\alpha}\mathcal{G}^0_{0,p}(\Pi_n\mathcal{U})\,.$$

Taking $N_0=N_0'+\beta+L$, for some $L=L(m)$ depending only on m, we get a bound in

$$\langle\xi\rangle^{m+1-N_0'-\beta}|n|^{\mu'+N_0'+\alpha+\beta}\mathcal{G}^0_{0,p}(\Pi_n\mathcal{U})$$

where $\mu'=\mu+L$. Consequently (2.2.12) satisfies (2.2.10) for any A (up to a change of notation for μ).

Consider next (2.2.11). We Taylor expand up to order $2A-1$ the symbol $a\big(\mathcal{U};x+\frac{z}{2},\xi-\zeta\big)$ at the point (x,ξ), writing the integrand in (2.2.11) as the product of $e^{-iz\zeta}$ and

$$\begin{aligned}&\sum_{\alpha+\beta\leq 2A-1}\frac{1}{\alpha!\beta!}\big(\partial_x^\alpha\partial_\xi^\beta a\big)(\mathcal{U};x,\xi)\Big(\frac{z}{2}\Big)^\alpha(-\zeta)^\beta\chi\Big(\frac{\zeta}{\langle\xi\rangle}\Big)\\ &+\sum_{\alpha+\beta=2A}\frac{2A}{\alpha!\beta!}\int_0^1(1-\lambda)^{2A}\big(\partial_x^\alpha\partial_\xi^\beta a\big)(\mathcal{U};x+\frac{\lambda z}{2},\xi-\lambda\zeta)\\ &\qquad\qquad\times\Big(\frac{z}{2}\Big)^\alpha(-\zeta)^\beta\chi\Big(\frac{\zeta}{\langle\xi\rangle}\Big)\,d\lambda.\end{aligned} \tag{2.2.13}$$

Since the oscillatory integral

$$\frac{1}{2\pi}\int_{\mathbb{R}^2}e^{-iz\zeta}(iz)^\alpha\zeta^\beta\chi(\zeta\langle\xi\rangle^{-1})d\zeta dz=\alpha!\delta_{\alpha\beta}$$

where $\delta_{\alpha\beta} = 0$ for $\alpha \neq \beta$ and $\delta_{\alpha\beta} = 1$ for $\alpha = \beta$, when we plug the sum in (2.2.13) for $\alpha + \beta \leq 2A - 1$ inside the integral in (2.2.11), we get the sum

$$\sum_{\alpha=0}^{A-1} \frac{(-1)^\alpha}{2^\alpha \alpha!} \big(\partial_x^\alpha D_\xi^\alpha a\big)(\mathcal{U}; x, \xi)$$

in the right hand side of (2.2.9).

We are left with showing that the last term of (2.2.13), where we replace $\mathcal{U}$ by $\Pi_n \mathcal{U}$, induces in the integral (2.2.11) contributions satisfying estimates (2.2.10). For each $\lambda \in [0, 1]$, the integrals we have to study are of the form

$$\int_{\mathbb{R}^2} e^{-iz\zeta} c(\Pi_n \mathcal{U}; x, \xi, z, \zeta) z^\alpha \zeta^\beta \, dz d\zeta \tag{2.2.14}$$

where c is supported for $|\zeta| < \delta\langle\xi\rangle$, and, by (2.1.5), for any $\alpha', \beta' \in \mathbb{N}$, it satisfies estimates as

$$|\partial_z^{\alpha'} \partial_\zeta^{\beta'} c(\Pi_n \mathcal{U}; x, \xi, z, \zeta)| \leq C|n|^{\mu+\alpha+\alpha'} \langle\xi\rangle^{m-\beta-\beta'} \mathcal{G}^0_{0,p}(\Pi_n \mathcal{U}) \tag{2.2.15}$$

with $\alpha + \beta = 2A$. Let us consider first the case $\alpha \leq \beta$. We write $z^\alpha e^{-iz\zeta} = (-D_\zeta)^\alpha e^{-iz\zeta}$ and perform integrations by parts in ζ to write (2.2.14) as

$$\begin{aligned} &\int_{\mathbb{R}^2} e^{-iz\zeta} D_\zeta^\alpha \big[\zeta^\beta c(\Pi_n \mathcal{U}; x, \xi, z, \zeta)\big] \, dz d\zeta \\ &\qquad = \sum_{\alpha_1 + \alpha_2 = \alpha} C_{\alpha_1, \alpha_2, \beta} \int_{\mathbb{R}^2} e^{-iz\zeta} \zeta^{\beta - \alpha_1} D_\zeta^{\alpha_2} c(\Pi_n \mathcal{U}; x, \xi, z, \zeta) \, dz d\zeta \, . \end{aligned}$$

In each integral above, we write $\zeta^{\beta-\alpha_1} e^{-iz\zeta} = (-D_z)^{\beta-\alpha_1} e^{-iz\zeta}$ and perform integrations by parts in z. We end up with integrals of the form

$$\int_{\mathbb{R}^2} e^{-iz\zeta} D_z^{\beta-\alpha_1} D_\zeta^{\alpha_2} c(\Pi_n \mathcal{U}; x, \xi, z, \zeta) \, dz d\zeta \, .$$

This is of the form

$$\int_{\mathbb{R}^2} e^{-iz\zeta} \tilde{c}(\Pi_n \mathcal{U}; x, \xi, z, \zeta) \, dz d\zeta \tag{2.2.16}$$

where $\tilde{c}$ is supported for $|\zeta| < \delta\langle\xi\rangle$ and by (2.2.15), and since $\alpha + \beta = 2A$ and $\alpha_1 + \alpha_2 = \alpha$, it satisfies, for any $\beta_1 \in \mathbb{N}$,

$$\begin{aligned} |\partial_\zeta^{\beta_1} \tilde{c}(\Pi_n \mathcal{U}; x, \xi, z, \zeta)| &\leq C|n|^{\mu+\alpha+\beta-\alpha_1} \langle\xi\rangle^{m-\beta-\alpha_2-\beta_1} \mathcal{G}^0_{0,p}(\Pi_n \mathcal{U}) \\ &\leq C\left(\frac{|n|}{\langle\xi\rangle}\right)^{2A-\alpha_1} |n|^\mu \langle\xi\rangle^{m-\beta_1} \mathcal{G}^0_{0,p}(\Pi_n \mathcal{U}) \, . \end{aligned} \tag{2.2.17}$$

Notice that, since $\alpha \leq \beta$, the exponent $2A - \alpha_1$ is between A and $2A$, for any $0 \leq \alpha_1 \leq \alpha$. We define next the differential operator

$$L = \big(1 + z^2 \langle\xi\rangle^2\big)^{-1} \big(1 - z\langle\xi\rangle^2 D_\zeta\big)$$

so that $L(e^{-iz\zeta}) = e^{-iz\zeta}$. We write $e^{-iz\zeta} = L^2(e^{-iz\zeta})$ in (2.2.16) and we integrate by parts twice. Since $L^t = \big(1+z^2\langle\xi\rangle^2\big)^{-1}\big(1+z\langle\xi\rangle^2 D_\zeta\big)$, by (2.2.17), we gain a factor $(1+|z|\langle\xi\rangle)^{-2}$, and taking the cut-off χ in (2.2.13) into account, we bound (2.2.16) with

$$C\int_{\mathbb{R}^2}(1+|z|\langle\xi\rangle)^{-2}\mathbf{1}_{\{|\zeta|<\delta\langle\xi\rangle\}}\,dzd\zeta\left(\frac{|n|}{\langle\xi\rangle}\right)^{2A-\alpha_1}|n|^\mu\langle\xi\rangle^m\mathcal{G}^0_{0,p}(\Pi_n\mathcal{U})$$
$$\leq C'\left(\frac{|n|}{\langle\xi\rangle}\right)^{\gamma}|n|^\mu\langle\xi\rangle^m\mathcal{G}^0_{0,p}(\Pi_n\mathcal{U})$$

for some $A\leq\gamma\leq 2A$. We have proved that (2.2.14) satisfies a bound of the form (2.2.10) (with derivation indices $\alpha=\beta=0$). The case $\beta\leq\alpha$ is treated in the same way, as well as similar estimates when derivatives act on the symbol. □

If a is a symbol in $\tilde\Gamma^m_p$, it follows from (2.2.7) and (2.2.9), that

$$\frac{1}{2\pi}\int_{\mathbb{R}} e^{i(x-y)\xi}a(\mathcal{U};x,\xi)\,d\xi = \sum_{\alpha=0}^{A-1} Op^{\mathrm{W}}(b_\alpha) + Op^{\mathrm{W}}(\tilde b) \tag{2.2.18}$$

with

$$b_\alpha(\mathcal{U};x,\xi) = \frac{(-1)^\alpha}{\alpha!2^\alpha}\big(\partial_x^\alpha D_\xi^\alpha a\big)(\mathcal{U};x,\xi). \tag{2.2.19}$$

Finally notice that

$$\overline{\mathrm{Op}(a)[u]} = \mathrm{Op}(\bar a^\vee)[\bar u] \qquad \text{where} \qquad a^\vee(x,\xi)\overset{\text{def}}{=} a(x,-\xi)\,. \tag{2.2.20}$$

We now define the *para-differential* quantization. Given $p\in\mathbb{N}^*$, we fix some smooth functions, even with respect to each of their arguments,

$$\chi_p : \mathbb{R}^p\times\mathbb{R}\to\mathbb{R},\quad p\in\mathbb{N}\,,$$
$$\chi : \mathbb{R}\times\mathbb{R}\to\mathbb{R},$$

satisfying, for some small $\delta>0$, the support conditions

$$\begin{aligned}&\mathrm{Supp}\,\chi_p\subset\{(\xi',\xi)\in\mathbb{R}^p\times\mathbb{R};|\xi'|\leq\delta\langle\xi\rangle\},\chi_p\equiv 1\text{ for }|\xi'|\leq\frac{\delta}{2}\langle\xi\rangle,\\ &\mathrm{Supp}\,\chi\subset\{(\xi',\xi)\in\mathbb{R}\times\mathbb{R};|\xi'|\leq\delta\langle\xi\rangle\},\chi\equiv 1\text{ for }|\xi'|\leq\frac{\delta}{2}\langle\xi\rangle.\end{aligned} \tag{2.2.21}$$

For $p=0$ we set $\chi_0\equiv 1$. We assume moreover that for any derivation indices α,β,

$$\begin{aligned}&|\partial_\xi^\alpha\partial_{\xi'}^\beta\chi_p(\xi',\xi)|\leq C_{\alpha,\beta}\langle\xi\rangle^{-\alpha-|\beta|},\ \forall\alpha\in\mathbb{N},\forall\beta\in\mathbb{N}^p\\ &|\partial_\xi^\alpha\partial_{\xi'}^\beta\chi(\xi',\xi)|\leq C_{\alpha,\beta}\langle\xi\rangle^{-\alpha-\beta},\ \forall\alpha,\beta\in\mathbb{N}\,.\end{aligned} \tag{2.2.22}$$

An example of a function satisfying the last condition above, that will be used extensively below, is $\chi(\xi',\xi)=\tilde\chi(\xi'/\langle\xi\rangle)$, where $\tilde\chi$ is a $C_0^\infty(\mathbb{R})$ function, with small enough support, equal to one on a neighborhood of zero.

Definition 2.2.2 (Bony-Weyl quantization) *If a is a symbol in $\tilde\Gamma^m_p$, respectively in $\Gamma^m_{K,K',p}[r]$, we set, using notation (2.1.4),*

$$\begin{aligned} a_{\chi_p}(\mathcal{U};x,\xi) &= \sum_{n\in(\mathbb{N}^*)^p} \chi_p(n,\xi)a(\Pi_n\mathcal{U};x,\xi) \\ a_\chi(U;t,x,\xi) &= \mathcal{F}^{-1}_{\xi'}[\chi(\xi',\xi)\hat a(U;t,\xi',\xi)] = \chi(D,\xi)a(U;t,x,\xi) \end{aligned} \tag{2.2.23}$$

where, in the last equality, $\hat a$ stands for the Fourier transform relatively to the x variable and $\mathcal{F}^{-1}_{\xi'}$ denotes the inverse Fourier transform relatively to the ξ' variable. We get symbols a_{χ_p}, a_χ that are smooth in x. If a is a symbol in $\tilde\Gamma^m_p$, respectively in $\Gamma^m_{K,K',p}[r]$, we define its Bony-Weyl quantization as the operators

$$\begin{aligned} Op^{\rm BW}(a(\mathcal{U};\cdot)) &= Op^{\rm W}(a_{\chi_p}(\mathcal{U};\cdot)) \\ Op^{\rm BW}(a(U;t,\cdot)) &= Op^{\rm W}(a_\chi(U;t,\cdot)). \end{aligned} \tag{2.2.24}$$

Finally, if a is a symbol in $\Sigma\Gamma^m_{K,K',p}[r,N]$, that we decompose as in (2.1.12), we define its Bony-Weyl quantization as the operator

(2.2.25)

$$Op^{\rm BW}(a(U;t,\cdot)) = \sum_{q=p}^{N-1} Op^{\rm BW}(a_q(U,\dots,U;\cdot)) + Op^{\rm BW}(a_N(U;t,\cdot))\,.$$

For symbols belonging to the autonomous subclass $\Sigma\Gamma^m_{K,0,p}[r,N,{\rm aut}]$, we shall not write the time dependence in equality (2.2.25).

Remarks: • When $p=0$, elements of $\tilde\Gamma^m_0$ are constant coefficient symbols $\xi\to a(\xi)$ (see Definition 2.1.1), and their Bony-Weyl quantization $Op^{\rm BW}(a)=a(D)$ is nothing but the Fourier multiplier associated to $a(\xi)$.

• If a is a symbol in $\tilde\Gamma^m_p$ then a_{χ_p} is in $\tilde\Gamma^m_p$ as well.

• Since the symbol $a(U;t,x,\xi)$ is periodic in x, the regularized function a_χ in the second line in (2.2.23) can be written, by (2.2.4), as

$$a_\chi(U;t,x,\xi) = \chi(D,\xi)[a(x,\xi)] = \sum_{n\in\mathbb{Z}} \chi(n,\xi)\hat a(U;t,n,\xi)\frac{e^{inx}}{\sqrt{2\pi}} \tag{2.2.26}$$

where $\hat a(U;t,n,\xi)$ denotes the Fourier coefficient of a with respect to x. We can also write $a_\chi(U;t,x,\xi)=K(\cdot,\xi)*a(U;t,\cdot,\xi)$ where the kernel $K(x,\xi)=\mathcal{F}^{-1}_{\xi'}(\chi(\xi',\xi))$ satisfies, by (2.2.21)-(2.2.22),

$$\|\partial^\gamma_\xi K(\cdot,\xi)\|_{L^1(\mathbb{R})} \le C_\gamma\langle\xi\rangle^{-\gamma}\,,\quad \forall\gamma\in\mathbb{N}\,.$$

It follows that a_χ satisfies estimates of the form (2.1.10) as the symbol a.

• We have $(a_{\chi_p})^\vee = a_{\chi_p}$, $(a_\chi)^\vee = a_\chi$ and so, by (2.2.20),

$$\overline{Op^{\rm BW}(a)[u]} = Op^{\rm BW}(\bar a^\vee)[\bar u]\,. \tag{2.2.27}$$

• If the symbol $a(U;t,x)$ does not depend on ξ then $Op^{\rm BW}(a)$ is called the para-product operator for the function $a(U;t,x)$.

The above definitions are independent of the cut-off functions χ, χ_p satisfying (2.2.21), (2.2.22), up to smoothing operators that we define now.

For a family $(n_1,\dots,n_{p+1})$ of $(\mathbb{N}^*)^{p+1}$ we denote by

$$\max_2(n_1,\dots,n_{p+1})$$

the second largest among the integers $n_1,\dots,n_{p+1}$.

Definition 2.2.3 *(i)* **(p-homogeneous smoothing operator)** *Let p be in $\mathbb{N}^*$, ρ in $\mathbb{R}_+$. We denote by $\tilde{\mathcal{R}}_p^{-\rho}$ the space of $(p+1)$-linear maps from the space $\dot{H}^\infty(\mathbb{T}^1,\mathbb{C}^2)^p\times\dot{H}^\infty(\mathbb{T}^1,\mathbb{C})$ to the space $\dot{H}^\infty(\mathbb{T}^1,\mathbb{C})$, symmetric in $(U_1,\dots,U_p)$, of the form*

$$(U_1,\dots,U_{p+1})\to R(U_1,\dots,U_p)U_{p+1}$$

that satisfy the following:

There is some $\mu\geq 0$, some $C>0$ and, for any $\mathcal{U}=(U_1,\dots,U_p)$ in $\dot{H}^\infty(\mathbb{T}^1,\mathbb{C}^2)^p$, any U_{p+1} in $\dot{H}^\infty(\mathbb{T}^1,\mathbb{C})$, any $n=(n_1,\dots,n_p)$ in $(\mathbb{N}^*)^p$, any n_0,n_{p+1} in $\mathbb{N}^*$,

$$\begin{aligned}(2.2.28)\quad &\|\Pi_{n_0}R(\Pi_n\mathcal{U})\Pi_{n_{p+1}}U_{p+1}\|_{L^2}\\ &\qquad\leq C\frac{\max_2(n_1,\dots,n_{p+1})^{\rho+\mu}}{\max(n_1,\dots,n_{p+1})^\rho}\mathcal{G}^0_{0,p+1}(\Pi_n\mathcal{U},\Pi_{n_{p+1}}U_{p+1})\,.\end{aligned}$$

Moreover, if

$$(2.2.29)\qquad \Pi_{n_0}R(\Pi_{n_1}U_1,\dots,\Pi_{n_p}U_p)\Pi_{n_{p+1}}U_{p+1}\not\equiv 0$$

then there is a choice of signs $\epsilon_0,\dots,\epsilon_{p+1}\in\{-1,1\}$ such that $\sum_0^{p+1}\epsilon_j n_j=0$.

(ii) **(Non-homogeneous smoothing operator)** *Let N be in $\mathbb{N}^*$, $K'\leq K$ in $\mathbb{N}$, $r>0$. We denote by $\mathcal{R}^{-\rho}_{K,K',N}[r]$ the space of maps $(V,t,U)\to R(V;t)U$ defined on $B^K_\sigma(I,r)\times I\times C^K_*(I,\dot{H}^\sigma(\mathbb{T}^1,\mathbb{C}))$ for some $\sigma>0$, that are linear in U, and such that for any s with $s>\sigma$, there are $C>0$, $r(s)\in]0,r[$ and for any V in $B^K_\sigma(I,r(s))\cap C^K_*(I,\dot{H}^s(\mathbb{T}^1,\mathbb{C}^2))$, any U in $C^K_*(I,\dot{H}^s(\mathbb{T}^1,\mathbb{C}))$, any $0\leq k\leq K-K'$, any $t\in I$, one has the estimate*

$$\begin{aligned}(2.2.30)\quad \|\partial_t^k(R(V;t)U)(t,\cdot)\|_{\dot{H}^{s+\rho-\frac{3}{2}k}}&\leq C\sum_{k'+k''=k}\Big(\mathcal{G}^\sigma_{k'+K',N}(V,t)\mathcal{G}^s_{k'',1}(U,t)\\ &\quad+\mathcal{G}^\sigma_{k'+K',N-1}(V,t)\mathcal{G}^\sigma_{k'',1}(U,t)\mathcal{G}^s_{k'+K',1}(V,t)\Big).\end{aligned}$$

(iii) **(Smoothing operator)** *Let p,N be in $\mathbb{N}^*$, with $p\leq N$, $K'\leq K$ in $\mathbb{N}$, ρ in $\mathbb{R}_+$. We denote by $\Sigma\mathcal{R}^{-\rho}_{K,K',p}[r,N]$ the space of maps $(V,t,U)\to R(V;t)U$ that may be written as*

$$(2.2.31)\qquad R(V;t)U=\sum_{q=p}^{N-1}R_q(V,\dots,V)U+R_N(V;t)U$$

for some R_q in $\tilde{\mathcal{R}}^{-\rho}_q$, $q = p, \dots, N-1$ and R_N in $\mathcal{R}^{-\rho}_{K,K',N}[r]$ (for $p = N$ we mean that $R = R_N$ is a purely non-homogeneous smoothing operator).

(iv) **(Autonomous smoothing operators)** *We define, with the notations of (ii) and (iii) above, the class of autonomous non-homogeneous smoothing operators $\mathcal{R}^{-\rho}_{K,0,N}[r, \mathrm{aut}]$ as the subspace of $\mathcal{R}^{-\rho}_{K,0,N}[r]$ made of those maps $(U, V) \to R(U)V$ satisfying estimates (2.2.30) with $K' = 0$, the time dependence being only through $U = U(t)$. In the same way, we denote by $\Sigma\mathcal{R}^{-\rho}_{K,0,p}[r, N, \mathrm{aut}]$ the space of maps $(U, V) \to R(U)V$ of the form (2.2.31) with $K' = 0$, where the last term belongs to $\mathcal{R}^{-\rho}_{K,0,N}[r, \mathrm{aut}]$.*

We have the following simple inclusions

$$\begin{aligned}
&\rho_1 \le \rho_2\,, && \Sigma\mathcal{R}^{-\rho_2}_{K,K',p}[r,N] \subseteq \Sigma\mathcal{R}^{-\rho_1}_{K,K',p}[r,N]\,,\\
&p_1 \le p_2\,, && \Sigma\mathcal{R}^{-\rho}_{K,K',p_2}[r,N] \subseteq \Sigma\mathcal{R}^{-\rho}_{K,K',p_1}[r,N]\,,\\
&K'_1 \le K'_2\,, && \Sigma\mathcal{R}^{-\rho}_{K,K'_1,p}[r,N] \subseteq \Sigma\mathcal{R}^{-\rho}_{K,K'_2,p}[r,N]\,.
\end{aligned}$$

Remarks: • Notice that if R is in $\tilde{\mathcal{R}}^{-\rho}_p$, $p \ge N$, then $(V, U) \to R(V, \dots, V)U$ is in $\mathcal{R}^{-\rho}_{K,0,N}[r, \mathrm{aut}]$. Actually, for any $0 \le k \le K$, by the multi-linearity of R in each argument and (2.2.28), we have

$$\|\partial_t^k \Pi_{n_0} R(V, \dots, V)U\|_{L^2} \le C \sum_{\substack{n_1,\dots,n_{p+1}\\ k_1+\dots+k_{p+1}=k}} \|R(\partial_t^{k_1}\Pi_{n_1}V, \dots, \partial_t^{k_p}\Pi_{n_p}V)\partial_t^{k_{p+1}}\Pi_{n_{p+1}}U\|_{L^2}$$

$$\le C \sum_{\substack{n_1,\dots,n_{p+1}\\ k_1+\dots+k_{p+1}=k}} \frac{\max_2(n_1,\dots,n_{p+1})^{\rho+\mu}}{\max(n_1,\dots,n_{p+1})^{\rho}} \prod_1^p \|\partial_t^{k_j}\Pi_{n_j}V\|_{L^2} \|\partial_t^{k_{p+1}}\Pi_{n_{p+1}}U\|_{L^2}\,.$$

Let us bound each term in the right hand side of the above sum with for instance $n_1 \ge n_2 \ge \cdots \ge n_{p+1}$. Recalling the definition of the norms (2.1.2)-(2.1.3), we get, for $s \ge \sigma > \frac{3}{2}k$,

(2.2.32)

$$\begin{aligned}
\|\partial_t^k \Pi_{n_0} R(V, \dots, V)U\|_{L^2} \le C \sum_{n_1,\dots,n_{p+1}} & \frac{n_2^{\rho+\mu}}{n_1^{\rho-\frac{3}{2}k}} n_1^{-s} n_2^{-\sigma} \prod_3^{p+1} n_j^{-\sigma} \prod_1^{p+1} c^j_{n_j}\\
& \times \sum_{k'+k''=k} \mathcal{G}^s_{k',1}(V,t)\mathcal{G}^\sigma_{k',p-1}(V,t)\mathcal{G}^\sigma_{k'',1}(U,t)
\end{aligned}$$

where $(c^j_{n_j})_{n_j}$ stands for some ℓ^2 sequences. The assumption (2.2.29) implies that $n_0 \le (p+1)n_1$. Then, if $\sigma \ge \max\{\rho + \mu + \frac{1}{2}, \frac{3K}{2}\}$, Young inequality for convolution of sequences implies that the product of the left hand side of (2.2.32) by $n_0^{s+\rho-\frac{3}{2}k}$ gives a ℓ^2-sequence, and

$$\begin{aligned}
\|\partial_t^k R(V, \dots, V)U\|_{\dot{H}^{s+\rho-\frac{3}{2}k}} &\le C \sum_{k'+k''=k} \mathcal{G}^\sigma_{k',p-1}(V,t)\mathcal{G}^\sigma_{k'',1}(U,t)\mathcal{G}^s_{k',1}(V,t)\\
&\le C \sum_{k'+k''=k} \mathcal{G}^\sigma_{k',N-1}(V,t)\mathcal{G}^\sigma_{k'',1}(U,t)\mathcal{G}^s_{k',1}(V,t)
\end{aligned}$$

since $p \geq N$ and $\sup_{t\in I} |||V(t,\cdot)|||_{K,\sigma} < r$ is small. This is the second term in the right hand side of (2.2.30) with $K' = 0$. The first term appears when for instance $n_{p+1} \geq \cdots \geq n_1$.

• The composition of smoothing operators R_1 and R_2 belonging, respectively, to the spaces $\Sigma\mathcal{R}^{-\rho}_{K,K',p_1}[r,N]$ and $\Sigma\mathcal{R}^{-\rho}_{K,K',p_2}[r,N]$, is a smoothing operator R_1R_2 in $\Sigma\mathcal{R}^{-\rho}_{K,K',p_1+p_2}[r,N]$. A similar statement holds for the autonomous subclasses.

We study now the boundedness properties on Sobolev spaces of the paradifferential operators (2.2.24).

Proposition 2.2.4 (Action of a paradifferential operator) *(i) Let $m \in \mathbb{R}$, $p \in \mathbb{N}$. There is $\sigma > 0$ such that for any symbol a in $\tilde\Gamma^m_p$, the map*

$$(U_1,\dots,U_{p+1}) \to Op^{\rm BW}(a(U_1,\dots,U_p;\cdot))U_{p+1} \tag{2.2.33}$$

extends, for any s in $\mathbb{R}$, as a continuous $(p+1)$-linear map

$$\dot H^\sigma(\mathbb{T}^1,\mathbb{C}^2)^p \times \dot H^s(\mathbb{T}^1,\mathbb{C}) \to \dot H^{s-m}(\mathbb{T}^1,\mathbb{C})\,.$$

Moreover, there is a constant C, depending only on s and on (2.1.5) with $\alpha = \beta = 0$, such that

$$\|Op^{\rm BW}(a(\mathcal{U};\cdot))U_{p+1}\|_{\dot H^{s-m}} \leq C\mathcal{G}^\sigma_{0,p}(\mathcal{U})\|U_{p+1}\|_{\dot H^s} \tag{2.2.34}$$

where $\mathcal{U} = (U_1,\dots,U_p)$. Finally, if for some $(n_0,\dots,n_{p+1}) \in (\mathbb{N}^*)^{p+2}$,

$$\Pi_{n_0}Op^{\rm BW}(a(\Pi_n\mathcal{U};\cdot))\Pi_{n_{p+1}}U_{p+1} \not\equiv 0 \tag{2.2.35}$$

then there is a choice of the signs $\epsilon_j \in \{-1,1\}$, $j = 0,\dots,p+1$, such that $\sum_0^{p+1}\epsilon_j n_j = 0$, and the indices satisfy

$$n_0 \sim n_{p+1}\,,\quad n_j \leq C\delta n_0,\quad n_j \leq C\delta n_{p+1},\quad j = 1,\dots,p\,. \tag{2.2.36}$$

(ii) Let $r > 0$, $m \in \mathbb{R}$, $p \in \mathbb{N}$, $K' \leq K \in \mathbb{N}$, a in $\Gamma^m_{K,K',p}[r]$. There is $\sigma > 0$ and for any U in $B^K_\sigma(I,r)$, the operator $Op^{\rm BW}(a(U;t,\cdot))$ extends, for any s in $\mathbb{R}$, as a bounded linear operator from the space

$$C^{K-K'}_*(I,\dot H^s(\mathbb{T}^1,\mathbb{C})) \to C^{K-K'}_*(I,\dot H^{s-m}(\mathbb{T}^1,\mathbb{C}))\,.$$

Moreover, there is a constant C, depending only on s, r and on (2.1.10) with $0 \leq \alpha \leq 2, \beta = 0$, such that, for any t in I, any $0 \leq k \leq K - K'$,

$$\|Op^{\rm BW}(\partial_t^k a(U;t,\cdot))\|_{\mathcal{L}(\dot H^s,\dot H^{s-m})} \leq C\mathcal{G}^{\sigma_0}_{k+K',p}(U,t)\,, \tag{2.2.37}$$

so that

$$|||Op^{\rm BW}(a(U;t,\cdot))V(t)|||_{s-m} \leq C\mathcal{G}^\sigma_{K,p}(U,t)|||V(t)|||_s\,.$$

Remarks: • The relation $n_0 \sim n_{p+1}$ in (2.2.36) shows that the action of $Op^{\rm BW}(a)$ does not enlarge much the support of the Fourier transform of functions.

• Notice that the paradifferential operator $Op^{\rm BW}(a)$ acts on homogeneous spaces of functions modulo constants.

Proof: (i) Let us prove (2.2.34). Fixed $\mathcal{U} = (U_1, \dots, U_p)$ we denote by

$$b(x,\xi) = a_{\chi_p}(\mathcal{U}; x, \xi) = \sum_{n \in (\mathbb{N}^*)^p} \chi_p(n,\xi) a(\Pi_n \mathcal{U}; x, \xi) \tag{2.2.38}$$

the regularized symbol defined in (2.2.23) (without indicating the time dependence on t). Then $b(x,\xi)$ is a periodic function of x whose Fourier coefficients $\hat b(\ell,\xi) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{T}^1} e^{-i\ell x} b(x,\xi)\, dx$ ($\ell \in \mathbb{Z}$) satisfy, according to (2.1.5) for $\alpha = 0$, $\beta = 0$, the bound

$$\begin{aligned} |\hat b(\ell,\xi)| &\le C \sum_{n=(n_1,\dots,n_p)\in(\mathbb{N}^*)^p} |n|^\mu \mathcal{G}^0_{0,p}(\Pi_n \mathcal{U}) \langle \xi \rangle^m \\ &\le C \sum_{n \in (\mathbb{N}^*)^p} |n|^\mu \prod_1^p n_j^{-\sigma} \mathcal{G}^\sigma_{0,p}(\mathcal{U}) \langle \xi \rangle^m . \end{aligned} \tag{2.2.39}$$

Moreover, as $\hat b(\ell,\xi) = \widehat{\Pi_{|\ell|} b}(\ell,\xi)$, it follows from (2.1.6) that in the sum (2.2.39), the indices satisfy, for some choice of the signs $\epsilon_j \in \{-1, 1\}$,

$$\sum_{j=1}^p \epsilon_j n_j = \pm \ell \,. \tag{2.2.40}$$

If we take $\sigma - \mu > 1$, we deduce from (2.2.39) and (2.2.40) the estimate

$$|\hat b(\ell,\xi)| \le c_\ell \langle \xi \rangle^m \mathcal{G}^\sigma_{0,p}(\mathcal{U}) \tag{2.2.41}$$

for some ℓ^1-sequence $(c_\ell)_\ell$. Notice also that there is $\delta' = O(\delta)$ such that

$$\hat b(\ell,\xi) \neq 0 \quad \Longrightarrow \quad |\ell| \le \delta' \langle \xi \rangle \,. \tag{2.2.42}$$

Indeed, by (2.2.21), (2.2.23), in the sum (2.2.39), in addition to the restriction $\sum_1^p \epsilon_j n_j = \pm \ell$, the indices satisfy also $|n| \le \delta \langle \xi \rangle$. Thus $|\ell| \le C|n| \le C\delta \langle \xi \rangle$ holds on the support of $\hat b(\ell,\xi)$.

By (2.2.24) and (2.2.38) we have $Op^{\rm BW}(a(\mathcal{U};\cdot)) = Op^{\rm W}(b)$. Denoting by $\hat u(n)$ with n in $\mathbb{Z}$ the Fourier coefficients of $u \in C^\infty(\mathbb{T}^1)$, it follows from (2.2.3) that

$$Op^{\rm W}(b) u = \frac{1}{\sqrt{2\pi}} \sum_{k \in \mathbb{Z}} \Big(\sum_{n' \in \mathbb{Z}} \hat b\Big(k - n', \frac{k+n'}{2}\Big) \hat u(n') \Big) \frac{e^{ikx}}{\sqrt{2\pi}} \tag{2.2.43}$$

and, for the property on the support of $\hat b(\ell,\xi)$ proved in (2.2.42), the sum (2.2.43) is restricted to the indices

$$|k-n'|\leq \delta'\Big\langle\frac{k+n'}{2}\Big\rangle\,. \tag{2.2.44}$$

This implies, in particular, that, if u is constant, so is $Op^{\rm W}(b)u$, i.e. $Op^{\rm W}(b)$ acts on homogeneous spaces of functions modulo constants. Moreover, according to (2.2.41), (2.2.43), and the fact that, by (2.2.44), the indices in the sum (2.2.43) satisfy $k\sim n'$, the $\dot H^{s-m}(\mathbb{T}^1)$ norm of $Op^{\rm W}(b)u$ is bounded from above by the product of $C\mathcal{G}^\sigma_{0,p}(\mathcal{U})$ and of

$$\Big\|\langle k\rangle^{s-m}\sum_{n'\in\mathbb{Z}}c_{k-n'}\langle n'\rangle^m|\hat u(n')|\Big\|_{\ell^2_k}\leq C\Big\|\sum_{n'\in\mathbb{Z}}|\hat u(n')|\langle n'\rangle^s c_{k-n'}\Big\|_{\ell^2_k}\,.$$

If u is in $\dot H^s$ then the sequence $(|\hat u(n')|\langle n'\rangle^s)_{n'\in\mathbb{Z}}$ is in ℓ^2, and, since $(c_\ell)_{\ell\in\mathbb{Z}}$ is in ℓ^1, the Young inequality for sequences implies (2.2.34).

By (2.2.43) we have that, for any $n_0,n_{p+1}\in\mathbb{N}^*$,

$$\Pi_{n_0}Op^{\rm BW}(a(\Pi_n\mathcal{U};\cdot))\Pi_{n_{p+1}}U_{p+1}= \frac{1}{\sqrt{2\pi}}\sum_{|k|=n_0}\Big(\sum_{|n'|=n_{p+1}}\hat b\Big(k-n',\frac{k+n'}{2}\Big)\hat U_{p+1}(n')\Big)\frac{e^{ikx}}{\sqrt{2\pi}} \tag{2.2.45}$$

with the restriction $\pm(k-n')=\sum_1^p\epsilon_jn_j$ on the indices due to (2.2.40). Thus if (2.2.35) holds then $\sum_0^{p+1}\epsilon_jn_j=0$. Let us finally prove (2.2.36). The sum in (2.2.45) is restricted to the indices satisfying (2.2.44), so that $k\sim n'$, and therefore $n_0=|k|\sim|n'|=n_{p+1}$. Moreover $|n|\leq\delta\big\langle\frac{k+n'}{2}\big\rangle$ and the inequalities (2.2.36) follow by

$$|n|\leq C\delta|k|\leq C\delta n_0 \tag{2.2.46}$$

since $|k|=n_0$, and $n_0\sim n_{p+1}$. This concludes the proof.

(ii) The proof is similar. Setting $b(U;t,x,\xi)=\partial_t^k a_\chi(U;t,x,\xi)$ for any $0\leq k\leq K-K'$, we use that (2.1.10) with $|\alpha|\leq 2$, $\beta=0$, implies the bound

$$|\hat b(\ell,\xi)|\leq\frac{C}{\langle\ell\rangle^2}\langle\xi\rangle^m\mathcal{G}^\sigma_{k+K',p}(\mathcal{U})$$

which replaces (2.2.41). Then (2.2.37) is proved in the same way. □

Remarks: • In the above proof, we did not use any ξ derivatives of the symbol a. The statement of Proposition 2.2.4-(i) applies when $a\in\tilde\Gamma^m_p$ satisfies (2.1.5) with just $\alpha=\beta=0$, and item (ii) applies when $a\in\Gamma^m_{K,K',p}[r]$ satisfies (2.1.10) for just $|\alpha|\leq 2$, $\beta=0$.

Notice also that we did not use that $\chi_p(n,\xi)=1$ for $|n|\langle\xi\rangle^{-1}\leq\delta/2$, so that the above estimates hold with any cut-off function χ_p, χ satisfying just the support conditions introduced in (2.2.21).

• The proof of the preceding proposition shows that the definition (2.2.24) of $Op^{\rm BW}(a)$ where a is a symbol of $\tilde\Gamma^m_p$ depends on the choice of χ_p only modulo a smoothing operator in $\tilde{\mathcal R}^{-\rho}_p$ for any ρ. Actually, let $\chi_p^{(1)}$, $\chi_p^{(2)}$ be cut-off functions satisfying (2.2.21) with two different small $\delta_2>\delta_1>0$. We want to prove that, calling the difference $\chi_p=\chi_p^{(2)}-\chi_p^{(1)}$, the operator

$$R(\mathcal U)=Op^{\rm W}\Big(\sum_{n\in(\mathbb N^*)^p}\chi_p(n,\xi)a(\Pi_n\mathcal U;\cdot)\Big) \tag{2.2.47}$$

satisfies (2.2.28). Applying (2.2.34) with $s=m$ we derive

$$\|\Pi_{n_0}R(\Pi_n\mathcal U)\Pi_{n_{p+1}}U_{p+1}\|_{L^2}\leq Cn_1^\sigma\dots n_p^\sigma n_{p+1}^m\mathcal G^0_{0,p+1}(\Pi_n\mathcal U,\Pi_{n_{p+1}}U_{p+1})\,. \tag{2.2.48}$$

If $\Pi_{n_0}R(\Pi_n\mathcal U)\Pi_{n_{p+1}}U_{p+1}\neq0$, then, by (2.2.45) and since the indices in (2.2.47) satisfy $\delta_1\langle\xi\rangle\leq|n|\leq\delta_2\langle\xi\rangle$, we deduce (arguing as for (2.2.46)) that

$$C_1\delta n_{p+1}\leq|n|\leq C_2\delta n_{p+1}\,.$$

Consequently $\max_2(n_1,\dots,n_{p+1})\sim\max(n_1,\dots,n_{p+1})$ and by (2.2.48) the estimate (2.2.28) holds for any ρ and $\mu=p\sigma+m$. The property (2.2.29) follows as in the above proof of Proposition 2.2.4.

A similar statement holds for $Op^{\rm BW}(a)$ when a is in $\Gamma^m_{K,K',p}[r]$: when we compare two definitions coming from different choices of χ in (2.2.23), the difference between both operators is ρ-regularizing i.e. (2.2.30) is satisfied when σ is large enough relatively to ρ.

A homogeneous symbol a of $\tilde\Gamma^m_p$ induces the non-homogeneous symbol $a(U,\dots,U;x,\xi)$ in $\Gamma^m_{K,0,p}[r,{\rm aut}]$, and, according to (2.2.23), we can consider the regularized symbols $a_{\chi_p}(U,\dots,U;x,\xi)$ and $a_\chi(U,\dots,U;x,\xi)$. It follows that

$$Op^{\rm W}(a_{\chi_p}(U,\dots,U;x,\xi))-Op^{\rm W}(a_\chi(U,\dots,U;x,\xi))$$

is the restriction at $U_1=\dots=U_p=U$ of a smoothing operator in $\tilde{\mathcal R}^{-\rho}_p$.

• Proposition 2.2.4 implies that, if $a(U;t,\cdot)$ is in $\Sigma\Gamma^{-\rho}_{K,K',p}[r,N]\otimes\mathcal M_2(\mathbb C)$, for some $\rho\geq0$, then $Op^{\rm BW}(a(U;t,\cdot))$ defines a smoothing operator in the space $\Sigma\mathcal R^{-\rho}_{K,K',p}[r,N]\otimes\mathcal M_2(\mathbb C)$. Let us prove for example that, if $a(\mathcal U;\cdot)$ is in $\tilde\Gamma^{-\rho}_p$, then $Op^{\rm BW}(a(\mathcal U;\cdot))$ is in $\tilde{\mathcal R}^{-\rho}_p$. By (2.2.34) with $s=-\rho$ we get

$$\|\Pi_{n_0}Op^{\rm BW}(a(\Pi_n\mathcal U;\cdot))\Pi_{n_{p+1}}U_{p+1}\|_{L^2}\leq C\frac{n_1^\sigma\dots n_p^\sigma}{n_{p+1}^\rho}\mathcal G^0_{0,p+1}(\Pi_n\mathcal U,\Pi_{n_{p+1}}U_{p+1})$$

and therefore, in order to prove (2.2.28), it is sufficient to show that

$$\frac{n_1^\sigma\dots n_p^\sigma}{n_{p+1}^\rho}\leq C\frac{\max_2(n_1,\dots,n_{p+1})^{\rho+\mu}}{\max(n_1,\dots,n_{p+1})^\rho} \tag{2.2.49}$$

for some $\mu \geq 0$. If $n_{p+1} = \max(n_1, \dots, n_{p+1})$ then (2.2.49) is immediate. Otherwise $n_{p+1} \leq \max_2(n_1, \dots, n_{p+1})$ and the second inequality in (2.2.36) implies that for any $j = 1, \dots, p$, $n_j \leq C\delta n_{p+1} \leq C\delta \max_2(n_1, \dots, n_{p+1})$ and $\max(n_1, \dots, n_{p+1}) \sim n_{p+1}$. Also in this case (2.2.49) follows.

In some instances, we shall use that operators of the form $Op^{\mathrm{BW}}(a(V, \cdot))$ or $R(V)$ satisfy some boundedness properties, without having to keep track of the number of lost derivatives in a very precise way. We introduce a notation for such classes.

Definition 2.2.5 *Let p be in $\mathbb{N}$, m in $\mathbb{R}_+$, $K' \leq K$ in $\mathbb{N}$, $N \in \mathbb{N}^*$, $r > 0$.*

(i) We denote by $\widetilde{\mathcal{M}}_p^m$ the space of $(p+1)$-linear maps from the space $\dot{H}^\infty(\mathbb{T}^1, \mathbb{C}^2)^p \times \dot{H}^\infty(\mathbb{T}^1, \mathbb{C})$ to the space $\dot{H}^\infty(\mathbb{T}^1, \mathbb{C})$, symmetric in $(U_1, \dots, U_p)$, of the form

$$(U_1, \dots, U_{p+1}) \to M(U_1, \dots, U_p)U_{p+1}$$

such that there is $C > 0$ and, for any $\mathcal{U} = (U_1, \dots, U_p)$ in $\dot{H}^\infty(\mathbb{T}^1, \mathbb{C}^2)^p$, any U_{p+1} in $\dot{H}^\infty(\mathbb{T}^1, \mathbb{C})$, any n_0, n_{p+1} in $\mathbb{N}^*$, $n = (n_1, \dots, n_p)$ in $(\mathbb{N}^*)^p$,

$$\begin{aligned} (2.2.50) \quad &\|\Pi_{n_0} M(\Pi_n \mathcal{U}) \Pi_{n_{p+1}} U_{p+1}\|_{L^2} \\ &\qquad \leq C(n_0 + \cdots + n_{p+1})^m \mathcal{G}^0_{0,p+1}(\Pi_n \mathcal{U}, \Pi_{n_{p+1}} U_{p+1}). \end{aligned}$$

Moreover, we assume that if

$$(2.2.51) \qquad \Pi_{n_0} M(\Pi_{n_1} U_1, \dots, \Pi_{n_p} U_p) \Pi_{n_{p+1}} U_{p+1} \not\equiv 0$$

then there is a choice of signs $\epsilon_0, \dots, \epsilon_{p+1} \in \{-1, 1\}$ such that $\sum_0^{p+1} \epsilon_j n_j = 0$. (When $p = 0$, the above conditions just mean that M is a linear map from $\dot{H}^\infty(\mathbb{T}^1, \mathbb{C})$ to itself, satisfying estimate (2.2.50) and (2.2.51)).

(ii) We denote by $\mathcal{M}^m_{K,K',N}[r]$ the space of maps $(V, t, U) \to M(V; t)U$, defined on $B^K_\sigma(I, r) \times I \times C^K_*(I, \dot{H}^\sigma(\mathbb{T}^1, \mathbb{C}))$ for some $\sigma > 0$, that are linear in U, and such that for any s with $s > \sigma$, there are $C > 0$, $r(s) \in]0, r[$ and for any V in $B^K_\sigma(I, r(s)) \cap C^K_*(I, \dot{H}^s(\mathbb{T}^1, \mathbb{C}^2))$, any U in $C^K_*(I, \dot{H}^s(\mathbb{T}^1, \mathbb{C}))$, any $0 \leq k \leq K - K'$, any $t \in I$, one has the estimate

$$\begin{aligned} (2.2.52) \quad \|\partial_t^k (M(V; t)U)(\cdot)\|_{\dot{H}^{s - \frac{3}{2}k - m}} \leq C \sum_{k' + k'' = k} \Big(&\mathcal{G}^\sigma_{k'+K',N}(V, t) \mathcal{G}^s_{k'',1}(U, t) \\ &+ \mathcal{G}^\sigma_{k'+K',N-1}(V, t) \mathcal{G}^\sigma_{k'',1}(U, t) \mathcal{G}^s_{k'+K',1}(V, t) \Big). \end{aligned}$$

We denote by $\mathcal{M}^m_{K,0,N}[r, \mathrm{aut}]$ the subspace of $\mathcal{M}^m_{K,0,N}[r]$ made of maps $(V, U) \to M(V)U$ that satisfy estimates (2.2.52) with $K' = 0$, the time dependence being only through $V = V(t)$.

(iii) Assume $p \leq N$. We denote by $\Sigma\mathcal{M}^m_{K,K',p}[r, N]$ the space of maps $(V, t, U) \to M(V; t)U$ that may be written as

$$(2.2.53) \qquad M(V; t)U = \sum_{q=p}^{N-1} M_q(V, \dots, V)U + M_N(V; t)U$$

for some M_q *in* $\widetilde{\mathcal{M}}^m_q$, $q = p, \dots, N-1$ *and* M_N *in* $\mathcal{M}^m_{K,K',N}[r]$ *(for* $p = N$ *we mean that* $M = M_N$ *is purely non-homogeneous).*

We set $\Sigma\mathcal{M}^m_{K,0,p}[r, N, \mathrm{aut}]$ *for the subspace of* $\Sigma\mathcal{M}^m_{K,0,p}[r, N]$ *made of sums of the form (2.2.53) with* $K' = 0$ *and* M_N *taken in* $\mathcal{M}^m_{K,0,N}[r, \mathrm{aut}]$.

Finally we denote
- $\widetilde{\mathcal{M}}_p = \bigcup_m \widetilde{\mathcal{M}}^m_p$
- $\mathcal{M}_{K,K',p}[r] = \bigcup_m \mathcal{M}^m_{K,K',p}[r]$
- $\Sigma\mathcal{M}_{K,K',p}[r, N] = \bigcup_m \Sigma\mathcal{M}^m_{K,K',p}[r, N]$.

We shall use similar notations for autonomous maps.

Remarks: • If M is in $\widetilde{\mathcal{M}}^m_N$ then the map $(V, U) \to M(V, \dots, V)U$ is in $\mathcal{M}^m_{K,0,N}[r, \mathrm{aut}]$. Indeed conditions (2.2.50), (2.2.51) imply that (2.2.52) is satisfied with $K' = 0$ for $s > \sigma \gg 1$. In particular, an element of $\Sigma\mathcal{M}^m_{K,K',p}[r, N]$ sends a couple $(V, U) \in B^K_\sigma(I, r(s)) \cap C^{K-K'}_*(I, \dot{H}^s(\mathbb{T}^1, \mathbb{C}))$ to an element $M(V; t)U$ in $C^{K-K'}_*(I, \dot{H}^{s-m}(\mathbb{T}^1, \mathbb{C}))$ for $s > \sigma \gg 1$.

• By Definition 2.2.3, any smoothing operator R in $\Sigma\mathcal{R}^{-\rho}_{K,K',p}[r, N]$ defines an element of $\Sigma\mathcal{M}^m_{K,K',p}[r, N]$ for some $m \geq 0$.

• If a is in $\Sigma\Gamma^m_{K,K',p}[r, N]$, then the map $(V, U) \to Op^{\mathrm{BW}}(a(V; t, \cdot))U$ defined by (2.2.25) is in $\Sigma\mathcal{M}^m_{K,K',p}[r, N]$ according to Proposition 2.2.4.

• Let a be a homogeneous symbol in $\tilde{\Gamma}^m_p$ and let M be an operator in $\Sigma\mathcal{M}_{K,K',q}[r, N-p]$ with $p + q \leq N$. Consider the function $U \to a(U, \dots, U, M(U; t)U; t, x, \xi)$. Decomposing M as in (2.2.53), we deduce by (2.1.5), (2.1.6), (2.2.50), (2.2.51) that

$$(U_1, \dots, U_{p+q}) \to a(U_1, \dots, U_{p-1}, M_q(U_p, \dots, U_{p+q-1})U_{p+q}; x, \xi)$$

defines a homogeneous symbol in $\tilde{\Gamma}^m_{p+q}$, and by (2.1.5), (2.1.6), (2.2.52) that $U \to a(U, \dots, U, M_{N-p}(t; U)U; \cdot)$ is in $\Gamma^m_{K,K',N}[r]$.

• If M is in $\Sigma\mathcal{M}_{K,K'_1,p}[r, N]$ and $\tilde{M}$ in $\Sigma\mathcal{M}_{K,K'_2,1}[r, N-p]$, then the map $(V, t, U) \to M(V + \tilde{M}(V; t)V; t)[U]$ is in $\Sigma\mathcal{M}_{K,K',p}[r, N]$ with $K' = K'_1 + K'_2$.

• If M is in $\Sigma\mathcal{M}_{K,K',p}[r, N]$ and $\tilde{M}$ in $\Sigma\mathcal{M}_{K,K',q}[r, N]$ then the map $(V, t, U) \to \pi(M(V; t)V)\pi(\tilde{M}(V; t)U)$ is in $\Sigma\mathcal{M}_{K,K',p+q+1}[r, N]$ where π denotes the canonical identification of $\dot{H}^\sigma$ and H^σ_0, so that the product makes sense.

• Finally, if M is in $\Sigma\mathcal{M}^m_{K,K',p}[r, N]$ and M' in $\Sigma\mathcal{M}^{m'}_{K,K',q}[r, N]$, then $M(V; t) \circ M'(V; t)$, namely the map

$$(V, t, U) \to M(V; t)[M'(V; t)U]\,,$$

is in $\Sigma\mathcal{M}^{m+m'}_{K,K',p+q}[r, N]$.

We end this section with a lemma that will be useful below, and that follows from the above remarks.

Lemma 2.2.6 *Let $C(U;t,\cdot)$ be a matrix of symbols in $\Sigma\Gamma^m_{K,K',p}[r,N]\otimes\mathcal{M}_2(\mathbb{C})$ for some m in $\mathbb{R}$, $K'\leq K-1$, $0\leq p\leq N$, and assume that U is a solution of an equation*

$$D_tU=\tilde{M}(U;t)U \tag{2.2.54}$$

for some $\tilde{M}$ in the class $\Sigma\mathcal{M}_{K,1,0}[r,N]\otimes\mathcal{M}_2(\mathbb{C})$. Then $D_tC(U;t,\cdot)$ belongs to $\Sigma\Gamma^m_{K,K'+1,p}[r,N]\otimes\mathcal{M}_2(\mathbb{C})$.

Proof: Decompose according to (2.1.12)

$$C(U;t,x,\xi)=\sum_{q=p}^{N-1}C_q(U,\dots,U;x,\xi)+C_N(U;t,x,\xi)\,.$$

Consider first the non-homogeneous symbol C_N. By the second remark after Definition 2.1.2 the symbol D_tC_N belongs to $\Gamma^m_{K,K'+1,N}[r]\otimes\mathcal{M}_2(\mathbb{C})$. Consider next the homogeneous contributions

$$D_tC_q(U,\dots,U;x,\xi)=\sum_{\ell=1}^{q}C_q(\underbrace{U,\dots,D_tU}_{\ell},\dots,U;x,\xi)\,.$$

Replacing D_tU by $\tilde{M}(U;t)U$ given by (2.2.54) and using the fourth remark after the statement of Definition 2.2.5, we obtain the wanted conclusion. □

2.3 Symbolic calculus

We study now the composition properties of the preceding operators. Define the differential operator

$$\sigma(D_x,D_\xi,D_y,D_\eta)=D_\xi D_y-D_xD_\eta \tag{2.3.1}$$

where $D_x=\frac{1}{i}\partial_x$ and D_ξ, D_y, D_ξ are similarly defined.

Definition 2.3.1 (Asymptotic expansion of composition symbol) *Let $K'\leq K,\rho,p,q$ be in $\mathbb{N}$, m,m' be in $\mathbb{R}$, $r>0$.*

(i) Let a be an homogeneous symbol in $\tilde{\Gamma}^m_p$ and b in $\tilde{\Gamma}^{m'}_q$. For

$$\mathcal{U}'=(U_1,\dots,U_p)\,,\quad \mathcal{U}''=(U_{p+1},\dots,U_{p+q})$$

with U_j in $\dot{H}^\infty(\mathbb{T}^1,\mathbb{C}^2)$, $1\leq j\leq p+q$, set $\mathcal{U}=(\mathcal{U}',\mathcal{U}'')$. We define the formal series

$$(a\#b)_\infty(\mathcal{U};x,\xi) \tag{2.3.2}$$
$$=\sum_{\ell=0}^{+\infty}\frac{1}{\ell!}\Big(\frac{i}{2}\sigma(D_x,D_\xi,D_y,D_\eta)\Big)^\ell\Big[a(\mathcal{U}';x,\xi)b(\mathcal{U}'';y,\eta)\Big]_{\substack{|x=y,\\ \xi=\eta}}$$

and the sum

$$(a\#b)_\rho = \sum_{0\le\ell<\rho} \frac{1}{\ell!}\Big(\frac{i}{2}\sigma(D_x,D_\xi,D_y,D_\eta)\Big)^\ell \Big[a(\mathcal{U}';x,\xi)b(\mathcal{U}'';y,\eta)\Big]|_{\substack{x=y,\\ \xi=\eta}}$$

modulo symbols in $\tilde\Gamma^{m+m'-\rho}_{p+q}$.

(ii) Let a *be a non-homogeneous symbol in* $\Gamma^m_{K,K',p}[r]$ *and* b *in* $\Gamma^{m'}_{K,K',q}[r]$. *For* U *in* $B^K_\sigma(I,r)$ *we define when* $\rho<\sigma-\sigma_0$

$$(a\#b)_\rho(U;t,x,\xi) = \sum_{0\le\ell<\rho} \frac{1}{\ell!}\Big(\frac{i}{2}\sigma(D_x,D_\xi,D_y,D_\eta)\Big)^\ell \Big[a(U;t,x,\xi)b(U;t,y,\eta)\Big]|_{\substack{x=y,\\ \xi=\eta}} \tag{2.3.3}$$

modulo symbols in $\Gamma^{m+m'-\rho}_{K,K',p+q}[r]$.

Remarks: • We shall use several times below that the first terms of the asymptotic expansion (2.3.2) are

$$a(\mathcal{U}';\cdot)b(\mathcal{U}'';\cdot)+\frac{1}{2i}\{a(\mathcal{U}';\cdot),b(\mathcal{U}'';\cdot)\}+\cdots$$

with the usual definition of the Poisson bracket $\{a,b\}=\partial_\xi a\partial_x b-\partial_x a\partial_\xi b$.

• It follows from (2.3.2) and the third remark after Definition 2.1.1 that in (i) of the definition, the symbol $(a\#b)_\rho$ is in $\tilde\Gamma^{m+m'}_{p+q}$ modulo $\tilde\Gamma^{m+m'-\rho}_{p+q}$ (with an exponent μ in (2.1.5) for $(a\#b)_\rho$ large enough in function of ρ). This just reflects the fact that the larger ρ (i.e. the more precise the symbolic calculus), the smoother the functions U_j in the coefficients must be.

In the same way, using the remarks following Definition 2.1.3, if a is in $\Gamma^m_{K,K',p}[r]$ and b is in $\Gamma^{m'}_{K,K',q}[r]$, then the symbol $(a\#b)_\rho$ belongs to $\Gamma^{m+m'}_{K,K',p+q}[r]$ modulo $\Gamma^{m+m'-\rho}_{K,K',p+q}[r]$ (taking σ in Definition 2.1.2 large enough with respect to ρ).

• If a is in $\Sigma\Gamma^m_{K,K',p}[r,N]$, b in $\Sigma\Gamma^{m'}_{K,K',p'}[r,N]$ and c in $\Sigma\Gamma^{m''}_{K,K',p''}[r,N]$ then

$$(a\#b\#c)_\rho\in\Sigma\Gamma^{m+m'+m''}_{K,K',p+p'+p''}[r,N]$$

modulo a symbol in $\Sigma\Gamma^{m+m'+m''-\rho}_{K,K',p+p'+p''}[r,N]$.

• In the sequel we shall use the observation that if a is in $\Sigma\Gamma^m_{K,K',p}[r,N]$, b is in $\Sigma\Gamma^{m'}_{K,K',p'}[r,N]$ and c is in $\Sigma\Gamma^{m''}_{K,K',p''}[r,N]$ then

$$(a\#b\#c)_\rho+(c\#b\#a)_\rho-2abc \tag{2.3.4}$$

is a symbol in $\Sigma\Gamma^{m+m'+m''-2}_{K,K',p+p'+p''}[r,N]$.

The main goal of this section is to extend to our classes the well known result asserting that $(a\#b)_\rho$ is the symbol of the composition, up to smoothing operators.

Proposition 2.3.2 (Composition of Bony-Weyl operators) *(i) With the notations of (i) of the preceding Definition 2.3.1,*

$$Op^{\mathrm{BW}}(a(\mathcal{U}';\cdot))\circ Op^{\mathrm{BW}}(b(\mathcal{U}'';\cdot))-Op^{\mathrm{BW}}((a\#b)_\rho(\mathcal{U};\cdot)) \tag{2.3.5}$$

belongs to $\tilde{\mathcal{R}}^{-\rho+m+m'}_{p+q}$.

(ii) With the notations of (ii) of the preceding Definition 2.3.1,

$$Op^{\mathrm{BW}}(a(U;t,\cdot))\circ Op^{\mathrm{BW}}(b(U;t,\cdot))-Op^{\mathrm{BW}}((a\#b)_\rho(U;t,\cdot)) \tag{2.3.6}$$

belongs to $\mathcal{R}^{-\rho+m+m'}_{K,K',p+q}[r]$.

If a and b are symbols in the autonomous classes of Definition 2.1.3, then $(a\#b)_\rho$ is in $\Gamma^{m+m'}_{K,0,p+q}[r,\mathrm{aut}]$, modulo a symbol in $\Gamma^{m+m'-\rho}_{K,0,p+q}[r,\mathrm{aut}]$, and (2.3.6) is an autonomous smoothing operator.

Remark: According to Definition 2.3.1, the symbol $(a\#b)_\rho(\mathcal{U};\cdot)$ in case (i), resp. $(a\#b)_\rho(U;t,\cdot)$ in case (ii), is defined modulo a symbol in $\tilde{\Gamma}^{m+m'-\rho}_{p+q}$, resp. $\Gamma^{m+m'-\rho}_{K,K',p+q}[r]$. By the last remark following Proposition 2.2.4, the paradifferential operator $Op^{\mathrm{BW}}((a\#b)_\rho(\mathcal{U};\cdot))$, resp. $Op^{\mathrm{BW}}((a\#b)_\rho(U;t,\cdot))$, is thus defined modulo a smoothing remainder in $\tilde{\mathcal{R}}^{-\rho+m+m'}_{p+q}$, resp. $\mathcal{R}^{-\rho+m+m'}_{K,K',p+q}[r]$.

To prove Proposition 2.3.2, we need lemmas 2.3.3 and 2.3.4. Lemma 2.3.4 proves that the composition of $Op^{\mathrm{BW}}(a)\circ Op^{\mathrm{BW}}(b)$ can be written as the $Op^{\mathrm{W}}(c)$ of a suitable symbol $c=a\#b$ and Lemma 2.3.3 provides its asymptotic expansion.

In order to treat at the same time conditions (i) and (ii), let us introduce the following notation. Let $a(x,\xi),b(x,\xi)$ be two tempered distributions in x, smoothly depending on ξ. Assume that their x-Fourier transforms $\hat{a}(\eta,\xi)$, $\hat{b}(\eta,\xi)$ are supported for $|\eta|\leq\delta\langle\xi\rangle$ for a small enough $\delta>0$. Then the integral

$$a\#b(x,\xi)\overset{\mathrm{def}}{=}\frac{1}{(2\pi)^2}\int_{\mathbb{R}^2}e^{ix(\xi^*+\eta^*)}\hat{a}\big(\eta^*,\xi+\frac{\xi^*}{2}\big)\hat{b}\big(\xi^*,\xi-\frac{\eta^*}{2}\big)\,d\xi^*d\eta^* \tag{2.3.7}$$

is well defined as a distribution in (ξ^*,η^*), compactly supported in $|\xi^*|+|\eta^*|\leq\delta'\langle\xi\rangle$ for a small $\delta'>0$, smoothly depending on ξ and acting on the smooth function $(\xi^*,\eta^*)\to e^{ix(\xi^*+\eta^*)}$. Assume also that for some $\rho\in\mathbb{N}$, any $0\leq\alpha\leq\rho$, any $\beta\in\mathbb{N}$, we have for some constant $M_{\alpha,\beta}(\cdot)$

$$\begin{aligned}&|\partial_x^\alpha\partial_\xi^\beta a(x,\xi)|\leq M_{\alpha,\beta}(a)\langle\xi\rangle^{m-\beta}\\&|\partial_x^\alpha\partial_\xi^\beta b(x,\xi)|\leq M_{\alpha,\beta}(b)\langle\xi\rangle^{m'-\beta}.\end{aligned} \tag{2.3.8}$$

Then $a\#b$ is also given by the oscillatory integral

$$\begin{aligned}&a\#b(x,\xi)=\\&\frac{1}{\pi^2}\int_{\mathbb{R}^4}e^{-2i\sigma(x^*,\xi^*,y^*,\eta^*)}a(x+x^*,\xi+\xi^*)b(x+y^*,\xi+\eta^*)\,dx^*dy^*d\xi^*d\eta^*\end{aligned} \tag{2.3.9}$$

where $\sigma(x^*,\xi^*,y^*,\eta^*) = \xi^* y^* - x^*\eta^*$ (which has a meaning if we remember that, by the support assumptions on $\hat{a}$, $\hat{b}$, we have that $|\xi^*| + |\eta^*| \ll |\xi|$, so that, to give a meaning to (2.3.9), it is enough to perform ∂_{ξ^*} and ∂_{η^*} integrations by parts to gain x^* and y^* decay). We also have that

$$a\#b(x,\xi) = e^{\frac{i}{2}\sigma(D_x,D_\xi,D_y,D_\eta)}[a(x,\xi)b(y,\eta)]_{|x=y,\xi=\eta}\,.$$

Let us prove:

Lemma 2.3.3 *Assume that $a(x,\xi), b(x,\xi)$ satisfy (2.3.8) for some ρ in $\mathbb{N}^*$ and their x-Fourier transforms $\hat{a}(\eta,\xi)$, $\hat{b}(\eta,\xi)$ are supported for $|\eta| \le \delta\langle\xi\rangle$ for a small enough $\delta > 0$. Then for any ℓ in $\mathbb{N}$, $\ell \le \rho$,*

$$|\partial_x^\ell[(a\#b - (a\#b)_{\rho-\ell})](x,\xi)| \le C_{\rho,\ell}\langle\xi\rangle^{m+m'-(\rho-\ell)}, \tag{2.3.10}$$

where, for some universal constants $K_{\rho,\ell}$,

$$C_{\rho,\ell} = K_{\rho,\ell}\sum_{\alpha'+\alpha''=\rho}\ \sum_{\beta'+\beta''=\rho-\ell} M_{\alpha',\beta'}(a)M_{\alpha'',\beta''}(b)\,. \tag{2.3.11}$$

Proof: Notice first that $\partial_x^\ell(a\#b)$ (resp. $\partial_x^\ell(a\#b)_{\rho-\ell}$) is a linear combination of $(\partial_x^{\ell'}a)\#(\partial_x^{\ell''}b)$ (resp. $\big((\partial_x^{\ell'}a)\#(\partial_x^{\ell''}b)\big)_{\rho-\ell}$ with $\ell'+\ell''=\ell$. Consequently, it suffices to prove (2.3.10) when $\ell = 0$. Define for τ in $[0,1]$, $X^* = (x^*,\xi^*)$, $Y^* = (y^*,\eta^*)$ in $\mathbb{R}^2$, the function

$$\begin{aligned} &c(\tau,x,\xi)\\ &\quad= \frac{1}{\pi^2}\int_{\mathbb{R}^4}[a(x',\xi')b(y',\eta')]|_{\substack{x'=x+\sqrt{\tau}x^*,y'=x+\sqrt{\tau}y^*\\ \xi'=\xi+\sqrt{\tau}\xi^*,\eta'=\xi+\sqrt{\tau}\eta^*}} e^{-2i\sigma(X^*,Y^*)}\,dX^*dY^*\,. \end{aligned}$$

Notice that $c(1,x,\xi) = a\#b(x,\xi)$ is the function defined in (2.3.9). By integrations by parts, one sees that the k-th τ-derivative of $c(\tau,x,\xi)$ is

$$\begin{aligned} \partial_\tau^k c(\tau,x,\xi) = \frac{1}{\pi^2}\int_{\mathbb{R}^4}\Big(\frac{i}{2}\sigma(D_{x'},D_{\xi'},D_{y'},D_{\eta'})\Big)^k[a(x',\xi')b(y',\eta')]\\ ((x,\xi)+\sqrt{\tau}X^*,(x,\xi)+\sqrt{\tau}Y^*)e^{-2i\sigma(X^*,Y^*)}\,dX^*dY^*, \end{aligned}$$

so that Taylor formula gives, denoting the derivative $\partial_\tau^k c = c^{(k)}$,

$$a\#b(x,\xi) - (a\#b)_\rho(x,\xi) = \frac{1}{(\rho-1)!}\int_0^1 c^{(\rho)}(\tau,x,\xi)(1-\tau)^{\rho-1}\,d\tau\,.$$

We may write

$$\begin{aligned} &c^{(\rho)}(\tau,x,\xi)\\ &\qquad= \frac{1}{\pi^2}\int_{\mathbb{R}^4} e^{-2i\sigma(X^*,Y^*)}e((x,\xi)+\sqrt{\tau}X^*,(x,\xi)+\sqrt{\tau}Y^*)\,dX^*dY^* \end{aligned} \tag{2.3.12}$$

where the symbol

$$e(x',\xi',y',\eta') = \Big(\frac{i}{2}\sigma(D_{x'},D_{\xi'},D_{y'},D_{\eta'})\Big)^{\rho}[a(x',\xi')b(y',\eta')] \tag{2.3.13}$$

satisfies, recalling (2.3.8), the estimates

$$\begin{aligned}|\partial_{\xi'}^{\gamma'}\partial_{\eta'}^{\gamma''}e(x',\xi',y',\eta')| \\ \leq K_{\rho,\gamma}\sum_{\substack{\alpha'+\alpha''=\rho\\ \beta'+\beta''=\rho}} M_{\alpha',\beta'+\gamma'}(a)M_{\alpha'',\beta''+\gamma''}(b)\langle\xi'\rangle^{m-\beta'-\gamma'}\langle\eta'\rangle^{m'-\beta''-\gamma''}.\end{aligned} \tag{2.3.14}$$

The symbol e in (2.3.13) may be written as a linear combination of expressions $a_1(x',\xi')b_1(y',\eta')$, where a_1, b_1 may be expressed from derivatives of a, b. Then, in the same way as (2.3.9) may be written under the form (2.3.7), we obtain a representation of $c^{(\rho)}$ in terms of a combination of integrals of the form

$$\int_{\mathbb{R}^2} e^{ix(\xi^*+\eta^*)}\hat{a}_1\big(\eta^*,\xi+\tau\frac{\xi^*}{2}\big)\hat{b}_1\big(\xi^*,\xi-\tau\frac{\eta^*}{2}\big)\,d\xi^*d\eta^*, \tag{2.3.15}$$

where, because of the support properties of $\hat{a},\hat{b}$, the integrand is supported on $|\xi^*|+|\eta^*|\leq C\delta\langle\xi\rangle$. Therefore we may insert in the integral (2.3.15) a cut-off $\chi((\xi^*,\eta^*)/\langle\xi\rangle)$, where χ is a compactly supported $C^\infty(\mathbb{R})$ function, equal to one close to zero. Now, expressing the Fourier transforms $\hat{a}_1,\hat{b}_1$ from a_1, b_1, we get an integral of the form (2.3.12), with moreover the cut-off $\chi((\xi^*,\eta^*)/\langle\xi\rangle)$. This shows that in (2.3.12) we may insert $\chi((\xi^*,\eta^*)/\langle\xi\rangle)$ under the integral. Performing in (2.3.12) two integrations by parts in ∂_{ξ^*} and in ∂_{η^*}, we get an integrating factor $(1+\langle\xi\rangle|y^*|)^{-2}(1+\langle\xi\rangle|x^*|)^{-2}$. Finally, using (2.3.14), we bound (2.3.12) by

$$C_\rho\langle\xi\rangle^{m+m'-\rho}\int_{\substack{|\xi^*|\ll\langle\xi\rangle\\ |\eta^*|\ll\langle\xi\rangle}}(1+\langle\xi\rangle|y^*|)^{-2}(1+\langle\xi\rangle|x^*|)^{-2}\,dX^*dY^*,$$

which gives (2.3.10) with $C_{\rho,0}$ as in (2.3.11) (with other constants $M_{\alpha,\beta}$). □

Before proceeding, we make the following remarks:

Let $a(x,\xi)$ satisfy (2.3.8) with $\alpha=0$, β in $\mathbb{N}$, with x-Fourier transform $\hat{a}(\hat{x},\xi)$ supported for $|\hat{x}|\leq\delta\langle\xi\rangle$ for some small $\delta>0$. Then $\hat{a}(\hat{x},\xi)$ may be considered as an $\hat{x}$-compactly supported distribution, for each fixed ξ, which is smooth in ξ, and the Definition (2.2.2) of Op^{W} shows that, if u is in $\mathcal{S}(\mathbb{R})$

$$\widehat{Op^{\mathrm{W}}(a)u}(\xi) = \frac{1}{2\pi}\int_{\mathbb{R}}\hat{a}\big(\eta,\xi-\frac{\eta}{2}\big)\hat{u}(\xi-\eta)\,d\eta \tag{2.3.16}$$

where the integral has to be interpreted as an η-compactly supported distribution depending smoothly on ξ acting on $\eta\to\hat{u}(\xi-\eta)$. Since $|\eta|\ll\langle\xi\rangle$ on the support of (2.3.16), it follows that $\widehat{Op^{\mathrm{W}}(a)u}$ is in $\mathcal{S}(\mathbb{R})$, and setting

$$a_R(x,\xi) = a(x,\xi)\theta(\xi/R) \tag{2.3.17}$$

for some cut-off function $\theta \in C_0^\infty(\mathbb{R})$ equal to one close to zero,

$$Op^{\mathrm{W}}(a_R)u \to Op^{\mathrm{W}}(a)u \quad \text{in} \quad \mathcal{S}(\mathbb{R})\,,$$

when R goes to $+\infty$. Moreover, formula (2.3.7) shows that the x-Fourier transform of $a\#b$ is

$$\widehat{a\#b}(\eta,\xi) = \frac{1}{2\pi}\int_{\mathbb{R}} \hat{a}\big(\xi^*, \xi + \frac{\eta}{2} - \frac{\xi^*}{2}\big)\hat{b}\big(\eta - \xi^*, \xi - \frac{\xi^*}{2}\big)\, d\xi^* \tag{2.3.18}$$

and, by the support conditions $|\xi^*| \le \delta\langle \xi + \frac{\eta}{2} - \frac{\xi^*}{2}\rangle$, $|\eta - \xi^*| \le \delta\langle \xi - \frac{\xi^*}{2}\rangle$, the integrand in (2.3.18) is supported, for δ small enough, on

$$|\xi^*| + |\eta| \ll |\xi|\,. \tag{2.3.19}$$

In addition, recalling (2.3.16), (2.3.17), (2.3.18), we have that, for any u in $\mathcal{S}(\mathbb{R})$,

$$\widehat{Op^{\mathrm{W}}(a_R\#b_R)u}(\xi) = \frac{1}{(2\pi)^2}\int_{\mathbb{R}^2} \widehat{a_R}\big(\xi^*, \xi - \frac{\xi^*}{2}\big)\widehat{b_R}\big(\eta - \xi^*, \xi - \frac{\eta}{2} - \frac{\xi^*}{2}\big) \times \hat{u}(\xi - \eta)\, d\xi^* d\eta$$

where, by the conditions $|\xi^*| \le \delta\langle \xi - \frac{\xi^*}{2}\rangle$, $|\eta - \xi^*| \le \delta\langle \xi - \frac{\eta}{2} - \frac{\xi^*}{2}\rangle$, we have, on the support of the integrand,

$$\Big|\xi - \frac{\xi^*}{2}\Big| \sim |\xi - \eta| \sim \Big|\xi - \frac{\eta}{2} - \frac{\xi^*}{2}\Big|\,.$$

As a consequence, as R goes to infinity,

$$\widehat{Op^{\mathrm{W}}(a_R\#b_R)u}(\xi) \;\to\; \widehat{Op^{\mathrm{W}}(a\#b)u}(\xi) \quad \text{in} \quad \mathcal{S}'(\mathbb{R})\,.$$

We shall exploit these remarks in the proof of the following lemma.

Lemma 2.3.4 *Let a, b be as in (i) (resp. (ii)) of Definition 2.3.1. Then*

$$Op^{\mathrm{BW}}(a) \circ Op^{\mathrm{BW}}(b) = Op^{\mathrm{W}}(c) \tag{2.3.20}$$

where

$$c(\mathcal{U}; x, \xi) = a_{\chi_p}(\mathcal{U}'; x, \xi)\#b_{\chi_q}(\mathcal{U}''; x, \xi), \tag{2.3.21}$$

with a_{χ_p}, b_{χ_q} defined by the first formula (2.2.23),
(resp. where

$$c(U; t, x, \xi) = a_\chi(U; t, x, \xi)\#b_\chi(U; t, x, \xi), \tag{2.3.22}$$

with a_χ, b_χ defined by the second formula (2.2.23)).

Proof: Let us treat the case (i). By (2.2.24), we have

$$Op^{\mathrm{BW}}(a)\circ Op^{\mathrm{BW}}(b)=Op^{\mathrm{W}}(a_{\chi_p})\circ Op^{\mathrm{W}}(b_{\chi_q})\,.$$

Since a is an homogeneous symbol in $\tilde\Gamma^m_p$ and b in $\tilde\Gamma^{m'}_q$ the symbols a_{χ_p}, b_{χ_q} satisfy, by (2.1.5) and (2.2.21), (2.2.22), the following estimates: there is σ such that for any $\alpha,\beta\in\mathbb{N}$, we have

$$\begin{aligned}|\partial_x^\alpha\partial_\xi^\beta a_{\chi_p}(\mathcal{U}';x,\xi)|&\le C_{\alpha,\beta}\langle\xi\rangle^{m-\beta+\alpha}\mathcal{G}^\sigma_{0,p}(\mathcal{U}')\,,\\ |\partial_x^\alpha\partial_\xi^\beta b_{\chi_q}(\mathcal{U}'';x,\xi)|&\le C_{\alpha,\beta}\langle\xi\rangle^{m'-\beta+\alpha}\mathcal{G}^\sigma_{0,q}(\mathcal{U}'')\,.\end{aligned}$$

Moreover, by (2.1.6), the support of their space Fourier transforms

$$\mathrm{supp}\big(\widehat{a_{\chi_p}(\mathcal{U}';\cdot)}(\eta,\xi)\big)\,,\ \mathrm{supp}\big(\widehat{b_{\chi_q}(\mathcal{U}'';\cdot)}(\eta,\xi)\big)\ \subseteq\ \{|\eta|\le C\delta\langle\xi\rangle\}\tag{2.3.23}$$

where δ is the small constant in (2.2.21). As a consequence we have that $a_{\chi_p}(\mathcal{U}';x,\xi)\#b_{\chi_q}(\mathcal{U}'';x,\xi)$ is well defined as the oscillatory integral (2.3.9).

We are thus left with showing that

$$Op^{\mathrm{W}}(a_{\chi_p}(\mathcal{U}';\cdot))\circ Op^{\mathrm{W}}(b_{\chi_q}(\mathcal{U}'';\cdot))=Op^{\mathrm{W}}((a_{\chi_p}\#b_{\chi_q})(\mathcal{U};\cdot)).\tag{2.3.24}$$

If we replace a_{χ_p}, b_{χ_q} by $a_{\chi_p,R}, b_{\chi_q,R}$ defined as in (2.3.17), we obtain symbols that satisfy the assumptions of Theorem 7.9 of [32]. Indeed, by (2.1.5), for any α,β in $\mathbb{N}$, we have the estimates

$$|\partial_x^\alpha\partial_\xi^\beta a_{\chi_p,R}(\mathcal{U}';x,\xi)|\le C_{\alpha,\beta,R}\langle\xi\rangle^m\,,\quad |\partial_x^\alpha\partial_\xi^\beta a_{\chi_q,R}(\mathcal{U}'';x,\xi)|\le C_{\alpha,\beta,R}\langle\xi\rangle^{m'}\,,$$

and the symbol $a_{\chi_p,R}$, resp. $b_{\chi_q,R}$, can be regarded as a function in $S_\delta(m_1)$, resp. $S_\delta(m_2)$, for any $0\le\delta\le\frac12$, with order functions $m_1(x,\xi)=\langle\xi\rangle^m$, resp. $m_2(x,\xi)=\langle\xi\rangle^{m'}$. Theorem 7.9 of [32] with $h=1$ implies that formula (2.3.24) holds for $a_{\chi_p,R}, b_{\chi_q,R}$. By the remarks preceding the statement of the lemma, we may pass to the limit when R goes to infinity proving (2.3.24) and therefore (2.3.20). The proof of case (ii) is similar. □

Proof of Proposition 2.3.2: We prove statement (i). By lemma 2.3.4, we may write (2.3.5) as

$$Op^{\mathrm{W}}\big(a_{\chi_p}(\mathcal{U}';\cdot)\#b_{\chi_q}(\mathcal{U}'';\cdot)\big)-Op^{\mathrm{BW}}((a\#b)_\rho(\mathcal{U};\cdot))=Op^{\mathrm{W}}(r(\mathcal{U};\cdot))$$

where $r=r_1+r_2$ with

$$\begin{aligned}r_1(\mathcal{U};\cdot)&=a_{\chi_p}(\mathcal{U}';\cdot)\#b_{\chi_q}(\mathcal{U}'';\cdot)-(a_{\chi_p}\#b_{\chi_q})_\rho(\mathcal{U};\cdot)\\ r_2(\mathcal{U};\cdot)&=(a_{\chi_p}\#b_{\chi_q})_\rho(\mathcal{U};\cdot)-(a\#b)_{\rho,\chi_{p+q}}(\mathcal{U};\cdot).\end{aligned}\tag{2.3.25}$$

By (2.3.10) with $\ell=0$, (2.3.11) and (2.1.5) (recall (2.3.8)), we have

$$|r_1(\Pi_n\mathcal{U};x,\xi)|\le C|n|^{\mu+\rho}\langle\xi\rangle^{m+m'-\rho}\mathcal{G}^0_{0,p+q}(\Pi_n\mathcal{U})\,.\tag{2.3.26}$$

Notice that the symbol r_1 satisfies, as a consequence of its definition (2.3.25), (2.3.23), (2.3.18), (2.3.19), and the Definition 2.3.1 of $(a_{\chi_p}\#b_{\chi_q})_\rho$, spectral localization properties that ensure that $Op^{\mathrm{W}}(r_1)$ coincides with $Op^{\mathrm{BW}}(r_1)$, if χ_p, χ_q in the definition of r_1 have been taken with small enough support. More precisely, the space Fourier transform of r_1 is supported for $|\eta| \ll \langle\xi\rangle$, if δ in (2.2.21) is small enough. Moreover by the first remark following the proof of Proposition 2.2.4 the estimate (2.3.26) is enough to prove that $Op^{\mathrm{W}}(r_1(\Pi_n\mathcal{U};\cdot)$ satisfies (2.2.34) (with m replaced by $m+m'-\rho$ and $\sigma \sim \rho$). Therefore applying (2.2.34) with $s = m + m' - \rho$, we have, up to changing the definition of μ, for any $n = (n_1, \dots, n_{p+q})$ in $(\mathbb{N}^*)^{p+q}$, any n_{p+q+1}, the bound

$$\begin{aligned}(2.3.27)\quad &\|Op^{\mathrm{W}}(r_1(\Pi_n\mathcal{U};\cdot))\Pi_{n_{p+q+1}}U_{p+q+1}\|_{L^2}\\ &\qquad\le C|n|^{\mu+\rho}n_{p+q+1}^{m+m'-\rho}\mathcal{G}^0_{0,p+q+1}(\Pi_n\mathcal{U},\Pi_{n_{p+q+1}}U_{p+q+1})\,.\end{aligned}$$

Moreover, since $Op^{\mathrm{W}}(a_{\chi_p}(\Pi_{n'}\mathcal{U}';\cdot))$ and $Op^{\mathrm{W}}(b_{\chi_q}(\Pi_{n''}\mathcal{U}'';\cdot))$ satisfy (2.2.35) (since a_{χ_p}, b_{χ_q} satisfy (2.1.6)), the composed operator

$$Op^{\mathrm{W}}(a_{\chi_p}(\Pi_{n'}\mathcal{U}';\cdot))\circ Op^{\mathrm{W}}(b_{\chi_q}(\Pi_{n''}\mathcal{U}'';\cdot))$$

satisfies the same property as well, and, by (2.2.36) in Proposition 2.2.4, if

$$Op^{\mathrm{W}}(a_{\chi_p}(\Pi_{n'}\mathcal{U}';\cdot))\circ Op^{\mathrm{W}}(b_{\chi_q}(\Pi_{n''}\mathcal{U}'';\cdot)\Pi_{n_{p+q+1}}U_{p+q+1}\neq 0,$$

then $n = (n', n'')$ satisfies $|n| \ll n_{p+q+1}$ (if δ in (2.2.21) is small enough). A similar statement holds for

$$Op^{\mathrm{W}}\big((a_{\chi_p}\#b_{\chi_q})_\rho(\Pi_n\mathcal{U};\cdot)\big)\Pi_{n_{p+q+1}}U$$

so that when (2.3.27) does not vanish identically, we may bound its right hand side as

$$(2.3.28)\qquad C\frac{\max_2(n_1,\dots,n_{p+q+1})^{\mu+\rho}}{\max(n_1,\dots,n_{p+q+1})^{\rho-m-m'}}\mathcal{G}^0_{0,p+q+1}(\Pi_n\mathcal{U},\Pi_{n_{p+q+1}}U_{p+q+1})\,.$$

Indeed, if $\max(n_1,\dots,n_{p+q}) < n_{p+q+1}$ then (2.3.28) directly follows by (2.3.27). On the other hand, if $\max(n_1,\dots,n_{p+q}) \ge n_{p+q+1}$, then, by $|n| \ll n_{p+q+1}$, we have that $\max_2(n_1,\dots,n_{p+q+1}) \sim \max(n_1,\dots,n_{p+q+1})$ and (2.3.28) follows as well. We have proved a bound of the form (2.2.28), showing that $Op^{\mathrm{W}}(r_1)$ is a smoothing operator of $\tilde{\mathcal{R}}^{-\rho+m+m'}_{p+q}$.

We have to prove a similar result for $Op^{\mathrm{W}}(r_2(\mathcal{U};\cdot))$ where $r_2(\mathcal{U};\cdot)$ is defined in (2.3.25). Recalling Definition 2.3.1, we notice that $r_2(\Pi_n\mathcal{U};\cdot)$ is a

combination of symbols of the form, for $0 \le \ell < \rho$

$$\Big(\frac{i}{2}\sigma(D_x, D_\xi, D_y, D_\eta)\Big)^\ell \big[a(\Pi_{n'}\mathcal{U}'; x, \xi) b(\Pi_{n''}\mathcal{U}''; y, \eta) \times \chi_p(n', \xi)\chi_q(n'', \eta)\big]|_{\substack{x=y\\ \xi=\eta}}$$
$$-\chi_{p+q}(n', n'', \xi)\Big(\frac{i}{2}\sigma(D_x, D_\xi, D_y, D_\eta)\Big)^\ell \big[a(\Pi_{n'}\mathcal{U}'; x, \xi) b(\Pi_{n''}\mathcal{U}''; y, \eta)\big]|_{\substack{x=y\\ \xi=\eta}}.$$

Because of the properties (2.2.21) of $\chi_p, \chi_q, \chi_{p+q}$, the above symbol is supported in the region where

$$\delta_1\langle\xi\rangle \le |(n', n'')| \le \delta_2\langle\xi\rangle, \quad \text{thus} \quad \max(n_1, \dots, n_{p+q}) \sim \langle\xi\rangle\,,$$

and, since a is in $\tilde\Gamma^m_p$ and b in $\tilde\Gamma^{m'}_q$, it is bounded in modulus by

$$C|n|^\sigma\langle\xi\rangle^{m+m'-\rho}\mathcal{G}^0_{0,p+q}(\Pi_n\mathcal{U})$$

for some $\sigma \sim \rho$. Arguing as for r_1 we deduce that $Op^{\mathrm{W}}(r_2(\Pi_n\mathcal{U}; \cdot))$ satisfies bounds of the form (2.2.28) (with ρ replaced by $\rho - m - m'$ and $p+1$ replaced by $p+q+1$) and therefore also $Op^{\mathrm{W}}(r_2)$ is a smoothing operator of $\tilde{\mathcal{R}}^{-\rho+m+m'}_{p+q}$. This concludes the proof of (i) of the proposition.

We now prove statement (ii). By (2.3.22) and the second line of (2.2.24) we may write (2.3.6) as

$$Op^{\mathrm{W}}(r_1(U; t, \cdot) + r_2(U; t, \cdot))$$

where

$$\begin{aligned} r_1(U; t, \cdot) &= a_\chi(U; t, \cdot)\#b_\chi(U; t, \cdot) - (a_\chi\#b_\chi)_\rho(U; t, \cdot)\\ r_2(U; t, \cdot) &= (a_\chi\#b_\chi)_\rho(U; t, \cdot) - (a\#b)_{\rho,\chi}(U; t, \cdot). \end{aligned} \tag{2.3.29}$$

The symbol a is in $\Gamma^m_{K,K',p}[r]$ and b in $\Gamma^{m'}_{K,K',q}[r]$. Taking σ large enough, depending on ρ, we have that (2.1.10) holds up to the index $0 \le \alpha \le \rho$ (and similarly for b). We notice that (2.3.10), (2.3.11) and (2.1.10) imply that the symbol r_1 satisfies bounds

$$|\partial_t^k\partial_x^\ell r_1(U; t, x, \xi)| \le C\langle\xi\rangle^{m+m'-(\rho-\ell)}\mathcal{G}^\sigma_{k+K',p+q}(U, t)$$

for $0 \le \ell \le 2$. By the first remark following the proof of Proposition 2.2.4 the estimate (2.2.37) holds, i.e.

$$\|Op^{\mathrm{BW}}(\partial_t^k r_1(U; t, \cdot))\|_{\mathcal{L}(\dot H^s, \dot H^{s-m-m'+\rho-2})} \le C\mathcal{G}^{\sigma_0}_{k+K',p+q}(U, t)\,,$$

and, for any $0 \leq k \leq K - K'$,

$$\|\partial_t^k (Op^{\mathrm{W}}(r_1(U;t,\cdot))V)\|_{\dot{H}^{s-m-m'+\rho-2-\frac{3}{2}k}} \leq C \sum_{k'+k''=k} \mathcal{G}^{\sigma}_{k'+K',p+q}(U,t)\mathcal{G}^{s}_{k'',1}(V,t)$$

which implies the estimate (2.2.30) of a smoothing operator of $\mathcal{R}^{-\rho+m+m'}_{K,K',p+q}[r]$, by renaming ρ as $\rho+2$. (We used again that Weyl and Bony-Weyl quantizations coincide for the symbol at hand, since the space Fourier transform of $r_1(U;t,x,\xi)$ is supported for $|\eta| \ll |\xi|$). The study of $Op^{\mathrm{BW}}(r_2)$ is similar, as r_2 satisfies the same bounds as r_1 above.

The statement concerning autonomous classes follows directly from the proof. □

Remark: The proof shows that the remainders in (2.3.5), (2.3.6) have actually better estimates than the general bounds (2.2.28), (2.2.30) satisfied by smoothing operators. In (2.3.25) the symbols r_1, r_2 are supported so that in (2.3.27), $|n| \ll n_{p+q+1}$. Consequently, for any ρ, there is $\sigma > 0$, and for any $\mathcal{U} = (U, \ldots, U)$ with $\|U\|_{\dot{H}^\sigma}$ smaller than some r, the operator (2.3.5) is bounded from $\dot{H}^s$ to $\dot{H}^{s+\rho-(m+m')}$ for *any* s. The same assertion holds for (2.3.6).

2.4 Composition theorems

We state in this section the composition results that will be used systematically in the rest of the monograph. Let us consider symbols

$$a = \sum_{q=p}^{N} a_q \in \Sigma\Gamma^m_{K,K',p}[r,N]\,, \quad b = \sum_{q'=p'}^{N} b_{q'} \in \Sigma\Gamma^{m'}_{K,K',p'}[r,N]$$

decomposed as in (2.1.12). We define the symbol of $\Sigma\Gamma^{m+m'}_{K,K',p+p'}[r,N]$,

$$(a\#b)_{\rho,N} = \sum_{q''=p+p'}^{N-1} c_{q''}(U,\ldots,U;x,\xi) + c_N(U;t,x,\xi)\,, \tag{2.4.1}$$

where, for $q'' \leq N-1$, one sets

$$c_{q''}(\mathcal{U};x,\xi) = \sum_{q+q'=q''} (a_q\#b_{q'})_\rho(\mathcal{U};x,\xi)$$

and

$$c_N(U;t,x,\xi) = \sum_{q+q'\geq N} (a_q\#b_{q'})_\rho(U;t,x,\xi)\,,$$

the factors $a_q, b_{q'}$, $q, q' \leq N-1$, in the expression of c_N being considered as symbols of $\Gamma^m_{K,0,q}[r]$, $\Gamma^{m'}_{K,0,q'}[r]$, according to the first remark following Definition 2.1.3.

The following proposition is a direct consequence of Proposition 2.3.2.

Proposition 2.4.1 (Composition) *Let m, m' be in $\mathbb{R}$, $K' \leq K$ in $\mathbb{N}$, p, p', N in $\mathbb{N}$ with $p+p' \leq N$, ρ in $\mathbb{N}$, $r > 0$. If a is a symbol in $\Sigma\Gamma^m_{K,K',p}[r, N]$ and b is a symbol in $\Sigma\Gamma^{m'}_{K,K',p'}[r, N]$, then*

(2.4.2)
$$Op^{\mathrm{BW}}(a(U;t,\cdot)) \circ Op^{\mathrm{BW}}(b(U;t,\cdot)) = Op^{\mathrm{BW}}((a\#b)_{\rho,N}(U;t,\cdot)) + R(U;t)$$

where R is a smoothing operator in $\Sigma\mathcal{R}^{-\rho+m+m'}_{K,K',p+p'}[r, N]$. A similar statement holds for autonomous classes.

We may compose as well smoothing operators and paradifferential ones. The outcome is another smoothing operator.

Proposition 2.4.2 *Let a be a symbol in $\Sigma\Gamma^m_{K,K',p}[r, N]$ with $m \geq 0$ and R a smoothing operator in $\Sigma\mathcal{R}^{-\rho}_{K,K',p'}[r, N]$. Then the composition operators*

$$Op^{\mathrm{BW}}(a(U;t,\cdot)) \circ R(U;t)\,, \quad R(U;t) \circ Op^{\mathrm{BW}}(a(U;t,\cdot))$$

are in $\Sigma\mathcal{R}^{-\rho+m}_{K,K',p+p'}[r, N]$. A similar statement holds for autonomous classes.

Proof: We prove the proposition for $Op^{\mathrm{BW}}(a(U;t,\cdot)) \circ R(U;t)$, the other is similar. Decomposing $a = \sum_{q=p}^{N} a_q$ as in (2.1.12) and $R = \sum_{q'=p'}^{N} R_{q'}$ according to (2.2.31), we have to show, on the one hand, that

$$Op^{\mathrm{BW}}(a_q(U_1, \ldots, U_q;\cdot)) \circ R_{q'}(U_{q+1}, \ldots, U_{q+q'}) \tag{2.4.3}$$

is an homogeneous smoothing operator in $\tilde{\mathcal{R}}^{-\rho+m}_{q''}$ if $q'' = q + q' \leq N-1$ and, on the other hand, that

$$\begin{aligned}
&Op^{\mathrm{BW}}(a_q(U, \ldots, U;\cdot)) \circ R_{q'}(U, \ldots, U),\ q+q' \geq N, q, q' \leq N-1\\
&Op^{\mathrm{BW}}(a_N(U;t,\cdot)) \circ R_{q'}(U, \ldots, U),\ 0 \leq q' \leq N-1\\
&Op^{\mathrm{BW}}(a_q(U, \ldots, U;\cdot)) \circ R_N(U;t),\ 0 \leq q \leq N-1\\
&Op^{\mathrm{BW}}(a_N(U;t,\cdot)) \circ R_N(U;t)
\end{aligned} \tag{2.4.4}$$

are non-homogeneous smoothing operators in $\mathcal{R}^{-\rho+m}_{K,K',N}[r]$.

Let us first study (2.4.3). Replacing U_j by $\Pi_{n_j} U_j$ we have to consider expressions of the form

$$\begin{aligned}
\sum_{n_0'} \Pi_{n_0} Op^{\mathrm{BW}}\big(a_q(\Pi_{n_1}U_1, \ldots, \Pi_{n_q}U_q;\cdot)\big)\Pi_{n_0'}\\
\circ \Pi_{n_0'} R_{q'}\big(\Pi_{n_{q+1}}U_{q+1}, \ldots, \Pi_{n_{q+q'}}U_{q+q'}\big)\Pi_{n_{q+q'+1}}U_{q+q'+1}\,.
\end{aligned} \tag{2.4.5}$$

By (2.2.35) for a_q and (2.2.29) for $R_{q'}$, the indices in the sum (2.4.5) satisfy, for some choice of the signs ϵ_j, $j = 0, \dots, q+q'+1$,

$$\sum_{j=0}^{q} \epsilon_j n_j = n_0' , \qquad n_0' = \sum_{j=q+1}^{q+q'+1} \epsilon_j n_j . \tag{2.4.6}$$

As a consequence, (2.4.3) satisfies the corresponding condition (2.2.29). We are left with checking that (2.4.3) satisfies estimates of the form (2.2.28). Combining the bound (2.2.34) (with $s = m$) for a_q, with (2.2.28) for $R_{q'}$ we deduce, using also the second restriction to the indices in (2.4.6), that the L^2 norm of (2.4.5) is bounded by

$$C n_1^{\sigma} \dots n_q^{\sigma} |n_0'|^m \frac{\max_2(n_{q+1}, \dots, n_{q+q'+1})^{\rho+\mu}}{\max(n_{q+1}, \dots, n_{q+q'+1})^{\rho}} \mathcal{G}^0_{0,q+q'+1}(\Pi_n \mathcal{U}) \tag{2.4.7}$$

where $n = (n_1, \dots, n_{q+q'+1})$ and $\mathcal{U} = (U_1, \dots, U_{q+q'+1})$. By the first identity in (2.4.6) we bound $n_0' \le n_0 + n_1 + \dots + n_{q+1}$. By the property (2.2.36) of $Op^{\mathrm{BW}}(a_q)$, the indices in the sum (2.4.5) satisfy $n_1, \dots, n_q \ll n_0 \sim n_0'$. Moreover, the second identity in (2.4.6) implies that $n_0' \le n_{q+1} + \dots + n_{q+q'+1}$, and one deduces that

$$\begin{aligned} \max(n_{q+1}, \dots, n_{q+q'+1}) &\sim \max(n_1, \dots, n_{q+q'+1}) \\ \max(n_1, \dots, n_q) &\le C \max_2(n_1, \dots, n_{q+q'+1}) . \end{aligned} \tag{2.4.8}$$

Inserting these inequalities in (2.4.7) we get that (2.4.5) satisfies an estimate of the form (2.2.28), with ρ replaced by $\rho - m$, a new value of μ, and p replaced by $q + q'$. This proves that (2.4.5) is a smoothing operator of $\tilde{\mathcal{R}}^{-\rho+m}_{q+q'}$.

One has next to check that each operator in (2.4.4) belongs to $\mathcal{R}^{-\rho+m}_{K,K',N}[r]$. This follows from the combination of estimates (2.2.30) and (2.2.37), writing for instance $\partial_t^k[Op^{\mathrm{BW}}(a_N(U;t,\cdot)) \circ R_N(U;t)V]$ as a combination of expressions $Op^{\mathrm{BW}}(\partial_t^{k'} a_N(U;t,\cdot))[\partial_t^{k''}(R_N(U;t)V)]$ with $k' + k'' = k$. This concludes the proof. □

Let us study next "inner compositions" where we substitute to one of the coefficients of a symbol, or of a smoothing operator, one of the maps of Definition 2.2.5.

Proposition 2.4.3 *Let K_1', K_2' be integers with $K_1' + K_2' = K' \le K$, and $N, p, p' \in \mathbb{N}$ with $p + p' \le N$.*

(i) Let a be in $\tilde{\Gamma}^m_p$ and M be in $\Sigma\mathcal{M}_{K,K',p'}[r, N-p]$. Then

$$U \to a(U, \dots, U, M(U;t)U; t, x, \xi) \tag{2.4.9}$$

is in $\Sigma\Gamma^m_{K,K',p+p'}[r, N]$. If the symbol a is independent of ξ (i.e. a is in $\tilde{\mathcal{F}}_p$ according to Definition 2.5.1), so is the symbol in (2.4.9) (thus it is a

function in $\Sigma\mathcal{F}_{K,K',p}[r,N]$*). Moreover if* a *is a symbol in* $\Gamma^m_{K,K',N}[r]$ *then the symbol in (2.4.9) is in* $\Gamma^m_{K,K',N}[r]$*.*

(ii) Let R *be in* $\tilde{\mathcal{R}}^{-\rho}_p$ *and let* M *be in* $\widetilde{\mathcal{M}}^{m'}_{p'}$ *for some* $m' \geq 0$*. Then the composed operator*

$$R(U_1,\dots,U_p)M(U_{p+1},\dots,U_{p+p'}) \tag{2.4.10}$$

is in $\tilde{\mathcal{R}}^{-\rho+m'}_{p+p'}$*.*

(iii) Let R *be in* $\mathcal{R}^{-\rho}_{K,K',p}[r]$ *and let* M *be in* $\mathcal{M}^{m'}_{K,K',p'}[r]$ *for some* $m' \geq 0$*. Then the composition* $R(U;t)M(U;t)$ *is in* $\mathcal{R}^{-\rho+m'}_{K,K',p+p'}[r]$*.*

(iv) Let $R(V,W;t)$ *be a smoothing operator of* $\Sigma\mathcal{R}^{-\rho}_{K,K'_1,p}[r,N]$*, depending linearly on* W*, i.e.*

$$R(V,W;t) = \sum_{q=p}^{N-1} R_q(V,\dots,V,W) + R_N(V,W;t)$$

where R_q *belongs to* $\tilde{\mathcal{R}}^{-\rho}_q$ *and* R_N *satisfies, instead of (2.2.30), for any* $0 \leq k \leq K - K'_1$*,*

$$\begin{aligned}\|\partial_t^k R_N(V,W;t)U(t,\cdot)\|_{\dot{H}^{s+\rho-\frac{3}{2}k}} &\\ \leq C \sum_{k'+k''=k} \Big(&\mathcal{G}^\sigma_{k'+K'_1,N-1}(V,t)\mathcal{G}^\sigma_{k'+K'_1,1}(W,t)\mathcal{G}^s_{k'',1}(U,t)\\ &+ \mathcal{G}^\sigma_{k'+K'_1,N-1}(V,t)\mathcal{G}^s_{k'+K'_1,1}(W,t)\mathcal{G}^\sigma_{k'',1}(U,t)\\ &+ \mathcal{G}^\sigma_{k'+K'_1,N-2}(V,t)\mathcal{G}^\sigma_{k'+K'_1,1}(W,t)\mathcal{G}^s_{k'+K'_1,1}(V,t)\mathcal{G}^\sigma_{k'',1}(U,t)\Big).\end{aligned} \tag{2.4.11}$$

Let M *be in* $\Sigma\mathcal{M}^{m'}_{K,K'_2,p'}[r,N]$*. Then* $R(V,M(V;t)W;t)$ *is in* $\Sigma\mathcal{R}^{-\rho+m'}_{K,K',p+p'}[r,N]$*, and it is linear in* W*.*

All statements of the proposition have their counterpart for autonomous operators.

Proof: (i) is nothing but the fourth remark following Definition 2.2.5.

(ii) We may write (2.4.10) in which we replace U_j by $\Pi_{n_j}U_j$, acting on $\Pi_{n_{p+p'+1}}U_{p+p'+1}$, and composed at the left by Π_{n_0} as

$$\begin{aligned}\sum_{n'_{p+1}} &\Pi_{n_0}R(\Pi_{n_1}U_1,\dots,\Pi_{n_p}U_p)\\ &\cdot\Pi_{n'_{p+1}}M(\Pi_{n_{p+1}}U_{p+1},\dots,\Pi_{n_{p+p'}}U_{p+p'})\Pi_{n_{p+p'+1}}U_{p+p'+1}\,.\end{aligned} \tag{2.4.12}$$

By (2.2.51), the summation is limited to those n'_{p+1} satisfying $\epsilon'_{p+1}n'_{p+1} + \sum_{j=p+1}^{p+p'+1}\epsilon_j n_j = 0$ for some choice of the signs $\epsilon'_{p+1},\epsilon_j$. Moreover, by (2.2.28),

(2.2.50) and $n'_{p+1} \le C\max(n_{p+1},\dots,n_{p+p'+1})$, the L^2 norm of (2.4.12) is bounded from above by the sum in n'_{p+1} of

$$C\frac{\max_2(n_1,\dots,n_p,n'_{p+1})^{\rho+\mu}}{\max(n_1,\dots,n_p,n'_{p+1})^{\rho}}\max(n_{p+1},\dots,n_{p+p'+1})^{m'} \times \mathcal{G}^0_{0,p+p'+1}(\Pi_n\mathcal{U}) \tag{2.4.13}$$

where $n=(n_1,\dots,n_{p+p'+1})$ and $\Pi_n\mathcal{U}=(\Pi_{n_1}U_1,\dots,\Pi_{n_{p+p'+1}}U_{p+p'+1})$. Assume first that one of the indices among $n_1,\dots,n_{p+p'+1}$ is much larger than all the other ones, say n_j. If $p+1\le j\le p+p'+1$, then $n'_{p+1}\sim n_j$, so that (2.4.13) is smaller than

$$C\frac{\max_2(n_1,\dots,n_{p+p'+1})^{\rho+\mu}}{\max(n_1,\dots,n_{p+p'+1})^{\rho-m'}}\mathcal{G}^0_{0,p+p'+1}(\Pi_n\mathcal{U})\,.$$

If, on the other hand, $1\le j\le p$, we have $n_j\gg n'_{p+1}$ and a similar bound follows. Finally, if the second largest among $n_1\dots,n_{p+p'+1}$ is of the same magnitude as the largest one, i.e. $\max_2(n_1\dots,n_{p+p'+1})\sim\max(n_1\dots,n_{p+p'+1})$, a bound of the form (2.2.28) follows immediately (with μ replaced by $\mu+m'$). Finally, by (2.2.29) and (2.2.51), if (2.4.12) is non zero then $\sum_{\ell=0}^{p+p'+1}\epsilon_\ell n_\ell=0$ for some choice of signs $\epsilon_\ell\in\{-1,1\}$. Consequently, (2.4.10) belongs to $\tilde{\mathcal{R}}^{-\rho+m'}_{p+p'}$.

(iii) To study $R(U;t)\circ M(U;t)$, we use that for $0\le k\le K-K'$, we have by (2.2.30),

$$\begin{aligned}\|\partial_t^k(R(U;t)\circ M(U;t)V)\|_{\dot H^{s+\rho-m'-\frac32 k}} &\le C\sum_{k'+k''=k}\Big(\mathcal{G}^\sigma_{k'+K',p}(U,t)\mathcal{G}^{s-m'}_{k'',1}(M(U;t)V,t)\\ &\quad+\mathcal{G}^\sigma_{k'+K',p-1}(U,t)\mathcal{G}^{s-m'}_{k'+K',1}(U,t)\mathcal{G}^\sigma_{k'',1}(M(U;t)V,t)\Big)\,.\end{aligned} \tag{2.4.14}$$

Since, recalling (2.1.3), (2.1.2),

$$\mathcal{G}^{s-m'}_{k'',1}(M(U;t)V,t)=\sum_{k'''\le k''}\|\partial_t^{k'''}M(U;t)V\|_{\dot H^{s-m'-\frac32 k'''}},$$

we may use (2.2.52) to bound the first term in the right hand side of (2.4.14) by

$$C\sum_{k'+k''=k}\Big(\mathcal{G}^\sigma_{k'+K',p+p'}(U,t)\mathcal{G}^s_{k'',1}(V,t) + \mathcal{G}^\sigma_{k'+K',p+p'-1}(U,t)\mathcal{G}^\sigma_{k'',1}(V,t)\mathcal{G}^s_{k'+K',1}(U,t)\Big). \tag{2.4.15}$$

The last term in (2.4.14) satisfies a similar estimate. Therefore, for a new value of σ, (2.4.14) is bounded by (2.4.15), proving that $R(U;t)M(U;t)$ is in $\mathcal{R}^{-\rho+m'}_{K,K',p+p'}[r]$.

(iv) We have to consider on the one hand composition of multilinear terms of the form, for $q \in \{p, \dots, N-1\}$, $q' \in \{p', \dots, N-1\}$,

$$R_q(U_1, \dots, U_{q-1}, M_{q'}(U_{q+1}, \dots, U_{q+q'})U_{q+q'+1})U_q\,. \tag{2.4.16}$$

By (2.2.28), (2.2.50) we get that the L^2 norm of

$$\sum_{n'_q} \Pi_{n_0} R_q\big(\Pi_{n_1}U_1, \dots, \Pi_{n_{q-1}}U_{q-1}, \Pi_{n_{q'}} M_{q'}(\Pi_{n_{q+1}}U_{q+1}, \dots, \Pi_{n_{q+q'}}U_{q+q'})\Pi_{n_{q+q'+1}}U_{q+q'+1}\big)\Pi_{n_q}U_q \tag{2.4.17}$$

is bounded by

$$C\frac{\max_2(n_1, \dots, n_{q-1}, n'_q, n_q)^{\rho+\mu}}{\max(n_1, \dots, n_{q-1}, n'_q, n_q)^{\rho}} \max(n'_q, n_{q+1}, \dots, n_{q+q'+1})^{m'} \times \mathcal{G}^0_{0,q+q'+1}(\Pi_n \mathcal{U})\,.$$

Then following the same arguments after (2.4.13), using (2.2.51) and (2.2.29), we prove that (2.4.16) is an element of $\tilde{\mathcal{R}}^{-\rho+m'}_{p+q}$.

On the other hand, we have to study the composition of

$$R_q(V, M_N(V;t)W;t)U\,, \quad q = p, \dots, N-1\,,$$
$$R_N(V, M_{q'}(V;t)W;t)U\,, \quad q' = p', \dots, N-1\,,$$

and

$$R_N(V, M_N(V;t)W;t)U\,.$$

If one combines (2.4.11) and (2.2.52) as in the proof of (iii) above, one gets for the composition an estimate of the form (2.4.11), with ρ replaced by $\rho - m'$, for a new value of σ, and paying attention to the fact that the losses of time derivatives K'_1 for R_q and K'_2 for $M_{q'}$ cumulate. This shows that we obtain an element of $\mathcal{R}^{-\rho+m'}_{K,K',N}[r]$. □

Let us state a consequence of the preceding proposition.

Proposition 2.4.4 *Let a be a symbol in $\tilde{\Gamma}^m_p$ and M in $\Sigma\mathcal{M}_{K,K',p'}[r, N-p]$, $p + p' \le N$, as in (i) of Proposition 2.4.3. Then*

$$Op^{\mathrm{BW}}(a(U, \dots, U, W;\cdot))|_{W=M(U;t)U} = Op^{\mathrm{BW}}(c(U;t,\cdot)) + R(U;t) \tag{2.4.18}$$

where c is the symbol in $\Sigma\Gamma^m_{K,K',p+p'}[r, N]$ given by (2.4.9),

$$c(U;t,\cdot) = a(U, \dots, U, M(U;t)U;t,x,\xi)\,,$$

and $R(U;t)$ is a smoothing operator in $\Sigma\mathcal{R}^{-\rho}_{K,K',p+p'}[r, N]$ for any ρ. If M is autonomous, so is R in the right hand side of (2.4.18).

Proof: We decompose $M(U;t) = \sum_{q'=p'}^{N-p-1} M_{q'}(U,\dots,U) + M_{N-p}(U;t)$ with $M_{q'}$ in $\widetilde{\mathcal{M}}_{q'}$ and M_{N-p} in $\mathcal{M}_{K,K',N-p}[r]$. We consider first the homogeneous terms. For $q' = p',\dots,N-p-1$ we define

$$c_{q'}(U_1,\dots,U_{p+q';}\cdot) = a(U_1,\dots,U_{p-1},M_{q'}(U_p,\dots,U_{p+q'-1})U_{p+q'};\cdot)$$

that, by the fourth remark following Definition 2.2.5, is an homogeneous symbol in $\tilde{\Gamma}^m_{p+q'}$. We want to prove that the difference

$$Op^{\mathrm{BW}}(a(U_1,\dots,U_{p-1},W))|_{M_{q'}(U_p,\dots,U_{p+q'-1})U_{p+q'}} - Op^{\mathrm{BW}}(c_{q'}(U_1,\dots,U_{p+q'}))$$

is a smoothing operator in $\tilde{\mathcal{R}}^{-\rho}_{p+q'}$. Let

$$R_{q'}(U_1,\dots,U_{p+q'}) = Op^{\mathrm{W}}[a_{\chi_p}(U_1,\dots,U_{p-1},M_{q'}(U_p,\dots,U_{p+q'-1})U_{p+q'};\cdot) - c_{q',\chi_{p+q'}}(U_1,\dots,U_{p+q'};\cdot)]\,.$$

By the definition (2.2.23) of a_{χ_p} and $c_{q',\chi_{p+q'}}$ the symbol in the above expression computed at $\Pi_{n_1}U_1,\dots,\Pi_{n_{p+q'}}U_{p+q'}$ may be written as

$$\begin{aligned}(2.4.19)\quad &\sum_{n'_p} a(\Pi_{n_1}U_1,\dots,\Pi_{n_{p-1}}U_{p-1},\\ &\Pi_{n'_p}M_{q'}(\Pi_{n_p}U_p,\dots,\Pi_{n_{p+q'-1}}U_{p+q'-1})\Pi_{n_{p+q'}}U_{p+q'};x,\xi)\\ &\qquad\times[\chi_p(n_1,\dots,n_{p-1},n'_p,\xi) - \chi_{p+q'}(n_1,\dots,n_{p+q'},\xi)]\,.\end{aligned}$$

The two cut-offs above are equal to one for $n_1+\dots+n_{p+q'} \le \delta'\langle\xi\rangle$ if δ' is small enough. Indeed, for $\chi_{p+q'}$ this directly follows by (2.2.21). Moreover, on the support of the n'_p summation, (2.2.51) implies that $n'_p = \sum_{j=p}^{p+q'} \epsilon_j n_j$ for some signs $\epsilon_p,\dots,\epsilon_{p+q'}$. As a consequence

$$n_1+\dots n_{p-1}+n'_p \le n_1+\dots+n_{p+q'} \le \delta'\langle\xi\rangle$$

and, if δ' is small enough, $\chi_p(n_1,\dots,n_{p-1},n'_p,\xi) = 1$, by (2.2.21). Consequently, on (2.4.19), we have $\max(n_1,\dots,n_{p+q'}) \sim \langle\xi\rangle$. According to the remarks after the proof of Proposition 2.2.4, this implies that $R_{q'}$ is an element of $\tilde{\mathcal{R}}^{-\rho}_{p+q'}$ for any ρ.

Concerning the non-homogeneous contribution, we define

$$c_{N-p}(U;t,\cdot) = a(U,\dots,U,M_{N-p}(U;t)U;\cdot)$$

that, by the fourth remark following Definition 2.2.5, is a symbol in $\Gamma^m_{K,K',N}[r]$.

Recalling (2.2.24), the associated paradifferential operator $Op^{\mathrm{BW}}(c_{N-p})$ is equal to $Op^{\mathrm{W}}(c_{N-p,\chi})$ where, by (2.2.23),

$$\begin{aligned}(2.4.20)\quad &c_{N-p,\chi}(U;t,x,\xi) = \chi(D,\xi)[c_{N-p}(U;t,x,\xi)]\\ &= \sum_{n_1,\dots,n_p} \chi(D,\xi)[a(\Pi_{n_1}U,\dots,\Pi_{n_{p-1}}U,\Pi_{n_p}M_{N-p}(U;t)U;t,x,\xi)]\,.\end{aligned}$$

If we restrict the sum in (2.4.20) to $n_1+\cdots+n_p \geq c\langle\xi\rangle$ for some c, it follows from (2.2.34), (2.2.52) that the associated operator satisfies (2.2.30) for any ρ, as soon as σ is large enough relatively to ρ, and therefore it is a smoothing operator in $\mathcal{R}^{-\rho}_{K,K',N}[r]$. We consider next the sum in (2.4.20) restricted to the indices satisfying $n_1+\cdots+n_p < c\langle\xi\rangle$. By (2.1.6), (2.2.26) and (2.2.21) we deduce that, for c small enough,

$$\sum_{n_1+\cdots+n_p<c\langle\xi\rangle} \chi(D,\xi)[a(\Pi_{n_1}U,\dots,\Pi_{n_{p-1}}U,\Pi_{n_p}M_{N-p}(U;t)U;t,x,\xi)] = \sum_{n_1+\cdots+n_p<c\langle\xi\rangle} a(\Pi_{n_1}U,\dots,\Pi_{n_{p-1}}U,\Pi_{n_p}M_{N-p}(U;t)U;t,x,\xi)\,.$$

Consequently, by (2.2.21), the difference between the above symbol and

$$a_{\chi_p}(U,\dots,U,M(U;t)U;t,x,\xi) = \sum_{n_1,\dots,n_p} \chi_p(n_1,\dots,n_p,\xi) \times a(\Pi_{n_1}U,\dots,\Pi_{n_{p-1}}U,\Pi_{n_p}M_{N-p}(U;t)U;t,x,\xi) \tag{2.4.21}$$

is supported on indices satisfying $n_1+\cdots+n_p \geq \delta'|\xi|$ for some $\delta'>0$. In conclusion, the difference between (2.4.20) and (2.4.21) is a smoothing operator of $\mathcal{R}^{-\rho}_{K,K',N}[r]$. □

To end this subsection, we restate the Bony paralinearization formula of the composition operator, that will be used in Chapter 6.

Lemma 2.4.5 (Bony Paralinearization of the composition operator) *Let f be a smooth $\mathbb{C}$-valued function defined on a neighborhood of zero in $\mathbb{C}^2$, vanishing at zero at order $q\in\mathbb{N}^*$. There is $r>0$, a 1×2 matrix of symbols $a(U;\cdot)$ in $\Gamma^0_{K,0,q-1}[r,\mathrm{aut}]$ and a 1×2 matrix of smoothing operators $R(U)$ of $\mathcal{R}^{-\rho}_{K,0,q'}[r,\mathrm{aut}]$, $q'=\max(q-1,1)$, for any ρ, such that*

$$f(U) = Op^{\mathrm{BW}}(a(U;\cdot))U + R(U)U. \tag{2.4.22}$$

Moreover, $a(U;\cdot)$ is given by the symbol $Df(U)$ which is independent of ξ.

Remark: $Op^{\mathrm{BW}}(a(U;\cdot))$ is the para-product for the function $a(U;x) = Df(U(x))$, see Definition 2.5.1.

Proof: Notice first that if f is linear, (2.4.22) is trivial, with $R=0$, so there is no restriction in assuming that $Df(0)=0$, i.e. $q\geq 2$, since this amounts to add a linear term to f. By the paralinearization formula of Bony, we know that $f(U) = T_{Df(U)}U + R(U)U$, where $R(U)$ satisfies (2.2.30) and where

$$T_{Df(U)}U = \frac{1}{2\pi}\int e^{i(x-y)\xi}c(U;x,\xi)U(y)\,dyd\xi$$

with $c(U;x,\xi)=\chi(\langle\xi\rangle^{-1}D)[Df(U)]$ for some $\chi\in C_0^\infty(\mathbb{R})$ with small enough support, equal to one close to zero. Defining the x-periodic function $b(U;x,\xi)$ through its Fourier coefficients

$$\hat b(U;n,\xi)=\hat c\Big(U;n,\xi-\frac{n}{2}\Big),\ n\in\mathbb{Z}\,,$$

we deduce by (2.2.5) that $T_{Df(U)}U=Op^{\rm W}(b(U;\cdot))U$. We may write

$$\begin{aligned}\hat b(U;n,\xi)&=\chi\Big(n\Big\langle\xi-\frac{n}{2}\Big\rangle^{-1}\Big)\widehat{Df(U)}(n)\\&=\chi(n\langle\xi\rangle^{-1})\widehat{Df(U)}(n)+\Big(\chi\Big(n\Big\langle\xi-\frac{n}{2}\Big\rangle^{-1}\Big)-\chi(n\langle\xi\rangle^{-1})\Big)\widehat{Df(U)}(n).\end{aligned}$$

If the support of χ is small enough, we thus obtain

$$T_{Df(U)}U=Op^{\rm W}(\chi(\langle\xi\rangle^{-1}D)[Df(U)])U+R_1(U)U$$

where the first term is $Op^{\rm BW}(Df(U))U$, since the cut off $\chi(\langle\xi\rangle^{-1}D)$ satisfies the assumptions made after (2.2.21), and the second one contributes to $R(U)U$ in (2.4.22). Notice that according to the second remark after the proof of Proposition 2.2.4, the definition of $Op^{\rm BW}(Df(U))U$ depends on the choice of χ only modulo a contribution to $R_1(U)U$. □

2.5 Paracomposition

We define in this section a paracomposition operator associated to a diffeomorphism $x\to x+\beta(x)$ of $\mathbb{T}^1$. The first property of a paracomposition operator is to be a bounded map between any Sobolev spaces, requiring for $\beta(x)$ just limited smoothness. The second property is to conjugate a paradifferential operator into another para-differential one, up to smoothing remainders. Paracomposition operators have been introduced by Alinhac in [8]. We propose below an alternative definition of paracomposition operator using flows.

We first define the subspace of the functions formed by the symbols of $\Sigma\Gamma^0_{K,K',p}[r,N]$ introduced in Definition 2.1.4, which are independent of ξ.

Definition 2.5.1 (Functions) *Let $K'\le K$ be in $\mathbb{N}$, p in $\mathbb{N}$, $r>0$, N in $\mathbb{N}$ with $p\le N$. We denote by*

- *$\tilde{\mathcal{F}}_p$ (resp. $\mathcal{F}_{K,K',p}[r]$, resp. $\Sigma\mathcal{F}_{K,K',p}[r,N]$) the subspace of $\tilde\Gamma^0_p$ (resp. $\Gamma^0_{K,K',p}[r]$, resp. $\Sigma\Gamma^0_{K,K',p}[r,N]$) made of those symbols that are independent of ξ;*
- *$\Sigma\mathcal{F}^{\mathbb{R}}_{K,K',p}[r,N]$ the restriction of elements of $\Sigma\mathcal{F}_{K,K',p}[r,N]$ to the subspace $C^K_{*\mathbb{R}}(I,\dot H^\sigma(\mathbb{T}^1,\mathbb{C}^2))$ of $C^K_*(I,\dot H^\sigma(\mathbb{T}^1,\mathbb{C}^2))$ defined after (2.1.2) that are moreover* real valued*;*
- *$\Sigma\mathcal{F}_{K,0,p}[r,N,{\rm aut}]=\Sigma\mathcal{F}_{K,0,p}[r,N]\cap\Sigma\Gamma^0_{K,0,p}[r,N,{\rm aut}]$ the subspace of functions whose dependence on t enters only through $U=U(t)$.*

We now consider a diffeomorphism $x \to x + \beta(U;t,x)$ of $\mathbb{T}^1$ with a function $\beta(U;t,\cdot)$ in the previous class.

Lemma 2.5.2 (Inverse diffeomorphism) *Let $0 \le K' \le K$ be in $\mathbb{N}$ and $\beta(x) \stackrel{\rm def}{=} \beta(U;t,x)$ be a real valued function*

$$\beta(U;t,\cdot) \in \Sigma\mathcal{F}_{K,K',1}[r,N] \tag{2.5.1}$$

for U in the space $C^K_{*\mathbb{R}}(I,\dot H^\sigma(\mathbb{T}^1,\mathbb{C}^2))$. If σ is large enough, and U stays in the ball of center zero and radius r in $C^K_{*\mathbb{R}}(I,\dot H^\sigma(\mathbb{T}^1,\mathbb{C}^2))$, the map

$$\Phi_U : x \to x + \beta(U;t,x) \tag{2.5.2}$$

is, for r small enough, a diffeomorphism from $\mathbb{T}^1$ to itself, and its inverse diffeomorphism may be written as

$$\Phi_U^{-1} : y \to y + \gamma(U;t,y) \tag{2.5.3}$$

for some γ in $\Sigma\mathcal{F}^{\mathbb{R}}_{K,K',1}[r,N]$.

Proof: By Definition 2.5.1 and (2.1.10), functions of $\mathcal{F}_{K,K',1}[r]$ satisfy bounds of the form

$$|\partial_t^k\partial_x^\alpha\beta(U;t,x)| \le C\mathcal{G}^\sigma_{k+K',1}(U,t)\,,\ 0\le k\le K-K',\ \alpha\le\sigma-\sigma_0\,, \tag{2.5.4}$$

if $\sigma \ge \sigma_0 \gg 1$ and U is in $B^K_{\sigma_0}(I,r(\sigma)) \cap C^{k+K'}_*(I,\dot H^\sigma(\mathbb{T}^1,\mathbb{C}^2))$.

Look first for a function $\gamma(U;t,x)$, continuous in $(t,x)\in I\times\mathbb{T}^1$, solving the fixed point problem

$$F_\beta(\gamma) = \gamma\,,\quad \text{where}\quad F_\beta(\gamma) \stackrel{\rm def}{=} -\beta\circ(\mathrm{Id}+\gamma)\,. \tag{2.5.5}$$

By (2.5.4) applied with $\alpha=1, k=0, \sigma=\sigma_0+1$, we get

$$\|F_\beta(\gamma)-F_\beta(\gamma')\|_{L^\infty} \le C\sup_{t\in I}\mathcal{G}^{\sigma_0+1}_{K',1}(U,t)\|\gamma-\gamma'\|_{L^\infty}$$

where $\|\gamma\|_{L^\infty} = \sup_{(t,x)\in I\times\mathbb{T}^1}|\gamma(t,x)|$, and

$$\|F_\beta(\gamma)\|_{L^\infty} \le \|\beta\|_{L^\infty} + C\sup_{t\in I}\mathcal{G}^{\sigma_0+1}_{K',1}(U,t)\|\gamma\|_{L^\infty}\,.$$

If $C\sup_{t\in I}\mathcal{G}^{\sigma_0+1}_{K',1}(U,t)\le 1/2$, the contraction mapping theorem implies the existence of a unique continuous function $\gamma(U;t,x)$ solving (2.5.5) and satisfying $\|\gamma\|_{L^\infty} = 2\|\beta\|_{L^\infty} = O(\sup_{t\in I}\mathcal{G}^{\sigma_0+1}_{K',1}(U,t))$. The solution γ is actually more regular. Indeed, consider the function $G: I\times\mathbb{T}^1\times\mathbb{R}\to\mathbb{R}$ defined by

$$G(t,x,y) \stackrel{\rm def}{=} y + \beta(U;t,x+y)\,,$$

so that $G(t,x,\gamma(U;t,x))=0$. Condition (2.5.4) implies that the function β is C^1 in (t,x), thus G is C^1 in (t,x,y). Moreover

$$(\partial_y G)(t,x,y)=1+(\partial_x\beta)(U;t,x+y)\geq 1-C\mathcal{G}^{\sigma_0+1}_{K',1}(U,t)\geq\frac{1}{2}\,.$$

As a consequence, the implicit function theorem implies that the function $\gamma(U;t,x)$ is C^1 in (t,x) and

$$\begin{aligned}\partial_t\gamma(U;t,x)&=-\frac{(\partial_t\beta)(U;t,x+\gamma(U;t,x))}{1+(\partial_x\beta)(U;t,x+\gamma(U;t,x))}\\ \partial_x\gamma(U;t,x)&=-\frac{(\partial_x\beta)(U;t,x+\gamma(U;t,x))}{1+(\partial_x\beta)(U;t,x+\gamma(U;t,x))}\,.\end{aligned}$$

By induction we deduce that there exist all the derivatives $\partial_t^k\partial_x^\alpha\gamma(U;t,x)$ for any $0\leq k\leq K-K'$ and α such that $k+\alpha\leq\sigma-\sigma_0$. Moreover $\partial_t^k\partial_x^\alpha\gamma$ may be written, when $k+\alpha\geq 1$, as a combination of functions of the form

$$\begin{aligned}(2.5.6)\quad &Q\big(\partial_t\beta(U;t,x+\gamma(U;t,x)),\partial_x\beta(U;t,x+\gamma(U;t,x))\big)\\ &\qquad\times\prod_{i=1}^m(\partial_t^{p_i+\ell_i'}\partial_x^{q_i+\ell_i''}\beta)(U;t,x+\gamma(U;t,x)),\end{aligned}$$

where Q is some rational map of its arguments, m is a positive integer, and p_i,q_i,ℓ_i',ℓ_i'' are non-negative integers satisfying

$$(2.5.7)\qquad \ell_i'+\ell_i''=1\,,\quad\sum_{i=1}^m(p_i+q_i)\leq k+\alpha-1\,,\quad p_i+\ell_i'\leq k\,.$$

We shall use the interpolation inequality

$$(2.5.8)\qquad \|(\partial_x^{q_1}u_1)\cdots(\partial_x^{q_m}u_m)\|_{L^\infty}\leq C\sum_{i=1}^m\prod_{j\neq i}\|u_j\|_{L^\infty}\|u_i\|_{W^{Q,\infty}}$$

where $Q=q_1+\ldots+q_m$, which stems by Gagliardo-Nirenberg inequality

$$\begin{aligned}\|(\partial_x^{q_1}u_1)\cdots(\partial_x^{q_m}u_m)\|_{L^\infty}&\leq C\|u_1\|_{L^\infty}^{1-\frac{q_1}{Q}}\|u_1\|_{W^{Q,\infty}}^{\frac{q_1}{Q}}\ldots\|u_m\|_{L^\infty}^{1-\frac{q_m}{Q}}\|u_m\|_{W^{Q,\infty}}^{\frac{q_m}{Q}}\\ &\leq C\|u_1\|_{L^\infty}^{\sum_{j\neq1}\frac{q_j}{Q}}\|u_1\|_{W^{Q,\infty}}^{\frac{q_1}{Q}}\ldots\|u_m\|_{L^\infty}^{\sum_{j\neq m}\frac{q_j}{Q}}\|u_m\|_{W^{Q,\infty}}^{\frac{q_m}{Q}}\\ &\leq C\prod_{i=1}^m\Big(\prod_{j\neq i}\|u_j\|_{L^\infty}\|u_i\|_{W^{Q,\infty}}\Big)^{\frac{q_i}{Q}}\leq C\sum_{i=1}^m\prod_{j\neq i}\|u_j\|_{L^\infty}\|u_i\|_{W^{Q,\infty}}\,.\end{aligned}$$

Let us estimate the modulus of (2.5.6). By (2.5.4) applied with $(k,\alpha)=(1,0)$ or $(0,1)$, we have

$$\begin{aligned}&|\partial_t\beta(U;t,x+\gamma(U;t,x))|+|\partial_x\beta(U;t,x+\gamma(U;t,x))|\\ &\qquad\qquad\leq C\mathcal{G}^{\sigma_0+1}_{1+K',1}(U,t)\leq Cr\,.\end{aligned}$$

Then, (2.5.8), (2.5.7) and (2.5.4) imply that the modulus of (2.5.6) is bounded by

$$C\sum_{i=1}^{m}\prod_{j\neq i}\|\partial_t^{p_j+\ell'_j}\partial_x^{\ell''_j}\beta\|_{L^\infty}\|\partial_t^{p_i+\ell'_i}\partial_x^{\ell''_i}\beta\|_{W^{Q,\infty}}$$
$$\leq C\sum_{i=1}^{m}\prod_{j\neq i}\mathcal{G}^{\sigma_0+1}_{k+K',1}(U,t)\mathcal{G}^{\sigma_0+Q+1}_{k+K',1}(U,t)$$

and, since $Q\leq k-1+\alpha\leq K-K'+\alpha-1$ by (2.5.7), we get a bound in

$$C\mathcal{G}^{\sigma_0+1}_{k+K',m-1}(U,t)\mathcal{G}^{\sigma}_{k+K',1}(U,t)\leq C'\mathcal{G}^{\sigma}_{k+K',1}(U,t)$$

if $\sigma\geq\alpha+\sigma'_0$ where $\sigma'_0=\sigma_0+K-K'$. In conclusion, we have proved that γ satisfies bounds of the form (2.5.4) with σ_0 replaced by σ'_0, i.e. γ belongs to $\mathcal{F}_{K,K',1}[r]$.

Assume now that β is in $\Sigma\mathcal{F}^{\mathbb{R}}_{K,K',1}[r,N]$. Then

$$\beta(U;t,x)=\sum_{p'=1}^{N-1}\beta_{p'}(U,\dots,U;x)+\beta_N(U;t,x)$$

with $\beta_{p'}$ in $\tilde{\mathcal{F}}_{p'}$ and β_N in $\mathcal{F}_{K,K',N}[r]$. In particular, β is in $\mathcal{F}_{K,K',1}[r]$, so that, as proved above, there exists a solution $\gamma(U;t,x)$ in $\mathcal{F}_{K,K',1}[r]$ of the fixed point problem

$$\begin{aligned}\gamma(U;t,x)&=-\beta(U;t,x+\gamma(U;t,x))\\&=-\sum_{p'=1}^{N-1}\beta_{p'}(U,\dots,U;x+\gamma(U;t,x))+\beta_N(U;t,x+\gamma(U;t,x))\,.\end{aligned}$$

By Taylor expansion of each $\beta_{p'}(U,\dots,U;x+\gamma(U;t,x))$ at the point x, and substituting iteratively the same expression of γ, we conclude, using the remarks following Definitions 2.1.1 and 2.1.3, that γ is in $\Sigma\mathcal{F}^{\mathbb{R}}_{K,K',1}[r,N]$. Finally the function $\beta_N(U;t,x+\gamma(U;t,x))$ is in $\mathcal{F}_{K,K',N}[r]$, as it can be verified using the formula for the derivatives of composite functions. This concludes the proof of the lemma. □

We want to associate to the composition operator

$$u(x)\mapsto u(x+\beta(x))$$

a paracomposition operator. The idea is the following. We consider the family of associated composition operators

$$u(x)\mapsto u(x+\theta\beta(x))\,,\quad\theta\in\mathbb{R}\,. \tag{2.5.9}$$

The group of transformations (2.5.9) is the flow of the transport equation

$$\begin{cases} \partial_\theta u = b(U;\theta,t,x)\partial_x u \\ u(0) = u(x) \end{cases} \tag{2.5.10}$$

where the function

$$b(U;\theta,t,x) = \frac{\beta(U;t,x)}{1+\theta\beta_x(U;t,x)} \in \Sigma\mathcal{F}_{K,K',1}[r,N]\,. \tag{2.5.11}$$

Indeed, differentiating (2.5.9) we get $\partial_\theta(u(x+\theta\beta(x))) = \beta(x)u_x(x+\theta\beta(x))$. Moreover $\partial_x(u(x+\theta\beta(x))) = (1+\theta\beta_x(x))\,u_x(x+\theta\beta(x))$ and we deduce that

$$\partial_\theta(u(x+\theta\beta(x))) = \frac{\beta(x)}{1+\theta\beta_x(x)}\partial_x(u(x+\theta\beta(x)))\,,$$

namely $u(x+\theta\beta(x))$ solves (2.5.10).

The transport equation (2.5.10) can be written as $\partial_\theta u = i\mathrm{Op}(B(U;\theta,\cdot))u$ with the real valued symbol

$$B = B(U;\theta,\cdot) \stackrel{\mathrm{def}}{=} B(U;\theta,t,x,\xi) = b(U;\theta,t,x)\xi\,. \tag{2.5.12}$$

Notice that $B(U;\theta,\cdot)$ is in $\Sigma\Gamma^1_{K,K',1}[r,N]$. For simplicity of notation we shall not denote in the sequel of this section the explicit t dependence of the symbols.

As a good candidate of a para-composition operator associated to (2.5.9), we consider the flow $\Omega_{B(U)}(\theta)$ of the linear para-differential equation

$$\begin{cases} \frac{d}{d\theta}\Omega_{B(U)}(\theta) = i\mathrm{Op}^{BW}\big(B(U;\theta)\big)\Omega_{B(U)}(\theta) \\ \Omega_{B(U)}(0) = \mathrm{Id}\,. \end{cases} \tag{2.5.13}$$

The operator $\Omega_{B(U)}(1)$ has the first property required by a para-composition operator, namely to be a bounded map between any Sobolev spaces. Indeed one can prove the following:

Lemma 2.5.3 *Let $0 \le K' \le K$ be in $\mathbb{N}$ and a function $\beta(U;t,\cdot)$ in the space $\Sigma\mathcal{F}_{K,K',1}[r,N]$ which is real valued for U in the space $C^K_{*\mathbb{R}}(I,\dot{H}^\sigma(\mathbb{T}^1,\mathbb{C}^2))$. Then there is $\sigma \in \mathbb{R}_+$ and, for any $U \in C^K_{*\mathbb{R}}(I,\dot{H}^\sigma(\mathbb{T}^1,\mathbb{R}))$, system (2.5.13) has a unique solution $\Omega_{B(U)}(\theta)$ defined for all $|\theta| \le 1$. Moreover the flow operator $\Omega_{B(U)}(\theta)$ is bounded on $\dot{H}^s$ for any s, and there is $r > 0$, and, for any $s \in \mathbb{R}$, a constant $C_s > 0$ such that, for any $U \in B^K_\sigma(I,r)$, for any $W \in \dot{H}^s$ we have*

$$C_s^{-1}\|W\|_{\dot{H}^s} \le \|\Omega_{B(U)}(\theta)W\|_{\dot{H}^s} \le C_s\|W\|_{\dot{H}^s}\,. \tag{2.5.14}$$

Moreover, for any $0 \le k \le K-K'$, we have

$$\|\partial_t^k\big[\Omega_{B(U)}(\theta)W\big]\|_{\dot{H}^{s-\frac{3}{2}k}} \le C\mathcal{G}^s_{k,1}(W,t)\,. \tag{2.5.15}$$

Proof: The existence of the flow operator $\Omega_{B(U)}(\theta)$ follows from the fact that the symbol $B(U;\theta)$ defined in (2.5.12) is real. Actually, set for $\lambda > 1$,

$$B_\lambda(U;\theta,t,x,\xi) = B(U;\theta,t,x,\xi)\chi(\xi/\lambda)$$

with χ in $C_0^\infty(\mathbb{R})$ equal to one close to zero. Define for any given W in $\dot{H}^s$ for some s, the function $W_\lambda(\theta)$ as the solution of the Banach space ODE

$$\begin{aligned} \frac{d}{d\theta}W_\lambda(\theta) &= iOp^{\mathrm{BW}}(B_\lambda(U;\theta))W_\lambda(\theta) \\ W_\lambda|_{\theta=0} &= W. \end{aligned} \tag{2.5.16}$$

Let us show that we have a bound, uniform in $\theta \in [-1,1]$ and $\lambda > 1$, as

$$C_s^{-1}\|W\|_{\dot{H}^s} \leq \|W_\lambda(\theta)\| \leq C_s\|W\|_{\dot{H}^s}\,. \tag{2.5.17}$$

Set $\Lambda_s(\xi) = (1+\xi^2)^{\frac{s}{2}}$, so that $\|Op^{\mathrm{BW}}(\Lambda_s(\xi))W\|_{\dot{H}^0} = \|\Lambda_s(D)W\|_{\dot{H}^0}$ is equivalent to the Sobolev norm $\|W\|_{\dot{H}^s}$. Notice that, since $B(U;\theta)$ is a symbol of order one, Propositions 2.3.2, 2.2.4 and the remark following the proof of Proposition 2.3.2, imply that

$$[Op^{\mathrm{BW}}(\Lambda_s), Op^{\mathrm{BW}}(B_\lambda(U;\theta))]Op^{\mathrm{BW}}(\Lambda_s^{-1})$$

is bounded on $\dot{H}^0$ for any s, with operator norm $O(\|U\|_{\dot{H}^\sigma})$ for some large enough σ, uniformly for $\lambda > 1$. As (2.5.16) implies

$$\begin{aligned} \frac{d}{d\theta}\Lambda_s(D)W_\lambda(\theta) = iOp^{\mathrm{BW}}(B_\lambda(U;\theta))[\Lambda_s(D)W_\lambda(\theta)] \\ + O_{L^2}(\|\Lambda_s(D)W_\lambda(\theta)\|_{L^2}\|U\|_{\dot{H}^\sigma}), \end{aligned}$$

we deduce (2.5.17) from the L^2 energy inequality

$$\|\Lambda_s(D)W_\lambda(\theta)\|_{L^2} \leq \|\Lambda_s(D)W\|_{L^2} + C\|U\|_{\dot{H}^\sigma}\left|\int_0^\theta \|\Lambda_s(D)W_\lambda(\theta')\|_{L^2}\,d\theta'\right|.$$

The estimate is uniform in λ since $Op^{\mathrm{BW}}(B_\lambda) - Op^{\mathrm{BW}}(B_\lambda)^*$ is uniformly bounded on L^2, as a consequence of the fact that the symbol B_λ is a real.

We thus get a family of functions $(W_\lambda)_\lambda$ uniformy bounded in the space $C^0([-1,1],\dot{H}^s)$ that is equicontinuous in $C^0([-1,1],\dot{H}^{s'})$ for any $s' < s$. By Ascoli theorem, we may extract a subsequence that converges in the latter space to a limit W that solves (2.5.16) with B_λ replaced by B and provides the definition of $\Omega_{B(U)}(\theta)W = W(\theta)$. Since

$$\frac{\partial}{\partial\theta}\|\Omega_{B(U)}(\theta)W\|_{\dot{H}^s}^2 = 2\mathrm{Re}\,\langle iOp^{\mathrm{BW}}(B(U;\theta))\Omega_{B(U)}(\theta)W, \Omega_{B(U)}(\theta)W\rangle_{\dot{H}^s}$$

the fact that $Op^{\mathrm{BW}}(B(U;\theta))$ is a self-adjoint operator of order one, and symbolic calculus, imply that $\|\Omega_{B(U)}(\theta)W\|_{\dot{H}^s}$ converges to $\|\Omega_{B(U)}(\theta')W\|_{\dot{H}^s}$,

when θ goes to θ', from which one deduces easily that $\Omega_{B(U)}(\theta)W$ is continuous on $[-1,1]$ with values in $\dot{H}^s$. Let us notice that (2.5.17) applied to W instead of W_λ shows that (2.5.14) holds true.

By a similar reasoning the norms $\|\sum_{k'=0}^k \partial_t^{k'}\cdot\|_{\dot{H}^{s-\frac{3}{2}k'}}$ satisfy an analogous inequality. More precisely, if we write

$$\frac{d}{d\theta}\partial_t\Omega_{B(U)}(\theta) = iOp^{\mathrm{BW}}(B(U;\theta))\partial_t\Omega_{B(U)}(\theta) + iOp^{\mathrm{BW}}(\partial_t B(U;\theta))\Omega_{B(U)}(\theta)$$

we deduce by Duhamel principle that

$$\partial_t\Omega_{B(U)}(\theta) = \Omega_{B(U)}(\theta)\int_0^\theta \Omega_{B(U)}(\tau)^{-1} iOp^{\mathrm{BW}}(\partial_t B(U;\theta))\Omega_{B(U)}(\tau)\,d\tau\,.$$

Applying the estimate (2.5.14) for $\Omega_{B(U)}(\theta)$ and for $\Omega_{B(U)}(\theta)^{-1}$, and using Proposition 2.2.4 (in particular (2.2.37)), we get that

$$\|(\partial_t\Omega_{B(U)}(\theta))W\|_{\dot{H}^{s-1}} \le C\|W\|_{\dot{H}^s}\mathcal{G}^\sigma_{1+K',1}(U,t)\,.$$

Since $1 \le \frac{3}{2}$ the estimate (2.5.15) for $k=1$ follows. Iterating the above reasoning we get, for any $1 \le k \le K-K'$, the estimate

$$\|(\partial_t^k\Omega_{B(U)}(\theta))W\|_{\dot{H}^{s-k}} \le C\|W\|_{\dot{H}^s}\mathcal{G}^\sigma_{k+K',1}(U,t)$$

and, since $U \in B^K_\sigma(I,r)$, we deduce (2.5.15). □

We prove below that, conjugating a para-differential operator

$$\mathrm{Op}^{BW}(a(U;\cdot))\,,\qquad a(U;\cdot)\in\Sigma\Gamma^m_{K,K',q}[r,N]\,, \tag{2.5.18}$$

with the flow $\Omega_B(\theta)$, we still get a para-differential operator. The conjugated operator

$$A(U)(\theta) := \Omega_{B(U)}(\theta)\mathrm{Op}^{BW}(a(U;\cdot))\big(\Omega_{B(U)}(\theta)\big)^{-1} \tag{2.5.19}$$

solves the Heisenberg equation

$$\begin{cases} \frac{d}{d\theta}A(U)(\theta) = i\big[\mathrm{Op}^{BW}\big(B(U;\theta)\big), A(U)(\theta)\big] \\ A(U)(0) = \mathrm{Op}^{BW}(a(U;\cdot))\,. \end{cases} \tag{2.5.20}$$

We now solve (2.5.20) in decreasing orders showing that it admits an approximate solution of the form

$$A(U)(\theta) = \mathrm{Op}^{BW}(a_0(\theta) + a_1(\theta) + \cdots)$$

with

$$a_0(\theta)\in\Sigma\Gamma^m_{K,K',q}[r,N]\,,\quad a_1(\theta)\in\Sigma\Gamma^{m-2}_{K,K',q+1}[r,N]\,,\ \ldots$$

and $a_0(\theta)$ is given in (2.5.30). In particular we will prove that

(2.5.21)
$$a_0(U;1,t,x,\xi) = a\Big(U;t,x+\beta(U;t,x),\xi\big(1+\partial_y\gamma(U;t,y)\big)_{|y=x+\beta(U;t,x)}\Big)$$

which is the usual formula in Alinhac para-composition theorem, see Theorem 2.5.8. Notice that the symbol a_1 has order $m-2$, twice smaller than a_0. This is an outcome of the Weyl quantization.

Before starting the proof, we first check the stability of our classes of symbols under composition by a change of variables.

Lemma 2.5.4 *Let $K' \le K$ in $\mathbb{N}$, m in $\mathbb{R}$, p in $\mathbb{N}$, N in $\mathbb{N}$ with $p \le N$, $r > 0$ small enough. Consider a symbol a in $\Sigma\Gamma^m_{K,K',p}[r,N]$ and functions b, c in $\Sigma\mathcal{F}^{\mathbb{R}}_{K,K',1}[r,N]$. Then*

$$a\big(V;t,x+b(V;t,x),\xi(1+c(V;t,x))\big) \tag{2.5.22}$$

is in $\Sigma\Gamma^m_{K,K',p}[r,N]$. In particular, if the symbol a is independent of ξ, i.e. a is a function in $\Sigma\mathcal{F}_{K,K',p}[r,N]$, we obtain that $a(V;t,x+b(V;t,x))$ is in $\Sigma\mathcal{F}_{K,K',p}[r,N]$.

Proof: We decompose

$$\begin{aligned} a &= \sum_{q=p}^{N-1} a_q(V,\dots,V;\cdot) + a_N(V;t,\cdot)\\ b &= \sum_{q'=1}^{N-1} b_{q'}(V,\dots,V;\cdot) + b_N(V;t,\cdot)\\ c &= \sum_{q''=1}^{N-1} c_{q''}(V,\dots,V;\cdot) + c_N(V;t,\cdot) \end{aligned}$$

with a_q (resp. $b_{q'}$, resp. $c_{q''}$) in $\tilde\Gamma^m_q$ (resp. $\tilde{\mathcal{F}}_{q'}$, resp. $\tilde{\mathcal{F}}_{q''}$) and a_N (resp. b_N, c_N) in $\Gamma^m_{K,K',N}[r]$ (resp. $\mathcal{F}_{K,K',N}[r]$). By a Taylor expansion at order N in (x,ξ), we may write (2.5.22) as a linear combination of terms of the form

$$(\partial_x^\alpha\partial_\xi^\beta a)(V;t,x,\xi)b(V;t,x)^\alpha\xi^\beta c(V;t,x)^\beta \tag{2.5.23}$$

with α,β in $\mathbb{N}$, $0 \le \alpha+\beta \le N-1$, plus an integral remainder

$$\int_0^1 (1-\lambda)^{N-1}(\partial_x^\alpha\partial_\xi^\beta a)(V;t,x+\lambda b(V;t,x),\xi(1+\lambda c(V;t,x))\,d\lambda \times b(V;t,x)^\alpha(\xi c(V;t,x))^\beta \tag{2.5.24}$$

with $\alpha+\beta = N$. Substituting in (2.5.23) the homogeneous and non-homogeneous symbols in the expansion of a,b,c, we obtain the following contributions:

• For any α, β in $\mathbb{N}$, $0 \leq \alpha + \beta \leq N-1$,

$$(\partial_x^\alpha \partial_\xi^\beta a_q)(V;x,\xi) b_{q'}(V;x)^\alpha \xi^\beta c_{q''}(V;x)^\beta \tag{2.5.25}$$

with indices $p \leq q \leq N-1$, $1 \leq q', q'' \leq N-1$. By the remarks following Definition 2.1.1, (2.5.25) is the restriction to $V_1 = \cdots = V_{\tilde{q}} = V$ of an homogeneous symbol in $\tilde{\Gamma}^m_{\tilde{q}}$ with $\tilde{q} = q + q'\alpha + q''\beta \geq q$. If $\tilde{q} \leq N-1$ this gives an homogeneous term in the expansion of (2.5.22). If $\tilde{q} \geq N$, it gives a non-homogeneous symbol in $\Gamma^m_{K,0,N}[r] \subset \Gamma^m_{K,K',N}[r]$.

• Symbols $(\partial_x^\alpha \partial_\xi^\beta a_q) b^\alpha_{q'} \xi^\beta c^\beta_{q''}$ where $\alpha + \beta \leq N-1$, but with at least one index among q, q', q'' equal to N. By the remarks following Definition 2.1.3, $(\partial_x^\alpha \partial_\xi^\beta a_q)\xi^\beta$ is in $\Gamma^m_{K,K',q}[r]$ and $b^\alpha_{q'} c^\beta_{q''}$ is in $\Gamma^0_{K,K',\alpha q' + \beta q''}[r]$, so that their product gives a contribution in $\Gamma^m_{K,K',N}[r]$.

• Also the integral remainder (2.5.24) gives a contribution to $\Gamma^m_{K,K',N}[r]$. Indeed the symbol $\xi^\beta b^\alpha c^\beta$ is in $\Gamma^\beta_{K,K',N}[r]$ and it is sufficient to prove that $(\partial_x^\alpha \partial_\xi^\beta a)(V;t,x+\lambda b(V;t,x), \xi(1+\lambda c(V;t,x))$ is in $\Gamma^{m-\beta}_{K,K',0}[r]$, uniformly in $\lambda \in [0,1]$. This can be verified by the formula for the derivative of the composition of functions.

This concludes the proof. □

We use the following asymptotic expansion formula for the commutator, which is derived by the composition results of para-differential operators for the Weyl quantization.

Lemma 2.5.5 (Commutator expansion) *Let a be in $\Sigma\Gamma^m_{K,K',q}[r,N]$ and B in $\Sigma\Gamma^1_{K,K',1}[r,N]$. Then*

$$\begin{aligned}\left[\mathrm{Op}^{BW}(iB(U;\cdot)), \mathrm{Op}^{BW}(a)\right] &= \mathrm{Op}^{BW}(\{B(U;\cdot), a\}) \\ &\quad + \mathrm{Op}^{BW}\left(r_{-3,\rho}(B,a)\right) + R\end{aligned} \tag{2.5.26}$$

with symbols

$$\{B(U;\cdot), a\} \in \Sigma\Gamma^m_{K,K',q+1}[r,N]\,,\ r_{-3,\rho}(B,a) \in \Sigma\Gamma^{m-2}_{K,K',q+1}[r,N]\,, \tag{2.5.27}$$

and a smoothing remainder

$$R \in \Sigma\mathcal{R}^{-\rho+m+1}_{K,K',q+1}[r,N]\,. \tag{2.5.28}$$

Note that the symbol $r_{-3,\rho}(B,a)$ has order $m-2$. If B is a bounded family of symbols depending on some parameter $\theta \in [-1,1]$, then the above conclusion holds uniformly in θ.

Proof: It follows by Proposition 2.4.1 and formulas (2.3.2) and (2.3.3). Note that the terms of even (resp. odd) rank in the asymptotic expansion of the

symbol $a\#B$ in the Weyl quantization are symmetric (resp. antisymmetric) in (a, B). Consequently the terms of even rank vanish in the symbol of the commutator. □

By (2.5.26), the equation (2.5.20) is solved, at the highest order, by

$$A_0(U)(\theta) = \mathrm{Op}^{BW}(a_0(U;\theta,\cdot))$$

where $a_0(U;\theta,\cdot) = a_0(\theta)$ is the solution of the transport equation

$$\begin{cases} \frac{d}{d\theta} a_0(\theta) = \{B(U;\cdot), a_0(\theta)\} \\ a_0(U)(0) = a(U;\cdot)\,. \end{cases} \tag{2.5.29}$$

This transport equation may be solved by the methods of characteristics.

Lemma 2.5.6 *The solution of the transport equation (2.5.29) is*

$$a_0(U;\theta,x,\xi) = a(U;\phi^{\theta,0}(x,\xi)) \tag{2.5.30}$$

where

$$\phi^{\theta,0}(x,\xi) = \Big(x + \theta\beta(U;x), \xi\big(1 + \partial_y\gamma(U;\theta,y)\big)_{|y=x+\theta\beta(U;x)}\Big) \tag{2.5.31}$$

are the solutions of the characteristic system

$$\begin{cases} \frac{d}{ds} x(s) = -b(s,x(s)) \\ \frac{d}{ds}\xi(s) = b_x(s,x(s))\xi(s)\,. \end{cases} \tag{2.5.32}$$

In particular for $\theta = 1$ we get (2.5.21). Moreover $a_0(\theta,\cdot) \in \Sigma\Gamma^m_{K,K',q}[r,N]$, with estimates uniform in $|\theta| \le 1$.

Proof: The function $s \mapsto a_0(s,x(s),\xi(s))$ is constant along the solutions of the Hamiltonian system generated by $b(s,x)\xi$, namely (2.5.32). Hence, denoting by $\phi^{\theta_0,\theta}(x,\xi)$ the solution of (2.5.32) with initial condition $\phi^{\theta_0,\theta_0}(x,\xi) = (x,\xi)$, the solution of the transport equation (2.5.29) is

$$a_0(\theta,x,\xi) = a(\phi^{\theta,0}(x,\xi))$$

proving (2.5.30). The solutions of the Hamiltonian system (2.5.32) are directly given by the path of diffeomorphisms

$$y = x + \theta\beta(x) \quad \Leftrightarrow \quad x = y + \gamma(\theta,y)\,, \quad \theta \in [-1,1]\,.$$

Indeed, recalling the definition of $b(\theta,x)$ in (2.5.11), $(x(\theta),\xi(\theta))$ defined by

$$x(\theta) := y + \gamma(\theta,y)\,, \quad \xi(\theta) := \xi_0(1+\partial_x\beta)(x(\theta)) = \frac{\xi_0}{1+\partial_y\gamma(\theta,y)}$$

is the solution of (2.5.32) with initial condition

$$x(0) = y\,, \qquad \xi(0) = \xi_0\,.$$

Taking the inverse flow one obtains (2.5.31).

Finally, since a is in $\Sigma\Gamma^m_{K,K',q}[r,N]$ and β in $\Sigma\mathcal{F}_{K,K',1}[r,N]$, lemma 2.5.4 implies that the symbol $a_0(\theta,\cdot)$ defined in (2.5.30) is in $\Sigma\Gamma^m_{K,K',q}[r,N]$, with estimates uniform in $|\theta|\leq 1$. □

Let us now quantify how approximately $A^{(0)}(\theta) = \mathrm{Op}^{BW}(a_0(\theta))$ solves (2.5.20). By (2.5.29) and (2.5.26) we have

(2.5.33)

$$\begin{aligned}\partial_\theta A^{(0)}(\theta) &= \mathrm{Op}^{BW}(\partial_\theta a_0(\theta)) = \mathrm{Op}^{BW}(\{B(U;\cdot), a_0(\theta)\})\\ &= \big[\mathrm{Op}^{BW}(iB(U;\theta,\cdot), \mathrm{Op}^{BW}(a_0(\theta))\big] - \mathrm{Op}^{BW}\big(r_{-3,\rho}(B,a_0(\theta))\big) - R\\ &= \big[\mathrm{Op}^{BW}(iB(U;\theta,\cdot), A^{(0)}(\theta)\big] - \mathrm{Op}^{BW}\big(r_{-3,\rho}(B,a_0(\theta))\big) - R(\theta)\,.\end{aligned}$$

Since $a_0(\theta,\cdot)\in\Sigma\Gamma^m_{K,K',q}[r,N]$, for any $\theta\in[-1,1]$, we get, by (2.5.27) and (2.5.28),

$$r_{-3,\rho}(B,a_0(\theta))\in\Sigma\Gamma^{m-2}_{K,K',q+1}[r,N]\,,\quad R(\theta)\in\Sigma\mathcal{R}^{-\rho+m+1}_{K,K',q+1}[r,N]\,. \tag{2.5.34}$$

Thus $A^{(0)}(\theta)$ is an approximate solution of (2.5.20) up to a symbol of order $m-2$. We define next the further approximation

$$A^{(1)}(\theta)\stackrel{\mathrm{def}}{=}\mathrm{Op}^{BW}(a_0+a_1) = A^{(0)}(\theta) + \mathrm{Op}^{BW}(a_1(\theta)) \tag{2.5.35}$$

of the equation (2.5.20). We choose the symbol $a_1(U;\theta,\cdot)$ to solve the non-homogeneous transport equation

$$\begin{cases}\partial_\theta a_1(\theta) - \{B(U;\theta,\cdot), a_1(\theta)\} = r_{-3,\rho}(B,a_0(\theta))\\ a_1(0) = 0\,.\end{cases} \tag{2.5.36}$$

By (2.5.33) we have

$$\begin{aligned}\partial_\theta A^{(1)}(\theta) &= \partial_\theta A^{(0)}(\theta) + \mathrm{Op}^{BW}(\partial_\theta a_1(\theta))\\ &= \big[\mathrm{Op}^{BW}(iB(U;\theta,\cdot), A^{(0)}(\theta)\big] - \mathrm{Op}^{BW}\big(r_{-3,\rho}(B,a_0(\theta))\big)\\ &\quad - R(\theta) + \mathrm{Op}^{BW}(\partial_\theta a_1(\theta))\,.\end{aligned}$$

Hence, recalling (2.5.35) and using (2.5.26), we get

$$\begin{aligned}&\partial_\theta A^{(1)}(\theta)\\ &= i\big[\mathrm{Op}^{BW}(B(U;\theta), A^{(1)}(\theta)\big] + \mathrm{Op}^{BW}\big(\partial_\theta a_1 - r_{-3,\rho}(B,a_0)\big))\\ &\quad - i\big[\mathrm{Op}^{BW}(B(U;\theta), \mathrm{Op}^{BW}(a_1)\big] - R(\theta)\\ &= i\big[\mathrm{Op}^{BW}(B(U;\theta), A^{(1)}(\theta)\big] + \mathrm{Op}^{BW}\big(\partial_\theta a_1 - r_{-3,\rho}(B,a_0)\\ &\quad - \{B(U;\theta,\cdot), a_1\}\big) + \mathrm{Op}^{BW}\big(r_{-3,\rho}(B,a_1)\big) + R'\end{aligned}$$

where the symbol $r_{-3,\rho}(B,a_1)$ and the operator R' are lower order terms that we estimate below. Since a_1 solves (2.5.36) we get

(2.5.37)
$$\begin{cases} \partial_\theta A^{(1)}(\theta) = i\big[\mathrm{Op}^{BW}(B(U;\theta,\cdot),A^{(1)}(\theta)\big] + \mathrm{Op}^{BW}\big(r_{-3,\rho}(B,a_1)\big) + R'(\theta) \\ A^{(1)}(0) = \mathrm{Op}^{BW}(a)\,. \end{cases}$$

The non-homogeneous transport equation (2.5.36) can be solved by the method of characteristics.

Lemma 2.5.7 *The solution of (2.5.36) may be written as*

$$a_1(\theta,x,\xi) = \int_0^\theta r_{-3,\rho}(B(s,\cdot),a_0(s,\cdot))(\phi^{\theta,s}(x,\xi))\,ds \tag{2.5.38}$$

where $\phi^{\theta,s}(x,\xi)$ is the solution of (2.5.32) that satisfies the initial condition $\phi^{\theta,\theta}(x,\xi)=(x,\xi)$. In particular

$$a_1(1,x,\xi) = \int_0^1 r_{-3,\rho}(B(s,\cdot),a_0(s,\cdot))(\phi^{1,s}(x,\xi))\,ds\,. \tag{2.5.39}$$

Moreover the symbol $a_1(\theta,x,\xi)$ is in $\Sigma\Gamma^{m-2}_{K,K',q+1}[r,N]$, with estimates uniform in $|\theta|\leq 1$.

Proof: Let $a_1(\theta,x,\xi)$ be a solution of (2.5.36) and $\phi^{0,s}(x_0,\xi_0)=(x(s),\xi(s))$ be the solution of the characteristic system (2.5.32) with initial condition $\phi^{0,0}(x_0,\xi_0)=(x_0,\xi_0)$. Thus the derivative of the scalar function $s\mapsto a_1(s,x(s),\xi(s))$ satisfies

$$\begin{aligned}\frac{d}{ds}a_1(s,x(s),\xi(s)) &= \partial_s a_1(s,x(s),\xi(s)) - \{B(U;s,\cdot),a_1(s,\cdot)\}(x(s),\xi(s)) \\ &= r_{-3,\rho}(B(s,\cdot),a_0(s,\cdot))(x(s),\xi(s))\,.\end{aligned}$$

Since $a_1(0,x_0,\xi_0)=0$ we deduce that

$$\begin{aligned} a_1(\theta,\phi^{0,\theta}(x_0,\xi_0)) &= a_1(\theta,x(\theta),\xi(\theta)) \\ &= \int_0^\theta r_{-3,\rho}(B(s,\cdot),a_0(s,\cdot))(x(s),\xi(s))\,ds \qquad (2.5.40)\\ &= \int_0^\theta r_{-3,\rho}(B(s,\cdot),a_0(s,\cdot))(\phi^{0,s}(x_0,\xi_0)))\,ds\,. \qquad (2.5.41)\end{aligned}$$

Setting

$$(x,\xi)=\phi^{0,\theta}(x_0,\xi_0) \quad\Longleftrightarrow\quad (x_0,\xi_0)=\phi^{\theta,0}(x,\xi)$$

we deduce by (2.5.41), and using the composition rule of the flow $\phi^{0,s}\phi^{\theta,0}=\phi^{\theta,s}$, that

$$a_1(\theta,x,\xi) = \int_0^\theta r_{-3,\rho}(B(s,\cdot),a_0(s,\cdot))(\phi^{\theta,s}(x,\xi)))\,ds \tag{2.5.42}$$

which is (2.5.38). Now (2.5.34), lemma 2.5.4 and lemma 2.5.6 imply that the symbol $r_{-3,\rho}(B,a_0)(\phi^{\theta,s}(x,\xi))$ is in $\Sigma\Gamma^{m-2}_{K,K',q+1}[r,N]$, with estimates uniform in $|\theta|,|s|\leq 1$. Thus integrating (2.5.42) we finally deduce that $a_1(\theta)\in\Sigma\Gamma^{m-2}_{K,K',q+1}[r,N]$, with estimates uniform in $|\theta|\leq 1$. □

Let us now quantify how approximately $A^{(1)}(\theta)$ defined in (2.5.35)-(2.5.36) solves (2.5.20), i.e. estimate the lower order terms in (2.5.37). By lemma 2.5.6 the symbol $a_1(\theta,\cdot)$ is in $\Sigma\Gamma^{m}_{K,K',q}[r,N]$, with estimates uniform in $|\theta|\leq 1$, and by (2.5.27) (with $m-2$ instead of m, and $q+1$ instead of q), we get

$$r_{-3,\rho}(B,a_1)\in\Sigma\Gamma^{m-4}_{K,K',q+2}[r,N]\,,$$

uniformly in $|\theta|\leq 1$. On the other hand the remainder R' in (2.5.37) is the sum of the remainder R in $\Sigma\mathcal{R}^{-\rho+m+1}_{K,K',q+1}[r,N]$ defined in (2.5.34) plus a smaller contribution in $\Sigma\mathcal{R}^{m-1-\rho}_{K,K',q+2}$, thus R' is in $\Sigma\mathcal{R}^{-\rho+m+1}_{K,K',q+1}[r,N]$.

Repeating ℓ times ($\ell\sim\rho/2$) the above procedure, until the new paracomposition term may be incorporated into the remainder R, we find a solution

$$A^{(\ell)}(\theta)=\mathrm{Op}^{BW}(a_0(\theta)+a_1(\theta)+\ldots+a_\ell(\theta)) \tag{2.5.43}$$

of the equation

$$\begin{cases}\partial_\theta A^{(\ell)}(\theta)=i\big[\mathrm{Op}^{BW}(B(U;\theta,\cdot),A^{(\ell)}(\theta)\big]+R(\theta)\\ A^{(\ell)}(0)=\mathrm{Op}^{BW}(a)\end{cases} \tag{2.5.44}$$

where $R(\theta)$ is in $\Sigma\mathcal{R}^{-\rho+m+1}_{K,K',q+1}[r,N]$, uniformly in $\theta\in[-1,1]$.

Finally, we estimate the difference between the approximate solution $A^{(\ell)}(\theta)$ and the true conjugated operator $A(\theta)$ in (2.5.19). We write

$$A^{(\ell)}(\theta)-A(\theta)=\big(A^{(\ell)}(\theta)\Omega_B(\theta)-\Omega_B(\theta)\mathrm{Op}^{BW}(a)\big)(\Omega_B(\theta))^{-1}\,. \tag{2.5.45}$$

The operator

$$V(\theta)\stackrel{\mathrm{def}}{=}A^{(\ell)}(\theta)\Omega_B(\theta)-\Omega_B(\theta)\mathrm{Op}^{BW}(a)$$

solves the non-homogeneous equation

$$\begin{cases}\partial_\theta V(\theta)=i\mathrm{Op}^{BW}(B)V(\theta)+R(\theta)\Omega_B(\theta)\\ V(0)=0\,.\end{cases}$$

Thus, by Duhamel principle and (2.5.13) we get

$$V(\theta):=\Omega_B(\theta)\int_0^\theta(\Omega_B(\tau))^{-1}R(\tau)\Omega_B(\tau)\,d\tau$$

and, substituting in (2.5.45) we get

$$A^{(\ell)}(\theta)-A(\theta)=\Omega_B(\theta)\Big(\int_0^\theta(\Omega_B(\tau))^{-1}R(\tau)\Omega_B(\tau)\,d\tau\Big)(\Omega_B(\theta))^{-1}\,. \tag{2.5.46}$$

We finally deduce the main result for the paracomposition operator $\Omega_B(1)$ associated to the diffeomorphism Φ_U in (2.5.2).

Theorem 2.5.8 (Paracomposition) *Let q be in $\mathbb{N}$, $K' \leq K$, $N \in \mathbb{N}$ with $q \leq N-1$, $r > 0$. Let $\beta(U;t,\cdot)$ be a function in $\Sigma\mathcal{F}_{K,K',1}[r,N]$ which is real valued for U in the space $C^K_{*\mathbb{R}}(I, \dot{H}^\sigma(\mathbb{T}^1,\mathbb{C}^2))$ and consider the diffeomorphism $\Phi_U : x \mapsto x + \beta(U;t,x)$. Let $a(U;\cdot)$ be a symbol in $\Sigma\Gamma^m_{K,K',q}[r,N]$. If σ is large enough, and U stays in the ball of center zero and radius r small enough in $C^K_{*\mathbb{R}}(I, \dot{H}^\sigma(\mathbb{T}^1,\mathbb{C}^2))$, then the flow $\Omega_{B(U)}(\theta)$ defined in (2.5.13) is well defined for $|\theta| \leq 1$, and, for any ρ large enough, there is a symbol $a_\Phi \in \Sigma\Gamma^m_{K,K',q}[r,N]$ such that the conjugated operator*

$$\Omega_{B(U)}(1)\mathrm{Op}^{BW}(a(U;\cdot))\big(\Omega_{B(U)}(1)\big)^{-1} = \mathrm{Op}^{BW}(a_\Phi(U;\cdot)) + R(U;t) \tag{2.5.47}$$

where $R(U;t)$ is in $\Sigma\mathcal{R}^{-\rho+m}_{K,K',q+1}[r,N]$. Moreover we have an expansion

$$a_\Phi(U;t,x,\xi) = a^0_\Phi(U;t,x,\xi) + a^1_\Phi(U;t,x,\xi) \tag{2.5.48}$$

where the principal symbol a^0_Φ is given by, denoting the inverse diffeomorphism of Φ_U as in (2.5.3),

$$a^0_\Phi(U;t,x,\xi) = a\Big(U;\Phi_U(t,x),\xi\partial_y\big(\Phi_U^{-1}(t,y)\big)_{|y=\Phi_U(t,x)}\Big)\,, \tag{2.5.49}$$

a^0_Φ is in $\Sigma\Gamma^m_{K,K',q}[r,N]$ and the symbol a^1_Φ is in $\Sigma\Gamma^{m-2}_{K,K',q+1}[r,N]$. Finally $a_\Phi \equiv 1$ if $a \equiv 1$.

The operator

$$\Phi^\star_U \stackrel{def}{=} \Omega_{B(U)}(1) \tag{2.5.50}$$

is by definition the paracomposition operator by the diffeomorphism Φ_U. There are multilinear operators M_p in $\widetilde{\mathcal{M}}_p$ for $p = 1,\dots,N-1$ and an operator M_N in $\mathcal{M}_{K,K',N}[r]$ such that

$$\Phi^\star_U W = W + \sum_{p=1}^{N-1} M_p(U,\dots,U)W + M_N(U;t)W\,. \tag{2.5.51}$$

Proof: Recalling (2.5.43) we set

$$a^1_\Phi \stackrel{\mathrm{def}}{=} a_1(1) + \ldots + a_\ell(1) \in \Sigma\Gamma^{m-2}_{K,K',q+1}[r,N]\,.$$

Moreover we know by (2.5.2) and by (2.5.21) that a^0_Φ is given by (2.5.49). Then we estimate the difference (2.5.46) taken at $\theta = 1$. The remainder $R(\tau)$ is in the class $\Sigma\mathcal{R}^{-\rho+m+1}_{K,K',q+1}[r,N]$. In order to prove that the operator in

(2.5.46) is a smoothing remainder, decomposed in homogeneous operators plus an $O(U^N)$ term, we have to Taylor expand the flow

(2.5.52)
$$\Omega_{B(U)}(\theta) = \mathrm{Id} + \sum_{\ell=1}^{N-q-1} \int \mathbf{1}_{\theta_\ell<\cdots<\theta_1<\theta} \prod_{j=1}^{\ell} \big[iOp^{\mathrm{BW}}(B(U;\theta_j,\cdot))\big]\, d\theta_1 \dots d\theta_\ell$$
$$+ \int \mathbf{1}_{\theta_{N-q}<\cdots<\theta_1<\theta} \prod_{j=1}^{N-q} \big[iOp^{\mathrm{BW}}(B(U;\theta_j,\cdot))\big] \Omega_{B(U)}(\theta_{N-q})\, d\theta_1 \dots d\theta_{N-q}\,.$$

Consider the integral term $I(U)$ in (2.5.52). We may write

$$\partial_t^k\Big(\prod_{j=1}^{N-q}\big[Op^{\mathrm{BW}}(B(U;\theta_j,\cdot))\big]\Omega_{B(U)}(\theta_{N-q})W\Big) = \tag{2.5.53}$$
$$\sum_{k_1+\ldots+k_{N-q+1}=k} C_{k_1,\ldots,k_{N-q+1}}\Big(\prod_{j=1}^{N-q}\big[Op^{\mathrm{BW}}(\partial_t^{k_j}B)\big]\partial_t^{k_{N-q+1}}[\Omega_{B(U)}W]\Big)$$

for suitable coefficients $C_{k_1,\ldots,k_{N-q+1}}$. Since $B(U;\theta)$ is in $\Sigma\Gamma^1_{K,K',1}[r,N]$, it follows from Proposition 2.2.4 and (2.5.15) that, for any $k \le K-K'$,

$$\|\partial_t^k(I(U)W)\|_{\dot H^{s-\frac{3}{2}k-(N-q)}} \le C \sum_{k'+k''=k} \mathcal{G}^\sigma_{k'+K',N-q}(U,t)\mathcal{G}^s_{k'',1}(W,t)\,.$$

We replace in (2.5.46) $\Omega_{B(U)}(\theta)$, $\Omega_{B(U)}(\tau)$ by the right hand side of (2.5.52), eventually taken at some order smaller than $N-q$, and do the same for the analogous expansions of $(\Omega_{B(U)}(\theta))^{-1}$, $(\Omega_{B(U)}(\tau))^{-1}$. The terms containing at least one integral remainder provide elements of $\mathcal{R}^{-\rho+m+1+N}_{K,K',N}[r]$. By the results of section 2.4 the polynomial contributions to the expansion may be expressed from compositions of the smoothing term in (2.5.46) with paradifferential operators. We deduce that the difference $A^{(\ell)}(\theta)-A(\theta)$ in (2.5.46) is in $\Sigma\mathcal{R}^{-\rho+m+1+N}_{K,K',q+1}[r,N]$, and renaming $\rho-N-1$ by ρ, we obtain that it is in $\Sigma\mathcal{R}^{-\rho+m}_{K,K',q+1}[r,N]$.

Finally, recalling (2.5.50), the formula (2.5.51) follows from (2.5.52) at $\theta=1$ acting on W. Actually, the right hand side of this last quantity may be written as (2.5.51) by Proposition 2.4.1, the remarks following Definition 2.2.5 and lemma 2.5.3. □

Remark: The above proof shows that, if $R(U;t)$ is a smoothing operator in the class $\Sigma\mathcal{R}^{-\rho}_{K,K',1}[r,N]$, then $\Omega_{B(U)}(1)\circ R(U;t)\circ(\Omega_{B(U)}(1))^{-1}$ belongs to $\Sigma\mathcal{R}^{-\rho+N}_{K,K',1}[r,N]$.

Theorem 2.5.8 shows that the time 1 flow $\Omega_{B(U)}(1)$ generated by (2.5.13) defines a para-composition operator associated to the composition operator

(2.5.9), providing the usual formula (2.5.47), (2.5.49) in Alinhac theorem. An advantage of the Weyl quantization is that the symbol a^1_Φ in (2.5.48) has two orders than less the principal symbol.

The conjugation operator $\Omega_{B(U)}(\theta)\circ\partial_t\circ\Omega_{B(U)}(\theta)^{-1}$ can be analyzed in a similar way.

Proposition 2.5.9 *Use the assumptions and notations of Theorem 2.5.8. Let U be a solution of equation (2.2.54) of lemma 2.2.6. Then for any ρ large enough, we have*

$$\begin{aligned}(2.5.54)\qquad \Omega_{B(U)}(\theta)\circ\partial_t\circ\Omega_{B(U)}(\theta)^{-1} &= \partial_t+\Omega_{B(U)}(\theta)\circ\big(\partial_t\Omega_{B(U)}(\theta)^{-1}\big)\\ &= \partial_t+\mathrm{Op}^{BW}(e(U;\cdot))+R(U;t)\end{aligned}$$

where e is a symbol in $\Sigma\Gamma^1_{K,K'+1,1}[r,N]$ with $\mathrm{Re}(e)$ in $\Sigma\Gamma^{-1}_{K,K'+1,1}[r,N]$ and $R(U;t)$ is a smoothing operator in $\Sigma\mathcal{R}^{-\rho}_{K,K'+1,1}[r,N]$.

Proof: Recall that the symbol $B:=B(U;\theta)$ depends on the time t through the function U and that the flow $\Omega_B(\theta)$ solves (2.5.13). Setting $\Psi(\theta)=\Omega_B(\theta)\circ\partial_t\circ\Omega_B(\theta)^{-1}$ we have

$$\begin{aligned}\partial_\theta\Psi(\theta) &= \partial_\theta\big(\Omega_B(\theta)\circ\partial_t\circ\Omega_B(\theta)^{-1}\big)\\ &= \big(\partial_\theta\Omega_B(\theta)\big)\circ\partial_t\circ\Omega_B(\theta)^{-1}+\Omega_B(\theta)\circ\partial_t\circ\big(\partial_\theta\Omega_B(\theta)^{-1}\big)\\ &= i\mathrm{Op}^{BW}(B(\theta))\Omega_B(\theta)\circ\partial_t\circ\Omega_B(\theta)^{-1}\\ &\quad-\Omega_B(\theta)\circ\partial_t\circ\Omega_B(\theta)^{-1}\big(\partial_\theta\Omega_B(\theta)\big)\Omega_B(\theta)^{-1}\\ &= i\mathrm{Op}^{BW}(B(\theta))\Psi(\theta)-\Psi(\theta)i\mathrm{Op}^{BW}(B(\theta))\end{aligned}$$

and therefore $\Psi(\theta)$ solves the Heisenberg equation

$$\begin{cases}\frac{d}{d\theta}\Psi(\theta)=i\big[\mathrm{Op}^{BW}\big(B(U;\theta)\big),\Psi(\theta)\big]\\ \Psi(0)=\partial_t\,.\end{cases}$$

Comparing with (2.5.54) we write

$$\Psi(\theta)=\partial_t+Q(\theta)\,,\quad Q(\theta)=\Omega_B(\theta)\circ\big(\partial_t\Omega_B(\theta)^{-1}\big)\,,$$

and we find out that $Q(\theta)$ solves

$$(2.5.55)\qquad \begin{cases}\frac{d}{d\theta}Q(\theta)=i\big[\mathrm{Op}^{BW}\big(B(U;\theta)\big),Q(\theta)\big]-i\mathrm{Op}^{BW}\big(\partial_tB(U;\theta)\big)\\ Q(0)=0\,.\end{cases}$$

This forced equation can be analyzed as above and we find that its solution is a paradifferential operator

$$Q(\theta)=\mathrm{Op}^{BW}(q_0+q_1+\ldots)\,,$$

up to a regularizing operator, where the principal symbol $q_0(\theta, t, \cdot)$ solves

$$\begin{cases} \frac{d}{d\theta} q_0(\theta) = \{B(U;\theta), q_0(\theta)\} - i\partial_t B(U;\theta) \\ q_0(0) = 0\,. \end{cases} \tag{2.5.56}$$

By lemma 2.5.7 the solution of this non-homogeneous transport equation is

$$q_0(\theta, x, \xi) = -i \int_0^\theta \partial_t B(U;\theta, \phi^{\theta,s}(x,\xi))\, ds\,. \tag{2.5.57}$$

Recalling the definition of B in (2.5.12), and lemma 2.5.4, we get that the function $B(U;\theta,\phi^{\theta,s}(x,\xi))$ is in $\Sigma\Gamma^1_{K,K',1}[r,N]$, with estimates uniform in $|\theta|, |s| \leq 1$. By lemma 2.2.6, we have also that $\partial_t B(U;\theta,\phi^{\theta,s}(x,\xi))$ is in $\Sigma\Gamma^1_{K,K'+1,1}[r,N]$ and by (2.5.57)

$$q_0(\theta,\cdot) \in \Sigma\Gamma^1_{K,K'+1,1}[r,N]\,,$$

with estimates uniform in $|\theta| \leq 1$. Let us quantify how approximately $Q^{(0)}(\theta) = \mathrm{Op}^{BW}(q_0(\theta))$ solves (2.5.55). By (2.5.56) and (2.5.26) we have

$$\begin{aligned} \partial_\theta Q^{(0)}(\theta) &= \mathrm{Op}^{BW}(\{B(U;\cdot), q_0(\theta)\}) - i\mathrm{Op}^{BW}(B_t(U;\theta)) \\ &= \big[\mathrm{Op}^{BW}(iB(U;\theta,\cdot), \mathrm{Op}^{BW}(q_0(\theta))\big] - \mathrm{Op}^{BW}\big(r_{-3,\rho}(B, q_0(\theta))\big) \\ &\quad - R(\theta) - i\mathrm{Op}^{BW}(B_t(U;\theta)) \\ &= \big[\mathrm{Op}^{BW}(iB(U;\theta,\cdot), Q^{(0)}(\theta)\big] - i\mathrm{Op}^{BW}(\partial_t B(U;\theta)) \\ &\quad - \mathrm{Op}^{BW}\big(r_{-3,\rho}(B, q_0(\theta))\big) - R(\theta) \end{aligned}$$

where, by (2.5.27), (2.5.28) (with $m = 1$ and $q = 1$), for any $\theta \in [-1,1]$,

$$r_{-3,\rho}(B, q_0(\theta)) \in \Sigma\Gamma^{-1}_{K,K'+1,2}[r,N]\,, \quad R(\theta) \in \Sigma\mathcal{R}^{-\rho+2}_{K,K'+1,2}[r,N]\,.$$

Notice that, thanks to the Weyl quantization, the symbol $r_{-3,\rho}(B, q_0(\theta))$ has order -1. Repeating the same argument as for Theorem 2.5.8 and renaming $\rho - N$ by ρ, we deduce the Proposition. □

Chapter 3

Complex formulation of the equation and diagonalization of the matrix symbol

In chapter 6 below, we shall obtain an equivalent form of the water waves equations (1.1.4) in terms of a complex unknown, suitable to prove energy estimates. In this chapter, after introducing some algebraic properties of the water waves equations, we present the general form of such paradifferential system, see (3.2.2)-(3.2.5), and we state the main Theorem 3.2.1 concerning almost global existence of its solutions with a small initial datum. The proof of this result is then provided in the last section of Chapter 3 and in Chapter 4.

3.1 Reality, parity and reversibility properties

We denote by S the linear involution, i.e. $S^2 = \mathrm{Id}$,

$$S : \mathbb{C}^2 \to \mathbb{C}^2 , \quad \text{with matrix } - \left[\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right] . \tag{3.1.1}$$

This is the translation in the present complex framework of the map introduced in (1.2.3), and denoted by the same letter. We shall consider the action of the group $\mathbb{Z}/2\mathbb{Z}$, identified to $\{\mathrm{Id}, S\}$, on the balls of functions $B^k_\sigma(I, r)$ defined in (2.1.9),

$$\begin{aligned} (g, U) &\longrightarrow U_g \\ \mathbb{Z}/2\mathbb{Z} \times B^k_\sigma(I, r) &\to B^k_\sigma(I, r) \end{aligned} \tag{3.1.2}$$

defined by

$$U_S(t) \stackrel{\text{def}}{=} SU(-t) , \quad \forall t \in I . \tag{3.1.3}$$

Notice that the spaces $C^K_{*\mathbb{R}}(I, \dot{H}^\sigma(\mathbb{T}^1, \mathbb{C}^2))$ defined after (2.1.2) are left invariant under this action. In our application to the proof of our main theorem, the function $(t, x) \to U(t, x)$ will be a solution to the system (3.2.2) below that translates system (1.1.4) on the unknown U. Since any solution of (1.1.4) satisfies (1.2.5), the solutions of (3.2.2) will satisfy the corresponding property, namely the solution of that system with initial datum $SU(0)$ coincides with $SU(-t)$ at any time. In other words, if $U(t)$ is a solution of (3.2.2), the function $U_S(t)$ defined by (3.1.3) is nothing but the solution of the same system (3.2.2) with initial datum $SU(0)$.

We first define some algebraic properties for a matrix of symbols that will play a crucial role in the long time existence proof.

Definition 3.1.1 *Let A be an element of the class $\Sigma\Gamma^m_{K,K',p}[r, N] \otimes \mathcal{M}_2(\mathbb{C})$ i.e. a 2×2 matrix whose entries are symbols in $\Sigma\Gamma^m_{K,K',p}[r, N]$. Let U satisfy $SU = -\bar{U}$. We set*

$$A(U; t, x, \xi)^\vee = A(U; t, x, -\xi)\,.$$

The matrix A satisfies the **reality** *condition if*

$$\bar{A}(U; t, x, \xi)^\vee = -SA(U; t, x, \xi)S\,, \tag{3.1.4}$$

the **anti-reality** *condition if*

$$\bar{A}(U; t, x, \xi)^\vee = SA(U; t, x, \xi)S\,, \tag{3.1.5}$$

the **parity preserving** *condition if*

$$A(U; t, -x, -\xi) = A(U; t, x, \xi)\,. \tag{3.1.6}$$

Finally, we say that A satisfies the **reversibility** *condition if*

$$A(U; -t, x, \xi)S = -SA(U_S; t, x, \xi) \tag{3.1.7}$$

and the **anti-reversibility** *condition if*

$$A(U; -t, x, \xi)S = SA(U_S; t, x, \xi). \tag{3.1.8}$$

Below, we shall use these properties for elements U of $C^K_{*\mathbb{R}}(I, \dot{H}^\sigma(\mathbb{T}^1, \mathbb{C}^2))$ for some large enough σ and K.

Remarks: • The product $A(U; t, x, \xi)B(U; t, x, \xi)$ of matrices of symbols A, B satisfying (3.1.5), resp. (3.1.6), resp. (3.1.8), satisfy (3.1.5), resp. (3.1.6), resp. (3.1.8), as well. In addition, if A satisfies (3.1.4), resp. (3.1.7), and B satisfies (3.1.5), resp. (3.1.8), then AB satisfies (3.1.4), resp. (3.1.7).

• If a matrix $A(U; t, x, \xi)$ satisfy one among the properties (3.1.4)-(3.1.8) and it is invertible, then $A(U; t, x, \xi)^{-1}$ satisfies the same property.

The reversibility and antireversibility conditions (3.1.7), (3.1.8) may be expressed more explicitly for the homogeneous components of the symbols, for which the time dependence enters through $U(t)$, see Definition 2.1.1.

Definition 3.1.2 *Let A_q be a matrix of homogeneous symbols in $\tilde{\Gamma}^m_q \otimes \mathcal{M}_2(\mathbb{C})$. Let U_j, $j = 1, \dots, q$ satisfy $SU_j = -\bar{U}_j$. We say that A_q satisfies the reversibility condition if*

$$A_q(SU_1, \dots, SU_q; x, \xi)S = -SA_q(U_1, \dots, U_q; x, \xi)\,, \tag{3.1.9}$$

and the anti-reversibility condition if

$$A_q(SU_1, \dots, SU_q; x, \xi)S = SA_q(U_1, \dots, U_q; x, \xi)\,. \tag{3.1.10}$$

We have the following lemma.

Lemma 3.1.3 *Let A be in $\Sigma\Gamma^m_{K,K',p}[r, N] \otimes \mathcal{M}_2(\mathbb{C})$. Decompose A according to (2.1.12) as*

$$A(U; t, x, \xi) = \sum_{q=p}^{N-1} A_q(U, \dots, U; x, \xi) + A_N(U; t, x, \xi) \tag{3.1.11}$$

with A_q in $\tilde{\Gamma}^m_q \otimes \mathcal{M}_2(\mathbb{C})$ for $q = p, \dots, N-1$ and A_N in $\Gamma^m_{K,K',N}[r] \otimes \mathcal{M}_2(\mathbb{C})$.

If A_q, $q = p, \dots, N-1$ satisfies (3.1.9) (resp. (3.1.10)) and if A_N satisfies (3.1.7) (resp. (3.1.8)), then A satisfies (3.1.7) (resp. (3.1.8)).

Conversely, if A satisfies (3.1.7) (resp. (3.1.8)), we may find symbols A'_q, $q = p, \dots, N-1$, satisfying (3.1.9) (resp. (3.1.10)) such that, for any $U = U(t)$ we have

$$A(U; t, x, \xi) = \sum_{q=p}^{N-1} A'_q(U, \dots, U; x, \xi) + A_N(U; t, x, \xi). \tag{3.1.12}$$

Proof: Let us write the proof for conditions (3.1.7) and (3.1.9). If A_q satisfies (3.1.9), then according to (2.1.7) and to the definition (3.1.3) of U_S, we have

$$\begin{aligned} A_q(U, \dots, U; -t, x, \xi)S &= A_q(U(-t), \dots, U(-t); x, \xi)S \\ &= A_q(SU_S(t), \dots, SU_S(t); x, \xi)S \\ &= -SA_q(U_S(t), \dots, U_S(t); x, \xi) \\ &= -SA_q(U_S, \dots, U_S; t, x, \xi) \end{aligned}$$

so that $A_q(U, \dots, U; t, x, \xi)$ satisfies (3.1.7). Since also A_N satisfies (3.1.7) then A defined in (3.1.11) satisfies (3.1.7).

Conversely, assume that the matrix of symbols A in (3.1.11) satisfies (3.1.7). Thus each $A_q(U, \dots, U; t, x, \xi)$ for $q = p, \dots, N-1$, satisfies (3.1.7) as well. Define from A_q a new multilinear symbol

$$A'_q(U_1, \dots, U_q; x, \xi) = \frac{1}{2}\big[A_q(U_1, \dots, U_q; x, \xi) - SA_q(SU_1, \dots, SU_q; x, \xi)S\big].$$

Then, by construction, A'_q satisfies (3.1.9). In particular, if we replace $U_1, \ldots, U_q$ by U we get by (3.1.3)

(3.1.13)
$$A'_q(U, \ldots, U; t, x, \xi) = \frac{1}{2}\big[A_q(U, \ldots, U; t, x, \xi) - SA_q(U_S, \ldots, U_S; -t, x, \xi)S\big] = A_q(U, \ldots, U; t, x, \xi)$$

by the condition (3.1.7). Hence we deduce (3.1.12). □

Remark: From now on, each time we shall consider symbols A of the form (3.1.11) satisfying the reversibility (3.1.7) or antireversibility (3.1.8) condition, we shall assume that the homogeneous contribution in the decomposition of A satisfy (3.1.9) or (3.1.10).

Let us introduce analogous conditions at the level of operators.

Definition 3.1.4 *Let $M(U;t)$ be a linear operator, depending on U satisfying $SU = -\bar{U}$. We say that M satisfies the* **reality** *condition if*

$$\overline{M(U;t)V} = -SM(U;t)S\overline{V}\,, \tag{3.1.14}$$

the **anti-reality** *condition if*

$$\overline{M(U;t)V} = SM(U;t)S\overline{V}\,. \tag{3.1.15}$$

Define the map τ, acting on functions of x, by

$$(\tau V)(x) \stackrel{def}{=} V(-x)\,. \tag{3.1.16}$$

We say that M is **parity preserving** *if*

$$M(U;t) \circ \tau = \tau \circ M(U;t). \tag{3.1.17}$$

Finally, we say that M satisfies the **reversibility** *condition if*

$$M(U;-t)S = -SM(U_S;t) \tag{3.1.18}$$

and the **anti-reversibility** *condition if*

$$M(U;-t)S = SM(U_S;t)\,. \tag{3.1.19}$$

Remarks: • The operator $M(U;t) = Op^{\mathrm{BW}}(A(U;t,\cdot))$ satisfies (3.1.14) (resp. (3.1.15), resp. (3.1.17), resp. (3.1.18), resp. (3.1.19)) if the matrix symbol $A(U;t,\cdot)$ satisfies the corresponding property (3.1.4) (resp. (3.1.5), resp. (3.1.6), resp. (3.1.7), resp. (3.1.8)). Notice that since the functions χ, χ_p used in (2.2.23) are even with respect to each of their argument, as soon

as (3.1.4), (3.1.5), (3.1.6) hold, the same is true for the Fourier truncated symbols (2.2.23).

• An operator $M(U;t)$ satisfying the anti-reality condition (3.1.15) maps the subspace of 2-vectors $\{SV = -\overline{V}\}$ (i.e. V of the form $\begin{bmatrix} v \\ \bar{v} \end{bmatrix}$) into itself.

• The operator $M(U;t)$ satisfies the reality property (3.1.14) if and only if $iM(U;t)$ satisfies the anti-reality property (3.1.15), and viceversa.

• If $M(U;t)$ satisfies (3.1.18), respectively (3.1.19), we say that $F(U;t) = M(U;t)U$ is reversible, respectively antireversible.

• If $M(U;t)$ satisfies one among the properties (3.1.14)-(3.1.19) and it is invertible, then $M(U;t)^{-1}$ satisfies the same property.

Arguing as in the proof of lemma 3.1.3 we deduce the following lemma.

Lemma 3.1.5 *If M is decomposed as in (2.2.31) or (2.2.53) as a sum*

$$M(U;t) = \sum_{q=p}^{N-1} M_q(U,\dots,U) + M_N(U;t)$$

in terms of homogeneous operators M_q, $q = p,\dots,N-1$, and if M satisfies the reversibility condition (3.1.18), respectively anti-reversibility (3.1.19), we may assume that M_q, $q = p,\dots,N-1$, satisfy the reversibility property

$$M_q(SU_1,\dots,SU_q)S = -SM(U_1,\dots,U_q)\,, \tag{3.1.20}$$

respectively the anti-reversibility property

$$M_q(SU_1,\dots,SU_q)S = SM(U_1,\dots,U_q). \tag{3.1.21}$$

These algebraic properties behave as follows under composition:

Lemma 3.1.6 *Composition of an operator satisfying the reality property (3.1.14) (resp. the reversibility property (3.1.18)) with one or several operators satisfying the antireality property (3.1.15) (resp. the anti-reversibility property (3.1.19)) still satisfies the reality property (3.1.14) (resp. reversibility property (3.1.18)). Composition of operators satisfying the parity preserving property (3.1.17) satisfy as well the parity preserving property (3.1.17). Composition of operators satisfying the anti-reality property (3.1.15) satisfy the anti-reality property (3.1.15) as well.*

3.2 Complex formulation of the capillarity-gravity water waves equations

We fix from now on large integers ρ, N. The time (resp. space) smoothness of the solution we shall consider will be measured by an integer K (resp. a real number s), and we shall assume

$$s \gg \sigma \gg K \gg \rho \gg N \tag{3.2.1}$$

where N is the exponent of $1/\epsilon$ in the lower bound of the time of existence of the solutions ($T_\epsilon \geq c\epsilon^{-N}$) that we want to prove (see the statement of Theorem 1.2.1).

We shall prove in Proposition 6.3.1 that the water waves system (1.1.4) is equivalent, for small solutions, even in x, to the following system

$$\begin{aligned}(D_t - Op^{\mathrm{BW}}(A(U;t,x,\xi)))U &= R(U;t)U \\ U|_{t=0} &= \epsilon U_0\end{aligned} \tag{3.2.2}$$

where $A(U;t,\cdot)$ is a two by two matrix of symbols, that may be decomposed in the basis of $\mathcal{M}_2(\mathbb{C})$ given by

$$\mathcal{I}_2 = \left[\begin{smallmatrix}1&0\\0&1\end{smallmatrix}\right], \quad \mathcal{K} = \left[\begin{smallmatrix}1&0\\0&-1\end{smallmatrix}\right], \quad \mathcal{J} = \left[\begin{smallmatrix}0&-1\\1&0\end{smallmatrix}\right], \quad \mathcal{L} = \left[\begin{smallmatrix}0&1\\1&0\end{smallmatrix}\right] \tag{3.2.3}$$

as follows:

$$\begin{aligned}A(U;t,x,\xi) = &\big(m_\kappa(\xi)(1+\zeta(U;t,x)) + \lambda_{\frac12}(U;t,x,\xi)\big)\mathcal{K} \\ &+ \big(m_\kappa(\xi)\zeta(U;t,x) + \lambda_{-\frac12}(U;t,x,\xi)\big)\mathcal{J} \\ &+ \lambda_1(U;t,x,\xi)\mathcal{I}_2 + \lambda_0(U;t,x,\xi)\mathcal{L}\end{aligned} \tag{3.2.4}$$

where

$$m_\kappa(\xi) = (\xi\tanh\xi)^{\frac12}(1+\kappa\xi^2)^{\frac12}(1-\chi(\xi)), \tag{3.2.5}$$

$\chi(\xi)$ being an even $C_0^\infty(\mathbb{R})$ function equal to one close to zero, supported for $|\xi| < \frac12$;

where λ_j, $j = 1, \frac12, 0, -\frac12$ are symbols of $\Sigma\Gamma^j_{K,1,1}[r,N]$ for some small enough $r > 0$, satisfying $\operatorname{Im}\lambda_j \in \Sigma\Gamma^{j-1}_{K,1,1}[r,N]$ if $j = 1$ or $\frac12$, and such that $A(U;t,x,\xi)$ satisfies the reality, parity preserving and reversibility properties (3.1.4), (3.1.6) and (3.1.7). Moreover $R(U;t)$ in the right hand side of (3.2.2) is a smoothing operator of $\Sigma\mathcal{R}^{-\rho}_{K,1,1}[r,N]\otimes\mathcal{M}_2(\mathbb{C})$ that satisfies the reality, parity preserving and reversibility properties (3.1.14), (3.1.17), (3.1.18). Finally the function ζ defined in (6.3.7) is an element of $\Sigma\mathcal{F}^{\mathbb{R}}_{K,0,1}[r,N]$ (actually even of the subspace $\Sigma\mathcal{F}^{\mathbb{R}}_{K,0,2}[r,N]$).

The above paradifferential formulation of the water waves system is suitable to prove energy estimates. In some instances we shall need to write the water waves system (3.2.2) just as

$$D_tU = m_\kappa(D)\mathcal{K}U + M(U;t)U \tag{3.2.6}$$

where the operator $M(U;t)$ is the element of $\Sigma\mathcal{M}_{K,1,1}[r,N]\otimes\mathcal{M}_2(\mathbb{C})$ defined by the sum of $Op^{\mathrm{BW}}\big(A(U;t,\cdot) - m_\kappa(\xi)\mathcal{K}\big)$ and the smoothing operator $R(U;t)$ in (3.2.2). Thus $M(U;t)U$ collects all the terms of the water waves system (3.2.2) which are at least quadratic in U (but the operator $M(U;t)$

loses derivatives). The fact that $M(U;t)$ is in $\Sigma\mathcal{M}_{K,1,1}[r,N]\otimes\mathcal{M}_2(\mathbb{C})$ follows by the remarks after Definition 2.2.5. Moreover, since the matrix of symbols A satisfies (3.1.4), (3.1.5), (3.1.7) and since $R(U;t)$ in (3.2.2) satisfies (3.1.14), (3.1.17), (3.1.18), the operator $M(U;t)$ satisfies as well the reality, parity preserving, and reversibility properties (3.1.14), (3.1.17), (3.1.18).

We look for solutions of (3.2.2) with Cauchy data U_0 in $\dot{H}^s_{\rm ev}(\mathbb{T}^1,\mathbb{C}^2)$ satisfying

$$SU_0=-\bar{U}_0\,,\quad i.e.\ U_0=\left[\begin{smallmatrix}u_0\\ \bar{u}_0\end{smallmatrix}\right].$$

Notice that $SU_0=-\bar{U}_0$ implies that for any t, $\bar{U}(t)=-SU(t)$ since the reality condition (3.1.14) implies that these two functions satisfy the same equation with the same Cauchy data. This amounts to look for real valued solutions of the water waves system (1.1.4).

Theorem 3.2.1 (Almost global existence for (3.2.2)) *There is a zero measure subset $\mathcal{N}$ of $]0,+\infty[$ and for any N in $\mathbb{N}$, any κ in $]0,+\infty[-\mathcal{N}$, there are K in $\mathbb{N}$, s_0 in $\mathbb{N}$ such that, for any $s\geq s_0$, there are positive constants ϵ_0, c, C such that for any U_0 in the unit ball of $\dot{H}^s_{\rm ev}(\mathbb{T}^1,\mathbb{C}^2)$, with $SU_0=-\bar{U}_0$, for any ϵ in $]0,\epsilon_0[$, system (3.2.2) has a unique classical solution*

$$U\in\cap_{k=0}^K C^k(]-T_\epsilon,T_\epsilon[,\dot{H}^{s-\frac{3}{2}k}_{\rm ev}(\mathbb{T}^1,\mathbb{C}^2))\,,\quad \textit{with}\quad T_\epsilon\geq c\epsilon^{-N}\,.$$

Moreover, the solution satisfies

$$\sup_{]-T_\epsilon,T_\epsilon[}\|\partial_t^k U(t,\cdot)\|_{\dot{H}^{s-\frac{3}{2}k}(\mathbb{T}^1,\mathbb{C}^2)}\leq C\epsilon,\ 0\leq k\leq K. \tag{3.2.7}$$

Theorem 3.2.1 and Proposition 6.3.1 then imply Theorem 1.2.1.

The first part of the proof of Theorem 3.2.1 will be to reduce (3.2.2) to an equivalent system, where the non diagonal part of the matrix symbol (3.2.4), namely $\left(m_\kappa\zeta+\lambda_{-\frac{1}{2}}\right)\mathcal{J}+\lambda_0\mathcal{L}$ will be replaced by a symbol of very negative order, whose associated operator may be incorporated to the smoothing term $R(U;t)$.

3.3 Diagonalization of the system

The goal of this subsection is to transform the matrix symbol A in the left hand side of (3.2.2) into a diagonal one, up to a smoothing term $R(U;t)$. We introduce first the class of matrices of symbols we shall use.

Definition 3.3.1 *Let m be in $\mathbb{R}_-$, N in $\mathbb{N}$, $K'\leq K$ in $\mathbb{N}$, $r>0$. We denote by $\mathcal{E}^m_{K,K',1}[r,N]$ the set of matrices of symbols of the form*

$$P(U;\cdot)=P(U;t,\cdot)=(1+\alpha)\mathcal{I}_2+\beta\mathcal{L}+\theta\mathcal{J}+\gamma\mathcal{K} \tag{3.3.1}$$

where

$$\alpha, \beta \in \Sigma\Gamma^{m}_{K,K',1}[r,N], \quad \gamma, \theta \in \Sigma\Gamma^{\min\left(m-\frac{1}{2},-\frac{3}{2}\right)}_{K,K',1}[r,N] \tag{3.3.2}$$

are such that P satisfies the anti-reality, parity preserving, anti-reversibility properties (3.1.5), (3.1.6), (3.1.8), and

$$\operatorname{Im}\alpha, \operatorname{Im}\beta \in \Sigma\Gamma^{\min\left(m,-\frac{3}{2}\right)}_{K,K',1}[r,N]. \tag{3.3.3}$$

Our goal is to prove:

Proposition 3.3.2 (Diagonalization of the matrix symbol (3.2.4)) *Assume that $K \geq \rho+1$, $\rho \in \mathbb{N}$, and that $r>0$ is small enough. There exist*

- *matrices of symbols $P(U;t,\cdot), Q(U;t,\cdot)$ in $\mathcal{E}^{0}_{K,\rho,1}[r,N]$, such that the operator $Op^{\mathrm{BW}}(P)\circ Op^{\mathrm{BW}}(Q)-\mathrm{Id}$ is in $\Sigma\mathcal{R}^{-\rho}_{K,\rho,1}[r,N]\otimes\mathcal{M}_2(\mathbb{C})$ and satisfies (3.1.15), (3.1.17) and (3.1.19),*
- *symbols $\lambda^{(1)}_j$ in $\Sigma\Gamma^{j}_{K,\rho+1,1}[r,N]$, $j=1,\frac{1}{2}$ satisfying*

$$\operatorname{Im}\lambda^{(1)}_j \in \Sigma\Gamma^{j-1}_{K,\rho+1,1}[r,N] \tag{3.3.4}$$

and a function $\zeta^{(1)}(U;t,x)$ in $\Sigma\mathcal{F}^{\mathbb{R}}_{K,\rho+1,1}[r,N]$, so that the diagonal matrix

$$\begin{aligned} A^{(1)}(U;t,x,\xi) = \big(&m_\kappa(\xi)(1+\zeta^{(1)}(U;t,x)) \\ &+\lambda^{(1)}_{1/2}(U;t,x,\xi)\big)\mathcal{K} + \lambda^{(1)}_{1}(U;t,x,\xi)\mathcal{I}_2 \end{aligned} \tag{3.3.5}$$

satisfies the reality (3.1.4), parity preserving (3.1.6), reversibility (3.1.7) properties,

- *smoothing operators $R'(U;t)$, respectively $R''(U;t)$, belonging to the space $\Sigma\mathcal{R}^{-\rho+\frac{3}{2}}_{K,\rho+1,1}[r,N]\otimes\mathcal{M}_2(\mathbb{C})$, resp. belonging to $\mathcal{R}^{-\rho+\frac{3}{2}}_{K,\rho+1,N}[r]\otimes\mathcal{M}_2(\mathbb{C})$, satisfying the reality (3.1.14), parity preserving (3.1.17) and reversibility (3.1.18) properties,*

such that the function $W=Op^{\mathrm{BW}}(Q)U$ solves the system

$$\big(D_t - Op^{\mathrm{BW}}(A^{(1)}(U;t,\cdot)\big)W = R'(U;t)W + R''(U;t)U \tag{3.3.6}$$

on the time interval over which a solution U of system (3.2.2) is defined.

We shall prove three lemmas.

Lemma 3.3.3 (Parametrix) *Take a matrix of symbols P in $\mathcal{E}^{m}_{K,K',1}[r,N]$ and let ρ be in $\mathbb{N}$. Then, if r is small enough, there is a matrix of symbols $Q(U;\cdot)$ in $\mathcal{E}^{m}_{K,K',1}[r,N]$ such that*

$$\begin{aligned} \big(P(U;\cdot)\#Q(U;\cdot)\big)_{\rho,N} - \mathcal{I}_2 &\in \mathcal{E}^{-\rho}_{K,K',1}[r,N] \\ P(U;\cdot)^{-1} - Q(U;\cdot) &\in \Sigma\Gamma^{\min\left(m-\frac{3}{2},-2\right)}_{K,K',1}[r,N]\otimes\mathcal{M}_2(\mathbb{C}). \end{aligned} \tag{3.3.7}$$

Proof: As $\alpha, \beta, \theta, \gamma$ in (3.3.1) go to zero when U is in $B^K_\sigma(I, r)$ defined by (2.1.9) and r goes to zero, we see that, for r small enough (and σ fixed large enough) the matrix of symbols (recall (3.2.3))

$$P(U;\cdot) = \begin{bmatrix} 1+\alpha+\gamma & \beta-\theta \\ \beta+\theta & 1+\alpha-\gamma \end{bmatrix}$$

is invertible, with inverse

$$P(U;\cdot)^{-1} = (1+\alpha')\mathcal{I}_2 + \beta'\mathcal{L} + \theta'\mathcal{J} + \gamma'\mathcal{K}$$

where, setting $\Delta = (1+\alpha)^2 - \beta^2 - \gamma^2 + \theta^2$,

$$\alpha' = (1+\alpha)\Delta^{-1} - 1,\ \beta' = -\beta\Delta^{-1},\ \gamma' = -\gamma\Delta^{-1},\ \theta' = -\theta\Delta^{-1}. \tag{3.3.8}$$

We claim that the matrix P^{-1} belongs to $\mathcal{E}^m_{K,K',1}[r, N]$. Indeed, since the order of the symbols α, β, γ, δ is $m \le 0$, a Taylor expansion of Δ^{-1} when $(\alpha, \beta, \gamma, \theta)$ goes to zero implies that

$$\alpha', \beta' \in \Sigma\Gamma^m_{K,K',1}[r, N], \quad \gamma', \theta' \in \Sigma\Gamma^{\min\left(m-\frac{1}{2}, -\frac{3}{2}\right)}_{K,K',1}[r, N]$$

which is condition (3.3.2). Moreover, since $\alpha, \beta, \gamma, \delta$ satisfy (3.3.2) and (3.3.3) the symbols α', β' satisfy also (3.3.3).

Finally, as $S^{-1} = S$, the fact that P satisfies the anti-reality, parity preserving and anti-reversibility conditions (3.1.5), (3.1.6), (3.1.8), implies that the inverse matrix P^{-1} satisfies the same properties. Consequently, P^{-1} belongs to $\mathcal{E}^m_{K,K',1}[r, N]$.

Let us compute $(P \# P^{-1})_{\rho,N}$ using the composition formulas (2.4.1), (2.3.2), (2.3.3). The first term of the asymptotic expansions (2.3.2), (2.3.3), corresponding to $\ell = 0$, is the identity $\mathcal{I}_2$. The next term, corresponding to $\ell = 1$, is

$$\begin{aligned} &\frac{1}{2i}\big[(\partial_\xi P)(\partial_x P^{-1}) - (\partial_x P)(\partial_\xi P^{-1})\big] \\ &\qquad = \frac{1}{2i}\big[-(\partial_\xi P)P^{-1}(\partial_x P)P^{-1} + (\partial_x P)P^{-1}(\partial_\xi P)P^{-1}\big]. \end{aligned} \tag{3.3.9}$$

Write then the matrix of symbols P in (3.3.1) as

$$P = P_0 + \theta\mathcal{J} + \gamma\mathcal{K}, \quad P_0 = (1+\alpha)\mathcal{I}_2 + \beta\mathcal{L},$$

so that, by (3.3.2), $P - P_0$ is in $\Sigma\Gamma^{\min\left(m-\frac{1}{2}, -\frac{3}{2}\right)}_{K,K',1}[r, N] \otimes \mathcal{M}_2(\mathbb{C})$ as well as $P^{-1} - P_0^{-1} = -P^{-1}(P - P_0)P_0^{-1}$. If in (3.3.9), we replace P by P_0, and P^{-1} by P_0^{-1}, the error we generate is then in the space of symbols $\Sigma\Gamma^{\min\left(m-\frac{3}{2}, -\frac{5}{2}\right)}_{K,K',1}[r, N] \otimes \mathcal{M}_2(\mathbb{C})$. On the other hand, the right hand side

of (3.3.9) with P replaced by P_0 vanishes, since P_0 and P_0^{-1} and their derivatives are linear combinations of $\mathcal{I}_2$ and $\mathcal{L}$, so that commute between themselves. Finally, terms corresponding to $\ell \geq 2$ in (2.3.2), (2.3.3) applied to compute $P\#P^{-1}$ involve at least two ∂_ξ derivatives, so belong to $\Sigma\Gamma^{m-2}_{K,K',1}[r,N] \otimes \mathcal{M}_2(\mathbb{C})$. Consequently, as $m \leq 0$, the matrix of symbols

$$R = \mathcal{I}_2 - (P\#P^{-1})_{\rho,N} \in \Sigma\Gamma^{\min\left(m-\frac{3}{2},-2\right)}_{K,K',1}[r,N] \otimes \mathcal{M}_2(\mathbb{C})\,. \tag{3.3.10}$$

This result differs from the first condition in (3.3.7) by the more modest smoothing properties of R. In order to get (3.3.7) we have to construct a matrix of symbols Q, close to P^{-1} in the sense specified in (3.3.7), such that $P\#Q$ differs from the identity $\mathcal{I}_2$ up to a smoother symbol. We define

(3.3.11)
$$Q = P^{-1} + (P^{-1}\#R)_{\rho,N} + (P^{-1}\#R\#R)_{\rho,N} + \cdots + (P^{-1}\#\underbrace{R\#\cdots\#R}_{\ell})_{\rho,N}$$

for some ℓ to be chosen. In view of (3.3.10), the term involving ℓ' factors R ($0 \leq \ell' \leq \ell$) in the sum (3.3.11) belongs to $\Sigma\Gamma^{\ell'\min\left(m-\frac{3}{2},-2\right)}_{K,K',1}[r,N] \otimes \mathcal{M}_2(\mathbb{C})$. By (3.3.11) and (3.3.10) the matrix of symbols $Q - P^{-1}$ is in

$$\Sigma\Gamma^{\min\left(m-\frac{3}{2},-2\right)}_{K,K',1}[r,N] \otimes \mathcal{M}_2(\mathbb{C})$$

and therefore the second condition (3.3.7) holds. Moreover Q satisfies (3.3.2), (3.3.3) as this holds for P^{-1}.

Finally, it follows from the fact that P, P^{-1} satisfy (3.1.5), (3.1.6), (3.1.8), from the definition of R, and the remark after Definition 3.1.1, that the same holds true for R. Moreover, again by composition, conditions (3.1.5), (3.1.6), (3.1.8) are satisfied also by Q. Thus Q is in $\mathcal{E}^m_{K,K',1}[r,N]$. To conclude the proof we just have to prove that the first property (3.3.7) holds, if ℓ has been taken large enough in (3.3.11). By (3.3.11) and (3.3.10) we get

$$(P\#Q)_{\rho,N} = \sum_{\ell'=0}^{\ell} \left((\mathcal{I}_2 - R)\#\underbrace{R\#\cdots\#R}_{\ell'}\right)_{\rho,N} = \mathcal{I}_2 - (\underbrace{R\#\cdots\#R}_{\ell+1})_{\rho,N}$$

which is equal to $\mathcal{I}_2$ modulo symbols of order $(\ell+1)\min(m-\frac{3}{2},-2)$ as negative as we want if ℓ is taken large enough. This concludes the proof. □

Lemma 3.3.4 (Step of diagonalization) *Let m be in $]-\infty,\frac{3}{2}]$, K,K' in $\mathbb{N}$ with $K-1 \geq K' \geq 0$, N in $\mathbb{N}^*$, ρ in $\mathbb{N}$, $r>0$. Consider symbols*

$$\begin{aligned} &\lambda' \in \Sigma\Gamma^{\frac{3}{2}}_{K,K',1}[r,N], \quad b' \in \Sigma\Gamma^{m}_{K,K',1}[r,N] \\ &c' \in \Sigma\Gamma^{\min\left(m-\frac{1}{2},0\right)}_{K,K',1}[r,N], \quad d' \in \Sigma\Gamma^{1}_{K,K',1}[r,N] \end{aligned} \tag{3.3.12}$$

satisfying

$$\text{Im}\,\lambda'\,,\text{Im}\,b'\in\Sigma\Gamma^{-\frac{1}{2}}_{K,K',1}[r,N],\quad \text{Im}\,d'\in\Sigma\Gamma^{0}_{K,K',1}[r,N]. \tag{3.3.13}$$

Define

$$A'(U;t,x,\xi)=(m_\kappa(\xi)+\lambda')\mathcal{K}+b'\mathcal{J}+c'\mathcal{L}+d'\mathcal{I}_2 \tag{3.3.14}$$

and assume that A' satisfies the reality, parity preserving and reversibility properties (3.1.4), (3.1.6) and (3.1.7). Then there are

- *A matrix of symbols Q in $\mathcal{E}^{m-\frac{3}{2}}_{K,K',1}[r,N]$,*
- *A matrix P in $\mathcal{E}^{m-\frac{3}{2}}_{K,K',1}[r,N]$ such that $(P\#Q)_{\rho,N}-\mathcal{I}_2$ is in $\mathcal{E}^{-\rho}_{K,K',1}[r,N]$,*
- *Symbols*

$$\begin{aligned}&\lambda''\in\Sigma\Gamma^{\frac{3}{2}}_{K,K'+1,1}[r,N],\quad b''\in\Sigma\Gamma^{m-1}_{K,K'+1,1}[r,N],\\&c''\in\Sigma\Gamma^{m-\frac{3}{2}}_{K,K'+1,1}[r,N],\quad d''\in\Sigma\Gamma^{1}_{K,K'+1,1}[r,N],\end{aligned} \tag{3.3.15}$$

with

$$\text{Im}\,\lambda''\,,\ \text{Im}\,b''\in\Sigma\Gamma^{-\frac{1}{2}}_{K,K'+1,1}[r,N],\quad \text{Im}\,d''\in\Sigma\Gamma^{0}_{K,K'+1,1}[r,N]\,, \tag{3.3.16}$$

such that

$$A''=(m_\kappa(\xi)+\lambda'')\mathcal{K}+b''\mathcal{J}+c''\mathcal{L}+d''\mathcal{I}_2$$

satisfies the reality, parity preserving and reversibility properties (3.1.4), (3.1.6) and (3.1.7),

- *A smoothing operator $R(U;t)$ in $\Sigma\mathcal{R}^{-\rho+\frac{3}{2}}_{K,K'+1,1}[r,N]\otimes\mathcal{M}_2(\mathbb{C})$, satisfying conditions (3.1.14), (3.1.17) and (3.1.18),*

such that

$$Op^{\text{BW}}(Q)[D_t-Op^{\text{BW}}(A')]=[D_t-Op^{\text{BW}}(A'')]Op^{\text{BW}}(Q)+R(U;t). \tag{3.3.17}$$

Finally, in the case $m=\frac{3}{2}$, if there are ζ_1',ζ_2' in $\Sigma\mathcal{F}^{\mathbb{R}}_{K,K',1}[r,N]$ such that

$$\begin{aligned}&\lambda'(U;t,x,\xi)-m_\kappa(\xi)\zeta_1'(U;t,x)\in\Sigma\Gamma^{\frac{1}{2}}_{K,K',1}[r,N]\\&b'(U;t,x,\xi)-m_\kappa(\xi)\zeta_2'(U;t,x)\in\Sigma\Gamma^{\frac{1}{2}}_{K,K',1}[r,N],\end{aligned} \tag{3.3.18}$$

then, for some function ζ'' of $\Sigma\mathcal{F}^{\mathbb{R}}_{K,K'+1,1}[r,1]$, one has

$$\lambda''(U;t,x,\xi)-m_\kappa(\xi)\zeta''(U;t,x)\in\Sigma\Gamma^{\frac{1}{2}}_{K,K'+1,1}[r,N]. \tag{3.3.19}$$

Note that the off-diagonal symbols b'', c'' of A'' in (3.3.15) are of lower order (actually 1-smoother) with respect to those of A', namely b', c' in (3.3.12). On the other hand the order of the diagonal symbols is unchanged. This step "consumes 1-time derivative", i.e. note that the second index $K'+1$ in (3.3.15) is larger than K' in (3.3.12).

Proof: The eigenvalues of the matrix $A'(U;t,x,\xi)$ in (3.3.14) are given by

$$d'(U;t,x,\xi) \pm \Big((m_\kappa + \lambda'(U;t,x,\xi))^2 + c'(U;t,x,\xi)^2 - b'(U;t,x,\xi)^2\Big)^{\frac{1}{2}}$$

when ξ stays outside some fixed neighborhood of zero, and when U belongs to $B^K_\sigma(I,r)$, for some small enough r so that the argument under the square root stays in the domain where the principal determination of that function is well defined. Take a cut-off function χ_1 in $C^\infty(\mathbb{R})$, even, real valued, $0 \leq \chi_1 \leq 1$, with χ_1 equal to zero on a convenient neighborhood of zero and equal to one outside a larger neighborhood of zero. Define

$$\begin{aligned}(3.3.20)\quad \omega_\pm(U;t,x,\xi) &= d'(U;t,x,\xi)\\ &\pm (m_\kappa(\xi) + \lambda'(U;t,x,\xi))\left[1 + \frac{c'^2 - b'^2}{(m_\kappa(\xi)+\lambda')^2}\chi_1(\xi)\right]^{\frac{1}{2}}\end{aligned}$$

which coincide with the eigenvalues of A' outside a neighborhood of zero. By (3.3.12), (3.3.13), the symbols $\omega_\pm \mp m_\kappa(\xi)$ are in $\Sigma\Gamma^{\frac{3}{2}}_{K,K',1}[r,N]$ and the imaginary part of the term under the square root in (3.3.20) is of order at most -2. Together with the assumption (3.3.13) on λ', this shows that $\mathrm{Im}\,(\omega_\pm - d')$ is in $\Sigma\Gamma^{-\frac{1}{2}}_{K,K',1}[r,N]$. It follows that the diagonal matrix of symbols

$$\left[\begin{smallmatrix}\omega_+ & 0\\ 0 & \omega_-\end{smallmatrix}\right]$$

may be written as a contribution to $(m_\kappa(\xi)+\lambda'')\mathcal{K} + d''\mathcal{I}_2$ in A'', with λ'', d'' satisfying (3.3.15), (3.3.16). Moreover, if $m = \frac{3}{2}$ and λ', b' are of the form (3.3.18), we get that λ'' is of the form (3.3.19).

The fact that by assumption $A'(U;t,x,\xi)$ satisfies (3.1.4) may be written

$$\begin{aligned}(3.3.21)\quad \bar m^\vee_\kappa(\xi) = m_\kappa(\xi),\ \bar\lambda'^\vee(U;\cdot) = \lambda'(U;\cdot),\ \bar b'^\vee(U;\cdot) = b'(U;\cdot)\\ \bar c'^\vee(U;\cdot) = -c'(U;\cdot),\ \bar d'^\vee(U;\cdot) = -d'(U;\cdot).\end{aligned}$$

In the same way, the reversibility condition (3.1.7) translates as

$$\begin{aligned}(3.3.22)\quad \lambda'(U;-t,\cdot) = \lambda'(U_S;t,\cdot),\quad b'(U;-t,\cdot) = b'(U_S;t,\cdot)\\ c'(U;-t,\cdot) = -c'(U_S;t,\cdot),\quad d'(U;-t,\cdot) = -d'(U_S;t,\cdot)\,.\end{aligned}$$

As a consequence the symbol

$$(3.3.23)\quad D(U;t,x,\xi) = \Big(m_\kappa(\xi)+\lambda'(U;t,x,\xi)\Big)\left[1+\Big(1+\frac{c'^2-b'^2}{(m_\kappa+\lambda')^2}\chi_1(\xi)\Big)^{\frac{1}{2}}\right]$$

satisfies

$$\bar{D}^{\vee}(U;\cdot) = D(U;\cdot),\ D(U;-t,\cdot) = D(U_S;t,\cdot)\,. \tag{3.3.24}$$

A matrix which diagonalizes A' on the domain where $\chi_1(\xi) = 1$ is

$$P(U;t,x,\xi) = \frac{\chi_1(\xi)}{\Big(1 - \frac{b'^2 - c'^2}{D^2}\Big)^{\frac{1}{2}}} \begin{bmatrix} 1 & \frac{b'-c'}{D} \\ \frac{b'+c'}{D} & 1 \end{bmatrix}.$$

We claim that P belongs to $\mathcal{E}^{m-\frac{3}{2}}_{K,K',1}[r,N]$. Indeed we may write P as

$$P(U;t,x,\xi) = (1+\alpha)\mathcal{I}_2 + \beta\mathcal{L} + \theta\mathcal{J} \tag{3.3.25}$$

with symbols α, β in $\Sigma\Gamma^{m-\frac{3}{2}}_{K,K',1}[r,N]$ and θ in $\Sigma\Gamma^{\min\left(m-2,-\frac{3}{2}\right)}_{K,K',1}[r,N]$. Thus α, β, θ satisfy (3.3.2) with m replaced by $m - \frac{3}{2}$. Moreover, it follows from (3.3.21), (3.3.22), (3.3.24) that

$$\begin{aligned} \bar{\alpha}^{\vee}(U;\cdot) = \alpha(U;\cdot), \quad \alpha(U;-t,\cdot) = \alpha(U_S;t,\cdot) \\ \bar{\beta}^{\vee}(U;\cdot) = \beta(U;\cdot), \quad \beta(U;-t,\cdot) = \beta(U_S;t,\cdot) \\ \bar{\theta}^{\vee}(U;\cdot) = -\theta(U;\cdot), \quad \theta(U;-t,\cdot) = -\theta(U_S;t,\cdot) \end{aligned}$$

and therefore P satisfies (3.1.5) and (3.1.8). It obviously satisfies as well (3.1.6). We have seen that $\operatorname{Im} D = \operatorname{Im}(\omega_+ - d') + \operatorname{Im}\lambda'$ is in $\Sigma\Gamma^{-\frac{1}{2}}_{K,K',1}[r,N]$. It follows that $\operatorname{Im}\alpha, \operatorname{Im}\beta$ are of order $\min\left(m-\frac{3}{2}, -\frac{3}{2}\right)$, so that (3.3.3) is satisfied with m replaced by $m - \frac{3}{2}$. Consequently P is in $\mathcal{E}^{m-\frac{3}{2}}_{K,K',1}[r,N]$. Also the matrix

$$P^{-1} = (1+\alpha)\mathcal{I}_2 - \beta\mathcal{L} - \theta\mathcal{J} \tag{3.3.26}$$

(when $\chi_1 \equiv 1$) is in $\mathcal{E}^{m-\frac{3}{2}}_{K,K',1}[r,N]$. By construction

$$P^{-1}A'P = \begin{bmatrix} \omega_+ & 0 \\ 0 & \omega_- \end{bmatrix} = (m_\kappa(\xi) + \tilde{\lambda}'(U;t,x,\xi))\mathcal{K} + d'\mathcal{I}_2 \tag{3.3.27}$$

on the domain where $\chi_1 \equiv 1$, for a symbol $\tilde{\lambda}'$ in $\Sigma\Gamma^{\frac{3}{2}}_{K,K',1}[r,N]$ with $\operatorname{Im}\tilde{\lambda}'$ in $\Sigma\Gamma^{-\frac{1}{2}}_{K,K',1}[r,N]$. Moreover, the matrix (3.3.27) satisfies (3.1.4), (3.1.6) and (3.1.7), and (3.3.19) is satisfied by $\tilde{\lambda}'$ if we assume (3.3.18).

We apply next to P lemma 3.3.3, with m replaced by $m - \frac{3}{2}$. We obtain Q in $\mathcal{E}^{m-\frac{3}{2}}_{K,K',1}[r,N]$ satisfying (3.3.7) (with m replaced by $m - \frac{3}{2}$). We write

$$(Q\#A'\#P)_{\rho,N} = ((Q - P^{-1})\#A'\#P)_{\rho,N} + (P^{-1}\#A'\#P)_{\rho,N}. \tag{3.3.28}$$

Since $Q - P^{-1}$ is of order $\min(m-3,-2)$, A' is of order $3/2$ and P of order $m - \frac{3}{2} \leq 0$, the first term in the right hand side has coefficients in $\Sigma\Gamma_{K,K',1}^{\min\left(m-\frac{3}{2},-\frac{1}{2}\right)}[r,N]$ and we may write the matrix

$$((Q-P^{-1})\#A'\#P)_{\rho,N} = \lambda''\mathcal{K} + b''\mathcal{J} + c''\mathcal{L} + d''\mathcal{I}_2 \tag{3.3.29}$$

with λ'', b'', c'', d'' satisfying conditions (3.3.15). Next we write the last term of (3.3.28) as

$$(P^{-1}\#A'\#P)_{\rho,N} = P^{-1}A'P + R\,, \tag{3.3.30}$$

where R is obtained from the terms indexed by $1 \leq \ell < \rho$ in the composition formulas (2.3.2), (2.3.3). By (3.3.25), (3.3.14) and (3.3.26) the matrix R is given by the terms of rank $1 \leq \ell < \rho$ in the expansion

$$\begin{aligned}\Big(((1+\alpha)\mathcal{I}_2 - \beta\mathcal{L} - \theta\mathcal{J})\#((m_\kappa+\lambda')\mathcal{K} + b'\mathcal{J} + c'\mathcal{L} + d'\mathcal{I}_2) \\ \#((1+\alpha)\mathcal{I}_2 + \beta\mathcal{L} + \theta\mathcal{J})\Big)_\rho\,.\end{aligned} \tag{3.3.31}$$

To finish the proof of lemma 3.3.4, we need the following result.

Lemma 3.3.5 *The matrix R in (3.3.30) may be written as*

$$R = \lambda''\mathcal{K} + b''\mathcal{J} + c''\mathcal{L} + d''\mathcal{I}_2 \tag{3.3.32}$$

where λ'', b'', c'', d'' satisfy conditions (3.3.15), (3.3.16) and the matrix R satisfies (3.1.4), (3.1.6), (3.1.7). Moreover, λ'' satisfies (3.3.19).

Proof: Consider first those contributions to (3.3.31) coming from one of the $\theta\mathcal{J}$ factors. Since θ is of order $\min\left(m-2,-\frac{3}{2}\right)$, and the other factors are at most of order $\frac{3}{2}$ (resp. 0) if they come from the middle (resp. one of the extreme) terms, the overall order would be $\min\left(m-\frac{1}{2},0\right)$. But since we are interested only on those contributions coming from the terms of the asymptotic expansion except the first one, we gain at least one extra order, i.e. we obtain a contribution to (3.3.32) of order $\min\left(m-\frac{3}{2},-1\right)$, so that conditions (3.3.15), (3.3.16) are largely satisfied. We are thus reduced to the study of elements of rank at least one in the asymptotic expansion (3.3.31) with θ removed. Notice also that c' being of order $\min\left(m-\frac{1}{2},0\right)$, the contribution to those elements of the asymptotic expansion (3.3.31) satisfy as well (3.3.15), (3.3.16). We may as well discard c', and are reduced to examine terms of rank at least one in the following asymptotic expansions:

$$\Big(((1+\alpha)\mathcal{I}_2 - \beta\mathcal{L})\#((m_\kappa+\lambda')\mathcal{K})\#((1+\alpha)\mathcal{I}_2 + \beta\mathcal{L})\Big)_\rho \tag{3.3.33}$$

$$\Big(((1+\alpha)\mathcal{I}_2 - \beta\mathcal{L})\#(b'\mathcal{J})\#((1+\alpha)\mathcal{I}_2 + \beta\mathcal{L})\Big)_\rho \tag{3.3.34}$$

$$\Big(((1+\alpha)\mathcal{I}_2-\beta\mathcal{L})\#(d'\mathcal{I}_2)\#((1+\alpha)\mathcal{I}_2+\beta\mathcal{L})\Big)_\rho . \tag{3.3.35}$$

Consider first (3.3.33), that may be written using the relations $\mathcal{L}\mathcal{K}=-\mathcal{K}\mathcal{L}=\mathcal{J}$, $\mathcal{L}\mathcal{K}\mathcal{L}=-\mathcal{K}$ as

$$\Big[(1+\alpha)\#(m_\kappa+\lambda')\#(1+\alpha)+\beta\#(m_\kappa+\lambda')\#\beta\Big]_\rho\mathcal{K} - \Big[(1+\alpha)\#(m_\kappa+\lambda')\#\beta+\beta\#(m_\kappa+\lambda')\#(1+\alpha)\Big]_\rho\mathcal{J}.$$

The expansions (2.3.2), (2.3.3) (see in particular (2.3.4)) show that the terms of rank one in the coefficients of $\mathcal{K}$ and $\mathcal{J}$ above vanish. Consequently, the contribution to R come only from terms of rank at least two in the expansion. As $m_\kappa+\lambda'$ is of order $\frac{3}{2}$ and α,β of order $m-\frac{3}{2}$, we get expressions of order $m-2\leq-\frac{1}{2}$, so that conditions (3.3.15), (3.3.16) are satisfied.

In the same way, using $\mathcal{L}\mathcal{J}=-\mathcal{J}\mathcal{L}=\mathcal{K}$, $\mathcal{L}\mathcal{J}\mathcal{L}=-\mathcal{J}$, we write (3.3.34) as

$$\Big[(1+\alpha)\#b'\#(1+\alpha)+\beta\#b'\#\beta\Big]_\rho\mathcal{J}-\Big[\beta\#b'\#(1+\alpha)+(1+\alpha)\#b'\#\beta\Big]_\rho\mathcal{K}$$

and conclude that the corresponding contribution to R are of order $m-2$. Finally, using that $\mathcal{L}^2=\mathcal{I}_2$, write (3.3.35) as

$$\Big[(1+\alpha)\#d'\#(1+\alpha)-\beta\#d'\#\beta\Big]_\rho\mathcal{I}_2 + \Big[(1+\alpha)\#d'\#\beta-\beta\#d'\#(1+\alpha)\Big]_\rho\mathcal{L}.$$

Again, by (2.3.4) applied with $a=c$, we see that the coefficient of $\mathcal{I}_2$ is of order -1, since d' is of order 1, so that the conditions imposed to d'' in (3.3.15), (3.3.16) are largely satisfied. In the coefficients of $\mathcal{L}$, we just use that α,β are of order $m-\frac{3}{2}\leq 0$, and d' of order 1, to conclude that terms of rank one of the expansion are at most of order $m-\frac{3}{2}$, which is the condition imposed to c'' in (3.3.15).

To finish the proof of the lemma, we have to check that conditions (3.1.4), (3.1.6), (3.1.7) hold true for the matrix R. This follows from the fact that these conditions are satisfied by A' and that P, P^{-1} satisfy (3.1.5), (3.1.6), (3.1.8). Notice also that (3.3.19) is satisfied by λ'' since the contributions to this symbol are all of non positive order. □

End of the proof of lemma 3.3.4: Recall that we have constructed matrices P,Q in $\mathcal{E}^{m-\frac{3}{2}}_{K,K',1}[r,N]$ such that, according to (3.3.28), (3.3.30), (3.3.29), (3.3.27) and (3.3.32),

$$(Q\#A'\#P)_{\rho,N}=A''\stackrel{\rm def}{=}(m_\kappa(\xi)+\lambda'')\mathcal{K}+b''\mathcal{J}+c''\mathcal{L}+d''\mathcal{I}_2 \tag{3.3.36}$$

for new values of the symbols λ'',b'',c'',d'' satisfying (3.3.15), (3.3.16), the matrix A'' satisfying as well (3.1.5), (3.1.6) and (3.1.7). We have checked

also that (3.3.19) holds under the assumptions (3.3.18). We are left with showing that (3.3.17) holds, up to a modification of the matrix A''. Compute from (3.3.36)

$$
\begin{aligned}
(3.3.37)\quad & (D_t - Op^{\mathrm{BW}}(A'')) \circ Op^{\mathrm{BW}}(Q) \\
& = Op^{\mathrm{BW}}(Q) D_t - Op^{\mathrm{BW}}\big((Q\#A'\#P)_{\rho,N}\big) \circ Op^{\mathrm{BW}}(Q) + Op^{\mathrm{BW}}(D_t Q) \\
& = Op^{\mathrm{BW}}(Q)\big[D_t - Op^{\mathrm{BW}}(A')\big] - Op^{\mathrm{BW}}\big((Q\#A'\#(P\#Q - \mathcal{I}_2))_{\rho,N}\big) \\
& \qquad + Op^{\mathrm{BW}}(D_t Q) + R(U;t)
\end{aligned}
$$

where $R(U;t)$, the smoothing remainder coming from (2.3.5), (2.3.6), may be taken in $\Sigma\mathcal{R}^{-\rho+\frac{3}{2}}_{K,K',1}[r,N] \otimes \mathcal{M}_2(\mathbb{C})$. Moreover, $R(U;t)$ will satisfy conditions (3.1.14), (3.1.17), (3.1.18), as it is obtained from the composition of one operator satisfying these conditions and of operators obeying conditions (3.1.15), (3.1.17) and (3.1.19). Because of (3.3.7), the fact that A' is of order $3/2$, and Q is of order ≤ 0, the term

$$
(3.3.38)\qquad Op^{\mathrm{BW}}\big((Q\#A'\#(P\#Q - \mathcal{I}_2))_{\rho,N}\big) \in \Sigma\mathcal{R}^{-\rho+\frac{3}{2}}_{K,K',1}[r,N] \otimes \mathcal{M}_2(\mathbb{C})
$$

may be incorporated as well to $R(U;t)$. To conclude the proof of (3.3.17), it remains to write

$$
(3.3.39)\qquad Op^{\mathrm{BW}}(D_t Q) = Op^{\mathrm{BW}}(\tilde{A}'') \circ Op^{\mathrm{BW}}(Q) + R(U;t)
$$

for another smoothing operator $R(U;t)$ in $\Sigma\mathcal{R}^{-\rho+\frac{3}{2}}_{K,K'+1,1}[r,N]$, and for a matrix of symbols $\tilde{A}''$ satisfying the same conditions as A''. In conclusion (3.3.37), (3.3.38), (3.3.39) will imply (3.3.17) with A'' replaced by $A'' + \tilde{A}''$ in its right hand side. In order to prove (3.3.39) we remark first that, since Q is in $\mathcal{E}^{m-\frac{3}{2}}_{K,K',1}[r,N]$, then, recalling Definition 3.3.1, Q is in $\Sigma\Gamma^{m-\frac{3}{2}}_{K,K',1}[r,N] \otimes \mathcal{M}_2(\mathbb{C})$, and so, by lemma 2.2.6,

$$
(3.3.40)\qquad D_t Q \in \Sigma\Gamma^{m-\frac{3}{2}}_{K,K'+1,1}[r,N] \otimes \mathcal{M}_2(\mathbb{C})\,.
$$

Moreover, if Q satisfies (3.1.5) (resp. (3.1.6), resp. (3.1.8)), then $D_t Q$ satisfies (3.1.4) (resp. (3.1.6), resp. (3.1.7)). If we apply (3.3.7), we may write

$$
Op^{\mathrm{BW}}(D_t Q) = Op^{\mathrm{BW}}\big((D_t Q\#P)_{\rho,N}\big) \circ Op^{\mathrm{BW}}(Q)
$$

modulo an operator $R(U;t)$ in $\Sigma\mathcal{R}^{-\rho+\frac{3}{2}}_{K,K'+1,1}[r,N] \otimes \mathcal{M}_2(\mathbb{C})$, satisfying (3.1.14), (3.1.17), (3.1.18). Thus (3.3.39) holds with $\tilde{A}'' = (D_t Q\#P)_{\rho,N}$ and we are left with checking that $\tilde{A}''$ satisfies the same properties as A''. By (3.3.40) and since P is of order zero, conditions (3.3.15) hold. We have to check

(3.3.16). Notice that, as P, Q are in $\mathcal{E}^{m-\frac{3}{2}}_{K,K',1}[r,N]$, conditions (3.3.2), (3.3.3) imply that we may write

$$P = (1+\alpha)\mathcal{I}_2 + \beta\mathcal{L} + \text{ order } (-\frac{3}{2}),\ Q = (1+\alpha')\mathcal{I}_2 + \beta'\mathcal{L} + \text{ order } (-\frac{3}{2}).$$

It follows that in $(D_tQ\#P)_{\rho,N}$ the only contributions of order strictly larger than $-\frac{3}{2}$ are multiples of $\mathcal{I}_2$ and $\mathcal{L}$, and are of non positive order. This shows that (3.3.16) holds and concludes the proof, since (3.1.4), (3.1.6), (3.1.7) hold, as they do for D_tQ. □

Proof of Proposition 3.3.2: We shall apply iteratively ρ-times lemma 3.3.4, obtaining off-diagonal para-differential operators which are regularizing enough to be incorporated to the smoothing term in (3.3.6). Consider a solution U of the system (3.2.2). Apply lemma 3.3.4 to the matrix $A' = A$ of equation (3.2.2) and $m = \frac{3}{2}$, $K' = 1$. We get a matrix $A_1 = A''$ whose coefficients satisfy (3.3.15), (3.3.16) with $m = \frac{3}{2}$, $K' = 1$, and (3.1.4), (3.1.6), (3.1.7), and a matrix Q_1 in $\mathcal{E}^0_{K,1,1}[r,N]$ such that by (3.3.17)

$$(D_t - Op^{\mathrm{BW}}(A_1)) \circ Op^{\mathrm{BW}}(Q_1) = Op^{\mathrm{BW}}(Q_1)[D_t - Op^{\mathrm{BW}}(A)] + R_1(U;t)$$

with R_1 in $\Sigma\Gamma^{-\rho+\frac{3}{2}}_{K,2,1}[r,N] \otimes \mathcal{M}_2(\mathbb{C})$, satisfying (3.1.14), (3.1.17) and (3.1.18). We iterate the construction, getting matrices

$$Q_\ell \in \mathcal{E}^{1-\ell}_{K,\ell,1}[r,N]\,, \quad A_\ell = (m_\kappa(\xi) + \lambda_\ell)\mathcal{K} + b_\ell\mathcal{J} + c_\ell\mathcal{L} + d_\ell\mathcal{I}_2$$

where the coefficients $\lambda_\ell, b_\ell, c_\ell, d_\ell$ satisfy (3.3.15), (3.3.16) with m replaced by $\frac{3}{2} - (\ell - 1)$, and K' replaced by ℓ, A_ℓ satisfying also (3.1.4), (3.1.6) and (3.1.7). We get

$$\begin{aligned}(3.3.41)\quad &\big(D_t - Op^{\mathrm{BW}}(A_\ell)\big)Op^{\mathrm{BW}}(Q_\ell) \circ \cdots \circ Op^{\mathrm{BW}}(Q_1) \\ &\qquad = Op^{\mathrm{BW}}(Q_\ell) \circ \cdots \circ Op^{\mathrm{BW}}(Q_1)\big(D_t - Op^{\mathrm{BW}}(A)\big) + R_\ell(U;t)\end{aligned}$$

for some new $R_\ell(U;t)$ in $\Sigma\mathcal{R}^{-\rho+\frac{3}{2}}_{K,\ell+1,1}[r,N] \otimes \mathcal{M}_2(\mathbb{C})$. Since

$$b_\ell\mathcal{J} \in \Sigma\Gamma^{\frac{3}{2}-\ell}_{K,\ell+1,1}[r,N] \otimes \mathcal{M}_2(\mathbb{C})\,, \quad c_\ell\mathcal{L} \in \Sigma\Gamma^{\frac{3}{2}-\ell-\frac{1}{2}}_{K,\ell+1,1}[r,N] \otimes \mathcal{M}_2(\mathbb{C})\,,$$

we see, by the last remark after Proposition 2.2.4, that for $\ell = \rho$ we obtain contributions that may be as well incorporated to the smoothing term in (3.3.6). We thus define $A^{(1)}$ in (3.3.5) taking only the remaining components of A_ρ, i.e.

$$A^{(1)}(U;t,x,\xi) = (m_\kappa(\xi) + \lambda_\rho)\mathcal{K} + d_\rho\mathcal{I}_2$$

with symbols λ_ρ that may be written by (3.3.19) as $m_\kappa\zeta^{(1)} + \lambda^{(1)}_{1/2}$ with $\lambda^{(1)}_{1/2}$ in $\Sigma\Gamma^{\frac{1}{2}}_{K,\rho+1,1}[r,N]$ satisfying (3.3.4), and with d_ρ that brings $\lambda^{(1)}_1$ in (3.3.5). We define

$$(3.3.42)\qquad W = Op^{\mathrm{BW}}(Q_\rho) \circ \cdots \circ Op^{\mathrm{BW}}(Q_1)U.$$

If we make act (3.3.41) with $\ell = \rho$ on U, we deduce from (3.2.2)

$$(D_t - Op^{\mathrm{BW}}(A^{(1)}(U;t,\cdot))W = R(U;t)U \tag{3.3.43}$$

for some new $R(U;t)$ in $\Sigma\mathcal{R}^{-\rho+\frac{3}{2}}_{K,\rho+1,1}[r,N]\otimes\mathcal{M}_2(\mathbb{C})$. To finish the proof of the proposition, we still have to check that one may write the right hand side of the equation (3.3.43) as in (3.3.6) and to construct the matrix P. We define

$$P = (P_1 \# \cdots \# P_\rho)_{\rho,N}$$

where P_ℓ is the matrix associated to Q_ℓ by lemma 3.3.3. By (3.3.7), the operator

$$Op^{\mathrm{BW}}((P_\ell \# Q_\ell)_{\rho,N}) - \mathrm{Id} = R_\ell(U;t) \in \Sigma\mathcal{R}^{-\rho}_{K,\ell,1}[r,N]\otimes\mathcal{M}_2(\mathbb{C})$$

and satisfies (3.1.15), (3.1.17), (3.1.19). It follows that

$$Op^{\mathrm{BW}}((P\#Q)_{\rho,N}) - \mathrm{Id} = \tilde{R}(U;t) \in \Sigma\mathcal{R}^{-\rho}_{K,\rho,1}[r,N]\otimes\mathcal{M}_2(\mathbb{C})$$

and satisfies as well (3.1.15), (3.1.17), (3.1.19). By (3.3.42), we may write

$$U = Op^{\mathrm{BW}}(P)W - \tilde{R}(U;t)U.$$

Inserting this expression of U in the right hand side of (3.3.43) repeatedly, we see, by Proposition 2.4.2 and the last remark following Definition 2.2.3, that we may write $R(U;t)U$ as $R'(U;t)W + R''(U;t)U$ with R', R'' satisfying the properties of the statement. This concludes the proof. □

Chapter 4

Reduction to a constant coefficients operator and proof of the main theorem

In this chapter we shall reduce the operator $Op^{\mathrm{BW}}(A^{(1)}(U;t,\cdot))$ in (3.3.6), given in terms of the diagonal matrix $A^{(1)}$ defined in (3.3.5), to a constant coefficients operator, up to smoothing operators. We shall do that first for terms of higher order, and then for the lower order contributions.

4.1 Reduction to constant coefficients of the highest order part

We apply Proposition 3.3.2 with ρ replaced by $\rho' = \rho + m$, for some integer m to be chosen in function of N (in Proposition 4.1.1 we require $m \geq N + \frac{3}{2}$). Recalling the form (3.3.5) of the diagonal matrix $A^{(1)}$, we may rewrite equation (3.3.6) as the system

$$D_t W - Op^{\mathrm{BW}} \begin{bmatrix} m_\kappa(\xi)(1+\zeta^{(1)}(U;\cdot)) + \lambda^{(1)}_{1/2} + \lambda^{(1)}_1 & 0 \\ 0 & -m_\kappa(\xi)(1+\zeta^{(1)}(U;\cdot)) - \lambda^{(1)}_{1/2} + \lambda^{(1)}_1 \end{bmatrix} W = R'(U;t)W + R''(U;t)U \tag{4.1.1}$$

with $R'(U;t)$ (resp. $R''(U;t)$) in the class $\Sigma\mathcal{R}^{-\rho'+\frac{3}{2}}_{K,\rho'+1,1}[r,N] \otimes \mathcal{M}_2(\mathbb{C})$ (resp. in the class $\mathcal{R}^{-\rho'+\frac{3}{2}}_{K,\rho'+1,N}[r] \otimes \mathcal{M}_2(\mathbb{C})$), satisfying the reality, parity preserving, reversibility conditions (3.1.14), (3.1.17) and (3.1.18). Moreover $A^{(1)}$ satisfies the reality, parity preserving, reversibility conditions (3.1.4), (3.1.6), (3.1.7), with coefficients verifying (3.3.4). We rewrite explicitly all these

conditions as

$$
\begin{aligned}
&\lambda_1^{(1)} \in \Sigma\Gamma^1_{K,\rho'+1,1}[r,N]\,, && \lambda_{1/2}^{(1)} \in \Sigma\Gamma^{\frac12}_{K,\rho'+1,1}[r,N],\\
&\operatorname{Im}\lambda_1^{(1)} \in \Sigma\Gamma^0_{K,\rho'+1,1}[r,N]\,, && \operatorname{Im}\lambda_{1/2}^{(1)} \in \Sigma\Gamma^{-\frac12}_{K,\rho'+1,1}[r,N],\\
&\zeta^{(1)} \in \Sigma\mathcal{F}^{\mathbb{R}}_{K,\rho'+1,1}[r,N]\,, && \operatorname{Im}\zeta^{(1)} = 0\,,
\end{aligned}
\tag{4.1.2}
$$

and, recalling that $m_\kappa(\xi)$ is real valued, even, and $S\mathcal{K} = -\mathcal{K}S$, $S^2 = I$,

$$
\begin{aligned}
\bar\lambda_1^{(1)}(U;t,x,-\xi) &= -\lambda_1^{(1)}(U;t,x,\xi)\\
\bar\lambda_{1/2}^{(1)}(U;t,x,-\xi) &= \lambda_{1/2}^{(1)}(U;t,x,\xi)
\end{aligned}
\tag{4.1.3}
$$

$$
\begin{aligned}
\lambda_1^{(1)}(U;t,-x,-\xi) &= \lambda_1^{(1)}(U;t,x,\xi)\\
\lambda_{1/2}^{(1)}(U;t,-x,-\xi) &= \lambda_{1/2}^{(1)}(U;t,x,\xi)\\
\zeta^{(1)}(U;t,-x) &= \zeta^{(1)}(U;t,x)
\end{aligned}
\tag{4.1.4}
$$

$$
\begin{aligned}
\lambda_1^{(1)}(U;-t,x,\xi) &= -\lambda_1^{(1)}(U_S;t,x,\xi)\\
\lambda_{1/2}^{(1)}(U;-t,x,\xi) &= \lambda_{1/2}^{(1)}(U_S;t,x,\xi)\\
\zeta^{(1)}(U;-t,x) &= \zeta^{(1)}(U_S;t,x)\,.
\end{aligned}
\tag{4.1.5}
$$

In Proposition 4.1.1 we shall conjugate system (4.1.1) under the paracomposition operator $\Phi_U^\star = \Omega_{B(U)}\mathcal{I}_2$ defined in section 2.5 induced by a diffeomorphism of $\mathbb{T}^1$,

$$
\Phi_U(t,x) = x + \beta(U;t,x)\,,
\tag{4.1.6}
$$

for a small periodic function $\beta(U;t,x)$ to be chosen, in such a way that in (4.1.13) the highest order coefficient in front of $m_\kappa(\xi)$ is constant in the space variable. It turns out that, denoting the inverse diffeomorphism of Φ_U by

$$
\Phi_U^{-1}(t,y) = y + \gamma(U;t,y)
\tag{4.1.7}
$$

we have to choose $\gamma(U;t,y)$ as the unique primitive with zero average of the zero mean periodic function

$$
(1+\underline{\zeta}(U;t))^{\frac23}[1+\zeta^{(1)}(U;t,y)]^{-\frac23} - 1
\tag{4.1.8}
$$

where

$$
\underline{\zeta}(U;t) = \left[\frac{1}{2\pi}\int_{\mathbb{T}^1}(1+\zeta^{(1)}(U;t,y))^{-\frac23}\,dy\right]^{-\frac32} - 1\,.
\tag{4.1.9}
$$

Notice that, as $\zeta^{(1)}(U;\cdot)$ vanishes at $U=0$, the above functions are well defined for U small enough. Moreover $\underline{\zeta}(U;t)$ is in $\Sigma\mathcal{F}^{\mathbb{R}}_{K,\rho'+1,1}[r,N]$ and, by the third condition in (4.1.5), satisfies $\underline{\zeta}(U;-t) = \underline{\zeta}(U_S;t)$.

Proposition 4.1.1 (Reduction of the highest order) *Let $\rho' = \rho+N+2$. There are:*
• *A function $\underline{\zeta}(U;t)$ in $\Sigma\mathcal{F}^{\mathbb{R}}_{K,\rho'+1,1}[r,N]$, independent of x, satisfying*

$$\underline{\zeta}(U;-t) = \underline{\zeta}(U_S;t), \tag{4.1.10}$$

• *A symbol $\tilde{\lambda}(U;t,\cdot)$ in $\Sigma\Gamma^{\frac{1}{2}}_{K,\rho'+1,1}[r,N]$ such that $\operatorname{Im}\tilde{\lambda}(U;t,\cdot)$ belongs to $\Sigma\Gamma^{-\frac{1}{2}}_{K,\rho'+1,1}[r,N]$, satisfying*

$$\begin{aligned} \overline{\tilde{\lambda}}^{\vee}(U;t,x,\xi) &= \tilde{\lambda}(U;t,x,\xi)\\ \tilde{\lambda}(U;t,-x,-\xi) &= \tilde{\lambda}(U;t,x,\xi)\\ \tilde{\lambda}(U;-t,x,\xi) &= \tilde{\lambda}(U_S;t,x,\xi), \end{aligned} \tag{4.1.11}$$

• *A symbol $\mu(U;t,\cdot)$ in $\Sigma\Gamma^{1}_{K,\rho'+2,1}[r,N]$ such that $\operatorname{Im}\mu(U;t,\cdot)$ belongs to the class $\Sigma\Gamma^{0}_{K,\rho'+2,1}[r,N]$ and*

$$\begin{aligned} \overline{\mu}^{\vee}(U;t,x,\xi) &= -\mu(U;t,x,\xi)\\ \mu(U;t,-x,-\xi) &= \mu(U;t,x,\xi)\\ \mu(U;-t,x,\xi) &= -\mu(U_S;t,x,\xi), \end{aligned} \tag{4.1.12}$$

• *Operators $R'(U;t)$ (resp. $R''(U;t)$) belonging to $\Sigma\mathcal{R}^{-\rho+\frac{3}{2}}_{K,\rho'+2,1}[r,N]\otimes\mathcal{M}_2(\mathbb{C})$ (resp. $\mathcal{R}^{-\rho+\frac{3}{2}}_{K,\rho'+2,N}[r]\otimes\mathcal{M}_2(\mathbb{C})$) satisfying the reality, parity preserving and reversibility conditions (3.1.14), (3.1.17) and (3.1.18),*

such that, if we set $V = \Omega_B(1)W = \Phi^{\star}_U W$ with W a solution to (4.1.1), then V satisfies the system

$$\begin{aligned} \Big(D_t - Op^{\mathrm{BW}}\big[\big((1+\underline{\zeta}(U;t))m_\kappa(\xi) + \tilde{\lambda}(U;t,\cdot)\big)\mathcal{K} + \mu(U;t,\cdot)\mathcal{I}_2\big]\Big)V\\ = R'(U;t)V + R''(U;t)U. \end{aligned} \tag{4.1.13}$$

Proof: Applying Proposition 2.5.9 with $K' = \rho'+1$, under the change of variable $V = \Omega_B(1)W$, system (4.1.1) transforms into

$$\begin{aligned} &D_tV =\\ &\Omega_B(1)Op^{\mathrm{BW}}\big[\big(m_\kappa(\xi)(1+\zeta^{(1)}(U;\cdot)) + \lambda^{(1)}_{1/2}(U;\cdot)\big)\mathcal{K} + \lambda^{(1)}_{1}(U;\cdot)\mathcal{I}_2\big]\Omega_B^{-1}(1)V\\ &+\big(\mathrm{Op}^{BW}(e_1(U;\cdot))\mathcal{I}_2\big)V + R(U;t)V + \Omega_B(1)R'(U;t)\Omega_B^{-1}(1)V + \Omega_B(1)R''(U;t)U \end{aligned} \tag{4.1.14}$$

where e_1 is in $\Sigma\Gamma^{1}_{K,\rho'+2,1}[r,N]$ with $\operatorname{Im} e_1$ in $\Sigma\Gamma^{-1}_{K,\rho'+2,1}[r,N]$ and $R(U;t)$ is in the space $\Sigma\mathcal{R}^{-\rho}_{K,\rho'+2,1}[r,N]$. Thus the operator

$$\mathrm{Op}^{BW}(e_1(U;\cdot))\mathcal{I}_2 \tag{4.1.15}$$

may be incorporated to $\mathrm{Op}^{BW}(\mu)\mathcal{I}_2$ in (4.1.13). Consider next the term

$$\Omega_B(1)R'(U;t)\Omega_B^{-1}(1)V + \Omega_B(1)R''(U;t)U. \tag{4.1.16}$$

Applying the remark following the proof of Theorem 2.5.8, since $R'(U;t)$ is in $\Sigma\mathcal{R}^{-\rho'+\frac{3}{2}}_{K,\rho'+1,1}[r,N]\otimes\mathcal{M}_2(\mathbb{C})$, we get that

$$\Omega_B(1)R'(U;t)\Omega_B^{-1}(1) \in \Sigma\mathcal{R}^{-\rho'+\frac{3}{2}+N}_{K,\rho'+1,1}[r,N]\otimes\mathcal{M}_2(\mathbb{C})$$

and, since $\rho' \geq \rho+N+\frac{3}{2}$, we get an operator of $\Sigma\mathcal{R}^{-\rho}_{K,\rho'+1,1}[r,N]\otimes\mathcal{M}_2(\mathbb{C})$. The last term in (4.1.16) belongs as well to $\mathcal{R}^{-\rho}_{K,\rho'+1,N}[r]\otimes\mathcal{M}_2(\mathbb{C})$. We finally analyze the contribution

$$\Omega_B(1)Op^{\mathrm{BW}}\Big[\big(m_\kappa(1+\zeta^{(1)})+\lambda^{(1)}_{1/2}\big)\mathcal{K}+\lambda^{(1)}_1\mathcal{I}_2\Big]\Omega_B^{-1}(1) \tag{4.1.17}$$

that, by Theorem 2.5.8 we may write as

$$Op^{\mathrm{BW}}\Big[\big(m_\kappa(1+\zeta^{(1)})+\lambda^{(1)}_{1/2}\big)_\Phi\mathcal{K}+(\lambda^{(1)}_1)_\Phi\mathcal{I}_2\Big]+R_2(U;t) \tag{4.1.18}$$

where R_2 belongs to $\Sigma\mathcal{R}^{-\rho+\frac{3}{2}}_{K,\rho'+1,1}[r,N]\otimes\mathcal{M}_2(\mathbb{C})$. According to (2.5.48)-(2.5.49) the symbol $\big(m_\kappa(1+\zeta^{(1)})+\lambda^{(1)}_{1/2}\big)_\Phi$ may be expanded as

$$m_\kappa\Big(\xi\partial_y(\Phi_U^{-1}(t,y))|_{y=\Phi_U(t,x)}\Big)(1+\zeta^{(1)}(U;t,\Phi_U(t,x)))+(\lambda^{(1)}_{1/2})^0_\Phi \tag{4.1.19}$$

modulo a symbol in $\Sigma\Gamma^{-\frac{1}{2}}_{K,\rho'+1,1}[r,N]$. But, according to (4.1.7), the definition of γ as a primitive of (4.1.8),

$$\partial_y(\Phi_U)^{-1}(t,y) = 1+\partial_y\gamma(U;t,y) = (1+\underline{\zeta}(U;t))^{\frac{2}{3}}(1+\zeta^{(1)}(U;t,y))^{-\frac{2}{3}}.$$

Since by (3.2.5), $m_\kappa(\xi)=\sqrt{\kappa}|\xi|^{\frac{3}{2}}(1+O(\xi^{-2}))$ when $|\xi|$ goes to $+\infty$, one checks that

$$m_\kappa(\xi)-m_\kappa\big(\xi(1+\underline{\zeta})^{\frac{2}{3}}(1+\zeta^{(1)})^{-\frac{2}{3}}\big)(1+\underline{\zeta})^{-1}(1+\zeta^{(1)})$$

belongs to $\Sigma\Gamma^{-\frac{1}{2}}_{K,\rho'+1,1}[r,N]$. It follows that

$$\big(m_\kappa(1+\zeta^{(1)})+\lambda^{(1)}_{1/2}\big)_\Phi = m_\kappa(\xi)(1+\underline{\zeta}(U;t))+(\lambda^{(1)}_{1/2})^0_\Phi$$

modulo a symbol of $\Sigma\Gamma^{-\frac{1}{2}}_{K,\rho'+1,1}[r,N]$. Since by (4.1.2) the imaginary term $\mathrm{Im}\,\lambda^{(1)}_{1/2}$ is of order $-\frac{1}{2}$, we see that $(\lambda^{(1)}_{1/2})^0_\Phi$ is a symbol of order $\frac{1}{2}$ whose imaginary part is of order $-\frac{1}{2}$. In the same way $(\lambda^{(1)}_1)_\Phi$ is equal to $(\lambda^{(1)}_1)^0_\Phi$ modulo a symbol in $\Sigma\Gamma^{-1}_{K,\rho'+1,1}[r,N]$. Hence (4.1.2) implies that $(\lambda^{(1)}_1)_\Phi$ is a

symbol of $\Sigma\Gamma^1_{K,\rho'+1,1}[r,N]$ whose imaginary part belongs to $\Sigma\Gamma^0_{K,\rho'+1,1}[r,N]$. We have thus written (4.1.18) as

$$Op^{\rm BW}\Big((m_\kappa(\xi)(1+\underline{\zeta}(U;t))+\tilde\lambda_{\frac12}(U;\cdot))\mathcal{K}+\tilde\lambda_1(U;\cdot)\mathcal{I}_2\Big)+R_2(U;t) \tag{4.1.20}$$

for symbols $\tilde\lambda_j\in\Sigma\Gamma^j_{K,\rho'+1,1}[r,N]$, $j=\frac12,1$, satisfying

$$\mathrm{Im}\,\tilde\lambda_{\frac12}\in\Sigma\Gamma^{-\frac12}_{K,\rho'+1,1}[r,N]\,,\quad \mathrm{Im}\,\tilde\lambda_1\in\Sigma\Gamma^0_{K,\rho'+1,1}[r,N]\,,$$

and a smoothing operator $R_2(U;t)$ in $\Sigma\mathcal{R}^{-\rho+\frac32}_{K,\rho'+1,1}[r,N]\otimes\mathcal{M}_2(\mathbb{C})$. Thus also (4.1.17) gives contributions of the form (4.1.13).

It remains to check that the algebraic properties of reality, parity and reversibility are preserved.

Since $\zeta^{(1)}(U;\cdot)$ is even in x, by the third condition (4.1.4), the real valued function $\gamma(U;\cdot)$ defined as a primitive of (4.1.8) is an odd element of $\Sigma\mathcal{F}^{\mathbb{R}}_{K,\rho'+1,1}[r,N]$. Moreover $\gamma(U;-t,y)=\gamma(U_S;t,y)$ by the third identity in (4.1.5), (4.1.9), and its definition as a primitive of (4.1.8). Consequently also the function $\beta(U;t,x)$ defined in (4.1.6) by the inverse diffeomorphism is odd, real valued and satisfies

$$\beta(U;-t,x)=\beta(U_S;t,x)\,. \tag{4.1.21}$$

It follows that the matrix symbol $B(U;\theta,t,x,\xi)\mathcal{I}_2$, with

$$B(U;\theta,t,x,\xi)=b(U;t,\theta,x)\xi=\frac{\beta(U;t,x)}{1+\theta\partial_x\beta(U;t,x)}\xi\,, \tag{4.1.22}$$

satisfies the reality, parity preserving and anti-reversibility properties (3.1.4), (3.1.6), (3.1.8). Hence, as the flow $\Omega_{B(U)}(1)$ defined in (2.5.13) is generated by the vector field $\mathrm{Op}^{BW}(iB(U;\cdot))$, we deduce that

$$\Omega_{B(U)}(1)\mathcal{I}_2\,,\ \Omega_{B(U)}(1)^{-1}\mathcal{I}_2\quad\text{satisfy}\quad(3.1.15),(3.1.17),(3.1.19)\,. \tag{4.1.23}$$

These elementary properties will be checked in the proof of lemma 4.2.2 below. As a consequence, since $R'(U;t)$, $R''(U;t)$ satisfy the conditions (3.1.14), (3.1.17) and (3.1.18), lemma 3.1.6 implies that $\Omega_B(1)R'(U;t)\Omega_B^{-1}(1)$ as well as $\Omega_B(1)R''(U;t)$ in (4.1.16) satisfy also the reality, parity, reversibility properties (3.1.14), (3.1.17) and (3.1.18).

Similarly, since the matrix $A^{(1)}$ in (3.3.5) satisfies (3.1.4), (3.1.6) and (3.1.7), the associated paradifferential operator, and thus the operator in (4.1.17) obtained by its conjugation with the flow $\Omega_B(1)$, satisfies the reality (3.1.14), parity preserving (3.1.17) and reversibility (3.1.18) properties. Thus the same operator in (4.1.18) and (4.1.20) satisfies as well these properties. We now write (4.1.20) as the sum of a paradifferential operator and a smoothing remainder that satisfy, each of them separately, the properties

of reality, parity preserving and reversibility. For the reversibility property, we write (4.1.20) as the sum of the paradifferential operator

$$\begin{aligned}
&\frac{1}{2}\Big[Op^{\mathrm{BW}}\Big((m_\kappa(1+\underline{\zeta}(U;t))+\tilde\lambda_{\frac12}(U;t,\cdot))\mathcal{K}+\tilde\lambda_1(U;t,\cdot)\mathcal{I}_2\Big)\\
&\quad - SOp^{\mathrm{BW}}\Big((m_\kappa(1+\underline{\zeta}(U_S;-t))+\tilde\lambda_{\frac12}(U_S;-t,\cdot))\mathcal{K}+\tilde\lambda_1(U_S;-t,\cdot)\mathcal{I}_2\Big)S\Big]\\
&= Op^{\mathrm{BW}}\Big(\Big(m_\kappa\big(1+\frac{\underline{\zeta}(U;t)+\underline{\zeta}(U_S;-t)}{2}\big)+\frac{1}{2}\big(\tilde\lambda_{\frac12}(U;t,\cdot)+\tilde\lambda_{\frac12}(U_S;-t,\cdot)\big)\Big)\mathcal{K}\\
&\qquad\qquad + \frac{1}{2}(\tilde\lambda_1(U;t,\cdot)-\tilde\lambda_1(U_S;-t,\cdot))\mathcal{I}_2\Big)
\end{aligned}$$

whose matrix symbol satisfies (3.1.7) and the smoothing operator $\frac12\big[R_2(U;t)-SR_2(U_S;-t)S\big]$ that satisfies (3.1.18). Then we decompose this operator arguing in a similar way to ensure also the reality and the parity conditions. This shows that the contribution of (4.1.17) to (4.1.14) may be written as in (4.1.13), with symbols satisfying conditions (4.1.10), (4.1.11), (4.1.12), that are a translation of (3.1.4), (3.1.6), (3.1.7) for the corresponding matrices, and with a smoothing term $R'(U;t)$ satisfying (3.1.14), (3.1.17), (3.1.18).

Consider now (4.1.15) and the remainder $R(U;t)$ in (4.1.14). By Proposition 2.5.9 we have written

$$\Omega_{B(U)}(1)D_t\Omega^{-1}_{B(U)}(1)\mathcal{I}_2 = \mathrm{Op}^{BW}(e_1(U;\cdot))\mathcal{I}_2 + R(U;t) \tag{4.1.24}$$

as the sum of a paradifferential and a smoothing operator. We first show that

$$\Omega_{B(U)}(1)D_t\Omega^{-1}_{B(U)}(1)\mathcal{I}_2 = -D_t\Omega_{B(U)}(1)\Omega^{-1}_{B(U)}(1)\mathcal{I}_2$$

satisfies the reality, parity preserving and reversibility properties (3.1.14), (3.1.17) and (3.1.18). By (4.1.23) and lemma 3.1.6 it is sufficient to prove that $D_t\Omega_{B(U)}(1)\mathcal{I}_2$ satisfies (3.1.14), (3.1.17), (3.1.18). Recalling the definition of $\Omega_{B(U)}(1)$ in (2.5.13) this is implied by the fact that $D_t\mathrm{Op}^{BW}(iB(U;\cdot))\mathcal{I}_2$ satisfies (3.1.14), (3.1.17), (3.1.18), namely that $\partial_t B(U;\theta,t,x,\xi)\mathcal{I}_2$ satisfies (3.1.4), (3.1.6), (3.1.7). But this follows from the fact, seen after (4.1.22), that $B(U;\theta,t,x,\xi)\mathcal{I}_2$, satisfies (3.1.4), (3.1.6), (3.1.8) and the definition of these properties. Finally we can repeat the previous argument for (4.1.20) to ensure a decomposition as in (4.1.24) with $\mathrm{Op}^{BW}(e_1(U;\cdot))\mathcal{I}_2$ and $R(U;t)$ that satisfy separately the reality, parity preserving and reversibility properties (3.1.14), (3.1.17) and (3.1.18). □

4.2 Reduction to constant coefficient symbols

In Proposition 4.1.1 we have obtained a diagonal system of the form

$$\big(D_t - Op^{\mathrm{BW}}(D(U;t,\cdot))\big)V = R'(U;t)V + R''(U;t)U \tag{4.2.1}$$

where $V = \Phi_U^\star W$, the smoothing operators $R'(U;t), R''(U;t)$ are, respectively, in $\Sigma\mathcal{R}^{-\rho+\frac{3}{2}}_{K,\rho'+2,1}[r,N]\otimes\mathcal{M}_2(\mathbb{C})$ and $\mathcal{R}^{-\rho+\frac{3}{2}}_{K,\rho'+2,N}[r]\otimes\mathcal{M}_2(\mathbb{C})$, and where $D(U;t,\cdot)$ is a diagonal matrix of symbols of the form

$$D(U;t,x,\xi) = (1+\underline{\zeta}(U;t))m_\kappa(\xi)\mathcal{K} + \tilde\lambda(U;t,x,\xi)\mathcal{K} + \mu(U;t,x,\xi)\mathcal{I}_2 \tag{4.2.2}$$

where $\underline{\zeta}$ is a function of $\Sigma\mathcal{F}^{\mathbb{R}}_{K,\rho'+1,1}[r,N]$, independent of x, the symbol $\tilde\lambda$ is in $\Sigma\Gamma^{\frac12}_{K,\rho'+1,1}[r,N]$ with $\operatorname{Im}\tilde\lambda$ in $\Sigma\Gamma^{-\frac12}_{K,\rho'+1,1}[r,N]$, the symbol μ is in $\Sigma\Gamma^{1}_{K,\rho'+2,1}[r,N]$ with $\operatorname{Im}\mu$ in $\Sigma\Gamma^{0}_{K,\rho'+2,1}[r,N]$. Moreover the matrix of symbols D satisfies the reality, parity preserving and reversibility properties (3.1.4), (3.1.6), (3.1.7), and the operators R', R'' satisfy the reality, parity preserving and reversibility properties (3.1.14), (3.1.17), (3.1.18).

Our goal is to perform a new conjugation of the system (4.2.1) in order to replace in (4.2.2), $\tilde\lambda$ and μ by *constant coefficients* symbols, up to remainders of very negative order. We shall conjugate (4.2.1) by the flow generated by the linear system

$$\begin{aligned} \frac{d}{d\theta}\Omega_{F(U)}(\theta) &= iOp^{\mathrm{BW}}(F(U))\Omega_{F(U)}(\theta)\\ \Omega_{F(U)}(0) &= \mathrm{Id}\end{aligned} \tag{4.2.3}$$

where the matrix symbol $F(U) = F(U;t)$ is self-adjoint (if $F(U)$ has positive order). We first provide some properties of such auxiliary flow, proved as in lemma 2.5.3.

Lemma 4.2.1 (Auxiliary flow) *Let $0 \le K' \le K$ be integers, $m \le 1$, and let F be a symbol in $\Sigma\Gamma^m_{K,K',1}[r,N]\otimes\mathcal{M}_2(\mathbb{C})$, modulo a symbol of order zero, valued in the space of self-adjoint matrices (when U belongs to $C^K_{*\mathbb{R}}(I,\dot H^\sigma(\mathbb{T}^1,\mathbb{C}^2))$).*

Then there is σ in $\mathbb{R}_+$, and for any U in $C^K_{*\mathbb{R}}(I,\dot H^\sigma(\mathbb{T}^1,\mathbb{C}^2))$, the system (4.2.3) has a unique solution $\Omega_{F(U)}(\theta)$ defined for $\theta\in[-1,1]$. The linear operator $\Omega_{F(U)}(\theta)$ is bounded on $\dot H^s$ for any s, and there is $r>0$ and for any $s\in\mathbb{R}$ a constant $C_s>0$ such that, for any U in $B^K_\sigma(I,r)$, any W in $\dot H^s$, any θ in $[-1,1]$

$$C_s^{-1}\|W\|_{\dot H^s} \le \|\Omega_{F(U)}(\theta)W\|_{\dot H^s} \le C_s\|W\|_{\dot H^s}. \tag{4.2.4}$$

Moreover, for any $k \le K-K'$, we have

$$\|\partial_t^k[\Omega_{F(U)}(\theta)W]\|_{\dot H^{s-\frac32 k}} \le C\mathcal{G}^s_{k,1}(W,t)\,. \tag{4.2.5}$$

Finally $\Omega_{F(U)}(-\theta) = \big(\Omega_{F(U)}(\theta)\big)^{-1}$.

Remark: We shall apply below this lemma for $m \leq 1/2$. Clearly if $m \leq 0$ the statement follows by just an ODE argument (and the matrix $F(U)$ does not need to be self-adjoint). The property $\Omega_{F(U)}(-\theta) = \Omega_{F(U)}(\theta)^{-1}$ holds because $F(U)$ is autonomous in the variable θ.

The flow $\Omega_{F(U)}(\theta)$ inherits the following algebraic properties of the operator $Op^{\mathrm{BW}}(F(U))$, where S is the involution defined in (3.1.1) and τ the map in (3.1.16). Since $F(U) = F(U;t)$ depends on the time variable t, we write below also $\Omega_{F(U;t,\cdot)}(\theta)$ to make appear explicitly this time dependence.

Lemma 4.2.2 *If $Op^{\mathrm{BW}}(F(U))$ satisfies the reality (3.1.14), resp. parity preserving (3.1.17), anti-reversibility (3.1.19), property, then the flow operator $\Omega_{F(U)}(\theta)$ satisfies, for any θ, the anti-reality (3.1.15), resp. parity preserving (3.1.17), anti-reversibility (3.1.19), property i.e.*

$$\begin{aligned} \overline{\Omega_{F(U)}(\theta)W} &= S\Omega_{F(U)}(\theta)S\overline{W} \\ \Omega_{F(U)}(\theta)\circ\tau &= \tau\circ\Omega_{F(U)}(\theta) \\ \Omega_{F(U;-t,\cdot)}(\theta) &= S\Omega_{F(U_S;t,\cdot)}(\theta)S\,. \end{aligned} \tag{4.2.6}$$

Proof: By assumption $Op^{\mathrm{BW}}(F(U))$ satisfies respectively

$$\begin{aligned} \overline{Op^{\mathrm{BW}}(F(U))W} &= -SOp^{\mathrm{BW}}(F(U))S\overline{W} \\ Op^{\mathrm{BW}}(F(U))\circ\tau &= \tau\circ Op^{\mathrm{BW}}(F(U)) \\ Op^{\mathrm{BW}}(F(U;-t,\cdot)) &= SOp^{\mathrm{BW}}(F(U_S;t,\cdot))S\,. \end{aligned}$$

Then $\tilde{\Omega}_{F(U)}(\theta)$ defined respectively by

$$\begin{aligned} \tilde{\Omega}_{F(U)}(\theta)W &= \overline{\Omega_{F(U)}(\theta)W} - S\Omega_{F(U)}(\theta)S\overline{W} \\ \tilde{\Omega}_{F(U)}(\theta) &= \Omega_{F(U)}(\theta)\circ\tau - \tau\circ\Omega_{F(U)}(\theta) \\ \tilde{\Omega}_{F(U;t,\cdot)}(\theta) &= \Omega_{F(U;-t,\cdot)}(\theta) - S\Omega_{F(U_S;t,\cdot)}(\theta)S \end{aligned}$$

satisfies respectively

$$\begin{aligned} \frac{d}{d\theta}\tilde{\Omega}_{F(U)}(\theta) &= iSOp^{\mathrm{BW}}(F(U))S\tilde{\Omega}_{F(U)}(\theta) \\ \frac{d}{d\theta}\tilde{\Omega}_{F(U)}(\theta) &= iOp^{\mathrm{BW}}(F(U))\tilde{\Omega}_{F(U)}(\theta) \\ \frac{d}{d\theta}\tilde{\Omega}_{F(U;t,\cdot)}(\theta) &= iSOp^{\mathrm{BW}}(F(U_S;t))S\tilde{\Omega}_{F(U;t,\cdot)}(\theta) \end{aligned}$$

with zero initial condition. Consequently $\tilde{\Omega}_{F(U)}(\theta) = 0$ for all θ and (4.2.6) follows. □

Let us write down the analogous properties satisfied by the paracomposition operator $\Phi_U^\star = \Omega_{B(U)}(1)\mathcal{I}_2$ defined in Theorem 2.5.8. Since the

change of variables Φ_U depends on the time variable t, we shall write below $\Phi_U(t;\cdot)^\star = \Omega_{B(U;t,\cdot)}(1)\mathcal{I}_2$ instead of $\Phi_U^\star$ when we want to make appear explicitly this time dependence.

Lemma 4.2.3 *We have*

$$\begin{aligned} &\overline{\Phi_U(t;\cdot)^\star W} = \Phi_U(t;\cdot)^\star \overline{W} = \Phi_{U_S}(-t;\cdot)^\star \overline{W} \\ &\Phi_U(t;\cdot)^\star \circ \tau = \tau \circ \Phi_U(t;\cdot)^\star \\ &S\Phi_U(t;\cdot)^\star = \Phi_U(t;\cdot)^\star S = \Phi_{U_S}(-t;\cdot)^\star S\,. \end{aligned} \tag{4.2.7}$$

Proof: Recall that $\Omega_{B(U;t,\cdot)}(\theta)$ is the scalar flow defined in (2.5.13). Thus $\Omega_{B(U;t,\cdot)}(\theta)\mathcal{I}_2$, in particular $\Phi_U(t;\cdot)^\star$, commutes with $S = -\left[\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right]$. The operator $Op^{\rm BW}(B(U;t,\cdot))\mathcal{I}_2$ satisfies (3.1.14), (3.1.17), (3.1.19), because $B(U;t,\cdot)\mathcal{I}_2$ satisfies the reality, parity preserving and antireversibility properties (3.1.4), (3.1.6), (3.1.8) as proved after (4.1.22). Thus the first identity (4.2.6) (with F replaced by B) and the fact that $\Omega_{B(U;t,\cdot)}(\theta)\mathcal{I}_2$ commutes with S, imply $\overline{\Phi_U(t;\cdot)^\star W} = \Phi_U(t;\cdot)^\star \overline{W}$. The second identity in (4.2.6) directly proves $\Phi_U(t;\cdot)^\star \circ \tau = \tau \circ \Phi_U(t;\cdot)^\star$. Finally $\Phi_U(t;\cdot)^\star = \Phi_{U_S}(-t;\cdot)^\star$ follows by the third identity in (4.2.6) and the fact that $\Phi_U(t;\cdot)^\star$ commutes with S, or, more directly, by $B(U;-t,\cdot) = B(U_S;t,\cdot)$ according to (4.1.21) and (4.1.22). All the identities in (4.2.7) are proved. □

The main result of this section is the following one:

Proposition 4.2.4 (Reduction to constant coefficients of (4.2.1)) *For any integers N, ρ in $\mathbb{N}$, we denote by ρ' the number introduced at the beginning of section 4.1 (and in Proposition 4.1.1). Set*

$$\begin{aligned} &\underline{K} = \textit{integer part of } \rho' + 2\rho + \frac{3}{2}N + 2\,, \\ &L = \max(2\rho - 1, N - 1)\,, \end{aligned} \tag{4.2.8}$$

and take $K \geq \underline{K}$. Let $U \in C^K_{*\mathbb{R}}(I, \dot{H}^\sigma(\mathbb{T}^1,\mathbb{C}^2))$ be in $B^K_\sigma(I,r)$ for r small enough provided by lemma 4.2.1. There are

- *A family of diagonal matrices of symbols $F_\ell(U)$ in $\Sigma\Gamma^{\frac{1}{2}-\frac{\ell}{2}}_{K,\underline{K},1}[r,N]\otimes\mathcal{M}_2(\mathbb{C})$, $\ell = 0,\dots,L$, with $\operatorname{Im} F_0 = 0$, satisfying the reality (3.1.4), parity preserving (3.1.6) and anti-reversibility (3.1.8) properties,*
- *Smoothing operators $R_1(U;t)$ in the space $\Sigma\mathcal{R}^{-\rho+\frac{3}{2}}_{K,\underline{K},1}[r,N]\otimes\mathcal{M}_2(\mathbb{C})$, $R_2(U;t)$ in $\mathcal{R}^{-\rho+\frac{3}{2}}_{K,\underline{K},N}[r]\otimes\mathcal{M}_2(\mathbb{C})$, satisfying the reality (3.1.14), parity preserving (3.1.17), reversibility (3.1.18) properties,*
- *A diagonal matrix $H(U;t,\xi)$ of symbols in $\Sigma\Gamma^1_{K,\underline{K},1}[r,N]$, independent of x, with $\operatorname{Im} H$ in $\Sigma\Gamma^0_{K,\underline{K},1}[r,N]\otimes\mathcal{M}_2(\mathbb{C})$, satisfying the reality (3.1.4), parity preserving (3.1.6) and reversibility (3.1.7) properties,*

such that, if V solves the system (4.2.1), then $\tilde V = \Omega_{F(U)}(-1)V$ where $F = \sum_0^L F_\ell$, solves the system

$$(4.2.9)\quad \Big(D_t - Op^{\rm BW}\big((1+\underline{\zeta}(U;t))m_\kappa(\xi)\mathcal{K} - H(U;t,\xi)\big)\Big)\tilde V = R_1(U;t)\tilde V + R_2(U;t)U.$$

Remark: Actually the constant coefficient symbols of the diagonal matrix $H(U;t,\xi)$ are in $\Sigma\Gamma^{1/2}_{K,\underline{K},1}[r,N]$ and not just in $\Sigma\Gamma^{1}_{K,\underline{K},1}[r,N]$. Indeed the potential order one contribution to $H(U;t,\xi)$ will be given by the x-average of the matrix symbol $\mu(U;t,x,\xi)\mathcal{I}_2$ in (4.1.13). Since the imaginary part of μ is of order zero, we have to cope just with the average in x of the real part. But by the first (resp. second) condition (4.1.12), the average of $\mathrm{Re}\,\mu$ is an odd (resp. even) function of ξ, so has to vanish. This remark is not necessary for the subsequent arguments but it provides the expected asymptotic expansion of the Floquet exponents in the periodic and quasi-periodic case [1], [16].

The proof of Proposition 4.2.4 is based on an iterative algorithm which replaces the variable coefficient diagonal symbols $D(U;t,x,\xi)$ in (4.2.2) into constant coefficient symbols up to terms of very negative order whose associated paradifferential operator may be incorporated to the smoothing remainders $R_1(U;t)$. At each step of the iteration we get constant coefficient symbols up to $1/2$ smoother ones. Thus we shall perform $L \sim 2\rho$ transformations, see (4.2.8).

In order to prove (4.2.9) we study separately in the next two lemmas the conjugation of $Op^{\rm BW}(D(U;\cdot))$ and D_t, respectively, under the flow map $\Omega_{F(U)}(1)$ generated by (4.2.3). For simplicity of notation in the sequel we neglect to write the explicit t dependence of $D(U;\cdot) = D(U;t,\cdot)$.

Lemma 4.2.5 *Let $r > 0$, $N \in \mathbb{N}$, $\rho \in \mathbb{N}$ be given. Let $K' = \rho' + 2$, $L = \max(2\rho - 1, N-1)$ and assume $L \le K - K'$. Consider a family of diagonal matrices of symbols F_ℓ in $\Sigma\Gamma^{\frac12-\frac{\ell}{2}}_{K,K'+\ell,1}[r,N]\otimes\mathcal{M}_2(\mathbb{C})$, $\ell = 0,\dots,L$, with $\mathrm{Im}\,F_0 = 0$, satisfying (3.1.4), (3.1.6) and (3.1.8). Let $D(U;\cdot)$ be the diagonal matrix (4.2.2).*

Then there are diagonal matrices G_ℓ of symbols in $\Sigma\Gamma^{1-\frac{\ell}{2}}_{K,K'+\ell,1}[r,N]$, $\ell = 1,\dots,L$, with $\mathrm{Im}\,G_1 \equiv 0$, satisfying (3.1.4), (3.1.6) and (3.1.7), G_ℓ depending only on $F_{\ell'}$, $\ell' < \ell$, such that, setting $F = \sum_0^L F_\ell$, we have

$$(4.2.10)\quad \Omega_{F(U)}(-1)Op^{\rm BW}(D(U;\cdot))\Omega_{F(U)}(1) = Op^{\rm BW}(D(U;\cdot)) + \sum_{\ell=1}^{L} Op^{\rm BW}\big[G_\ell(U;\cdot) + (1+\underline{\zeta}(U;t))\{F_\ell(U;\cdot), m_\kappa(\xi)\mathcal{K}\}\big] + Op^{\rm BW}((1+\underline{\zeta}(U;t))\{F_0(U;\cdot), m_\kappa(\xi)\})\mathcal{K} + R(U;t)$$

where $R(U;t)$ is an operator in $\Sigma\mathcal{R}^{-\rho+\frac{3}{2}}_{K,K'+L,1}[r,N]\otimes\mathcal{M}_2(\mathbb{C})$, satisfying the properties (3.1.14), (3.1.17) and (3.1.18).

Proof: Since

$$\frac{d}{d\theta}\Big(\Omega_{F(U)}(-\theta)Op^{\rm BW}(D(U;\cdot))\Omega_{F(U)}(\theta)\Big) = -i\Omega_{F(U)}(-\theta)\Big[Op^{\rm BW}(F(U)),Op^{\rm BW}(D(U;\cdot))\Big]\Omega_{F(U)}(\theta)$$

we may write, iterating this identity, and applying Taylor formula

$$\begin{aligned}(4.2.11)\quad &\Omega_{F(U)}(-1)Op^{\rm BW}(D(U;\cdot))\Omega_{F(U)}(1)\\ &= Op^{\rm BW}(D(U;\cdot)) + \sum_{q=1}^{L}\frac{(-i)^q}{q!}{\rm Ad}^q_{Op^{\rm BW}(F(U))}Op^{\rm BW}(D(U;\cdot))\\ &+\frac{(-i)^{L+1}}{L!}\int_0^1\Omega_{F(U)}(-\theta){\rm Ad}^{L+1}_{Op^{\rm BW}(F(U))}Op^{\rm BW}(D(U;\cdot))\Omega_{F(U)}(\theta)\\ &\qquad\qquad\times(1-\theta)^L\,d\theta\end{aligned}$$

with the notation

$${\rm Ad}_{Op^{\rm BW}(F(U))}B = [Op^{\rm BW}(F(U)),B]\,.$$

Notice that since F is a diagonal matrix in $\Sigma\Gamma^{\frac{1}{2}}_{K,K'+L,1}[r,N]\otimes\mathcal{M}_2(\mathbb{C})$, each commutator $[Op^{\rm BW}(F(U)),\cdot\,]$ gains $1/2$ unit on the order of the operators and one order of vanishing as U goes to zero. It follows that (4.2.11) is an expansion in operators with decreasing orders and increasing degree of homogeneity. More precisely, according to the composition result of Proposition 2.4.1, formulas (2.3.2), (2.3.3), and Proposition 2.4.2 (see also lemma 2.5.5), where we replace the smoothing index ρ by some $\tilde\rho$ to be chosen below, we may write

$${\rm Ad}^q_{Op^{\rm BW}(F(U))}Op^{\rm BW}(D(U;\cdot)) = Op^{\rm BW}(C_q(U;\cdot)) + R_q(U;t)$$

where $C_q(U;\cdot)$ is a diagonal matrix of symbols in $\Sigma\Gamma^{\frac{3}{2}-\frac{q}{2}}_{K,K'+L,q}[r,N]$ and R_q is a diagonal smoothing operator in

$$\Sigma\mathcal{R}^{-\tilde\rho+\frac{3}{2}+\frac{q}{2}}_{K,K'+L,q}[r,N]\otimes\mathcal{M}_2(\mathbb{C})$$

for $\tilde\rho$ as large as we want. If the level L at which we stop the Taylor expansion (4.2.11) is large enough, namely $L\geq 2\rho-1$ and $L\geq N-1$, then, the last remark following Proposition 2.2.4 implies that the operator

$$Op^{\rm BW}(C_{L+1}(U;\cdot))\in\mathcal{R}^{-\rho+\frac{3}{2}}_{K,K'+L,N}[r]\otimes\mathcal{M}_2(\mathbb{C})\,.$$

The same is true for the R_q's for $q = 1, \dots, L+1$ if we take $\tilde{\rho} \geq \rho + \frac{L+1}{2}$. Therefore

$$\mathrm{Ad}^{L+1}_{Op^{\mathrm{BW}}(F(U))} Op^{\mathrm{BW}}(D(U;\cdot)) \in \mathcal{R}^{-\rho+\frac{3}{2}}_{K,K'+L,N}[r] \otimes \mathcal{M}_2(\mathbb{C})\,.$$

Using also (4.2.5), if follows that the integral term in (4.2.11) is a smoothing operator in $\mathcal{R}^{-\rho+\frac{3}{2}}_{K,K'+L,N}[r] \otimes \mathcal{M}_2(\mathbb{C})$, so contributes to the last operator R in (4.2.10).

Consider now the general term in the sum (4.2.11). This is a combination of operators of the form

$$i^q \Big[Op^{\mathrm{BW}}(F_{\ell_1}), \Big[Op^{\mathrm{BW}}(F_{\ell_2}), \Big[\cdots \Big[Op^{\mathrm{BW}}(F_{\ell_q}), Op^{\mathrm{BW}}(D) \Big] \cdots \Big] \Big] \Big] \tag{4.2.12}$$

with $1 \leq q \leq L$, $0 \leq \ell_1, \dots, \ell_q \leq L$. Set $l = (\ell_1, \dots, \ell_q)$. Again by Proposition 2.4.1, formulas (2.3.2), (2.3.3) and Proposition 2.4.2, each operator in (4.2.12) may be written as $Op^{\mathrm{BW}}(G_{q,l}(U;\cdot)) + R_{q,l}(U;t)$, where $G_{q,l}$ is a diagonal matrix of symbols in

$$\Sigma\Gamma^{\frac{3}{2}-\frac{1}{2}\left(\ell_1+\cdots+\ell_q\right)-\frac{q}{2}}_{K,K'+\max(\ell_j),q}[r,N]$$

and $R_{q,l}$ is a diagonal matrix in $\Sigma\mathcal{R}^{-\rho+\frac{3}{2}}_{K,K'+\max(\ell_j),q}[r,N] \otimes \mathcal{M}_2(\mathbb{C})$ (using again that $\tilde{\rho} \geq \rho + \frac{L+1}{2}$). Notice that all the symbols $G_{q,l}$ such that

$$\ell_1 + \cdots + \ell_q + q = \ell + 1$$

have order $1 - \frac{\ell}{2}$, and that, if $\ell_1 + \cdots + \ell_q + q = \ell + 1$, then $\ell_1, \dots, \ell_q$ are strictly smaller than ℓ, except if $q = 1$, $\ell_1 = \ell$. We define the symbol G_ℓ in the right hand side of (4.2.10), for $\ell = 1, \dots, L$, as the sum of the $G_{q,l}$ for which $\ell_1 + \cdots + \ell_q + q = \ell + 1$ and $\ell_1, \dots, \ell_q < \ell$. The remaining symbol of order $1 - \frac{\ell}{2}$ is

$$i[Op^{\mathrm{BW}}(F_\ell), Op^{\mathrm{BW}}(D)]\,. \tag{4.2.13}$$

Replace in (4.2.13) D by its expression (4.2.2). By Proposition 2.4.1 and formulas (2.3.2), (2.3.3), the commutator $i[Op^{\mathrm{BW}}(F_\ell), Op^{\mathrm{BW}}(\tilde{\lambda}\mathcal{K} + \mu\mathcal{I}_2)]$ has order $\frac{1}{2} - \frac{\ell}{2} = 1 - \frac{\ell+1}{2}$, up to a smoothing remainder, and so it may be incorporated to $G_{\ell+1}$. The remaining operator

$$[iOp^{\mathrm{BW}}(F_\ell), Op^{\mathrm{BW}}\big((1 + \underline{\zeta}(U;t))m_\kappa(\xi)\big)\mathcal{K}] \tag{4.2.14}$$

may be written as the sum of

$$Op^{\mathrm{BW}}((1 + \underline{\zeta}(U;t))\{F_\ell, m_\kappa(\xi)\}\mathcal{K})\,,$$

a contribution to $Op^{\mathrm{BW}}(G_{\ell+2})$, and a smoothing operator $R(U;t)$ as above. The operator in (4.2.14) (of order $1 - \frac{\ell}{2}$) is the term written in the right

hand side of (4.2.10), where we have distinguished the cases $\ell = 0$ and $\ell = 1, \dots, L$.

Finally, the other operators $Op^{\mathrm{BW}}(G_{q,l})$ of order $1 - \frac{\ell}{2}$ with $\ell \geq L+1$, may be incorporated as above into the remainder R in (4.2.10).

Let us verify that $\mathrm{Im}\, G_1 = 0$. Remark that the symbol G_1 is the sum of the $G_{q,l}$ such that $\ell_1 + \cdots + \ell_q + q = 2$ and $\ell_1, \dots, \ell_q < 1$. Hence $q = 2$ and $\ell_1 = \ell_2 = 0$ (the case $q = 1, \ell_1 = 1$ is not allowed). So we just have to consider

$$-[Op^{\mathrm{BW}}(F_0), [Op^{\mathrm{BW}}(F_0), Op^{\mathrm{BW}}(D)]].$$

Replacing D by its expression (4.2.2), we get the operator of nonpositive order $-[Op^{\mathrm{BW}}(F_0), [Op^{\mathrm{BW}}(F_0), Op^{\mathrm{BW}}(\tilde{\lambda}\mathcal{K} + \mu\mathcal{I}_2)]]$, that contribute to G_ℓ, $\ell \geq 2$, and the operator

$$-[Op^{\mathrm{BW}}(F_0), [Op^{\mathrm{BW}}(F_0), Op^{\mathrm{BW}}\big((1 + \underline{\zeta}(U;t))m_\kappa(\xi)\mathcal{K}\big)]].$$

The principal part of this operator has symbol $\{F_0, \{F_0, m_\kappa\}\}(1+\underline{\zeta})\mathcal{K}$ which is a diagonal matrix with real entries since m_κ, F_0, $\underline{\zeta}$ are real valued. This shows that $\mathrm{Im}\, G_1 = 0$.

We still have to check that G_ℓ satisfies (3.1.4), (3.1.6), (3.1.7) and that $R(U;t)$ obeys (3.1.14), (3.1.17), (3.1.18). It suffices to see that these last three properties are verified by the left hand side of (4.2.10). As F_ℓ satisfies (3.1.4), (3.1.6), (3.1.8) for any ℓ, $Op^{\mathrm{BW}}(F_\ell)$ satisfies (3.1.14), (3.1.17), (3.1.19). Hence lemma 4.2.2 implies that $\Omega_{F(U)}(\pm\theta)$ obeys (3.1.15), (3.1.17), (3.1.19), and, since $Op^{\mathrm{BW}}(D)$ satisfies (3.1.14), (3.1.17), (3.1.18), lemma 3.1.6 implies that the left hand side of (4.2.10) satisfies the same properties. □

Lemma 4.2.6 *Let F_ℓ, $\ell = 0, \dots, L$, be a family of diagonal matrices of symbols as in lemma 4.2.5. Then there are diagonal matrices of symbols G_ℓ, $\ell = 1, \dots, L$ as in the statement of the preceding lemma such that*

$$\Omega_{F(U)}(-1) D_t \Omega_{F(U)}(1) = D_t + \sum_{\ell=1}^{L} Op^{\mathrm{BW}}(G_\ell(U;\cdot)) + R(U;t) \tag{4.2.15}$$

where $R(U;t)$ is in $\Sigma\mathcal{R}^{-\rho+\frac{3}{2}}_{K,K'+L+1,1}[r,N] \otimes \mathcal{M}_2(\mathbb{C})$ and satisfies the reality (3.1.14), parity preserving (3.1.17) and reversibility (3.1.18) properties.

Proof: We have

$$\frac{d}{d\theta}\Big(\Omega_{F(U)}(-\theta) \circ D_t \circ \Omega_{F(U)}(\theta)\Big) = -i\Omega_{F(U)}(-\theta)\big[Op^{\mathrm{BW}}(F(U)), D_t\big]\Omega_{F(U)}(\theta)\,.$$

Since the commutator $\big[Op^{\rm BW}(F(U)), D_t\big] = -Op^{\rm BW}(D_tF(U))$, we derive, iterating the above identity and applying Taylor formula, the expansion

$$
\begin{aligned}
(4.2.16)\quad \Omega_{F(U)}(-1)D_t\Omega_{F(U)}(1) &= D_t + \sum_{q=1}^{L}\frac{(-i)^q}{q!}{\rm Ad}^q_{Op^{\rm BW}(F(U))}D_t \\
&+ \frac{(-i)^{L+1}}{L!}\int_0^1 \Omega_{F(U)}(-\theta)\big({\rm Ad}^{L+1}_{Op^{\rm BW}(F(U))}D_t\big)\Omega_{F(U)}(\theta)(1-\theta)^L\,d\theta \\
&= D_t - \sum_{q=1}^{L}\frac{(-i)^q}{q!}{\rm Ad}^{q-1}_{Op^{\rm BW}(F(U))}Op^{\rm BW}(D_tF(U)) \\
&- \frac{(-i)^{L+1}}{L!}\int_0^1 \Omega_{F(U)}(-\theta){\rm Ad}^{L}_{Op^{\rm BW}(F(U))}Op^{\rm BW}(D_tF(U))\Omega_{F(U)}(\theta) \\
&\qquad\qquad\qquad\qquad\qquad\qquad \times(1-\theta)^L\,d\theta
\end{aligned}
$$

with the convention ${\rm Ad}^0_{Op^{\rm BW}(F(U))} = {\rm Id}$.

Since F_ℓ is a diagonal matrix of symbols in $\Sigma\Gamma^{\frac12-\frac{\ell}{2}}_{K,K'+\ell,1}[r,N]\otimes\mathcal{M}_2(\mathbb{C})$, and since U solves equation (3.2.6), it follows from lemma 2.2.6 that D_tF_ℓ is a diagonal matrix with entries in $\Sigma\Gamma^{\frac12-\frac{\ell}{2}}_{K,K'+\ell+1,1}[r,N]$. The general term of the sum in the right hand side of (4.2.16) is the combination of operators of the form

$$
i^{q+1}\Big[Op^{\rm BW}(F_{\ell_1}),\Big[Op^{\rm BW}(F_{\ell_2}),\Big[\cdots\Big[Op^{\rm BW}(F_{\ell_q}),Op^{\rm BW}(D_tF_{\ell_{q+1}})\Big]\cdots\Big]\Big]\Big]
$$

with $0\le q\le L-1$ and $0\le \ell_1,\dots,\ell_{q+1}\le L$. Set $l=(\ell_1,\dots,\ell_{q+1})$. By Proposition 2.4.1, formulas (2.3.2), (2.3.3) and Proposition 2.4.2 (applied with the smoothing index ρ replaced by some $\tilde\rho$ to be chosen), each of these operators has the form $Op^{\rm BW}(G_{q,l}(U;\cdot))+R_{q,l}(U;t)$, where $G_{q,l}$ is a diagonal matrix of symbols belonging to

$$
\Sigma\Gamma^{\frac12-\frac12(\ell_1+\cdots+\ell_{q+1})-\frac{q}{2}}_{K,K'+\max(\ell_j)+1,q+1}[r,N]
$$

and $R(U;t)$ is in $\Sigma\mathcal{R}^{-\tilde\rho+\frac{q+1}{2}-\frac12(\ell_1+\cdots+\ell_{q+1})}_{K,K'+\max(\ell_j)+1,q+1}[r,N]\otimes\mathcal{M}_2(\mathbb{C})$. Choosing $\tilde\rho-\rho$ large enough (depending on L), we get that $R(U;t)$ is in

$$
\Sigma\mathcal{R}^{-\rho+\frac32}_{K,K'+\max(\ell_j)+1,q+1}[r,N]\otimes\mathcal{M}_2(\mathbb{C})\,.
$$

For $\ell=1,\dots,L$, we denote by G_ℓ, the sum of the symbols $G_{q,l}$ such that $\ell_1+\cdots+\ell_{q+1}+q+1=\ell$. Thus the symbol G_ℓ is in $\Sigma\Gamma^{1-\frac{\ell}{2}}_{K,K'+\ell,1}[r,N]\otimes\mathcal{M}_2(\mathbb{C})$ and, since all the ℓ_j are strictly smaller than ℓ, G_ℓ depends only on $F_{\ell'}$, $\ell'<\ell$.

The other operators $Op^{\rm BW}(G_{q,l})$ of order $1-\frac{\ell}{2}$ with $\ell\ge L+1$, may be incorporated, by the last remark following Proposition 2.2.4, into the remainder R in (4.2.15). Finally, arguing in the same way, as we fixed L

large enough ($L \geq 2\rho - 1$, $L \geq N - 1$), the integral term in (4.2.16) gives a contribution belonging to $\mathcal{R}^{-\rho+\frac{3}{2}}_{K,K'+L+1,N}[r] \otimes \mathcal{M}_2(\mathbb{C})$.

Let us check that

$$i^q \mathrm{Ad}^{q-1}_{Op^{\mathrm{BW}}(F(U))} Op^{\mathrm{BW}}(D_t F(U))$$
$$i^{L+1} \Omega_{F(U)}(-\theta) \mathrm{Ad}^{L}_{Op^{\mathrm{BW}}(F(U))}(Op^{\mathrm{BW}}(D_t F)) \Omega_{F(U)}(\theta)$$

satisfy (3.1.14), (3.1.17), (3.1.18).

Notice first that if F satisfies (3.1.8), it follows immediately that $D_t F$ satisfies (3.1.7). In the same way, we see that if F satisfies (3.1.4), $D_t F$ satisfies (3.1.5). Consequently, $Op^{\mathrm{BW}}(D_t F)$ satisfies (3.1.15), (3.1.17) and (3.1.18). Moreover, by assumption, $Op^{\mathrm{BW}}(iF)$ satisfies (3.1.15), (3.1.17), (3.1.19). Remark that if an operator M satisfies (3.1.15) (resp. (3.1.17), resp. (3.1.18)) and an operator M' satisfies (3.1.15) (resp. (3.1.17), resp. (3.1.19)), then $[M, M']$ satisfies (3.1.15) (resp. (3.1.17), resp. (3.1.18)), so that the general term of the sum in the right hand side of (4.2.16), or the term below the integral, satisfies (3.1.14) – taking into account the extra power of i that is present – (resp. (3.1.17), (3.1.18)). We are thus left with showing that these properties are preserved through conjugation by $\Omega_{F(U)}(\theta)$ in the integral term. This follows from the fact that $\Omega_{F(U)}(\theta)$ satisfies (3.1.15), (3.1.17) and (3.1.19), as it has already be seen.

Finally, let us check that $\mathrm{Im}\, G_1 = 0$. Actually, G_1 is the sum of the symbols $G_{q,l}$ such that $\ell_1 + \cdots + \ell_{q+1} + q = 0$, so that $q = 0$, $\ell_1 = 0$, i.e. $G_1 = iD_t F_0 = \partial_t F_0$. Since, by assumption F_0 is real, so is G_1. This concludes the proof. □

Proof of Proposition 4.2.4: We set as in lemma 4.2.5, $K' = \rho' + 2$, $L = \max(2\rho - 1, N - 1)$ and fix $K \geq K' + L$.

For some $F = \sum_0^L F_\ell$ to be determined, satisfying the assumptions of lemma 4.2.5, we set $\tilde{V} = \Omega_{F(U)}(-1)V$. Conjugating (4.2.1) with the flow $\Omega_{F(U)}(-1)$, we get

$$\begin{aligned}(4.2.17)\quad &\Omega_{F(U)}(-1)\big(D_t - Op^{\mathrm{BW}}(D(U;\cdot))\big)\Omega_{F(U)}(1)\tilde{V} \\ &\qquad = \Omega_{F(U)}(-1)R'(U;t)\Omega_{F(U)}(1)\tilde{V} + \Omega_{F(U)}(-1)R''(U;t)U.\end{aligned}$$

According to lemmas 4.2.5 and 4.2.6, and recalling the definition of $D(U;\cdot)$ in (4.2.2), we may write (4.2.17) as

$$\begin{aligned}(4.2.18)\quad &\Big[D_t - Op^{\mathrm{BW}}((1 + \underline{\zeta}(U;t))m_\kappa(\xi))\mathcal{K} \\ &\qquad - \sum_{\ell=0}^{L} Op^{\mathrm{BW}}\big(G_\ell + (1 + \underline{\zeta}(U;t))\{F_\ell, m_\kappa(\xi)\mathcal{K}\}\big)\Big]\tilde{V} \\ &= \Omega_{F(U)}(-1)R'(U;t)\Omega_{F(U)}(1)\tilde{V} + \Omega_{F(U)}(-1)R''(U;t)U + R(U;t)\tilde{V}\end{aligned}$$

where:
• $R'(U;t)$ is in $\Sigma\mathcal{R}^{-\rho+\frac{3}{2}}_{K,K',1}[r,N]\otimes\mathcal{M}_2(\mathbb{C})$, $R''(U;t)$ is in $\mathcal{R}^{-\rho+\frac{3}{2}}_{K,K',N}[r]\otimes\mathcal{M}_2(\mathbb{C})$, $R(U;t)$ is in $\Sigma\mathcal{R}^{-\rho+\frac{3}{2}}_{K,K'+L+1,1}[r,N]\otimes\mathcal{M}_2(\mathbb{C})$,
• $G_0=\mathrm{Re}\,\mu(U;t,x,\xi)\mathcal{I}_2$ in (4.2.2), so that G_0 is in $\Sigma\Gamma^1_{K,K',1}[r,N]\otimes\mathcal{M}_2(\mathbb{C})$, and satisfies $\mathrm{Im}\,G_0=0$,
• G_1 is the sum of the diagonal matrices of symbols denoted by that letter in (4.2.10), (4.2.15) plus $\mathrm{Re}\,\tilde\lambda(U;t,x,\xi)\mathcal{K}$ coming from (4.2.2). lemmas 4.2.5 and 4.2.6, and the properties of $\tilde\lambda$ in Proposition 4.1.1, imply that G_1 is a diagonal matrix of symbols in $\Sigma\Gamma^{\frac{1}{2}}_{K,K'+1,1}[r,N]\otimes\mathcal{M}_2(\mathbb{C})$ with $\mathrm{Im}\,G_1(U;t,x,\xi)=0$.
• G_2 is made from the contributions denoted by this letter in (4.2.10), (4.2.15), plus the symbol $\mathrm{Im}\,\mu(U;t,x,\xi)\mathcal{I}_2$ coming from (4.2.2). This is a diagonal matrix of symbols in $\Sigma\Gamma^0_{K,K'+2,1}[r,N]\otimes\mathcal{M}_2(\mathbb{C})$.
• G_3 is made of the similar contributions coming from (4.2.10), (4.2.15) plus the symbol $i\mathrm{Im}\,\tilde\lambda(U;t,x,\xi)\mathcal{K}$ that we discarded from G_1. This is a diagonal matrix of symbols in $\Sigma\Gamma^{-\frac{1}{2}}_{K,K'+3,1}[r,N]\otimes\mathcal{M}_2(\mathbb{C})$.
• Finally, G_ℓ, $\ell\geq 4$ come from the corresponding terms in (4.2.10), (4.2.15).

All these symbols satisfy the reality, parity-preserving and reversibility properties (3.1.4), (3.1.6), (3.1.7), as follows from lemmas 4.2.5 and 4.2.6 and the properties of the matrix D in (4.2.2).

We construct now F_ℓ so that $G_\ell+(1+\underline{\zeta}(U;t))\{F_\ell,m_\kappa(\xi)\mathcal{K}\}$ has constant coefficients, which are moreover real valued when $\ell=0,1$, i.e. when the order of the symbol is positive. The diagonal matrix G_ℓ may be written as

$$G_\ell=g'_\ell(U;t,x,\xi)\mathcal{I}_2+g''_\ell(U;t,x,\xi)\mathcal{K}$$

with g'_ℓ,g''_ℓ in $\Sigma\Gamma^{1-\frac{\ell}{2}}_{K,K'+\ell,1}[r,N]$. Moreover, since G_0 and G_1 are real,

$$\mathrm{Im}\,g'_0,\mathrm{Im}\,g''_0=0\,,\quad \mathrm{Im}\,g'_1,\mathrm{Im}\,g''_1=0\,.$$

We may also, up to a modification of the smoothing operators, assume that g'_ℓ,g''_ℓ vanish for $|\xi|\leq\frac{1}{2}$. We decompose

$$G_\ell=G^D_\ell+G^{ND}_\ell\quad\text{where}\quad G^{\mathrm{D}}_\ell=g'^{\mathrm{D}}_\ell\mathcal{I}_2+g''^{\mathrm{D}}_\ell\mathcal{K}$$

and

$$g'^{\mathrm{D}}_\ell(U;t,\xi)=\frac{1}{2\pi}\int_{\mathbb{T}^1}g'_\ell(U;t,x,\xi)dx\ ,\ \ g''^{\mathrm{D}}_\ell(U;t,\xi)=\frac{1}{2\pi}\int_{\mathbb{T}^1}g''_\ell(U;t,x,\xi)dx$$
$$g'^{\mathrm{ND}}_\ell=g'_\ell-g'^{\mathrm{D}}_\ell\,,\ \ g''^{\mathrm{ND}}_\ell=g''_\ell-g''^{\mathrm{D}}_\ell\,.$$

To eliminate the variable coefficients part in the left hand side of (4.2.18), we need to find matrices of symbols

$$F_\ell=f'_\ell(U;t,x,\xi)\mathcal{K}+f''_\ell(U;t,x,\xi)\mathcal{I}_2$$

such that

$$G_\ell^{\rm ND} + (1+\underline{\zeta}(U;t))\{F_\ell, m_\kappa(\xi)\mathcal{K}\} = 0\,,$$

i.e. expanding the Poisson bracket

$$\begin{aligned} &g_\ell'^{\rm ND}(U;t,x,\xi) - (1+\underline{\zeta}(U;t))m_\kappa'(\xi)\frac{\partial f_\ell'}{\partial x}(U;t,x,\xi) = 0\\ &g_\ell''^{\rm ND}(U;t,x,\xi) - (1+\underline{\zeta}(U;t))m_\kappa'(\xi)\frac{\partial f_\ell''}{\partial x}(U;t,x,\xi) = 0. \end{aligned} \tag{4.2.19}$$

As $g_\ell'^{\rm ND}, g_\ell''^{\rm ND}$ have zero x-average, and $1+\underline{\zeta}(U;t) \geq \frac{1}{2}$ if U is small enough, we may find a unique pair (f_ℓ', f_ℓ'') of functions with zero x-average, solving these equations. Notice that m_κ' vanishes close to zero, but we may always truncate $g_\ell'^{\rm ND}, g_\ell''^{\rm ND}$ outside a neighborhood of $\xi = 0$, as smoothing symbols contribute to the right hand side of (4.2.9). Since g_0', g_0'' are real valued, we get that f_0', f_0'', and thus F_0, are real valued. As m_κ' is elliptic of order $1/2$, we get that f_ℓ', f_ℓ'', and thus F_ℓ, belong to $\Sigma\Gamma^{\frac{1}{2}-\frac{\ell}{2}}_{K,K'+\ell,1}[r,N]$. We have obtained symbols F_ℓ such that the only remaining terms in the sum in the left hand side of (4.2.18) is

$$-\Big(\sum_{\ell=0}^{L} Op^{\rm BW}\big(g_\ell'^{\rm D}\mathcal{I}_2 + g_\ell''^{\rm D}\mathcal{K}\big)\Big)\tilde{V}.$$

This contributes to $Op^{\rm BW}(H)\tilde{V}$ in (4.2.9), since $g_\ell'^{\rm D}, g_\ell''^{\rm D}$ have constant coefficients by construction, and since their imaginary part vanishes if they are of positive order, i.e. when $\ell = 0, 1$. Actually, since $G_0 = \mathrm{Re}\,\mu(U;t,x,\xi)\mathcal{I}_2$ and $\mu(U;t,x,\xi)$ satisfies (4.1.12) we deduce that its x-average vanishes at the order 1. This proves the remark stated after Proposition 4.2.4.

We are left with proving that (3.1.4), (3.1.6), (3.1.8) hold for F_ℓ, and that the smoothing terms may be written as in (4.2.9). Notice that since G_ℓ satisfies (3.1.4), we have

$$\overline{(G_\ell^{\rm ND})}^\vee = -S(G_\ell^{\rm ND})S\,. \tag{4.2.20}$$

Since (4.2.19) may be written as

$$G_\ell^{\rm ND}(U;t,\cdot) = (1+\underline{\zeta}(U;t))m_\kappa'(\xi)\mathcal{K}\frac{\partial F_\ell}{\partial x}\,,$$

it follows from (4.2.20), the fact that $m_\kappa'(\xi)$ is odd and the relation $S\mathcal{K}S = -\mathcal{K}$ that (3.1.4) holds for F_ℓ. In the same way, condition (3.1.6) for $G_\ell^{\rm ND}$, together with oddness of $m_\kappa'(\xi)$ implies that

$$\big(\frac{\partial F_\ell}{\partial x}\big)(U;t,-x,-\xi) = -\big(\frac{\partial F_\ell}{\partial x}\big)(U;t,x,\xi)$$

from which the property (3.1.6) for F_ℓ follows by integration. Finally, since $G_\ell^{\rm ND}$ satisfies (3.1.7), it follows from (4.1.10), $\mathcal{K}^2 = I$, $\mathcal{K}S\mathcal{K} = -S$ that $\frac{\partial F_\ell}{\partial x}(U;-t,\cdot) = S\frac{\partial F_\ell}{\partial x}(U_S;t,\cdot)S$, so that F_ℓ satisfies (3.1.8).

Consider now the smoothing terms in the right hand side of (4.2.18). Since $R'(U;t), R''(U;t), R(U;t)$ satisfy the reality, parity preserving and reversibility properties (3.1.14), (3.1.17), (3.1.18), and since the operator $\Omega_{F(U)}(\theta)$ satisfies (3.1.15), (3.1.17) and (3.1.19), we conclude that the operators in the right hand side of (4.2.18) satisfy as well (3.1.14), (3.1.17) and (3.1.18).

Moreover, since $R(U;t)$ is in $\Sigma\mathcal{R}^{-\rho+\frac{3}{2}}_{K,K'+L+1,1}[r,N]\otimes\mathcal{M}_2(\mathbb{C})$ and $\underline{K}\geq K'+L+1$, as follows by the definition of $\underline{K}$ in (4.2.8) (taking into account also the translation on ρ that we shall perform below), we conclude that $R(U;t)$ is in $\Sigma\mathcal{R}^{-\rho+\frac{3}{2}}_{K,\underline{K},1}[r,N]\otimes\mathcal{M}_2(\mathbb{C})$ and so it contributes to $R_1(U;t)$ in (4.2.9).

In addition $R''(U;t)$ is in $\mathcal{R}^{-\rho+\frac{3}{2}}_{K,K',N}[r]\otimes\mathcal{M}_2(\mathbb{C})$, $K'\leq\underline{K}$, and, by (4.2.5) and recalling the Definition 2.2.3-(ii), we deduce that $\Omega_{F(U)}(-1)R''(U;t)$ is in $\mathcal{R}^{-\rho+\frac{3}{2}}_{K,\underline{K},N}[r]\otimes\mathcal{M}_2(\mathbb{C})$ as it satisfies (2.2.30). Thus the smoothing operator $\Omega_{F(U)}(-1)R''(U;t)$ contributes to $R_2(U;t)$ in (4.2.9).

Let us finally show that the operator $\Omega_{F(U)}(-1)R'(U;t)\Omega_{F(U)}(1)$ where $R'(U;t)$ is in $\Sigma\mathcal{R}^{-\rho+\frac{3}{2}}_{K,K',1}[r,N]\otimes\mathcal{M}_2(\mathbb{C})$, contributes to $R_1(U;t)$ in the right hand side of (4.2.9). By (4.2.3) and applying Taylor formula

$$\Omega_{F(U)}(1)=\mathrm{Id}+\sum_{\ell=1}^{N-1}\frac{1}{\ell!}(iOp^{\mathrm{BW}}(F(U)))^{\ell}+\frac{1}{(N-1)!}\int_0^1 Op^{\mathrm{BW}}(iF(U))^N\Omega_{F(U)}(\theta)(1-\theta)^{N-1}\,d\theta\,. \tag{4.2.21}$$

Consider the integral term $I(U)$ in (4.2.21). Since $F(U)$ is a matrix of symbols in $\Sigma\Gamma^{\frac{1}{2}}_{K,K'+L,1}[r,N]\otimes\mathcal{M}_2(\mathbb{C})$, it follows from Proposition 2.2.4, in particular (2.2.37), and (4.2.5) that for any $k\leq K-(K'+L)$,

$$\|\partial_t^k(I(U)W)\|_{\dot{H}^{s-\frac{3}{2}k-\frac{N}{2}}}\leq C\sum_{k'+k''=k}\mathcal{G}^{\sigma}_{k'+K'+L,N}(U,t)\mathcal{G}^{s}_{k'',1}(W,t)\,.$$

Consequently, the terms obtained replacing in $\Omega_{F(U)}(-1)R'(U;t)\Omega_{F(U)}(1)$, $\Omega_{F(U)}(1)$ by $I(U)$ will provide an operator $R'''(U;t)$ belonging to the space $\mathcal{R}^{-\rho+\frac{3}{2}+\frac{N}{2}}_{K,K'+L,N}[r]\otimes\mathcal{M}_2(\mathbb{C})$. If we replace ρ by $\rho+\frac{N}{2}$, and change accordingly $K'=\rho'+2$ into K' given by the integer part of $\rho'+2+\frac{N}{2}$, (so that, recalling definition (4.2.8), $K'\leq\underline{K}$), we get a smoothing operator of $\mathcal{R}^{-\rho+\frac{3}{2}}_{K,\underline{K},N}[r]\otimes\mathcal{M}_2(\mathbb{C})$ that contributes to $R_1(U;t)$ in (4.2.9). By the composition results of section 2.4, in particular Proposition 2.4.1, we may write, for each $\ell=1,\dots,N-1$,

$$(iOp^{\mathrm{BW}}(F(U)))^{\ell}=Op^{\mathrm{BW}}(M_\ell(U;t,\cdot))+\tilde{R}_\ell(U;t) \tag{4.2.22}$$

where M_ℓ is a matrix of symbols in the space $\Sigma\Gamma^{\ell/2}_{K,K'+L,\ell}[r,N]\otimes\mathcal{M}_2(\mathbb{C})$ and $\tilde R_\ell(U;t)$ is a matrix of smoothing operators in $\Sigma\mathcal{R}^{-\rho+\frac{\ell}{2}}_{K,K'+L,\ell}[r,N]\otimes\mathcal{M}_2(\mathbb{C})$. Then we compose each operator in (4.2.22) at the left with $\Omega_{F(U)}(-1)R'(U;t)$, where we expand also the flow $\Omega_{F(U)}(-1)$ as in (4.2.21) at the order $N-\ell-1$, instead of N. By the previous arguments we get again contributions to the smoothing term in the right hand side of (4.2.9) performing as above a translation in ρ. This concludes the proof. □

4.3 Normal forms

In Proposition 4.2.4 we have obtained the system (4.2.9) which is diagonal, up to smoothing terms, and the symbol $(1+\underline{\zeta}(U;t))m_\kappa(\xi)\mathcal{K}-H(U;t,\xi)$ has constant coefficients. The associated operator commutes thus to derivatives, so that getting a Sobolev energy inequality is equivalent to getting an L^2 (or $\dot H^0$)-energy inequality. If the symbol H were real valued, the associated operator would be self-adjoint on $\dot H^0$, so that, forgetting for a while the smoothing operators in the the right hand side of (4.2.9), we would get preservation of the $\dot H^0$ norm (and the $\dot H^s$ norm) of $\tilde V$. It turns out that the imaginary part of H is not zero, but only given by an operator of order zero. In this section we shall perform a normal form construction to replace $\operatorname{Im} H$ by a symbol, still of order zero, but vanishing like $\|U\|^N_{\dot H^\sigma}$ when U goes to zero. Consequently, the remaining non self-adjoint part of the equation will not affect energy estimates up to a time of order ϵ^{-N}, where ϵ is the size of the Cauchy data. In the next section 4.4 we shall perform another normal form procedure to replace the smoothing operator $R_1(U;t)$ in the right hand side of (4.2.9), by an operator vanishing at order N at $U=0$, like $R_2(U;t)$, modulo again remainders that do not contribute to the energy inequality (actually we shall only construct modified Sobolev energies).

Remark: From now on, our symbols will be always computed at U belonging to $C^K_{*\mathbb{R}}(I,\dot H^\sigma(\mathbb{T}^1,\mathbb{C}^2))$, with σ,K large enough. In particular, the argument U is always of the form $U=\left[\begin{smallmatrix}u\\ \bar u\end{smallmatrix}\right]$. When we consider a matrix of symbols A in $\Sigma\Gamma^m_{K,K',1}[r,N]\otimes\mathcal{M}_2(\mathbb{C})$ for some $K'\le K$, we denote

$$\operatorname{Im}\left(A(U;t,\cdot)\right)=\frac{1}{2i}[A(U;t,\cdot)-\overline{A(U;t,\cdot)}]\,.$$

Recalling Definition 2.1.4, we may decompose $A=\sum_{q=1}^{N-1}A_q+A_N$ where A_q is in $\tilde\Gamma^m_q\otimes\mathcal{M}_2(\mathbb{C})$ and A_N in $\Gamma^m_{K,K',N}[r]\otimes\mathcal{M}_2(\mathbb{C})$. We shall denote by $\operatorname{Im}A_q$ the matrix of q-linear forms

$$\frac{1}{2i}\left[A_q(U_1,\dots,U_q;\cdot)-\overline{A_q(-S\bar U_1,\dots,-S\bar U_q;\cdot)}\right] \tag{4.3.1}$$

so that, when restricted to $U_j=U$ for any j, with U in the above space, we do get $\operatorname{Im}\left(A_q(U,\dots,U;\cdot)\right)$, since we have the relation $\bar U=-SU$.

Proposition 4.3.1 (Reduction of $\mathrm{Im}\, H$) *Assume that the parameter κ is outside the subset of zero measure of Proposition 7.1.1, so that estimate (7.1.3) holds. Then there are*

• *a family $(B_q(\mathcal{U};\xi))_{q=1,\dots,N-1}$ of diagonal matrices of homogeneous symbols in $\tilde\Gamma^0_q$, with constant coefficients in x, whose restrictions to $U_1=\cdots=U_q=U$ with $\bar U=-SU$ satisfy conditions (3.1.5), (3.1.6), (3.1.8) (Notice that by lemma 3.1.3 the last anti-reversibility condition may be expressed equivalently through condition (3.1.10)),*

• *a family $(H^1_q(\mathcal{U};\xi))_{q=1,\dots,N}$ of diagonal matrices of symbols with constant coefficients in x,*

$$\begin{aligned} &H^1_q(\mathcal{U};\xi)\in\tilde\Gamma^1_q\otimes\mathcal{M}_2(\mathbb{C})\,,\ q=1,\dots,N-1\,,\\ &H^1_N(U;t,\xi)\in\Gamma^1_{K,\underline{K},N}[r]\otimes\mathcal{M}_2(\mathbb{C})\,, \end{aligned}\tag{4.3.2}$$

such that the homogeneous symbols H^1_q, $q=1,\dots,N-1$, are real valued, and $\mathrm{Im}\, H^1_N$ is in $\Gamma^0_{K,\underline{K},N}[r]\otimes\mathcal{M}_2(\mathbb{C})$, these matrices verifying the reality, parity preserving and reversibility properties (3.1.4), (3.1.6), (3.1.7), when restricted to $U_1=\cdots=U_q=U$ as above,

such that, if we set

$$\begin{aligned} B(U;t,\xi)&=\sum_{q=1}^{N-1}B_q(U,\dots,U;\xi),\ \tilde V_1=\exp(Op^{\mathrm{BW}}(B(U;t,\xi)))\tilde V\\ H^1(U;t,\xi)&=\sum_{q=1}^{N-1}H^1_q(U\dots,U;\xi)+H^1_N(U;t,\xi), \end{aligned}\tag{4.3.3}$$

then $\tilde V^1$ solves the equation

$$\begin{aligned} \Big(D_t-Op^{\mathrm{BW}}\Big((1+\underline{\zeta}(U;t))m_\kappa(\xi)\mathcal{K}+H^1(U;t,\xi)\Big)\Big)\tilde V^1\\ =R_1(U;t)\tilde V^1+R_2(U;t)U \end{aligned}\tag{4.3.4}$$

where the smoothing operator $R_1(U;t)$ is in $\Sigma\mathcal{R}^{-\rho+\frac32}_{K,\underline{K},1}[r,N]\otimes\mathcal{M}_2(\mathbb{C})$ and $R_2(U;t)$ is in $\mathcal{R}^{-\rho+\frac32}_{K,\underline{K},N}[r]\otimes\mathcal{M}_2(\mathbb{C})$, $R_1(U;t)$, $R_2(U;t)$ satisfying the reality, parity preserving and reversibility conditions (3.1.14), (3.1.17) and (3.1.18). Moreover, we may write

$$\tilde V^1=\tilde V+M(U;t)\tilde V\tag{4.3.5}$$

for some operator $M(U;t)$ in $\Sigma\mathcal{M}_{K,\underline{K},1}[r,N]\otimes\mathcal{M}_2(\mathbb{C})$ satisfying conditions (3.1.15), (3.1.17) and (3.1.19). Finally, for a large enough σ and any s, one has the bound

$$\|\tilde V^1-\tilde V\|_{\dot H^s}\le C_s\|U\|_{\dot H^\sigma}\|\tilde V\|_{\dot H^s}.\tag{4.3.6}$$

Before starting the proof of the proposition, we need to exhibit some structural properties of the symbol H in the left hand side of (4.2.9).

For p in $\mathbb{N}^*$, p even, we define the set

$$\mathcal{C}_{p,\frac{p}{2}} = \big\{(n_1,\dots,n_p)\in(\mathbb{N}^*)^p; \{n_1,\dots,n_{\frac{p}{2}}\} = \{n_{\frac{p}{2}+1},\dots,n_p\}\big\} \tag{4.3.7}$$

collecting those integer vectors $(n_1,\dots,n_p)$ of $(\mathbb{N}^*)^p$ such that there is a bijection from the subset of the first $p/2$ components of $(n_1,\dots,n_p)$ onto the subset of the last $p/2$ ones. If p is odd, or p is even and $\ell\neq\frac{p}{2}$, define $\mathcal{C}_{p,\ell}$ to be the empty set.

For any n in $\mathbb{N}^*$, define

$$\Pi_n^+ = \left[\begin{smallmatrix}1&0\\0&0\end{smallmatrix}\right]\Pi_n, \qquad \Pi_n^- = \left[\begin{smallmatrix}0&0\\0&1\end{smallmatrix}\right]\Pi_n\,, \tag{4.3.8}$$

the composition of the spectral projectors Π_n with projection from $\mathbb{C}^2$ to $\mathbb{C}\times\{0\}$ (resp. to $\{0\}\times\mathbb{C}$). For U satisfying $\bar U = -SU$, namely U of the form $U = \left[\begin{smallmatrix}u\\ \bar u\end{smallmatrix}\right]$, the projectors $\Pi_n^\pm$ can be written as follows. Denote by $(\varphi_n)_{n\in\mathbb{N}^*}$ a real valued Hilbert basis of the space of even L^2 functions with zero mean, with φ_n in the range of Π_n, i.e. $\varphi_n(x) = \frac{1}{\sqrt{\pi}}\cos(nx)$. Then if we set $\hat u(n) = \frac{1}{\sqrt{\pi}}\int_{\mathbb{T}^1} u(x)\cos(nx)\,dx$, we have

$$\Pi_n U = \left[\begin{smallmatrix}\hat u(n)\\ \overline{\hat u(n)}\end{smallmatrix}\right]\varphi_n, \quad \Pi_n^+ U = \hat u(n)e_+\varphi_n\,, \quad \Pi_n^- U = \overline{\hat u(n)}e_-\varphi_n\,, \tag{4.3.9}$$

where $e_+ = \left[\begin{smallmatrix}1\\0\end{smallmatrix}\right]$, $e_- = \left[\begin{smallmatrix}0\\1\end{smallmatrix}\right]$.

Lemma 4.3.2 *Let $H_p(\mathcal{U};\xi)$ be a matrix of* constant *coefficients homogeneous symbols in $\tilde\Gamma_p^m\otimes\mathcal{M}_2(\mathbb{C})$ for some m in $\mathbb{R}$, p in $\mathbb{N}^*$, satisfying the reality, parity preserving and reversibility properties (3.1.4), (3.1.6) and (3.1.9). Then for any function U even in x, satisfying $SU=-\bar U$, for any even p in $\mathbb{N}^*$ and $(n_1,\dots,n_\ell)$ in $(\mathbb{N}^*)^\ell$, $\ell=\frac{p}{2}$, we have*

$$\operatorname{Im} H_p(\Pi_{n_1}^+U,\dots,\Pi_{n_\ell}^+U,\Pi_{n_1}^-U,\dots,\Pi_{n_\ell}^-U;\xi)\equiv 0\,. \tag{4.3.10}$$

As a consequence

$$\begin{aligned}\operatorname{Im} H_p(U,\dots,U;\xi)&\\ = \sum_{\ell=0}^{p}\sum_{\substack{(n_1,\dots,n_p)\\ \notin\mathcal{C}_{p,\ell}}}&\binom{p}{\ell}\operatorname{Im} H_p(\Pi_{n_1}^+U,\dots,\Pi_{n_\ell}^+U,\Pi_{n_{\ell+1}}^-U,\dots,\Pi_{n_p}^-U;\xi)\,.\end{aligned} \tag{4.3.11}$$

Proof: Notice first that (4.3.11) is trivial when one omits the restriction in the summation on $n_1,\dots,n_p$: this is just what one obtains using that $\operatorname{Im} H_p$ is a symmetric function of its arguments and writing $\Pi_{n_j} = \Pi_{n_j}^+ + \Pi_{n_j}^-$ for any $j=1,\dots,p$. Therefore (4.3.11) follows by (4.3.10).

Recalling (4.3.9) we have to check by $\mathbb{C}$-linearity of $\operatorname{Im} H_p$ that

$$\operatorname{Im} H_p(\varphi_{n_1}e_+,\dots,\varphi_{n_\ell}e_+,\varphi_{n_1}e_-,\dots,\varphi_{n_\ell}e_-;\xi)\equiv 0.$$

By (3.1.4), we have

$$(\operatorname{Im} H_p)(U,\dots,U;-\xi)=S(\operatorname{Im} H_p)(U,\dots,U;\xi)S$$

and by (3.1.9)

$$(\operatorname{Im} H_p)(SU,\dots,SU;\xi)=-S(\operatorname{Im} H_p)(U,\dots,U;\xi)S$$

for any U satisfying $SU=-\bar{U}$. Consequently, as p is even

$$(\operatorname{Im} H_p)(U,\dots,U;-\xi)=-(\operatorname{Im} H_p)(\bar{U},\dots,\bar{U};\xi)\,.$$

Decompose $U=\sum\Pi_{n_j}U$, with $\Pi_{n_j}U$ given by (4.3.9). The above equality may be written, using symmetry and $\mathbb{C}$-linearity, as

$$\begin{aligned}\sum_{n_1,\dots,n_p}\sum_{\ell=0}^{p}\tbinom{p}{\ell}&\hat{u}(n_1)\cdots\hat{u}(n_\ell)\overline{\hat{u}(n_{\ell+1})}\cdots\overline{\hat{u}(n_p)}\\ &\times(\operatorname{Im} H_p)(\varphi_{n_1}e_+,\dots,\varphi_{n_\ell}e_+,\varphi_{n_{\ell+1}}e_-,\dots,\varphi_{n_p}e_-;-\xi)=\\ &-\sum_{n_1,\dots,n_p}\sum_{\ell=0}^{p}\tbinom{p}{\ell}\overline{\hat{u}(n_{\ell+1})}\cdots\overline{\hat{u}(n_p)}\hat{u}(n_1)\cdots\hat{u}(n_\ell)\\ &\qquad\times(\operatorname{Im} H_p)(\varphi_{n_{\ell+1}}e_+,\dots,\varphi_{n_p}e_+,\varphi_{n_1}e_-,\dots,\varphi_{n_\ell}e_-;\xi)\,.\end{aligned}$$

Identifying the coefficients of $\hat{u}(n_1)\cdots\hat{u}(n_\ell)\overline{\hat{u}(n_{\ell+1})}\cdots\overline{\hat{u}(n_p)}$ on each side, we get

$$\begin{aligned}(\operatorname{Im} H_p)&(\varphi_{n_1}e_+,\dots,\varphi_{n_\ell}e_+,\varphi_{n_{\ell+1}}e_-,\dots,\varphi_{n_p}e_-;-\xi)\\ &=-(\operatorname{Im} H_p)(\varphi_{n_{\ell+1}}e_+,\dots,\varphi_{n_p}e_+,\varphi_{n_1}e_-,\dots,\varphi_{n_\ell}e_-;\xi)\,.\end{aligned}$$

In particular, if $\ell=\frac{p}{2}$ and $n_{\ell+1}=n_1,\dots,n_p=n_\ell$, we obtain that

$$\xi\to(\operatorname{Im} H_p)(\varphi_{n_1}e_+,\dots,\varphi_{n_\ell}e_+,\varphi_{n_{\ell+1}}e_-,\dots,\varphi_{n_p}e_-;\xi)$$

is an odd function. Making the same reasoning starting from (3.1.6)

$$(\operatorname{Im} H_p)(U,\dots,U;-\xi)=\operatorname{Im} H_p(U,\dots,U;\xi)$$

we conclude that $(\operatorname{Im} H_p)(\varphi_{n_1}e_+,\dots,\varphi_{n_\ell}e_+,\varphi_{n_{\ell+1}}e_-,\dots,\varphi_{n_p}e_-;\xi)$ is also an even function, so that it vanishes identically. This proves the lemma. □

We shall need a second lemma to prove Proposition 4.3.1.

Lemma 4.3.3 *Let $B(\mathcal{U};\xi)$ be a constant coefficients matrix with entries in $\tilde\Gamma^0_p$, satisfying*

$$\overline{B(U,\dots,U;\xi)}^{\vee} = \pm SB(U,\dots,U;\xi)S \tag{4.3.12}$$

for any function $U(x)$, even in x and satisfying $SU = -\bar U$. Then for any such U, any indices $n_1,\dots,n_p$ in $\mathbb{N}^*$, any $\ell = 0,\dots,p$

$$\begin{aligned}\overline{B(\Pi^+_{n_1}U,\dots,\Pi^+_{n_\ell}U,\Pi^-_{n_{\ell+1}}U,\dots,\Pi^-_{n_p}U;\xi)}^{\vee}\\ = \pm SB(\Pi^-_{n_1}U,\dots,\Pi^-_{n_\ell}U,\Pi^+_{n_{\ell+1}}U,\dots,\Pi^+_{n_p}U;\xi)S\,.\end{aligned} \tag{4.3.13}$$

In the same way, if we assume

$$B(SU,\dots,SU;\xi) = -SB(U,\dots,U;\xi)S\,,$$

then

$$\begin{aligned}B(\Pi^+_{n_1}SU,\dots,\Pi^+_{n_\ell}SU,\Pi^-_{n_{\ell+1}}SU,\dots,\Pi^-_{n_p}SU;\xi)\\ = -SB(\Pi^-_{n_1}U,\dots,\Pi^-_{n_\ell}U,\Pi^+_{n_{\ell+1}}U,\dots,\Pi^+_{n_p}U;\xi)S\,.\end{aligned} \tag{4.3.14}$$

Proof: We may write using (4.3.9)

$$\begin{aligned}&B(U,\dots,U;\xi)\\ &\quad=\sum_{\ell=0}^{p}\tbinom{p}{\ell}\sum_{n_1,\dots,n_p} B(\Pi^+_{n_1}U,\dots,\Pi^+_{n_\ell}U,\Pi^-_{n_{\ell+1}}U,\dots,\Pi^-_{n_p}U;\xi)\\ &\qquad=\sum_{\ell=0}^{p}\tbinom{p}{\ell}\sum_{n_1,\dots,n_p}\hat u(n_1)\cdots\hat u(n_\ell)\overline{\hat u(n_{\ell+1})}\cdots\overline{\hat u(n_p)}\\ &\qquad\qquad\times B(e_+\varphi_{n_1},\dots,e_+\varphi_{n_\ell},e_-\varphi_{n_{\ell+1}},\dots,e_-\varphi_{n_p};\xi)\,.\end{aligned}$$

Plug this expression inside (4.3.12), and identify on both sides the coefficients of $\overline{\hat u(n_1)}\dots\overline{\hat u(n_\ell)}\hat u(n_{\ell+1})\dots\hat u(n_p)$. We get

$$\begin{aligned}\overline{B(e_+\varphi_{n_1},\dots,e_+\varphi_{n_\ell},e_-\varphi_{n_{\ell+1}},\dots,e_-\varphi_{n_p};\xi)}^{\vee}\\ = \pm SB(e_-\varphi_{n_1},\dots,e_-\varphi_{n_\ell},e_+\varphi_{n_{\ell+1}},\dots,e_+\varphi_{n_p};\xi)S\end{aligned}$$

using that B is p-linear symmetric in its first argument. This gives (4.3.13).

To prove the last statement of the lemma, we write using again (4.3.9)

$$\begin{aligned}&B(SU,\dots,SU;\xi)\\ &\quad=\sum_{\ell=0}^{p}\tbinom{p}{\ell}\sum_{n_1,\dots,n_p} B(S\Pi^+_{n_1}U,\dots,S\Pi^+_{n_\ell}U,S\Pi^-_{n_{\ell+1}}U,\dots,S\Pi^-_{n_p}U;\xi)\\ &\qquad=(-1)^p\sum_{\ell=0}^{p}\tbinom{p}{\ell}\sum_{n_1,\dots,n_p}\hat u(n_1)\cdots\hat u(n_\ell)\overline{\hat u(n_{\ell+1})}\cdots\overline{\hat u(n_p)}\\ &\qquad\qquad\times B(e_-\varphi_{n_1},\dots,e_-\varphi_{n_\ell},e_+\varphi_{n_{\ell+1}},\dots,e_+\varphi_{n_p};\xi).\end{aligned}$$

By identification with the expansion of $SB(U,\dots,U;\xi)S$ we get

$$\begin{aligned}B(e_-\varphi_{n_1},\dots,e_-\varphi_{n_\ell},e_+\varphi_{n_{\ell+1}},\dots,e_+\varphi_{n_p};\xi)\\ = -(-1)^p SB(e_+\varphi_{n_1},\dots,e_+\varphi_{n_\ell},e_-\varphi_{n_{\ell+1}},\dots,e_-\varphi_{n_p};\xi)S\end{aligned}$$

which implies (4.3.14). □

Proof of Proposition 4.3.1: We conjugate the equation (4.2.9) by the operator $\exp(Op^{\rm BW}(B(U;t,\xi)))$ where $B(U;t,\xi)$ is a diagonal matrix of constant coefficient symbols as in (4.3.3), with $B_q(\mathcal{U};\xi)$ in $\tilde\Gamma^0_q\otimes\mathcal{M}_2(\mathbb{C})$ to be chosen. The left hand side operator in (4.2.9) transforms into

(4.3.15)
$$\begin{aligned}&\exp(Op^{\rm BW}(B(U;t,\xi)))\big[D_t - Op^{\rm BW}\big((1+\underline{\zeta}(U;t))m_\kappa(\xi)\mathcal{K} - H(U;t,\xi)\big)\big]\\ &\qquad\qquad\times\exp(Op^{\rm BW}(-B(U;t,\xi)))\\ &= D_t - Op^{\rm BW}\big[D_tB(U;t,\xi) + (1+\underline{\zeta}(U;t))m_\kappa(\xi)\mathcal{K} - H(U;t,\xi)\big].\end{aligned}$$

We now choose

$$B(U;t,\xi)=\sum_{q=1}^{N-1}B_q(U,\dots,U;\xi) \tag{4.3.16}$$

so that the symbol

$$D_tB(U;t,\xi)-H(U;t,\xi)=H^1(U;t,\xi) \tag{4.3.17}$$

has the form $H^1=\sum_{q=1}^{N-1}H^1_q+H^1_N$ in (4.3.3), with homogeneous symbols H^1_q which are *real* valued, (4.3.2) holds, and $\operatorname{Im}H^1_N$ is in $\Gamma^0_{K,\underline{K},N}[r]\otimes\mathcal{M}_2(\mathbb{C})$. First we notice that, differentiating each B_q in (4.3.16) and inserting the expression (3.2.6) of D_tU, we get

$$\begin{aligned}D_t(B_q(U,\dots,U;\xi)) &= \sum_{q'=1}^{q}B_q(\underbrace{U,\dots,U}_{q'-1},D_tU,\dots,U;\xi)\\ &= \sum_{q'=1}^{q}B_q(\underbrace{U,\dots,U}_{q'-1},m_\kappa(D)\mathcal{K}U,\dots,U;\xi)\\ &\qquad+\sum_{q'=1}^{q}B_q(\underbrace{U,\dots,U}_{q'-1},M(U;t)U,\dots,U;\xi)\,.\end{aligned} \tag{4.3.18}$$

Since we look for $B_q(U,\dots,U;\xi)$ satisfying (3.1.5) (resp. (3.1.6), resp. (3.1.8)), these properties imply that $\partial_tB_q(U,\dots,U;\xi)$ satisfies (3.1.5) (resp. (3.1.6), resp. (3.1.7)). Consequently, the left hand side of (4.3.18) satisfies (3.1.4)

(resp. (3.1.6), resp. (3.1.7)). Decomposing the right hand side in homogeneous contributions, we see that each of them satisfies conditions (3.1.4), (3.1.6) and (3.1.9), by lemma 3.1.3.

According to Proposition 4.2.4, the matrix H in the left hand side of (4.3.17) may be written as

$$H(U;t,\xi) = \sum_{q=1}^{N-1} H_q(U,\dots,U;\xi) + H_N(U;t,\xi) \tag{4.3.19}$$

where H_q is a diagonal matrix of symbols of $\tilde{\Gamma}^1_q$ with imaginary part in $\tilde{\Gamma}^0_q$, and where H_N is a diagonal matrix with entries in $\Gamma^1_{K,\underline{K},N}[r]$ and imaginary part in $\Gamma^0_{K,\underline{K},N}[r]$. Moreover, these matrices of symbols satisfy the reality, parity preserving and reversibility properties (3.1.4), (3.1.6) and (3.1.7) (or (3.1.9) for the homogeneous components by lemma 3.1.3).

By (4.3.16), (4.3.18), (4.3.19) we may write the left hand side of (4.3.17) as

$$\begin{aligned}\sum_{q=1}^{N-1}\Big[\sum_{q'=1}^{q} B_q(\underbrace{U,\dots,U}_{q'-1}, m_\kappa(D)\mathcal{K}U,\dots,U;\xi) - H_q(U,\dots,U;\xi) \\ + \sum_{q'=1}^{q} B_q(\underbrace{U,\dots,U}_{q'-1}, M(U;t)U,\dots,U;\xi)\Big] - H_N\,.\end{aligned} \tag{4.3.20}$$

Since each B_q is in $\tilde{\Gamma}^0_q$ and $M(U;t)$ is in $\Sigma\mathcal{M}_{K,1,1}[r,N]\otimes\mathcal{M}_2(\mathbb{C})$, Proposition 2.4.3-(i) implies that each symbol $B_q(U,\dots,M(U;t)U,\dots,U;\xi)$ is in $\Sigma\Gamma^0_{K,\underline{K},q+1}[r,N]\otimes\mathcal{M}_2(\mathbb{C})$. We may group these symbols together with H_q and H_N, according to their degree of homogeneity, and rewrite (4.3.20) as

$$\begin{aligned}\sum_{q=1}^{N-1}\Big[\sum_{q'=1}^{q} B_q(\underbrace{U,\dots,U}_{q'-1}, m_\kappa(D)\mathcal{K}U,\dots,U;\xi) - \tilde{H}_q(U,\dots,U;\xi)\Big] \\ - \tilde{H}_N(U;\xi)\end{aligned} \tag{4.3.21}$$

where $\tilde{H}_q$ is a diagonal matrix of symbols in $\tilde{\Gamma}^1_q$ with imaginary part in $\tilde{\Gamma}^0_q$, and where $\tilde{H}_N$ is a diagonal matrix with entries in $\Gamma^1_{K,\underline{K},N}[r]$ and imaginary part in $\Gamma^0_{K,\underline{K},N}[r]$. Note that $\tilde{H}_q$ depends only on $B_{q'}$, $q'<q$. Moreover $\tilde{H}_q$ and $\tilde{H}_N$ satisfy, like H_q and H_N, the reality, parity preserving and reversibility properties (3.1.4), (3.1.6), (3.1.7) (or (3.1.9) for the homogeneous component $\tilde{H}_q$), as the last sum in (4.3.18) satisfies these properties as well.

To obtain (4.3.17), we have to choose B_q in order to compensate the imaginary part of $\tilde{H}_q$ in (4.3.21), so that we shall be left with only the real part of $\tilde{H}_q$, $q=1,\dots,N-1$, that provides the contribution to H^1_q in (4.3.17),

(4.3.3). In other words, we want to find B_q so that

$$\sum_{q'=1}^{q} B_q(U_1,\dots,m_\kappa(D)\mathcal{K}U_{q'},\dots,U_q;\xi) = i\mathrm{Im}\,\tilde{H}_q(U_1,\dots,U_q;\xi)\,.$$

We now determine

$$B_q(\Pi^+_{n_1}U_1,\dots,\Pi^+_{n_\ell}U_\ell,\Pi^-_{n_{\ell+1}}U_{\ell+1},\dots,\Pi^-_{n_q}U_q;\xi)$$

for any $\ell = 1,\dots,q$, and $n_1,\dots,n_q \in \mathbb{N}^*$. We decompose $U_j = \sum_{n_j\in\mathbb{N}^*}\Pi^+_{n_j}U_j + \Pi^-_{n_j}U_j$, $j=1,\dots,q$, and since

$$m_\kappa(D)\mathcal{K}\Pi^\pm_{n_j}U_j = \pm m_\kappa(n_j)\Pi^\pm_{n_j}U_j\,,$$

we write this equation

$$\begin{aligned}\Big[\sum_{q'=1}^{\ell} m_\kappa(n_{q'}) - \sum_{q'=\ell+1}^{q} m_\kappa(n_{q'})\Big]&\\ \times B_q(\Pi^+_{n_1}U_1,\dots,\Pi^+_{n_\ell}U_\ell,\Pi^-_{n_{\ell+1}}U_{\ell+1},\dots,\Pi^-_{n_q}U_q;\xi)&\\ = i\mathrm{Im}\,\tilde{H}_q(\Pi^+_{n_1}U_1,\dots\Pi^+_{n_\ell}U_\ell,\Pi^-_{n_{\ell+1}}U_{\ell+1},\dots,\Pi^-_{n_q}U_q;\xi)\,.&\end{aligned}$$

As $\tilde{H}_q$ satisfies the assumptions of lemma 4.3.2, the right hand side vanishes if q is even, $\ell = \frac{q}{2}$ and $\{n_1,\dots,n_\ell\} = \{n_{\ell+1},\dots,n_q\}$. In all the other cases, by Proposition 7.1.1, when the parameter κ is fixed outside a subset of zero measure, the quantity

$$\mathcal{D}_\ell(n_1,\dots,n_q) = \sum_{q'=1}^{\ell} m_\kappa(n_{q'}) - \sum_{q'=\ell+1}^{q} m_\kappa(n_{q'}) \tag{4.3.22}$$

does not vanish, and its absolute value is actually bounded from below by $c|n|^{-N_0}$, $n=(n_1,\dots,n_q)$, for some integer N_0. We may thus define

$$\begin{aligned}B_q(U,\dots,U;\xi) = \sum_{\ell=0}^{q}\binom{q}{\ell}\sum_{(n_1,\dots,n_q)\notin\mathcal{C}_{q,\ell}}&\mathcal{D}_\ell(n_1,\dots,n_q)^{-1}\\ \times i\mathrm{Im}\,\tilde{H}_q(\Pi^+_{n_1}U,\dots,\Pi^+_{n_\ell}U,\Pi^-_{n_{\ell+1}}U,\dots,\Pi^-_{n_q}U;\xi)\,.&\end{aligned} \tag{4.3.23}$$

The lower bound $|\mathcal{D}_\ell(n)| \geq c|n|^{-N_0}$ implies that the constant coefficients symbol B_q satisfies estimates of the form (2.1.5) with $m=0$ as $\mathrm{Im}\,\tilde{H}_q$ does, changing the value of μ depending on N_0 (actually we have to check (2.1.5) just for $\alpha = 0$ since B_q is constant in x). By the last remark after Definition 2.1.1 we obtain symbols B_q, $q=1,\dots,N-1$, which act on $\dot{H}^\sigma$ taking σ large enough with respect to μ, i.e. large with respect to the number of steps N and the loss of derivatives N_0 produced by the small divisors $\mathcal{D}_\ell(n)$.

Therefore B_q is a diagonal matrix of homogeneous symbols in $\tilde{\Gamma}^0_q \otimes \mathcal{M}_2(\mathbb{C})$. It satisfies (3.1.6) as $\tilde{H}_q$ does. Let us check that (3.1.8) holds. Since $\tilde{H}_q$ satisfies (3.1.9), we deduce from the last statement in lemma 4.3.3 that (4.3.14) holds with B replaced by $i\mathrm{Im}\,\tilde{H}_q$. We deduce that

$$B_q(SU,\dots,SU;\xi) = -\sum_{\ell=0}^{q}\binom{q}{\ell}\sum_{(n_1,\dots,n_q)\notin\mathcal{C}_{q,\ell}}\mathcal{D}_\ell(n_1,\dots,n_q)^{-1} \times iS\mathrm{Im}\,\tilde{H}_q(\Pi^-_{n_1}U,\dots,\Pi^-_{n_\ell}U,\Pi^+_{n_{\ell+1}}U,\dots,\Pi^+_{n_q}U;\xi)S\,.$$

Since $\tilde{H}_q$ is symmetric in its first q arguments and

$$\mathcal{D}_\ell(n_1,\dots,n_\ell,n_{\ell+1},\dots,n_q) = -\mathcal{D}_\ell(n_{\ell+1},\dots,n_q,n_1,\dots,n_\ell),$$

we obtain that

$$B_q(SU,\dots,SU;\xi) = SB_q(U,\dots,U;\xi)S.$$

This is condition (3.1.10) (applied to the arguments $U_1 = \cdots = U_q = U$), which, if we consider $B_q(U,\dots,U;\xi)$ as a non-homogeneous symbol, implies the anti-reversibility condition (3.1.8).

Let us check that B_q satisfies as well (3.1.5). The matrix $i\mathrm{Im}\,\tilde{H}_q$ satisfies (3.1.4) by assumption i.e. condition (4.3.12) with the minus sign. It follows from lemma 4.3.3 that

$$\overline{i\mathrm{Im}\,\tilde{H}_q(\Pi^+_{n_1}U,\dots,\Pi^+_{n_\ell}U,\Pi^-_{n_{\ell+1}}U,\dots,\Pi^-_{n_q}U;\xi)}^{\vee} = -Si\mathrm{Im}\,\tilde{H}_q(\Pi^-_{n_1}U,\dots,\Pi^-_{n_\ell}U,\Pi^+_{n_{\ell+1}}U,\dots,\Pi^+_{n_q}U;\xi)S,$$

so that, by (4.3.23)

$$\begin{aligned}\overline{B_q(U,\dots,U;\xi)}^{\vee} &= -i\sum_{\ell=0}^{q}\binom{q}{\ell}\sum_{(n_1,\dots,n_q)\notin\mathcal{C}_{q,\ell}}\mathcal{D}_\ell(n_1,\dots,n_\ell,n_{\ell+1},\dots,n_q)^{-1}\\ &\quad\times S\mathrm{Im}\,\tilde{H}_q(\Pi^-_{n_1}U,\dots,\Pi^-_{n_\ell}U,\Pi^+_{n_{\ell+1}}U,\dots,\Pi^+_{n_q}U;\xi)S\\ &= i\sum_{\ell=0}^{q}\binom{q}{\ell}\sum_{(n_1,\dots,n_q)\notin\mathcal{C}_{q,\ell}}\mathcal{D}_\ell(n_{\ell+1},\dots,n_q,n_1,\dots,n_\ell)^{-1}\\ &\quad\times S\mathrm{Im}\,\tilde{H}_q(\Pi^+_{n_{\ell+1}}U,\dots,\Pi^+_{n_q}U,\Pi^-_{n_1}U,\dots,\Pi^-_{n_\ell}U;\xi)S\\ &= SB_q(U,\dots,U;\xi)S\,.\end{aligned}$$

In other words, B_q satisfies (3.1.5).

We are left with checking that (4.3.4) holds. Under the change of variables $\tilde{V}^1 = \exp\big(Op^{\mathrm{BW}}(B(U;t;\xi))\big)\tilde{V}$, taking into account (4.3.15) and

(4.3.17), the system (4.2.9) transforms into

$$\begin{aligned}(4.3.24)\quad &\Big(D_t - Op^{\mathrm{BW}}\big[(1+\underline{\zeta}(U;t))m_\kappa(\xi)\mathcal{K} + H_1(U;t,\cdot)\big]\Big)\tilde V_1 \\ &= \exp\big(Op^{\mathrm{BW}}(B(U;t,\xi))\big)\big[R_1(U;t)\exp\big(-Op^{\mathrm{BW}}(B(U;t,\xi))\big)\tilde V_1\big] \\ &\qquad + \exp\big(Op^{\mathrm{BW}}(B(U;t,\xi))\big)[R_2(U;t)U].\end{aligned}$$

Let us expand

$$(4.3.25)\quad \exp\big(Op^{\mathrm{BW}}(B(U;t,\xi))\big) = \sum_{q=0}^{N-1}\frac{(Op^{\mathrm{BW}}(B(U;t,\xi)))^q}{q!} + S_N(U;t).$$

As S_N is the remainder of an absolutely convergent series of bounded operators on $\dot H^s$ for any s, as well as its ∂_t^k-derivatives, $k \le K-K'$, we see that the replacement of one of the exponentials in the right hand side of (4.3.24) by $S_N(U;t)$ brings operators satisfying the bounds (2.2.30) defining $\mathcal{R}^{-\rho+\frac{3}{2}}_{K,\underline{K},N}[r]$. The corresponding terms may be incorporated to $R_1(U;t)\tilde V^1 + R_2(U;t)U$ in (4.3.4). Consider next the sum in (4.3.25). By the definition (4.3.3) of B, and the symbolic calculus properties of Proposition 2.4.1 and Proposition 2.4.2 (which are quite trivial here since we deal with constant coefficient symbols), we may write that sum as

$$\mathrm{Id} + \sum_{q=1}^{N-1}\Big(Op^{\mathrm{BW}}(\tilde B_q(U,\dots,U;\cdot)) + R'_q(U;t)\Big)$$

modulo terms of the same form as $S_N(U;t)$ in (4.3.25). Replacing the exponentials in (4.3.24) by such a sum brings again smoothing expressions contributing to $R_1(U;t)\tilde V^1 + R_2(U;t)U$ in the right hand side of (4.3.4), by Propositions 2.4.2 and 2.4.3.

Moreover, since B satisfies (3.1.5), (3.1.6) and (3.1.8), the operators S_N, $Op^{\mathrm{BW}}(\tilde B_q)$, R'_q satisfy (3.1.15), (3.1.17) and (3.1.19). Since in (4.3.24), R_1 and R_2 satisfy (3.1.14), (3.1.17) and (3.1.18), it follows by composition that the similar operators in (4.3.4) satisfy these properties as well. This concludes the proof of (4.3.4). Finally, the definition (4.3.3) of $\tilde V^1$ and (4.3.25) show that $\tilde V^1$ may be written as (4.3.5), and that (4.3.6) holds, as a consequence of the boundedness of paradifferential operators on Sobolev spaces. □

4.4 Proof of Theorem 3.2.1

The proof of Theorem 3.2.1 will rely on a normal form construction of modified energy Sobolev norms, which are quasi-invariant. The small divisors,

which produce losses of derivatives, are compensated by the smoothing character of the operators R_1, R_2 in the right hand side of (4.3.4). We define first the classes of multilinear forms that will be used to construct a modified energy for (4.3.4).

Definition 4.4.1 (Multilinear energy forms) *Let ρ, s be in $\mathbb{R}_+$, p in $\mathbb{N}$. One denotes by $\tilde{\mathcal{L}}^{s,-\rho}_{p,\pm}$ the space of symmetric $(p+2)$-linear forms*

$$(U_0, \dots, U_{p+1}) \to L(U_0, \dots, U_{p+1})$$

defined on $\dot{H}^\infty(\mathbb{T}^1, \mathbb{C}^2)$ and satisfying for some μ in $\mathbb{R}_+$ and all $n_0, \dots, n_{p+1}$ in $(\mathbb{N}^*)^{p+2}$, all $(U_0, \dots, U_{p+1})$ in $\dot{H}^\infty(\mathbb{T}^1, \mathbb{C}^2)^{p+2}$,

$$|L(\Pi_{n_0}U_0, \dots, \Pi_{n_{p+1}}U_{p+1})| \le C \max(n_0, \dots, n_{p+1})^{2s-\rho} \times \max_3(n_0, \dots, n_{p+1})^{\rho+\mu} \prod_0^{p+1} \|\Pi_{n_\ell}U_\ell\|_{L^2} \tag{4.4.1}$$

where $\max_3(n_0, \dots, n_{p+1})$ is the third largest among those integers, and such that

$$L(\Pi_{n_0}U_0, \dots, \Pi_{n_{p+1}}U_{p+1}) \not\equiv 0 \Rightarrow \sum_0^{p+1} \epsilon_\ell n_\ell = 0 \text{ for some choice of the signs } \epsilon_\ell \in \{-1, 1\} \tag{4.4.2}$$

and, for any $U_0, \dots, U_{p+1}$ satisfying $SU_j = -\bar{U}_j$,

$$L(SU_0, \dots, SU_{p+1}) = \pm L(U_0, \dots, U_{p+1}). \tag{4.4.3}$$

Remark: To check (4.4.3), it is enough to prove that

$$L(SU, \dots, SU) = \pm L(U, \dots, U) \tag{4.4.4}$$

for any U satisfying $SU = -\bar{U}$, as follows from the $\mathbb{R}$-linearity and symmetry of L.

We define below a multilinear form L of $\tilde{\mathcal{L}}^{s,-\rho}_{p,\pm}$ that will be used in the proof of lemma 4.4.3.

Example: Consider R an element of $\tilde{\mathcal{R}}^{-\rho}_p$ and define $L(U_0, \dots, U_{p+1})$ to be the symmetrization of

$$(U_0, \dots, U_{p+1}) \to \int_{\mathbb{T}^1} (|D|^s SU_0)(|D|^s R(U_1, \dots, U_p)U_{p+1})\, dx. \tag{4.4.5}$$

Let us show that if $s \gg \rho$, we get an element of $\tilde{\mathcal{L}}^{s,-\rho}_{p,+}$ (resp. $\tilde{\mathcal{L}}^{s,-\rho}_{p,-}$) if $R(U_1, \dots, U_p)$ satisfies (3.1.21) (resp. (3.1.20)). First, we have the estimate

$$|L(\Pi_{n_0}U_0, \dots, \Pi_{n_{p+1}}U_{p+1})| \le C n_0^{2s} \|\Pi_{n_0}U_0\|_{L^2} \|\Pi_{n_0}R(\Pi_{n_1}U_1, \dots, \Pi_{n_p}U_p)\Pi_{n_{p+1}}U_{p+1}\|_{L^2}. \tag{4.4.6}$$

By condition (2.2.29), we see that (4.4.2) holds. Moreover (2.2.28) implies that the right hand side in (4.4.6) is bounded by

$$Cn_0^{2s}\max\nolimits_2(n_1,\dots,n_{p+1})^{\rho+\mu}\max(n_1,\dots,n_{p+1})^{-\rho}\prod_0^{p+1}\|\Pi_{n_\ell}U_\ell\|_{L^2}. \tag{4.4.7}$$

We may assume $n_1\geq n_2\geq\cdots\geq n_{p+1}$ and because of (4.4.2), $|n_0-n_1|\leq Cn_2$. If $n_0\geq\frac{1}{2}n_1$, we have that $n_0\sim n_1$ and $\max_2(n_1,\dots,n_{p+1})\sim\max_3(n_0,\dots,n_{p+1})$ so that (4.4.7) implies (4.4.1). If $n_0\leq\frac{1}{2}n_1$, then $n_1\sim n_2$, so that $n_0\leq C\max_3(n_0,\dots,n_{p+1})$. Then we estimate (4.4.7), and thus (4.4.6), by

$$C\max(n_0,\dots,n_{p+1})^{2s+\mu}\Big(\frac{\max_3(n_0,\dots,n_{p+1})}{\max(n_0,\dots,n_{p+1})}\Big)^{2s}\prod_0^{p+1}\|\Pi_{n_\ell}U_\ell\|_{L^2}$$

which, for $2s\geq\rho+\mu$, is bounded by the right hand side in (4.4.1).

Finally we check that the form defined in (4.4.5) satisfies (4.4.4). Since the matrix S is symmetric we have

$$\begin{aligned}\int_{\mathbb{T}^1}(|D|^sS(SU))(|D|^sR(SU,\dots,SU)SU)\,dx\\ =\int_{\mathbb{T}^1}(|D|^sSU)(|D|^sSR(SU,\dots,SU)SU)\,dx.\end{aligned} \tag{4.4.8}$$

If $R(U_1,\dots,U_p)$ satisfies (3.1.21) (resp. (3.1.20)) then the right hand side in (4.4.8) is equal to (4.4.5) (resp. minus (4.4.5)) computed at $U_0=\cdots=U_{p+1}=U$. This concludes the proof.

We shall need the following properties of the multilinear forms of the class $\tilde{\mathcal{L}}^{s,-\rho}_{p,\pm}$.

Lemma 4.4.2 *(i) Let L be in $\tilde{\mathcal{L}}^{s,-\rho}_{p,\pm}$. Then for any $m\geq 0$ such that $\rho>m+\frac{1}{2}$ and any $s>\rho+\mu+m+\frac{1}{2}$, L extends as a continuous $(p+2)$-linear form on $\dot{H}^s\times\cdots\times\dot{H}^s\times\dot{H}^{s-m}\times\dot{H}^s\times\cdots\times\dot{H}^s$.*

(ii) Assume that $p=2\ell$ is even, and let L be in $\tilde{\mathcal{L}}^{s,-\rho}_{p,-}$. If U is an even function of x, satisfying $SU=-\bar{U}$, then for any $n_0,\dots,n_\ell$ in $\mathbb{N}^*$,

$$L(\Pi^+_{n_0}U,\dots,\Pi^+_{n_\ell}U,\Pi^-_{n_0}U,\dots,\Pi^-_{n_\ell}U)=0 \tag{4.4.9}$$

where $\Pi^\pm_n$ are defined in (4.3.8)-(4.3.9).

(iii) Assume that the parameter κ is outside the subset of zero measure of Proposition 7.1.1, so that estimate (7.1.3) holds, namely using notation (4.3.22), for some $c>0$ and $N_0\in\mathbb{N}$,

$$|\mathcal{D}_\ell(n_0,\dots,n_{p+1})|\geq c\max(n_0,\dots,n_{p+1})^{-N_0}$$

for any $(n_0,\dots,n_{p+1})$ *in* $(\mathbb{N}^*)^{p+2}$ *if* p *is odd or* p *is even and* $\ell\neq\frac{p}{2}$, *and for any* $(n_0,\dots,n_{p+1})$ *in* $(\mathbb{N}^*)^{p+2}$ *such that*

$$\{n_0,\dots,n_\ell\}\neq\{n_{\ell+1},\dots,n_{p+1}\}$$

when p *is even and* $\ell=\frac{p}{2}$. *Then, for any* L *in* $\tilde{\mathcal{L}}^{s,-\rho}_{p,-}$, *there is* $\tilde L$ *in* $\tilde{\mathcal{L}}^{s,-\rho+N_0}_{p,+}$ *such that*

$$\sum_{j=0}^{p+1}\tilde L(U,\dots,m_\kappa(D)\mathcal{K}U,\dots,U)=iL(U,\dots,U) \tag{4.4.10}$$

(where $m_\kappa(D)\mathcal{K}$ *acts on the argument corresponding to* U_j *in the above sum).*

(iv) Let L *be a multilinear form in* $\tilde{\mathcal{L}}^{s,-\rho}_{p,\pm}$ *and let* M *be an operator in* $\Sigma\mathcal{M}^m_{K,K',q}[r,N]\otimes\mathcal{M}_2(\mathbb{C})$ *(see Definition 2.2.5) satisfying conditions (3.1.15) and (3.1.18) (resp. and (3.1.19)). Then*

$$U\to L(U,\dots,U,M(U;t)U,U,\dots,U) \tag{4.4.11}$$

may be written as the sum $\sum_{q'=0}^{N-p-q-1}L_{q'}(U,\dots,U)$ *where* $L_{q'}$ *are suitable multilinear forms* $L_{q'}$ *in* $\tilde{\mathcal{L}}^{s,-\rho+m}_{p+q+q',\mp}$ *(resp.* $\tilde{\mathcal{L}}^{s,-\rho+m}_{p+q+q',\pm}$*), plus a term that, at any time* t, *is*

$$O\big(\|U(t,\cdot)\|^{p+2}_{\dot H^s}\mathcal{G}^\sigma_{K',N-p}(U,t)+\mathcal{G}^\sigma_{K',N}(U,t)\mathcal{G}^s_{K',1}(U,t)\|U(t,\cdot)\|^{p+1}_{\dot H^s}\big) \tag{4.4.12}$$

if $s>\sigma\gg\rho$, *and if* $\mathcal{G}^\sigma_{K',1}(U,t)$ *is bounded.*

Proof: (i) Fix an index $\ell_0\in\{0,\dots,p+1\}$ and assume that U_ℓ is in $\dot H^s$, $\ell=0,\dots,p+1$, $\ell\neq\ell_0$, and $U_{\ell_0}\in\dot H^{s-m}$. The left hand side of (4.4.1) evaluated at $n_0\geq n_1\geq\cdots\geq n_{p+1}$ is bounded by

$$Cn_0^{2s-\rho+m}n_2^{\rho+\mu}\prod_0^{p+1}n_\ell^{-s}c^\ell_{n_\ell}\Big(\prod_{\substack{\ell\neq\ell_0\\0\leq\ell\leq p+1}}\|U_\ell\|_{\dot H^s}\Big)\|U_{\ell_0}\|_{\dot H^{s-m}}$$

where $(c^\ell_{n_\ell})_{n_\ell}$ is a sequence in ℓ^2 and, by (4.4.2), $n_0\sim n_1$. Summing over all the indices $n_0,\dots,n_{p+1}$ satisfying (4.4.2) one gets the conclusion.

(ii) Write condition (4.4.3) for L (with the minus sign because L is in $\tilde{\mathcal{L}}^{s,-\rho}_{p,-}$) with the vectors valued functions

$$U_j=\Big[\frac{\hat u(n_j)}{\overline{\hat u(n_j)}}\Big]\varphi_{n_j}=\hat u(n_j)\varphi_{n_j}e_+ +\overline{\hat u(n_j)}\varphi_{n_j}e_-\,,\quad j=0,\dots,p+1\,,$$

having used the notation introduced in (4.3.9). Then, since $Se_+=-e_-$ and $Se_-=-e_+$, we get

$$\begin{aligned}&L(\varphi_{n_0}\hat u(n_0)e_-+\varphi_{n_0}\overline{\hat u(n_0)}e_+,\dots,\varphi_{n_{p+1}}\hat u(n_{p+1})e_-+\varphi_{n_{p+1}}\overline{\hat u(n_{p+1})}e_+)\\&=-(-1)^{p+2}L(\varphi_{n_0}\hat u(n_0)e_+ +\varphi_{n_0}\overline{\hat u(n_0)}e_-,\dots,\varphi_{n_{p+1}}\hat u(n_{p+1})e_+ +\varphi_{n_{p+1}}\overline{\hat u(n_{p+1})}e_-).\end{aligned}$$

Using the $\mathbb{C}$-linearity of L and identifying the coefficients of

$$\hat{u}(n_0)\cdots\hat{u}(n_\ell)\overline{\hat{u}(n_{\ell+1})}\cdots\overline{\hat{u}(n_{p+1})}$$

on each side, we get

$$\begin{aligned}(4.4.13)\quad & L(\varphi_{n_0}e_-,\dots,\varphi_{n_\ell}e_-,\varphi_{n_{\ell+1}}e_+,\dots,\varphi_{n_{p+1}}e_+)\\ &\qquad = -(-1)^{p+2}L(\varphi_{n_0}e_+,\dots,\varphi_{n_\ell}e_+,\varphi_{n_{\ell+1}}e_-,\dots,\varphi_{n_{p+1}}e_-).\end{aligned}$$

If p is even, $\ell=\frac{p+2}{2}$ and $n_0=n_{\ell+1},\dots,n_\ell=n_{p+1}$, we get by symmetry

$$L(\varphi_{n_0}e_+,\dots,\varphi_{n_\ell}e_+,\varphi_{n_0}e_-,\dots,\varphi_{n_\ell}e_-)=0$$

which, recalling (4.3.9), implies (4.4.9).

(iii) Decompose

$$L(U,\dots,U)=\sum_{\ell=-1}^{p+1}\tbinom{p+2}{\ell+1}\sum_{n_0,\dots,n_{p+1}}L(\Pi^+_{n_0}U,\dots,\Pi^+_{n_\ell}U,\Pi^-_{n_{\ell+1}}U,\dots,\Pi^-_{n_{p+1}}U).$$

According to (ii), if p is even, and $\ell=p/2$, we may assume that in the corresponding sum $\{n_0,\dots,n_\ell\}\neq\{n_{\ell+1},\dots,n_{p+1}\}$. In order to solve (4.4.10) we define $\tilde{L}$ as the symmetric $(p+2)$-linear form associated to the homogeneous map

$$\begin{aligned}(4.4.14)\quad U\to i\sum_{\ell=-1}^{p+1}\tbinom{p+2}{\ell+1}\sum_{n_0,\dots,n_{p+1}}&\mathcal{D}_\ell(n_0,\dots,n_{p+1})^{-1}\\ &\times L(\Pi^+_{n_0}U,\dots,\Pi^+_{n_\ell}U,\Pi^-_{n_{\ell+1}}U,\dots,\Pi^-_{n_{p+1}}U)\end{aligned}$$

with $\mathcal{D}_\ell(n_0,\dots,n_{p+1})$ given by (4.3.22). By (7.1.3) we have an estimate

$$|\mathcal{D}_\ell(n_0,\dots,n_{p+1})|^{-1}\le C\max(n_0,\dots,n_{p+1})^{N_0}$$

for all indices $n_0,\dots,n_{p+1}$, except those for which p is even, $\ell=p/2$ and $\{n_0,\dots,n_\ell\}=\{n_{\ell+1},\dots,n_{p+1}\}$, which are anyway excluded in the sum (4.4.14) by the property (4.4.9). This shows that $\tilde{L}$ satisfies (4.4.1) with ρ,μ replaced by $(\rho-N_0,\mu+N_0)$. Moreover $\tilde{L}$ also satisfies (4.4.2). To prove that $\tilde{L}$ is in $\tilde{\mathcal{L}}^{s,-\rho+N_0}_{p,+}$, it remains to show that (4.4.4) holds for $\tilde{L}$ with the $+$ sign. Write the $n_0,\dots,n_{p+1}$ sum in (4.4.14) expressing $\Pi^+_{n_j}U=\hat{u}(n_j)\varphi_{n_j}e_+$, $\Pi^-_{n_j}U=\overline{\hat{u}(n_j)}\varphi_{n_j}e_-$. We get

$$\begin{aligned}(4.4.15)\quad \sum_{n_0,\dots,n_{p+1}}&\mathcal{D}_\ell(n_0,\dots,n_{p+1})^{-1}\hat{u}(n_0)\cdots\hat{u}(n_\ell)\overline{\hat{u}(n_{\ell+1})}\cdots\overline{\hat{u}(n_{p+1})}\\ &\times L(\varphi_{n_0}e_+,\dots,\varphi_{n_\ell}e_+,\varphi_{n_{\ell+1}}e_-,\dots,\varphi_{n_{p+1}}e_-).\end{aligned}$$

The corresponding expression for $\tilde{L}(SU,\dots,SU)$ is (changing the indices in the summation)

$$(-1)^{p+2}\sum_{n_0,\dots,n_{p+1}}\mathcal{D}_\ell(n_{\ell+1},\dots,n_{p+1},n_0,\dots,n_\ell)^{-1}\\ \times\overline{\hat{u}(n_{\ell+1})}\cdots\overline{\hat{u}(n_{p+1})}\hat{u}(n_0)\cdots\hat{u}(n_\ell)\\ \times L(\varphi_{n_{\ell+1}}e_+,\dots,\varphi_{n_{p+1}}e_+,\varphi_{n_0}e_-,\dots,\varphi_{n_\ell}e_-).$$

If we use (4.4.13) and the fact that

$$\mathcal{D}_\ell(n_0,\dots,n_\ell,n_{\ell+1},\dots,n_{p+1})=-\mathcal{D}_\ell(n_{\ell+1},\dots,n_{p+1},n_0,\dots,n_\ell),$$

we see that we recover expression (4.4.15) i.e. that (4.4.4) holds. Finally, the fact that (4.4.10) holds trus follows from the definition of $\tilde{L}$ and $\mathcal{D}_\ell$.

(iv) By the first remark after Definition 2.2.5 and (2.2.52) for $k=0$, any operator $M(U;t)$ in the class $\Sigma\mathcal{M}^m_{K,K',q}[r,N]\otimes\mathcal{M}_2(\mathbb{C})$, $0\le q\le N$, is bounded from $\dot{H}^s$ to $\dot{H}^{s-m}$ for any s large enough, and

$$\|M(U;t)U(t,\cdot)\|_{\dot{H}^{s-m}}\le C\mathcal{G}^\sigma_{K',q}(U,t)\|U(t,\cdot)\|_{\dot{H}^s}+C\mathcal{G}^\sigma_{K',N}(U,t)\mathcal{G}^s_{K',1}(U,t)\,.$$

Notice that the last term comes only from the non-homogeneous part of $M(U;t)$. Combining this estimate with (i) of the lemma, we get that, at any time t,

$$|L(U,\dots,M(U;t)U,\dots,U)(t,\cdot)|\le C\|U(t,\cdot)\|^{p+1}_{\dot{H}^s}\|M(U;t)U(t,\cdot)\|_{\dot{H}^{s-m}}\\ \le C\mathcal{G}^\sigma_{K',q}(U,t)\|U(t,\cdot)\|^{p+2}_{\dot{H}^s}+C\mathcal{G}^\sigma_{K',N}(U,t)\mathcal{G}^s_{K',1}(U,t)\|U(t,\cdot)\|^{p+1}_{\dot{H}^s}$$

so that all the terms with $p+q\ge N$ contribute to (4.4.12). Thus, it is sufficient to check that if $\tilde{M}$ is in $\widetilde{\mathcal{M}}^m_{\tilde{q}}\otimes\mathcal{M}_2(\mathbb{C})$, $q\le\tilde{q}<N-p$, then

$$(U_0,\dots,U_{p+\tilde{q}+1})\to L(U_0,\dots,U_p,\tilde{M}(U_{p+1},\dots,U_{p+\tilde{q}})U_{p+\tilde{q}+1})$$

defines after symmetrization a multilinear form $\tilde{L}$ satisfying the estimates of Definition 4.4.1 for elements of $\tilde{\mathcal{L}}^{s,-\rho+m}_{p+\tilde{q},\pm}$. The multilinear form $\tilde{L}$ is denoted by $L_{q'}$, $q'=\tilde{q}-q$, $0\le q'\le N-p-q-1$, in the statement of (iv). By (2.2.50)-(2.2.51) and since L satisfies (4.4.1), we get

$$|L(\Pi_{n_0}U_0,\dots,\Pi_{n_p}U_p,\tilde{M}(\Pi_{n_{p+1}}U_{p+1},\dots,\Pi_{n_{p+\tilde{q}}}U_{p+\tilde{q}})\Pi_{n_{p+\tilde{q}+1}}U_{p+\tilde{q}+1})|\\ \le C\sum_{n'}\max(n_0,\dots,n_p,n')^{2s-\rho}\,\mathrm{max}_3(n_0,\dots,n_p,n')^{\rho+\mu}\\ \times(n'+n_{p+1}+\ldots+n_{p+\tilde{q}+1})^m\prod_0^{p+\tilde{q}+1}\|\Pi_{n_\ell}U_\ell\|_{L^2}$$

where the sum is restricted to the indices $n' = \sum_{p+1}^{p+\tilde{q}+1} \epsilon_\ell n_\ell$ for some choice of the signs $\epsilon_\ell \in \{-1, 1\}$. The right hand side of the previous formula is bounded by

$$C \max(n_0, \dots, n_{p+\tilde{q}+1})^{2s-\rho+m} \max_3(n_0, \dots, n_{p+\tilde{q}+1})^{\rho+\mu} \prod_0^{p+\tilde{q}+1} \|\Pi_{n_\ell} U_\ell\|_{L^2}$$

and therefore $\tilde{L}$ satisfies (4.4.1) with ρ and p replaced by $\rho - m$ and $p + \tilde{q}$. Since L satisfies (4.4.2) and M satisfies (2.2.51) we derive as well that (4.4.11) satisfies the corresponding condition (4.4.2). We have still to check that condition (4.4.3) (or (4.4.4)) holds with the $\mp$ (resp. $\pm$) sign if M satisfies (3.1.15) and (3.1.18) (resp. (3.1.15) and (3.1.19)). Let us treat the first case, i.e. show that if $\tilde{M}$ denotes some multilinear component of M, then

(4.4.16)
$$L(SU, \dots, SU, \tilde{M}(SU, \dots, SU)SU) = \mp L(U, \dots, U, \tilde{M}(U, \dots, U)U)$$

for any U such that $SU = -\bar{U}$. Notice first that by (3.1.15) and $S\bar{U} = -U$,

$$\overline{\tilde{M}(U, \dots, U)U} = S\tilde{M}(U, \dots, U)S\bar{U} = -S(\tilde{M}(U, \dots, U)U),$$

so that $\tilde{M}(U, \dots, U)U$ satisfies the same property as U. Moreover since M satisfies the reversibility condition (3.1.18), lemma 3.1.5 implies that the homogeneous component $\tilde{M}$ satisfies the reversibility condition (3.1.20), namely $\tilde{M}(SU, \dots, SU)SU = -S(\tilde{M}(U, \dots, U)U)$. In conclusion (4.4.16) follows by (4.4.3) applied with $U_0 = \cdots = U_p = U$, $U_{p+1} = \tilde{M}(U, \dots, U)U$. □

Finally, let us state a Sobolev energy inequality that will be the starting point of the normal forms reduction giving Theorem 3.2.1.

Lemma 4.4.3 (First energy inequality) *Let $\rho \in \mathbb{N}$, $\rho \gg N$. There are indices $s \geq s_0 \gg \sigma \gg K \gg \rho$ as in (2.1.1) and a family $(L_p)_{1 \leq p \leq N-1}$ of multilinear forms belonging to $\tilde{\mathcal{L}}^{s,-\rho}_{p,-}$ for any $s \geq s_0$, such that the following holds:*

Let U be a solution of system (3.2.2) defined for all times $]-T, T[$ satisfying the properties of Theorem 3.2.1, in particular, $U(t)$ is an even function of x satisfying $SU(t) = -\bar{U}(t)$, for any t. Let $W = Op^{\mathrm{BW}}(Q(U; t, \cdot))U$ defined in Proposition 3.3.2, $V = \Phi^\star_U W$ given by Proposition 4.1.1, $\tilde{V} = \Omega_{F(U)}(-1)V$ introduced in Proposition 4.2.4 and finally

$$\tilde{V}^1 = \exp(Op^{\mathrm{BW}}(B(U; t, \xi)))\tilde{V}$$

given by Proposition 4.3.1. Then for any $s \geq s_0$,

$$\frac{d}{dt} \int_{\mathbb{T}^1} \left||D|^s \tilde{V}^1(t, x)\right|^2 dx = \sum_1^{N-1} L_p(U, \dots, U) + O\big(\|U(t, \cdot)\|^{N+2}_{\dot{H}^s}\big) \tag{4.4.17}$$

as long as $\|U(t,\cdot)\|_{\dot H^s}$ *stays small enough. Moreover*

$$C_s^{-1}\|\tilde V^1\|_{\dot H^s} \leq \|U\|_{\dot H^s} \leq C_s\|\tilde V^1\|_{\dot H^s} \tag{4.4.18}$$

for some $C_s > 0$.

Remark: Since $SU(t) = -\bar U(t)$ and each of the operators $Op^{\rm BW}(Q)$, $\Phi_U^\star$, $\Omega_{F(U)}(-1)$, $\exp(Op^{\rm BW}(B(U;t,\xi)))$, satisfies the anti-reality condition (3.1.15) (see Proposition 3.3.2, Lemmas 4.2.3 and 4.2.2, Proposition 4.3.1), then the function

$$\tilde V^1 = \exp(Op^{\rm BW}(B(U;t,\xi))) \circ \Omega_{F(U)}(-1) \circ \Phi_U^\star \circ Op^{\rm BW}(Q(U;t,\cdot))U$$

satisfies $S\tilde V^1(t) = -\overline{\tilde V^1(t)}$ as well, by the second remark after Definition 3.1.4.

Proof: Notice first that by Proposition 6.3.2 and recalling (2.1.2)-(2.1.3), we have the bound

$$\mathcal{G}^s_{K,1}(U,t) = |\!|\!|U(t,\cdot)|\!|\!|_{K,s} \leq C_s\|U(t,\cdot)\|_{\dot H^s}\,. \tag{4.4.19}$$

According to system (4.3.4), the left hand side of (4.4.17) is equal to

(4.4.20)

$$\begin{aligned} 2{\rm Re}\, i\int_{\mathbb{T}^1} \overline{(|D|^s\tilde V^1)}\Big[Op^{\rm BW}\big((1+\underline{\zeta}(U;t))m_\kappa(\xi)\mathcal{K} + H^1(U;t,\xi)\big)|D|^s\tilde V^1\Big]\,dx \\ + 2{\rm Re}\, i\int_{\mathbb{T}^1} \overline{(|D|^s\tilde V^1)}(|D|^s R_1(U;t)\tilde V^1)\,dx \\ + 2{\rm Re}\, i\int_{\mathbb{T}^1} \overline{(|D|^s\tilde V^1)}(|D|^s R_2(U;t)U)\,dx\,. \end{aligned}$$

In the first integral above, the contribution of $Op^{\rm BW}\big((1+\underline{\zeta}(U;t))m_\kappa(\xi)\mathcal{K}\big)$ is zero, as this operator is self-adjoint on L^2. In the same way, since H^1_q, $1 \leq q \leq N-1$ is real valued in the decomposition (4.3.3) of $H^1(U;t,\xi)$, the corresponding contribution to (4.4.20) vanishes. The same is true for ${\rm Re}\, H^1_N(U;t,\xi)$, so that the first integral in (4.4.20) is actually equal to the contribution coming from $i({\rm Im}\, H^1_N)(U;t,\xi)$. Since by Proposition 4.3.1 ${\rm Im}\, H^1_N$ is in $\Gamma^0_{K,\underline{K},N}[r]\otimes\mathcal{M}_2(\mathbb{C})$ we get by (2.2.37) (with $k=0$, $s=0$ and $m=0$) and (4.4.19) that the first term in (4.4.20) is, for s large enough and $K \geq \underline{K}$, bounded by

$$O\big(\mathcal{G}^\sigma_{\underline{K},N}(U,t)\|\tilde V^1(t,\cdot)\|^2_{\dot H^s}\big) = O\big(\|U(t,\cdot)\|^N_{\dot H^s}\|\tilde V^1(t,\cdot)\|^2_{\dot H^s}\big)\,.$$

Consider the last integral in (4.4.20). Since $R_2(U;t)$ is in $\mathcal{R}^{-\rho+\frac32}_{K,\underline{K},N}[r]\otimes\mathcal{M}_2(\mathbb{C})$, we get, by (2.2.30) with $k=0$, and (4.4.19), a bound in

$$O\big(\|U(t,\cdot)\|^{N+1}_{\dot H^s}\|\tilde V^1(t,\cdot)\|_{\dot H^s}\big)$$

for s, σ satisfying (2.1.1). In the second integral in (4.4.20), we may decompose the smoothing operator

$$R_1(U;t) = \sum_{p=1}^{N-1} R_{1,p}(U,\dots,U) + R_{1,N}(U;t)$$

with $R_{1,p}$ in $\tilde{\mathcal{R}}_p^{-\rho+\frac{3}{2}}$ and $R_{1,N}(U;t)$ in $\mathcal{R}_{K,\underline{K},N}^{-\rho+\frac{3}{2}}[r]$. By (2.2.30) with $k=0$, and (4.4.19), the contribution coming from the last term is bounded by

$$O\big(\|\tilde{V}^1(t,\cdot)\|_{\dot{H}^s}^2 \mathcal{G}^s_{\underline{K},N}(U,t)\big) = O\big(\|\tilde{V}^1(t,\cdot)\|_{\dot{H}^s}^2 \|U(t,\cdot)\|_{\dot{H}^s}^N\big)\,.$$

On the other hand, as we have remarked after the statement of lemma 4.4.3 that $\overline{\tilde{V}^1} = -S\tilde{V}^1$, we may write the contribution of the term $R_{1,p}$ to the second integral in (4.4.20) as (4.4.5). By lemma 3.1.5, since since $R_1(U,t)$ satisfies the reversibility property (3.1.18) the homogeneous smoothing operators $R_{1,p}$ satisfy (3.1.20). We have seen in the example following Definition 4.4.1 that we get an expression $\tilde{L}_p(\tilde{V}^1,U,\dots,U,\tilde{V}^1)$ with $\tilde{L}_p$ in $\tilde{\mathcal{L}}^{s,-\rho}_{p,-}$. Consequently, we have written the left hand side of (4.4.17) as

$$\begin{aligned}(4.4.21)\quad &\sum_{p=1}^{N-1} \tilde{L}_p(\tilde{V}^1,U,\dots,U,\tilde{V}^1)\\ &\qquad + O\big(\|\tilde{V}^1(t,\cdot)\|_{\dot{H}^s}^2 \|U(t,\cdot)\|_{\dot{H}^s}^N + \|U(t,\cdot)\|_{\dot{H}^s}^{N+1} \|\tilde{V}^1(t,\cdot)\|_{\dot{H}^s}\big)\,.\end{aligned}$$

To deduce (4.4.17) from (4.4.21), it is sufficient to show that (4.4.18) holds and that the above terms $\tilde{L}_p(\tilde{V}^1,U,\dots,U,\tilde{V}^1)$ may be expressed as in the right hand side of (4.4.17). Recall that by (4.3.3), Proposition 4.2.4, Proposition 4.1.1, Proposition 3.3.2,

$$(4.4.22)\quad \tilde{V}^1 = \exp(Op^{\mathrm{BW}}(B(U;t,\xi)))\Omega_{F(U)}(-1)\Phi_U^\star Op^{\mathrm{BW}}(Q(U;t,\cdot))U\,.$$

Let us show that $\tilde{V}^1 = U + M(U;t)U$ for some M in $\Sigma\mathcal{M}_{K,\underline{K},1}[r,N]\otimes\mathcal{M}_2(\mathbb{C})$ satisfying (3.1.15) and (3.1.19), and that $\|\tilde{V}^1\|_{\dot{H}^s} \sim \|U\|_{\dot{H}^s}$ for all large enough s, when $\|U(t,\cdot)\|_{\dot{H}^s}$ is small enough.

By Proposition 3.3.2, and the Definition 3.3.1 of $\mathcal{E}^0_{K,\rho,1}[r,N]$, there is a matrix of symbols $Q_1(U;t,\cdot)$ in $\Sigma\Gamma^0_{K,\rho,1}[r,N]\otimes\mathcal{M}_2(\mathbb{C})$ satisfying (3.1.5) and (3.1.8) such that

$$(4.4.23)\quad W = U + Op^{\mathrm{BW}}(Q_1(U;t,\cdot))U\,.$$

By the third remark following Definition 2.2.5, $Op^{\mathrm{BW}}(Q_1(U;t,\cdot))$ is an element of $\Sigma\mathcal{M}^0_{K,\rho,1}[r,N]\otimes\mathcal{M}_2(\mathbb{C})$ and it satisfies moreover (3.1.15) and (3.1.19). By Proposition 2.2.4, a bound of the form $\|Op^{\mathrm{BW}}(Q_1(U;t,\cdot))U\|_{\dot{H}^s} = O\big(\|U\|_{\dot{H}^s}^2\big)$ holds if s is large enough, so that $\|W\|_{\dot{H}^s} \sim \|U\|_{\dot{H}^s}$, as long as $\|U\|_{\dot{H}^s}$ stays small enough.

Formula (2.5.51) allows to write $V = \Phi_U^\star W$ as

$$V = W + \sum_{p=1}^{N-1} M_p(U,\dots,U)W + M_N(U;t)W \tag{4.4.24}$$

with M_p in $\widetilde{\mathcal{M}}_p \otimes \mathcal{M}_2(\mathbb{C})$ and M_N in $\mathcal{M}_{K,\rho,N}[r] \otimes \mathcal{M}_2(\mathbb{C})$. Moreover, it follows from the definition (2.5.50) of $\Phi_U^\star$ and (2.5.14) that $\|V\|_{\dot{H}^s} \sim \|W\|_{\dot{H}^s}$. Inserting in (4.4.24) the expression of W obtained in (4.4.23), and using the last remark after Definition 2.2.5, we deduce that V may be written as

$$V = U + M(U;t)U = U + \sum_{p=1}^{N-1} M_p(U,\dots,U)U + M_N(U;t) \tag{4.4.25}$$

for some other M_p in $\widetilde{\mathcal{M}}_p \otimes \mathcal{M}_2(\mathbb{C})$ and M_N in $\mathcal{M}_{K,\rho,N}[r] \otimes \mathcal{M}_2(\mathbb{C})$, i.e. for some operator $M(U;t)$ in the space $\Sigma\mathcal{M}_{K,\rho,1}[r,N] \otimes \mathcal{M}_2(\mathbb{C})$. Let us check that $M(U;t)$ satisfies (3.1.15) and (3.1.19). Recalling that $V = \Phi_U(t;\cdot)^\star W$ and that $\Phi_U(t;\cdot)^\star$ satisfies (4.2.7), we get that M_p and M_N in (4.4.24) satisfy (3.1.19) and (3.1.15). Since also $Op^{\mathrm{BW}}(Q_1(U;t,\cdot))$ in (4.4.23) satisfies (3.1.19) and (3.1.15) we get that also the operators M_p and M_N in (4.4.25) satisfy (3.1.19) and (3.1.15).

Consider next $\tilde{V} = \Omega_{F(U)}(-1)V$. By (4.2.4), we have

$$\|\tilde{V}\|_{\dot{H}^s} \sim \|V\|_{\dot{H}^s} \sim \|U\|_{\dot{H}^s}$$

if $\|U\|_{\dot{H}^s}$ is small enough. Moreover, $\Omega_{F(U)}(-1)V$ may be expressed as in (4.2.21), (4.2.22), so that, combining with (4.4.25), we get a similar expression for $\tilde{V}$ in terms of U, with a new M_p belonging to $\widetilde{\mathcal{M}}_p \otimes \mathcal{M}_2(\mathbb{C})$, M_N in $\mathcal{M}_{K,\underline{K},N}[r] \otimes \mathcal{M}_2(\mathbb{C})$. Since the F_ℓ in Proposition 4.2.4 satisfy (3.1.4) and (3.1.8), $Op^{\mathrm{BW}}(iF(U))$ satisfies (3.1.15) and (3.1.19), and $\Omega_{F(U)}(-1)$ does so as well. By composition, we deduce that the M_p's satisfy also (3.1.15) and (3.1.19). Finally, we consider $\tilde{V}^1$ given by the expression $\tilde{V}^1 = \exp(Op^{\mathrm{BW}}(B(U;t,\xi)))\tilde{V}$. By (4.3.6), (4.4.18) holds. Moreover (4.3.5) and the above expression of $\tilde{V}$ in terms of U allows us to obtain an expression for $\tilde{V}^1$ in terms of U similar to (4.4.25). To conclude the proof of (4.4.17), we plug these expressions inside (4.4.21). By (iv) of lemma 4.4.2, we express (4.4.21) in terms of new multilinear forms $L_p(U,\dots,U)$ belonging to $\tilde{\mathcal{L}}^{s,-\rho}_{p,-}$ and of remainders satisfying $O(\|U(t;\cdot)\|^{N+2}_{\dot{H}^s})$. This concludes the proof. □

Proof of Theorem 3.2.1: The proof is based on a bootstrap argument. Let κ be fixed in $]0,+\infty[-\mathcal{N}$ where the set $\mathcal{N}$ is provided by Proposition 7.1.1. Take N an arbitrary positive integer. We shall prove that there are constants $s_0 \gg K \gg N$ and for any $s \geq s_0$, there are constants

$$\epsilon_0 > 0\,, \quad c > 0\,, \quad 0 < A_0 < A_1 < \cdots < A_K\,,$$

such that if a solution $U(t)$ of system (3.2.2) exists on some time interval $I =]-T, T[$ and satisfies, together with the properties of Theorem 3.2.1, the bounds

$$\sup_{]-T,T[} \|\partial_t^k U(t;\cdot)\|_{\dot{H}^{s-\frac{3}{2}k}} \leq A_k \epsilon, \quad k = 0, \dots, K, \tag{4.4.26}$$

then, for $\epsilon \in]0, \epsilon_0[$ and $T < c\epsilon^{-N}$, the solution $U(t)$ actually satisfies on the same time interval the improved bounds

$$\sup_{]-T,T[} \|\partial_t^k U(t;\cdot)\|_{\dot{H}^{s-\frac{3}{2}k}} \leq \frac{A_k}{2} \epsilon, \quad k = 0, \dots, K. \tag{4.4.27}$$

This directly implies, taking into account the classical results of local existence (see Schweizer [60]) that the solution U may be extended up to an interval of length $c\epsilon^{-N}$, and that (3.2.7) holds.

To prove that (4.4.26) implies (4.4.27), let us show first that we may find recursively multilinear forms

$$\begin{aligned} \tilde{L}_p &\in \tilde{\mathcal{L}}_{p,+}^{s,-\rho+(N_0+m)(p-1)+N_0}, \quad 1 \leq p \leq N-1, \\ L_p^{(q)} &\in \tilde{\mathcal{L}}_{p,-}^{s,-\rho+(N_0+m)q}, \quad q+1 \leq p \leq N-1, \end{aligned} \tag{4.4.28}$$

for $s \gg K \gg \rho \gg N$, such that, for $q = 0, \dots, N-1$

$$\begin{aligned} \frac{d}{dt}\Big[\int_{\mathbb{T}^1} \big||D|^s \tilde{V}^1(t;\cdot)\big|^2 dx + \sum_{p=1}^{q} \tilde{L}_p(U(t;\cdot), \dots, U(t;\cdot))\Big] \\ = \sum_{p=q+1}^{N-1} L_p^{(q)}(U(t;\cdot), \dots, U(t;\cdot)) + O\Big(\|U(t;\cdot)\|_{\dot{H}^s}^{N+2}\Big). \end{aligned} \tag{4.4.29}$$

The number $m \geq 3/2$ in (4.4.28) is the order of the operator $M(U;t)$ in $\Sigma\mathcal{M}_{K,1,1}^m[r,N] \otimes \mathcal{M}_2(\mathbb{C})$ defined in (3.2.6). Notice that in (4.4.28) at each step of the iteration there is a loss of derivatives proportional to N_0 due to the small divisors. It will be compensated by taking the regularizing index ρ large enough with respect to the number of steps N and N_0.

Notice that if $q = 0$, then (4.4.29) follows from (4.4.17) with $L_p^{(0)} = L_p$. Then we proceed by induction. Assume that (4.4.29) holds at rank $q-1$ for some $q \geq 1$, and define $\tilde{L}_q$ to be the multilinear form of $\tilde{\mathcal{L}}_{q,+}^{s,-\rho+(N_0+m)(q-1)+N_0}$ given by (iii) of lemma 4.4.2 applied to $L = L_q^{(q-1)}$. Using (3.2.6) we have

$$\begin{aligned} \frac{d}{dt}\tilde{L}_q(U(t;\cdot), \dots, U(t;\cdot)) &= i\sum_{j=0}^{q+1} \tilde{L}_q(\underbrace{U, \dots, U}_{j}, m_\kappa(D)\mathcal{K}U, \dots, U) \\ &\quad + i\sum_{j=0}^{q+1} \tilde{L}_q(\underbrace{U, \dots, U}_{j}, M(U;t)U, \dots, U) \end{aligned} \tag{4.4.30}$$

for some operator $M(U;t)$ in $\Sigma\mathcal{M}^m_{K,1,1}[r,N]\otimes\mathcal{M}_2(\mathbb{C})$ for some $m\geq 3/2$, satisfying (3.1.18). Using (iv) of lemma 4.4.2 and (4.4.19), we write the last sum in (4.4.30) as a contribution to the right hand side of (4.4.29) at rank q, so that adding (4.4.29) at rank $q-1$ and (4.4.30), and using (4.4.10), we get (4.4.29) at rank q.

We integrate next (4.4.29) at rank $q=N-1$ from 0 to t. We get (in the case $t\geq 0$)

$$\|\tilde{V}^1(t;\cdot)\|^2_{\dot{H}^s}+\sum_{p=1}^{N-1}\tilde{L}_p(U(t;\cdot),\dots,U(t;\cdot))$$
$$\leq\|\tilde{V}^1(0;\cdot)\|^2_{\dot{H}^s}+\sum_{p=1}^{N-1}\tilde{L}_p(U(0;\cdot),\dots,U(0;\cdot))+C\int_0^t\|U(\tau;\cdot)\|^{N+2}_{\dot{H}^s}\,d\tau$$

as long as (4.4.26) holds. Taking into account (4.4.18) and (i) of lemma 4.4.2, that applies to the multilinear forms $\tilde{L}_p$, $1\leq p\leq N-1$, in (4.4.28), if ρ is large enough relatively to N_0,N (i.e. $\rho>(N_0+m)N+\frac{1}{2}$), we get, for some new constant C,

$$\|U(t;\cdot)\|^2_{\dot{H}^s}\leq C\|U(0;\cdot)\|^2_{\dot{H}^s}+C\|U(t;\cdot)\|^3_{\dot{H}^s}+C\int_0^t\|U(\tau;\cdot)\|^{N+2}_{\dot{H}^s}\,d\tau \tag{4.4.31}$$

as long as (4.4.26) holds. Taking A_0 large enough relatively to C, we may assume that

$$\|U(0;\cdot)\|^2_{\dot{H}^s}\leq\frac{A_0^2}{8C}\epsilon^2\,. \tag{4.4.32}$$

Moreover, by (4.4.26), the integrand in the last term in the right hand side of (4.4.31) is bounded at any time τ by $(A_0\epsilon)^{N+2}$. If $\epsilon<\epsilon_0$ is small enough, and

$$T<c\epsilon^{-N}\qquad\text{with}\qquad cCA_0^N<\frac{1}{16}\,,$$

it follows by (4.4.31), (4.4.32) and (4.4.26) that estimate (4.4.27) with $k=0$ holds, for any $t\in]-T,T[$. Finally, by (4.4.19), we also deduce that the derivatives $\partial_t^kU(t;\cdot)$ satisfy (4.4.27), if ϵ is small enough, $s\gg K$, and taking A_{k+1} large enough relatively to A_k, $k=0,\dots,K-1$. □

Chapter 5

The Dirichlet-Neumann paradifferential problem

In order to be able to derive system (3.2.2) from the water waves equations (1.1.4) in Chapter 6, we shall need a paradifferential a representation of the Dirichlet-Neumann operator $G(\eta)$. This will be obtained in section 6.1 following the method of Alazard-Métivier [7], Alazard-Burq-Zuily [2, 3] (see also Alazard-Delort [6]). A difference with the above references is that we need to obtain such a representation in terms of the classes of paradifferential operators that have been introduced in Chapter 2. To do so, we shall study in the present chapter the Dirichlet-Neumann boundary value problem in a strip, introducing paradifferential versions of the Poisson operators of Boutet de Monvel [18, 19, 20].

5.1 Paradifferential and para-Poisson operators

We shall work on the strip

$$\mathcal{B} = \left\{(z,x) \in \mathbb{R} \times \mathbb{T}^1 \,; -1 \leq z \leq 0\right\}.$$

The functions we shall consider will depend on (z,x) in $\mathcal{B}$ and on the time parameter t staying in some interval $[-T,T]$. We shall use the notations of Chapter 2 for paradifferential and smoothing operators, except that here our functions and symbols will also depend on the parameter $z \in [-1,0]$. The spaces measuring the smoothness of the functions at hand are the following.

Definition 5.1.1 (Space of functions on a strip) *For j in $\mathbb{N}$ and s in $\mathbb{R}$, we define the space*

$$\begin{aligned} F_j^s = \big\{ &\Phi \in L^\infty([-1,0], \dot{H}^s(\mathbb{T}^1;\mathbb{C})) \text{ such that} \\ &\partial_z^{j'}\Phi \in L^\infty([-1,0], \dot{H}^{s-j'}(\mathbb{T}^1;\mathbb{C})),\ \forall j' = 1,\dots,j \big\} \end{aligned} \tag{5.1.1}$$

together with the natural norm

$$\|\Phi\|_{F_j^s} = \sum_{j'=0}^{j} \|\partial_z^{j'}\Phi\|_{L^\infty([-1,0],\dot{H}^{s-j'})}\,. \tag{5.1.2}$$

We set $F_\infty^\infty \stackrel{def}{=} \bigcap_{s,j} F_j^s$.

We extend first the Definitions 2.1.1 and 2.1.3 to symbols that depend now also on the variable z. We shall use in this section only classes of autonomous symbols, and not versions in a strip of the class of non-homogeneous symbols of Definition 2.1.2. In particular, the time dependence of our (non-homogeneous) symbols will be only through their dependence in the arguments Φ, Ψ, U,... even if we do not repeat this condition in all statements.

We denote by $C_*^K(I, F_j^\sigma) = \bigcap_0^K C^k(I, F_j^{\sigma-\frac{3}{2}k})$. We generalize notations (2.1.2), (2.1.3) to functions depending on z setting

$$\begin{aligned} |\!|\!|\Phi(t,\cdot)|\!|\!|_{K,\sigma,j} &= \sum_{k=0}^{K} \|\partial_t^k \Phi(t,\cdot)\|_{F_j^{\sigma-\frac{3}{2}k}} \\ \mathcal{G}_{K,p}^{\sigma,j}(\Phi,t) &= |\!|\!|\Phi(t,\cdot)|\!|\!|_{K,\sigma,j}^p \\ \mathcal{G}_{0,p}^{\sigma,j}(\Phi) &= \prod_{p'=1}^{p} \|\Phi_{p'}\|_{F_j^\sigma} \end{aligned} \tag{5.1.3}$$

where in the last formula Φ stands for a family $\Phi = (\Phi_1, \dots, \Phi_p)$ of functions of (z, x) in $[-1, 0] \times \mathbb{T}^1$. When $p = 0$, we set $\mathcal{G}_{K,0}^{\sigma,j} = 1$ by convention.

Definition 5.1.2 (Symbols in a strip) *(i) Let m be in $\mathbb{R}$, p in $\mathbb{N}^*$. We denote by $\tilde{\Gamma}_p^m[\mathcal{B}]$ the space of symmetric p-linear maps from $(F_\infty^\infty)^p$ to the space of functions that are C^∞ in $(z, x, \xi) \in [-1, 0] \times \mathbb{T}^1 \times \mathbb{R}$*

$$\Phi = (\Phi_1, \dots, \Phi_p) \to \big((z, x, \xi) \to a(\Phi; (z, x, \xi))\big) \tag{5.1.4}$$

satisfying the following: There is $\mu \geq 0$ and for any j, α, β in $\mathbb{N}$, there is $C > 0$ and for any Φ in $(F_\infty^\infty)^p$, any $n = (n_1, \dots, n_p)$ in $(\mathbb{N}^*)^p$

$$|\partial_z^j \partial_x^\alpha \partial_\xi^\beta a(\Pi_n \Phi; z, x, \xi)| \leq C|n|^{\mu+\alpha+j} \langle\xi\rangle^{m-\beta} \mathcal{G}_{0,p}^{0,j}(\Phi). \tag{5.1.5}$$

Moreover, we assume that if $(n_0, \dots, n_p)$ is in $\mathbb{N} \times (\mathbb{N}^*)^p$ and if

$$\Pi_{n_0} a(\Pi_{n_1}\Phi_1, \dots, \Pi_{n_p}\Phi_p; \cdot) \not\equiv 0, \tag{5.1.6}$$

then there is a choice of signs $\epsilon_j \in \{-1, 1\}$ such that $\sum_0^p \epsilon_j n_j = 0$.

When $p = 0$, we denote by $\tilde{\Gamma}_0^m[\mathcal{B}]$ the space of constant coefficient symbols $(z, \xi) \to a(z, \xi)$, that satisfy inequality (5.1.5) for $\alpha = 0$, with in the right hand side the $|n|$ factor replaced by one.

(ii) Let $r > 0$, K in $\mathbb{N}$. We denote by $\Gamma^m_{K,0,j,p}[\mathcal{B}, r]$ the space of maps $(\Phi; z, x, \xi) \to a(\Phi; t, z, x, \xi)$ defined for

$$\Phi \in B^K_{\sigma_0}(I, r, j) \stackrel{def}{=} \left\{ U \in C^K_*(I, F^{\sigma_0}_j)\,;\ \sup_{t\in I} |\!|\!| \Phi(t,\cdot) |\!|\!|_{K,\sigma,j} < r \right\}$$

for some large enough σ_0, $x \in \mathbb{T}^1$, $\xi \in \mathbb{R}$, $z \in [-1, 0]$, with complex values, such that for any $0 \le k \le K$, any $0 \le j' \le j$, any $\sigma \ge \sigma_0$, there are $C > 0$, $r(\sigma) \in]0, r[$, and for any Φ in $B^K_{\sigma_0}(I, r(\sigma), j) \cap C^K_*(I, F^\sigma_j)$, any $\alpha, \beta \in \mathbb{N}$ with $\alpha \le \sigma - \sigma_0$, any $t \in I$,

$$|\partial_t^k \partial_z^{j'} \partial_x^\alpha \partial_\xi^\beta a(\Phi; z, x, \xi)| \le C \langle \xi \rangle^{m-\beta} \mathcal{G}^{\sigma_0, j'}_{k,p-1}(\Phi, t) \mathcal{G}^{\sigma, j'}_{k,1}(\Phi, t) \tag{5.1.7}$$

(where, in the case $p = 0$, the right hand side should be read as $C\langle\xi\rangle^{m-\beta}$).

(iii) We denote by $\Sigma\Gamma^m_{K,0,j,p}[\mathcal{B}, r, N]$ the space of functions

$$(\Phi; z, x, \xi) \to a(\Phi; z, x, \xi)$$

that may be written as

$$a(\Phi; z, x, \xi) = \sum_{q=p}^{N-1} a_q(\underbrace{\Phi, \dots, \Phi}_{q}; z, x, \xi) + a_N(\Phi; z, x, \xi) \tag{5.1.8}$$

for some a_q in $\tilde\Gamma^m_q[\mathcal{B}]$, a_N in $\Gamma^m_{K,0,j,N}[\mathcal{B}, r]$.

Remarks: • We shall have to consider occasionally symbols of $\Gamma^m_{K,0,j,p}[\mathcal{B}, r]$ depending on a couple of functions (Φ, Φ') with linear dependence on Φ', instead of a single function Φ (for example in the Proof of Proposition 6.1.4). In that case, if we replace in the left hand side of (5.1.7) $a(\Phi; z, x, \xi)$ by $a(\Phi, \Phi'; z, x, \xi)$, the right hand side of this inequality should read

$$\begin{aligned} C\langle\xi\rangle^{m-\beta} \Big[& \mathcal{G}^{\sigma_0,j'}_{k,p-1}(\Phi, t) \mathcal{G}^{\sigma,j'}_{k,1}(\Phi', t) \\ & + \mathcal{G}^{\sigma_0,j'}_{k,p-2}(\Phi, t) \mathcal{G}^{\sigma_0,j'}_{k,1}(\Phi', t) \mathcal{G}^{\sigma,j'}_{k,1}(\Phi, t) \Big]. \end{aligned} \tag{5.1.9}$$

In the same way, we shall consider symbols of $\Sigma\Gamma^m_{K,0,j,p}[\mathcal{B}, r, N]$, depending on Φ, Φ', linear in Φ', given by

$$\sum_{q=p}^{N-1} a_q(\underbrace{\Phi, \dots, \Phi}_{q-1}, \Phi'; z, x, \xi) + a_N(\Phi, \Phi'; z, x, \xi)$$

with a_q in $\tilde\Gamma^m_q[\mathcal{B}]$ and a_N in $\Gamma^m_{K,0,j,N}[\mathcal{B}, r]$, linear in Φ'.

• We shall also use the notation $a(\Phi, \psi; z, x, \xi)$ for symbols depending on Φ and linearly on some function $\psi(t, x)$, independent of z. In this case (5.1.9) has to be replaced by a similar estimate, but with $\mathcal{G}^{\sigma,j'}_{k,1}(\Phi', t)$ and

$\mathcal{G}^{\sigma_0,j'}_{k,1}(\Phi',t)$ replaced respectively by $\mathcal{G}^{\sigma}_{k,1}(\psi,t)$ and $\mathcal{G}^{\sigma_0}_{k,1}(\psi,t)$, and a similar modification of the notation for the elements of $\tilde{\Gamma}^m_q[\mathcal{B}]$.

• If a is a symbol in $\Sigma\Gamma^m_{K,0,j,p}[\mathcal{B},r,N]$, we may define the associated paradifferential operator $Op^{\rm BW}(a(\Phi;\cdot))$ as in (2.2.25), z playing the role of a parameter. According to Proposition 2.2.4, if Φ is in $B^K_\sigma(I,r,j)$ for some σ such that $\sigma-j$ is large enough (independently of s), then the paradifferential operator $Op^{\rm BW}(a(\Phi;\cdot))$ is bounded from F^s_j to F^{s-m}_j for any s.

• In Definition 5.1.2, the functions Φ_j (resp. Φ) belong to F^∞_∞ (resp. are functions of time with values in a space $F^{\sigma_0}_j$), so are $\mathbb{C}$ valued. We shall use also the same notation for the above spaces of symbols, when we consider function Φ_j,Φ that are $\mathbb{C}^2$ valued instead of $\mathbb{C}$ valued.

Let us define as well smoothing operators acting on the spaces F^s_j.

Definition 5.1.3 (Smoothing operator in a strip) *Let K in $\mathbb{N}$, p,N in $\mathbb{N}^*$, $p\le N$, $\rho\ge 0$, j in $\mathbb{N}$.*

(i) One defines $\tilde{\mathcal{R}}^{-\rho}_p[\mathcal{B}]$ as the space of $(p+1)$-linear maps from the space $(F^\infty_\infty)^{p+1}$ to F^∞_∞, symmetric in $(\Phi_1,\dots,\Phi_p)$, of the form

$$(\Phi_1,\dots,\Phi_{p+1})\to R(\Phi_1,\dots,\Phi_p)\Phi_{p+1}\,,$$

satisfying condition (2.2.29) and such that there is some $\mu\ge 0$, and for any $j\in\mathbb{N}$, some $C>0$ such that, for any $\Phi=(\Phi_1,\dots,\Phi_p)$ in $(F^\infty_\infty)^p$, any Φ_{p+1} in F^∞_∞, any $n=(n_1,\dots,n_p)$ in $(\mathbb{N}^*)^p$, any n_0,n_{p+1} in $\mathbb{N}^*$,

$$\sup_{z\in[-1,0]}\|\partial^j_z\Pi_{n_0}R(\Pi_n\Phi)\Pi_{n_{p+1}}\Phi_{p+1}\|_{\dot H^0}\le C\frac{\max_2(n_1,\dots,n_{p+1})^{\rho+\mu}}{\max(n_1,\dots,n_{p+1})^{\rho-j}}\mathcal{G}^{0,j}_{0,p+1}(\Phi,\Phi_{p+1}).\tag{5.1.10}$$

(ii) One defines for N in $\mathbb{N}$, $r>0$, $\mathcal{R}^{-\rho}_{K,0,j,N}[\mathcal{B},r]$ as the space of maps $(\Phi,\Psi)\to R(\Phi)\Psi$, that are defined on $B^K_\sigma(I,r,j)\times C^K_*(I,F^\sigma_j)$ for some $\sigma>0$, that are linear in Ψ and such that, for any s with $s>\sigma$, there are $C>0$ and $r(s)\in]0,r[$, and for any Φ in $B^K_\sigma(I,r(s),j)\cap C^K_*(I,F^s_j)$, any Ψ in $C^K_*(I,F^s_j)$, any $0\le k\le K$, any $0\le j'\le j$, one has the estimate

$$\|\partial^k_t R(\Phi)\Psi(t,\cdot)\|_{F^{s+\rho-\frac{3}{2}k}_{j'}}\le C\sum_{k'+k''=k}\big(\mathcal{G}^{\sigma,j'}_{k',N}(\Phi,t)\mathcal{G}^{s,j'}_{k'',1}(\Psi,t)+\mathcal{G}^{\sigma,j'}_{k',N-1}(\Phi,t)\mathcal{G}^{\sigma,j'}_{k',1}(\Psi,t)\mathcal{G}^{s,j'}_{k'',1}(\Phi,t)\big)\,.\tag{5.1.11}$$

(iii) One denotes by $\Sigma\mathcal{R}^{-\rho}_{K,0,j,p}[\mathcal{B},r,N]$ the space of maps $(\Phi,\Psi)\to R(\Phi)\Psi$ that may be written as

$$R(\Phi)\Psi=\sum_{q=p}^{N-1}R_q(\Phi,\dots,\Phi)\Psi+R_N(\Phi)\Psi\tag{5.1.12}$$

with R_q in $\tilde{\mathcal{R}}^{-\rho}_q[\mathcal{B}]$ and R_N in $\mathcal{R}^{-\rho}_{K,0,j,N}[\mathcal{B},r]$.

Remarks: • If a is a symbol in the class $\Sigma\Gamma^m_{K,0,j,p}[\mathcal{B},r,N]$ and b in $\Sigma\Gamma^{m'}_{K,0,j,p'}[\mathcal{B},r,N]$, then the composition proposition of symbolic calculus 2.4.1 applies and formula (2.4.2) holds with a smoothing operator in the class $\Sigma\mathcal{R}^{-\rho+m+m'}_{K,0,j,p+p'}[\mathcal{B},r,N]$ of Definition 5.1.3.

• Let R be an homogeneous smoothing operator of $\tilde{\mathcal{R}}^{-\rho}_p$ as defined in (i) of Definition 2.2.3. We claim that R defines also a smoothing operator of $\tilde{\mathcal{R}}^{-\rho}_p[\mathcal{B}]$. Indeed, since R is multilinear, given functions Φ_j, $j=1,\dots,p+1$ in F^∞_∞, we may compute

$$\partial_z^j R(\Phi_1,\dots,\Phi_p)\Phi_{p+1}=\sum_{j_1+\cdots+j_{p+1}=j} C_{j_1\dots j_{p+1}} R(\partial_z^{j_1}\Phi_1,\dots,\partial_z^{j_p}\Phi_p)\partial_z^{j_{p+1}}\Phi_{p+1}$$

for suitable binomial coefficients $C_{j_1\dots j_{p+1}}$. Using (2.2.28) and the smoothing estimates $\|\Pi_{n_\ell}\partial_z^{j_\ell}\Phi_\ell\|_{\dot H^{-j}}\le n_\ell^{-j_\ell}\|\Phi_\ell\|_{F^0_j}$, $\ell=1,\dots,p+1$, one deduces (5.1.10).

• If R is an homogeneous smoothing operator of $\tilde{\mathcal{R}}^{-\rho}_p$ then $R(\Phi,\dots,\Phi)\Psi$ satisfies (5.1.11) with $\rho-\alpha$, for any $\alpha>1/2$, instead of ρ.

• As in the case of symbols, we shall use the same notation for classes of smoothing operators when we allow the arguments $\Phi_1,\dots,\Phi_p$ (resp. Φ) in (i) (resp. (ii)) of the above definition to be $\mathbb{C}^2$ valued instead of $\mathbb{C}$ valued.

• Occasionally, we shall have to consider elements of $\tilde{\mathcal{R}}^{-\rho}_p[\mathcal{B}]$ acting on functions $(\Phi_1,\dots,\Phi_{p+1})$ of $(F^\infty_\infty)^{p+1}$, where at least one of the arguments is a function of F^∞_∞ that does not depend on z, i.e. it belongs to $\dot H^\infty(\mathbb{T}^1;\mathbb{C})$ (for example in the proof of Proposition 6.1.4). Of course, such a function is also in F^∞_∞, so that the estimates (5.1.10) remain meaningful. In the same way, we shall have to consider smoothing operators of $\Sigma\mathcal{R}^{-\rho}_{K,0,j,p}[\mathcal{B},r,N]$ where the argument Ψ of (5.1.12) is a function that does not depend on z, so that (5.1.11) holds with $\mathcal{G}^{\sigma,j'}_{k',1}(\Psi,t)=\mathcal{G}^{\sigma}_{k',1}(\Psi,t)$. For simplicity, even in such cases, we shall denote these classes of operators using the same notations.

We shall need as well para-Poisson operators, sending functions defined on one of the boundaries of the strip $\mathcal{B}$ to functions defined on the whole $\mathcal{B}$. We first define the Poisson symbols.

Definition 5.1.4 (Poisson symbols) *Let m be in $\mathbb{R}$, $p\le N,K$ in $\mathbb{N}$.*

(i) **(Homogeneous Poisson symbols)** *One denotes by $\tilde{\mathcal{P}}^{m,+}_p$ (resp. $\tilde{\mathcal{P}}^{m,-}_p$) the space of symmetric p-linear maps defined on $\dot H^\infty(\mathbb{T}^1,\mathbb{C}^2)^p$*

$$\mathcal{U}=(U_1,\dots,U_p)\to\big((z,x,\xi)\to a(\mathcal{U};z,x,\xi)\big)$$

such that for any ℓ,j in $\mathbb{N}$, $z^\ell\partial_z^j a(\mathcal{U};z,x,\xi)$ (resp. $(1+z)^\ell\partial_z^j a(\mathcal{U};z,x,\xi)$) belongs to the class of symbols $\tilde\Gamma^{m+j-\ell}_p$ of Definition 2.1.1, and satisfies bounds (2.1.5) with m replaced by $m+j-\ell$, and some μ independent of ℓ, uniformly in $z\in[-1,0]$.

In the same way, $\tilde{\mathcal{P}}_p^{m,\mathrm{int}}$ denotes the space of symmetric p-linear maps

$$\mathcal{U} = (U_1, \dots, U_p) \to \big((z, z', x, \xi) \to a(\mathcal{U}; z, z', x, \xi)\big)$$

with values in functions of $z, z' \in [-1, 0]$, $x \in \mathbb{T}^1$, $\xi \in \mathbb{R}$ that may be written as

$$a_+(\mathcal{U}; z, z', x, \xi)\mathbf{1}_{z-z'>0} + a_-(\mathcal{U}; z, z', x, \xi)\mathbf{1}_{z-z'<0} \tag{5.1.13}$$

where for any ℓ, j, j' in $\mathbb{N}$, $(z - z')^\ell \partial_z^j \partial_{z'}^{j'} a_\pm(\mathcal{U}; z, z', \cdot)$ belongs to $\tilde{\Gamma}_p^{m+j+j'-\ell}$ and satisfies (2.1.5) with m replaced by $m+j+j'-\ell$ and some μ independent of ℓ, uniformly in z, z'.

(ii) **(Non-homogeneous Poisson symbols)** *Let $r > 0$. One denotes by $\mathcal{P}_{K,0,p}^{m,+}[r]$ (resp. $\mathcal{P}_{K,0,p}^{m,-}[r]$) the space of functions*

$$(U, z, x, \xi) \to a(U; z, x, \xi)$$

such that for any j, ℓ in $\mathbb{N}$, the symbol $z^\ell \partial_z^j a(U; z, \cdot)$ (resp. $(1+z)^\ell \partial_z^j a(U; z, \cdot)$) belongs to $\Gamma_{K,0,p}^{m+j-\ell}[r, \mathrm{aut}]$ (see Definition 2.1.3), and satisfies (2.1.10) with m replaced by $m+j-\ell$, $K' = 0$, and some σ, σ_0 independent of ℓ, uniformly in $z \in [-1, 0]$.

One defines $\mathcal{P}_{K,0,p}^{m,\mathrm{int}}[r]$ as the space of functions

$$a(U; z, z', x, \xi) = \sum_{+,-} a_\pm(U; z, z', x, \xi)\mathbf{1}_{\pm(z-z')>0}, \tag{5.1.14}$$

where for any integers ℓ, j, j', $(z-z')^\ell \partial_z^j \partial_{z'}^{j'} a_\pm(U; t, z, z', \cdot)$ belongs to the class $\Gamma_{K,0,p}^{m+j+j'-\ell}[r, \mathrm{aut}]$ and satisfies (2.1.10) with m replaced by $m + j + j' - \ell$, $K' = 0$, and some σ independent of ℓ, uniformly in z, z'.

(iii) **(Poisson symbols)** *One defines the Poisson symbols $\Sigma\mathcal{P}_{K,0,p}^{m,\pm}[r, N]$ (resp. $\Sigma\mathcal{P}_{K,0,p}^{m,\mathrm{int}}[r, N]$) as the sum of homogeneous Poisson symbols a_q in $\tilde{\mathcal{P}}_q^{m,\pm}$, $q = p, \dots, N-1$, evaluated at $(U, \dots, U)$, plus a non-homogeneous Poisson symbol a_N of $\mathcal{P}_{K,0,N}^{m,\pm}[r]$,*

$$a = \sum_{q=p}^{N-1} a_q(U, \dots, U; z, z', x, \xi) + a_N(U; z, z', x, \xi)\,. \tag{5.1.15}$$

We denote $\Sigma\mathcal{P}_{K,0,p}^{-\infty,\pm}[r, N]$ the intersection $\bigcap_m \Sigma\mathcal{P}_{K,0,p}^{m,\pm}[r, N]$.

Remarks: • If a is a Poisson symbol in $\tilde{\mathcal{P}}_p^{m,\pm}$, resp. $\Sigma\mathcal{P}_{K,0,p}^{m,\pm}[r, N]$, then $\partial_z a$ is in $\tilde{\mathcal{P}}_p^{m+1,\pm}$, resp. $\Sigma\mathcal{P}_{K,0,p}^{m+1,\pm}[r, N]$. Similarly if a is a Poisson symbol in $\tilde{\mathcal{P}}_p^{m,\mathrm{int}}$, resp. $\Sigma\mathcal{P}_{K,0,p}^{m,\mathrm{int}}[r, N]$, then $\partial_z a$ and $\partial_{z'} a$ are in $\tilde{\mathcal{P}}_p^{m+1,\mathrm{int}}$, resp. $\Sigma\mathcal{P}_{K,0,p}^{m+1,\mathrm{int}}[r, N]$.

• If a is a Poisson symbol in $\Sigma\mathcal{P}^{m,\pm}_{K,0,p}[r,N]$, resp. $\Sigma\mathcal{P}^{m,\mathrm{int}}_{K,0,p}[r,N]$, then $\partial_\xi a$ is in $\Sigma\mathcal{P}^{m-1,\pm}_{K,0,p}[r,N]$, resp. $\Sigma\mathcal{P}^{m-1,\mathrm{int}}_{K,0,p}[r,N]$.

• If a is a Poisson symbol in $\tilde{\mathcal{P}}^{m,+}_p$, resp. $\tilde{\mathcal{P}}^{m,-}_p$, then za is in $\tilde{\mathcal{P}}^{m-1,+}_p$, resp. $(1+z)a$ is in $\tilde{\mathcal{P}}^{m-1,-}_p$. Similarly if a is a Poisson symbol in $\tilde{\mathcal{P}}^{m,\mathrm{int}}_p$, resp. $\Sigma\mathcal{P}^{m,\mathrm{int}}_{K,0,p}[r,N]$, then $(z-z')a$ is in $\tilde{\mathcal{P}}^{m-1,\mathrm{int}}_p$, resp. $\Sigma\mathcal{P}^{m-1,\mathrm{int}}_{K,0,p}[r,N]$.

• Let $\theta\in]0,1[$. If a is in $\Sigma\mathcal{P}^{m,\mathrm{int}}_{K,0,p}[r,N]$, then $|z-z'|^\theta a$ is in $\Sigma\mathcal{P}^{m-\theta,\mathrm{int}}_{K,0,p}[r,N]$. Indeed, for any z,z' in $[-1,0]$, for any $\alpha,\beta\in\mathbb{N}$,

$$||z-z'|^\theta\partial_x^\alpha\partial_\xi^\beta a_\pm| = |(z-z')\partial_x^\alpha\partial_\xi^\beta a_\pm|^\theta|\partial_x^\alpha\partial_\xi^\beta a_\pm|^{1-\theta}$$

satisfies the estimates of a symbol in $\Sigma\Gamma^{m-\theta}_{K,0,p}[r,N,\mathrm{aut}]$, uniformly in z,z'.

• The following Poisson symbols

$$\mathcal{C}(z,\xi)=\frac{\cosh((z+1)\xi)}{\cosh\xi}\in\tilde{\mathcal{P}}^{0,+}_0\,,\quad \mathcal{S}(z,\xi)=\frac{\sinh(z\xi)}{\xi\cosh\xi}\in\tilde{\mathcal{P}}^{-1,-}_0\,, \tag{5.1.16}$$

and

$$K_0(z,z',\xi)=(\cosh\xi)\big(\mathcal{C}(z,\xi)\mathcal{S}(z',\xi)\mathbf{1}_{z-z'<0}+\mathcal{S}(z,\xi)\mathcal{C}(z',\xi)\mathbf{1}_{z-z'>0}\big)\in\tilde{\mathcal{P}}^{-1,\mathrm{int}}_0$$

will arise in the next section.

• We associate to a Poisson symbol $a(U;z,x,\xi)$ in $\Sigma\mathcal{P}^{m,\pm}_{K,0,p}[r,N]$ the corresponding paradifferential operator sending functions defined on $\mathbb{T}^1$ to functions of $(z,x)\in\mathcal{B}$ by applying (2.2.25) for each z. We call $Op^{\mathrm{BW}}(a(U;z,\cdot))$ the para-Poisson operator associated to a. On the other hand, to a Poisson symbol a in $\Sigma\mathcal{P}^{m,\mathrm{int}}_{K,0,p}[r,N]$ we associate the para-Poisson operator

$$V\to\int_{-1}^0 Op^{\mathrm{BW}}(a(U;z,z',\cdot))V(z',\cdot)\,dz'\,,$$

which sends a function V defined on the strip $\mathcal{B}$ into another function of $\mathcal{B}$.

According to the Definition 5.1.4 of Poisson symbols, the boundedness properties of a paradifferential operator given in Proposition 2.2.4 imply the following lemma.

Lemma 5.1.5 (Action of a para-Poisson operator from the boundary to the interior) *Let a be a Poisson symbol in $\Sigma\mathcal{P}^{m,+}_{K,0,p}[r,N]$ (resp. $\Sigma\mathcal{P}^{m,-}_{K,0,p}[r,N]$). If U is in $B^K_{\sigma_0}(I,r)$ for some large enough σ_0, then, for any s in $\mathbb{R}$, j in $\mathbb{N}$, any V belonging to $\dot{H}^s(\mathbb{T}^1,\mathbb{C})$, the function*

$$(z,t,x)\to z^\ell\partial_z^j Op^{\mathrm{BW}}(a(U;z,\cdot))V$$

(resp. $(z,t,x)\to(1+z)^\ell\partial_z^j Op^{\mathrm{BW}}(a(U;z,\cdot))V$) is in $C^K_*(I,\dot{H}^{s-m+\ell-j}(\mathbb{T}^1,\mathbb{C}))$ uniformly in $z\in[-1,0]$, and for any $0\le k\le K$

$$\|z^\ell\partial_z^j Op^{\mathrm{BW}}(\partial_t^k a(U;z,\cdot))V\|_{L^\infty([0,1],\dot{H}^{s-m+\ell-j})}\le C\|V\|_{\dot{H}^s}$$

uniformly in $t \in I$, the constant C depending only on $|\!|\!| U(t,\cdot) |\!|\!|_{K,\sigma_0}$. In particular $\|Op^{\rm BW}(\partial_t^k a(U;z,\cdot))V\|_{F_j^{s-m}} \leq C\|V\|_{\dot H^s}$.

To study the action of para-Poisson operators associated to symbols in the space $\Sigma\mathcal{P}^{m,\rm int}_{K,0,p}[r,N]$, we introduce another scale of spaces.

Definition 5.1.6 (Space of functions on a strip) *We denote by E_j^s the subspace of F_j^s (introduced in Definition 5.1.1) given by*

$$E_j^s = \left\{ \Phi \in F_j^s \,;\; \partial_z^{j'}\Phi \in L^2([-1,0],\ \dot H^{s-j'+\frac12}(\mathbb{T}^1)) ;\; j' = 0,\dots,j \right\} \tag{5.1.17}$$

together with the natural norm

$$\|\Phi\|_{E_j^s} = \|\Phi\|_{F_j^s} + \sum_{j'=0}^{j} \|\partial_z^{j'}\Phi\|_{L^2([-1,0],\dot H^{s-j'+\frac12})} \,. \tag{5.1.18}$$

We denote by $C^K_*(I,E_j^s) = \bigcap_0^K C^k(I, E_j^{s-\frac32 k})$ and by $|\!|\!| \Phi |\!|\!|^E_{K,s,j}$ the norm defined by the first line in (5.1.3) with $F_j^{\sigma-\frac32 k}$ replaced by $E_j^{s-\frac32 k}$. We set $E_\infty^\infty \stackrel{def}{=} \bigcap_{s,j} E_j^s$.

Remark: We have the continuous inclusions $E_j^s \subset F_j^s \subset E_j^{s-\frac12}$.

These spaces are characterized in terms of Littlewood-Paley decomposition, that we now recall. Consider a Littlewood-Paley partition of unity,

$$1 = \psi(\xi) + \sum_{\ell\geq 1} \varphi(2^{-\ell}\xi)\,,$$

where the C^∞ function φ, resp. ψ, is supported for $C^{-1} \leq |\xi| \leq C$ for some constant $C > 2$, resp. for $|\xi| \leq c$ for some $c > 0$, ψ and φ being even in ξ. We denote

$$\Delta_0 = \psi(D)\,, \quad \Delta_\ell = \varphi(2^{-\ell}D)\,,\ \ell \geq 1\,. \tag{5.1.19}$$

We have the following characterization of the spaces E_j^s.

Lemma 5.1.7 (Littlewood-Paley characterization of E_j^s) *A function Φ is in E_j^s if and only if, for any $0 \leq j' \leq j$, $z \in [-1,0]$,*

$$\begin{aligned} \|\Delta_\ell \partial_z^{j'}\Phi(z,\cdot)\|_{L^2(\mathbb{T}^1)} &\leq c_\ell(z) 2^{-\ell(s-j')} \\ \|\Delta_\ell \partial_z^{j'}\Phi\|_{L^2([-1,0],L^2(\mathbb{T}^1))} &\leq c_\ell 2^{-\ell(s+\frac12-j')} \end{aligned} \tag{5.1.20}$$

for a sequence $(c_\ell)_\ell$ in $\ell^2(\mathbb{N},\mathbb{R})$ and a sequence $(c_\ell(z))_\ell$ in $\ell^2(\mathbb{N},\mathbb{R})$ such that

$$\|\Phi\|^2_{F_j^s} \simeq \sup_{z\in[-1,0]} \sum_\ell |c_\ell(z)|^2\,, \quad \|\Phi\|^2_{E_j^s} \simeq \|\Phi\|^2_{F_j^s} + \sum_{\ell\geq 0} c_\ell^2\,.$$

We also have

$$\|\Delta_\ell \partial_z^{j'} \Phi(z,\cdot)\|_{L^2(\mathbb{T}^1)} \leq c'_\ell(z) 2^{-\ell(s-j'+\frac{1}{2})} \tag{5.1.21}$$

for a sequence $(c'_\ell(z))_\ell$ *such that* $\sum_\ell \|c'_\ell(z)\|^2_{L^2([-1,0],\mathbb{R})} = \sum_{\ell\geq 0} c_\ell^2$.

Proof: By the characterization of the Sobolev spaces with a Paley-Littlewood decomposition we have, for any $0 \leq j' \leq j$,

$$\|\partial_z^{j'} \Phi(z,\cdot)\|^2_{\dot{H}^{s-j'}} \simeq \sum_\ell \left(\|\Delta_\ell \partial_z^{j'} \Phi(z,\cdot)\|_{L^2(\mathbb{T}^1)} 2^{\ell(s-j')} \right)^2$$

$$\|\partial_z^{j'} \Phi\|^2_{L^2([-1,0],\dot{H}^{s-j'+\frac{1}{2}})} \simeq \sum_\ell \left(\|\Delta_\ell \partial_z^{j'} \Phi\|_{L^2([-1,0],L^2(\mathbb{T}^1))} 2^{\ell(s-j'+\frac{1}{2})} \right)^2$$

and (5.1.20) follows with

$$c_\ell(z) \stackrel{\text{def}}{=} \max_{0\leq j'\leq j} \|\Delta_\ell \partial_z^{j'} \Phi(z,\cdot)\|_{L^2(\mathbb{T}^1)} 2^{\ell(s-j')}$$

$$c_\ell \stackrel{\text{def}}{=} \max_{0\leq j'\leq j} \|\Delta_\ell \partial_z^{j'} \Phi\|_{L^2([-1,0],L^2(\mathbb{T}^1))} 2^{\ell(s-j'+\frac{1}{2})}\,.$$

Finally (5.1.21) follows with $c'_\ell(z) = c_\ell(z) 2^{\ell/2}$. □

The para-Poisson operators associated to a symbol of $\Sigma\mathcal{P}^{m,\mathrm{int}}_{K,0,p}[r,N]$ gain one derivative in the E^s_j scale (we shall use this property for instance in the proof of lemma 5.3.2).

Lemma 5.1.8 (Action of a para-Poisson operator from the interior to the interior) *Let* p, K *be in* $\mathbb{N}$, $r > 0$, $N \geq p$ *and let* a *be a Poisson symbol of* $\Sigma\mathcal{P}^{m,\mathrm{int}}_{K,0,p}[r,N]$. *There is* $\sigma_0 > 0$ *such that if* U *is in* $B^K_{\sigma_0}(I,r)$, *for any* s *in* $\mathbb{R}$, *any* j *in* $\mathbb{N}$, *any* $k \leq K$, *the para-Poisson operator*

$$V \to \int_{-1}^0 Op^{\mathrm{BW}}(\partial_t^k a(U;z,z',\cdot)) V(z',\cdot)\, dz' \tag{5.1.22}$$

is bounded from E^s_j *to* E^{s+1-m}_j *uniformly in* $t \in I$, *the bound depending only on* $|\!|\!| U(t,\cdot) |\!|\!|_{K,\sigma_0}$.

Proof: We may assume $k = 0$ and we do not write time dependence. By Definition 5.1.4, the right hand side of (5.1.22) may be written as

$$\int_{-1}^z Op^{\mathrm{BW}}(a_+(U;z,z',\cdot)) V(z')\, dz' + \int_z^0 Op^{\mathrm{BW}}(a_-(U;z,z',\cdot)) V(z')\, dz'. \tag{5.1.23}$$

Denote by $\Delta_\ell = \varphi(2^{-\ell} D_x)$ the elements of some Littlewood-Paley decomposition as in (5.1.19). It follows from the definition of Op^{BW} that the

action of $Op^{\rm BW}(a_\pm)$ does not enlarge much the support of the Fourier transform of functions, see (2.2.36), (2.2.42)-(2.2.43) and the first remark after Proposition 2.2.4. We may thus find a compactly supported function $\tilde\varphi$ of $C_0^\infty(\mathbb{R}^*)$, equal to one in a large enough compact subset of $\mathbb{R}^*$, so that, setting $\tilde\Delta_\ell = \tilde\varphi(2^{-\ell}D_x)$, it results

$$\Delta_\ell Op^{\rm BW}(a_\pm) = \Delta_\ell Op^{\rm BW}(a_\pm)\tilde\Delta_\ell\,.$$

By the boundedness properties of paradifferential operators of Proposition 2.2.4, if σ_0 is large enough, we have, for any N,

$$\begin{aligned}(5.1.24)\quad |z-z'|^N \|\Delta_\ell Op^{\rm BW}(a_\pm(U;z,z',\cdot))V(z',\cdot)\|_{L^2(\mathbb{T}^1)} \\ \le C2^{\ell(m-N)}\|\tilde\Delta_\ell V(z',\cdot)\|_{L^2(\mathbb{T}^1)}\end{aligned}$$

the constant C depending only on $|\!|\!|U(t,\cdot)|\!|\!|_{0,\sigma_0}$. Since the function $V(z',\cdot)$ is in E_0^s we get by (5.1.21) that

$$\|\tilde\Delta_\ell V(z',\cdot)\|_{L^2(\mathbb{T}^1)} \le c'_\ell(z')2^{-\ell(s+\frac12)}$$

for some sequence $(c'_\ell(z'))_\ell$ that satisfies

$$(5.1.25)\qquad \sum_\ell \|c'_\ell(z')\|^2_{L^2(dz')} < C\|V\|^2_{E_0^s}\,.$$

Therefore (5.1.24) is bounded by $C2^{\ell(m-N-s-\frac12)}c'_\ell(z')$. Using (5.1.24) with $N=0$ and $N=2$, according if $|z-z'|2^\ell<1$ or $|z-z'|2^\ell\ge1$, we may thus bound the $L^2(dx)$ norm of the action of Δ_ℓ on (5.1.23), obtaining

$$(5.1.26)\quad \Big\|\Delta_\ell\int_{-1}^0 Op^{\rm BW}(a(U;z,z',\cdot))V(z',\cdot)\,dz'\Big\|_{L^2(\mathbb{T}^1)} \le C2^{\ell(m-s-\frac12)}\tilde c'_\ell(z)$$

with

$$\tilde c'_\ell(z) = \int_{-1}^0 (1+2^\ell|z-z'|)^{-2}c'_\ell(z')\,dz'\,.$$

By the Cauchy-Schwartz inequality

$$\begin{aligned}|\tilde c'_\ell(z)|^2 \le \Big(\int_{-1}^0 \frac{dz'}{(1+2^\ell|z-z'|)^2}\Big)\Big(\int_{-1}^0 \frac{|c'_\ell(z')|^2}{(1+2^\ell|z-z'|)^2}dz'\Big) \\ \le C2^{-\ell}\int_{-1}^0 \frac{|c'_\ell(z')|^2}{(1+2^\ell|z-z'|)^2}dz'\end{aligned}$$

and we deduce that

$$\begin{aligned}(5.1.27)\quad \sup_{z\in[-1,0]}\sum_\ell 2^\ell|\tilde c'_\ell(z)|^2 \;+\; \int_{-1}^0\sum_\ell 2^{2\ell}|\tilde c'_\ell(z)|^2\,dz \\ \le C\sum_\ell \|c'_\ell(z')\|^2_{L^2(dz')} \le C'\|V\|^2_{E_0^s}\end{aligned}$$

where the last inequality is given by (5.1.25). By (5.1.26) and (5.1.27) we deduce that (5.1.22) (for $k=0$) satisfies estimates (5.1.20) with $j'=0$, the Sobolev index s replaced by $s+1-m$, and sequences $c_\ell(z)=\tilde{c}'_\ell(z)2^{\ell/2}$, $c_\ell^2=2^{2\ell}\int_{-1}^0|\tilde{c}'_\ell(z)|^2dz$, and hence

$$\int_{-1}^0 Op^{\rm BW}(a(U;z,z',\cdot))V(z',\cdot)\,dz'\in E_0^{s+1-m}\,,$$

and its E_0^{s+1-m}-norm is bounded by $\|V\|_{E_0^s}$.

We have next to study the z-derivatives of (5.1.23). Notice that the first derivative is given by (5.1.23) with $a_\pm$ replaced by $\partial_z a_\pm$, and this term may be treated as above, plus the contribution

$$Op^{\rm BW}(a_+(U;z,z,\cdot))V(z,\cdot)-Op^{\rm BW}(a_-(U;z,z,\cdot))V(z,\cdot)\,. \tag{5.1.28}$$

Since the symbols $a_\pm(U;z,z,\cdot)$ are in $\Sigma\Gamma^m_{K,0,p}[r,N,{\rm aut}]$, uniformly in z, and V is in $L^\infty([-1,0],\dot{H}^s)\cap L^2([-1,0],\dot{H}^{s+\frac12})$, Proposition 2.2.4 implies that (5.1.28) belongs to $L^\infty([-1,0],\dot{H}^{s-m})\cap L^2([-1,0],\dot{H}^{s+\frac12-m})$ with a norm bounded by $C\|V\|_{E_0^s}$. Higher order ∂_z derivatives are treated in the same way. □

Arguing in a similar way to Lemma 5.1.8 we have also the following lemma concerning the action of a para-Poisson operator from the boundary to the interior with values in the scale E_j^s.

Lemma 5.1.9 (Action of a para-Poisson operator from the boundary to the interior) *Let p,K be in $\mathbb{N}$, $r>0$, $N\geq p$ and let a be a Poisson symbol of $\Sigma\mathcal{P}^{m,\pm}_{K,0,p}[r,N]$. There is $\sigma_0>0$ such that if U is in $B^K_{\sigma_0}(I,r)$, for any s in $\mathbb{R}$, any j in $\mathbb{N}$, any $k\leq K$, the para-Poisson operator*

$$V\to Op^{\rm BW}(\partial_t^k a(U;z,\cdot))V$$

is bounded from $\dot{H}^s(\mathbb{T}^1,\mathbb{C})$ to E_j^{s-m}, uniformly in $t\in I$ and for any j, the bound depending only on $|\!|\!|U(t,\cdot)|\!|\!|_{K,\sigma_0}$.

We shall define next the natural classes of smoothing operators that will give remainders in the symbolic calculus associated to Poisson symbols of $\Sigma\mathcal{P}^{m,\pm}_{K,0,p}[r,N]$ and $\Sigma\mathcal{P}^{m,{\rm int}}_{K,0,p}[r,N]$. Let us introduce a variant of notation (5.1.3), namely define

$$\tilde{\mathcal{G}}^{\sigma,j}_{K,1}(U,t)=|\!|\!|U(t,\cdot)|\!|\!|^E_{K,\sigma,j}=\sum_{k=0}^K\|\partial_t^kU(t,\cdot)\|_{E_j^{\sigma-\frac32k}}\,. \tag{5.1.29}$$

Definition 5.1.10 (Smoothing operators from the boundary to a strip and on a strip) *Let ρ be in $\mathbb{R}_+$, p,K in $\mathbb{N}$, $p\leq N$.*

(i) We denote by $\tilde{\mathcal{R}}_p^{-\rho,+}$ *(resp.* $\tilde{\mathcal{R}}_p^{-\rho,-}$*) the space of* $(p+1)$*-linear operators defined on* $\dot{H}^\infty(\mathbb{T}^1,\mathbb{C}^2)^p \times \dot{H}^\infty(\mathbb{T}^1,\mathbb{C})$*, depending on* $z \in [-1,0]$*,*

$$(U_1,\dots,U_{p+1}) \to R(U_1,\dots,U_p;z)U_{p+1}$$

symmetric in $(U_1,\dots,U_p)$*, such that for any* j,ℓ *in* $\mathbb{N}$ *with* $j-\ell \le \rho$*,* $(z^\ell \partial_z^j R(U_1,\dots,U_p;z))_{z\in[-1,0]}$ *(resp.* $((1+z)^\ell \partial_z^j R(U_1,\dots,U_p;z))_{z\in[-1,0]}$*) is a bounded family of smoothing operators of* $\tilde{\mathcal{R}}_p^{-\rho+j-\ell}$ *defined in Definition 2.2.3, in the sense that (2.2.28) holds with* ρ *replaced by* $\rho+\ell-j$ *and some* μ *independent of* ℓ*, uniformly in* z*.*

We denote by $\tilde{\mathcal{R}}_p^{-\rho,\mathrm{int}}$ *the space of* $(p+1)$*-linear operators, defined for* $(U_1,\dots,U_p,V)$ *in* $\dot{H}^\infty(\mathbb{T}^1,\mathbb{C}^2)^p \times E_\infty^\infty$*, of the form*

$$V \to \int_{-1}^0 R(U_1,\dots,U_p;z,z')V(z',\cdot)\,dz' \tag{5.1.30}$$

symmetric in $(U_1,\dots,U_p)$*, whose integral kernel depend on* $z,z' \in [-1,0]$*, and may be written as*

$$R(U_1,\dots,U_p;z,z') = \sum_{+,-} R_\pm(U_1,\dots,U_p;z,z')\mathbf{1}_{\pm(z-z')>0}$$

where, for any j,j',ℓ *in* $\mathbb{N}$ *with* $j+j'-\ell \le \rho-1$*,*

$$\big((z-z')^\ell \partial_z^j \partial_{z'}^{j'} R_\pm(U_1,\dots,U_p;z,z')\big)_{z,z'\in[-1,0]} \tag{5.1.31}$$

is a bounded family of smoothing operators of , in the sense that (2.2.28) holds with ρ *replaced by* $-\rho+j+j'-\ell+1$ *and some* μ *independent of* ℓ*, uniformly in* z,z'*.*

(ii) Let $r>0$*. We denote by* $\mathcal{R}_{K,0,N}^{-\rho,\pm}[r]$ *(resp.* $\mathcal{R}_{K,0,N}^{-\rho,\mathrm{int}}[r]$*) the space of maps* $(U,V) \to R(U)V$ *that are defined on* $B_\sigma^K(I,r) \times C_*^K(I,\dot{H}^\sigma(\mathbb{T}^1,\mathbb{C}))$ *(resp. on* $B_\sigma^K(I,r) \times C_*^K(I,E_j^\sigma)$*) for some* $\sigma>0$ *(resp. some* $\sigma \ge j \ge 0$*), with values in* $C_*^K(I,E_j^{\sigma+\rho})$*, that are linear in* V*, and such that for any* $s>\sigma$*, there are* $C>0$*,* $r(s) \in]0,r[$ *and for any* $U \in B_\sigma^K(I,r(s)) \cap C_*^K(I,\dot{H}^s(\mathbb{T}^1,\mathbb{C}^2))$*, any* V *in* $C_*^K(I,\dot{H}^s(\mathbb{T}^1,\mathbb{C}))$*, any* $0 \le k \le K$*, any* j *in* $\mathbb{N}$*, any* t *in* I*, one has a bound*

$$\begin{aligned}\|\partial_t^k R(U)V(t,\cdot)\|_{E_j^{s+\rho-\frac{3}{2}k}} \le C\Big(&\sum_{k'+k''=k} \mathcal{G}_{k',N}^\sigma(U,t)\mathcal{G}_{k'',1}^s(V,t)\\ &+ \mathcal{G}_{k',N-1}^\sigma(U,t)\mathcal{G}_{k'',1}^\sigma(V,t)\mathcal{G}_{k',1}^s(U,t)\Big)\end{aligned} \tag{5.1.32}$$

(resp. for any $0 \le k \le K$*, any* $0 \le j' \le j$*, any* $t \in I$*, one has a bound for any* V *in* $C_*^K(I,E_j^s)$

$$\begin{aligned}\|\partial_t^k R(U)V(t,\cdot)\|_{E_{j'}^{s+\rho-\frac{3}{2}k}} \le C\Big(&\sum_{k'+k''=k} \mathcal{G}_{k',N}^\sigma(U,t)\tilde{\mathcal{G}}_{k'',1}^{s,j'}(V,t)\\ &+ \mathcal{G}_{k',N-1}^\sigma(U,t)\tilde{\mathcal{G}}_{k'',1}^{\sigma,j'}(V,t)\mathcal{G}_{k',1}^s(U,t)\Big)\end{aligned} \tag{5.1.33}$$

where we have used notation (5.1.29)).

Moreover, the operator $R(U)$ is autonomous in the sense that the time dependence enters only through $U = U(t)$.

(iii) One denotes by $\Sigma\mathcal{R}^{-\rho,\pm}_{K,0,p}[r,N]$ (resp. $\Sigma\mathcal{R}^{-\rho,\mathrm{int}}_{K,0,p}[r,N]$) the space of sums of operators

$$V \to \sum_{q=p}^{N-1} R_q(U,\dots,U;z)V + R_N(U)V$$

(resp.

$$V \to \sum_{q=p}^{N-1} \int_{-1}^{0} R_q(U,\dots,U;z,z')V(z',\cdot)\,dz' + R_N(U)V \big) \tag{5.1.34}$$

with R_q in $\tilde{\mathcal{R}}^{-\rho,\pm}_q$ (resp. $\tilde{\mathcal{R}}^{-\rho,\mathrm{int}}_q$) $q = p,\dots,N-1$ and R_N in $\mathcal{R}^{-\rho,\pm}_{K,0,N}[r]$ (resp. $\mathcal{R}^{-\rho,\mathrm{int}}_{K,0,N}[r]$).

Remarks: • If R is in $\Sigma\mathcal{R}^{-\rho,\pm}_{K,0,p}[r,N]$, then $R_{|z=0}$ is in $\Sigma\mathcal{R}^{-\rho}_{K,0,p}[r,N,\mathrm{aut}]$, according to Definition 2.2.3.

• If R is in $\Sigma\mathcal{R}^{-\rho,\pm}_{K,0,p}[r,N]$, then $\partial_z R$ is in $\Sigma\mathcal{R}^{-\rho+1,\pm}_{K,0,p}[r,N]$.

• Consider a homogeneous smoothing operator in $\tilde{\mathcal{R}}^{-\rho,\mathrm{int}}_p$ with kernel R, as in (5.1.30). Then the operator with kernel $\partial_z R$ is in $\tilde{\mathcal{R}}^{-\rho+1,\mathrm{int}}_p$ and, for any $\theta \in [0,1]$, the operator with kernel $|z-z'|^\theta R$ is in $\tilde{\mathcal{R}}^{-\rho-\theta,\mathrm{int}}_p$.

• In the sequel we shall identify a non-homogeneous $R(U)V$ in $\mathcal{R}^{-\rho,\pm}_{K,0,N}[r]$ to the integral expression given in terms of its Schwartz kernel $R(U;z,z')$ i.e. write the action of this operator on V as

$$R(U)V = \int_{-1}^{0} R_N(U;z,z')V(z',\cdot)dz'.$$

Notice that we characterize the homogeneous operators of $\tilde{\mathcal{R}}^{-\rho,\mathrm{int}}_p$ by properties of the kernel $R(U_1,\dots,U_p;z,z')$, and the non-homogeneous ones in $\mathcal{R}^{-\rho,\pm}_{K,0,N}[r]$ by their action between the spaces E^s_j.

• If the homogeneous smoothing operator $\tilde{R}$ is in $\tilde{\mathcal{R}}^{-\rho,\pm}_p$ (resp. in $\tilde{\mathcal{R}}^{-\rho,\mathrm{int}}_p$) then $R(U)V = \tilde{R}(U,\dots,U;z)V$ (resp.

$$R(U)V = \int_{-1}^{0} \tilde{R}(U,\dots,U;z,z')V(z')\,dz' \tag{5.1.35}$$

defines a smoothing operator of $\mathcal{R}^{-\rho,\pm}_{K,0,p}[r]$ (resp. $\mathcal{R}^{-\rho,\mathrm{int}}_{K,0,p}[r]$) for any $r > 0$. Let us prove this claim in the case of interior operators. Let us show that the estimate (5.1.33) holds for the operator (5.1.35) when for instance $k = 0, j' = 1$. We have to bound

$$\|R(U)V\|_{E^{s+\rho}_0} + \|(\partial_z R(U))V\|_{E^{s+\rho-1}_0} + \|R(U)[\partial_z V]\|_{E^{s+\rho-1}_0}$$

by the right hand side of (5.1.33) with $k=0, j'=1$ (and N replaced by p). Below we prove in detail the estimate for $\|(\partial_z R(U))V\|_{E_0^{s+\rho-1}}$. According to (i) of Definition 5.1.10 the operator $\tilde R$ may be written as

$$\tilde R_+(U,\dots,U;z,z')\mathbf{1}_{z-z'>0}+\tilde R_-(U,\dots,U;z,z')\mathbf{1}_{z-z'<0}$$

so that

(5.1.36)
$$\begin{aligned}\partial_z\tilde R(U,\dots,U;z,z') &= \big(\tilde R_+(U,\dots,U;z,z)-\tilde R_-(U,\dots,U;z,z)\big)\delta(z-z')\\ &\quad+\partial_z\tilde R_+(U,\dots,U;z,z')\mathbf{1}_{z-z'>0}+\partial_z\tilde R_-(U,\dots,U;z,z')\mathbf{1}_{z-z'<0}\,.\end{aligned}$$

Recalling lemma 5.1.7, we have to bound by the right hand side of (5.1.33) the product of $2^{\ell(s+\rho-1)}$ (resp. $2^{\ell(s+\rho-\frac12)}$) times the $L^\infty_z(\ell^2_\ell L^2(\mathbb{T}^1))$ (resp. $L^2_z(\ell^2_\ell L^2(\mathbb{T}^1))$) norm of each of the expressions

(5.1.37)
$$\begin{aligned}&\Delta_\ell R^0(U,\dots,U;z)V(z,\cdot)\\ &\Delta_\ell\int_{-1}^0 R^1(U,\dots,U;z,z')V(z',\cdot)\,dz'\end{aligned}$$

where $R^0=(\tilde R_+-\tilde R_-)|_{z=z'}$ and R^1 denotes the operator in second line in (5.1.36). According to Definition 5.1.10 the family of operators R^0 satisfies (2.2.28) with ρ replaced by $\rho-1$, uniformly in z, and (2.2.29), and $(z-z')^qR^1$ satisfies (2.2.28) with ρ replaced by $\rho+q-2$, uniformly in (z,z') and (2.2.29). We estimate the $L^2(\mathbb{T}^1)$ norm of the second expression in (5.1.37) by

(5.1.38)
$$\sum_{n_0,\dots,n_{p+1}}\int_{-1}^0\|\Delta_\ell\Pi_{n_0}R^1(\Pi_{n_1}U,\dots,\Pi_{n_p}U;z,z')\Pi_{n_{p+1}}V(z',\cdot)\|_{L^2}\,dz'.$$

Since R^1 is symmetric in its p-arguments we may limit the above sum to indices satisfying $n_1\le\cdots\le n_p$. Moreover, recalling (5.1.19), the index n_0 is of magnitude 2^ℓ, and by condition (2.2.29) there is $c>0$ such that

$$\max(n_p,n_{p+1})\ge c2^\ell\,. \tag{5.1.39}$$

Consider first the case $n_{p+1}\ge n_p$. The fact that V is in E^s_0 implies that

$$\|\Pi_{n_{p+1}}V(z',\cdot)\|_{L^2}\le c_{n_{p+1}}(z')n_{p+1}^{-s}\tilde{\mathcal{G}}^{s,0}_{0,1}(V) \tag{5.1.40}$$

for some sequence $(c_{n_{p+1}}(z'))_{n_{p+1}}$ satisfying

(5.1.41)
$$\sup_{z'\in[-1,0]}\sum_{n_{p+1}}|c_{n_{p+1}}(z')|^2<+\infty\,,\quad\sum_{n_{p+1}}n_{p+1}\int_{-1}^0|c_{n_{p+1}}(z')|^2\,dz'<+\infty\,.$$

Moreover $\|\Pi_{n_j}U\|_{L^2}\le c_{n_j}n_j^{-\sigma}\mathcal{G}^\sigma_{0,1}(U)$ for an ℓ^2 sequence $(c_{n_j})_{n_j}$. Consequently, applying (2.2.28) to R^1 and to $(z-z')^2R^1$ (with a gain of two units

on ρ in that case), and taking into account that $2^\ell \sim n_0$ and $n_0 \le Cn_{p+1}$, we get that (5.1.38) is bounded by $C\mathcal{G}^\sigma_{0,p}(U)\tilde{\mathcal{G}}^{s,0}_{0,1}(V)$ times

$$\sum_{n_0}\cdots\sum_{n_{p+1}}\int_{-1}^{0}(1+2^\ell|z-z'|)^{-2}\prod_{j=1}^{p}n_j^{-\sigma}c_{n_j}\,n_{p+1}^{-s}c_{n_{p+1}}(z')\,dz' \times\Big(\frac{n_p}{n_{p+1}}\Big)^{\rho-2}n_p^{\mu}\,\mathbf{1}_{|n_0-n_{p+1}|\le Cn_p,n_0\sim 2^\ell}\mathbf{1}_{n_1\le\cdots\le n_p\le n_{p+1}} \tag{5.1.42}$$

for some μ independent of s,σ,ρ, and noticing that the support condition $|n_0-n_{p+1}|\le Cn_p$ comes from (2.2.29). For $\sigma\ge\rho+\mu+1$, summing in $n_1,\dots,n_{p-1}$ and using (5.1.39), we bound (5.1.42) by

$$C2^{-\ell(s+\rho-2)}\sum_{n_0,n_p,n_{p+1}}n_p^{-3}\mathbf{1}_{|n_0-n_{p+1}|\le Cn_p,n_0\sim 2^\ell} \times\int_{-1}^{0}(1+2^\ell|z-z'|)^{-2}c_{n_{p+1}}(z')\,dz'\,. \tag{5.1.43}$$

The $L^2(dz)$ norm of (5.1.43) is bounded, using the Cauchy-Scwhartz inequality and performing a change of variable in the integral, by

$$C2^{-\ell(s+\rho-1)}\sum_{n_0,n_p,n_{p+1}}n_p^{-3}\mathbf{1}_{|n_0-n_{p+1}|\le Cn_p,n_0\sim 2^\ell}\|c_{n_{p+1}}(z')\|_{L^2(dz')}\,. \tag{5.1.44}$$

We may write $\|c_{n_{p+1}}(z')\|_{L^2(dz')}=2^{-\ell/2}\delta_{n_{p+1}}$ where, by (5.1.41) and the fact that $n_{p+1}\ge c2^\ell$ (see (5.1.39)) the sequence $(\delta_{n_{p+1}})_{n_{p+1}}$ is in $\ell^2(\mathbb{N})$. Then we write (5.1.44) as $C2^{-\ell(s+\rho-\frac12)}d_\ell$ where

$$d_\ell=\sum_{n_0,n_p,n_{p+1}}n_p^{-3}\mathbf{1}_{|n_0-n_{p+1}|\le Cn_p,n_0\sim 2^\ell}\delta_{n_{p+1}}\,.$$

Since $(d_\ell)_\ell$ is an ℓ^2 sequence, we get a contribution to the second expression (5.1.37) satisfying the second estimate (5.1.20) of a function of $E_0^{s+\rho-1}$. The first estimate is obtained taking the $L^\infty(dz)$ instead of $L^2(dz)$ norms of (5.1.43), which leads to (5.1.44) multiplied by $2^{\ell/2}$.

Consider now (5.1.38) in the case $n_{p+1}\le n_p$. In this case, we use instead of (5.1.40)

$$\|\Pi_{n_{p+1}}V(z',\cdot)\|_{L^2}\le c_{n_{p+1}}(z')n_{p+1}^{-\sigma}\tilde{\mathcal{G}}^{\sigma,0}_{0,1}(V),$$

the estimate $\|\Pi_{n_p}U\|_{L^2}\le c_{n_p}n_p^{-s}\mathcal{G}^s_{0,1}(U)$, that $n_p\ge c2^\ell$ (see (5.1.39)) and $|n_0-n_p|\le C\max(n_{p+1},n_{p-1})$. We get then for (5.1.38) an estimate by the product of $\mathcal{G}^\sigma_{0,p-1}(U)\tilde{\mathcal{G}}^{\sigma,0}_{0,1}(V)\mathcal{G}^s_{0,1}(U)$ and of an expression similar to (5.1.42). By the same computations as above, we obtain that the estimates (5.1.20) of an element of $E_0^{s+\rho-1}$ are satisfied. The second expression (5.1.37) is thus in $E_0^{s+\rho-1}$.

The first expression (5.1.37) is bounded by (5.1.42) where we remove the integral, take $z = z'$ and replace ρ by $\rho + 1$. Similar computations as above show that $\|R^0(U, \dots, U; z)V(z, \cdot)\|_{E_0^{s+\rho-1}}$, as well as $\|R(U)[\partial_z V]\|_{E_0^{s+\rho-1}}$, are bounded by the right hand side of (5.1.33) with $k = 0, j' = 1$ and $N = p$.

If one takes further ∂_z derivatives of (5.1.36), one gets similarly that $\int_{-1}^0 \partial_z^j \tilde{R}(U, \dots, U; z, z')V(z')\, dz'$ is in $E_0^{s+\rho-j}$ if V is in E_j^s. This proves the remark for interior operators. The corresponding statement for the classes $\mathcal{R}^{-\rho,\pm}_{K,0,p}[r]$ is proved in a similar and easier way.

• By the third remark after Proposition 2.2.4, if a is in $\Sigma\Gamma^{-\rho}_{K,0,p}[r, N]$, then $Op^{\mathrm{BW}}(a)$ defines an element of $\Sigma\mathcal{R}^{-\rho}_{K,0,p}[r, N]$. As a consequence, if a is a Poisson symbol in $\Sigma\mathcal{P}^{-\rho,\mathrm{int}}_{K,0,p}[r, N]$ as in (5.1.15), then $Op^{\mathrm{BW}}(a)$ is the kernel of the smoothing operator in $\Sigma\mathcal{R}^{-\rho,\mathrm{int}}_{K,0,p}[r, N]$,

$$V \to \int_{-1}^0 Op^{\mathrm{BW}}(a(U; z, z'))V(z', \cdot)dz' .$$

Identifying an operator with the kernel, we shall also simply write that $Op^{\mathrm{BW}}(a)$ is in $\Sigma\mathcal{R}^{-\rho,\mathrm{int}}_{K,0,p}[r, N]$. Similarly if a is in $\Sigma\mathcal{P}^{-\rho,\pm}_{K,0,p}[r, N]$ then $Op^{\mathrm{BW}}(a)$ is a smoothing operator in $\Sigma\mathcal{R}^{-\rho,\pm}_{K,0,p}[r, N]$.

Let us study now composition of operators associated to the classes of Poisson symbols of Definition 5.1.4.

Proposition 5.1.11 (Composition of para-Poisson operators) *Let m, m' be in $\mathbb{R}$, p, p', K, N in $\mathbb{N}$, with $N \geq p + p'$, $\rho \geq 0$.*

(i) Let $a(U; \cdot)$ be a symbol in $\Sigma\Gamma^m_{K,0,p}[r, N, \mathrm{aut}]$ and $e(U; z, x, \xi)$ be a Poisson symbol in $\Sigma\mathcal{P}^{m',\pm}_{K,0,p'}[r, N]$. Set

$$\tilde{e}(U; z, x, \xi) = (a\#e)_{\rho,N} \qquad (resp.\ (e\#a)_{\rho,N}) , \tag{5.1.45}$$

defined in (2.4.1) where $z \in [-1, 0]$ is considered as a parameter. Then $\tilde{e}$ is a Poisson symbol in $\Sigma\mathcal{P}^{m+m',\pm}_{K,0,p+p'}[r, N]$, and

$$Op^{\mathrm{BW}}(a(U; \cdot)) \circ Op^{\mathrm{BW}}(e(U; z, \cdot)),\ Op^{\mathrm{BW}}(e(U; z, \cdot)) \circ Op^{\mathrm{BW}}(a(U; \cdot))$$

may be written as

$$Op^{\mathrm{BW}}(\tilde{e}(U; z, \cdot)) + R(U; z)$$

where $R(U; \cdot)$ is a smoothing remainder in $\Sigma\mathcal{R}^{-\rho+m+m',\pm}_{K,0,p+p'}[r, N]$.

(ii) Let a be a symbol of $\Sigma\Gamma^m_{K,0,p}[r, N, \mathrm{aut}]$ and let $c(U; z, z', \cdot)$ be a Poisson symbol in $\Sigma\mathcal{P}^{m',\mathrm{int}}_{K,0,p'}[r, N]$. Set

$$\tilde{c}(U; z, z', \cdot) = (a\#c)_{\rho,N} \qquad (resp.\ (c\#a)_{\rho,N}) ,$$

where (z,z') is considered as a parameter. Then $\tilde c$ is a Poisson symbol in $\Sigma\mathcal{P}^{m+m',\mathrm{int}}_{K,0,p+p'}[r,N]$ and the operator $R(U)$ defined by

$$R(U)V=\int_{-1}^{0} Op^{\mathrm{BW}}(a(U;\cdot))\circ Op^{\mathrm{BW}}(c(U;z,z',\cdot))V(z')\,dz' - \int_{-1}^{0} Op^{\mathrm{BW}}(\tilde c(U;z,z',\cdot))V(z')\,dz',$$

(resp. by

$$R(U)V=\int_{-1}^{0} Op^{\mathrm{BW}}(c(U;z,z',\cdot))\circ Op^{\mathrm{BW}}(a(U;\cdot))V(z')\,dz' - \int_{-1}^{0} Op^{\mathrm{BW}}(\tilde c(U;z,z',\cdot))V(z')\,dz'\big)$$

belongs to $\Sigma\mathcal{R}^{-\rho+m+m'-1,\mathrm{int}}_{K,0,p+p'}[r,N]$. Identifying R with its kernel we simply write that $R=Op^{\mathrm{BW}}(a(U;\cdot))\circ Op^{\mathrm{BW}}(c(U;z,z',\cdot))-Op^{\mathrm{BW}}(\tilde c(U;z,z',\cdot))$ is in $\Sigma\mathcal{R}^{-\rho+m+m'-1,\mathrm{int}}_{K,0,p+p'}[r,N]$.

(iii) Let $c_j(U;z,z',\cdot)$ be in $\Sigma\mathcal{P}^{m_j,\mathrm{int}}_{K,0,p_j}[r,N]$ for $j=1,2$, with $m_j\in\mathbb{R}$, p_j in $\mathbb{N}$ with $p_1+p_2\le N$. Then

$$\tilde c(U;z,z',\cdot)\overset{def}{=}\int_{-1}^{0}\big(c_1(U;z,z'',\cdot)\#c_2(U;z'',z',\cdot)\big)_{\rho,N}\,dz'' \tag{5.1.46}$$

is a symbol in $\Sigma\mathcal{P}^{m_1+m_2-1,\mathrm{int}}_{K,0,p_1+p_2}[r,N]$ and the operator $R(U)$ defined by

(5.1.47)

$$R(U)V=\int_{-1}^{0}\int_{-1}^{0} Op^{\mathrm{BW}}(c_1(U;z,z'',\cdot))\circ Op^{\mathrm{BW}}(c_2(U;z'',z',\cdot))V(z')\,dz''dz' - \int_{-1}^{0} Op^{\mathrm{BW}}(\tilde c(U;z,z',\cdot))V(z')\,dz'$$

is in $\Sigma\mathcal{R}^{-\rho+m_1+m_2-1,\mathrm{int}}_{K,0,p_1+p_2}[r,N]$.

(iv) Let c be in $\Sigma\mathcal{P}^{m,\mathrm{int}}_{K,0,p}[r,N]$ and e be in $\Sigma\mathcal{P}^{m',\pm}_{K,0,p'}[r,N]$. Define

$$\tilde c(U;z,\cdot)=\int_{-1}^{0}\big(c(U;z,z',\cdot)\#e(U;z',\cdot)\big)_{\rho,N}\,dz'\,. \tag{5.1.48}$$

Then $\tilde c$ is in $\Sigma\mathcal{P}^{m+m'-1,\pm}_{K,0,p+p'}[r,N]$ and the operator $R(U)$ defined by

$$R(U)=\int_{-1}^{0} Op^{\mathrm{BW}}(c(U;z,z',\cdot))\circ Op^{\mathrm{BW}}(e(U;z',\cdot))\,dz' - Op^{\mathrm{BW}}(\tilde c(U;z,\cdot))$$

is in $\Sigma\mathcal{R}^{m+m'-1-\rho,\pm}_{K,0,p+p'}[r,N]$.

Proof: (i) We have to consider on the one hand the case when e and a are replaced by multilinear symbols in $\tilde{\mathcal{P}}^{m',\pm}_{q'}$ and $\tilde{\Gamma}^m_q$ respectively. The composition result of Proposition 2.3.2 together with Definitions 5.1.4 and 5.1.10 bring the conclusion.

On the other hand, we have to check that, if a is in $\Gamma^m_{K,0,q}[r,\mathrm{aut}]$ and e is in $\mathcal{P}^{m',\pm}_{K,0,q'}[r]$, then the operator

$$R(U;z) = Op^{\mathrm{BW}}(a(U;\cdot)) \circ Op^{\mathrm{BW}}(e(U;z,\cdot)) - Op^{\mathrm{BW}}(\tilde{e}(U;z,\cdot))\,, \tag{5.1.49}$$

where $\tilde{e} = (a\#e)_{\rho,N}$, belongs to $\mathcal{R}^{m+m'-\rho,\pm}_{K,0,q+q'}[r]$. In the case of sign $+$, for instance, for any integer i, the symbol $z^i e$ is in $\Gamma^{m'-i}_{K,0,q'}[r,\mathrm{aut}]$ and Proposition 2.3.2 implies that

$$z^i R(U;z) = Op^{\mathrm{BW}}(a) \circ Op^{\mathrm{BW}}(z^i e) - Op^{\mathrm{BW}}((a\#z^i e)_{\rho,N})$$

is a bounded family in z of smoothing operators of $\mathcal{R}^{-\rho+m+m'-i}_{K,0,q+q'}[r]$. We now prove that R is also a smoothing operator in $\mathcal{R}^{-\rho+m+m',+}_{K,0,q+q'}[r]$ according to Definition 5.1.10-(ii), i.e. R satisfies bounds of the form (5.1.32). We first notice that, since the remainder $R(U;z)$ in (5.1.49) is the difference of paradifferential operators, then, by (2.2.36), it does not enlarge much the support of the Fourier transform of functions, namely there is a function $\tilde{\varphi}$ in $C^\infty_0(\mathbb{R}^*)$, equal to one on a large enough compact ring, such that, setting $\tilde{\Delta}_\ell = \tilde{\varphi}(2^{-\ell}D)$, for $\ell > 0$, we have

$$\Delta_\ell R(U;z) = \Delta_\ell R(U;z)\tilde{\Delta}_\ell\,, \tag{5.1.50}$$

and a similar property for the zero frequency. Using that the operators $z^i R$, $i = 0, 1$, are in $\mathcal{R}^{-\rho+m+m'-i}_{K,0,q+q'}[r]$, it follows from (5.1.50) and estimates (2.2.30) that

$$\begin{aligned} \|\Delta_\ell R(U;z)V\|_{\dot{H}^{s+\rho-m-m'}} &\\ &\le C(1+2^\ell|z|)^{-1}\mathcal{G}^\sigma_{0,q+q'}(U,t)\|\tilde{\Delta}_\ell V\|_{L^2}2^{\ell s} \\ &\quad + C(1+2^\ell|z|)^{-1}\mathcal{G}^\sigma_{0,q+q'-1}(U,t)\mathcal{G}^s_{0,1}(U,t)\|\tilde{\Delta}_\ell V\|_{L^2}2^{\ell\sigma}\,. \end{aligned} \tag{5.1.51}$$

We deduce by

$$\|\Delta_\ell R(U;z)V\|_{L^2} \le 2^{-\ell(s+\rho-m-m')}\|\Delta_\ell R(U;z)V\|_{\dot{H}^{s+\rho-m-m'}}\,,$$

(5.1.51) and lemma 5.1.7, the estimate

$$\begin{aligned} \|R(U;z)V\|_{L^\infty([-1,0],\dot{H}^{s+\rho-m-m'})} + \|R(U;z)V\|_{L^2([-1,0],\dot{H}^{s+\frac{1}{2}+\rho-m-m'})} &\\ \le C\big[\mathcal{G}^\sigma_{0,q+q'}(U,t)\mathcal{G}^s_{0,1}(V,t) + \mathcal{G}^\sigma_{0,q+q'-1}(U,t)\mathcal{G}^\sigma_{0,1}(V,t)\mathcal{G}^s_{0,1}(U,t)\big] \end{aligned}$$

i.e. a bound of the form (5.1.32) when $j = 0$, $k = 0$ and s is replaced by $s+\rho-m-m'$ and N by $q+q'$. One obtains similarly the estimates involving ∂_z or ∂_t derivatives.

(ii) We decompose a (resp. c) as a sum of homogeneous symbols a_q of $\tilde\Gamma^m_q$ (resp. $c_{q'}(z;\cdot)$ of $\tilde{\mathcal{P}}^{m',\mathrm{int}}_{q'}$) and of a non-homogeneous symbol of $\Gamma^m_{K,0,N}[r,\mathrm{aut}]$ (resp. of $\mathcal{P}^{m',\mathrm{int}}_{K,0,N}[r]$). By the composition Proposition 2.3.2 we get in the expression of $R(U)$ on the one hand multilinear contributions

$$\sum_{\pm}\int_{-1}^{0} R_{q'',\pm}(U,\dots,U;z,z',\cdot)\mathbf{1}_{\pm(z-z')>0}V(z',\cdot)\,dz'\,, \tag{5.1.52}$$

where $(z-z')^\ell\partial_z^j\partial_{z'}^{j'}R_{\pm,q''}$ is in $\tilde{\mathcal{R}}^{-\rho+m+m'-\ell+j+j'}_{q''}$ uniformly in z,z'. Thus, according to (i) of Definition 5.1.10, we get that (5.1.52) is a smoothing operator of $\tilde{\mathcal{R}}^{-\rho+m+m'-1,\mathrm{int}}_{q''}$. In the same way, the contributions with at least one nonhomogeneous component have the form

$$\sum_{\pm}\int_{-1}^{0} R_{N,\pm}(U;z,z',\cdot)\mathbf{1}_{\pm(z-z')>0}V(z',\cdot)\,dz' \tag{5.1.53}$$

where $(z-z')^\ell\partial_z^j\partial_{z'}^{j'}R_{\pm,N}$ is in $\mathcal{R}^{-\rho+m+m'-\ell+j+j'}_{K,0,N}[r]$ uniformly in z,z'. In order to prove that (5.1.53) defines an operator in $\mathcal{R}^{-\rho+m+m'-1,\mathrm{int}}_{K,0,N}[r]$ we have to show that it satisfies bounds of the form (5.1.33) with ρ replaced by $\rho-m-m'+1$. Let us consider for instance the case $k=0$, $j'=0$. Using (5.1.50) and (2.2.30) we get

$$\begin{aligned}
&\int_{-1}^{0}\|\Delta_\ell R_{N,\pm}(U;z,z')V(z',\cdot)\|_{\dot H^{s+\rho+1-m-m'}}\mathbf{1}_{\pm(z-z')>0}\,dz'\\
&\quad\le C2^\ell\int_{-1}^{0}\|\Delta_\ell R_{N,\pm}(U;z,z')\tilde\Delta_\ell V(z',\cdot)\|_{\dot H^{s+\rho-m-m'}}\mathbf{1}_{\pm(z-z')>0}\,dz'\\
&\quad\le C\int_{-1}^{0}(1+2^\ell|z-z'|)^{-2}\mathcal{G}^\sigma_{0,N}(U,t)\|\tilde\Delta_\ell V(z',\cdot)\|_{L^2}\,dz'2^{\ell(s+1)}\\
&+C\int_{-1}^{0}(1+2^\ell|z-z'|)^{-2}\mathcal{G}^\sigma_{0,N-1}(U,t)\mathcal{G}^s_{0,1}(U,t)\|\tilde\Delta_\ell V(z',\cdot)\|_{L^2}\,dz'2^{\ell(\sigma+1)}\,.
\end{aligned}$$

The dz' integration makes gain one power of $2^{-\ell}$, that shows, using lemma 5.1.7, that (5.1.53) satisfies (5.1.33) with ρ replaced by $\rho+1-m-m'$ (in the case $k=0$, $j'=0$).

(iii) We decompose the Poisson symbols $c_j=\sum_{+,-}c_{j,\pm}\mathbf{1}_{\pm(z-z')>0}$ according to in Definition 5.1.4. Then (5.1.46) may be written as the sum of

the following contributions:

$$
\begin{aligned}
(5.1.54)\quad \mathbf{1}_{z-z'>0}\Big[&\int_{z'}^{z}\big(c_{1,+}(U;z,z'',\cdot)\#c_{2,+}(U;z'',z',\cdot)\big)_{\rho,N}\,dz''\\
&+\int_{-1}^{z'}\big(c_{1,+}(U;z,z'',\cdot)\#c_{2,-}(U;z'',z',\cdot)\big)_{\rho,N}\,dz''\\
&+\int_{z}^{0}\big(c_{1,-}(U;z,z'',\cdot)\#c_{2,+}(U;z'',z',\cdot)\big)_{\rho,N}\,dz''\Big]\\
+\mathbf{1}_{z-z'<0}\Big[&\int_{-1}^{z}\big(c_{1,+}(U;z,z'',\cdot)\#c_{2,-}(U;z'',z',\cdot)\big)_{\rho,N}\,dz''\\
&+\int_{z'}^{0}\big(c_{1,-}(U;z,z'',\cdot)\#c_{2,+}(U;z'',z',\cdot)\big)_{\rho,N}\,dz''\\
&+\int_{z}^{z'}\big(c_{1,-}(U;z,z'',\cdot)\#c_{2,-}(U;z'',z',\cdot)\big)_{\rho,N}\,dz''\Big].
\end{aligned}
$$

Recalling Definition 5.1.4, the symbols

$$(1+|z-z''|\langle\xi\rangle)^2 c_{1,\pm}(U;z,z'',x,\xi)$$

(resp. $(1+|z''-z'|\langle\xi\rangle)^2 c_{2,\pm}(U;z'',z',x,\xi)$) are uniformly bounded in the class $\Sigma\Gamma^{m_1}_{K,0,p_1}[r,N,\mathrm{aut}]$ (resp. $\Sigma\Gamma^{m_2}_{K,0,p_2}[r,N,\mathrm{aut}]$). Combining Definition 2.3.1 of $(a\#b)_{\rho,N}$ together with the fact that the dz''-integration brings a $O(\langle\xi\rangle^{-1})$, each of the above integrals, and thus $\tilde{c}$ defined by (5.1.46), is a Poisson symbol in $\Sigma\mathcal{P}^{m_1+m_2-1,\mathrm{int}}_{K,0,p_1+p_2}[r,N]$. The assertion that the operator defined in (5.1.47) is in $\Sigma\mathcal{R}^{-\rho+m_1+m_2-1,\mathrm{int}}_{K,0,p_1+p_2}[r,N]$ is a consequence of Proposition 2.3.2 and the definition of smoothing operators, as in the proof of (i) and (ii) above.

(iv) We write the symbol $\tilde{c}$ in (5.1.48) as

$$\int_{-1}^{z}\big(c_{+}(U;z,z',\cdot)\#e(U;z',\cdot)\big)_{\rho,N}\,dz'+\int_{z}^{0}\big(c_{-}(U;z,z',\cdot)\#e(U;z',\cdot)\big)_{\rho,N}\,dz'.$$

Using that $(1+|z-z'|\langle\xi\rangle)^2 c_{\pm}(U;z,z',x,\xi)$, resp. $e(U;z',x,\xi)$, are symbols uniformly bounded in the class $\Sigma\Gamma^{m}_{K,0,p}[r,N,\mathrm{aut}]$, resp. $\Sigma\Gamma^{m'}_{K,0,p'}[r,N,\mathrm{aut}]$, we conclude, arguing as in (iii), that $\tilde{c}$ is in $\Sigma\mathcal{P}^{m+m'-1,\pm}_{K,0,p+p'}[r,N]$ (gaining one on the order because of the dz' integration). The claim that $R(U)$ is in $\Sigma\mathcal{R}^{m+m'-1-\rho,\pm}_{K,0,p+p'}[r,N]$ follows by the composition result from Proposition 2.3.2 arguing as in the proof of (i) and (ii) above. □

We may as well compose paradifferential and para-Poisson operators with smoothing operators.

Proposition 5.1.12 *(i) Let a be a symbol in $\Sigma\Gamma^{m}_{K,0,p_1}[r,N,\mathrm{aut}]$ and R be a smoothing operator in $\Sigma\mathcal{R}^{-\rho,\pm}_{K,0,p_2}[r,N]$ (resp. $\Sigma\mathcal{R}^{-\rho,\mathrm{int}}_{K,0,p_2}[r,N]$). Then $Op^{\mathrm{BW}}(a(U;\cdot))\circ R(U)$ is in $\Sigma\mathcal{R}^{-\rho+m,\pm}_{K,0,p_1+p_2}[r,N]$ (resp. $\Sigma\mathcal{R}^{-\rho+m,\mathrm{int}}_{K,0,p_1+p_2}[r,N]$).*

(ii) Let c_1 be a Poisson symbol in $\Sigma\mathcal{P}^{m,\mathrm{int}}_{K,0,p_1}[r,N]$ and R be in $\Sigma\mathcal{R}^{-\rho,\mathrm{int}}_{K,0,p_2}[r,N]$. Then

$$V \to \int_{-1}^{0} Op^{\mathrm{BW}}(c_1(U;z,z'',\cdot))\circ R(U)V(z'',\cdot)\,dz'' \tag{5.1.55}$$

is in $\Sigma\mathcal{R}^{-\rho+m-1,\mathrm{int}}_{K,0,p_1+p_2}[r,N]$.

(iii) Let c_1 be in $\Sigma\mathcal{P}^{m,\mathrm{int}}_{K,0,p_1}[r,N]$ as above and R be in $\Sigma\mathcal{R}^{-\rho,\pm}_{K,0,p_2}[r,N]$. Then

$$\int_{-1}^{0} Op^{\mathrm{BW}}(c_1(U;z,z',\cdot))\circ R(U;z')\,dz'$$

is in $\Sigma\mathcal{R}^{-\rho+m-1,\pm}_{K,0,p_1+p_2}[r,N]$.

Proof: In the case when R is in $\Sigma\mathcal{R}^{-\rho,\pm}_{K,0,p_2}[r,N]$, (i) follows from the fact that the multilinear contributions to the composition are of the form (2.4.5), so they form a uniformly bounded family of elements of $\tilde{\mathcal{R}}^{-\rho+m}_{p}$ for some p between p_1+p_2 and N. If we make act $z^\ell\partial_z^j$ or $(1+z)^\ell\partial_z^j$, we get a similar conclusion with ρ replaced by $\rho+\ell-j$. This shows that the multilinear expressions belong to $\tilde{\mathcal{R}}^{-\rho+m,\pm}_{p}$, $p_1+p_2\le p\le N-1$. Concerning the contributions vanishing at order N at $U=0$, one has just to use the boundedness of $Op^{\mathrm{BW}}(a(U;\cdot))$ from E^s_j to E^{s-m}_j for any s and j, to deduce from the estimates (5.1.32) similar ones for the composition with $Op^{\mathrm{BW}}(a(U;\cdot))$. The case of R in $\Sigma\mathcal{R}^{-\rho,\mathrm{int}}_{K,0,p_2}[r,N]$ is similar.

(ii) Let c_{1,p'_1} be an homogeneous Poisson symbol in $\tilde{\mathcal{P}}^{m,\mathrm{int}}_{p'_1}$ for some $p_1\le p'_1\le N-1$ and $R_{p'_2}$ be the kernel of an homogeneous smoothing operator in $\tilde{\mathcal{R}}^{-\rho,\mathrm{int}}_{p'_2}$ for some $p_2\le p'_2\le N-1$. The contribution of these multilinear terms to the operator in (5.1.55) is

$$\begin{aligned} V\to\int_{-1}^{0}\Big\{\int_{-1}^{0} Op^{\mathrm{BW}}\big(c_{1,p'_1}(U_1,\dots,U_{p'_1};z,z'',\cdot)\big)\\ \circ R_{p'_2}(U_{p'_1+1},\dots,U_{p'_1+p'_2};z'',z')\,dz''\Big\}V(z',\cdot)\,dz'. \end{aligned} \tag{5.1.56}$$

According to (5.1.13) and to Definition 5.1.10, we may decompose

$$c_{1,p'_1}=c_{1,p'_1,+}\mathbf{1}_{z>z''}+c_{1,p'_1,-}\mathbf{1}_{z<z''},\ R_{p'_2}=R_{p'_2,+}\mathbf{1}_{z''>z'}+R_{p'_2,-}\mathbf{1}_{z''<z'}\,.$$

Proceeding as in (5.1.54), the kernel of the operator (5.1.56) may be written as the product of $\mathbf{1}_{\pm(z-z')>0}$ and of integrals of the form

$$\begin{aligned} \int_{I(z,z')} Op^{\mathrm{BW}}\big(c_{1,p'_1,\pm}(U_1,\dots,U_{p'_1};z,z'',\cdot)\big)\\ \circ R_{p'_2,\pm}(U_{p'_1+1},\dots,U_{p'_1+p'_2};z'',z')\,dz'' \end{aligned} \tag{5.1.57}$$

where $I(z,z')$ are intervals of integration like those in (5.1.54), and we have to show that if we make act $(z-z')^\ell \partial_z^j \partial_{z'}^{j'}$ on (5.1.57), we get a bounded family of operators of $\tilde{\mathcal{R}}^{-(\rho-m+1)+j+j'-\ell+1}_{p'_1+p'_2}$ uniformly in z,z'. Let us just treat the case $\ell=j=j'=0$. Recalling Definition 2.2.3 we have to prove bounds as in (2.2.28) for

$$\begin{aligned}(5.1.58)\quad &\sum_{n'_0}\int_{I(z,z')} \Pi_{n_0} Op^{\rm BW}\big(c_{1,p'_1,\pm}(\Pi_{n_1}U_1,\dots,\Pi_{n_{p'_1}}U_{p'_1};z,z'',\cdot)\big)\\ &\qquad\circ \Pi_{n'_0} R_{p'_2,\pm}(\Pi_{n_{p'_1+1}}U_{p'_1+1},\dots,\Pi_{n_{p'_1+p'_2}}U_{p'_1+p'_2};z'',z')\\ &\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\times \Pi_{n_{p'_1+p'_2+1}} V\,dz''\end{aligned}$$

(similarly to (2.4.5)) where, by (2.2.35) (applied to $c_{1,p'_1,\pm}$) and (2.2.29), the indices in the sum (2.4.5) satisfy, for some choice of the signs ϵ_j

$$(5.1.59)\qquad \sum_{j=0}^{p'_1}\epsilon_j n_j = n'_0\,,\qquad n'_0=\sum_{j=p'_1+1}^{p'_1+p'_2+1}\epsilon_j n_j\,.$$

As a consequence the operator in (5.1.58) satisfies, for any z,z', the corresponding condition (2.2.29). Since c_{1,p'_1} is in $\tilde{\mathcal{P}}^{m,\rm int}_{p'_1}$ (see Definition 5.1.4) we apply (2.2.34) (with $s=m$) for the first operator in (5.1.58). Then, we apply (2.2.28) to $R_{p'_2,\pm}$ and to $(z'-z'')^2 R_{p'_2,\pm}$ that, according to Definition 5.1.10, is a family of smoothing operators in $\tilde{\mathcal{R}}^{-\rho-2+1}_{p'_2}$, uniformly in z',z''. Consequently we estimate the L^2 norm of (5.1.58) by

$$\begin{aligned}&Cn_1^\sigma\dots n_{p'_1}^\sigma |n'_0|^m \frac{\max_2(n_{p'_1+1},\dots,n_{p'_1+p'_2+1})^{\rho+\mu}}{\max(n_{p'_1+1},\dots,n_{p'_1+p'_2+1})^{\rho-1}}\prod_{\ell=1}^{p'_1+p'_2}\|\Pi_{n_\ell}U_\ell\|_{L^2}\\ &\quad\times\|\Pi_{n_{p'_1+p'_2+1}}V\|_{L^2}\int_{-1}^0\big(1+\max(n_{p'_1+1},\dots,n_{p'_1+p'_2+1})|z'-z''|\big)^{-2}dz''\,.\end{aligned}$$

Changing variable in the integral we get a factor $\max(n_{p'_1+1},\dots,n_{p'_1+p'_2+1})^{-1}$, and $n_1,\dots,n_{p'_1}\le C\max_2(n_1,\dots,n_{p'_1+p'_2+1})$ and $\max(n_{p'_1+1},\dots,n_{p'_1+p'_2+1})\sim \max(n_1,\dots,n_{p'_1+p'_2+1})$ (that we deduce as (2.4.8) by $n_1,\dots,n_{p'_1}\ll n_0\sim n'_0$ and (5.1.59)), imply a bound in

$$C\frac{\max_2(n_1,\dots,n_{p'_1+p'_2+1})^{\rho-m+\mu'}}{\max(n_1,\dots,n_{p'_1+p'_2+1})^{\rho-m}}\prod_{\ell=1}^{p'_1+p'_2}\|\Pi_{n_\ell}U_\ell\|_{L^2}\|\Pi_{n_{p'_1+p'_2+1}}V\|_{L^2}$$

for some new μ'. In conclusion (5.1.58) is in $\tilde{\mathcal{R}}^{-(\rho-m+1)+1}_{p'_1+p'_2}$.

For the terms of (5.1.55) vanishing at order N, one uses lemma 5.1.8 together with estimates (5.1.33) defining $\mathcal{R}^{-\rho,\rm int}_{K,0,p'_2}[r]$.

(iii) is proved in a similar way as (ii). □

To conclude this section, we shall establish several composition results involving the smoothing operators introduced in Definition 5.1.3, that will be useful in Chapter 6.

In the sequel, we shall consider sometimes functions of $(t,x) \in I \times \mathbb{T}^1$ as functions of $(z,t,x) \in [-1,0] \times I \times \mathbb{T}^1$ independent of z. In particular, a function in $C^K_*(I, \dot{H}^s(\mathbb{T}^1,\mathbb{C}))$ may be considered as an element of $C^K_*(I,F^s_j)$, for any j, where the space F^s_j is defined in (5.1.1).

Lemma 5.1.13 *(i) Let m be in $\mathbb{R}$, ρ positive, K,p,q,N,j in $\mathbb{N}$, $p+q \leq N$, $r>0$. Let a be a Poisson symbol in $\Sigma\mathcal{P}^{m,\mathrm{int}}_{K,0,p}[r,N]$ and R a smoothing operator in the class $\Sigma\mathcal{R}^{-\rho}_{K,0,j,q}[\mathcal{B},r,N]$ of Definition 5.1.3. Consider*

$$\tilde{R}(U)V = \int_{-1}^{0} Op^{\mathrm{BW}}(a(U;z,z',\cdot))R(U;z')V\,dz' \tag{5.1.60}$$

defined for U in $B^K_\sigma(I,r)$ and V in $C^K_*(I,\dot{H}^\sigma(\mathbb{T}^1,\mathbb{C}))$, for some large enough σ (and are considered as functions of z', constant in the variable z'). Then, for any $j' \leq j+1$, $\partial_z^{j'}\tilde{R}(U)$ is a bounded family in z of smoothing operators of $\Sigma\mathcal{R}^{-\rho+m+j'}_{K,0,p+q}[r,N,\mathrm{aut}]$ (here we adopt the abuse of notation introduced in the last remark after Definition 5.1.3).

(ii) Let R_{int} be a smoothing operator in $\Sigma\mathcal{R}^{-\rho',\mathrm{int}}_{K,0,p}[r,N]$ for some $\rho' \geq 0$ (see (5.1.34)), and take R in $\Sigma\mathcal{R}^{-\rho}_{K,0,j,q}[\mathcal{B},r,N]$ with $\rho \geq \frac{3}{2}$. Define

$$\tilde{R}(U)V = R_{\mathrm{int}}(U)[R(U)V] = \int_{-1}^{0} R_{\mathrm{int}}(U;z,z')[R(U)V](z',\cdot)dz' \tag{5.1.61}$$

for $U \in B^K_\sigma(I,r)$ and V in $C^K_*(I,\dot{H}^\sigma(\mathbb{T}^1,\mathbb{C}))$, the last expression in (5.1.61) above being the one in terms of the Schwartz kernel of the operator.

Then, for any $j' \leq j$, $\partial_z^{j'}\tilde{R}(U)$ is a bounded family in z of operators in $\Sigma\mathcal{R}^{-\rho'+j'+\epsilon}_{K,0,p+q}[r,N,\mathrm{aut}]$ for any $\epsilon>0$.

Proof: (i) We consider on the one hand multilinear contributions, and on the other hand those terms vanishing at least at order N at $U=0$. Let us first consider the homogeneous terms. Let a be a symbol in $\tilde{\mathcal{P}}^{m,\mathrm{int}}_p$ for some p and let R be in $\tilde{\mathcal{R}}^{-\rho}_q[\mathcal{B}]$ for some q. Let us compute for instance

$$\partial_z \int_{-1}^{0} Op^{\mathrm{BW}}\big(a(\Pi_{n'}\mathcal{U}';z,z',\cdot)\big) \circ R(\Pi_{n''}\mathcal{U}'';z')\Pi_{n_{p+q+1}}U_{p+q+1}\,dz' \tag{5.1.62}$$

where $\mathcal{U}' = (U_1,\dots,U_p)$, $n' = (n_1,\dots,n_p)$, $\mathcal{U}'' = (U_{p+1},\dots,U_{p+q})$, $n'' = (n_{p+1},\dots,n_{p+q})$ and U_j are functions only of x. With the notation $a = a_+\mathbf{1}_{z-z'>0} + a_-\mathbf{1}_{z-z'<0}$ in (5.1.13), we may write (5.1.62) as the sum of

$$Op^{\mathrm{BW}}(a_+(\Pi_{n'}\mathcal{U}';z,z,\cdot)R(\Pi_{n''}\mathcal{U}'';z)\Pi_{n_{p+q+1}}U_{p+q+1} \tag{5.1.63}$$

$$- Op^{\rm BW}(a_-(\Pi_{n'}\mathcal{U}';z,z,\cdot)R(\Pi_{n''}\mathcal{U}'';z)\Pi_{n_{p+q+1}}U_{p+q+1} \tag{5.1.64}$$

$$\int_{-1}^{0} Op^{\rm BW}(a_1(\Pi_{n'}\mathcal{U}';z,z',\cdot)R(\Pi_{n''}\mathcal{U}'';z')\Pi_{n_{p+q+1}}U_{p+q+1}\,dz' \tag{5.1.65}$$

where $a_1 = (\partial_z a_+)\mathbf{1}_{z-z'>0} + (\partial_z a_-)\mathbf{1}_{z-z'<0}$. We make act at the left of (5.1.63), (5.1.64), (5.1.65) another projector Π_{n_0} and bound the $L^2(\mathbb{T}^1)$ norm at fixed z. Let us make in detail the estimate for (5.1.65). We write

$$\sum_{n_0'} \int_{-1}^{0} \Pi_{n_0} Op^{\rm BW}(a_1(\Pi_{n'}\mathcal{U}';z,z',\cdot)\Pi_{n_0'}R(\Pi_{n''}\mathcal{U}'';z')\Pi_{n_{p+q+1}}U_{p+q+1}\,dz' \tag{5.1.66}$$

where (because of conditions (2.2.35) and (2.2.29)) the index n_0' is such that

$$\sum_{j=0}^{p} \epsilon_j n_j + \epsilon_0' n_0' = 0 \ , \quad \epsilon_0'' n_0' + \sum_{j=p+1}^{p+q+1} \epsilon_j n_j = 0\,, \tag{5.1.67}$$

for some choice of the signs $\epsilon_j, \epsilon_0', \epsilon_0'' \in \{-1,1\}$. By (5.1.67) we deduce that, if (5.1.66) is not zero, then $\sum_{j=0}^{p+q+1} \epsilon_j n_j = 0$ for some choice of signs ϵ_j so that (5.1.65) satisfies the corresponding condition (2.2.29).

Since a is a Poisson symbol in $\tilde{\mathcal{P}}_p^{m,\rm int}$, the remarks following Definition 5.1.4 imply that $a_1 = \partial_z a$ is in $\tilde{\mathcal{P}}_p^{m+1,\rm int}$ and $|z-z'|^\theta a_1$ is in $\tilde{\mathcal{P}}_p^{m+1-\theta,\rm int}$, for any $\theta \in]0,1[$. Thus $|z-z'|^\theta a_1$ is in $\tilde{\Gamma}_p^{m+1-\theta}$, uniformly in z, z', and Proposition 2.2.4 implies that the L^2 norm of (5.1.66) is bounded by

$$C|n'|^\mu n_0^{m+1-\theta} \prod_1^p \|U_j\|_{L^2} \int_{-1}^{0} \|\Pi_{n_0'}R(\Pi_{n''}\mathcal{U}'';z')\Pi_{n_{p+q+1}}U_{p+q+1}\|_{L^2} \frac{dz'}{|z-z'|^\theta}\,.$$

Applying (5.1.10) to the term R that belongs to $\tilde{\mathcal{R}}_q^{-\rho}[\mathcal{B}]$, and integrating in dz', we obtain a bound in

$$C|n'|^\mu n_0^{m+1-\theta} \frac{\max_2(n_{p+1},\dots,n_{p+q+1})^{\rho+\mu'}}{\max(n_{p+1},\dots,n_{p+q+1})^\rho} \prod_1^{p+q+1} \|U_j\|_{L^2}\,.$$

By (2.2.36), the sum (5.1.66) is restricted to indices $n_0' \sim n_0$ and (5.1.67) implies that $n_0' \le \max(n_{p+1},\dots,n_{p+q+1})$. Moreover, arguing as for (2.4.8),

$$\max(n_{p+1},\dots,n_{p+q+1}) \sim \max(n_1,\dots,n_{p+q+1})$$
$$|n'| \le C\max_2(n_1,\dots,n_{p+q+1})$$

and one gets an estimate of the form (2.2.28) with ρ replaced by $\rho-m-1+\theta$, $p+1$ replaced by $p+q+1$, and for a new value of μ. This proves that (5.1.65) is in $\tilde{\mathcal{R}}_{p+q}^{-\rho+m+\epsilon}$ with $\epsilon = 1-\theta$ in $]0,1[$, uniformly in z, and the same holds

for the terms (5.1.63), (5.1.64). Higher order ∂_z derivatives may be treated in the same way obtaining that $\partial_z^{j'} R$ is in $\tilde{\mathcal{R}}^{-\rho+m+j'-1+\epsilon}_{p+q}$.

Consider now the case when a is a non-homogeneous symbol in $\mathcal{P}^{m,\mathrm{int}}_{K,0,p}[r]$ and R is a smoothing operator in $\mathcal{R}^{-\rho}_{K,0,j,N}[\mathcal{B}, r]$. Writing $a = a_+ \mathbf{1}_{z-z'>0} + a_- \mathbf{1}_{z-z'<0}$ we have that

$$\partial_z \int_{-1}^{0} Op^{\mathrm{BW}}(a(U; z, z', \cdot)) R(U; z') V \, dz' \tag{5.1.68}$$

is equal to

$$\begin{aligned} Op^{\mathrm{BW}}(a_+(U; z, z, \cdot)) R(U; z) V - Op^{\mathrm{BW}}(a_-(U; z, z, \cdot)) R(U; z) V \\ + \int_{-1}^{0} Op^{\mathrm{BW}}(a_1(U; z, z', \cdot)) R(U; z') V \, dz' \end{aligned} \tag{5.1.69}$$

where $a_1 = (\partial_z a_+)\mathbf{1}_{z-z'>0} + (\partial_z a_-)\mathbf{1}_{z-z'<0}$. Combining estimates (5.1.11) for R, with the boundedness of $Op^{\mathrm{BW}}(a_\pm)$ (resp. $Op^{\mathrm{BW}}(|z-z'|^\theta a_1)$) from $\dot{H}^{s+\rho}$ to the space $\dot{H}^{s+\rho-m}$ (resp. from $\dot{H}^{s+\rho}$ to $\dot{H}^{s+\rho-m-1+\theta}$), that follows by Proposition 2.2.4 and the fact that these symbols are in $\Sigma\Gamma^m_{K,0,p}[r, N, \mathrm{aut}]$ (resp. $\Sigma\Gamma^{m+1-\theta}_{K,0,p}[r, N, \mathrm{aut}]$), we get for (5.1.69) bounds of the form (2.2.30) with ρ replaced by $\rho - (m+1) + \theta$, uniformly in z. We have proved that $\partial_z \tilde{R}(U)$ is in $\mathcal{R}^{-\rho+m+\epsilon}_{K,0,N}[r, \mathrm{aut}]$ with $\epsilon = 1 - \theta$ in $]0, 1[$. Higher ∂_z derivatives are treated in the same way.

Finally, we have to consider the contributions $\mathrm{Op}^{BW}(a_N) \circ R_q$ where R_q is an homogeneous smoothing term in $\tilde{\mathcal{R}}^{-\rho}_q[\mathcal{B}]$. Because of the third remark after Definition 5.1.3, the associated operator $R_q(U, \dots, U)$ satisfies (5.1.11) with $\rho - \alpha$ instead of ρ for $\alpha > 1/2$ and we obtain that the operator (5.1.68) is a remainder in $\mathcal{R}^{-\rho+m+\epsilon+\alpha}_{K,0,N}[r, \mathrm{aut}] \subset \mathcal{R}^{-\rho+m+1}_{K,0,N}[r, \mathrm{aut}]$, taking $\alpha + \epsilon \leq 1$. This concludes the proof of (i).

(ii) Let us study first the multilinear contributions. According to Definition 5.1.10-(i) we write the homogeneous smoothing operator R_{int} as

$$V \to \int_{-1}^{0} \big(R_{\mathrm{int},+} \mathbf{1}_{z-z'>0} + R_{\mathrm{int},-} \mathbf{1}_{z-z'<0} \big) V(z', \cdot) dz'$$

where $R_{\mathrm{int},\pm}(U_1, \dots, U_p; z, z')$ are homogeneous smoothing operators of $\tilde{\mathcal{R}}^{-\rho'+1}_p$, uniformly in z, z'. The analogous of (5.1.62) with $Op^{\mathrm{BW}}(a)$ replaced by $R_{\mathrm{int}}(\Pi_{n_1} U_1, \dots, \Pi_{n_p} U_p; z, z')$ is the sum (similarly to (5.1.63)-(5.1.65)) of

$$\begin{aligned} & R_{\mathrm{int},+}(\Pi_{n'} \mathcal{U}'; z, z) R(\Pi_{n''} \mathcal{U}''; z) \Pi_{n_{p+q+1}} U_{p+q+1} \\ & - R_{\mathrm{int},-}(\Pi_{n'} \mathcal{U}'; z, z) R(\Pi_{n''} \mathcal{U}''; z) \Pi_{n_{p+q+1}} U_{p+q+1} \end{aligned}$$

$$\int_{-1}^{0} R_{\mathrm{int},1}(\Pi_{n'} \mathcal{U}'; z, z') R(\Pi_{n''} \mathcal{U}''; z') \Pi_{n_{p+q+1}} U_{p+q+1} \, dz' \tag{5.1.70}$$

where $R_{\text{int},1} = (\partial_z R_{\text{int},+})\mathbf{1}_{z-z'>0} + (\partial_z R_{\text{int},-})\mathbf{1}_{z-z'<0}$. Recalling the third remark following Definition 5.1.10 we have that $|z-z'|^\theta R_{\text{int},1}$ is in $\tilde{\mathcal{R}}_p^{-\rho'+1-\theta,\text{int}}$, that is $(|z-z'|^\theta R_{\text{int},1}(\Pi_{n'}\mathcal{U}';z,z'))_{z,z'\in[-1,0]}$ is a bounded family of smoothing operators of $\tilde{\mathcal{R}}_p^{-\rho'+2-\theta}$, uniformly in z, z'. By (5.1.10) and (2.2.28) we estimate the L^2 norm of

$$\sum_{n'_p} \Pi_{n_0} \int_{-1}^{0} R_{\text{int},1}(\Pi_{n'}\mathcal{U}';z,z')\Pi_{n'_p} R(\Pi_{n''}\mathcal{U}'';z')\Pi_{n_{p+q+1}} U_{p+q+1}\, dz'$$

by the product of $\prod_{j=1}^{p+q+1}\|U_j\|_{L^2}$ times

$$C\frac{\max_2(n_1,\dots,n_p,n'_p)^{\rho'-2+\theta+\mu}}{\max(n_1,\dots,n_p,n'_p)^{\rho'-2+\theta}}\frac{\max_2(n_{p+1},\dots,n_{p+q+1})^{\rho+\mu}}{\max(n_{p+1},\dots,n_{p+q+1})^{\rho}}\,. \tag{5.1.71}$$

By (2.2.29) we know that, for some choice of the signs $\epsilon'_j, \epsilon''_j$,

$$\epsilon'_0 n_0 + \sum_1^p \epsilon'_j n_j + \epsilon'_p n'_p = 0\,, \quad \epsilon''_p n'_p + \sum_{p+1}^{p+q+1} \epsilon''_j n_j = 0\,. \tag{5.1.72}$$

Assume first that one among $n_1,\dots,n_p,n_{p+1},\dots,n_{p+q+1}$ is much larger than all the other ones, say n_j. If $1\le j\le p$ then, using (5.1.72) we derive that $n'_p \le n_p$ and (5.1.71) is bounded by

$$C\frac{\max_2(n_1,\dots,n_{p+q+1})^{\rho'-2+\theta+\mu}}{\max(n_1,\dots,n_{p+q+1})^{\rho'-2+\theta}} \tag{5.1.73}$$

for some new value of μ. If $p+1\le j\le p+q+1$ then $n'_p\sim\max(n_1,\dots,n_{p+q+1})$ and (5.1.73) follows as well. When the largest two among $n_1,\dots,n_{p+q+1}$ are of the same magnitude, i.e $\max_2(n_1,\dots,n_{p+q+1}) \sim \max(n_1,\dots,n_{p+q+1})$, the bound (5.1.73) follows immediately. This proves that (5.1.70) is a bounded family in z of smoothing operators of $\tilde{\mathcal{R}}_p^{-\rho'+1+\epsilon}$ where $\epsilon = 1-\theta$.

Consider next the contributions that are non homogeneous, i.e. for $0\le j'\le j$ the operator

$$V \to \partial_z^{j'}(R_{\text{int},p}(U)\cdot R_q(U,\cdot)V)(z,\cdot) \tag{5.1.74}$$

where $R_{\text{int},p}$ is in $\mathcal{R}^{-\rho',\text{int}}_{K,0,p}[r]$ and $R_q \in \mathcal{R}^{-\rho}_{K,0,j,q}[\mathcal{B},r]$. We want to show that (5.1.74) defines a family indexed by z of operators of $\mathcal{R}^{-\rho'+j'}_{K,0,p+q}[r]$, with bounds uniform in z. According to (2.2.30), we have to show an estimate

$$\begin{aligned}&\|\partial_t^k\partial_z^{j'}(R_{\text{int},p}(U)\cdot R_q(U,\cdot)V)(z,\cdot)\|_{\dot H^{s+\rho'-\frac{3}{2}k-j'}} \\ &\le C\sum_{k'+k''=k}\Big[\mathcal{G}^\sigma_{k',p+q}(U,t)\mathcal{G}^s_{k'',1}(V,t) + \mathcal{G}^\sigma_{k',p+q-1}(U,t)\mathcal{G}^s_{k'',1}(U,t)\mathcal{G}^\sigma_{k',1}(V,t)\Big].\end{aligned} \tag{5.1.75}$$

By (5.1.33) applied to $R_{\mathrm{int},p}$ with ρ replaced by ρ', we have a bound of the left hand side of (5.1.75) by

$$(5.1.76)\quad C\Big[\sum_{k'+k''=k}\mathcal{G}^{\sigma}_{k',p}(U,t)\tilde{\mathcal{G}}^{s,j'}_{k'',1}(R_q(U,\cdot)V,t) + \sum_{k'+k''=k}\mathcal{G}^{\sigma}_{k',p-1}(U,t)\tilde{\mathcal{G}}^{\sigma,j'}_{k'',1}(R_q(U,\cdot)V,t)\mathcal{G}^{s}_{k',1}(U,t)\Big].$$

Recalling the notation (5.1.29), the general term in the first sum in (5.1.76) is bounded by

$$\mathcal{G}^{\sigma}_{k',p}(U,t)\Big(\sum_{k''_1=0}^{k''}\|\partial_t^{k''_1}R_q(U,\cdot)V\|_{E^{s-\frac{3}{2}k''_1}_{j'}}\Big) \le \mathcal{G}^{\sigma}_{k',p}(U,t)\Big(\sum_{k''_1=0}^{k''}\|\partial_t^{k''_1}R_q(U,\cdot)V\|_{F^{s+\frac{1}{2}-\frac{3}{2}k''_1}_{j'}}\Big)$$

(see the remark after Definition 5.1.6). The above expression is bounded according to (5.1.11) (that we apply with ρ replaced by $\rho-\alpha$ with $\alpha=3/4$, by the third remark following Definition 5.1.3), and using that $\rho-\alpha>\frac{1}{2}$ since $\rho\ge\frac{3}{2}$, by the right hand side of (5.1.75). The general term in the second sum in (5.1.76) is bounded in the same way (changing the definition of σ in the final result). □

Finally, we shall state a result of the same type as lemma 5.1.13, when one replaces in (5.1.60), (5.1.61) the smoothing map $R(U)$ by an operator $M(U;z)$ that may lose derivatives, but which is supported for z in a compact subset of $[-1,0[$. We shall estimate then the quantities (5.1.60), (5.1.61) only for z in a neighborhood of zero.

We now consider operators $M(U)=M(U;z)$ of the form

$$(5.1.77)\qquad M(U)V=\sum_{q'=q}^{N-1}M_{q'}(U,\dots,U)V+M_N(U)V\,,$$

where $(U_1,\dots,U_{q'+1})\to M_{q'}(U_1,\dots,U_{q'})U_{q'+1}$ is a family of $(q'+1)$-linear maps, defined on $\dot{H}^\infty(\mathbb{T}^1,\mathbb{C}^2)^{q'}\times\dot{H}^\infty(\mathbb{T}^1,\mathbb{C})$, satisfying for each fixed $z\in[-1,0]$ conditions (2.2.51), and, for some $m\ge0$, and for any $n_0,\dots,n_{q'+1}$, the estimate

$$(5.1.78)\quad \sup_{-1\le z\le0}\|\Pi_{n_0}M_{q'}(\Pi_n\mathcal{U})\Pi_{n_{q'+1}}U_{q'+1}\|_{\dot{H}^0} \le C(n_0+\cdots+n_{q'+1})^m\mathcal{G}^0_{0,q'+1}(\mathcal{U},U_{q'+1})$$

with the notation $\mathcal{U} = (U_1, \dots, U_{q'+1})$. Moreover, we suppose that for some $\sigma_0' \geq \sigma_0$ large enough, we have for $0 \leq k \leq K$ the estimate

$$\sup_{-1\leq z\leq 0} \|\partial_t^k M_N(U)V\|_{\dot{H}^{\sigma_0}} \leq C \sum_{k'+k''=k} \mathcal{G}^{\sigma_0'}_{k',N}(U,t)\mathcal{G}^{\sigma_0'}_{k'',1}(V,t)\,. \tag{5.1.79}$$

Finally, we assume that for some $\delta \in]0,1[$,

$$M(U)V \equiv 0 \quad \text{if} \quad -2\delta \leq z \leq 0\,. \tag{5.1.80}$$

We may now state the following lemma which is a variant of lemma 5.1.13.

Lemma 5.1.14 *Let m be in $\mathbb{R}$, ρ positive, K, p, q, N in $\mathbb{N}$, $p+q \leq N$, $r > 0$. Let a be a Poisson symbol in $\Sigma\mathcal{P}^{m,\mathrm{int}}_{K,0,p}[r,N]$ and R_{int} be a smoothing operator in $\Sigma\mathcal{R}^{-\rho,\mathrm{int}}_{K,0,p}[r,N]$. Let $(M(U;z))_{z\in[-1,0]}$ be a family of operators as in (5.1.77) satisfying (5.1.78)-(5.1.80). Define*

$$\tilde{R}(U)V = \int_{-1}^{0} Op^{\mathrm{BW}}(a(U;z,z',\cdot))M(U;z')V\,dz' \tag{5.1.81}$$

or

$$\tilde{R}(U)V = \int_{-1}^{0} R_{\mathrm{int}}(U;z,z')M(U;z')V\,dz'\,. \tag{5.1.82}$$

Then $\partial_z^j \tilde{R}(U)$ restricted to $z \in [-\delta, 0]$ is, for any $j \in \mathbb{N}$, a bounded family in z of smoothing operators of $\Sigma\mathcal{R}^{-\rho}_{K,0,p+q}[r,N,\mathrm{aut}]$ for any ρ.

Proof: Consider first (5.1.81). If z' stays in the support of $M(U;z')$ and $-\delta \leq z \leq 0$, we may write

$$a(U;z,z',\cdot) = (z-z')^{\ell} a(U;z,z',\cdot)\omega(z,z')$$

where ω is smooth for $z' \leq -2\delta, -\delta \leq z \leq 0$. According to Definition 5.1.4, for those values of z, z', we may write $a = \sum_{p'=p}^{N-1} a_{p'} + a_N$ where $a_{p'}$ is a family in (z,z') of homogeneous symbols in $\tilde{\Gamma}^{m-\ell}_{p'}$ satisfying estimates (2.1.5) with m replaced by $m-\ell$, and some μ independent of ℓ, and where a_N satisfies (2.1.10), with m, K' replaced by $m-\ell, 0$ and some σ, σ_0 independent of ℓ. By (2.2.34), and denoting $\mathcal{U}' = (U_1, \dots, U_{p'})$, $n' = (n_1, \dots, n_{p'})$, we have an estimate of the form

$$\begin{aligned}\|\Pi_{n_0} Op^{\mathrm{BW}}(a_{p'}(\Pi_{n'}\mathcal{U}';z,z',\cdot))\Pi_{n'_{p'+1}}V\|_{\dot{H}^0} \\ \leq C n_0^{m-\ell}|n'|^{\mu}\mathcal{G}^0_{0,p'}(\Pi_{n'}\mathcal{U}')\|\Pi_{n'_{p'+1}}V\|_{\dot{H}^0}\end{aligned} \tag{5.1.83}$$

for some μ independent of ℓ. Moreover (2.1.6) and (2.2.36) imply that

$$\sum_{0}^{p'} \epsilon_j n_j + \epsilon'_{p'+1} n'_{p'+1} = 0\,, \quad |n'| \ll n_0 \sim n'_{p'+1} \tag{5.1.84}$$

for some choice of signs ϵ_j and $\epsilon'_{p'+1}$. In (5.1.83) we replace V by the quantity $M_{q'}(\Pi_{n''}\mathcal{U}'')\Pi_{n_{p'+q'+1}}U_{p'+q'+1}$ where $\mathcal{U}'' = (U_{p'+1}, \dots, U_{p'+q'})$, $n'' = (n_{p'+1}, \dots, n_{p'+q'})$, that, by (2.2.51), is different from zero only if

$$n'_{p'+1} = \sum_{p'+1}^{p'+q'+1} \epsilon_j n_j \tag{5.1.85}$$

Thus, using (5.1.78) for $M_{q'}$, we get for the resulting expression a bound in

$$\begin{aligned} &Cn_0^{m-\ell}(n_1 + \cdots + n_{p'})^{\mu}(n_{p'+1} + \cdots + n_{p'+q'+1})^m \\ &\qquad \times \mathcal{G}^0_{0,p'+q'+1}(\Pi_{n'}\mathcal{U}', \Pi_{n''}\mathcal{U}'', \Pi_{n_{p'+q'+1}}U_{p'+q'+1}, t) \end{aligned} \tag{5.1.86}$$

where μ and m are independent of ℓ. If among the indices $n_1, \dots, n_{p'+q'+1}$ the largest two ones are of the same magnitude, an estimate of the form (2.2.28) with some μ, with p replaced by $p'+q'$, and an arbitrary ρ holds trivially. On the other hand, if the largest one among the indices $n_1, \dots, n_{p'+q'+1}$ is much larger than the second largest, then, by (5.1.84) and (5.1.85) we deduce that $\max(n_1, \dots, n_{p'+q'+1}) = \max(n_{p'+1}, \dots, n_{p'+q'+1})$ and $n_0 \leq C\max(n_1, \dots, n_{p'+q'+1})$. Then by (5.1.86) and taking $\ell \geq \rho + m$ we deduce an estimate of the form (2.2.28). This settles the case of multilinear contributions to (5.1.81).

If, on the other hand, we consider the contribution coming from the non-homogeneous symbol a_N term in the expression of a, and from a $M(U)$ satisfying (5.1.79), we use (2.2.37) with m, K' replaced by $m-\ell, 0$ and some σ independent of ℓ. We obtain

$$\begin{aligned} &\|\partial_t^k Op^{\mathrm{BW}}(a(U;\cdot))M(U)V\|_{\dot{H}^{\sigma_0 - m + \ell - \frac{3}{2}k}} \\ &\qquad \leq C \sum_{k'+k''=k} \mathcal{G}^{\sigma_0}_{k',N}(U)\mathcal{G}^{\sigma_0}_{k'',1}(M(U)V, t) \end{aligned}$$

which, combined with (5.1.79), implies an estimate of the form (2.2.30) if ℓ is taken large enough relatively to s and ρ. We obtain estimates for the ∂_z^j derivatives in the same way, noticing that for z in $[-\delta, 0]$, z' in the support of $M(U; z')$, $\partial_z^j a(U; z, z', \cdot)$ is a bounded function of z for any j. One treats (5.1.82) in the same way. □

5.2 Parametrix of Dirichlet-Neumann problem

We consider a real valued even function η in $C^K_*(I, \dot{H}^\sigma(\mathbb{T}^1, \mathbb{R}))$ for some K in $\mathbb{N}$ and some large enough $\sigma > \frac{3}{2}K$. We denote by η' the space derivative of η and by $\eta' \otimes \eta'$ the quadratic map associated to the bilinear map $(\eta_1, \eta_2) \to \eta'_1(x)\eta'_2(x)$. This defines an homogeneous symbol of $\tilde{\Gamma}^0_2$. In the same way the map $\eta' \to (\xi \to \eta'(x)\xi)$ defines an homogeneous symbol of $\tilde{\Gamma}^1_1$.

We want to construct a parametrix E of the paradifferential elliptic operator, acting on functions of $(z,x)\in[-1,0]\times\mathbb{T}^1$, given by

$$(1+Op^{\mathrm{BW}}(\eta'\otimes\eta'))\partial_z^2-2iOp^{\mathrm{BW}}(\eta'\xi)\partial_z-Op^{\mathrm{BW}}(\xi^2) \tag{5.2.1}$$

together with boundary conditions

$$E|_{z=0}=\mathrm{Id},\quad \partial_z E|_{z=-1}=\mathrm{Id} \tag{5.2.2}$$

up to smoothing operators.

Recall that we introduced after Definition 5.1.4 the Poisson symbols

(5.2.3)
$$\begin{aligned}\mathcal{C}(z,\xi)&=\frac{\cosh((z+1)\xi)}{\cosh\xi},\quad \mathcal{S}(z,\xi)=\frac{\sinh(z\xi)}{\xi\cosh\xi}\\ K_0(z,z',\xi)&=(\cosh\xi)\big(\mathcal{C}(z,\xi)\mathcal{S}(z',\xi)\mathbf{1}_{z-z'<0}+\mathcal{S}(z,\xi)\mathcal{C}(z',\xi)\mathbf{1}_{z-z'>0}\big)\end{aligned}$$

where $\mathcal{C}$ is in $\tilde{\mathcal{P}}_0^{0,+}$, $\mathcal{S}$ is in $\tilde{\mathcal{P}}_0^{-1,-}$ and K_0 is in $\tilde{\mathcal{P}}_0^{-1,\mathrm{int}}$. Notice that the function $K_0(z,z',\xi)=K_0(z',z,\xi)$ is symmetric in (z,z') and

$$\partial_z K_0(z,z',\xi)=(\cosh\xi)\big(\partial_z\mathcal{C}(z,\xi)\mathcal{S}(z',\xi)\mathbf{1}_{z-z'<0}+\partial_z\mathcal{S}(z,\xi)\mathcal{C}(z',\xi)\mathbf{1}_{z-z'>0}\big)$$

is a Poisson symbol in $\tilde{\mathcal{P}}_0^{0,-}$.

The definition of the parametrix E is made precise in the following Proposition.

Proposition 5.2.1 (Parametrix of Dirichlet-Neumann problem) *Let $N\in\mathbb{N}^*$, $\rho\in\mathbb{N}$ be given. There are $r>0$ and Poisson symbols $e_{+,1}(\eta;\cdot)$ in $\Sigma\mathcal{P}^{0,+}_{K,0,1}[r,N]$, $e_{-,1}(\eta;\cdot)$ in $\Sigma\mathcal{P}^{-1,-}_{K,0,1}[r,N]$, $K_1(\eta;\cdot)$ in $\Sigma\mathcal{P}^{-1,\mathrm{int}}_{K,0,1}[r,N]$, which are even as functions of (x,ξ) (for $\eta(x)$ even), such that, if we set*

$$\begin{aligned}e_+(\eta;z,x,\xi)&=\mathcal{C}(z,\xi)+e_{+,1}(\eta;z,x,\xi)\\ e_-(\eta;z,x,\xi)&=\mathcal{S}(z,\xi)+e_{-,1}(\eta;z,x,\xi)\\ K(\eta;z,z',x,\xi)&=K_0(z,z',\xi)+K_1(\eta;z,z',x,\xi)\end{aligned} \tag{5.2.4}$$

and if one defines, for functions $g_\pm(x)$ and $f(z,x)$,

$$\begin{aligned}&E(t,z)[g_+,g_-,f]\\ &\overset{def}{=}Op^{\mathrm{BW}}(e_+)g_+ + Op^{\mathrm{BW}}(e_-)g_- + \int_{-1}^0 Op^{\mathrm{BW}}\big(K(\eta;z,z',\cdot)\big)f(z',\cdot)\,dz',\end{aligned} \tag{5.2.5}$$

then the action of (5.2.1) on (5.2.5) may be written as

$$f+R_+(\eta;z)g_+ + R_-(\eta;z)g_- + \int_{-1}^0 R_{\mathrm{int}}(\eta;z,z')f(z',\cdot)\,dz' \tag{5.2.6}$$

where $R_+(\eta';z)$ *(resp.* $R_-(\eta';z)$*, resp.* $R_{\rm int}(\eta;z,z')$*) is in* $\Sigma\mathcal{R}^{-\rho+2,+}_{K,0,1}[r,N]$ *(resp.* $\Sigma\mathcal{R}^{-\rho+1,-}_{K,0,1}[r,N]$*, resp.* $\Sigma\mathcal{R}^{-\rho+1,{\rm int}}_{K,0,1}[r,N]$*). Moreover the following boundary conditions hold*

$$E(t,0)[g_+,g_-,f]=g_+,\quad (\partial_z E)(t,-1)[g_+,g_-,f]=g_-\,. \tag{5.2.7}$$

Finally

$$\partial_z e_{+,1}(\eta;z,x,\xi)|_{z=0}=i\frac{\eta'}{1+\eta'^2}\xi-(\tanh\xi)\xi\frac{\eta'^2}{1+\eta'^2} \tag{5.2.8}$$

modulo $\Sigma\Gamma^0_{K,0,1}[r,N,{\rm aut}]$.

In order to prove Proposition 5.2.1, we start solving the boundary value problem corresponding to (5.2.1)-(5.2.2) at the level of principal symbols. Notice that the Poisson symbols $\mathcal{C},\mathcal{S}$ and K_0 in (5.2.3) solve the following boundary value problems

$$\big(\partial_z^2-\xi^2\big)\mathcal{C}(z,\xi)=0\,,\quad \mathcal{C}(z,\xi)|_{z=0}=1\,,\quad \partial_z\mathcal{C}(z,\xi)|_{z=-1}=0\,,$$

$$\big(\partial_z^2-\xi^2\big)\mathcal{S}(z,\xi)=0\,,\quad \mathcal{S}(z,\xi)|_{z=0}=0\,,\quad \partial_z\mathcal{S}(z,\xi)|_{z=-1}=1\,,$$

and

$$\begin{aligned}&\big(\partial_z^2-\xi^2\big)K_0(z,z',\xi)=\delta(z-z')\\ &K_0(z,z',\xi)|_{z=0}=0\,,\ \partial_z K_0(z,z',\xi)|_{z=-1}=0\,.\end{aligned} \tag{5.2.9}$$

In the next lemma we solve the ODE (5.2.12) for a general small η. For simplicity of notation we do not write explicitly the t dependence which is irrelevant.

Lemma 5.2.2 *There are symbols* $e^0_{+,1}(\eta;z,\cdot)$ *in* $\Sigma\mathcal{P}^{0,+}_{K,0,1}[r,N]$*,* $e^0_{-,1}(\eta;z,\cdot)$ *in* $\Sigma\mathcal{P}^{-1,-}_{K,0,1}[r,N]$*,* $K^0_1(\eta;z,z',\cdot)$ *in* $\Sigma\mathcal{P}^{-1,{\rm int}}_{K,0,1}[r,N]$*, which are even functions of* (x,ξ) *if* η *is an even function in* x*, satisfying the boundary conditions*

$$\begin{aligned}&e^0_{+,1}|_{z=0}=\partial_z e^0_{+,1}|_{z=-1}=e^0_{-,1}|_{z=0}=\partial_z e^0_{-,1}|_{z=-1}=0\\ &K^0_1|_{z=0}=0,\ \partial_z K^0_1|_{z=-1}=0,\end{aligned} \tag{5.2.10}$$

such that, setting

$$\begin{aligned}e^0_+(\eta;z,\cdot)&=\mathcal{C}(z,\xi)+e^0_{+,1}(\eta;z,\cdot)\\ e^0_-(\eta;z,\cdot)&=\mathcal{S}(z,\xi)+e^0_{-,1}(\eta;z,\cdot)\\ K^0(\eta;z,z',\cdot)&=K_0(z,z',\cdot)+K^0_1(\eta;z,z',\cdot)\,,\end{aligned} \tag{5.2.11}$$

the unique solution of the ODE

$$\begin{aligned}Pu\stackrel{def}{=}\big((1+\eta'^2)\partial_z^2-2i\eta'\xi\partial_z-\xi^2\big)u&=f\\ u|_{z=0}&=g_+\\ \partial_z u|_{z=-1}&=g_-\end{aligned} \tag{5.2.12}$$

is given by

$$u(z,x,\xi) = e^0_+(\eta;z,x,\xi)g_+(x) + e^0_-(\eta;z,x,\xi)g_-(x) + \int_{-1}^{0} K^0(\eta;z,z',x,\xi)f(z',x)\,dz'. \tag{5.2.13}$$

Moreover

$$Pe^0_\pm = 0\,, \quad PK^0 = \delta(z-z')\,. \tag{5.2.14}$$

Finally $\partial_z e^0_+|_{z=0}$ *has the following expansion*

$$\partial_z e^0_+(\eta;z,x,\xi)|_{z=0} = \xi\tanh\xi + i\frac{\eta'}{1+\eta'^2}\xi - (\tanh\xi)\xi\frac{\eta'^2}{1+\eta'^2} \tag{5.2.15}$$

modulo a symbol of $\Sigma\Gamma^{-\infty}_{K,0,1}[r,N,\mathrm{aut}]$.

Proof: The solution u of the linear equation (5.2.12) may be written as the sum (5.2.13) where e^0_+, e^0_- and K^0 solve respectively

$$Pe^0_+ = 0\,, \quad e^0_+|_{z=0} = 1\,, \quad \partial_z e^0_+|_{z=-1} = 0 \tag{5.2.16}$$

$$Pe^0_- = 0\,, \quad e^0_-|_{z=0} = 0\,, \quad \partial_z e^0_-|_{z=-1} = 1 \tag{5.2.17}$$

$$PK^0 = \delta(z-z')\,, \quad K^0|_{z=0} = 0\,, \quad \partial_z K^0|_{z=-1} = 0\,.$$

Dividing P by $1+\eta'^2$, we see that e^0_+, e^0_- are the solutions of the homogeneous linear equation

$$\big(\partial_z^2 - 2ia\xi\partial_z - (1+b)\xi^2\big)e^0_\pm(z,x,\xi) = 0 \tag{5.2.18}$$

with the corresponding boundary conditions in (5.2.16), (5.2.17), where

$$a = a(\eta') = \frac{\eta'}{1+\eta'^2}\,, \quad b = b(\eta') = -\frac{\eta'^2}{1+\eta'^2}\,. \tag{5.2.19}$$

The solutions $w_\pm$ of the homogeneous equation (5.2.18) (where we consider a, b as real constants) with boundary values

$$w_+|_{z=0} = 1\,,\ \partial_z w_+|_{z=-1} = 0\,,\ w_-|_{z=0} = 0\,,\ \partial_z w_-|_{z=-1} = 1\,, \tag{5.2.20}$$

are, setting $c = c(a,b) = \sqrt{1+b-a^2} - 1$,

(5.2.21)

$$w_+(z,\xi,a,b) = e^{iza\xi}\frac{\cosh((z+1)\xi(1+c))}{\cosh(\xi(1+c))}\frac{1-\frac{ia}{1+c}\tanh((z+1)\xi(1+c))}{1-\frac{ia}{1+c}\tanh(\xi(1+c))}$$

$$w_-(z,\xi,a,b) = e^{i(z+1)a\xi}\frac{\cosh(z\xi(1+c))}{\cosh(\xi(1+c))}\frac{\tanh(z\xi(1+c))}{\xi(1+c)} \times \left(1-\frac{ia}{1+c}\tanh(\xi(1+c))\right)^{-1}.$$

For a, b in a neighborhood of zero, $z \in [-1, 0]$, $w_\pm$ are analytic functions of a, b that satisfy the estimates

$$\begin{aligned} |z^\ell \partial_z^j \partial_a^\alpha \partial_b^\beta \partial_\xi^\gamma w_+(z,\xi,a,b)| &\leq C\langle\xi\rangle^{j-\ell-\gamma} \\ |(1+z)^\ell \partial_z^j \partial_a^\alpha \partial_b^\beta \partial_\xi^\gamma w_-(z,\xi,a,b)| &\leq C\langle\xi\rangle^{-1+j-\ell-\gamma}. \end{aligned} \tag{5.2.22}$$

Actually, to check the first estimate (5.2.22), it suffices to remark that w_+ is of order $-\infty$ in ξ, as well as its derivatives, uniformly for z in $[-1, -\frac{1}{2}]$. On the other hand, for z in $[-\frac{1}{2}, 0]$ and for instance $\xi > 0$, we may write

$$w_+(z,\xi,a,b) = e^{iza\xi + z\xi(1+c)}(1 + G(z,\xi,a,b)) \tag{5.2.23}$$

where G as well as its derivatives is uniformly of order $-\infty$, while the exponential factor satisfies the wanted estimates as $1 + c \geq \frac{1}{2}$ for a, b small enough.

If we expand $w_+(z,\xi,a,b)$ by Taylor formula relatively to the variables (a, b) at $(0, 0)$, we get $\mathcal{C}(z,\xi)$ as the constant term, polynomial contributions in (a, b), whose coefficients satisfy estimates of the form (5.2.22), and remainders, vanishing at order N at $(0, 0)$, and verifying as well (5.2.22). Expressing in each polynomial term $a = a(\eta')$ and $b = b(\eta')$ by (5.2.19), and developing the functions $a(\eta'), b(\eta')$ as polynomials in η' plus a remainder vanishing at order N at $\eta' = 0$, we have that

$$e_+^0 = w_+(z,\xi,a(\eta'),b(\eta')) = \mathcal{C}(z,\xi) + e_{+,1}^0\,, \tag{5.2.24}$$

as in (5.2.11), where $e_{+,1}^0$ is a Poisson symbol in $\Sigma\mathcal{P}_{K,0,1}^{0,+}[r, N]$. We argue similarly for

$$e_-^0 = w_-(z,\xi,a(\eta'),b(\eta')) = \mathcal{S}(z,\xi) + e_{-,1}^0\,. \tag{5.2.25}$$

Let us prove the expansion (5.2.15). Differentiating (5.2.21) at $z = 0$ we get

$$(\partial_z w_+)_{|z=0} = ia\xi + \xi(1+c)\tanh(\xi(1+c)) + \frac{ia\xi\cosh^{-2}(\xi(1+c))}{1 - \frac{ia}{1+c}\tanh(\xi(1+c))}$$

The last term is in $\Sigma\Gamma_{K,0,1}^{-\infty}[r, N, \mathrm{aut}]$. Since $\xi\tanh(\xi(1+c)) - \xi\tanh(\xi)$ is in $\Sigma\Gamma_{K,0,1}^{-\infty}[r, N, \mathrm{aut}]$ and since, by (5.2.19),

$$c = \sqrt{1 + b - a^2} - 1 = -\frac{\eta'^2}{1+\eta'^2}\,,$$

we get

$$(\partial_z w_+)_{|z=0} = \xi\tanh(\xi) + ia\xi - \frac{\eta'^2}{1+\eta'^2}\xi\tanh(\xi)$$

modulo a symbol in $\Sigma\Gamma_{K,0,1}^{-\infty}[r, N, \mathrm{aut}]$, proving the expansion (5.2.15).

Let us obtain next the expression of $K^0(\eta;\cdot)$ in (5.2.11). Consider the wronskian of the solutions (w_+, w_-)

$$W(z,\xi,a,b) = \begin{vmatrix} w_+ & w_- \\ \partial_z w_+ & \partial_z w_- \end{vmatrix}.$$

Since the wronskian $W(z)$ solves the differential equation $\partial_z W(z) = 2ia\xi W(z)$ we have that $W(z) = e^{2ia\xi(z+1)}W(-1)$. By (5.2.20), $W(-1) = w_+(-1)$, and, inserting the value of $w_+(-1)$ obtained by (5.2.21), we get

(5.2.26)
$$W(z,\xi,a,b) = e^{i(2z+1)a\xi}(\cosh(\xi(1+c)))^{-1}\Big(1 - \frac{ia}{1+c}\tanh(\xi(1+c))\Big)^{-1}.$$

Let us define

(5.2.27)
$$\begin{aligned}\tilde K(z,z',\xi,a,b)\\ = \Big(w_+(z,\xi,a,b)w_-(z',\xi,a,b)\mathbf{1}_{z-z'<0} + w_-(z,\xi,a,b)w_+(z',\xi,a,b)\mathbf{1}_{z-z'>0}\Big)\\ \times W(z',\xi,a,b)^{-1}.\end{aligned}$$

Since $w_\pm$ are the solutions of the homogeneous linear equation (5.2.18), the action of the differential operator in (5.2.18) on $\tilde K$ gives $\delta(z-z')$, and the boundary conditions (5.2.20) imply $\tilde K|_{z=0} = 0, \partial_z\tilde K|_{z=-1} = 0$. Moreover, proceeding as in (5.2.23), one checks that the coefficients $\tilde K_\pm$ of $\mathbf{1}_{\pm(z-z')>0}$ in (5.2.27) satisfy estimates

$$|(z-z')^\ell \partial_z^j \partial_{z'}^{j'} \partial_a^\alpha \partial_b^\beta \partial_\xi^\gamma \tilde K_\pm(z,z',\xi,a,b)| \le C\langle\xi\rangle^{-1-\ell+j+j'-\gamma}.$$

Decomposing $\tilde K_\pm$ in Taylor series in the variables (a,b) at $(0,0)$, and expressing a, b from η' by (5.2.19), we conclude that

(5.2.28)
$$K^0(\eta;z,z',x,\xi) \stackrel{\rm def}{=} \tilde K(z,z',\xi,a,b)(1+\eta'^2)^{-1}$$

is a symbol of $\Sigma\mathcal{P}^{-1,\rm int}_{K,0,0}[r,N]$, which has the expansion (5.2.11), and the action of the operator $(1+\eta'^2)\partial_z^2 - 2i\eta'\xi\partial_z - \xi^2$ on K^0 gives $\delta(z-z')$, proving the last identity in (5.2.14).

Finally the symbols $e^0_\pm$, K^0 are even in (x,ξ), when η is even, as it follows by formulas (5.2.24), (5.2.25), (5.2.21), (5.2.27), noticing that, if η is even, then $b(\eta'), c(\eta'), a(\eta')\xi$ are even functions of (x,ξ). The proof of the lemma is concluded. □

Proof of Proposition 5.2.1: We look for $e_\pm(\eta;\cdot)$, $K(\eta;\cdot)$ in (5.2.4) as a sum of Poisson symbols in decreasing order (forgetting the time dependence in

the notation) of the form

$$
\begin{aligned}
e_+(\eta;z,x,\xi) &= \mathcal{C}(z,\xi)+\sum_{j=0}^{\rho-1} e^j_{+,1}(\eta;z,x,\xi)\\
e_-(\eta;z,x,\xi) &= \mathcal{S}(z,\xi)+\sum_{j=0}^{\rho-1} e^j_{-,1}(\eta;z,x,\xi)\\
K(\eta;z,z',x,\xi) &= K_0(z,z',\xi)+\sum_{j=0}^{\rho-1} K^j_1(\eta;z,z',x,\xi)
\end{aligned}
\tag{5.2.29}
$$

where $e^0_{+,1}\in\Sigma\mathcal{P}^{0,+}_{K,0,1}[r,N]$, $e^0_{-,1}\in\Sigma\mathcal{P}^{-1,-}_{K,0,1}[r,N]$, $K^0_1\in\Sigma\mathcal{P}^{-1,\mathrm{int}}_{K,0,1}[r,N]$ are defined in lemma 5.2.2, and, for $j=1,\dots,\rho-1$,

- $e^j_{+,1}(\eta;\cdot)$ are in $\Sigma\mathcal{P}^{-j,+}_{K,0,1}[r,N]$,
- $e^j_{-,1}(\eta;\cdot)$ are in $\Sigma\mathcal{P}^{-1-j,-}_{K,0,1}[r,N]$,
- $K^j_1(\eta;\cdot)$ are in $\Sigma\mathcal{P}^{-1-j,\mathrm{int}}_{K,0,1}[r,N]$.

We also require that these symbols satisfy the boundary conditions

$$
e^j_{\pm,1}|_{z=0}=0\,,\ \partial_z e^j_{\pm,1}|_{z=-1}=0\,,\ K^j_1|_{z=0}=0\,,\ \partial_z K^j_1|_{z=-1}=0\,.
\tag{5.2.30}
$$

As a consequence, if E is defined by (5.2.5), it follows by (5.2.3), (5.2.10) and (5.2.30) that the boundary conditions (5.2.7) are satisfied.

In order to prove the proposition we have thus to find symbols $e^j_{\pm,1}$, K^j_1, $1\le j\le\rho-1$, as above so that the action of the operator (5.2.1) on E gives (5.2.6).

Using that $\eta'\otimes\eta'$ is a symbol in $\tilde\Gamma^0_2$, $\eta'\xi$ is in $\tilde\Gamma^1_1$, and that the Poisson symbol $\partial_z^2 e_+$ is in $\Sigma\mathcal{P}^{2,+}_{K,0,0}[r,N]$ and $\partial_z^2 e_-$ is in $\Sigma\mathcal{P}^{1,+}_{K,0,0}[r,N]$, by the composition result of Proposition 5.1.11 we compute

$$
\begin{aligned}
&\big((1+Op^{\mathrm{BW}}(\eta'\otimes\eta'))\partial_z^2-2iOp^{\mathrm{BW}}(\eta'\xi)\partial_z-Op^{\mathrm{BW}}(\xi^2)\big)\circ Op^{\mathrm{BW}}(e_\pm)\\
&\quad=Op^{\mathrm{BW}}(Pe_\pm)+Op^{\mathrm{BW}}\big(((\eta'\otimes\eta')\#\partial_z^2 e_\pm)_{\rho,N}-\eta'^2\partial_z^2 e_\pm\big)\\
&\qquad-2iOp^{\mathrm{BW}}\big(((\eta'\xi)\#\partial_z e_\pm)_{\rho,N}-(\eta'\xi)\partial_z e_\pm\big)\\
&\qquad\qquad-Op^{\mathrm{BW}}\big((\xi^2\#e_\pm)_{\rho,N}-\xi^2 e_\pm\big)+R_\pm(\eta)
\end{aligned}
\tag{5.2.31}
$$

where P is defined in (5.2.12) and where $R_+(\eta)$ (resp. $R_-(\eta)$) belongs to $\Sigma\mathcal{R}^{-\rho+2,+}_{K,0,1}[r,N]$ (resp. $\Sigma\mathcal{R}^{-\rho+1,-}_{K,0,1}[r,N]$). Recalling the definition (5.2.11) of the symbols $e^0_\pm$, we have $e_\pm=e^0_\pm+\sum_{j=1}^{\rho-1}e^j_{\pm,1}$ and, since $Pe^0_\pm=0$ by (5.2.14), we deduce that $Pe_\pm$ in (5.2.31) is

$$
Pe_\pm=\sum_{j=1}^{\rho-1}Pe^j_{\pm,1}\,.
$$

The other symbols in the right hand side of (5.2.31) may be written as

$$\sum_{j=1}^{\rho-1} \mathcal{L}_{j,\pm}(e^0_\pm, \dots, e^{j-1}_{\pm,1}) + \mathcal{L}_{\rho,\pm}(e^0_\pm, \dots, e^{\rho-1}_{\pm,1})$$

where $\mathcal{L}_{j,+}$ (resp. $\mathcal{L}_{j,-}$) belongs to $\Sigma\mathcal{P}^{2-j,+}_{K,0,1}[r,N]$ (resp. to $\Sigma\mathcal{P}^{1-j,-}_{K,0,1}[r,N]$), and depends only on $e^\ell_{\pm,1}$, with $\ell < j$, because of the Definition 2.3.1 of $(\cdot\#\cdot)_\rho$ and of the fact that in the right hand side of (5.2.31), the first term of the asymptotic expansion is removed from each expression. By the last remark following Definition 5.1.10, the paradifferential operators $Op^{\mathrm{BW}}\big(\mathcal{L}_{\rho,\pm}(e^0_\pm, \dots, e^{\rho-1}_{\pm,1})\big)$ may be incorporated to the smoothing remainder $R_\pm(\eta)$, and thus we may write the right hand side of (5.2.31) as

$$\sum_{j=1}^{\rho-1} Op^{\mathrm{BW}}\big(Pe^j_{\pm,1} - \mathcal{L}_{j,\pm}(e^0_\pm, e^1_{\pm,1}, \dots, e^{j-1}_{\pm,1})\big) + R_\pm(\eta)\,.$$

In order to prove the Proposition we need to find recursively Poisson symbols $e^j_{\pm,1}$, $j = 1, \dots, \rho-1$ (in the classes described above), solving

$$\begin{aligned} & Pe^j_{\pm,1} = \mathcal{L}_{j,\pm}(e^0_\pm, e^1_{\pm,1}, \dots, e^{j-1}_{\pm,1}) \\ & e^j_{\pm,1}|_{z=0} = 0 \\ & \partial_z e^j_{\pm,1}|_{z=-1} = 0\,. \end{aligned} \tag{5.2.32}$$

By (5.2.13), the solution of (5.2.32) is given by

$$e^j_{\pm,1} = \int_{-1}^{0} K^0(\eta; z, z', x, \xi)\mathcal{L}_{j,\pm}(e^0_\pm, e^1_{\pm,1}, \dots, e^{j-1}_{\pm,1})(\eta; z', x, \xi)\, dz'.$$

By Proposition 5.1.11-(iv), since K^0 is a Poisson symbol in $\Sigma\mathcal{P}^{-1,\mathrm{int}}_{K,0,0}[r,N]$ and $\mathcal{L}_{j,+}$ (resp. $\mathcal{L}_{j,-}$) is in $\Sigma\mathcal{P}^{2-j,+}_{K,0,1}[r,N]$ (resp. to $\Sigma\mathcal{P}^{1-j,-}_{K,0,1}[r,N]$), we deduce that $e^j_{+,1}$ is in $\Sigma\mathcal{P}^{-j,+}_{K,0,1}[r,N]$ and $e^j_{-,1}$ is in $\Sigma\mathcal{P}^{-1-j,-}_{K,0,1}[r,N]$.

The formula (5.2.8) follows by (5.2.15) and the fact that $\sum_{j=1}^{\rho-1} e^j_{+,1}$ is a symbol in $\Sigma\mathcal{P}^{-1,+}_{K,0,1}[r,N]$.

Finally let us construct iteratively in a similar way the Poisson symbols $K^j_1 \in \Sigma\mathcal{P}^{-1-j,\mathrm{int}}_{K,0,1}[r,N]$, $j = 1, \dots, \rho-1$, in the last line of (5.2.29).

First of all, recall that the Poisson symbol K^0 in (5.2.11) satisfies $PK^0 = \delta(z-z')$ (see (5.2.14)) and that K^0 is in $\Sigma\mathcal{P}^{-1,\mathrm{int}}_{K,0,0}[r,N]$. Moreover by (5.2.28) and (5.2.27) we deduce that $\partial_z K^0(\eta;\cdot)$ is in $\Sigma\mathcal{P}^{0,\mathrm{int}}_{K,0,0}[r,N]$ as the coefficient of $\delta(z-z')$ in the ∂_z derivative vanishes. Finally, by the equation $PK^0 = \delta(z-z')$ and recalling the expression of P in (5.2.12), we deduce that

$$\partial_z^2 K_0 - (1+\eta'^2)^{-1}\delta(z-z') \quad \text{is in } \Sigma\mathcal{P}^{1,\mathrm{int}}_{K,0,0}[r,N]\,.$$

Therefore applying the composition result of Proposition 5.1.11-(ii) we get

$$Op^{\rm BW}(P)\circ Op^{\rm BW}(K^0)=\delta(z-z')+\mathcal{L}_1+R \tag{5.2.33}$$

where (recall the definition of P in (5.2.12))

$$\mathcal{L}_1\stackrel{\rm def}{=}\big(((1+\eta'^2)\partial_z^2-2i\eta'\xi\partial_z-\xi^2)\#K^0\big)_{\rho,N}-\delta(z-z') \tag{5.2.34}$$

belongs to $\Sigma\mathcal{P}^{0,\rm int}_{K,0,1}[r,N]$ and the smoothing operator R is in $\Sigma\mathcal{R}^{-\rho+1,\rm int}_{K,0,1}[r,N]$. Then we argue iteratively. Suppose that we have yet defined Poisson symbols $\mathcal{L}_1,\dots,\mathcal{L}_j$ with $\mathcal{L}_j\in\Sigma\mathcal{P}^{1-j,\rm int}_{K,0,1}[r,N]$, and $K_1^1,\dots,K_1^{j-1}$ with $K_1^{j-1}(\eta;\cdot)\in\Sigma\mathcal{P}^{-j,\rm int}_{K,0,1}[r,N]$. Then define, for $j\geq 1$, the K_1^j as the solution of

$$\begin{aligned}PK_1^j(\eta;z,z',x,\xi)&=-\mathcal{L}_j(\eta;z,z',x,\xi)\\ K_1^j|_{z=0}&=0\\ \partial_zK_1^j|_{z=-1}&=0\end{aligned} \tag{5.2.35}$$

(where P is the operator (5.2.12) acting relatively to the z variable, z' being a parameter) and

$$\mathcal{L}_{j+1}\stackrel{\rm def}{=}\big(((1+\eta'^2)\partial_z^2-2i\eta'\xi\partial_z-\xi^2)\#K_1^j\big)_{\rho,N}+\mathcal{L}_j\,. \tag{5.2.36}$$

By (5.2.13), the solution K_1^j of (5.2.35) is given by

$$K_1^j(\eta;z,z',x,\xi)=-\int_{-1}^0K^0(\eta;z,z'',x,\xi)\mathcal{L}_j(\eta;z'',z',x,\xi)\,dz'' \tag{5.2.37}$$

and, since K^0 is in $\Sigma\mathcal{P}^{-1,\rm int}_{K,0,0}[r,N]$ and $\mathcal{L}_j$ in $\Sigma\mathcal{P}^{1-j,\rm int}_{K,0,1}[r,N]$, Proposition 5.1.11-(iii) implies that K_1^j is a Poisson symbol in $\Sigma\mathcal{P}^{-1-j,\rm int}_{K,0,1}[r,N]$. In the same way, since $\partial_zK^0(\eta;\cdot)$ is in $\Sigma\mathcal{P}^{0,\rm int}_{K,0,0}[r,N]$, we deduce that

$$\partial_zK_1^j(\eta;\cdot)=-\int_{-1}^0\partial_zK^0(\eta;z,z'',x,\xi)\mathcal{L}_j(\eta;z'',z',x,\xi)\,dz''$$

is a Poisson symbol in $\Sigma\mathcal{P}^{-j,\rm int}_{K,0,1}[r,N]$, and, finally, using the first equation (5.2.35) and recalling the definition of P in (5.2.12), we get that $\partial_z^2K_1^j(\eta;\cdot)$ is in $\Sigma\mathcal{P}^{1-j,\rm int}_{K,0,1}[r,N]$. We conclude that the symbol $\mathcal{L}_{j+1}$ defined in (5.2.36) is in $\Sigma\mathcal{P}^{-j,\rm int}_{K,0,1}[r,N]$, because, by the first equation in (5.2.35), the first term in the expansion of $(P\#K_1^j)_{\rho,N}$ cancels out.

In conclusion, applying the composition result of Proposition 5.1.11, we deduce that the Poisson symbol $K(\eta;\cdot)=K^0+\sum_{j=1}^{\rho-1}K_1^j(\eta;\cdot)$ satisfies, by (5.2.33), (5.2.35), (5.2.36),

$$\begin{aligned}Op^{\rm BW}\Big((1+\eta'^2)\partial_z^2-2i\eta'\xi\partial_z-\xi^2\Big)\circ Op^{\rm BW}\big(K(\eta;z,z',\cdot)\big)&\\ =\delta(z-z')+Op^{\rm BW}(\mathcal{L}_\rho)+R(\eta;z,z')&\end{aligned}$$

for some smoothing remainder R in $\Sigma\mathcal{R}^{-\rho+1,\mathrm{int}}_{K,0,1}[r,N]$. As $Op^{\mathrm{BW}}(\mathcal{L}_\rho)$ may be incorporated to such a term (see the last remark after Definition 5.1.10), we obtain that the action of (5.2.1) on $\int_{-1}^{0} Op^{\mathrm{BW}}(K(\eta;z,z',\cdot))f(z')\,dz'$ gives the first and last terms in (5.2.6). This concludes the proof. □

5.3 Solving the Dirichlet-Neumann problem

The goal of this section is to deduce from the parametrix constructed in section 5.2 an expression for the solution of the Dirichlet-Neumann boundary value problem (5.3.1) below.

Proposition 5.3.1 (Dirichlet-Neumann problem) *Let ρ, N, K be given integers. There is $\sigma > 0$ such that, for any $s \in \mathbb{R}, s' \in \mathbb{N}$ with $s - s' \geq \sigma$, there is $r > 0$ and for any function η in $C^K_*(I, \dot{H}^\sigma(\mathbb{T}^1,\mathbb{R}))$ in the ball $B^K_\sigma(I,r)$ defined in (2.1.9), for any function f in the space $\bigcap_0^K C^k(I, E^{s-2-\frac{3}{2}k}_{s'})$ (with $E^s_{s'}$ defined in (5.1.17)), any*

$$g_+ \in C^K_*(I,\dot{H}^s(\mathbb{T}^1,\mathbb{R}))\,,\quad g_- \in C^K_*(I,\dot{H}^{s-1}(\mathbb{T}^1,\mathbb{R}))\,,$$

the boundary value problem

$$\begin{aligned} \big[(1+Op^{\mathrm{BW}}(\eta'\otimes\eta'))\partial_z^2 - 2iOp^{\mathrm{BW}}(\eta'\xi)\partial_z - Op^{\mathrm{BW}}(\xi^2)\big]u &= f \\ u|_{z=0} &= g_+ \\ \partial_z u|_{z=-1} &= g_- \end{aligned} \tag{5.3.1}$$

has a unique solution u in $\bigcap_0^K C^k(I, E^{s-\frac{3}{2}k}_{s'})$. Moreover, there are

- *Poisson symbols $e_+(\eta;\cdot)$ in $\Sigma\mathcal{P}^{0,+}_{K,0,0}[r,N]$, $e_-(\eta;\cdot)$ in $\Sigma\mathcal{P}^{-1,-}_{K,0,0}[r,N]$, $K(\eta;\cdot)$ in $\Sigma\mathcal{P}^{-1,\mathrm{int}}_{K,0,0}[r,N]$ (which are those defined in (5.2.4)),*
- *smoothing operators R_+ in $\Sigma\mathcal{R}^{-\rho,+}_{K,0,1}[r,N]$, R_- in $\Sigma\mathcal{R}^{-\rho-1,-}_{K,0,1}[r,N]$, R_{int} in $\Sigma\mathcal{R}^{-\rho-1,\mathrm{int}}_{K,0,1}[r,N]$,*

such that for any f, g_+, g_- as above, the solution u to (5.3.1) may be written as

$$\begin{aligned} u = {} & Op^{\mathrm{BW}}(e_+(\eta;z,\cdot))g_+ + Op^{\mathrm{BW}}(e_-(\eta;z,\cdot))g_- \\ & + \int_{-1}^{0} Op^{\mathrm{BW}}(K(\eta;z,z',\cdot))f(z',\cdot)\,dz' \\ & + R_+(\eta;z)g_+ + R_-(\eta;z)g_- + \int_{-1}^{0} R_{int}(\eta;z,z')f(z',\cdot)\,dz'. \end{aligned} \tag{5.3.2}$$

Moreover

$$\partial_z e_+(\eta;z,x,\xi)|_{z=0} = (\tanh\xi)\xi + i\frac{\eta'}{1+\eta'^2}\xi - (\tanh\xi)\xi\frac{\eta'^2}{1+\eta'^2} \tag{5.3.3}$$

modulo $\Sigma\Gamma^0_{K,0,1}[r,N,\mathrm{aut}]$.

Proof: In the proof, we shall ignore the time dependence i.e. argue like if $K = 0$. We look for the solution u of (5.3.1) as

$$u = E(t,z)[g_+, g_-, f] + W$$

where the first term is given by (5.2.5) (and the first two lines in (5.3.2)), and the remainder W satisfies, according to (5.2.6)-(5.2.7),

$$\begin{aligned} &\big[(1+Op^{\mathrm{BW}}(\eta'\otimes\eta'))\partial_z^2 - 2iOp^{\mathrm{BW}}(\eta'\xi)\partial_z - Op^{\mathrm{BW}}(\xi^2)\big]W \\ &\qquad = -R_+^2(\eta;z)g_+ - R_-^1(\eta;z)g_- - \int_{-1}^0 R'_{\mathrm{int}}(\eta;z,z')f(z')\,dz' \\ &W|_{z=0} = 0 \\ &\partial_z W|_{z=-1} = 0 \end{aligned} \tag{5.3.4}$$

with R_+^2 (resp. R_-^1, resp. R'_{int}) in $\Sigma\mathcal{R}^{-\rho+2,+}_{K,0,1}[r,N]$ (resp. $\Sigma\mathcal{R}^{-\rho+1,-}_{K,0,1}[r,N]$, resp. $\Sigma\mathcal{R}^{-\rho+1,\mathrm{int}}_{K,0,1}[r,N]$). We rewrite the equation (5.3.4) as

$$\begin{aligned} (\partial_x^2+\partial_z^2)W &= LW + G \\ W|_{z=0} &= 0 \\ \partial_z W|_{z=-1} &= 0 \end{aligned} \tag{5.3.5}$$

with

$$\begin{aligned} LW &= -Op^{\mathrm{BW}}(\eta'\otimes\eta')\partial_z^2 W + 2iOp^{\mathrm{BW}}(\eta'\xi)\partial_z W \\ G &= -R_+^2(\eta;z)g_+ - R_-^1(\eta;z)g_- - \int_{-1}^0 R'_{\mathrm{int}}(\eta;z,z')f(z')\,dz'. \end{aligned} \tag{5.3.6}$$

By lemma 5.2.2, applied with $\eta = 0$, $g_+ = g_- = 0$, we may rewrite (5.3.5) as

$$(\mathrm{Id} - M)W(z,x) = \int_{-1}^0 K_0(z,z',D_x)G(z',\cdot)\,dz' \tag{5.3.7}$$

where $K_0(z,z',\xi) \in \tilde{\mathcal{P}}_0^{-1,\mathrm{int}}$ is the Poisson symbol with constant coefficients (in x) defined in (5.2.3), and $M = M(\eta)$ is the operator

$$MW(z,x) \stackrel{\mathrm{def}}{=} \int_{-1}^0 K_0(z,z',D_x)LW(z')\,dz'. \tag{5.3.8}$$

In order to prove that W has the form described in the last line in (5.3.2), we have to invert the operator $\mathrm{Id} - M$. This will be a consequence of the next lemma. We denote by H the right hand side of (5.3.7).

Lemma 5.3.2 *(i) The operator M is bounded from the subspace*

$$\big\{w \in E^s_{s'} \,:\, w|_{z=0} = 0\,, \partial_z w|_{z=-1} = 0\big\}$$

to itself, and its operator norm is $O_{s,s'}(\|\eta\|_{\dot H^\sigma})$ *if* $\sigma > \frac{3}{2}$.

(ii) The right hand side H *of (5.3.7) has the same structure as* G *in (5.3.6) except that the smoothing operators* $R^2_+, R^1_-, R'_{\rm int}$ *are respectively in* $\Sigma\mathcal{R}^{-\rho,+}_{K,0,1}[r,N]$, $\Sigma\mathcal{R}^{-\rho-1,-}_{K,0,1}[r,N]$, $\Sigma\mathcal{R}^{-\rho-1,{\rm int}}_{K,0,1}[r,N]$. *Moreover* $H|_{z=0} = 0$ *and* $\partial_z H|_{z=-1} = 0$.

(iii) The operator M *sends an expression of the form* G *in (5.3.6), satisfying* $G|_{z=0} = 0$ *and* $\partial_z G|_{z=-1} = 0$, *to another expression of the same type. Moreover* $\sum_{k=1}^{+\infty} M^k$ *sends an expression of the form* G *satisfying* $G|_{z=0} = 0$ *and* $\partial_z G|_{z=-1} = 0$, *to a similar expression, with* R^2_+ *in* $\Sigma\mathcal{R}^{-\rho,+}_{K,0,1}[r,N]$, R^1_- *in* $\Sigma\mathcal{R}^{-\rho-1,-}_{K,0,1}[r,N]$, $R'_{\rm int}$ *in* $\Sigma\mathcal{R}^{-\rho-1,{\rm int}}_{K,0,1}[r,N]$.

Proof: (i) By (5.3.8) and the definition (5.3.6) of L, we may write

$$Mw(z,x) = \int_{-1}^{0} K_0(z,z',D_x)[\partial^2_{z'} w_1 + \partial_{z'} w_2](z',\cdot)\, dz' \tag{5.3.9}$$

where

$$w_1 = -Op^{\rm BW}(\eta'\otimes\eta')w\,, \quad w_2 = 2iOp^{\rm BW}(\eta'\xi)w\,. \tag{5.3.10}$$

By Proposition 2.2.4, if w is a function of $E^s_{s'}$ then w_1 is in $E^s_{s'}$ and w_2 in $E^{s-1}_{s'}$, with $\|w_1\|_{E^s_{s'}} + \|w_2\|_{E^{s-1}_{s'}} \le C\|\eta\|_{\dot H^\sigma}\|w\|_{E^s_{s'}}$ (if $\sigma > \frac{3}{2}$). Performing integrations by parts in (5.3.9), using the boundary conditions $w|_{z=0} = \partial_z w|_{z=-1} = 0$, and that K_0 defined in (5.2.3) satisfies $K_0(z,z',D_x)|_{z'=-1} = \mathcal{S}(z,D_x)$ and $\partial_{z'}K_0(z,z',D_x)|_{z'=0} = 0$, we rewrite (5.3.9) as

$$\begin{aligned} w_1(z) - \mathcal{S}(z,D_x)w_2(-1) + \int_{-1}^{0} K_0(z,z',D_x)D_x^2 w_1(z')\, dz' \\ - \int_{-1}^{0} (\partial_{z'}K_0)(z,z',D_x)w_2(z')\, dz'. \end{aligned} \tag{5.3.11}$$

We apply lemma 5.1.8 to $K_0 \in \tilde{\mathcal{P}}_0^{-1,{\rm int}}$ and to $\partial_{z'}K_0 \in \tilde{\mathcal{P}}_0^{0,{\rm int}}$, and lemma 5.1.5 to $\mathcal{S} \in \tilde{\mathcal{P}}_0^{-1,-}$. It follows from the previous bound on the functions w_1, w_2 defined in (5.3.10), that (5.3.11) belongs to $E^s_{s'}$, with a norm controlled from above by $C\|\eta\|_{\dot H^\sigma}\|w\|_{E^s_{s'}}$. Finally notice that the function Mw defined in (5.3.8) satisfies $(Mw)(z,x)|_{z=0} = 0$, $\partial_z(Mw)(z,x)|_{z=-1} = 0$ because $K_0(z,z',\xi)|_{z=0} = 0$, $\partial_z K_0(z,z',\xi)|_{z=-1} = 0$. This proves (i) of the lemma.

(ii) follows from Proposition 5.1.12-(ii)-(iii) as K_0 is in $\tilde{\mathcal{P}}_0^{-1,{\rm int}}$.

(iii) Since $G|_{z=0} = 0$ and $\partial_z G|_{z=-1} = 0$ the action of M on G is given by (5.3.11), where we replace w_1 (resp. w_2) by $-Op^{\rm BW}(\eta'\otimes\eta')G$ (resp. $2iOp^{\rm BW}(\eta'\xi)G$). If G has the form (5.3.6), Proposition 5.1.12-(i) implies that the function w_1 (resp. w_2) is of the same form (5.3.6) as well, with ρ replaced by $\rho - 1$ in the case w_2 (and actually with a cubic degree of

homogeneity). Using (ii) and (iii) of Proposition 5.1.12, we conclude that MG has an expression of the same form as G. Then we decompose

$$\sum_{k=1}^{+\infty} M(\eta)^k G = \sum_{k=1}^{N-2} M(\eta)^k G + \sum_{k=N-1}^{\infty} M(\eta)^k G$$

where, by what proved above, the first finite sum has the form of G in (5.3.6). To estimate the series, use that the operator norm of $M(\eta)$ acting on $E^s_{s'}$ is $O_s(\|\eta\|_{\dot H^\sigma})$ (for some σ independent of s), as we proved in item (i) of the present lemma. Consequently $\sum_{k=N-1}^{+\infty} M(\eta)^k$ sends an element of $\mathcal{R}^{-\rho,\pm}_{K,0,1}[r]$, $\mathcal{R}^{-\rho,\mathrm{int}}_{K,0,1}[r]$ to an element of $\mathcal{R}^{-\rho,\pm}_{K,0,N}[r]$, $\mathcal{R}^{-\rho,\mathrm{int}}_{K,0,N}[r]$, using estimates (5.1.32), (5.1.33), the above continuity property of $M(\eta)$ and the fact that we may choose the numbers $r(s)$ in (ii) of Definition 5.1.10 small enough. □

For $\|\eta\|_{\dot H^\sigma}$ small enough, lemma 5.3.2-(i) implies that the operator $\mathrm{Id}-M$ is invertible, and, denoting by H the right hand side of (5.3.7), we may write the solution W of (5.3.7) as the Neumann series

$$W(z,x) = H + \sum_{k=1}^{+\infty} M^k H\,. \tag{5.3.12}$$

End of the proof of Proposition 5.3.1: By lemma 5.3.2-(ii)-(iii) the function $W(z,x)$ in (5.3.12) may be written as

$$R_+(\eta;z)g_+ + R_-(\eta;z)g_- + \int_{-1}^{0} R_{\mathrm{int}}(\eta;z,z')f(z')\,dz'$$

in (5.3.2) for suitable smoothing operators R_+ in $\Sigma\mathcal{R}^{-\rho,+}_{K,0,1}[r,N]$, R_- in $\Sigma\mathcal{R}^{-\rho-1,-}_{K,0,1}[r,N]$, R_{int} in $\Sigma\mathcal{R}^{-\rho-1,\mathrm{int}}_{K,0,1}[r,N]$.

Finally the expansion (5.3.3) for $e_+ = \mathcal{C} + e_{+,1}$ defined in (5.2.4), follows by (5.2.3) and (5.2.8). This concludes the proof of the proposition. □

We shall need in the next chapter a variant of Proposition 5.3.1, when one solves, instead of (5.3.1), a differential boundary value problem. Let us introduce first a class of operators playing the same role, for boundary value problems, as the operators of Definition 2.2.5. In the sequel, we shall consider functions η with values in $\mathbb{R}$ as functions with values in $\mathbb{C}^2$, identifying η to the vector $\left[\begin{smallmatrix}\eta\\ \eta\end{smallmatrix}\right]$ of $\mathbb{C}^2$.

Definition 5.3.3 *Let m be in $\mathbb{R}_+$, K,p in $\mathbb{N}$ with $p\le N$, $r>0$, $j\in\mathbb{N}$.*

(i) We denote by $\tilde{\mathcal{N}}^m_p$ the space of $(p+1)$-linear maps of the form

$$(\eta_1,\dots,\eta_p,\Phi)\to M(\eta_1,\dots,\eta_p)\Phi\,,$$

symmetric in $(\eta_1,\dots,\eta_p)$, defined for $\eta_1,\dots,\eta_p$ in $\dot H^\infty(\mathbb{T}^1,\mathbb{C}^2)$, with values in the space of linear maps from E^∞_∞ to itself, such that, for some $\mu\ge 0$,

for any $(\eta_1,\dots,\eta_p)$ *in* $\dot H^\infty(\mathbb{T}^1,\mathbb{C}^2)^p$, *any* $\Phi\in C^K_*(I,E^\infty_\infty)$, *any* n_0,n_{p+1} *in* $\mathbb{N}^*$, *any* $n'=(n_1,\dots,n_p)$ *in* $(\mathbb{N}^*)^p$, *any* j *in* $\mathbb{N}$,

$$\|\Pi_{n_0}M(\Pi_{n_1}\eta_1,\dots,\Pi_{n_p}\eta_p)\Pi_{n_{p+1}}\Phi\|_{E^0_j} \le C(n_0+\cdots+n_{p+1})^m|n'|^\mu\mathcal{G}^0_{0,p}(\eta_1,\dots\eta_p)\tilde{\mathcal{G}}^{0,j}_{0,1}(\Phi) \tag{5.3.13}$$

(where we used notations (2.1.3) and (5.1.29)). Moreover, we assume that condition (2.2.51) holds.

(When $p=0$, *the above conditions just mean that* M *is a linear map from* E^∞_∞ *to itself, satisfying estimate (5.3.13) and (2.2.51)).*

(ii) We denote by $\mathcal{N}^m_{K,j,p}[r]$ *the space of maps* $(\eta,\Phi)\to M(\eta)\Phi$, *linear in* Φ *such that there are* σ_0,μ *in* $\mathbb{R}_+$ *so that, for any* $\sigma>\sigma_0$, *there is* $r(\sigma)\in]0,r[$, *and for any* $\eta\in B^K_{\sigma_0}(I,r(\sigma))\cap C^K_*(I,\dot H^{\sigma+\mu}(\mathbb{T}^1,\mathbb{C}^2))$, *any* Φ *in* $C^K_*(I,E^\sigma_j)$, *one has for any* $0\le k\le K$, *any* $0\le j'\le j$, *bounds*

$$\|\partial_t^k(M(\eta)\Phi)\|_{E^{\sigma-m-\frac{3}{2}k}_{j'}} \le C_\sigma\sum_{k'+k''=k}\big(\mathcal{G}^{\sigma_0}_{k',p-1}(\eta,t)\tilde{\mathcal{G}}^{\sigma_0,j'}_{k'',1}(\Phi,t)\mathcal{G}^{\sigma+\mu}_{k',1}(\eta,t) + \mathcal{G}^{\sigma_0}_{k',p}(\eta,t)\tilde{\mathcal{G}}^{\sigma,j'}_{k'',1}(\Phi,t)\big). \tag{5.3.14}$$

(iii) We denote by $\Sigma\mathcal{N}^m_{K,j,p}[r,N]$ *the space of sums*

$$M(\eta)=\sum_{q=p}^{N-1}M_q(\eta,\dots,\eta)+M_N(\eta) \tag{5.3.15}$$

where M_q *is in* $\tilde{\mathcal{N}}^m_q$ *for* $q=p,\dots,N-1$ *and* M_N *is in* $\mathcal{N}^m_{K,j,N}[r]$.

We denote by $\Sigma\mathcal{N}_{K,j,p}[r,N]$ *the union over* $m\ge 0$ *of the preceding spaces.*

Remarks: • If M is in $\tilde{\mathcal{N}}^m_p$ then $M(\eta,\dots,\eta)$ is in $\mathcal{N}^m_{K,j,p}[r]$ for any r, any j. Indeed, the fact that condition (2.2.51) holds for elements of $\tilde{\mathcal{N}}^m_p$ implies that, if (5.3.13) does not vanish identically, then $|n_0-n_{p+1}|\le C|n'|$. Using this inequality, one checks that, if M is in $\tilde{\mathcal{N}}^m_p$, then $M(\eta,\dots,\eta)$ satisfies bounds (5.3.14) for any j if σ_0 is large enough. Actually, if the largest frequency among $n_0,\dots,n_{p+1}$ is n_0 or n_{p+1}, and if it is much larger than $n_1,\dots,n_p$, then both of them are of the same magnitude, and one will get a contribution to the left hand side of (5.3.14) bounded from above by the last term in the right hand side. On the other hand, if one among $n_1,\dots,n_p$ is larger than all other integers among $n_0,\dots,n_{p+1}$, one will get a bound by the first term in the right hand side of (5.3.14), the last factor in that expression $\mathcal{G}^{\sigma+\mu}_{0,1}(\eta_1,\dots,\eta_p)$ (with a μ possibly different from the one in (5.3.13)) coming from that special index. Consequently $M(\eta,\dots,\eta)$ is in $\mathcal{N}^m_{K,j,p}[r]$ for any r and any j.

In particular, an operator $M(\eta)$ of $\Sigma\mathcal{N}^m_{K,j,p}[r,N]$ sends a couple (η,Φ) where η is in $B^K_{\sigma_0}(I,r(\sigma))\cap C^K_*(I,\dot H^{\sigma+\mu}(\mathbb{T}^1,\mathbb{C}^2))$ and Φ in $C^K_*(I,E^\sigma_j)$ to a function $M(\eta)\Phi$ in $C^K_*(I,E^{\sigma-m}_j)$.

• If $M(\eta)$ belongs to $\mathcal{N}^0_{K,j,1}[r]$ then the series $\sum_{\ell\geq p}M(\eta)^\ell$ defines an operator of $\mathcal{N}^0_{K,j,p}[r]$. Indeed, given $M(\eta)$ of $\mathcal{N}^0_{K,j,1}[r]$, iterating estimate (5.3.14) with $p=1$, $m=0$, one gets by induction, for $k\leq K$,

$$\begin{aligned}\|\partial_t^k[M(\eta)^\ell\Phi]\|_{E_j^{\sigma-\frac{3}{2}k}}&\leq(3A_KC_\sigma)^\ell\\&\quad\times\sum_{k'+k''=k}\big(\mathcal{G}^{\sigma_0'}_{k',1}(\eta,t)^{\ell-1}\mathcal{G}^{\sigma+\mu}_{k',1}(\eta,t)\tilde{\mathcal{G}}^{\sigma_0,j}_{k'',1}(\Phi,t)\\&\qquad+\mathcal{G}^{\sigma_0'}_{k',1}(\eta,t)^{\ell}\tilde{\mathcal{G}}^{\sigma,j}_{k'',1}(\Phi,t)\big)\end{aligned}$$

for some constant A_K depending only on K,j, and some σ_0' depending on σ_0,μ. If we assume that $\mathcal{G}^{\sigma_0'}_{K,1}(\eta)=|\!|\!|\eta|\!|\!|_{K,\sigma_0'}<r'(\sigma)$ for a small enough $r'(\sigma)$, we conclude that the series $\sum_{\ell\geq p}M(\eta)^\ell$ satisfies estimates of the form (5.3.14) (with $m=0$) and thus defines an element of $\mathcal{N}^0_{K,j,p}[r]$.

Let us consider a variant of the above spaces, obtained replacing in (5.3.13), (5.3.14) the function Φ by $\frac{\cosh((z+1)D)}{\cosh D}\psi$, for some function ψ depending only on x. Since $\frac{\cosh((z+1)\xi)}{\cosh\xi}$ is in $\tilde{\mathcal{P}}^{0,+}_0$ (see (5.1.16)), lemma 5.1.9 implies that, for any j,

$$\Big\|\frac{\cosh((z+1)D)}{\cosh D}\psi\Big\|_{E^s_j}\leq C\|\psi\|_{\dot H^s}\,. \tag{5.3.16}$$

The following definition of families of operators sending functions of x to functions of (z,x) is suggested by the estimates obtained plugging (5.3.16) inside (5.3.13), (5.3.14).

Definition 5.3.4 *Let m be in $\mathbb{R}_+$, K,p in $\mathbb{N}$ with $p\leq N$, $r>0$, $j\in\mathbb{N}$.*

(i) We denote by $\tilde{\mathcal{N}}^{\partial,m}_p$ the space of symmetric p-linear maps of the form $(\eta_1,\dots,\eta_p)\to M(\eta_1,\dots,\eta_p)$, defined on $\dot H^\infty(\mathbb{T}^1,\mathbb{C}^2)^p$, with values in the space of linear maps from $\dot H^\infty(\mathbb{T}^1,\mathbb{C})$ to E^∞_∞, such that there is $\mu>0$ and for any $(\eta_1,\dots,\eta_p)$ in $\dot H^\infty(\mathbb{T}^1,\mathbb{C}^2)$, any ψ in $\dot H^\infty(\mathbb{T}^1,\mathbb{C})$, any n_0,n_{p+1} in $\mathbb{N}^*$, any $n'=(n_1,\dots,n_p)$ in $(\mathbb{N}^*)^p$, any j in $\mathbb{N}$,

$$\begin{aligned}&\|\Pi_{n_0}M(\Pi_{n_1}\eta_1,\dots,\Pi_{n_p}\eta_p)\Pi_{n_{p+1}}\psi\|_{E^0_j}\\&\qquad\leq C(n_0+|n'|+n_{p+1})^m|n'|^\mu\mathcal{G}^0_{0,p}(\eta_1,\dots,\eta_p)\mathcal{G}^0_{0,1}(\psi)\end{aligned}\tag{5.3.17}$$

and such that condition (2.2.51) holds.

(When $p=0$, the above conditions just mean that M is a linear map from $\dot H^\infty(\mathbb{T}^1,\mathbb{C})$ to E^∞_∞, satisfying estimate (5.3.17) and (2.2.51)).

(ii) We denote by $\mathcal{N}^{\partial,m}_{K,j,N}[r]$ *the space of maps* $(\eta,\psi)\to M(\eta)\psi$*, linear in* ψ*, such that there are* σ_0 *and* μ *in* $\mathbb{R}_+$ *so that, for any* $\sigma>\sigma_0$*, there is* $r(\sigma)<r$ *and, for any* η *in* $B^K_{\sigma_0}(I,r(\sigma))\cap C^K_*(I,\dot H^{\sigma+\mu}(\mathbb{T}^1,\mathbb{C}^2))$*, any* ψ *in* $C^K_*(I,\dot H^\sigma(\mathbb{T}^1,\mathbb{C}))$*, one has for any* $0\le k\le K$*, any* $0\le j'\le j$*, the bounds*

(5.3.18)
$$\|\partial_t^k(M(\eta)\psi)\|_{E^{\sigma-m-\frac{3}{2}k}_{j'}}\le C\sum_{k'+k''=k}\big(\mathcal{G}^{\sigma_0}_{k',N-1}(\eta,t)\mathcal{G}^{\sigma+\mu}_{k',1}(\eta,t)\mathcal{G}^{\sigma_0}_{k'',1}(\psi,t)+\mathcal{G}^{\sigma_0}_{k',N}(\eta,t)\mathcal{G}^{\sigma}_{k'',1}(\psi,t)\big).$$

(iii) One denotes by $\Sigma\mathcal{N}^{\partial,m}_{K,j,p}[r,N]$ *the space of sums (5.3.15) with* M_q *in* $\tilde{\mathcal{N}}^{\partial,m}_q$ *for* $q=p,\dots,N-1$*, and* M_N *in* $\mathcal{N}^{\partial,m}_{K,j,N}[r]$*.*

Remarks: • By (5.3.16), the operator $\frac{\cosh((1+z)D)}{\cosh D}$ is in $\tilde{\mathcal{N}}^{\partial,0}_0$, and, if $M(\eta)$ is in $\Sigma\mathcal{N}^m_{K,j,p}[r,N]$ then the operator

$$\psi\to M(\eta)\frac{\cosh((1+z)D)}{\cosh D}\psi$$

is an element of $\Sigma\mathcal{N}^{\partial,m}_{K,j,p}[r,N]$.

• Composition at the left of an operator of $\Sigma\mathcal{N}^{\partial,m}_{K,j,p}[r,N]$ by an operator of $\Sigma\mathcal{N}^0_{K,j,0}[r,N]$ gives an element of $\Sigma\mathcal{N}^{\partial,m}_{K,j,p}[r,N]$.

• It follows from estimates (5.3.17), (5.3.18) that if $M(\eta)$ is in the space $\Sigma\mathcal{N}^{\partial,m}_{K,j,p}[r,N]$, then, for any $0\le j_0\le j$, the operator $\partial_z^{j_0}M(\eta)$ is in $\Sigma\mathcal{N}^{\partial,m+j_0}_{K,j-j_0,p}[r,N]$.

• If one multiplies at the left an element of the space $\Sigma\mathcal{N}^m_{K,j,p}[r,N]$ (resp. $\Sigma\mathcal{N}^{\partial,m}_{K,j,p}[r,N]$) by a smooth function of $(\eta,\eta',\dots,\eta^{(\ell)})$, defined on a neighborhood of zero, that vanishes with order p' at $\eta=0$, one gets an element of the space $\Sigma\mathcal{N}^m_{K,j,p+p'}[r,N]$ (resp. $\Sigma\mathcal{N}^{\partial,m}_{K,j,p+p'}[r,N]$).

• Let $(\Phi,\Phi')\to a(\Phi,\Phi';z,x,\xi)$ be an element of $\Sigma\Gamma^m_{K,0,j,p}[\mathcal{B},r,N]$ defined in Definition 5.1.2, linear in Φ' (see the remarks after Definition 5.1.2), and $M(\eta)$ be an operator in $\Sigma\mathcal{N}^{\partial,m'}_{K,j,1}[r,N]$ for some m'. Then, for any function $\psi(x)$, the map

$$(\Phi,\psi)\to\tilde a(\Phi,\psi;z,x,\xi)=a(\Phi,M(\eta)\psi;z,x,\xi)$$

defines an element of $\Sigma\Gamma^m_{K,0,j,p+q}[\mathcal{B},r,N]$, with linear dependence in ψ.

• If M is in $\tilde{\mathcal{N}}^{\partial,m}_p$ then $M(\eta,\dots,\eta)$ is in $\mathcal{N}^{\partial,m}_{K,j,p}[r]$ for any $r>0$, any j. In particular, an operator $M(\eta)$ of $\Sigma\mathcal{N}^m_{K,j,p}[r,N]$ sends a couple (η,ψ) where η is in $B^K_{\sigma_0}(I,r(\sigma))\cap C^K_*(I,\dot H^{\sigma+\mu}(\mathbb{T}^1,\mathbb{C}^2))$ and ψ in $C^K_*(I,\dot H^\sigma(\mathbb{T}^1,\mathbb{C}))$ to a function $M(\eta)\psi$ in $C^K_*(I,E^{\sigma-m}_j)$.

• If M is an operator in $\Sigma\mathcal{N}^{\partial,m}_{K,0,q}[r,N]$ then it has the form (5.1.77) and (5.1.78) holds (with m replaced by $m+\mu$) as well as (5.1.79) (with some σ_0' large depending on K and m).

• Consider a smoothing operator $R(\eta)$ in $\Sigma\mathcal{R}^{-\rho}_{K,0,j,p}[\mathcal{B},r,N]$ (see Definition 5.1.3) and let $M(\eta)$ be in $\Sigma\mathcal{N}^{\partial,m}_{K,j,q}[r,N]$. Then the composed operator $R(\eta)\circ M(\eta)$ defines a smoothing operator of $\Sigma\mathcal{R}^{-\rho+m'}_{K,0,j,p+q}[\mathcal{B},r,N]$ for some $m'\geq m$, acting on a function $\psi(x)$. The argument is the same as in Proposition 2.4.3-(ii)-(iii). Here we adopt the abuse of notation introduced in the last remark after Definition 5.1.3 to denote these classes with the same notations.

• If $R(\Phi,\Phi')$ is a smoothing operator of $\Sigma\mathcal{R}^{-\rho}_{K,0,j,p}[\mathcal{B},r,N]$ which is linear in Φ', and $M(\eta)$ in $\Sigma\mathcal{N}^{\partial,m}_{K,j,q}[r,N]$, then $R(\Phi,M(\eta)\psi)$ is a smoothing operator of $\Sigma\mathcal{R}^{-\rho+m'}_{K,0,j,p+q}[\mathcal{B},r,N]$ for some $m'\geq m$, and using again the abuse of notation introduced in the last remark after Definition 5.1.3. The proof is the same as for Proposition 2.4.3-(iv), composing as above (5.1.10) and (5.3.17), or (5.1.11) and (5.3.18).

• Let $\tilde M(\eta)$ be an operator in $\Sigma\mathcal{N}^{\partial,m}_{K,0,1}[r,N-1]$, for some m. Then the para-product operator $Op^{\rm BW}(\tilde M(\eta)\psi)$ defines an element of $\Sigma\mathcal{N}^{\partial,\frac12}_{K,0,1}[r,N]$, linear in ψ, i.e. the nonhomogeneous part of $Op^{\rm BW}(\tilde M(\eta)\psi)\tilde\psi$ satisfies estimates of the form (5.3.18) (with $m=1/2$ and $j'=0$), where the right hand side is replaced by

$$\begin{aligned}(5.3.19)\qquad \sum_{k'+k''=k}\Big(&\mathcal{G}^{\sigma_0}_{k',N-2}(\eta,t)\mathcal{G}^{\sigma_0}_{k',1}(\psi,t)\mathcal{G}^{\sigma+\mu}_{k',1}(\eta,t)\mathcal{G}^{\sigma_0}_{k'',1}(\tilde\psi,t)\\ &+\mathcal{G}^{\sigma_0}_{k',N-1}(\eta,t)\mathcal{G}^{\sigma+\mu}_{k',1}(\psi,t)\mathcal{G}^{\sigma_0}_{k'',1}(\tilde\psi,t)\\ &+\mathcal{G}^{\sigma_0}_{k',N-1}(\eta,t)\mathcal{G}^{\sigma_0}_{k',1}(\psi,t)\mathcal{G}^{\sigma}_{k'',1}(\tilde\psi,t)\Big)\,.\end{aligned}$$

We write the reasoning just for the nonhomogeneous term $\tilde M$ in $\mathcal{N}^{\partial,m}_{K,0,N-1}[r]$. At each fixed $z\in[-1,0]$, $Op^{\rm BW}\big((\tilde M(\eta)\psi)(z;\cdot)\big)\tilde\psi$ is the para-product of the function $(\tilde M(\eta)\psi)(z;\cdot)$ by $\tilde\psi$. Consequently, for any σ,

$$\begin{aligned}(5.3.20)\qquad &\|\partial_t^k Op^{\rm BW}\big((\tilde M(\eta)\psi)(z;\cdot)\big)\tilde\psi\|_{\dot H^{\sigma-\frac32 k}}\leq\\ &\qquad C\sum_{k_1+k_2=k}\|\partial_t^{k_1}(\tilde M(\eta)\psi)(z;\cdot)\|_{\dot H^1}\|\partial_t^{k_2}\tilde\psi\|_{\dot H^{\sigma-\frac32 k_2}}\,.\end{aligned}$$

By (5.3.18) (with $N-1$ replaced by $N-2$ and N by $N-1$), we may find some σ_0' large enough (depending on σ_0, m and K) such that, for any z in $[-1,0]$, any $0\leq k_1\leq K$,

$$\|\partial_t^{k_1}(\tilde M(\eta)\psi)(z;\cdot)\|_{\dot H^1}\leq C\sum_{k'+k''=k_1}\mathcal{G}^{\sigma_0'}_{k',N-1}(\eta,t)\mathcal{G}^{\sigma_0'}_{k'',1}(\psi,t)\,.$$

As a consequence, (5.3.20) is bounded by

$$C\sum_{k_1+k_2=k}\mathcal{G}^{\sigma_0'}_{k_1,N-1}(\eta,t)\mathcal{G}^{\sigma_0'}_{k_1,1}(\psi,t)\mathcal{G}^{\sigma}_{k_2,1}(\tilde\psi,t)\,,$$

uniformly in z. Therefore $\|\partial_t^k Op^{\rm BW}(\tilde M(\eta)\psi)\tilde\psi\|_{F_0^{\sigma-\frac32 k}}$ is bounded by the right hand side of (5.3.19) (with σ_0' instead of σ_0). Recalling the continuous embedding of $F_0^{\sigma-\frac32 k}$ into $E_0^{\sigma-\frac12-\frac32 k}$ (see the first remark after Definition 5.1.6) we have proved that $Op^{\rm BW}(\tilde M(\eta)\psi)\tilde\psi$ satisfies an estimate of the form (5.3.18) with $m=1/2$ and $j'=0$, as claimed.

Proposition 5.3.5 *Let K be in $\mathbb{N}$, N in $\mathbb{N}^*$, $m\geq 2$, $r>0$, j in $\mathbb{N}$. Consider an operator $\tilde M(\eta)$ of $\Sigma\mathcal{N}^{\partial,m}_{K,j,1}[r,N]$, and operators $G_\ell(\eta)$ of $\Sigma\mathcal{N}^{2-\ell}_{K,j,1}[r,N]$, $\ell=0,1,2$. Assume also that we are given a function $\tilde S_N(\eta,\psi)$ of $(z,x)\in\mathcal{B}$, supported for $z\in[-1,-\frac14]$ and such that, for any large enough σ_0, for any j, there is some $\sigma_0'>0$ with, for all $k=0,\dots,K$,*

$$\sum_{k'=0}^{k}\|\partial_t^{k'}\tilde S_N(\eta,\psi)\|_{E_j^{j+\sigma_0-\frac32 k'}}\leq C\sum_{k'+k''=k}\mathcal{G}^{\sigma_0'}_{k',N}(\eta,t)\mathcal{G}^{\sigma_0'}_{k'',1}(\psi,t). \tag{5.3.21}$$

Then, eventually increasing σ_0, for any $\sigma>\sigma_0$, there is $r(\sigma)<r$, such that for η in $B^K_{\sigma_0}(I,r(\sigma))\cap C^K_*(I,\dot H^{\sigma+\mu}(\mathbb{T}^1,\mathbb{C}^2))$, for ψ in $C^K_*(I,\dot H^{\sigma}(\mathbb{T}^1,\mathbb{C}))$, the elliptic problem

$$\begin{aligned}(\partial_x^2+\partial_z^2)\tilde\Phi&=G_0(\eta)\tilde\Phi+\partial_z[G_1(\eta)\tilde\Phi]+\partial_z^2[G_2(\eta)\tilde\Phi]+\tilde M(\eta)\psi+\tilde S_N\\ \tilde\Phi|_{z=0}&=\psi\\ \partial_z\tilde\Phi|_{z=-1}&=0\end{aligned} \tag{5.3.22}$$

has a unique solution $\tilde\Phi$ in $C^K_*(I,E_j^{\sigma-m+2})$. If $\tilde M\equiv 0$ and $\tilde S_N\equiv 0$, then $\tilde\Phi$ is in $C^K_*(I,E_j^\sigma)$. Moreover $\tilde\Phi$ may be written as

$$\tilde\Phi=\frac{\cosh((1+z)D)}{\cosh D}\psi+M(\eta)\psi+S_N(\eta,\psi) \tag{5.3.23}$$

where $M(\eta)$ is some operator in $\Sigma\mathcal{N}^{\partial,m-2}_{K,j,1}[r,N]$ and the function S_N is supported for $-1\leq z\leq-\frac18$ and satisfies estimates (5.3.21) with $E_j^{j+\sigma_0-\frac32 k}$ replaced by $E_j^{j+2+\sigma_0-\frac32 k}$ in the left hand side. Moreover, if $\tilde S_N\equiv 0$ in (5.3.22), then $S_N\equiv 0$ in (5.3.23) and if $\tilde M\equiv 0$, then $M(\eta)$ is in $\Sigma\mathcal{N}^{\partial,0}_{K,j,1}[r,N]$.

In order to prove the proposition we need the following lemma.

Lemma 5.3.6 *Let $G_\ell(\eta)$, $\ell=0,1,2$, be operators in $\Sigma\mathcal{N}^{2-\ell}_{K,j,1}[r,N]$, and η a function as in the proposition. Consider the operator*

$$M_0(\eta)\Phi=\int_{-1}^0 K_0(z,z',D_x)\big[\partial_{z'}^2[G_2(\eta)\Phi]+\partial_{z'}[G_1(\eta)\Phi]+G_0(\eta)\Phi\big]\,dz' \tag{5.3.24}$$

where $K_0(z,z',D_x)$ is defined in (5.2.3), acting on functions Φ satisfying

$$\Phi|_{z=0}=0\,,\quad \partial_z\Phi|_{z=-1}=0\,. \tag{5.3.25}$$

Then $M_0(\eta)$ is an operator of $\Sigma\mathcal{N}^0_{K,j,1}[r,N]$. Moreover we may express $M_0(\eta)$ from a distributional kernel $\tilde{K}_1(\eta;z,z')$ (with values in operators) by

$$M_0(\eta)\Phi = \int_{-1}^{0} \tilde{K}_1(\eta;z,z')\Phi(z',\cdot)\,dz' \tag{5.3.26}$$

and, more generally, for any $n \in \mathbb{N}$

$$M_0(\eta)^n\Phi = \int_{-1}^{0} \tilde{K}_n(\eta;z,z')\Phi(z',\cdot)\,dz'\,, \tag{5.3.27}$$

where $\tilde{K}_n(\eta;z,z')$, $n \geq 2$, are defined iteratively by

$$\tilde{K}_{n+1}(\eta;z,z') = \int_{-1}^{0} \tilde{K}_n(\eta;z,z'')\tilde{K}_1(\eta;z'',z')\,dz'' \tag{5.3.28}$$

and the following estimates hold: There are $\sigma_0 > 0$, $\mu > 0$, $\sigma_1 > 0$ with $\sigma_0 \geq \sigma_1 + \mu$, and for every ℓ in $\mathbb{N}$, any $\sigma \geq \sigma_0$, a constant $C_{\ell,\sigma} > 0$ such that for any n in $\mathbb{N}^*$, for any $0 \leq \ell' \leq \ell$, for $0 \leq k \leq K$,

$$\begin{aligned}\sum_{k'=0}^{k} \Big\| \partial_t^{k'} \int_{-1}^{0} \tilde{K}_n(\eta;z,z')(z-z')^{\ell}\Phi(z',\cdot)\,dz' \Big\|_{E_j^{\sigma+\ell'-\frac{3}{2}k'}} \\ \leq C^n_{\ell,\sigma}\mathcal{G}^{\sigma_0}_{k,n-1}(\eta,t) \sum_{k'+k''=k} \big[\mathcal{G}^{\sigma+\ell'+\mu}_{k',1}(\eta,t)\tilde{\mathcal{G}}^{\sigma_1,j}_{k'',1}(\Phi,t) \\ + \mathcal{G}^{\sigma_0}_{k',1}(\eta,t)\tilde{\mathcal{G}}^{\sigma,j}_{k'',1}(\Phi,t)\big].\end{aligned} \tag{5.3.29}$$

Proof: Since Φ satisfies the boundary conditions (5.3.25), by (5.3.9) and (5.3.11) we may write

$$\begin{aligned}M_0(\eta)\Phi = G_2(\eta)\Phi(z,\cdot) - \mathcal{S}(z,D_x)[G_1(\eta)\Phi(-1,\cdot)] \\ + \int_{-1}^{0} K_0(z,z',D_x)[D_x^2 G_2(\eta)\Phi(z',\cdot) + G_0(\eta)\Phi(z',\cdot)]\,dz' \\ - \int_{-1}^{0} \partial_{z'}K_0(z,z',D_x)[G_1(\eta)\Phi(z',\cdot)]\,dz'.\end{aligned} \tag{5.3.30}$$

Since K_0 is a Poisson symbol in $\tilde{\mathcal{P}}_0^{-1,\mathrm{int}}$, lemma 5.1.8 implies that the operator

$$U \to \int_{-1}^{0} K_0(z,z',D_x)U(z')\,dz' \quad \text{maps} \quad E_j^s \to E_j^{s+2}\,, \tag{5.3.31}$$

and, similarly, since $\partial_{z'}K_0$ is in $\tilde{\mathcal{P}}_0^{0,\mathrm{int}}$, $U \to \int_{-1}^{0} \partial_{z'}K_0(z,z',D_x)U(z')\,dz'$ acts from E_j^s to E_j^{s+1}. It follows that (5.3.30) defines an operator of $\Sigma\mathcal{N}^0_{K,j,1}[r,N]$.

Remark that (5.3.30) provides an integral expression of the form (5.3.26), with the operatorial valued kernel

$$\begin{aligned}(5.3.32)\quad \tilde K_1(\eta;z,z') &= \delta(z-z')G_2(\eta)+\delta(z'+1)\mathcal{S}(z,D_x)\circ G_1(\eta)\\ &+K_0(z,z',D_x)\circ[D_x^2G_2(\eta)+G_0(\eta)]-\partial_{z'}K_0(z,z',D_x)\circ G_1(\eta)\,.\end{aligned}$$

We claim that $\tilde K_1(\eta;z,z')$ satisfies for any $0\le k\le K$, $0\le\ell'\le\ell$, some $\mu>0$, some $\sigma_1>0$ and any $\sigma>\sigma_1$, the estimate

$$\begin{aligned}(5.3.33)\quad &\sum_{k'=0}^k\Big\|\partial_t^{k'}\int_{-1}^0\tilde K_1(\eta;z,z')(z-z')^\ell\Phi(z',\cdot)\,dz'\Big\|_{E_j^{\sigma+\ell'-\frac32k'}}\\ &\le B_{\ell+\sigma}\sum_{k'+k''=k}[\mathcal{G}^{\sigma+\ell'+\mu}_{k',1}(\eta,t)\tilde{\mathcal{G}}^{\sigma_1,j}_{k'',1}(\Phi,t)+\mathcal{G}^{\sigma_1}_{k',1}(\eta,t)\tilde{\mathcal{G}}^{\sigma,j}_{k'',1}(\Phi,t)]\end{aligned}$$

for some increasing sequence of constants $B_{\sigma+\ell}$. Actually, since $G_2(\eta)$ is in $\Sigma\mathcal{N}^0_{K,j,1}[r,N]$, the kernel $\delta(z-z')G_2(\eta)$ satisfies (5.3.33) when $\ell=0$, as follows by (5.3.14) where we replace σ_0 by σ_1. The estimate (5.3.33) when $\ell>0$ is trivial. Also $\delta(z'+1)\mathcal{S}(z,D_x)G_1(\eta)$ satisfies (5.3.33): Actually since $\mathcal{S}(z,\xi)$ is in $\tilde{\mathcal{P}}_0^{-1,-}$, the symbol $(z+1)^\ell\mathcal{S}(z,\xi)$ is in $\tilde{\mathcal{P}}_0^{-1-\ell,-}$ (see remarks after Definition 5.1.4), lemma 5.1.9 implies that the operator $(z+1)^\ell\mathcal{S}(z,D_x)$ sends $\dot H^{\sigma-1}$ to $E_\infty^{\sigma+\ell}$. Using that $G_1(\eta)$ satisfies (5.3.14) with $m=1$, we deduce that $\delta(z'+1)\mathcal{S}(z,D_x)G_1(\eta)$ satisfies (5.3.33) (with $\ell'=0$).

Moreover, since K_0 is a Poisson symbol in $\tilde{\mathcal{P}}_0^{-1,\mathrm{int}}$, then $(z-z')^\ell K_0$ is in $\tilde{\mathcal{P}}_0^{-1-\ell,\mathrm{int}}$ (see remarks after Definition 5.1.4), and lemma 5.1.8 implies that the operator

$$U\to\int_{-1}^0(z-z')^\ell K_0(z,z',D_x)U(z',\cdot)\,dz'$$

acts from E_j^σ to $E_j^{\sigma+\ell+2}$. Similarly, since $\partial_{z'}K_0$ is in $\tilde{\mathcal{P}}_0^{0,\mathrm{int}}$, the operator $U\to\int_{-1}^0(z-z')^\ell\partial_{z'}K_0(z,z',D_x)U(z',\cdot)\,dz'$ acts from E_j^σ to $E_j^{\sigma+\ell+1}$. Using that $G_2(\eta)$ is in $\Sigma\mathcal{N}^0_{K,j,1}[r,N]$ and $G_1(\eta)$ is in $\Sigma\mathcal{N}^1_{K,j,1}[r,N]$, we deduce from (5.3.14) that the two integral terms in (5.3.30) may be written under the form (5.3.26), with a kernel given by the last line in (5.3.32), satisfying the estimate (5.3.33).

The estimate (5.3.29) for $n=1$ follows by (5.3.33), setting

$$(5.3.34)\qquad C_{\ell,\sigma}=3^{\ell+1}(K+1)B_{\sigma+\ell}$$

and $\sigma_0=\sigma_1+\mu$ where σ_1 is the index in (5.3.33).

The expression (5.3.27) of $M_0(\eta)^n\Phi$ for $n\ge2$ follows by the formula (5.3.26) of $M_0(\eta)$ and the iterative definition (5.3.28) of $\tilde K_n$.

Let us prove by induction (5.3.29) for any n. Write

$$\begin{aligned} A^\ell_{n+1} &\stackrel{\text{def}}{=} \int_{-1}^0 \tilde K_{n+1}(\eta;z,z')(z-z')^\ell \Phi(z',\cdot)\,dz' \\ &= \sum_{0\le \ell_1\le \ell} \binom{\ell}{\ell_1} \int_{-1}^0 \tilde K_n(\eta;z,z'')(z-z'')^{\ell-\ell_1} \\ &\qquad \times \Big(\int_{-1}^0 \tilde K_1(\eta;z'',z')(z''-z')^{\ell_1}\Phi(z',\cdot)\,dz'\Big) dz''. \end{aligned}$$

We now apply the inductive estimate (5.3.29) to the first integral kernel. For any $\ell' \le \ell$, we decompose $\ell' = (\ell'-\ell'_1)+\ell'_1$ with $\ell'_1 \le \ell_1$, $\ell'-\ell'_1 \le \ell-\ell_1$ and we have apply (5.3.29) with σ (resp. ℓ') replaced by $\sigma+\ell'_1$ (resp. $\ell'-\ell'_1$), yielding

$$\begin{aligned} (5.3.35)\quad \sum_{k'=0}^k \|\partial_t^{k'} A^\ell_{n+1}\|_{E_j^{\sigma+\ell'-\frac32 k'}} &\le \sum_{\ell_1\le\ell} \binom{\ell}{\ell_1} \sum_{k'+k''=k} \mathcal{G}^{\sigma_0}_{k',1}(\eta,t)^{n-1} \\ &\times C^n_{\ell'-\ell'_1,\sigma+\ell'_1}\Big[\mathcal{G}^{\sigma+\ell'+\mu}_{k',1}(\eta,t)\tilde{\mathcal{G}}^{\sigma_1,j}_{k'',1}(A^{\ell_1}_1,t) + \mathcal{G}^{\sigma_0}_{k',1}(\eta,t)\tilde{\mathcal{G}}^{\sigma+\ell'_1,j}_{k'',1}(A^{\ell_1}_1,t)\Big]. \end{aligned}$$

Next, according to (5.3.33) with σ replaced by σ_1, $\ell'=0$, $\ell=\ell_1$, and since $\sigma_0 = \sigma_1+\mu$, we bound

$$\begin{aligned} \tilde{\mathcal{G}}^{\sigma_1,j}_{k'',1}(A^{\ell_1}_1,t) &= \sum_{k'''=0}^{k''} \|\partial_t^{k'''} A^{\ell_1}_1\|_{E_j^{\sigma_1-\frac32 k'''}} \\ &\le 2B_{\sigma_1+\ell_1} \sum_{k''_1+k''_2=k''} \mathcal{G}^{\sigma_0}_{k''_1,1}(\eta,t)\tilde{\mathcal{G}}^{\sigma_1,j}_{k''_2,1}(\Phi,t) \end{aligned}$$

and according to (5.3.33) with ℓ' replaced by ℓ'_1 and $\ell=\ell_1$, we get

$$\begin{aligned} \tilde{\mathcal{G}}^{\sigma+\ell'_1,j}_{k'',1}(A^{\ell_1}_1,t) &= \sum_{k'''=0}^{k''} \|\partial_t^{k'''} A^{\ell_1}_1\|_{E_j^{\sigma+\ell'_1-\frac32 k'''}} \\ &\le B_{\sigma+\ell_1} \sum_{k''_1+k''_2=k''} \Big[\mathcal{G}^{\sigma+\ell'_1+\mu}_{k''_1,1}(\eta,t)\tilde{\mathcal{G}}^{\sigma_1,j}_{k''_2,1}(\Phi,t) \\ &\qquad + \mathcal{G}^{\sigma_1}_{k''_1,1}(\eta,t)\tilde{\mathcal{G}}^{\sigma,j}_{k''_2,1}(\Phi,t)\Big]. \end{aligned}$$

Plugging these estimates inside (5.3.35) and using the definition of $C_{\ell,\sigma}$ in (5.3.34), we obtain that, for any $0\le k\le K$, the left hand side of (5.3.35) is bounded by

$$\begin{aligned} &2^\ell(3^{\ell+1}(K+1)B_{\sigma+\ell})^n(K+1)B_{\sigma+\ell} \sum_{k'+k''=k} \Big[3\mathcal{G}^{\sigma+\ell'+\mu}_{k',1}(\eta,t)\tilde{\mathcal{G}}^{\sigma_1,j}_{k'',1}(\Phi,t) \\ &\qquad + \mathcal{G}^{\sigma_0}_{k',1}(\eta,t)\tilde{\mathcal{G}}^{\sigma,j}_{k'',1}(\Phi,t)\Big]\mathcal{G}^{\sigma_0}_{k,1}(\eta,t)^n \end{aligned}$$

which implies (5.3.29) at rank $n+1$. This concludes the proof of the lemma. □

Proof of Proposition 5.3.5: Since $\frac{\cosh((1+z)D)}{\cosh D}\psi$ is the solution of

$$(\partial_x^2+\partial_z^2)\Phi=0\,,\quad \Phi|_{z=0}=\psi\,,\quad \partial_z\Phi|_{z=-1}=0\,,$$

the solution $\tilde\Phi$ of the problem (5.3.22) may be written as

$$\tilde\Phi=\frac{\cosh((1+z)D)}{\cosh D}\psi+\underline{\tilde\Phi}$$

where $\underline{\tilde\Phi}$ satisfies

$$\begin{aligned}(\partial_x^2+\partial_z^2)\underline{\tilde\Phi}&=G_0(\eta)\underline{\tilde\Phi}+\partial_z[G_1(\eta)\underline{\tilde\Phi}]+\partial_z^2[G_2(\eta)\underline{\tilde\Phi}]+\underline{\tilde M}(\eta)\psi+\tilde S_N\\ \underline{\tilde\Phi}|_{z=0}&=0\\ \partial_z\underline{\tilde\Phi}|_{z=-1}&=0\end{aligned}\tag{5.3.36}$$

where

$$\begin{aligned}\underline{\tilde M}(\eta)\psi\stackrel{\rm def}{=}G_0(\eta)\frac{\cosh((1+z)D)}{\cosh D}\psi+\partial_z\left[G_1(\eta)\frac{\cosh((1+z)D)}{\cosh D}\psi\right]\\ +\partial_z^2\left[G_2(\eta)\frac{\cosh((1+z)D)}{\cosh D}\psi\right]+\tilde M(\eta)\psi.\end{aligned}$$

By assumption $\tilde M(\eta)$ is an operator in $\Sigma\mathcal{N}^{\partial,m}_{K,j,1}[r,N]$ for some $m\geq 2$. Since $G_\ell(\eta)$ are operators in $\Sigma\mathcal{N}^{2-\ell}_{K,j,1}[r,N]$, $\ell=0,1,2$, the first and the third remarks following Definition 5.3.4, imply that $\underline{\tilde M}(\eta)$ is in $\Sigma\mathcal{N}^{\partial,m}_{K,j-2,1}[r,N]$. Notice that, if $\tilde M(\eta)=0$, then $\underline{\tilde M}(\eta)$ is in $\Sigma\mathcal{N}^{\partial,2}_{K,j-2,1}[r,N]$.

Notice also that since $\frac{\cosh((1+z)\xi)}{\cosh\xi}$ is in $\tilde{\mathcal{P}}_0^{0,+}$ (see (5.1.16)) and the function $\psi\in C^K_*(I,\dot H^\sigma(\mathbb{T}^1,\mathbb{C}))$, then, lemma 5.1.9 implies that $\frac{\cosh((1+z)D)}{\cosh D}\psi$ is in $C^K_*(I,E^\sigma_j)$, for any j.

By lemma 5.2.2 applied with $\eta=0$, $g_+=g_-=0$, and

$$f=G_0(\eta)\underline{\tilde\Phi}+\partial_z[G_1(\eta)\underline{\tilde\Phi}]+\partial_z^2[G_2(\eta)\underline{\tilde\Phi}]+\underline{\tilde M}(\eta)\psi+\tilde S_N\,,$$

the function $\underline{\tilde\Phi}$ solves (5.3.36) if and only if

$$(\mathrm{Id}-M_0(\eta))\underline{\tilde\Phi}=F\stackrel{\rm def}{=}\int_{-1}^0 K_0(z,z',D_x)[\underline{\tilde M}(\eta)\psi+\tilde S_N(\eta,\psi)]\,dz'$$

where $M_0(\eta)\underline{\tilde\Phi}$ is defined in (5.3.24) and $K_0(z,z',D_x)$ in (5.2.3). Notice that F satisfies the boundary conditions (5.3.25). Write

$$F=\tilde M_1(\eta)\psi+\int_{-1}^0 K_0(z,z',D_x)\tilde S_N(\eta,\psi)(z')\,dz'\tag{5.3.37}$$

where

$$\tilde{M}_1(\eta)\psi = \int_{-1}^{0} K_0(z,z',D_x)\underline{\tilde{M}}(\eta)\psi\, dz'. \tag{5.3.38}$$

We claim that, even if $\underline{\tilde{M}}(\eta)$ is in $\Sigma\mathcal{N}^{\partial,m}_{K,j-2,1}[r,N]$, the operator $\tilde{M}_1(\eta)$ belongs to $\Sigma\mathcal{N}^{\partial,m-2}_{K,j,1}[r,N]$. Actually, this follows from the fact that, by a direct computation, recalling (5.2.3),

$$\partial_z \int_{-1}^{0} K_0(z,z',D_x)U(z')\,dz' = \int_{-1}^{0} K_0'(z,z',D_x)D_xU(z')\,dz'$$

with a symbol K_0' in $\tilde{\mathcal{P}}_0^{-1,\mathrm{int}}$, and, according to (5.2.9),

$$\partial_z^2 \int_{-1}^{0} K_0(z,z',D_x)U(z')\,dz' = \int_{-1}^{0} K_0(z,z',D_x)D_x^2U(z')\,dz' + U(z)\,.$$

As a consequence, to estimate ∂_z^j-derivatives of $\tilde{M}_1(\eta)\psi$ in (5.3.38), we need only ∂_z^{j-2}-derivatives of $\underline{\tilde{M}}(\eta)\psi$. Recalling (5.3.31) and Definition 5.3.4 we deduce that $\tilde{M}_1(\eta)$ belongs to $\Sigma\mathcal{N}^{\partial,m-2}_{K,j,1}[r,N]$. If $\tilde{M}(\eta)=0$, then $\underline{\tilde{M}}(\eta)$ is in $\Sigma\mathcal{N}^{\partial,2}_{K,j-2,1}[r,N]$, and $\tilde{M}_1(\eta)$ is in $\Sigma\mathcal{N}^{\partial,0}_{K,j,1}[r,N]$.

We shall solve the equation $(\mathrm{Id}-M_0(\eta))\underline{\tilde{\Phi}} = F$ by the Neumann series

$$\underline{\tilde{\Phi}} = \sum_{n=0}^{\infty} M_0(\eta)^n F\,. \tag{5.3.39}$$

Since the operator $M_0(\eta)$ is in $\Sigma\mathcal{N}^{0}_{K,j,1}[r,N]$ as stated in lemma 5.3.6, the second remark after Definition 5.3.3 implies that the Neumann series $\sum_{n=0}^{\infty} M_0(\eta)^n$ defines an operator of $\Sigma\mathcal{N}^{0}_{K,j,0}[r,N]$. By the second remark after Definition 5.3.4, it follows that the composition

$$\Big(\sum_{n=0}^{\infty} M_0(\eta)^n\Big)\circ \tilde{M}_1(\eta)$$

with the operator $\tilde{M}_1(\eta)\in\Sigma\mathcal{N}^{\partial,m-2}_{K,j,1}[r,N]$ in (5.3.37)-(5.3.38), gives an operator $M(\eta)$ in $\Sigma\mathcal{N}^{\partial,m-2}_{K,j,1}[r,N]$ as in (5.3.23). By the sixth remark after Definition 5.3.4 the function $M(\eta)\psi$ is in $C^K_*(I,E_j^{\sigma-m+2})$. If $\tilde{M}\equiv 0$, then $\tilde{M}_1(\eta)$ is in $\Sigma\mathcal{N}^{\partial,0}_{K,j,1}[r,N]$, and the operator $M(\eta)$ is in $\Sigma\mathcal{N}^{0}_{K,j,1}[r,N]$, thus $M(\eta)\psi$ is in $C^K_*(I,E_j^{\sigma})$. Summarizing, if $\tilde{S}_N\equiv 0$, we have proved (5.3.23) with $S_N\equiv 0$.

We are left with (5.3.39) where we replace F by the integral term in (5.3.37), that is

$$\sum_{n=0}^{\infty} M_0(\eta)^n\Big(\int_{-1}^{0} K_0(z,z',D_x)\tilde{S}_N(\eta,\psi)(z')\,dz'\Big)\,. \tag{5.3.40}$$

By (5.3.27) we may write

$$M_0(\eta)^n\Phi = \int_{-1}^{0} \tilde{K}_n(\eta; z, z')\Phi(z', \cdot)\, dz'$$

so that, if we define $\tilde{K}(\eta; z, z') = \sum_{n\geq 0} \tilde{K}_n(\eta; z, z')$, we have

$$(\mathrm{Id} - M_0(\eta))^{-1}\Phi = \sum_{n=0}^{+\infty} M_0(\eta)^n\Phi = \int_{-1}^{0} \tilde{K}(\eta; z, z')\Phi(z', \cdot)\, dz', \tag{5.3.41}$$

the convergence of $\sum_{n\geq 0} \tilde{K}_n$ following from (5.3.29), if we assume that η is in $B^K_{\sigma_0}(I, r(\sigma, \ell))$, for a small enough $r(\sigma, \ell)$. For such η, we get therefore

$$\begin{aligned}\sum_{k'=0}^{k} \Big\| \partial_t^{k'} \int_{-1}^{0} (z - z')^\ell \tilde{K}(\eta; z, z')\Phi(z')\, dz' \Big\|_{E_j^{\sigma+\ell-\frac{3}{2}k'}} \\ \leq C'_{\ell,\sigma} \sum_{k'+k''=k} \big[\mathcal{G}^{\sigma+\ell+\mu}_{k',1}(\eta, t)\tilde{\mathcal{G}}^{\sigma_0,j}_{k'',1}(\Phi, t) \\ + \mathcal{G}^{\sigma_0}_{k',1}(\eta, t)\tilde{\mathcal{G}}^{\sigma,j}_{k'',1}(\Phi, t)\big].\end{aligned} \tag{5.3.42}$$

By (5.3.41), we decompose the term (5.3.40) as $I + II$ where

(5.3.43)

$$\begin{aligned} I &= \theta(z) \int_{-1}^{0}\int_{-1}^{0} \tilde{K}(\eta; z, z'')K_0(z'', z', D)\tilde{S}_N(\eta, \psi)(z', \cdot)\, dz'dz'' \\ II &= (1 - \theta(z)) \int_{-1}^{0}\int_{-1}^{0} \tilde{K}(\eta; z, z'')K_0(z'', z', D)\tilde{S}_N(\eta, \psi)(z', \cdot)\, dz'dz'' \end{aligned}$$

for $\theta \in C_0^\infty(\mathbb{R})$, equal to one on $[-\frac{1}{8}, 0]$ and vanishing close to $[-1, -\frac{1}{4}]$. Expression II is supported for $-1 \leq z \leq -1/8$ and may be written as

$$(1 - \theta)(z) \int_{-1}^{0} \tilde{K}'(\eta; z, z')\tilde{S}_N(\eta, \psi)(z', \cdot)\, dz' \,,$$

with the new kernel

$$\tilde{K}'(\eta; z, z') = \int_{-1}^{0} \tilde{K}(\eta; z, z'')K_0(z'', z', D)dz''$$

that satisfies (5.3.42) with $E_j^{\sigma+\ell-\frac{3}{2}k'}$ replaced by $E_j^{\sigma+2+\ell-\frac{3}{2}k'}$ in the left hand side (as integration against K_0 gains two derivatives). Applying (5.3.42) with $\ell = 0$ and $\sigma = j + \sigma_0$, we obtain that

$$\begin{aligned}\sum_{k'=0}^{k} \|\partial_t^{k'} II(t, \cdot)\|_{E_j^{j+\sigma_0+2-\frac{3}{2}k'}} \leq C \sum_{k'+k''=k} \big(\mathcal{G}^{j+\sigma_0+\mu}_{k',1}(\eta, t)\tilde{\mathcal{G}}^{\sigma_0,j}_{k'',1}(\tilde{S}_N, t) \\ + \mathcal{G}^{\sigma_0}_{k',1}(\eta, t)\tilde{\mathcal{G}}^{j+\sigma_0,j}_{k'',1}(\tilde{S}_N, t)\big).\end{aligned}$$

If we bound $\tilde{\mathcal{G}}^{\sigma_0,j}_{k,1}(\tilde{S}_N,t)$, $\tilde{\mathcal{G}}^{j+\sigma_0,j}_{k,1}(\tilde{S}_N,t)$ by the right hand side of (5.3.21) (recall (5.1.29)), we obtain that II satisfies a similar estimate for a convenient σ_0' (depending on j,μ). Thus II contributes to the term S_N in the right hand side of (5.3.23).

Finally we consider the term I in (5.3.43)

$$I=\int_{-1}^{0}\theta(z)\tilde{K}'(\eta;z,z')\tilde{S}_N(\eta,\psi)(z',\cdot)\,dz'\,.$$

Using that the support of θ and the support of $\tilde{S}_N(z';\cdot)$ stay at positive distance (recall that $\mathrm{Supp}\,(\theta)\subset]-1/4,0]$ and $\mathrm{Supp}\,(\tilde{S}_N)\subset[-1,-1/4]$), the integral I may be written as

$$\int_{-1}^{0}(z-z')^{\ell}\tilde{K}'(\eta;,z,z')\omega(z,z')\tilde{S}_N(z',\cdot)\,dz' \tag{5.3.44}$$

for a suitable function $\omega(z,z')$ in $C_0^\infty([-1,0]\times[-1,0])$ that vanishes close to $z=z'$. By the density of $C_0^\infty(X)\otimes C_0^\infty(Y)$ in $C_0^\infty(X\times Y)$ we write $\omega(z,z')$ as a rapidly convergent series $\sum_q\omega_q^1(z)\omega_q^2(z')$, with ω_q^j in $C_0^\infty(\mathbb{R})$. This allows us to apply the estimates (5.3.42) for $(z-z')^\ell\tilde{K}'$ (with $\sigma=\sigma_0$) to $(z-z')^\ell\tilde{K}'\omega(z,z')$ in (5.3.44). In conclusion we obtain

$$\sum_{k'=0}^{k}\|\partial_t^{k'}I(t,\cdot)\|_{E_j^{\sigma_0+\ell-\frac{3}{2}k'}}\le C\sum_{k'+k''=k}\mathcal{G}^{\sigma_0+\ell+\mu}_{k',1}(\eta,t)\tilde{\mathcal{G}}^{\sigma_0,j}_{k'',1}(\tilde{S}_N,t).$$

Plugging (5.3.21) in the right hand side of the above inequality, and choosing ℓ so that $\sigma-m+2\le\sigma_0+\ell\le\sigma-m+1$, we see that we obtain an estimate of the form (5.3.18), with σ_0 replaced by some convenient σ_0' and $m-2$ instead of m. Consequently, we have shown that I is in $C_*^K(I,E_j^{\sigma-m+2})$ and it contributes to the term $M(\eta)\psi$ in the right hand side of (5.3.23) (actually to its non-homogeneous term in $\mathcal{N}^{\partial,m-2}_{K,j,N}[r]$). This concludes the proof. □

Chapter 6

Dirichlet-Neumann operator and the good unknown

6.1 The good unknown

We introduce here the good unknown ω in terms of which we shall express the water waves equations. We follow mainly the approach of Alazard-Métivier [7], Alazard-Burq-Zuily [2, 3].

Recall that, given an integer K, a real number σ large enough, a function η of $C^K_*(I, \dot{H}^\sigma(\mathbb{T}^1, \mathbb{R}))$ with $\|\eta(t,\cdot)\|_{L^\infty} < 1$, we have defined in (1.1.1) the open set

$$\Omega_t = \big\{(x,y) \in \mathbb{T}^1 \times \mathbb{R}\,;\ -1 < y < \eta(t,x)\big\}$$

where we take the depth $h = 1$. In the sequel we identify $C^K_*(I, \dot{H}^\sigma(\mathbb{T}^1, \mathbb{R}))$ with the subspace of $C^K_{*\mathbb{R}}(I, \dot{H}^\sigma(\mathbb{T}^1, \mathbb{C}^2))$ of vector valued functions of the form $\begin{bmatrix} \eta \\ \eta \end{bmatrix}$ with η real. We shall always assume below that η belongs to $B^K_{\sigma_0}(I, r(\sigma_0))$ for some given σ_0 large and $r(\sigma_0) > 0$ small.

Let ψ be in $C^K_*(I, \dot{H}^\sigma(\mathbb{T}^1, \mathbb{R}))$ and consider the elliptic system (at any fixed time $t \in I$)

$$\begin{aligned} &\Delta \underline{\phi}(x,y) = 0 \ \text{ in } \Omega_t \\ &\underline{\phi}|_{y=\eta(t,x)} = \psi(t,x) \\ &\frac{\partial \underline{\phi}}{\partial y}|_{y=-1} = 0\,. \end{aligned} \tag{6.1.1}$$

Define new coordinates $(x, \tilde{z})$ by $y = \eta(t,x) + \tilde{z}(1+\eta(t,x))$ so that, in the new coordinates system, the closure of Ω_t becomes the strip

$$\mathcal{B} = \big\{(x,\tilde{z})\,;\, x \in \mathbb{T}^1, -1 \le \tilde{z} \le 0\big\}\,.$$

We consider the function $\underline{\varphi}(x,\tilde{z})$ defined for $(x,\tilde{z}) \in \mathcal{B}$ by

$$\underline{\varphi}(x,\tilde{z}) \stackrel{\text{def}}{=} \underline{\phi}(x, \eta(t,x) + \tilde{z}(1+\eta(t,x)))\,. \tag{6.1.2}$$

A direct calculus shows that $\underline{\varphi}$ solves the elliptic problem

$$\begin{aligned} (\partial_x^2+\partial_{\tilde z}^2)\underline{\varphi} &= \partial_{\tilde z}^2[G_2(\eta)\underline{\varphi}]+\partial_{\tilde z}[G_1(\eta)\underline{\varphi}]+G_0(\eta)\underline{\varphi} \\ \underline{\varphi}|_{\tilde z=0} &= \psi(t,x) \\ \partial_{\tilde z}\underline{\varphi}|_{\tilde z=-1} &= 0 \end{aligned} \tag{6.1.3}$$

with

$$\begin{aligned} G_2(\eta)\underline{\varphi} &= -(1+\tilde z)^2\eta'^2\underline{\varphi} \\ G_1(\eta)\underline{\varphi} &= 2(1+\tilde z)\eta'(1+\eta)\partial_x\underline{\varphi}+(1+\tilde z)[2\eta'^2+\eta''(1+\eta)]\underline{\varphi} \\ G_0(\eta)\underline{\varphi} &= -(2\eta+\eta^2)\partial_x^2\underline{\varphi}-2\eta'(1+\eta)\partial_x\underline{\varphi}-\eta''(1+\eta)\underline{\varphi}\,. \end{aligned} \tag{6.1.4}$$

According to Definition 5.3.3, the operators G_ℓ, $\ell=0,1,2$, are sums of elements of $\tilde{\mathcal{N}}_1^{2-\ell}$. By the first remark after Definition 5.3.3 we may regard G_ℓ as operators of $\mathcal{N}_{K,j,1}^{2-\ell}[r]$ for any j. Then we apply Proposition 5.3.5, with an index j that will be chosen later on large enough depending on N (see lemma 6.1.1) and with $\tilde S_N\equiv 0$, $\tilde M=0$. We deduce that there exists $\sigma_0>0$ and, for $\sigma>\sigma_0$, some $r(\sigma)<r$, such that, if η belongs to $B^K_{\sigma_0}(I,r(\sigma))\cap C^K_*(I,\dot H^\sigma(\mathbb{T}^1,\mathbb{R}))$ and ψ in $C^K_*(I,\dot H^\sigma(\mathbb{T}^1,\mathbb{R}))$, equation (6.1.3) has a unique solution $\underline{\varphi}$ in $C^K_*(I,E^\sigma_j)$, that may be written, by (5.3.23), as

$$\underline{\varphi}=\frac{\cosh((\tilde z+1)D)}{\cosh D}\psi+M(\eta)\psi \tag{6.1.5}$$

for some $M(\eta)$ in $\Sigma\mathcal{N}^{\partial,0}_{K,j,1}[r,N]$.

Remark: The equation (6.1.3) satisfied by the function $\underline{\varphi}$ is a linear ODE in $\partial_{\tilde z}$ with *variable* coefficients in $\tilde z$. It is technically convenient, for the constructions to follow, to have instead a solution of a linear *constant coefficients* ODE. For this reason we introduce in (6.1.10) below the function $\tilde\Phi$ that will satisfy the system (6.1.11).

We consider first the function $\varphi(x,z)$ defined from $\underline{\phi}$ through a change of variables that flattens the free boundary of Ω_t, but that modifies the bottom, namely

$$\varphi(x,z)\stackrel{\rm def}{=}\underline{\phi}(x,z+\eta(t,x))=\underline{\varphi}\big(x,\frac{z}{1+\eta(t,x)}\big) \tag{6.1.6}$$

which is well defined for $-(1+\eta(t,x))\le z\le 0$. Notice that $\varphi(x,z)$ solves the elliptic equation

$$\Big(\frac{\partial^2}{\partial x^2}+(1+\eta'^2)\frac{\partial^2}{\partial z^2}-2\eta'\frac{\partial^2}{\partial x\partial z}-\eta''\frac{\partial}{\partial z}\Big)\varphi=0 \tag{6.1.7}$$

and satisfies the Dirichlet boundary condition at the top $\varphi|_{z=0}=\psi(t,x)$. Then for $-1\le z\le 0$ we also consider the function

$$\varphi_0(x,z)=\frac{\cosh((z+1)D)}{\cosh D}\psi \tag{6.1.8}$$

which solves

$$(\partial_x^2+\partial_z^2)\varphi_0=0 \quad \text{in } \mathcal{B} \tag{6.1.9}$$

and satisfies the Neumann boundary condition at the bottom $\frac{\partial\varphi_0}{\partial z}|_{z=-1}=0$. Finally, taking a cut off function χ in $C_0^\infty(\mathbb{R})$, supported in $[-\frac12,\frac12]$, equal to one on $[-\frac14,\frac14]$, with values in $[0,1]$, we define the function

$$\tilde\Phi(x,z)\stackrel{\rm def}{=}\chi(z)\varphi(x,z)+(1-\chi)(z)\varphi_0(x,z)\,. \tag{6.1.10}$$

Notice that, if $\|\eta(t,\cdot)\|_{L^\infty}<\frac12$, the function $\varphi(x,z)$ in (6.1.6) is well defined for $-1/2\le z\le 0$ and therefore $\tilde\Phi(x,z)$ is well defined for $z\in[-1,0]$.

Lemma 6.1.1 *Let the index j chosen before (6.1.5) satisfy $j\ge N+5$. Then the function $\tilde\Phi$ satisfies the system*

$$\begin{aligned}\Big(\frac{\partial^2}{\partial x^2}+(1+\eta'^2)\frac{\partial^2}{\partial z^2}-2\eta'\frac{\partial^2}{\partial x\partial z}-\eta''\frac{\partial}{\partial z}\Big)\tilde\Phi&=\tilde M(\eta)\psi+\tilde S_N(\eta,\psi)\\ \tilde\Phi|_{z=0}&=\psi\\ \partial_z\tilde\Phi|_{z=-1}&=0\end{aligned} \tag{6.1.11}$$

where $\tilde M(\eta)$ is some operator of $\Sigma\mathcal{N}^{\partial,N}_{K,4,1}[r,N]$ (for N an arbitrary given integer) and where $\tilde S_N(\eta,\psi)$ is linear in ψ and, for any large enough σ_0, there is σ_0' such that, for any $0\le k\le K$,

$$\|\partial_t^k\tilde S_N(\eta,\psi)\|_{E_4^{\sigma_0+2-\frac32 k}}\le C\sum_{k'+k''=k}\mathcal{G}^{\sigma_0'}_{k',N}(\eta,t)\mathcal{G}^{\sigma_0'}_{k'',1}(\psi,t)\,. \tag{6.1.12}$$

Moreover $\tilde M(\eta)\psi$ and $\tilde S_N(\eta,\psi)$ are supported for $-1\le z\le-\frac14$.

Proof: By the definition of $\tilde\Phi$ in (6.1.10) and (6.1.7), (6.1.9), it follows that (6.1.11) is equal to

$$\begin{aligned}f=\big[(1+\eta'^2)\partial_z^2-2\eta'\partial_x\partial_z-\eta''\partial_z\,,\,\chi(z)\big](\varphi-\varphi_0)\\ +(1-\chi(z))\big(\eta'^2\partial_z^2\varphi_0-2\eta'\partial_x\partial_z\varphi_0-\eta''\partial_z\varphi_0\big).\end{aligned} \tag{6.1.13}$$

Moreover f is supported for $-1\le z\le-\frac14$ because, for $-1/4\le z\le 0$, the function $\tilde\Phi$ is equal to φ which is a solution of (6.1.7). Let us check that f may be written as the right hand side of the first line in (6.1.11). Applying Taylor formula to the last expression for φ given by (6.1.6), we have

$$\begin{aligned}\varphi(x,z)=\sum_{p=0}^{N-1}\frac{1}{p!}\partial_z^p\underline{\varphi}(x,z)\big(-\frac{z\eta(x)}{1+\eta(x)}\big)^p\\ +\frac{1}{(N-1)!}\int_0^1(1-\lambda)^{N-1}(\partial_z^N\underline{\varphi})\big(x,z-\frac{\lambda z\eta}{1+\eta}\big)\,d\lambda\big(-\frac{z\eta(x)}{1+\eta(x)}\big)^N.\end{aligned} \tag{6.1.14}$$

We express in the above formula $\underline{\varphi}$ using (6.1.5). The first term in (6.1.14), corresponding to $p = 0$ in the sum, is equal to

$$\underline{\varphi} = \frac{\cosh((z+1)D)}{\cosh D}\psi + M(\eta)\psi = \varphi_0 + M(\eta)\psi$$

where $M(\eta)$ is in $\Sigma\mathcal{N}^{\partial,0}_{K,j,1}[r,N]$. By the first, third and fourth remarks after Definition 5.3.4, each term

$$\frac{1}{p!}\Big(-\frac{z\eta(x)}{1+\eta(x)}\Big)^p\partial_z^p\Big(\frac{\cosh((z+1)D)}{\cosh D} + M(\eta)\Big)\psi$$

of the sum in (6.1.14), for $1 \le p \le N-1$, may be written as the action of an operator of $\Sigma\mathcal{N}^{\partial,N-1}_{K,5,1}[r,N]$ on ψ, if the index j chosen before (6.1.5) is taken large enough relatively to N, e.g. $j \ge N+4$.

In the same way, in the Taylor remainder in (6.1.14) the argument of the integral will be a path of operators in $\Sigma\mathcal{N}^{\partial,N}_{K,5,1}[r,N]$ if $j \ge N+5$. Thus using (5.3.18) (with m replaced by N and σ large depending on N) we see that the Taylor remainder in (6.1.14), and its ∂_z derivative, satisfy (6.1.12), for some σ_0' depending on N and σ_0.

Collecting the previous arguments we may write

$$\varphi - \varphi_0 = M(\eta)\psi + \tilde{S}_N(\eta,\psi)$$

where the operator $M(\eta)$ is in $\Sigma\mathcal{N}^{\partial,N-1}_{K,5,1}[r,N]$ and the functions $\tilde{S}_N(\eta,\psi)$, $\partial_z\tilde{S}_N(\eta,\psi)$ satisfy (6.1.12). Clearly $\tilde{S}_N(\eta,\psi)$ is linear in ψ. Consequently the first term in the right hand side of (6.1.13) may be written as

$$\big[(1+\eta'^2)\partial_z^2 - 2\eta'\partial_x\partial_z - \eta''\partial_z\,,\,\chi(z)\big](\varphi-\varphi_0) = \tilde{M}(\eta)\psi + \tilde{S}_N(\eta,\psi)$$

where $\tilde{M}(\eta)$ is in $\Sigma\mathcal{N}^{\partial,N}_{K,4,1}[r,N]$ (use the third and fourth remarks after Definition 5.3.4 and that the commutator $[\partial_{zz},\chi(z)]$ is the first order operator $(\partial_{zz}\chi) + 2(\partial_z\chi)\partial_z$) and $\tilde{S}_N(\eta,\psi)$ satisfies (6.1.12), as claimed in the statement of the lemma.

Finally, since the operator $\frac{\cosh((z+1)D)}{\cosh D}$ is in $\tilde{\mathcal{N}}_0^{\partial,0}$ (recall Definition 5.3.4-(i)) we conclude that the second term in the right hand side of (6.1.13) may be written as

$$(1-\chi(z))\big(\eta'^2\partial_z^2 - 2\eta'\partial_x\partial_z - \eta''\partial_z\big)\frac{\cosh((z+1)D)}{\cosh D}\psi = \tilde{M}(\eta)\psi$$

where $\tilde{M}(\eta)$ is the sum of homogeneous operators in $\tilde{\mathcal{N}}_1^{\partial,2}$. □

The good unknown that allows to express the capillarity-gravity water waves equations without artificial loss of derivatives is defined as

$$\omega \stackrel{\text{def}}{=} \Phi|_{z=0} = \psi - Op^{\text{BW}}(B)\eta \tag{6.1.15}$$

where

$$\Phi \stackrel{\text{def}}{=} \tilde{\Phi} - Op^{\mathrm{BW}}(\partial_z \tilde{\Phi})\eta\,, \quad B = \partial_z \tilde{\Phi}|_{z=0}\,, \tag{6.1.16}$$

$\tilde{\Phi}$ being the function (6.1.10). To obtain an expression for ω in terms of (η, ψ) alone, we need to deduce from (6.1.11) a paradifferential equation satisfied by Φ (see Lemma 6.1.3).

We shall use the following paralinearization lemma for the product of p functions (in the case $p = 2$ it gives the usual para-product decomposition formula).

Lemma 6.1.2 (Para-product) *Let $u_1, \dots, u_p$ be p elements in $H^\sigma_0(\mathbb{T}^1, \mathbb{C})$, with $\sigma > \frac{1}{2}$. Then we may write*

$$u_1 \cdots u_p = \sum_{j=1}^{p} \big(Op^{\mathrm{BW}}(u_1 \cdots \hat{u}_j \cdots u_p) u_j + R_j(u_1 \cdots \hat{u}_j \cdots u_p) u_j \big) \tag{6.1.17}$$

where R_j is in $\tilde{\mathcal{R}}^{-\rho}_{p-1}$ for any ρ, and where $\hat{u}_j$ means that the j-th term is excluded.

Proof: Let χ be an even function in $C^\infty_0(\mathbb{R})$, $\chi \equiv 1$ close to zero, with small enough support. We decompose the product of p functions as

$$\begin{aligned} u_1 \cdots u_p = \sum_{j=1}^{p} \sum_{n_1, \dots, n_p \in \mathbb{N}^*} \prod_{i \neq j} \chi\Big(\frac{n_i}{n_j}\Big) \prod_{\ell=1}^{p} \Pi_{n_\ell} u_\ell \\ + \sum_{n_1, \dots, n_p} \theta(n_1, \dots, n_p) \prod_{\ell=1}^{p} \Pi_{n_\ell} u_\ell \end{aligned} \tag{6.1.18}$$

where

$$\theta(n_1, \dots, n_p) = 1 - \sum_{j=1}^{p} \prod_{i \neq j} \chi\Big(\frac{n_i}{n_j}\Big)$$

is supported for $\max(n_1, \dots, n_p) \sim \max_2(n_1, \dots, n_p)$. As a consequence, recalling Definition 2.2.3-(i), the last term in (6.1.18) contributes to the smoothing operators in $\tilde{\mathcal{R}}^{-\rho}_{p-1}$ in the right hand side of (6.1.17).

All the terms in the sum (6.1.18) are the same, up to permutation of the indices. The one corresponding to $j = p$,

$$\prod_{i \neq p} \chi\Big(\frac{n_i}{n_p}\Big) \prod_{\ell=1}^{p} \Pi_{n_\ell} u_\ell\,,$$

may be written, using (2.2.4) and (2.2.1) for $\tau = 1$, as

$$\frac{1}{2\pi} \int_{\mathbb{R}\times\mathbb{R}} e^{i(x-y)\xi} a_{\chi_{p-1}}(u_1, \dots, u_{p-1}; x, \xi) u_p(y)\, dy d\xi \tag{6.1.19}$$

where

$$a_{\chi_{p-1}}(u_1,\dots,u_{p-1};x,\xi)=\sum_{n_1,\dots,n_{p-1}}\prod_{\ell=1}^{p-1}\chi\big(\frac{n_\ell}{\xi}\big)\Pi_{n_\ell}u_\ell(x) \tag{6.1.20}$$

has the form (2.2.23) (with $p-1$ instead of p), admissible cut-off function $\chi_{p-1}(n_1,\dots,n_{p-1};\xi)=\prod_{\ell=1}^{p-1}\chi\big(\frac{n_\ell}{\xi}\big)$ and $a=\prod_{\ell=1}^{p-1}u_\ell$. Notice that $a_{\chi_{p-1}}$ is an homogeneous symbol in $\tilde\Gamma^0_{p-1}$ supported for $|n'|\langle\xi\rangle^{-1}$ small where we denote $n'=(n_1,\dots,n_{p-1})$.

We apply Lemma 2.2.1 to the symbol $a_{\chi_{p-1}}\in\tilde\Gamma^0_{p-1}$. By (2.2.18), (2.2.19), (2.2.7), we may write (6.1.19), in terms of the Weyl quantization, as

$$Op^{\rm W}(b)u_p=\sum_{\alpha=0}^{A-1}Op^{\rm W}(b_\alpha)u_p+Op^{\rm W}(\tilde b)u_p \tag{6.1.21}$$

where

$$b_\alpha(u_1,\dots,u_{p-1};x,\xi)=\frac{(-1)^\alpha}{\alpha!2^\alpha}\partial_x^\alpha D_\xi^\alpha a_{\chi_{p-1}}(u_1,\dots,u_{p-1};x,\xi) \tag{6.1.22}$$

and where $\tilde b$ satisfies estimates of the form (2.2.10) with $m=0$ and the prefactor Q is $O\big(\big(\frac{|n'|}{\langle\xi\rangle}\big)^A\big)$ when $\frac{|n'|}{\langle\xi\rangle}$ is small.

By (6.1.22) and (6.1.20), and recalling the Definition 2.2.2 of a Bony-Weyl operator, the first term in the sum (6.1.21) is

$$Op^{\rm W}(b_0)u_p=Op^{\rm W}(a_{\chi_{p-1}})u_p=Op^{\rm BW}(u_1\cdots u_{p-1})u_p$$

see (2.2.24).

When $1\le\alpha\le A-1$, it follows from (6.1.22) and the definition of $a_{\chi_{p-1}}$ in (6.1.20) that b_α is supported for $|n'|\sim\langle\xi\rangle$. As a consequence each operator $Op^{\rm W}(b_\alpha)$ satisfies the condition (2.2.28) that defines an homogeneous smoothing operator of $\tilde{\mathcal R}^{-\rho}_{p-1}$ (recall formula (2.2.45)). In conclusion each $Op^{\rm W}(b_\alpha)u_p$, $1\le\alpha\le A-1$, in (6.1.21) has the form $R_p(u_1,\dots,u_{p-1})u_p$ in the right hand side of (6.1.17).

Finally we consider $Op^{\rm W}(\tilde b)u_p$. Since Q is $O\big(\big(\frac{|n'|}{\langle\xi\rangle}\big)^A\big)$ when $\frac{|n'|}{\langle\xi\rangle}$ is small, the function $\tilde b$ satisfies by (2.2.10) bounds with an $O(\langle\xi\rangle^{-A})$ gain, as soon as we assume enough smoothness on $u_1,\dots,u_{p-1}$ to compensate the loss $|n'|^A$. Moreover $\tilde b$ is supported for $|n'|\langle\xi\rangle^{-1}$ small as the symbol $a_{\chi_{p-1}}$ in (6.1.20). This follows from (2.2.5) for b, and then for $\tilde b=b-\sum_{\alpha=1}^{A-1}b_\alpha$, since each b_α is supported for $c_1\langle\xi\rangle\le|n'|\le c_2\langle\xi\rangle$ for some constants $0<c_1<c_2\ll1$. Taking $A\sim\rho$, we deduce that also $Op^{\rm W}(\tilde b)$ is a smoothing operator in $\tilde{\mathcal R}^{-\rho}_{p-1}$ and thus it contributes to the last term in (6.1.17) (with $j=p$). Actually, by the first remark after Proposition 2.2.4 and (2.2.10) with $\alpha=\beta=0$, and $Q=O\big(\big(\frac{|n'|}{\langle\xi\rangle}\big)^\rho\big)$, we may apply the statement of Proposition 2.2.4 with

$m = -\rho$ (even tough the estimates (2.2.10) do not imply that $\tilde{b}$ is a symbol, since each ∂_ξ^β derivative of $\tilde{b}$ involves a loss in $|n|^\beta$). Then the last remark after Proposition 2.2.4 implies that $Op^{\mathrm{W}}(\tilde{b})$ is in $\tilde{\mathcal{R}}^{-\rho}_{p-1}$. This concludes the proof. □

Remark: When we shall apply below lemma 6.1.2, we shall replace some arguments by η (or one of its derivatives), which belongs to a space of the form H^s_0. The remaining arguments will be replaced by either a space derivative of $\psi \in \dot{H}^s$, or by $\partial_z\tilde{\Phi}$, $\partial_z^2\tilde{\Phi}$, B, that may be considered as elements of H^s_0 (at fixed z), even though the function $\tilde{\Phi}$ is defined only modulo constants like ψ, since one makes always act at least one derivative on it. This will be used without further notice in the proof of lemma 6.1.3 below, as well as in formulas (6.1.57), (6.2.16), (6.2.18), (6.2.20).

Lemma 6.1.3 *Let $\tilde{\Phi}$ be a smooth enough solution of (6.1.11). Then the function Φ defined by (6.1.16) satisfies the paradifferential equation*

$$
\begin{aligned}
(6.1.23)\quad \big[(1 + Op^{\mathrm{BW}}(\eta' \otimes \eta'))\partial_z^2 - 2iOp^{\mathrm{BW}}(\eta'\xi)\partial_z - Op^{\mathrm{BW}}(\xi^2)\big]\Phi \\
= -Op^{\mathrm{BW}}[\partial_z\tilde{M}(\eta)\psi + \partial_z\tilde{S}_N(\eta,\psi)]\eta \\
+ \sum_{j=1}^{2}(R'_j(\eta)\partial_z^j\tilde{\Phi} + R''_j(\eta,\partial_z^j\tilde{\Phi})\eta) + \tilde{M}(\eta)\psi + \tilde{S}_N(\eta,\psi)
\end{aligned}
$$

where $\tilde{M}(\eta)$ is in $\Sigma\mathcal{N}^{\partial,N}_{K,4,1}[r,N]$, $\tilde{S}_N$ is linear in ψ, and such that, for any large enough σ_0, there is σ_0' so that, for any $0 \le k \le K$,

$$
(6.1.24)\qquad \|\partial_t^k\tilde{S}_N(\eta,\psi)\|_{E_4^{\sigma_0+2-\frac{3}{2}k}} \le C\sum_{k'+k''=k}\mathcal{G}^{\sigma_0'}_{k',N}(\eta,t)\mathcal{G}^{\sigma_0'}_{k'',1}(\psi,t),
$$

and where $R'_j(\eta)$ (resp. $R''_j(\eta,\partial_z^j\tilde{\Phi})$) is a sum of smoothing operators of $\tilde{\mathcal{R}}^{-\rho}_p[\mathcal{B}]$, $p = 1,2$, ρ being an arbitrary positive number. Moreover $R''_j(\eta,\partial_z^j\tilde{\Phi})$ is linear in $\partial_z^j\tilde{\Phi}$ and $R'_j(\eta)$, $R''_j(\eta,\partial_z^j\tilde{\Phi})$ are local relatively to the z variable. Finally, $\tilde{M}(\eta)\psi$ and $\tilde{S}_N(\eta,\psi)$ are supported for $-1 \le z \le -\frac{1}{4}$.

Remarks: • We recall that a local operator is an operator that does not increase the support of functions. Here the expression "local relatively to z" means that the z-support of functions does not increase.

• As stated in lemma 6.1.1 the terms $\tilde{M}(\eta)\psi$, $\tilde{S}_N(\eta,\psi)$, that come from the cut-off in (6.1.10), are supported for $-1 \le z \le -\frac{1}{4}$. Because of that, they will induce only smoothing contributions to the restriction of Φ at $z = 0$, see Proposition 6.1.4.

Proof: We apply the paralinearization lemma 6.1.2 to each term in the left

hand side of equation (6.1.11) at fixed z. We may write

$$\eta'^2\partial_z^2\tilde{\Phi} = Op^{\mathrm{BW}}(\eta'\otimes\eta')\partial_z^2\tilde{\Phi} + 2Op^{\mathrm{BW}}(\eta'\otimes\partial_z^2\tilde{\Phi})\eta' + R(\eta',\eta')\partial_z^2\tilde{\Phi} + R(\eta',\partial_z^2\tilde{\Phi})\eta' \tag{6.1.25}$$

and

$$\begin{aligned} 2\eta'\partial_x\partial_z\tilde{\Phi} + \eta''\partial_z\tilde{\Phi} = {} & Op^{\mathrm{BW}}(2\eta')\partial_x\partial_z\tilde{\Phi} + Op^{\mathrm{BW}}(\eta'')\partial_z\tilde{\Phi} \\ & + 2Op^{\mathrm{BW}}(\partial_x\partial_z\tilde{\Phi})\eta' + Op^{\mathrm{BW}}(\partial_z\tilde{\Phi})\eta'' \\ & + R(\eta')\partial_x\partial_z\tilde{\Phi} + R(\partial_x\partial_z\tilde{\Phi})\eta' \\ & + R(\eta'')\partial_z\tilde{\Phi} + R(\partial_z\tilde{\Phi})\eta'' \end{aligned} \tag{6.1.26}$$

where, using the second remark after Definition 5.1.3, R stands for a generic smoothing operator of $\tilde{\mathcal{R}}_p^{-\rho}[\mathcal{B}]$ as defined in (i) of Definition 5.1.3, with $p=1$ when R has only one argument, and $p=2$ if R has two arguments. By the composition formula (2.3.5) of Proposition 2.3.2, the sum of the first two terms in the right hand side of (6.1.26) is

$$Op^{\mathrm{BW}}(2\eta')\partial_x\partial_z\tilde{\Phi} + Op^{\mathrm{BW}}(\eta'')\partial_z\tilde{\Phi} = Op^{\mathrm{BW}}(2i\eta'\xi)\partial_z\tilde{\Phi} \tag{6.1.27}$$

up to a smoothing remainder $R(\eta)\partial_z\tilde{\Phi}$ with $R(\eta)$ in $\tilde{\mathcal{R}}_1^{-\rho}$, thus in $\tilde{\mathcal{R}}_1^{-\rho}[\mathcal{B}]$ by the second remark after Definition 5.1.3. By (6.1.25), (6.1.26), (6.1.27) we may then write (6.1.11) as

$$\begin{aligned} \big[(1+Op^{\mathrm{BW}}(\eta'\otimes\eta'))\partial_z^2 - 2iOp^{\mathrm{BW}}(\eta'\xi)\partial_z - Op^{\mathrm{BW}}(\xi^2)\big]\tilde{\Phi} & \\ = -2Op^{\mathrm{BW}}(\eta'\otimes\partial_z^2\tilde{\Phi})\eta' + 2Op^{\mathrm{BW}}(\partial_x\partial_z\tilde{\Phi})\eta' & \\ + Op^{\mathrm{BW}}(\partial_z\tilde{\Phi})\eta'' + \sum_{j=1}^{2} R_j'(\eta)\partial_z^j\tilde{\Phi} + \sum_{j=1}^{2} R_j''(\eta,\partial_z^j\tilde{\Phi})\eta & \\ + \tilde{M}(\eta)\psi + \tilde{S}_N(\eta,\psi) & \end{aligned} \tag{6.1.28}$$

where $R_j'(\eta)$ (resp. $R_j''(\eta,\partial_z^j\tilde{\Phi})$) is a sum of elements of $\tilde{\mathcal{R}}_p^{-\rho}[\mathcal{B}]$, $p=1,2$ (resp. which is moreover linear in $\partial_z^j\tilde{\Phi}$). These operators are local in z as we use only symbolic calculus relatively to the x variable, z being just a parameter. Moreover, as stated in lemma 6.1.1, the operator $\tilde{M}(\eta)$ is in $\Sigma\mathcal{N}_{K,4,1}^{\partial,N}[r,N]$ and $\tilde{S}_N(\eta,\psi)$ satisfy (6.1.12), i.e. (6.1.24).

On the other hand, we compute

$$\big[(1+Op^{\mathrm{BW}}(\eta'\otimes\eta'))\partial_z^2 - 2iOp^{\mathrm{BW}}(\eta'\xi)\partial_z - Op^{\mathrm{BW}}(\xi^2)\big]Op^{\mathrm{BW}}(\partial_z\tilde{\Phi})\eta\,.$$

Using again the composition formula (2.3.5), we get

$$\begin{aligned} & Op^{\mathrm{BW}}((1+(\eta'\otimes\eta'))\partial_z^3\tilde{\Phi})\eta \\ & - 2Op^{\mathrm{BW}}(\eta'\partial_z^2\tilde{\Phi})\eta' - 2Op^{\mathrm{BW}}(\eta'\partial_z^2\partial_x\tilde{\Phi})\eta - Op^{\mathrm{BW}}(\eta''\partial_z^2\tilde{\Phi})\eta \\ & + Op^{\mathrm{BW}}(\partial_z\tilde{\Phi})\eta'' + 2Op^{\mathrm{BW}}(\partial_z\partial_x\tilde{\Phi})\eta' + Op^{\mathrm{BW}}(\partial_z\partial_x^2\tilde{\Phi})\eta \end{aligned} \tag{6.1.29}$$

modulo again smoothing operators as in (6.1.28) and (6.1.29).

Making the difference between (6.1.28) and (6.1.29), and using the definition (6.1.16) of Φ, we get

$$\begin{aligned}
&\big[(1+Op^{\mathrm{BW}}(\eta'\otimes\eta'))\partial_z^2 - 2iOp^{\mathrm{BW}}(\eta'\xi)\partial_z - Op^{\mathrm{BW}}(\xi^2)\big]\Phi \\
&\quad = -Op^{\mathrm{BW}}\big[\big((1+(\eta'\otimes\eta'))\partial_z^2 - 2\eta'\partial_z\partial_x + \partial_x^2 - \eta''\partial_z\big)\partial_z\tilde\Phi\big]\eta \\
&\qquad + \sum_{j=1}^{2} R_j'(\eta)\partial_z^j\tilde\Phi + \sum_{j=1}^{2} R_j''(\eta,\partial_z^j\tilde\Phi)\eta \\
&\qquad\quad + \tilde M(\eta)\psi + \tilde S_N(\eta,\psi) \\
&\qquad = -Op^{\mathrm{BW}}\big[\partial_z\tilde M(\eta)\psi + \partial_z\tilde S_N(\eta,\psi)\big]\eta \\
&\qquad + \sum_{j=1}^{2} R_j'(\eta)\partial_z^j\tilde\Phi + \sum_{j=1}^{2} R_j''(\eta,\partial_z^j\tilde\Phi)\eta \\
&\qquad\qquad + \tilde M(\eta)\psi + \tilde S_N(\eta,\psi)
\end{aligned}$$

where the last equality follows from substitution of (6.1.11) in the argument of Op^{BW}. This concludes the proof of (6.1.23). □

We may deduce from the preceding results the following expression of the good unknown $\Phi|_{z=0}$ and its $\partial_z\Phi|_{z=0}$ derivative.

Proposition 6.1.4 (Good unknown) *Let N be in $\mathbb{N}^*$, $K\in\mathbb{N}$, $\rho>0$ be given. There is $r>0$ and a function $a_0(\eta,\psi)$ in $\Sigma\mathcal{F}^{\mathbb{R}}_{K,0,1}[r,N,\mathrm{aut}]$, linear in ψ, such that $B=\partial_z\tilde\Phi|_{z=0}$ defined in (6.1.16) is actually given by*

$$B(\eta,\psi) = a_0(\eta,\psi)\,. \tag{6.1.30}$$

Moreover there is a symbol a_1 in $\Sigma\Gamma^1_{K,0,1}[r,N,\mathrm{aut}]$ such that, setting

$$\omega = \psi - Op^{\mathrm{BW}}(a_0(\eta,\psi))\eta\,, \tag{6.1.31}$$

one has

$$\partial_z\Phi|_{z=0} = (D\tanh D)\omega + Op^{\mathrm{BW}}(a_1(\eta;\cdot))\omega + R_1(\eta)\omega + R_2(\eta,\omega)\eta \tag{6.1.32}$$

where $R_1(\eta), R_2(\eta,\omega)$ are in $\Sigma\mathcal{R}^{-\rho}_{K,0,1}[r,N,\mathrm{aut}]$, R_2 being linear in ω. The symbol a_1 is given by

$$a_1(\eta;x,\xi) = \frac{i\eta'}{1+\eta'^2}\xi - (\xi\tanh\xi)\frac{\eta'^2}{1+\eta'^2} \tag{6.1.33}$$

modulo a symbol in $\Sigma\Gamma^0_{K,0,1}[r,N,\mathrm{aut}]$.

Finally, there is $\sigma>0$ such that, for $\|\eta\|_{K,\sigma}$ small enough, there is a function $a(\eta,\omega)$ in $\Sigma\mathcal{F}^{\mathbb{R}}_{K,0,1}[r,N,\mathrm{aut}]$, linear in ω, even in x, such that

$$\psi = \omega + Op^{\mathrm{BW}}(a(\eta,\omega))\eta\,. \tag{6.1.34}$$

In the new variables (η,ω) the involution $S=\left[\begin{smallmatrix}1&0\\0&-1\end{smallmatrix}\right]$ is unchanged.

Proof: We have introduced in (6.1.10) the function $\tilde{\Phi}$ and we have proved in Lemma 6.1.1 that $\tilde{\Phi}$ satisfies equation (6.1.11). Then we defined in (6.1.16) $\Phi = \tilde{\Phi} - Op^{\mathrm{BW}}(\partial_z \tilde{\Phi})\eta$ and in (6.1.15) the good unknown $\omega = \psi - Op^{\mathrm{BW}}(B)\eta$ with $B = \partial_z \tilde{\Phi}|_{z=0}$. Let us show first that B may be written as a function $a_0(\eta, \psi)$ of $\Sigma\mathcal{F}^{\mathbb{R}}_{K,0,1}[r, N, \mathrm{aut}]$ as in the statement. To do so, we notice that equation (6.1.11) solved by $\tilde{\Phi}$ is of the form (5.3.22) with

$$G_2(\eta) = -\eta'^2\,, \quad G_1(\eta) = 2\eta'\partial_x + \eta''\,, \quad G_0(\eta) = 0\,,$$

and $\tilde{M}(\eta), \tilde{S}_N$ given by lemma 6.1.1. Since $\tilde{M}(\eta)$ is in $\Sigma\mathcal{N}^{\partial,N}_{K,4,1}[r, N]$ and $\tilde{S}_N$ satisfies (6.1.12), we apply Proposition 5.3.5 with $m = N$, $j = 4$. By (5.3.23), we may write

$$\tilde{\Phi} = \frac{\cosh((1+z)D)}{\cosh D}\psi + M(\eta)\psi + S_N(\eta, \psi) \tag{6.1.35}$$

where $M(\eta)$ is in $\Sigma\mathcal{N}^{\partial,N-2}_{K,4,1}[r, N]$ and S_N is supported for $-1 \le z \le -\frac{1}{8}$ and satisfies, for $0 \le k \le K$ and some σ_0',

$$\|\partial_t^k S_N(\eta, \psi)(t)\|_{E_4^{\sigma_0+2-\frac{3}{2}k}} \le C \sum_{k'+k''=k} \mathcal{G}^{\sigma_0'}_{k',N}(\eta, t)\mathcal{G}^{\sigma_0'}_{k'',1}(\psi, t)\,. \tag{6.1.36}$$

In particular, close to $z = 0$,

$$\partial_z \tilde{\Phi} = \frac{\sinh((1+z)D)}{\cosh D}(D\psi) + \partial_z M(\eta)\psi \tag{6.1.37}$$

and $\partial_z M(\eta)$ is in $\Sigma\mathcal{N}^{\partial,N-1}_{K,3,1}[r, N]$, by the third remark after Definition 5.3.4. Inserting (6.1.37) in (6.1.16) and using the fifth remark after Definition 5.3.4, we see that, for z close to zero, we may write

$$\Phi(z, \cdot) = \tilde{\Phi}(z, \cdot) - Op^{\mathrm{BW}}(a(\eta, \psi; z, x))\eta$$

for some symbol a in the class $\Sigma\Gamma^0_{K,0,3,1}[\mathcal{B}, r, N]$, independent of ξ. Restricting to $z = 0$, we get, by the definition of ω in (6.1.15),

$$\omega = \psi - Op^{\mathrm{BW}}(a_0(\eta, \psi; x))\eta$$

where a_0 is in $\Sigma\Gamma^0_{K,0,1}[r, N, \mathrm{aut}]$, and it is independent of ξ, i.e. it is a function of $\Sigma\mathcal{F}_{K,0,1}[r, N, \mathrm{aut}]$. Moreover, by construction, $a_0(\eta, \psi)$ is real valued when η, ψ are real valued. We have thus obtained the representation (6.1.31) of ω.

Let us now establish (6.1.32). The strategy will be to prove first that Φ satisfies a paradifferential equation of the form (5.3.1), see (6.1.42), then use Proposition 5.3.1 to express Φ from the data, see (6.1.43), and finally compute $\partial_z \Phi|_{z=0}$ from (5.3.2), see (6.1.47).

We recall that Φ solves the equation (6.1.23). In its right hand side we replace in the terms $R_j'(\eta)\partial_z^j\tilde\Phi$, $R_j''(\eta,\partial_z^j\tilde\Phi)\eta$ the function $\tilde\Phi$ by its expression (6.1.35). Recalling that $R_j''(\eta,\partial_z^j\tilde\Phi)$ is linear in $\partial_z^j\tilde\Phi$, we get the following terms, for $j=1,2$,

$$\begin{aligned} &R_j'(\eta)\partial_z^j\Big(\frac{\cosh((1+z)D)}{\cosh D}\psi+M(\eta)\psi\Big)\\ &R_j''\Big(\eta,\partial_z^j\Big(\frac{\cosh((1+z)D)}{\cosh D}\psi+M(\eta)\psi\Big)\Big)\eta\\ &R_j'(\eta)\partial_z^j(S_N(\eta,\psi))\\ &R_j''(\eta,\partial_z^j(S_N(\eta,\psi)))\eta\,. \end{aligned} \tag{6.1.38}$$

By lemma 6.1.3 the smoothing terms R_j', R_j'' belong to $\Sigma\mathcal{R}^{-\rho}_{K,0,\ell,1}[\mathcal{B},r,N]$ for any ρ and any ℓ, according to (iii) of Definition 5.1.3. Moreover, it has been seen after (6.1.35) that $M(\eta)$ is in $\Sigma\mathcal{N}^{\partial,N-2}_{K,4,1}[r,N]$, and, by the first remark after Definition 5.3.4, that $\frac{\cosh((z+1)D)}{\cosh D}$ is in $\tilde{\mathcal{N}}_0^{\partial,0}$. Thus, the third remark following Definition 5.3.4 implies that the operators

$$\partial_z^j\Big[\frac{\cosh((1+z)D)}{\cosh D}+M(\eta)\Big]\,,\quad j=1,2\,,$$

are in $\Sigma\mathcal{N}^{\partial,N}_{K,2,0}[r,N]$. It follows from the third-last and second last remarks following Definition 5.3.4 that the first two lines in (6.1.38) may be written as

$$R'(\eta)\psi+R''(\eta,\psi)\eta \tag{6.1.39}$$

where R' and R'' are smoothing operators in $\Sigma\mathcal{R}^{-\rho}_{K,0,2,1}[\mathcal{B},r,N]$, for any ρ (up to a renaming of ρ, and using the abuse of notation that has been introduced in the last remark after Definition 5.1.3).

The functions in the last two lines of (6.1.38) are supported for $-1\le z\le -\frac18$ as this is true for S_N, and the operators R_j', R_j'' are local in z. Moreover, by (5.1.11), (6.1.36) and the continuous embeddings $E_j^s\subset F_j^s\subset E_j^{s-\frac12}$, these terms may be written as a function $\tilde S_N^1(\eta,\psi)$ that satisfies, for any large enough σ_0, and a convenient $\sigma_0'>\sigma_0$, for any $0\le k\le K$,

$$\|\partial_t^k\tilde S_N^1(\eta,\psi)\|_{E_2^{\sigma_0+2-\frac32 k}}\le C\sum_{k'+k''=k}\mathcal{G}^{\sigma_0'}_{k',N}(\eta,t)\mathcal{G}^{\sigma_0'}_{k'',1}(\psi,t)\,. \tag{6.1.40}$$

We now consider the first term $Op^{\rm BW}[\partial_z\tilde M(\eta)\psi]\eta$ in the right hand side of (6.1.23). Since $\tilde M(\eta)$ is in $\Sigma\mathcal{N}^{\partial,N}_{K,4,1}[r,N]$ then $\partial_z\tilde M(\eta)$ is in $\Sigma\mathcal{N}^{\partial,N+1}_{K,3,1}[r,N]$ by the third remark following Definition 5.3.4, and the last remark after Definition 5.3.4 implies that $Op^{\rm BW}(\partial_z\tilde M(\eta)\psi)$ is in $\Sigma\mathcal{N}^{\partial,1/2}_{K,0,1}[r,N]$. In conclusion $Op^{\rm BW}[\partial_z\tilde M(\eta)\psi]$ is an operator of the form $M''(\eta,\psi)$ of $\Sigma\mathcal{N}^{\partial,1/2}_{K,0,1}[r,N]$, linear in ψ, supported for $-1\le z\le-\frac18$, as $\partial_z\tilde M(\eta)\psi$ satisfies such a property.

By (6.1.24), the function $\tilde{S}_N^{(2)}(\eta,\psi) = -Op^{\rm BW}[\partial_z \tilde{S}_N(\eta,\psi)]\eta$ in the right hand side of (6.1.23) satisfies estimates of the form

$$\|\partial_t^k \tilde{S}_N^{(2)}(\eta,\psi)\|_{E_3^{\sigma_0+2-\frac{3}{2}k}} \leq C \sum_{k'+k''=k} \mathcal{G}^{\sigma_0'}_{k',N}(\eta,t)\mathcal{G}^{\sigma_0'}_{k'',1}(\psi,t), \tag{6.1.41}$$

(for some σ_0 large depending on m, K and N) and it is supported for $-1 \leq z \leq -\frac{1}{8}$.

In conclusion, we have written the right hand side of (6.1.23) as

$$f = M'(\eta)\psi + M''(\eta,\psi)\eta + \tilde{S}_N(\eta,\psi) + R'(\eta)\psi + R''(\eta,\psi)\eta \tag{6.1.42}$$

where M', M'' are in $\Sigma\mathcal{N}^{\partial,m}_{K,0,1}[r,N]$ for some m, $M'(\eta)\psi$ and $M''(\eta,\psi)$ are supported in $-1 \leq z \leq -\frac{1}{8}$, where $\tilde{S}_N$ is a new function, which is the sum of $\tilde{S}_N$ in the right hand side of (6.1.23) (that satisfies (6.1.24)) and $\tilde{S}_N^1, \tilde{S}_N^{(2)}$ in (6.1.40), (6.1.41). This $\tilde{S}_N$ satisfies (6.1.40) and is supported for $-1 \leq z \leq -\frac{1}{8}$. Finally, the smoothing operators R', R'' in $\Sigma\mathcal{R}^{-\rho}_{K,0,2,1}[\mathcal{B},r,N]$ are given by (6.1.39).

Consequently, by (6.1.23) and (6.1.42), the function Φ solves an equation of the form (5.3.1), with f given by (6.1.42), and

$$g_- = Op^{\rm BW}\big(\frac{D^2}{\cosh D}\psi\big)\eta\,, \quad g_+ = \Phi|_{z=0} = \omega\,.$$

By (5.3.2), we may represent the solution Φ as

$$\begin{aligned}\Phi = {} & Op^{\rm BW}(e_+(\eta;z,\cdot))\omega + Op^{\rm BW}(e_-(\eta;z,\cdot))Op^{\rm BW}\big(\frac{D^2}{\cosh D}\psi\big)\eta \\ & + \int_{-1}^0 \big[Op^{\rm BW}(K(\eta;z,\tilde{z},\cdot)) + R_{\rm int}(\eta;z,\tilde{z})\big] f(\tilde{z})\, d\tilde{z} \\ & + R_+(\eta;z)\omega + R_-(\eta;z)Op^{\rm BW}\big(\frac{D^2}{\cosh D}\psi\big)\eta\end{aligned} \tag{6.1.43}$$

with Poisson symbols $e_\pm(\eta;\cdot)$ in $\Sigma\mathcal{P}^{0,\pm}_{K,0,0}[r,N]$, $K(\eta;\cdot)$ in $\Sigma\mathcal{P}^{-1,{\rm int}}_{K,0,0}[r,N]$ and smoothing operators $R_{\rm int}$ in $\Sigma\mathcal{R}^{-\rho-1,{\rm int}}_{K,0,1}[r,N]$ and $R_\pm$ in $\Sigma\mathcal{R}^{-\rho,\pm}_{K,0,1}[r,N]$.

We now substitute the expression of f in (6.1.42) into (6.1.43). We consider first the contribution of the last two terms $R'(\eta)\psi + R''(\eta,\psi)\eta$ to (6.1.43). By lemma 5.1.13, these two contributions may be written as

$$R_1'(\eta)\psi + R_1''(\eta,\psi)\eta \tag{6.1.44}$$

where $\partial_z^j R_1'$, $\partial_z^j R_1''$, $j = 1,2$, are smoothing operators in $\Sigma\mathcal{R}^{-\rho}_{K,0,1}[r,N,{\rm aut}]$ for any given ρ, uniformly in $z \in [-1,0]$ (thus in particular $\partial_z R_1'$ and $\partial_z R_1''$ are continuous in z and their restriction at $z=0$ is well defined). Moreover R_1'' is linear in ψ. Consider next

$$\int_{-1}^0 \big[Op^{\rm BW}(K(\eta;z,\tilde{z},\cdot)) + R_{\rm int}(\eta;z,\tilde{z})\big]\big[M'(\eta)\psi + M''(\eta,\psi)\eta\big]\, d\tilde{z}\,. \tag{6.1.45}$$

The operators $M'(\eta), M''(\eta,\psi) \in \Sigma\mathcal{N}^{\partial,m}_{K,0,1}[r,N]$ satisfy the assumptions of lemma 5.1.14, in particular the support condition (5.1.80). Consequently, (6.1.45) can be written as $R_1'(\eta)\psi + R_1''(\eta,\psi)\eta$ where, for z close to zero, the derivatives $\partial_z^j R_1'$, $\partial_z^j R_1''$ are smoothing operators of $\Sigma\mathcal{R}^{-\rho}_{K,0,1}[r,N,\mathrm{aut}]$, uniformly depending on z close enough to zero. Finally

$$\int_{-1}^{0} \big[Op^{\mathrm{BW}}(K(\eta;z,\tilde z,\cdot)) + R_{\mathrm{int}}(\eta;z,\tilde z)\big]\tilde S_N(\eta,\psi)\,d\tilde z \tag{6.1.46}$$

where $\tilde S_N(\eta,\psi)$ is supported for $-1 \le z \le -1/8$. This property, together with estimate (6.1.40) (with σ_0 large depending on K and m), shows that the assumptions (5.1.79), (5.1.80) of lemma 5.1.14 hold for $\tilde S_N(\eta,\psi)$ (notice that $\tilde S_N$ is linear in ψ and vanishes of order N in η), so that, applying again lemma 5.1.14, we conclude that (6.1.46) has the same structure as (6.1.45) (actually just $R_1'(\eta)\psi$).

Taking a ∂_z derivative of (6.1.43), and restricting to $z=0$, we conclude, using (5.3.3), that $\partial_z\Phi|_{z=0}$ is the sum of

$$\begin{aligned}\partial_z\Phi|_{z=0} = (D\tanh D)\omega + Op^{\mathrm{BW}}(a_1(\eta;\cdot))\omega \\ + \tilde R_1(\eta)\psi + \tilde R_2(\eta,\psi)\eta\end{aligned} \tag{6.1.47}$$

where a_1 is given by (6.1.33), modulo a symbol in $\Sigma\Gamma^0_{K,0,1}[r,N,\mathrm{aut}]$, and $\tilde R_1, \tilde R_2$ are smoothing operators in $\Sigma\mathcal{R}^{-\rho}_{K,0,1}[r,N,\mathrm{aut}]$, $\tilde R_2$ being linear in ψ, plus

$$\begin{aligned}Op^{\mathrm{BW}}(\partial_z e_-(\eta;z,\cdot)_{|z=0})Op^{\mathrm{BW}}\big(\frac{D^2}{\cosh D}\psi\big)\eta + \big(\partial_z R_+(\eta;z)\big)_{|z=0}\omega \\ + \big(\partial_z R_-(\eta;z)\big)_{|z=0}Op^{\mathrm{BW}}\big(\frac{D^2}{\cosh D}\psi\big)\eta\,.\end{aligned} \tag{6.1.48}$$

Since R_+ is in $\Sigma\mathcal{R}^{-\rho,+}_{K,0,1}[r,N]$, then $(\partial_z R_+)_{|z=0}$ is in $\Sigma\mathcal{R}^{-\rho+1}_{K,0,1}[r,N]$ for any ρ (see the first two remarks after Definition 5.1.10) and so it contributes to the smoothing operator $R_1(\eta)$ in (6.1.32). Moreover, since e_- is a Poisson symbol in $\Sigma\mathcal{P}^{0,-}_{K,0,0}[r,N]$, the derivative $\partial_z e_-(\eta;z,\cdot)_{|z=0}$ is in $\Sigma\Gamma^{-\rho}_{K,0,0}[r,N,\mathrm{aut}]$ for any ρ, and the third remark after Proposition 2.2.4 implies that the operator $Op^{\mathrm{BW}}((\partial_z e_-)_{|z=0})$ is in $\Sigma\mathcal{R}^{-\rho}_{K,0,0}[r,N,\mathrm{aut}]$. Therefore by Proposition 2.4.2, the first term in (6.1.48) contributes to $\tilde R_2(\eta,\psi)\eta$ in (6.1.47). The same conclusion holds for the last term in (6.1.48).

To finish the proof of (6.1.32), it is enough to prove (6.1.34). Actually (6.1.32) is a consequence of (6.1.47), (6.1.34) and the composition results of (ii), (iii) and (iv) of Proposition 2.4.3.

In order to invert (6.1.31), we consider the affine map

$$\psi \to \omega + Op^{\mathrm{BW}}(a_0(\eta,\psi))\eta \tag{6.1.49}$$

(recall that $a_0(\eta,\psi)$ is linear in ψ). Since $a_0(\eta,\psi)$ is in $\Sigma\mathcal{F}^{\mathbb{R}}_{K,0,1}[r,N,\mathrm{aut}]$, we deduce, by Proposition 2.2.4, that there is σ such that

$$\|\omega + Op^{\mathrm{BW}}(a_0(\eta,\psi))\eta\|_{K,\sigma} \leq \|\omega\|_{K,\sigma} + C\|\psi\|_{K,\sigma}\|\eta\|_{K,s}\,.$$

Therefore, for $\|\eta\|_{K,\sigma} < \frac{1}{2C}$, the map in (6.1.49) is a contraction of the ball of $C^K_*(I,\dot{H}^\sigma(\mathbb{T}^1,\mathbb{C}))$ with radius $R > 2\|\omega\|_{K,\sigma}$. Let $\psi = \Theta(\eta,\omega)$ denote the unique fixed point in such a ball. The map Θ is linear in ω, and, if η,ω are in $C^K_*(I,\dot{H}^s(\mathbb{T}^1,\mathbb{C}))$ with $s \geq \sigma$, then $\Theta(\eta,\omega)$ is in $C^K_*(I,\dot{H}^s(\mathbb{T}^1,\mathbb{C}))$, with an estimate

$$\|\Theta(\eta,\omega)\|_{K,s} \leq \|\omega\|_{K,s} + C_s\|\omega\|_{K,\sigma}\|\eta\|_{K,s}\,. \tag{6.1.50}$$

If we iterate the formula (6.1.31) giving ψ from ω, we get

$$\psi = \omega + Op^{\mathrm{BW}}(a_0(\eta,\omega))\eta + Op^{\mathrm{BW}}(\tilde{a}_0(\eta,\psi))\eta$$

where $\tilde{a}_0$ is obtained replacing in $a_0(\eta,\psi)$, ψ by $Op^{\mathrm{BW}}(a_0(\eta,\psi))\eta$. Iterating the process and using (i) of Proposition 2.4.3 and the fact that $Op^{\mathrm{BW}}(a_0(\eta,\psi))$ is in $\Sigma\mathcal{M}^0_{K,0,1}[r,N,\mathrm{aut}]$ by the third remark following Definition 2.2.5, we may write

$$\psi = \omega + \sum_{p=1}^{N-1} Op^{\mathrm{BW}}(a_{0,p}(\eta,\dots,\eta,\omega))\eta + Op^{\mathrm{BW}}(a_{0,N}(\eta,\psi))\eta, \tag{6.1.51}$$

where $a_{0,p}$ are functions in $\tilde{\mathcal{F}}_p$ and $a_{0,N}$ is a function in $\mathcal{F}_{K,0,N}[r,\mathrm{aut}]$, linear in ψ. If we replace in the symbol $a_{0,N}(\eta,\psi)$ the variable ψ by $\Theta(\eta,\omega)$, we deduce from (6.1.50) that estimates of the form (2.1.10) hold for the composition, so that the last term in (6.1.51) may be written as $Op^{\mathrm{BW}}(a'_{0,N}(\eta,\omega))\eta$ for some function $a'_{0,N}$ in $\mathcal{F}_{K,0,N}[r,\mathrm{aut}]$.

Finally, by (6.1.31) and since $a_0(\eta,\psi)$ is linear in ψ, the involution $S = \left[\begin{smallmatrix}1 & 0\\ 0 & -1\end{smallmatrix}\right]$ translates in the variables (η,ω) as the same map $(\eta,\omega) \to (\eta,-\omega)$. This concludes the proof. □

We may now provide a paradifferential expression of the Dirichlet-Neumann operator.

Proposition 6.1.5 (Dirichlet-Neumann) *Let us define*

$$V(\eta,\psi) = \partial_x\tilde{\Phi}|_{z=0} - \eta'\partial_z\tilde{\Phi}|_{z=0}\,. \tag{6.1.52}$$

If η,ψ are even real valued functions, then $V(\eta,\psi)$ is a function of the space $\Sigma\mathcal{F}^{\mathbb{R}}_{K,0,1}[r,N,\mathrm{aut}]$ of Definition 2.5.1 (i.e. a symbol of $\Sigma\Gamma^0_{K,0,1}[r,N,\mathrm{aut}]$ independent of ξ), for some $r > 0$ and any N in $\mathbb{N}$, which is linear in ψ, real valued, and odd as a function of x. Moreover, there are

• *symbols* $b^0(\eta;\cdot)$ *in* $\Sigma\Gamma^{-1}_{K,0,1}[r,N,\mathrm{aut}]$, $c^0(\eta,\omega;\cdot)$ *in* $\Sigma\Gamma^{0}_{K,0,1}[r,N,\mathrm{aut}]$, $c^0(\eta,\omega;\cdot)$ *being linear in* ω, *even in* (x,ξ), *satisfying*

$$(\bar b^0)^\vee = b^0, \quad (\bar c^0)^\vee = c^0\,, \tag{6.1.53}$$

• *smoothing operators* $R_1(\eta)$, $R_2(\eta,\omega)$ *in* $\Sigma\mathcal{R}^{-\rho}_{K,0,1}[r,N,\mathrm{aut}]$, *for* ρ *an arbitrary given number,* $R_2(\eta,\omega)$ *being linear in* ω, *satisfying*

$$\overline{R_j V} = R_j \overline{V}\,, \qquad R_j\circ\tau = \tau\circ R_j\,, \quad j=1,2\,, \tag{6.1.54}$$

where τ *is defined above (3.1.17), i.e. these operators send real valued (resp. even) functions to real valued (resp. even) functions,*

such that

$$\begin{aligned} G(\eta)\psi = {} & (D\tanh D)\omega - iOp^{\mathrm{BW}}\big(V(\eta,\psi)\xi\big)\eta \\ & + Op^{\mathrm{BW}}(b^0(\eta;\cdot))\omega + Op^{\mathrm{BW}}(c^0(\eta,\omega;\cdot))\eta \\ & + R_1(\eta)\omega + R_2(\eta,\omega)\eta\,. \end{aligned} \tag{6.1.55}$$

Proof: We shall prove here that (6.1.55) holds with b^0 in $\Sigma\Gamma^0_{K,0,1}[r,N,\mathrm{aut}]$. The fact that b^0 is actually of order -1 will be shown in section 7.2, computing explicitly its principal symbol.

According to (1.1.3), the Dirichlet-Neumann operator is given by

$$G(\eta)\psi = (\partial_y \underline{\phi} - \eta'\partial_x\underline{\phi})|_{y=\eta(t,x)}$$

where $\underline{\phi}$ is the velocity potential defined as the solution of (6.1.1). Using (6.1.6) and (6.1.10) ($\tilde\Phi = \varphi$ for z close to 0), we may then express

$$G(\eta)\psi = \big((1+\eta'^2)\partial_z\tilde\Phi - \eta'\partial_x\tilde\Phi\big)|_{z=0}\,. \tag{6.1.56}$$

We now provide a paradifferential expression of $G(\eta)\psi$ in (6.1.56). By lemma 6.1.2 we write

$$\begin{aligned} G(\eta)\psi = {} & \partial_z\tilde\Phi|_{z=0} + Op^{\mathrm{BW}}(\eta'\otimes\eta')\partial_z\tilde\Phi|_{z=0} + 2Op^{\mathrm{BW}}(\eta'\otimes\partial_z\tilde\Phi|_{z=0})\eta' \\ & - Op^{\mathrm{BW}}(\eta')\partial_x\tilde\Phi|_{z=0} - Op^{\mathrm{BW}}(\partial_x\tilde\Phi|_{z=0})\eta' \\ & + R'(\eta')(\partial\tilde\Phi|_{z=0}) + R''(\eta',\partial\tilde\Phi|_{z=0})\eta' \\ & = I + \cdots + VII \end{aligned} \tag{6.1.57}$$

where $R'(\eta')$, $R''(\eta',\cdot)$ are smoothing operators in $\Sigma\mathcal{R}^{-\rho}_{K,0,1}[r,N,\mathrm{aut}]$ (as in item (iv) of Proposition 2.4.3) for an arbitrary ρ, R'' being linear in the argument $\partial\tilde\Phi|_{z=0}$, and $\partial\tilde\Phi$ standing for either a ∂_x or ∂_z derivative of $\tilde\Phi$.

Let us consider $\partial_z\tilde\Phi|_{z=0}$. By differentiating (6.1.16) with respect to z we get

$$\partial_z\tilde\Phi = \partial_z\Phi + Op^{\mathrm{BW}}(\partial_z^2\tilde\Phi)\eta\,. \tag{6.1.58}$$

On the other hand, (6.1.37) shows that $\partial_z\tilde{\Phi}|_{z=0}$ and $\partial_z^2\tilde{\Phi}|_{z=0}$ may be written as $M(\eta)\psi$ for some M in $\Sigma\mathcal{M}^m_{K,0,0}[r,N,\mathrm{aut}]$ and some m (recalling the definition of the class $\Sigma\mathcal{N}^{\partial,m}_{\cdot}[r,N]$ of Definition 5.3.4). Then, expressing ψ as a function of (η,ω) as in (6.1.34), we may write

$$\partial_z^2\tilde{\Phi}|_{z=0} = M'(\eta)\omega + M''(\eta,\omega)\eta \tag{6.1.59}$$

for operators M', M'' in $\Sigma\mathcal{M}^m_{K,0,0}[r,N,\mathrm{aut}]$, sending real valued functions to real valued functions, with M'' linear in ω. Estimates (2.2.50), (2.2.52) imply that such functions define symbols of order zero, independent of ξ, i.e. that bounds (2.1.5), (2.1.10) with $m=0$ are satisfied. Consequently, the last term in (6.1.58) restricted at $z=0$ may be written as

$$Op^{\mathrm{BW}}(\partial_z^2\tilde{\Phi}_{|z=0})\eta = Op^{\mathrm{BW}}(c^0(\eta,\omega))\eta \tag{6.1.60}$$

where $c^0(\eta,\omega)$ is in $\Sigma\mathcal{F}_{K,0,1}[r,N,\mathrm{aut}]$, linear in ω, and so it gives a contribution to $Op^{\mathrm{BW}}(c^0)\eta$ in (6.1.55). (We shall verify condition (6.1.53) below). Moreover using (6.1.58), the expression of $\partial_z\Phi|_{z=0}$ given by (6.1.32), (6.1.60), we may write, using Propositions 2.4.1 and 2.4.2,

$$I + II = \big(\mathrm{Id} + Op^{\mathrm{BW}}(\eta'\otimes\eta')\big)\big[(D\tanh D)\omega + Op^{\mathrm{BW}}(a_1(\eta,\cdot))\omega\big]$$

modulo contributions of the form $Op^{\mathrm{BW}}(c^0)\eta$ (for a different c^0) and of the form of the smoothing terms in (6.1.55). By symbolic calculus, we may rewrite this as

$$Op^{\mathrm{BW}}\big((1+\eta'^2)(\xi\tanh\xi) + (1+\eta'^2)a_1(\eta;x,\xi)\big)\omega \tag{6.1.61}$$

modulo a contribution to the $Op^{\mathrm{BW}}(b^0(\eta,\cdot))\omega$ term in (6.1.55) (with b^0 in $\Sigma\Gamma^0_{K,0,1}[r,N,\mathrm{aut}]$). Taking into account the expression of the symbol a_1 in (6.1.33), we may finally write (6.1.61) as

$$(D\tanh D)\omega + iOp^{\mathrm{BW}}(\eta'\xi)\omega. \tag{6.1.62}$$

Consider next $III+IV$. Since $\tilde{\Phi}(x,0)=\psi(x)$ by (6.1.10), (6.1.6), (6.1.1), we get, according to (6.1.34),

$$\partial_x\tilde{\Phi}|_{z=0} = \partial_x\psi = \partial_x\big[\omega + Op^{\mathrm{BW}}(a(\eta,\omega;\cdot))\eta\big] \tag{6.1.63}$$

so that, again by symbolic calculus

$$III + IV = 2iOp^{\mathrm{BW}}(\eta'\partial_z\tilde{\Phi}|_{z=0}\xi)\eta - iOp^{\mathrm{BW}}(\eta'\xi)[\omega + Op^{\mathrm{BW}}(a(\eta,\omega;\cdot))\eta]$$

modulo again contributions to $Op^{\mathrm{BW}}(b^0)\omega + Op^{\mathrm{BW}}(c^0)\eta$ and to the smoothing terms in (6.1.55). Since the symbol $a(\eta,\omega)$ in (6.1.34) is nothing but $a_0(\eta,\psi) = B = \partial_z\tilde{\Phi}|_{z=0}$ (see (6.1.30)), we obtain

$$III + IV = iOp^{\mathrm{BW}}(\eta'\partial_z\tilde{\Phi}|_{z=0}\xi)\eta - iOp^{\mathrm{BW}}(\eta'\xi)\omega$$

still modulo the same contributions as above. Using again symbolic calculus, we get, up to such contributions,

$$III + IV + V = iOp^{\mathrm{BW}}\big((\eta'\partial_z\tilde{\Phi} - \partial_x\tilde{\Phi})|_{z=0}\xi\big)\eta - iOp^{\mathrm{BW}}(\eta'\xi)\omega\,. \tag{6.1.64}$$

Finally we consider the terms $VI + VII$ of (6.1.57). By (6.1.63) and (6.1.37) where we express ψ by (6.1.34), we may express $\partial_x\tilde{\Phi}|_{z=0}$ and $\partial_z\tilde{\Phi}|_{z=0}$ as the right hand side of (6.1.59), so that, using (ii) and (iv) of Proposition 2.4.3 (actually just for the homogeneous components of the smoothing operator), we deduce that

$$VI + VII = R_1(\eta)\omega + R_2(\eta,\omega)\eta \tag{6.1.65}$$

contribute to the last two terms in (6.1.55).

In conclusion, by (6.1.57), (6.1.62), (6.1.64), (6.1.65), (6.1.52) we get

$$G(\eta)\psi = (D\tanh D)\omega + iOp^{\mathrm{BW}}\big(V(\eta,\psi)\xi\big)\eta$$

modulo contributions to the last four terms in (6.1.55).

The fact that V is real valued follows from its expression (6.1.52). Moreover, if η and ψ are even, the solution $\underline{\phi}$ to (6.1.1) is even in x, so that φ and φ_0 given by (6.1.6) and (6.1.8) are even in x, as well as $\tilde{\Phi}$ given by (6.1.10). Then $\partial_z\tilde{\Phi}$ is even in x, $\partial_x\tilde{\Phi}$ is odd in x, so that (6.1.52) shows that $V(\eta,\psi)$ is odd in x as claimed.

Moreover, $G(\eta)\psi$ is even and real valued by (6.1.56) as well as the function $(D\tanh D)\omega - iOp^{\mathrm{BW}}\big(V(\eta,\psi)\xi\big)\eta$. It follows that the sum of the last four terms in (6.1.55) is even and real valued. Consequently, we may assume that b^0, c^0 satisfy (6.1.53) and are even functions of (x,ξ), and that the smoothing operators send real (resp. even) functions to real (resp. even) functions. As a consequence, we may always replace for instance $R_1(\eta)$ by $\frac{1}{2}[\tau\circ R_1(\eta) + R_1(\eta)\circ\tau]$, that ensures that the second condition in (6.1.54) is satisfied by $R_1(\eta)$, and argue similarly for the other smoothing terms, and for the first property in (6.1.54). This concludes the proof.

As mentioned at the beginning of the proof, the fact that b^0 is actually of order -1 will be a consequence of some explicit computations that we postpone to section 7.2. □

6.2 Paralinearization of the water waves system

This section is devoted to write system (1.1.4) (with $g = 1$) as a paradifferential system in the variables (η,ω). This has been done already for the first equation $\partial_t\eta = G(\eta)\psi$ in Proposition 6.1.5, see (6.1.55). In this section we deduce from (1.1.4) a paradifferential equation for $\partial_t\omega$.

We first recall that by (6.1.16), (6.1.30)

$$B(\eta,\psi) = \partial_z\tilde\Phi|_{z=0} = a_0(\eta,\psi)\,, \tag{6.2.1}$$

and, by (6.1.52) and $\tilde\Phi(x,0)=\psi(x)$ (see (6.1.10), (6.1.6), (6.1.3))

$$V(\eta,\psi) = \partial_x\tilde\Phi|_{z=0} - \eta'\partial_z\tilde\Phi|_{z=0} = \partial_x\psi - \eta' B\,. \tag{6.2.2}$$

We also recall the linear involution

$$S:\mathbb{R}^2\to\mathbb{R}^2\,,\quad \text{with matrix } \left[\begin{smallmatrix}1&0\\0&-1\end{smallmatrix}\right],$$

and we consider the $\mathbb{Z}/2\mathbb{Z}$ action of the group $\{\mathrm{Id}, S\}$ on functions $\left[\begin{smallmatrix}\eta(t,\cdot)\\ \psi(t,\cdot)\end{smallmatrix}\right]$ defined by

$$\left[\begin{smallmatrix}\eta\\ \psi\end{smallmatrix}\right]_S(t) \stackrel{\rm def}{=} S\left[\begin{smallmatrix}\eta\\ \psi\end{smallmatrix}\right](-t) = \left[\begin{smallmatrix}\eta(-t)\\ -\psi(-t)\end{smallmatrix}\right] \tag{6.2.3}$$

(which is is the analogue in the real variables of the action introduced in (3.1.2)-(3.1.3)).

The main result of this section is

Proposition 6.2.1 (Equation for $\partial_t\omega$) *Let N in $\mathbb{N}^*$. Let (η,ψ) be a solution of (1.1.4). For $r>0$ small enough, there exist*

• *Symbols $a_2(\eta,\cdot)$ in $\Sigma\Gamma^2_{K,0,1}[r,N,\mathrm{aut}]$, $a_1(\eta,\omega;\cdot)$ in $\Sigma\Gamma^1_{K,0,1}[r,N,\mathrm{aut}]$, and $a_0(\eta,\omega;t,\cdot)$ in $\Sigma\Gamma^0_{K,1,2}[r,N]$ satisfying*

$$\mathrm{Im}\, a_2\equiv 0,\quad \mathrm{Re}\, a_1\in\Sigma\Gamma^0_{K,0,1}[r,N,\mathrm{aut}],\quad \mathrm{Im}\, a_1 = -V(\eta,\psi)\xi \tag{6.2.4}$$

such that a_1 is linear in ω (or in ψ according if we express ψ from ω by (6.1.34)),

$$a_0(\eta,\omega;-t) = a_0([\eta,\omega]_S;t)\,, \tag{6.2.5}$$

the homogeneous terms of a_0 are quadratic in ω, and

$$\bar a_j^\vee = a_j,\quad a_j(\cdot;x,\xi) = a_j(\cdot;-x,-\xi),\quad j=0,1,2\,. \tag{6.2.6}$$

Moreover

$$a_2(\eta;x,\xi) = -\kappa\xi^2\big[(1+\eta'(t,x)^2)^{-\frac32}-1\big] \tag{6.2.7}$$

modulo a symbol in $\Sigma\Gamma^0_{K,0,1}[r,N,\mathrm{aut}]$,

• *A smoothing operator $R_1(\eta,\omega)$, resp. $R_2(\eta,\omega,\omega)$, resp. $R_0(\eta)$, which is linear in ω and belongs to $\Sigma\mathcal{R}^{-\rho}_{K,0,1}[r,N,\mathrm{aut}]$, resp. which is quadratic in ω and belongs to $\Sigma\mathcal{R}^{-\rho}_{K,0,2}[r,N,\mathrm{aut}]$, resp. which belongs to $\Sigma\mathcal{R}^{-\rho}_{K,0,1}[r,N,\mathrm{aut}]$, satisfying condition (6.1.54),*

such that

$$\begin{aligned}\partial_t\omega = &-(1+\kappa D_x^2)\eta + Op^{\mathrm{BW}}(a_2(\eta;\cdot))\eta + Op^{\mathrm{BW}}(a_1(\eta,\omega;\cdot))\omega\\ &+ Op^{\mathrm{BW}}(a_0(\eta,\omega;t,\cdot))\eta + R_1(\eta,\omega)\omega + R_2(\eta,\omega,\omega)\eta + R_0(\eta)\eta\,.\end{aligned} \tag{6.2.8}$$

Remark: The symbol a_0 is the only non-autonomous time dependent term in the right hand side of (6.2.8). All the other terms are autonomous, namely the time dependence enters only through $\eta(t)$ and $\omega(t)$.

Proof: As a consequence of (6.1.55) and (6.1.31), and since a_0 is linear in ψ and c_0 is linear in ω, we may write

$$\partial_t \eta = G(\eta)\psi = (D\tanh D)\psi + M(\eta)\psi \tag{6.2.9}$$

for some operator M in $\Sigma\mathcal{M}^1_{K,0,1}[r, N, \mathrm{aut}]$. We substitute this expression inside the right hand side of the second equation (1.1.4). Using the last remark after Definition 2.2.5, we may write

$$\partial_t \psi = M(\eta)\eta + M'(\eta,\psi)\psi + M''(\eta,\psi,\psi)\eta \tag{6.2.10}$$

for some operators

$$M \in \Sigma\mathcal{M}_{K,0,0}[r, N, \mathrm{aut}]\,, \quad M', M'' \in \Sigma\mathcal{M}_{K,0,1}[r, N, \mathrm{aut}]$$

with M' (resp. M'') linear (resp. quadratic) in ψ. By (6.2.1) and (6.1.37), we may write also

$$B(\eta,\psi) = \partial_z \tilde{\Phi}|_{z=0} = (D\tanh D)\psi + M'''(\eta)\psi \tag{6.2.11}$$

for some M''' in $\Sigma\mathcal{M}_{K,0,1}[r, N, \mathrm{aut}]$. Differentiating with respect to time we get

$$\partial_t[B(\eta,\psi)] = [(D\tanh D) + M'''(\eta)]\partial_t\psi + \partial_t[M'''(\eta)]\,\psi$$

and using the expression of $\partial_t\psi$ in (6.2.10), this implies that

$$\partial_t[B(\eta,\psi)] = M(\eta)\eta + M'(\eta,\psi)\psi + M''(\eta,\psi,\psi)\eta + \partial_t[M'''(\eta)]\,\psi \tag{6.2.12}$$

for new M in $\Sigma\mathcal{M}_{K,0,0}[r, N, \mathrm{aut}]$ and M', M'' in $\Sigma\mathcal{M}_{K,0,1}[r, N, \mathrm{aut}]$ with M' (resp. M'') linear (resp. quadratic) in ψ. Consider then the term $\partial_t[M'''(\eta)]\,\psi$ where we expand the operator $M'''(\eta)$ in $\Sigma\mathcal{M}_{K,0,1}[r, N, \mathrm{aut}]$ as

$$M'''(\eta) = M'''_1(\eta) + \ldots + M'''_{N-1}(\eta) + M'''_N(\eta)\,.$$

The homogeneous terms $M'''_j(\eta)$, $j = 1, \ldots, N-1$, are differentiable with respect to η. But, since we have not assumed differentiability with respect to η of the non-homogeneous autonomous term (just the estimates (2.2.52)), we deal $M'''_N(\eta)$ as a time dependent operator. We use the same argument of lemma 2.2.6 and the fact that $\partial_t\eta$ solves (6.2.9) to conclude that $\partial_t[M'''(\eta)]$ is in $\Sigma\mathcal{M}_{K,1,1}[r, N]$. We may insert the terms coming from the homogeneous contributions into the autonomous expressions (6.2.12) and we get

$$\partial_t[B(\eta,\psi)] = M(\eta)\eta + M'(\eta,\psi)\psi + M''(\eta,\psi,\psi)\eta + [\partial_t M'''_N(\eta)]\,\psi\,. \tag{6.2.13}$$

We remark that the only term that we deal as a time dependent one is the function $[\partial_t M_N'''(\eta)]\,\psi$. All the other terms are autonomous. The condition of reversibility of this non-homogeneous time dependent term is that the function $[\partial_t M_N'''(\eta)]\,\psi$ satisfies (6.2.5) (with ψ instead of ω) as it follows recalling (6.2.3) (remark that we do not say that $[\partial_t M_N'''(\eta)]\,\psi$ is quadratic in ψ because, in this time dependent term we do not substitute the equation for $\partial_t\eta$).

By (6.1.56), (6.2.1), and since $\tilde\Phi(x,0) = \psi(x)$ (see (6.1.10), (6.1.6), (6.1.3)) we have

$$G(\eta)\psi = (1+\eta'^2)B(\eta,\psi) - \eta'\partial_x\psi\,. \tag{6.2.14}$$

Thus differentiating the good unknown ω defined in (6.1.15), since η,ψ solve system (1.1.4) with $g=1$, and using (6.2.14), we get

$$\begin{aligned}\partial_t\omega &= \partial_t\psi - Op^{\rm BW}(B)\partial_t\eta - Op^{\rm BW}(\partial_t B)\eta \\ &= -\eta + \kappa\left(\frac{\eta'}{\sqrt{1+\eta'^2}}\right)' - \frac{1}{2}(\partial_x\psi)^2 + \frac{1}{2}(1+\eta'^2)B(\eta,\psi)^2 \\ &\quad - Op^{\rm BW}(B)[(1+\eta'^2)B(\eta,\psi) - \eta'\partial_x\psi] - Op^{\rm BW}(\partial_t B)\eta\,.\end{aligned} \tag{6.2.15}$$

We now paralinearize the different terms in the right hand side of (6.2.15). Applying lemma 6.1.2, and using the symbolic calculus formulas (2.3.2), (2.3.3) (which are exact and not asymptotic ones for the symbols considered here), we get

$$\begin{aligned}\eta'^2 B &= Op^{\rm BW}(\eta'^2)B + 2Op^{\rm BW}(\eta' B)\eta' + R_1'(\eta)B + R_1''(\eta,B)\eta \\ &= Op^{\rm BW}(\eta'^2)B + 2iOp^{\rm BW}(\eta' B\xi)\eta - Op^{\rm BW}(\partial_x(\eta' B))\eta \\ &\qquad + R'(\eta)\omega + R''(\eta,\omega)\eta\end{aligned} \tag{6.2.16}$$

where R', R'' (resp. R_1', R_1'') are elements of $\Sigma\mathcal{R}^{-\rho}_{K,0,2}[r,N,{\rm aut}]$, with R'' (resp. R_1'') linear in ω (resp. B) that are obtained from the smoothing operators of lemma 6.1.2 expressing B in terms of ω by (6.2.11) and (6.1.34), and using that by Proposition 2.4.3 the composition $R(M(U)U)$ or $R(U)M(U)$ of a smoothing operator R and of an element M of a class $\Sigma\mathcal{M}_{K,0,p}[r,N,{\rm aut}]$, gives again a smoothing operator (up to a change of the smoothing index ρ, which is in any case as large as we want). Making act $Op^{\rm BW}(B)$ on (6.2.16) and using symbolic calculus, we get

$$\begin{aligned}Op^{\rm BW}(B)[(1+\eta'^2)B] &= Op^{\rm BW}(B(1+\eta'^2))B + 2iOp^{\rm BW}(\eta' B^2\xi)\eta \\ &\quad - Op^{\rm BW}(\partial_x(B^2\eta'))\eta + R'(\eta,\omega)\omega + R''(\eta,\omega,\omega)\eta\end{aligned} \tag{6.2.17}$$

where R' (resp. R'') is in $\Sigma\mathcal{R}^{-\rho}_{K,0,3}[r,N,{\rm aut}]$ and it is linear in ω (resp. in $\Sigma\mathcal{R}^{-\rho}_{K,0,3}[r,N,{\rm aut}]$ and quadratic in ω).

In the same way, lemma 6.1.2 allows us to write

$$\eta'\partial_x\psi = Op^{\rm BW}(\eta')\partial_x\psi + Op^{\rm BW}(\partial_x\psi)\eta' + R'(\eta')\partial_x\psi + R''(\partial_x\psi)\eta' \tag{6.2.18}$$

with R', R'' in $\tilde{\mathcal{R}}_1^{-\rho}$. Applying $Op^{\rm BW}(B)$ on (6.2.18), using symbolic calculus and expressing in the smoothing operators ψ from (η,ω) by (6.1.34), we obtain

$$\begin{aligned} Op^{\rm BW}(B)(\eta'\partial_x\psi) &= iOp^{\rm BW}(\eta' B\xi)\psi + iOp^{\rm BW}(B(\partial_x\psi)\xi)\eta \\ &\quad - \frac{1}{2}Op^{\rm BW}(\partial_x(B\eta'))\psi - \frac{1}{2}Op^{\rm BW}\big(\partial_x[B(\partial_x\psi)]\big)\eta \\ &\quad + R'(\eta,\omega)\omega + R''(\eta,\omega,\omega)\eta \end{aligned} \tag{6.2.19}$$

with R', R'' in $\Sigma\mathcal{R}^{-\rho}_{K,0,2}[r,N,{\rm aut}]$, respectively linear and quadratic in ω.

Applying again lemma 6.1.2, we get also

$$\begin{aligned} \frac{1}{2}(1+\eta'^2)B^2 &= Op^{\rm BW}((1+\eta'^2)B)B + Op^{\rm BW}(B^2\eta')\eta' \\ &\quad + R_1'(\eta,B)B + R_1''(\eta,B,B)\eta \end{aligned} \tag{6.2.20}$$

with R_1' in $\Sigma\mathcal{R}^{-\rho}_{K,0,1}[r,N,{\rm aut}]$ linear in B, R_1'' in $\Sigma\mathcal{R}^{-\rho}_{K,0,3}[r,N,{\rm aut}]$ quadratic in B. Using symbolic calculus, and expressing B in the smoothing operators in terms of η, ω, by (6.2.11) and (6.1.34), we get

$$\begin{aligned} \frac{1}{2}(1+\eta'^2)B^2 &= Op^{\rm BW}((1+\eta'^2)B)B + iOp^{\rm BW}(B^2\eta'\xi)\eta \\ &\quad - \frac{1}{2}Op^{\rm BW}(\partial_x(B^2\eta'))\eta + R'(\eta,\omega)\omega + R''(\eta,\omega,\omega)\eta \end{aligned} \tag{6.2.21}$$

with R', R'' in $\Sigma\mathcal{R}^{-\rho}_{K,0,1}[r,N,{\rm aut}]$ and $\Sigma\mathcal{R}^{-\rho}_{K,0,2}[r,N,{\rm aut}]$ respectively, with R' (resp. R'') linear (resp. quadratic) in ω. In the same way, we write

$$\begin{aligned} \frac{1}{2}(\partial_x\psi)^2 &= iOp^{\rm BW}((\partial_x\psi)\xi)\psi - \frac{1}{2}Op^{\rm BW}(\partial_x^2\psi)\psi \\ &\quad + R'(\eta,\omega)\omega + R''(\eta,\omega,\omega)\eta\,. \end{aligned} \tag{6.2.22}$$

Moreover, by (2.4.22) applied to the remainder of Taylor formula for the function $\eta' \to \frac{\eta'}{\sqrt{1+\eta'^2}}$ (expanded at the order N), and using lemma 6.1.2 for the polynomial terms of the expansion, we may write

$$\frac{\eta'}{\sqrt{1+\eta'^2}} = Op^{\rm BW}(a(\eta'))\eta' + R(\eta)\eta$$

where $R(\eta)$ is in $\Sigma\mathcal{R}^{-\rho-1}_{K,0,2}[r,N,{\rm aut}]$ and

$$a(\eta') = (1+\eta'^2)^{-3/2}\,. \tag{6.2.23}$$

Consequently, using again symbolic calculus,

$$\partial_x\Big(\frac{\eta'}{\sqrt{1+\eta'^2}}\Big) = -Op^{\rm BW}\big(\xi^2 a(\eta') + \frac{1}{4}\partial_x^2(a(\eta'))\big)\eta + R(\eta)\eta \tag{6.2.24}$$

for some other smoothing operator $R(\eta)$ in $\Sigma\mathcal{R}^{-\rho}_{K,0,2}[r,N,\mathrm{aut}]$.

We now compute $\partial_t\omega$ from (6.2.15). Plugging in the right hand side of (6.2.15) the formulas (6.2.17), (6.2.19), (6.2.21), (6.2.22), (6.2.23), (6.2.24) and using (6.1.30), (6.1.34), we get, after simplification,

$$\begin{aligned}\partial_t\omega = &-\eta - \kappa Op^{\rm BW}\big((1+\eta'^2)^{-3/2}\xi^2\big)\eta - iOp^{\rm BW}\big((\partial_x\psi - \eta' B)\xi\big)\psi\\ &- iOp^{\rm BW}\big(B(\eta' B - (\partial_x\psi))\xi\big)\eta + Op^{\rm BW}(a_0^0(\eta))\eta\\ &+ Op^{\rm BW}(a_0^1(\eta,\omega))\psi + Op^{\rm BW}(a_0^2(\eta,\omega,\omega))\eta - Op^{\rm BW}(\partial_t B)\eta\\ &+ R(\eta)\eta + R'(\eta,\omega)\omega + R''(\eta,\omega,\omega)\eta\end{aligned} \tag{6.2.25}$$

where R and R'' are in $\Sigma\mathcal{R}^{-\rho}_{K,0,2}[r,N,\mathrm{aut}]$, R'' being quadratic in ω, R' is in $\Sigma\mathcal{R}^{-\rho}_{K,0,1}[r,N,\mathrm{aut}]$, linear in ω, the functions a_0^0, a_0^1, a_0^2 belong respectively to $\Sigma\mathcal{F}_{K,0,2}[r,N,\mathrm{aut}]$, $\Sigma\mathcal{F}_{K,0,1}[r,N,\mathrm{aut}]$, $\Sigma\mathcal{F}_{K,0,2}[r,N,\mathrm{aut}]$, a_0^1 (resp. a_0^2) being linear (resp. quadratic) in ω, these functions being real valued and even in x (as follows from the evenness of η and ψ). Moreover

$$a_0^0(\eta) = -\frac{\kappa}{4}\partial_x^2\big[(1+\eta'^2)^{-3/2}\big]\,.$$

Notice also that by (6.2.11), $\partial_t B$ is real valued and even in x, as $\tilde\Phi$ enjoys the same properties.

The sum of the third and fourth terms in the right hand side of (6.2.25) may be written, using that $V = \partial_x\psi - \eta' B$ by (6.2.2), as

$$-iOp^{\rm BW}(V\xi)\psi + iOp^{\rm BW}(BV\xi)\eta\,.$$

Replacing ψ by its value coming from (6.1.31) and (6.1.30), namely $\psi = \omega + Op^{\rm BW}(B)\eta$, and using symbolic calculus, we get

$$-iOp^{\rm BW}(V\xi)\omega - \frac{1}{2}Op^{\rm BW}(V\partial_x B)\eta \tag{6.2.26}$$

modulo smoothing terms as in (6.2.25). By (6.2.25), (6.2.26), expressing in $V(\eta,\psi)$, $B(\eta,\psi)$ the variable ψ in terms of η,ω by (6.1.34), and substituting the expression of $\partial_t B$ in (6.2.13), we finally get

$$\begin{aligned}\partial_t\omega = &-(1+\kappa D^2)\eta + Op^{\rm BW}(a_2(\eta;\cdot))\eta + Op^{\rm BW}(a_1(\eta,\omega;\cdot))\omega\\ &+ Op^{\rm BW}(a_0(\eta,\omega;t,\cdot))\eta + R(\eta)\eta + R'(\eta,\omega)\omega + R''(\eta,\omega,\omega)\eta\end{aligned} \tag{6.2.27}$$

where

• a_2 is a symbol in $\Sigma\Gamma^2_{K,0,1}[r,N,\mathrm{aut}]$, real valued, even in (x,ξ) and satisfies

$\bar{a}_2^\vee = a_2$.
• a_1 is a symbol in $\Sigma\Gamma^1_{K,0,1}[r, N, \text{aut}]$, even in (x,ξ) and satisfies $\bar{a}_1^\vee = a_1$. Moreover a_1 is linear in ω, $\operatorname{Re} a_1 \in \Sigma\Gamma^0_{K,0,1}[r,N]$ and $\operatorname{Im} a_1 = -V(\eta,\psi)\xi$.
• a_0 is in $\Sigma\Gamma^0_{K,1,2}[r,N]$, even in (x,ξ), satisfies $\bar{a}_0^\vee = a_0$. Moreover a_0 is even in t and its homogeneous part is autonomous and quadratic in ω.
• $R(\eta)$ (resp. $R'(\eta,\omega)$, resp. $R''(\eta,\omega,\omega)$) belongs to $\Sigma\mathcal{R}^{-\rho}_{K,0,1}[r,N,\text{aut}]$ (resp. $\Sigma\mathcal{R}^{-\rho}_{K,0,1}[r,N,\text{aut}]$, resp. $\Sigma\mathcal{R}^{-\rho}_{K,0,2}[r,N,\text{aut}]$), the operator R' (resp. R'') being linear (resp. quadratic) in ω.

Actually, a_2 is given by $\kappa[1-(1+\eta'^2)^{-3/2}]\xi^2 + a_0^0(\eta)$ plus the function $M(\eta)\eta$ that comes from the first term in the expression of $\partial_t B$ given by (6.2.13). It satisfies the above requirements since η is even. Moreover, it is given by (6.2.7) modulo a symbol of order zero.

The symbol a_1 is formed by an order one contribution given by $-iV\xi$ in (6.2.26), plus the symbol $a_0^1(\eta,\omega)$ that arises by the term $Op^{\text{BW}}(a_0^1(\eta,\omega))\psi$ in (6.2.25), where we express ψ by (6.1.34), and keep only $Op^{\text{BW}}(a_0^1(\eta,\omega))\omega$. Since V and a_0^1 are real valued, and since V is an odd function of x and a_0^1 an even one, we get that all conditions on a_1 are satisfied.

The symbol a_0 is computed from the contribution $-\frac{1}{2}Op^{\text{BW}}(V\partial_x B)\eta$ in (6.2.26), from $Op^{\text{BW}}(a_0^1(\eta,\omega))Op^{\text{BW}}(a(\eta,\omega;\cdot))\eta$ that comes from the sixth term in the right hand side of (6.2.25), where we replace ψ by (6.1.34), from $Op^{\text{BW}}(a_0^2(\eta,\omega,\omega))\eta$ in (6.2.25), and from the contribution to $Op^{\text{BW}}(\partial_t B)\eta$ coming from (6.2.13), that is $Op^{\text{BW}}([\partial_t M_N'''(\eta)]\psi)\eta$. Notice that this non-homogeneous term is the only non-autonomous one. As we proved that $[\partial_t M_N'''(\eta)]\psi$ satisfies (6.2.5) (with ψ instead of ω), we get that the function $[\partial_t M_N'''(\eta)](\omega + Op^{\text{BW}}(a(\eta,\omega))\eta)$ obtained substituting ψ by (6.1.34), satisfies (6.2.5) as well. Moreover all the functions appearing as symbols in the above operators are real valued and even in x, so that the requirements of the statement are satisfied.

The fact that we may assume that $R(\eta)$, $R'(\eta,\omega)$, $R''(\eta,\omega,\omega)$ satisfy (6.1.54) follows from the fact that $\partial_t\omega$ in the left hand side of (6.2.27) as well as the first four terms in the right hand side of (6.2.27) are even in x and real valued. This follows from the fact that η,ω are real valued even functions and $Op^{\text{BW}}(a_\ell)$ preserves these properties because a_ℓ is even in (x,ξ) and satisfies $\bar{a}_\ell^\vee = a_\ell$. As a consequence, we may always replace for instance $R(\eta)$ by $\frac{1}{2}[\tau\circ R(\eta) + R(\eta)\circ\tau]$, that ensures that the second condition in (6.1.54) is satisfied by $R(\eta)$, and argue similarly for the other smoothing terms and for the first property in (6.1.54). This concludes the proof. □

We summarize in a corollary the new form of the water waves system that we have obtained by Propositions 6.1.5 and 6.2.1.

Corollary 6.2.2 (Water-waves equations in (η,ω) variables) *Let ρ in $\mathbb{R}$, N in $\mathbb{N}^*$, K in $\mathbb{N}$ be given (large) numbers. There is $r > 0$ and there are*

• *symbols a_0, a_1, a_2 satisfying the properties of Proposition 6.2.1,*

• *symbols $b_1(\eta,\omega;\cdot)$ in $\Sigma\Gamma^1_{K,0,1}[r,N,\mathrm{aut}]$, linear in ω, $b_0(\eta;\cdot)$ belonging to $\Sigma\Gamma^{-1}_{K,0,1}[r,N,\mathrm{aut}]$, even in (x,ξ), satisfying*

$$\text{(6.2.28)}\quad \mathrm{Im}\, b_1 = -V(\eta,\psi)\xi\,,\ \mathrm{Re}\, b_1 \in \Sigma\Gamma^0_{K,0,1}[r,N],\quad \bar b_0^\vee = b_0,\quad \bar b_1^\vee = b_1\,,$$

• *smoothing operators $R_1(\eta)$, $R_1'(\eta,\omega)$, $R_2(\eta)$ and $R_2'(\eta,\omega)$, $R_2''(\eta,\omega,\omega)$ in $\Sigma\mathcal{R}^{-\rho}_{K,0,1}[r,N,\mathrm{aut}]$, with R_1', R_2' linear in ω, R_2'' quadratic in ω, satisfying (6.1.54),*

such that, for any large enough s, if a small enough function (η,ψ) of the space $C^K_*(I, H^{s+\frac14}_{0,\mathrm{ev}}\times \dot H^{s-\frac14}_{\mathrm{ev}})$ solves system (1.1.4), then (η,ω), with ω given by (6.1.31), solves on the same time interval I the system

$$\begin{aligned}\text{(6.2.29)}\quad \partial_t\eta &= (D\tanh D)\omega + Op^{\mathrm{BW}}(b_1(\eta,\omega;\cdot))\eta + Op^{\mathrm{BW}}(b_0(\eta;\cdot))\omega\\ &\quad + R_1(\eta)\omega + R_1'(\eta,\omega)\eta\\ \partial_t\omega &= -(1+\kappa D^2)\eta + Op^{\mathrm{BW}}(a_2(\eta;\cdot))\eta + Op^{\mathrm{BW}}(a_1(\eta,\omega;\cdot))\omega\\ &\quad + Op^{\mathrm{BW}}(a_0(\eta,\omega;t,\cdot))\eta + R_2(\eta)\eta + R_2'(\eta,\omega)\omega + R_2''(\eta,\omega,\omega)\eta\,.\end{aligned}$$

System (6.2.29) is reversible with respect to the involution $S = \left[\begin{smallmatrix}1 & 0\\ 0 & -1\end{smallmatrix}\right]$, namely writing (6.2.29) as

$$\text{(6.2.30)}\qquad \partial_t\left[\begin{smallmatrix}\eta\\ \omega\end{smallmatrix}\right] = M(\eta,\omega;t)\left[\begin{smallmatrix}\eta\\ \omega\end{smallmatrix}\right]$$

then $M(\eta,\omega;-t)S = -SM([\eta,\omega]_S;t)$ where $[\eta,\omega]_S$ is defined in (6.2.3) (this condition is like (3.1.18)). For the autonomous terms of the vector field this condition amounts to the reversibility property (1.2.4). As a consequence any solution of (6.2.29) satisfies (1.2.5). Moreover the operator $M(\eta,\omega;t)$ sends real valued functions into real valued functions and $M(\eta,\omega;t)\circ\tau = \tau\circ M(\eta,\omega;t)$ commutes with the map τ defined in (3.1.16).

Proof: If (η,ψ) is in a small enough ball of center zero in $C^K_*(I, H^{s+\frac14}_{0,\mathrm{ev}}\times \dot H^{s-\frac14}_{\mathrm{ev}})$ and ω is defined by (6.1.31), we may apply Proposition 6.1.4, that allows us to recover ψ from η,ω by (6.1.34). By (6.1.55), the first equation $\partial_t\eta = G(\eta)\psi$ of the water waves system may be written as the first equation in (6.2.29) with

$$b_1(\eta,\omega;\cdot) = c^0(\eta,\omega;\cdot) - iV(\eta,\psi)\xi\quad b_0 = b^0\,.$$

Then (6.1.53) and the fact that V is real valued and odd imply that (6.2.28) holds. Moreover b_1 is even in (x,ξ) because c^0 and $V(\eta,\psi)\xi$ are even in (x,ξ). The second equation in (6.2.29) is (6.2.8).

Finally, writing (6.2.29) as the system (6.2.30), since $b_1(\eta,\omega)$, $R_1'(\eta,\omega)$, $a_1(\eta,\omega)$, $R_2'(\eta,\omega)$ are linear in ω, $R_2''(\eta,\omega,\omega)$ is quadratic in ω, and the

symbol $a_0(\eta,\omega;t,\cdot)$ satisfies (6.2.5), we deduce that the operator $M(\eta,\omega;t)$ satisfies $M(\eta,\omega;-t)S=-SM([\eta,\omega]_S;t)$. Moreover the operator $M(\eta,\omega;t)$ sends real valued functions into real valued functions, and commutes with τ, because the symbols a_0,a_1,a_2 and b_0,b_1 satisfy $\bar{a}_j^\vee=a_j$, $\bar{b}_j^\vee=b_j$, are even in (x,ξ), and the smoothing operators R_1, R_1', R_2, R_2', R_2'' satisfy (6.1.54). This concludes the proof. □

6.3 The capillarity-gravity water waves equations in complex coordinates

We denote by $\Lambda_\kappa(D)$ the Fourier multiplier with symbol

$$\Lambda_\kappa(\xi)=\Big(\frac{\xi\tanh\xi}{1+\kappa\xi^2}\Big)^{1/4}(1-\chi(\xi)) \tag{6.3.1}$$

where χ is an even smooth function, equal to one close to zero and supported for $|\xi|<\frac{1}{2}$. Notice that, on the space of periodic functions with zero mean, or on periodic functions modulo constants, the operator $\Lambda_\kappa(D)$ is just given by

$$\Lambda_\kappa(D)=\Big(\frac{D\tanh D}{1+\kappa D^2}\Big)^{1/4}. \tag{6.3.2}$$

We denote by $\Lambda_\kappa(D)^{-1}$ the inverse of $\Lambda_\kappa(D)$ acting from the space of zero mean functions to itself. For η in $H^{s+\frac{1}{4}}_{0,\mathrm{ev}}(\mathbb{T}^1,\mathbb{R})$ and ω in $\dot{H}^{s-\frac{1}{4}}_{\mathrm{ev}}(\mathbb{T}^1,\mathbb{R})$ we consider the complex function

$$u=\Lambda_\kappa(D)\omega+i\Lambda_\kappa(D)^{-1}\eta \tag{6.3.3}$$

which belongs to the space

$$\dot{H}^s_0(\mathbb{T}^1,\mathbb{C})\stackrel{\mathrm{def}}{=}\Big\{u\in H^s_{\mathrm{ev}}(\mathbb{T}^1,\mathbb{C});\int_{\mathbb{T}^1}\mathrm{Im}\,u\,dx=0\Big\}\Big/\mathbb{R} \tag{6.3.4}$$

where $H^s_{\mathrm{ev}}(\mathbb{T}^1,\mathbb{C})$ denotes the subspace of even functions in $H^s(\mathbb{T}^1,\mathbb{C})$. This space is endowed with the norm $\|u\|_{\dot{H}^s}=\big(\sum_1^{+\infty}n^{2s}\|\Pi_n u\|^2_{L^2}\big)^{\frac{1}{2}}$. Inverting (6.3.3) we may express

$$\eta=\Lambda_\kappa(D)\Big[\frac{u-\bar{u}}{2i}\Big],\quad \omega=\Lambda_\kappa(D)^{-1}\Big[\frac{u+\bar{u}}{2}\Big], \tag{6.3.5}$$

where we make here a small abuse of notation, as, in contrast with (6.3.3), we consider here $\Lambda_\kappa(D)^{-1}$ as an operator sending a space of functions modulo constants to itself.

Notice that, since the symbol $\Lambda_\kappa(\xi)$ in (6.3.1) is real and even, the corresponding Fourier multipliers $\Lambda_\kappa(D), \Lambda_\kappa(D)^{-1}$ map real valued, resp. even, functions into real valued, resp. even, functions.

We now prove that system (6.2.29) in the real variables (η, ω), may be written, in the complex variable $U = \left[\begin{smallmatrix} u \\ \bar u \end{smallmatrix}\right]$, as system (3.2.2). Notice first that in these complex coordinates the real involution $\left[\begin{smallmatrix} \eta \\ \omega \end{smallmatrix}\right] \mapsto \left[\begin{smallmatrix} \eta \\ -\omega \end{smallmatrix}\right]$ considered for system (6.2.29) (or (1.1.4)) translates into the complex involution

$$S : \left[\begin{smallmatrix} u \\ \bar u \end{smallmatrix}\right] \mapsto -\left[\begin{smallmatrix} \bar u \\ u \end{smallmatrix}\right], \quad \text{with matrix} \quad S = -\left[\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right]$$

(that for simplicity of notation we denote with the same letter S). The real system (6.2.30) may be written in the new complex coordinates as

$$D_t U = X(U;t)U \tag{6.3.6}$$

where $D_t = \frac{1}{i}\partial_t$, the operator $X(U;t)$ satisfies the reality condition (3.1.14), it is parity preserving (according to definition (3.1.17)) and reversible (according to (3.1.18)), namely $X(U;-t) = -SX(U_S;t)S$ with the new involution $S = -\left[\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right]$.

We provide below the para-differential structure of system (6.3.6) which is suitable for proving energy estimates.

Proposition 6.3.1 (Water waves equations in complex coordinates) *Let (η, ω) be a solution of system (6.2.29) belonging to the space*

$$C^K_*(I, H^{s+\frac14}_{0,\mathrm{ev}}(\mathbb{T}^1, \mathbb{R})) \times C^K_*(I, \dot H^{s-\frac14}_{\mathrm{ev}}(\mathbb{T}^1, \mathbb{R}))$$

for some large enough s, K, some time interval I, symmetric with respect to $t = 0$, (η, ω) being small enough in that space. Define

$$\begin{aligned} \zeta(U;t,x) &= \frac12\big[(1+\eta'(t,x)^2)^{-3/2} - 1\big] \\ &= \frac12\Big[\Big(1 + \Big(\partial_x \Lambda_\kappa(D)\Big(\frac{u-\bar u}{2i}\Big)\Big)^2\Big)^{-3/2} - 1\Big]. \end{aligned} \tag{6.3.7}$$

Recall that we introduced the matrices

$$\mathcal{I}_2 = \left[\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right], \quad \mathcal{K} = \left[\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right], \quad \mathcal{J} = \left[\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right], \quad \mathcal{L} = \left[\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right] \tag{6.3.8}$$

and set

$$m_\kappa(\xi) = (\xi \tanh \xi)^{1/2}(1 + \kappa\xi^2)^{1/2}(1 - \chi(\xi)) \tag{6.3.9}$$

where χ is as in (6.3.1). Let N, ρ be given positive integers. Then, for $r > 0$ small enough, there are symbols λ_j in $\Sigma\Gamma^j_{K,1,1}[r, N]$, $j = -\frac12, 0, \frac12, 1$, satisfying

$$\operatorname{Im}\lambda_1 \in \Sigma\Gamma^0_{K,1,1}[r,N], \quad \operatorname{Im}\lambda_{\frac12} \in \Sigma\Gamma^{-\frac12}_{K,1,1}[r,N] \tag{6.3.10}$$

and

$$
\begin{aligned}
(\bar\lambda_j)^\vee = \lambda_j,\ j = -\frac{1}{2}, \frac{1}{2},\ (\bar\lambda_j)^\vee = -\lambda_j,\ j = 0, 1 \\
\lambda_j(U;t,x,\xi) = \lambda_j(U;t,-x,-\xi),\ j = -\frac{1}{2}, 0, \frac{1}{2}, 1 \\
\lambda_j(U;-t,x,\xi) = \lambda_j(U_S;t,x,\xi),\ j = -\frac{1}{2}, \frac{1}{2} \\
\lambda_j(U;-t,x,\xi) = -\lambda_j(U_S;t,x,\xi),\ j = 0, 1
\end{aligned}
\tag{6.3.11}
$$

such that $U = \left[\begin{smallmatrix} u \\ \bar u \end{smallmatrix}\right]$ *with* u *given by (6.3.3) solves the system*

$$
D_t U = Op^{\mathrm{BW}}(A(U;t,x,\xi))U + R(U;t)U \tag{6.3.12}
$$

where $A(U;t,x,\xi)$ *is the matrix of symbols*

$$
\begin{aligned}
A(U;t,x,\xi) = \big(m_\kappa(\xi)(1+\zeta(U;t,x)) + \lambda_{\frac{1}{2}}\big)\mathcal{K} \\
+ \big(m_\kappa(\xi)\zeta(U;t,x) + \lambda_{-\frac{1}{2}}\big)\mathcal{J} + \lambda_1 \mathcal{I}_2 + \lambda_0 \mathcal{L}\,,
\end{aligned}
\tag{6.3.13}
$$

that satisfies the reality, parity preserving, reversibility properties

$$
\begin{aligned}
\bar A^\vee(U;t,x,\xi) = -SA(U;t,x,\xi)S \\
A(U;t,-x,-\xi) = A(U;t,x,\xi) \\
A(U;-t,x,\xi) = -SA(U_S;t,x,\xi)S
\end{aligned}
\tag{6.3.14}
$$

where S *is the involution defined in (3.1.1), and* $R(U;t)$ *is a smoothing operator in* $\Sigma\mathcal{R}^{-\rho}_{K,1,1}[r,N] \otimes \mathcal{M}_2(\mathbb{C})$ *satisfying as well the reality, parity preserving, reversibility properties*

$$
\begin{aligned}
\overline{R(U;t)V} = -SR(U;t)S\overline{V} \\
R(U;t)\circ\tau = \tau \circ R(U;t) \\
R(U;-t) = -SR(U_S;t)S\,.
\end{aligned}
\tag{6.3.15}
$$

Remark: Some of the above symbols or smoothing operators are indeed autonomous and belong to the corresponding smaller class with the index $K' = 0$ instead of $K' = 1$. But since this fact will not be used later we shall systematically use this weaker statement. The paradifferential system (6.3.12) is the one presented in section 3.2 to start the proof of Theorem 3.2.1.

Proof: Recalling (6.3.3), to compute the equation satisfied by u, we insert (6.2.29) in

$$
\partial_t u = \Lambda_\kappa(D)\partial_t\omega + i\Lambda_\kappa(D)^{-1}\partial_t\eta\,. \tag{6.3.16}
$$

The linear contribution is

$$i(D\tanh D)^{\frac12}(1+\kappa D^2)^{\frac12}u = iOp^{\rm BW}(m_\kappa(\xi))u \tag{6.3.17}$$

as follows from (6.3.2) and the fact that $Op^{\rm BW}(m_\kappa(\xi))$, where $m_\kappa(\xi)$ is the symbol defined in (6.3.9), coincides with the operator $(D\tanh D)^{\frac12}(1+\kappa D^2)^{\frac12}$ when acting on periodic functions with zero mean or on periodic functions modulo constants.

The only other term of order $\frac32$ comes from $Op^{\rm BW}(a_2(\eta;\cdot))\eta$ in the formula of $\partial_t\omega$ in (6.2.29). Expressing η as a function of $(u,\bar u)$ as in (6.3.5), it is given by

$$\Lambda_\kappa(D)Op^{\rm BW}(a_2(\eta;\cdot))\Lambda_\kappa(D)\Big[\frac{u-\bar u}{2i}\Big]\,. \tag{6.3.18}$$

We now apply the composition results in Propositions 2.4.1 and 2.4.2. By the properties of # in Definition 2.3.1, the symbol $(\Lambda_\kappa(\xi)\#a_2\#\Lambda_\kappa(\xi))_\rho$ is equal to $a_2(\eta;x,\xi)\Lambda_\kappa(\xi)^2$ modulo a symbol of order $-\frac12$ (see in particular (2.3.4) applied with $a=c$), and (6.3.18) may be written as

$$\begin{aligned} iOp^{\rm BW}(\tilde a_{\frac32}(U;\cdot))(u-\bar u) + iOp^{\rm BW}(\tilde a_{-\frac12}(U;\cdot))(u-\bar u)\\ + R(U)(u-\bar u)\end{aligned} \tag{6.3.19}$$

where $\tilde a_{\frac32}$ is the symbol in $\Sigma\Gamma^{\frac32}_{K,0,1}[r,N,{\rm aut}]\subset\Sigma\Gamma^{\frac32}_{K,1,1}[r,N]$ given by

$$\tilde a_{\frac32} = -\frac12 a_2(\eta;x,\xi)\Lambda_\kappa(\xi)^2\,, \tag{6.3.20}$$

$\tilde a_{-\frac12}$ is a symbol in $\Sigma\Gamma^{-\frac12}_{K,0,1}[r,N,{\rm aut}]\subset\Sigma\Gamma^{-\frac12}_{K,1,1}[r,N]$ and $R(U)$ is a smoothing operator in $\Sigma\Gamma^{-\rho}_{K,0,1}[r,N,{\rm aut}]\subset\Sigma\Gamma^{-\rho}_{K,1,1}[r,N]$. The symbol a_2 is given by the right hand side of (6.2.7) plus a symbol $\tilde a_2^{(0)}$ of order zero. Since $\Lambda_\kappa(\xi)$ defined in (6.3.1) has order $-1/4$, the symbol $\tilde a_2^{(0)}(\eta;x,\xi)\Lambda_\kappa(\xi)^2$ may be added to $\tilde a_{-\frac12}$ in (6.3.19). By (6.2.7) the symbol $\tilde a_{\frac32}$ in (6.3.20) may be written as

$$\frac{\kappa}{2}(1-\chi(\xi))^2\xi^2(\xi\tanh\xi)^{\frac12}(1+\kappa\xi^2)^{-\frac12}[(1+\eta'^2)^{-\frac32}-1] = m_\kappa(\xi)\zeta(U;t,x)+\tilde r_{-1/2}$$

where ζ is defined in (6.3.7), $m_\kappa(\xi)$ in (6.3.9), and $\tilde r_{-1/2}$ is a symbol of order $-\frac12$ that may be incorporated to the $\tilde a_{-1/2}$ term in (6.3.19), up to a change of notation. In conclusion we have written (6.3.19), thus (6.3.18), as

$$\begin{aligned} iOp^{\rm BW}\big(m_\kappa(\xi)\zeta(U;t,x)\big)(u-\bar u) + iOp^{\rm BW}\big(a_{-1/2}(U;t,\cdot)\big)(u-\bar u)\\ + R(U;t)(u-\bar u)\end{aligned} \tag{6.3.21}$$

with $a_{-1/2}$ in $\Sigma\Gamma^{-\frac{1}{2}}_{K,1,1}[r,N]$.

The symbol a_2 satisfies (6.2.6) as well as the even real symbol $\Lambda_\kappa(\xi)$ defined in (6.3.1). Since these properties are preserved by the composition operator $\#$, we conclude that also the symbols $\tilde a_{\frac32}, \tilde a_{-\frac12}$ in (6.3.19) satisfy them. In addition, by (6.3.7) and (6.3.9), also $m_\kappa(\xi)\zeta(U;t,x)$ satisfies (6.2.6). It follows that also the symbol $a_{-\frac12}$ in (6.3.21) satisfy these conditions. Moreover, these symbols, in particular $a_{-\frac12}$, as functions of (η,ψ) do not depend on ψ (as a_2).

The contributions of order one to $\partial_t u$ in (6.3.16) come from the term $Op^{\rm BW}(a_1)\omega$ in the right hand side of the second equation (6.2.29) and from the term $Op^{\rm BW}(b_1)\eta$ of the first equation (6.2.29), and are given by

$$\Lambda_\kappa(D)Op^{\rm BW}(a_1)\omega + i\Lambda_\kappa(D)^{-1}Op^{\rm BW}(b_1)\eta\,. \tag{6.3.22}$$

By (6.2.4) and Corollary 6.2.2 we have that

$$a_1 = -iV(\eta,\psi)\xi\,,\quad b_1 = -iV(\eta,\psi)\xi \tag{6.3.23}$$

modulo a symbol of order zero. Moreover a_1, b_1 are linear in ψ, even in (x,ξ), and satisfy $\bar a_1^\vee = a_1$, $\bar b_1^\vee = b_1$, as stated in Proposition 6.2.1 and Corollary 6.2.2. Substituting (6.3.23) and the expressions (6.3.5) of η and ω, (6.3.22) may be written as

$$\begin{aligned}\Lambda_\kappa(D)Op^{\rm BW}(-iV(\eta,\psi)\xi)\Lambda_\kappa(D)^{-1}\Big(\frac{u+\bar u}{2}\Big)\\ + i\Lambda_\kappa(D)^{-1}Op^{\rm BW}(-iV(\eta,\psi)\xi)\Lambda_\kappa(D)\Big(\frac{u-\bar u}{2i}\Big)\\ + Op^{\rm BW}(e_{0,+})u + Op^{\rm BW}(e_{0,-})\bar u\end{aligned}$$

where $e_{0,\pm}$ are symbols of order zero, even in (x,ξ), satisfy $\bar e_{0,\pm}^\vee = e_{0,\pm}$, and they are linear in ψ. By symbolic calculus (i.e. Propositions 2.4.1 and 2.4.2), we may write this expression as

$$i\big[Op^{\rm BW}(\lambda_1(\eta,\psi))u + Op^{\rm BW}(\lambda_0(\eta,\psi))\bar u\big] \tag{6.3.24}$$

up to contributions $R_1(\eta,\psi)u + R_2(\eta,\psi)\bar u$ with smoothing operators R_1, R_2 in $\Sigma\mathcal{R}^{-\rho}_{K,0,1}[r,N,{\rm aut}] \subset \Sigma\mathcal{R}^{-\rho}_{K,1,1}[r,N]$, where

- λ_1 is in $\Sigma\Gamma^1_{K,0,1}[r,N,{\rm aut}] \subset \Sigma\Gamma^1_{K,1,1}[r,N]$, with $\operatorname{Im}\lambda_1$ in $\Sigma\Gamma^0_{K,1,1}[r,N]$,
- λ_0 is in $\Sigma\Gamma^0_{K,0,1}[r,N,{\rm aut}] \subset \Sigma\Gamma^0_{K,1,1}[r,N]$.

Moreover the symbols λ_0, λ_1 satisfy

$$\bar\lambda_\ell^\vee = -\lambda_\ell,\quad \lambda_\ell(\eta,\psi;t,-x,-\xi) = \lambda_\ell(\eta,\psi;t,x,\xi), \tag{6.3.25}$$

and $\lambda_\ell(\eta,\psi;t,x,\xi)$, $\ell = 0,1$, are linear in ψ.

The remaining terms involving paradifferential operators in (6.2.29) bring to $\partial_t u$ in (6.3.16) the following contributions

$$\Lambda_\kappa(D)Op^{\rm BW}(a_0(\eta,\omega;t,\cdot))\Lambda_\kappa(D)\Big[\frac{u-\bar u}{2i}\Big] + i\Lambda_\kappa(D)^{-1}Op^{\rm BW}(b_0(\eta;\cdot))\Lambda_\kappa(D)^{-1}\Big[\frac{u+\bar u}{2}\Big].$$

Since a_0 is a symbol of order zero, b_0 of order -1, and $\Lambda_\kappa(\xi)$ of order $-1/4$, by symbolic calculus we may write it as

$$iOp^{\rm BW}(\mu_{-1/2})u - iOp^{\rm BW}(\lambda_{-1/2})\bar u + R_1(\eta,\omega;t)u + R_2(\eta,\omega;t)\bar u \tag{6.3.26}$$

where $\mu_{-1/2}$ and $\lambda_{-1/2}$ are in $\Sigma\Gamma^{-\frac12}_{K,1,1}[r,N]$. In addition, according to the properties of a_0, b_0 listed in Proposition 6.2.1 and Corollary 6.2.2,

$$\begin{aligned}(\bar\mu_{-\frac12})^\vee = \mu_{-\frac12}, \quad \mu_{-\frac12}(\eta,\psi;t,-x,-\xi) = \mu_{-\frac12}(\eta,\psi;t,x,\xi),\\ (\bar\lambda_{-\frac12})^\vee = \lambda_{-\frac12}, \quad \lambda_{-\frac12}(\eta,\psi;t,-x,-\xi) = \lambda_{-\frac12}(\eta,\psi;t,x,\xi)\,,\end{aligned}$$

and, since a_0 satisfies (6.2.5), as well as b_0 which depends only on η,

$$\mu_{-\frac12}(\eta,\psi;-t) = \mu_{-\frac12}([\eta,\psi]_S;t)\,, \quad \lambda_{-\frac12}(\eta,\psi;-t) = \lambda_{-\frac12}([\eta,\psi]_S;t)\,,$$

where $[\eta,\psi]_S$ is defined in (6.2.3).

In conclusion (6.3.16), (6.3.17), (6.3.21), (6.3.24), (6.3.26) imply that

$$\begin{aligned}D_t u = Op^{\rm BW}[m_\kappa(\xi)(1+\zeta)+\lambda_{1/2}]u - Op^{\rm BW}[m_\kappa(\xi)\zeta+\lambda_{-1/2}]\bar u\\ + Op^{\rm BW}(\lambda_1)u + Op^{\rm BW}(\lambda_0)\bar u + R_1(U;t)u + R_2(U;t)\bar u\end{aligned} \tag{6.3.27}$$

where λ_j are new symbols in $\Sigma\Gamma^j_{K,1,1}[r,N]$ and satisfy

$$\begin{aligned}&\text{Im}\,\lambda_{1/2} \text{ is of order } -\frac12\,,\ \text{Im}\,\lambda_1 \text{ is of order } 0\\ &(\bar\lambda_{1/2})^\vee = \lambda_{1/2}\,,\ (\bar\lambda_{-1/2})^\vee = \lambda_{-1/2}\\ &(\bar\lambda_1)^\vee = -\lambda_1\,,\ (\bar\lambda_0)^\vee = -\lambda_0\\ &\lambda_j \text{ are even functions of } (x,\xi).\end{aligned} \tag{6.3.28}$$

Notice that we have actually proved that the symbol $\lambda_{1/2}$ is of order $-\frac12$, and not just its imaginary part, but we conserve only those properties of the symbols that are useful for us, and that are preserved under the reductions of Chapter 3 and Chapter 4. Moreover, as we have seen above

$$\begin{aligned}&\lambda_j(U;-t,x,\xi) = -\lambda_j(U_S;t,x,\xi), \quad j=0,1\,,\\ &\lambda_j(U;-t,x,\xi) = \lambda_j(U_S;t,x,\xi), \quad j=\frac12,-\frac12\end{aligned} \tag{6.3.29}$$

(where S is now the complex involution defined in (3.1.1)).

Taking the complex conjugate of (6.3.27), we get, because of (2.2.27), the fact that $m_\kappa(\xi)$ is even in ξ, and (6.3.28),

$$D_t\bar u = -Op^{\rm BW}[m_\kappa(\xi)(1+\zeta)+\lambda_{1/2}]\bar u + Op^{\rm BW}[m_\kappa(\xi)\zeta+\lambda_{-1/2}]u + Op^{\rm BW}(\lambda_1)\bar u + Op^{\rm BW}(\lambda_0)u - \overline{R_1(U;t)u} - \overline{R_2(U;t)\bar u}\,. \tag{6.3.30}$$

Finally, we write the equations (6.3.27) and (6.3.30) as a system in the variable $U = \left[\begin{smallmatrix} u \\ \bar u \end{smallmatrix}\right]$. We get, using notation (6.3.8),

$$D_tU = Op^{\rm BW}\Big([m_\kappa(\xi)(1+\zeta)+\lambda_{1/2}]\mathcal{K} + [m_\kappa(\xi)\zeta+\lambda_{-1/2}]\mathcal{J} + \lambda_1\mathcal{I}_2 + \lambda_0\mathcal{L}\Big)U + R(U;t)U \tag{6.3.31}$$

for some matrix R of smoothing operators in $\Sigma\mathcal{R}^{-\rho}_{K,1,1}[r,N]$. We have finally proved the paradifferential form (6.3.12)-(6.3.13) of system (6.3.6).

Since $S\mathcal{K}S = -\mathcal{K}, S\mathcal{J}S = -\mathcal{J}, S\mathcal{L}S = \mathcal{L}, S\mathcal{I}_2S = \mathcal{I}_2$, it follows from (6.3.28) that the matrix $A(U;t,\cdot)$ giving the symbol in the right hand side of (6.3.31) satisfies the reality and parity preserving conditions (3.1.4), (3.1.6). Moreover conditions (6.3.29) are equivalent to the reversibility condition (3.1.7). We have thus checked that all properties (6.3.10) to (6.3.14) hold.

Finally, by comparing (6.3.6) and (6.3.12) we have that

$$R(U;t) = X(U;t) - Op^{\rm BW}(A(U;t,\cdot))\,.$$

As stated after (6.3.6) the operator $X(U;t)$ satisfies the reality, parity preserving, reversibility properties (3.1.14), (3.1.17), (3.1.18). Since the operator $Op^{\rm BW}(A(U;t,\cdot))$ satisfies as well (3.1.14), (3.1.17), (3.1.18) (by (6.3.14)) we deduce that $R(U;t)$ satisfies (6.3.15). This concludes the proof. □

Remark: In the above system (6.3.12), (6.3.13), the symbols λ_j are in the class $\Sigma\Gamma^j_{K,1,1}[r,N]$ and the smoothing operator $R(U;t)$ belongs to the space $\Sigma\mathcal{R}^{-\rho}_{K,1,1}[r,N]\otimes\mathcal{M}_2(\mathbb{C})$. This implies, recalling (2.2.37) and (2.2.30) for $k=0$ and $K'=1$, that the Sobolev norms of the non-homogeneous terms in the right hand side of (6.3.12) may be estimated from the Sobolev norms of U and of ∂_tU. This dependence of the bounds on first order time derivatives is irrelevant to derive energy inequalities for the water waves system (6.3.12). On the other hand, we shall also need to estimate time derivatives of U by the space derivatives, as claimed in (6.3.32) of the next proposition. For that we use the water waves equation in its initial formulation (1.1.4).

Proposition 6.3.2 *Let (η,ψ) be a solution of system (1.1.4) defined on some interval I and belonging to the space*

$$C^0(I, H^{s+\frac14}_{0,\rm ev}(\mathbb{T}^1,\mathbb{R})\times \dot H^{s-\frac14}_{\rm ev}(\mathbb{T}^1,\mathbb{R}))\,.$$

Let $U = \left[\begin{smallmatrix} u \\ \bar{u} \end{smallmatrix}\right]$ where u is defined in (6.3.3). For any $0 \leq k \leq K$ there is a constant C_k such that, as long as $U(t,\cdot)$ stays in the unit ball of $\dot{H}^s(\mathbb{T}^1, \mathbb{C}^2)$ with $s \gg K$, one has the estimate

$$\|\partial_t^k U(t,\cdot)\|_{\dot{H}^{s-\frac{3}{2}k}} \leq C_k \|U(t,\cdot)\|_{\dot{H}^s}. \tag{6.3.32}$$

Proof: We proceed by induction. Assume that (6.3.32) has been proved for $k = 0, \dots, k' \leq K-1$. The assumption $\|U(t,\cdot)\|_{\dot{H}^s} \leq 1$ implies, in particular, that

$$\sum_{k=0}^{k'} \|\partial_t^k U(t,\cdot)\|_{\dot{H}^{s-\frac{3}{2}k'}} \leq \tilde{C}_{k'} \tag{6.3.33}$$

for some constant $\tilde{C}_{k'}$, uniformly for t in I. To prove (6.3.32) with $k = k'+1$, it is enough, according to (6.3.3) and (6.3.2), to show that

$$\|\partial_t^{k'+1}\omega\|_{\dot{H}^{s-\frac{1}{4}-\frac{3}{2}(k'+1)}} + \|\partial_t^{k'+1}\eta\|_{H_0^{s+\frac{1}{4}-\frac{3}{2}(k'+1)}} \leq C\|U(t,\cdot)\|_{\dot{H}^s}.$$

By (6.1.31), the fact that $a_0(\eta,\psi)$ is in $\Sigma\mathcal{F}_{K,0,1}[r,N]$ and the bounds for paradifferential operators of Proposition 2.2.4, it is enough to show that

$$\|\partial_t^{k'+1}\psi\|_{\dot{H}^{s-\frac{1}{4}-\frac{3}{2}(k'+1)}} + \|\partial_t^{k'+1}\eta\|_{H_0^{s+\frac{1}{4}-\frac{3}{2}(k'+1)}} \leq C\|U(t,\cdot)\|_{\dot{H}^s} \tag{6.3.34}$$

or, equivalently, that

$$\|\partial_t^{k'+1}\psi\|_{\dot{H}^{s-\frac{1}{4}-\frac{3}{2}(k'+1)}} + \|\partial_t^{k'+1}\eta\|_{H_0^{s+\frac{1}{4}-\frac{3}{2}(k'+1)}} \leq C\big(\|\psi\|_{\dot{H}^{s-\frac{1}{4}}} + \|\eta\|_{H_0^{s+\frac{1}{4}}}\big) \tag{6.3.35}$$

since $\|U\|_{\dot{H}^s}$ is equivalent to $\|\psi\|_{\dot{H}^{s-\frac{1}{4}}} + \|\eta\|_{H_0^{s+\frac{1}{4}}}$ by (6.3.3), (6.3.2) and (6.1.31), (6.1.34). One may rewrite (1.1.4) as

$$\begin{aligned} \partial_t \eta &= G(\eta)\psi \\ \partial_t \psi &= \mathcal{F}(\eta, \eta', \eta'', \partial_x\psi, G(\eta)\psi) \end{aligned}$$

where $\mathcal{F}$ is some analytic function vanishing at the origin. Thus we write

$$\partial_t^{k'+1}\psi = \partial_t^{k'}\big(\mathcal{F}(\eta,\eta',\eta'',\partial_x\psi, G(\eta)\psi)\big), \quad \partial_t^{k'+1}\eta = \partial_t^{k'}(G(\eta)\psi),$$

and, since for $s \gg K$, $H^{s-\frac{3}{2}K}$ is an algebra, we may estimate the left hand side of (6.3.34) in terms of

$$\|\partial_t^k(G(\eta)\psi)\|_{H_0^{s-\frac{3}{2}+\frac{1}{4}-\frac{3}{2}k}}, \quad \|\partial_t^k\eta\|_{H_0^{s+\frac{1}{4}-\frac{3}{2}k}}, \quad \|\partial_t^k\psi\|_{\dot{H}^{s+\frac{3}{4}-\frac{3}{2}(k+1)}}$$

for $k \leq k'$. Let us first prove that, for any $k \leq k'$,

$$\|\partial_t^k(G(\eta)\psi)\|_{H_0^{s-\frac{3}{2}+\frac{1}{4}-\frac{3}{2}k}} \leq C\|U(t,\cdot)\|_{\dot{H}^s}. \tag{6.3.36}$$

We use that $G(\eta)\psi$ is expressed from (6.1.55) where the function $V(\eta,\psi)$ is in $\Sigma\mathcal{F}_{K,0,1}[r,N]$, the symbol b^0 in $\Sigma\Gamma^{-1}_{K,0,1}[r,N]$, c^0 in $\Sigma\Gamma^{0}_{K,0,1}[r,N]$, and the smoothing remainders R_1, R_2 in $\Sigma\mathcal{R}^{-\rho}_{K,0,1}[r,N]$. According to Proposition 2.2.4 applied to symbols of order one, (2.2.30) with $k=0$, $K'=0$, and the first remark after Definition 2.2.3, we get for any $k\leq k'$

$$\|\partial_t^k(G(\eta)\psi)\|_{H_0^{s-\frac{3}{2}+\frac{1}{4}-\frac{3}{2}k}} \leq C\sum_{k''\leq k}\left(\|\partial_t^{k''}\psi\|_{\dot{H}^{s-\frac{1}{4}-\frac{3}{2}k''}} + \|\partial_t^{k''}\eta\|_{H_0^{s-\frac{1}{4}-\frac{3}{2}k''}}\right)$$

as long as (6.3.33) holds. Thus (6.1.34), the fact that $a(\eta,\psi)$ is in $\Sigma\mathcal{F}_{K,0,1}[r,N]$, (6.3.5) and the inductive assumption (6.3.32) for $k''\leq k\leq k'$, imply (6.3.36). Similarly we deduce that, for any $0\leq k\leq k'$,

$$\|\partial_t^k\eta\|_{H_0^{s+\frac{1}{4}-\frac{3}{2}k}} + \|\partial_t^k\psi\|_{\dot{H}^{s+\frac{3}{4}-\frac{3}{2}(k+1)}} \leq C\|U(t,\cdot)\|_{\dot{H}^s}\,,$$

and (6.3.34) is proved. □

Chapter 7

Proof of some auxiliary results

7.1 Non resonance condition

Recall that we defined in (3.2.5), for κ a positive number, and n a positive integer

$$m_\kappa(n) = (n \tanh n)^{\frac{1}{2}}(1 + \kappa n^2)^{\frac{1}{2}}. \tag{7.1.1}$$

Moreover, if p is in $\mathbb{N}$, $-1 \leq \ell \leq p+1$, $n_0, \ldots, n_{p+1}$ are in $\mathbb{N}^*$, we have defined the "small divisors"

$$\mathcal{D}_\ell(n_0, \ldots, n_{p+1}) = \sum_{j=0}^{\ell} m_\kappa(n_j) - \sum_{j=\ell+1}^{p+1} m_\kappa(n_j). \tag{7.1.2}$$

The goal of this section is to prove the following:

Proposition 7.1.1 (Non resonance condition) *There is a subset $\mathcal{N}$ of $]0, +\infty[$ with zero measure such that, for any compact interval $[a, b] \subset]0, +\infty[$, there is an integer $N_0 \in \mathbb{N}$, and for any κ in $[a, b] - \mathcal{N}$, there is a positive constant c such that the inequality*

$$|\mathcal{D}_\ell(n_0, \ldots, n_{p+1})| \geq c \max(n_0, \ldots, n_{p+1})^{-N_0} \tag{7.1.3}$$

holds for any $(n_0, \ldots, n_{p+1})$ in $(\mathbb{N}^*)^{p+2}$ if p is odd or p is even and $\ell \neq \frac{p}{2}$, and for any $(n_0, \ldots, n_{p+1})$ in $(\mathbb{N}^*)^{p+2}$ such that

$$\{n_0, \ldots, n_\ell\} \neq \{n_{\ell+1}, \ldots, n_{p+1}\} \tag{7.1.4}$$

when p is even and $\ell = \frac{p}{2}$.

To prove the proposition, we may fix a compact interval $[a, b]$. The assumption made in the statement about $n_0, \ldots, n_{p+1}$ means that we have to show

that, given q in $\mathbb{N}$, $q \geq 2$, and $(c_1, \ldots, c_q)$ in $(\mathbb{Z}^*)^q$, there exist N_0 in $\mathbb{N}$, $\delta > 0$, $c > 0$ such that for all $\gamma \in]0,1[$, there is a subset $\mathcal{N}_\gamma$ of $[a,b]$ of measure $O(\gamma^\delta)$ such that, for any κ in $[a,b] - \mathcal{N}_\gamma$, any integers $1 \leq j_1 < \cdots < j_q$, we have

$$\Big|\sum_{i=1}^{q} c_i m_\kappa(j_i)\Big| \geq c\gamma\Big(\sum_{i=1}^{q} |j_i|\Big)^{-N_0}. \tag{7.1.5}$$

We define $\tau(\xi) = (\xi \tanh \xi)^{1/2}$ and associate to $(j_1, \ldots, j_q)$ the points of the interval $[0,1]$ defined by

$$x_0 = \Big(\sum_{\ell=1}^{q}(|j_\ell| + \tau(j_\ell))\Big)^{-1},\ x_i = x_0 j_i,\ x_{i+q} = \zeta_i = x_0 \tau(j_i), \tag{7.1.6}$$

$i = 1, \ldots, q$. We denote $X = [0,1]^{2q+1}$ so that $(x_0, \ldots, x_q, \zeta_1, \ldots, \zeta_q) = (x_0, \ldots, x_q, x_{1+q}, \ldots, x_{2q}) \in X$. We shall deduce (7.1.5) from Theorem 5.1 in [30], applied with the preceding space X and $Y = [a,b]$ to the function

$$f(x,\kappa) = \sum_{i=1}^{q} c_i \zeta_i \sqrt{x_0^2 + \kappa x_i^2} \tag{7.1.7}$$

and taking

$$\rho(x) = x_0 \prod_{1 \leq i_1 < i_2 \leq q} (x_{i_1} - x_{i_2})\Big(\sum_{i=1}^{q} \zeta_i\Big). \tag{7.1.8}$$

Let us check that the assumptions of Theorem 5.1 of [30] hold. The function f is continuous and subanalytic. We have to check moreover that f is analytic on the set $\{x \in X; \rho(x) \neq 0\} \times Y$. Since

(7.1.9)

$$\{x \in X; \rho(x) \neq 0\} = \Big\{x \in X; x_0 \neq 0, x_{i_1} \neq x_{i_2}, 1 \leq i_1 < i_2 \leq q, \sum_{1}^{q} \zeta_i \neq 0\Big\}$$

is contained in $\{x_0 \neq 0\}$, this is clear. Moreover, we have to verify that for all x in X such that $\rho(x) \neq 0$, the function $\kappa \to f(x,\kappa)$ has only finitely many zeros in $[a,b]$. This is a consequence of the following lemma.

Lemma 7.1.2 *Let $\rho(x) \neq 0$. Then the analytic function $\kappa \to f(x,\kappa)$ is not identically zero on $[a,b]$.*

Proof: By (7.1.9), if $\rho(x) \neq 0$, we have that $x_0 \neq 0$, $x_i \neq x_j$ if $1 \leq i < j \leq q$ and $\sum_1^q \zeta_i \neq 0$. In particular, not all ζ_i vanish and we may assume $\zeta_{i_1} > 0, \ldots, \zeta_{i_m} > 0$ for some $1 \leq m \leq q$. We have to prove that the function

$$\kappa \to f(x,\kappa) = \sum_{\ell=1}^{m} c_{i_\ell} \zeta_{i_\ell} \sqrt{x_0^2 + \kappa x_{i_\ell}^2} \tag{7.1.10}$$

is not identically zero on $[a,b]$ or equivalently on $[0,+\infty[$. Since $x_0 \neq 0$, up to a change of notation, we may assume that $x_0^2 = 1$ in (7.1.10). Argue by contradiction assuming that (7.1.10) vanishes identically on $[0,+\infty[$.

Case 1: $x_{i_\ell} \neq 0$ for $\ell = 1,\dots,m$. We expand in Taylor series on a neighborhood of $\kappa = 0$

$$\sqrt{1+\kappa x_{i_\ell}^2} = \sum_{n=0}^{+\infty} a_n \kappa^n x_{i_\ell}^{2n} \tag{7.1.11}$$

with Taylor coefficients a_n that are all non zero. If we assume that (7.1.10) with $x_0^2 = 1$ vanishes identically for $\kappa \in [0,+\infty[$, plugging (7.1.11) in (7.1.10) and writing that the coefficients of κ^n in the resulting expression all vanish, we get

$$\sum_{\ell=1}^{m} c_{i_\ell} \zeta_{i_\ell} x_{i_\ell}^{2n} = 0 \tag{7.1.12}$$

for any $n \in \mathbb{N}^*$. As all c_{i_ℓ} are non zero, as well as the ζ_{i_ℓ}, this implies that the Van der Monde determinant

$$\begin{vmatrix} x_{i_1}^2 & \dots & x_{i_m}^2 \\ x_{i_1}^4 & \dots & x_{i_m}^4 \\ \vdots & \vdots & \vdots \\ x_{i_1}^{2m} & \dots & x_{i_m}^{2m} \end{vmatrix} \tag{7.1.13}$$

has to vanish, which is a contradiction as we assumed $x_{i_\ell} \neq 0$, for any ℓ and $x_i \neq x_j$ if $i \neq j$.

Case 2: One of the x_{i_ℓ} vanishes, for instance $x_{i_1} = 0$. Since the x_i are two by two distincts, the other x_{i_ℓ} are nonzero and (7.1.10) with $x_0^2 = 1$ may be written as

$$c_{i_1}\zeta_{i_1} + \sum_{\ell=2}^{m} c_{i_\ell}\zeta_{i_\ell}\sqrt{1+\kappa x_{i_\ell}^2}.$$

If this quantity vanishes, expanding as above in Taylor series at $\kappa = 0$, we conclude that again a Van der Monde determinant in the variables $(x_{i_2},\dots,x_{i_m})$ has to vanish, that contradicts the fact that $x_i \neq x_j$ for $i \neq j$. This concludes the proof. □

End of proof of Proposition 7.1.1: As we have already seen, we have to construct $\mathcal{N}_\gamma$ such that (7.1.5) holds if κ is in $[a,b]-\mathcal{N}_\gamma$. We apply Theorem 5.1 of [30], whose assumptions are satisfied because of lemma 7.1.2. There exists N_1 in $\mathbb{N}$, $\delta > 0$, $C > 0$ such that, for all $N \geq N_1$, all $x \in X = [0,1]^{2q+1}$ such that $\rho(x) \neq 0$,

$$\text{meas}\left\{\kappa \in [a,b]; |f(x,\kappa)| \leq \gamma|\rho(x)|^N\right\} \leq C\gamma^\delta |\rho(x)|^{N\delta}. \tag{7.1.14}$$

Define $\mathcal{N}_\gamma$ to be the set of κ in $[a,b]$ such that $|f(x,\kappa)| \le \gamma|\rho(x)|^N$ for some x defined by (7.1.6) from some $1 \le j_1 < \cdots < j_q$. Notice that

$$\begin{aligned}|\rho(x)| &= |x_0 \prod_{1\le i_1<i_2\le q} (x_{i_1} - x_{i_2}) \sum_1^q \zeta_i| \\ &\le \big(\sum(|j_i| + \tau(j_i))\big)^{-1} \prod_{1\le i_1<i_2\le q} \frac{|j_{i_1} - j_{i_2}|}{\sum(|j_i| + \tau(j_i))} \frac{\sum \tau(j_i)}{\sum(|j_i| + \tau(j_i))}\end{aligned}$$

satisfies

$$C^{-1}\Big(\sum_{i=1}^q |j_i|\Big)^{-N_2} \le |\rho(x)| \le C\Big(\sum_{i=1}^q |j_i|\Big)^{-1} \tag{7.1.15}$$

with $N_2 = \frac{q^2-q+4}{2}$ since $|j_{i_1} - j_{i_2}| \ge 1$ and $\tau(j_i) \ge (\tanh 1)^{1/2}$. Consequently, the measure of the set $\mathcal{N}_\gamma$ is bounded from above using (7.1.14) and (7.1.15) by

$$\sum_{1\le j_1<\cdots<j_q} C\gamma^\delta \Big(\sum_{i=1}^q |j_i|\Big)^{-N\delta} = O(\gamma^\delta)$$

if N is chosen large enough so that $N\delta > q$. Moreover, when κ is outside $\mathcal{N}_\gamma$, we have

$$|f(x,\kappa)| \ge \gamma|\rho(x)|^N \ge c\gamma\Big(\sum_{i=1}^q |j_i|\Big)^{-NN_2}$$

i.e. taking (7.1.7), (7.1.6) into account

$$\Big|\sum_{i=1}^q c_i m_\kappa(j_i)\Big|\Big(\sum_{\ell=1}^q(|j_\ell| + \tau(j_\ell))\Big)^{-2} \ge c\gamma\Big(\sum_{i=1}^q |j_i|\Big)^{-NN_2}$$

which implies (7.1.5) with $N_0 = NN_2 - 2$. In conclusion the set $\mathcal{N} = \bigcap_{0<\gamma<1} \mathcal{N}_\gamma$ has zero measure, and for all $\kappa \in [a,b] \setminus \mathcal{N}$ the bound (7.1.3) holds for some $c > 0$. □

7.2 Precise structure of the Dirichlet-Neumann operator

The Dirichlet-Neumann operator, in one dimension, is well known to be expressed as $G(\eta)\psi$ in (6.1.55) with a symbol b^0 of order as negative as desired, if η is smooth enough. In our framework, it is sufficient to know that b^0 is of order -1, and this is the assertion that is made in the statement of Proposition 6.1.5. We prove such property in this section.

Proposition 7.2.1 *With the notation of Proposition 6.1.5, the symbol* $b^0(\eta;\cdot)$ *belongs to* $\Sigma\Gamma^{-1}_{K,0,1}[r,N,\mathrm{aut}]$.

We shall use the following lemma. Recall that we have defined in (5.2.11) the symbol

$$e^0_+(\eta;z,x,\xi)=\mathcal{C}(z,\xi)+e^0_{+,1}(\eta;z,x,\xi),$$

whose explicit value is given by w_+ in (5.2.21), with $\mathcal{C}(z,\xi)$ defined in (5.2.3) and $e^0_{+,1}$ in $\Sigma\mathcal{P}^{0,+}_{K,0,1}[r,N]$. Thus the symbol e_+ in (5.2.29) may be written as

$$e_+(\eta;z,x,\xi)=e^0_+(\eta;z,x,\xi)+e^1_{+,1}(\eta;z,x,\xi)+\cdots+e^{\rho-1}_{+,1}(\eta;z,x,\xi) \tag{7.2.1}$$

where $e^j_{+,1}$ is in $\Sigma\mathcal{P}^{-j,+}_{K,0,1}[r,N]$.

Lemma 7.2.2 *The component* $e^1_{+,1}$ *of the symbol* e_+ *in (7.2.1) solves for say* $|\xi|\geq 1$ *the equation*

$$\begin{aligned} &Pe^1_{+,1}=-|\xi|(\operatorname{sgn}\xi-i\eta')^{-2}\eta''(x)e^0_+(\eta;z,x,\xi)\\ &e^1_{+,1}|_{z=0}=0\\ &\partial_z e^1_{+,1}|_{z=-1}=0 \end{aligned} \tag{7.2.2}$$

where $P=(1+\eta'^2)\partial_z^2-2i\eta'\xi\partial_z-\xi^2$ *is the operator introduced in (5.2.12).*

Proof: Recall that by definition $e^1_{+,1}$ solves equation (5.2.32) with $j=1$, whose right hand side $\mathcal{L}_{1,+}(e^0_+)$ is given by the opposite of the symbols of order one in those terms in the right hand side of (5.2.31) that follow the $Op^{\mathrm{BW}}(Pe_+)$ contribution. In other words, we obtain $\mathcal{L}_{1,+}(e^0_+)$ taking the opposite of the principal term in

$$\begin{aligned}\big((\eta'^2\#\partial_z^2e^0_+)_{\rho,N}-\eta'^2\partial_z^2e^0_+\big)-2i\big(((\eta'\xi)\#\partial_ze^0_+)_{\rho,N}-(\eta'\xi)\partial_ze^0_+\big)\\ -\big((\xi^2\#e^0_+)_{\rho,N}-\xi^2e^0_+\big)\end{aligned}$$

so that

$$Pe^1_{+,1}=-\left[\frac{1}{2i}\{\eta'^2,\partial_z^2e^0_+\}-\{\eta'\xi,\partial_ze^0_+\}-\frac{1}{2i}\{\xi^2,e^0_+\}\right] \tag{7.2.3}$$

for $|\xi|\geq 1$. The condition $Pe^0_+=0$ brings an expression for $\partial_z^2e^0_+$ as

$$\partial_z^2e^0_+=2i\frac{\eta'}{1+\eta'^2}\xi\partial_ze^0_+ +\frac{\xi^2}{1+\eta'^2}e^0_+. \tag{7.2.4}$$

Since e^0_+ is given by w_+ in (5.2.21), taking into account (5.2.19), we may write

$$\partial_ze^0_+=\frac{\xi}{1+\eta'^2}\left[i\eta'+\tanh\Big[\frac{(z+1)\xi}{1+\eta'^2}\Big]\right]e^0_+(\eta;z,x,\xi) \tag{7.2.5}$$

modulo a symbol in $\Sigma\mathcal{P}^{-\infty,+}_{K,0,1}[r,N]$ that may be discarded as it will play no role in the computation of the symbol $e^1_{+,1}$ of order -1. Plugging in (7.2.4), we get

$$\partial_z^2 e^0_+ = \frac{\xi^2}{(1+\eta'^2)^2}\Big[(1-\eta'^2)+2i\xi\eta'\tanh\Big[\frac{(z+1)\xi}{1+\eta'^2}\Big]\Big]e^0_+(\eta;z,x,\xi) \tag{7.2.6}$$

modulo a symbol in of order $-\infty$. Note that if z stays in $[-\frac{1}{2},0]$ and $|\xi|\geq 1$, $\tanh\Big[\frac{(z+1)\xi}{1+\eta'^2}\Big]-\operatorname{sgn}\xi$ is of order $-\infty$. Moreover, for z in $[-1,-\frac{1}{2}]$, the symbol e^0_+ is of order $-\infty$ by construction. We may thus rewrite (7.2.5), (7.2.6) as

$$\begin{aligned}\partial_z e^0_+ &= \xi(\operatorname{sgn}\xi - i\eta')^{-1}e^0_+\\ \partial_z^2 e^0_+ &= \xi^2(\operatorname{sgn}\xi - i\eta')^{-2}e^0_+\end{aligned} \tag{7.2.7}$$

modulo again symbols of $\Sigma\mathcal{P}^{-\infty,+}_{K,0,0}[r,N]$. Moreover, from (5.2.21), we deduce that e^0_+ may be written in the region $|\xi|>1$ as $e^{iza\xi+z|\xi|(1+c)}$ modulo symbols of order $-\infty$, so that

$$\begin{aligned}\{\xi,e^0_+\} &= iz\xi\{\xi,\eta'\}(\operatorname{sgn}\xi - i\eta')^{-2}e^0_+\\ \{\eta',e^0_+\} &= z\{\eta',\xi\}(\operatorname{sgn}\xi - i\eta')^{-1}e^0_+\end{aligned} \tag{7.2.8}$$

for $|\xi|\geq 1$, modulo symbols of $\Sigma\mathcal{P}^{-\infty,+}_{K,0,1}[r,N]$. If one plugs (7.2.7) inside (7.2.3) and uses (7.2.8), we get by a direct computation that (7.2.3) is given by

$$-|\xi|(\operatorname{sgn}\xi - i\eta')^{-2}\{\xi,\eta'\}e^0_+$$

for $|\xi|\geq 1$. This gives (7.2.2). □

We compute now the ∂_z derivative of $e^1_{+,1}$.

Lemma 7.2.3 *The solution $e^1_{+,1}$ of (7.2.2) satisfies for $|\xi|\geq 1$*

$$(1+\eta'^2)\partial_z e^1_{+,1}|_{z=0} = -\frac{1}{2}\eta''\frac{(\operatorname{sgn}\xi + i\eta')^2}{1+\eta'^2} \tag{7.2.9}$$

modulo a symbol in $\Sigma\Gamma^{-\infty}_{K,0,1}[r,N,\mathrm{aut}]$.

Proof: According to lemma 5.2.2, the solution $e^1_{+,1}$ of (7.2.2) is given by (7.2.10)

$$e^1_{+,1}(\eta;,z,x,\xi) = -|\xi|(\operatorname{sgn}\xi - i\eta')^{-2}\eta''(x)\int_{-1}^0 K^0(\eta;z,z',x,\xi)e^0_+(z';x,\xi)\,dz'$$

where K^0 is the kernel in (5.2.11). This kernel has been defined in the proof of lemma 5.2.2 as $K^0=(1+\eta'^2)^{-1}\tilde{K}$, with $\tilde{K}$ given by (5.2.27). It follows that (7.2.9) is given by the product of

$$-|\xi|(\operatorname{sgn}\xi - i\eta')^{-2}\eta''(x) \tag{7.2.11}$$

and of

$$\int_{-1}^{0} \partial_z w_-(0,\xi,a,b) w_+(z',\xi,a,b) W(z',\xi,a,b)^{-1} e_+^0(z';x,\xi)\, dz'. \tag{7.2.12}$$

According to (5.2.21) and (5.2.26)

$$w_+(z')W(z')^{-1} = e^{-i(z'+1)a\xi} \cosh((z'+1)\xi(1+c)) \\ \times \Big(1 - i\frac{a}{1+c}\tanh(\xi(z'+1)(1+c))\Big)$$

and

$$\partial_z w_-(0) = e^{ia\xi}\cosh(\xi(1+c))^{-1}\Big(1 - i\frac{a}{1+c}\tanh(\xi(1+c))\Big)^{-1}$$

so that the integral (7.2.12) is given by

$$\int_{-1}^{0} e^{-iaz'\xi} \frac{\cosh((z'+1)\xi(1+c))}{\cosh(\xi(1+c))} e_+^0(z',\xi)\, dz'$$

modulo a symbol of order $-\infty$. Replacing e_+^0 by its value w_+ given by (5.2.21), we reduce the above integral to

$$\int_{-1}^{0} e^{2z'|\xi|(1+c)}\, dz' = \frac{1+\eta'^2}{2|\xi|}$$

modulo a symbol of order $-\infty$. Multiplying this value of (7.2.12) by (7.2.11), we obtain for (7.2.9) an expression that may be written as the right hand side of this equality. □

Proof of Proposition 7.2.1: We want to show that, in the expression of $G(\eta)\psi$ given by (6.1.55), the symbol b^0 is of order -1 (and not just of order zero). We have thus to examine those contributions in (6.1.57) given by paradifferential operators acting on ψ (or on ω). They come from

$$I + II + IV = \partial_z\tilde{\Phi}|_{z=0} + Op^{\mathrm{BW}}(\eta'\otimes\eta')\partial_z\tilde{\Phi}|_{z=0} - Op^{\mathrm{BW}}(\eta')\partial_x\tilde{\Phi}|_{z=0}\,.$$

The smoothing term VI gives the last but one term in (6.1.55). Taking (6.1.58) into account, the paradifferential operators acting on ψ (or on ω) come from

$$\partial_z\Phi|_{z=0} + Op^{\mathrm{BW}}(\eta'\otimes\eta')\partial_z\Phi|_{z=0} - Op^{\mathrm{BW}}(\eta')\partial_x\Phi|_{z=0}\,. \tag{7.2.13}$$

Recall that Φ is given by (6.1.43) where the symbol e_+ in $\Sigma\mathcal{P}^{0,+}_{K,0,0}[r,N]$ and the integral expression and the last term in the right hand side provide smoothing contributions. Note in particular that the integral term $\int_{-1}^{0}\big[Op^{\mathrm{BW}}(K(\eta;z,\tilde{z},\cdot))_{|z=0} f(\tilde{z})\, d\tilde{z}$ in (6.1.43) is smoothing because the function $f(\tilde{z})$ given by (6.1.42) is supported for $\tilde{z}\le -1/8$ and then we use lemma

5.1.14 (condition (5.1.79) holds by (6.1.40), changing the value of σ_0). Modulo such smoothing terms, we may replace in (7.2.13) Φ by $Op^{\rm BW}(e_+(\eta;\cdot))\omega$. Using symbolic calculus we are thus left with showing that the operator

$$(7.2.14)\quad Op^{\rm BW}(1+\eta'^2)Op^{\rm BW}\big(\partial_z e_+|_{z=0}\big) - iOp^{\rm BW}(\eta'\xi)Op^{\rm BW}(e_+|_{z=0}) + \frac{1}{2}Op^{\rm BW}(\eta'')Op^{\rm BW}(e_+|_{z=0})$$

may be written as

$$(7.2.15)\qquad Op^{\rm BW}\big(\xi\tanh\xi + b_{-1}(\eta';\cdot)\big)$$

where b_{-1} is in $\Sigma\Gamma^{-1}_{K,0,1}[r,N,{\rm aut}]$.

Recall that the symbol e_+ in (7.2.14) has the form (7.2.1), namely may be written as

$$e_+ = e^0_+ + e^1_{+,1} + r$$
$$e^0_+ = \mathcal{C}(z,\xi) + e^0_{+,1}(\eta;z,x,\xi)$$

where $e^1_{+,1} \in \Sigma\mathcal{P}^{-1,+}_{K,0,1}[r,N]$ is the solution of (7.2.2), $r = e^2_{+,1} + \cdots + e^{\rho-1}_{+,1}$ is in $\Sigma\mathcal{P}^{-2,+}_{K,0,1}[r,N]$ and $e^0_+ \in \Sigma\mathcal{P}^{0,+}_{K,0,0}[r,N]$ is given by (5.2.11), actually is equal to w_+ in (5.2.21). The contribution coming from r to (7.2.14) may be written as $Op^{\rm BW}(b_{-1}(\eta;\cdot))\omega$ for a symbol of order -1, since r is of order -2. If we consider now the contributions of $e^0_+ + e^1_{+,1}$ to (7.2.14), we see using symbolic calculus that, up to expressions of the form $Op^{\rm BW}(b_{-1})$ with b_{-1} in $\Sigma\Gamma^{-1}_{K,0,1}[r,N,{\rm aut}]$, the operator (7.2.14) is given by a paradifferential operator with symbol

$$(7.2.16)\quad (1+\eta'^2)\partial_z(e^0_+ + e^1_{+,1})|_{z=0} + \frac{1}{2i}\big\{\eta'^2, \partial_z e^0_+|_{z=0}\big\} - i\eta'\xi(e^0_+ + e^1_{+,1})|_{z=0} - \frac{1}{2}\big\{\eta'\xi, e^0_+|_{z=0}\big\} + \frac{1}{2}\eta'' e^0_+|_{z=0}\,.$$

By (5.2.11), (5.2.3), (5.2.10), (7.2.2) we have $e^0_+|_{z=0} = 1, e^1_{+,1}|_{z=0} = 0$ and by (7.2.5),

$$\partial_z e^0_+|_{z=0} = \frac{\xi\tanh\xi + i\eta'\xi}{1+\eta'^2}$$

modulo symbols in $\Sigma\Gamma^{-1}_{K,0,1}[r,N,{\rm aut}]$. Inserting in (7.2.16) these formulas and (7.2.9) we get

$$\xi\tanh\xi + i\eta'\xi - \frac{1}{2}\eta''\frac{(\operatorname{sgn}\xi + i\eta')^2}{1+\eta'^2} + \frac{1}{2i}\frac{1}{1+\eta'^2}\{\eta'^2, |\xi| + i\eta'\xi\} - i\eta'\xi + \frac{1}{2}\eta'' = \xi\tanh\xi$$

modulo symbols in $\Sigma\Gamma^{-1}_{K,0,1}[r,N,{\rm aut}]$ (that depend on η'). We have thus obtained that (7.2.14) has the form (7.2.15), which concludes the proof. □

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