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ON LATTICE PATH MATROID POLYTOPES: INTEGER POINTS AND EHRHART POLYNOMIAL

KOLJA KNAUER, LEONARDO MARTÍNEZ-SANDOVAL,
AND JORGE LUIS RAMÍREZ ALFONSÍN

Abstract. In this paper we investigate the number of integer points lying in dilations of lattice path matroid polytopes. We give a characterization of such points as polygonal paths in the diagram of the lattice path matroid. Furthermore, we prove that lattice path matroid polytopes are affinely equivalent to a family of distributive polytopes. As applications we obtain two new infinite families of matroids verifying a conjecture of De Loera et. al. and present an explicit formula of the Ehrhart polynomial for one of them.

1. Introduction

For general background on matroids we refer the reader to [20, 27]. A *matroid* $M = (E, \mathcal{B})$ of *rank* $r = r(M)$ is a finite set $E = \{1, \ldots, n\}$ together with a non-empty collection $\mathcal{B} = \mathcal{B}(M)$ of r-subsets of E (called the *bases* of M) satisfying the following *basis exchange axiom*:

$$\text{if } B_1, B_2 \in \mathcal{B} \text{ and } e \in B_1 \setminus B_2, \text{ then there exists}$$
$$f \in B_2 \setminus B_1 \text{ such that } (B_1 - e) + f \in \mathcal{B}.$$

For a matroid $M = (E, \mathcal{B})$, the *matroid basis polytope* P_M of M is defined as the convex hull of the incidence vectors of bases of M, that is,

$$P_M := \operatorname{conv}\left\{\sum_{i \in B} e_i : B \text{ a base of } M\right\},$$

here e_i is the ith standard basis vector in $\mathbb{R}^n$. It is well-known [14] that $\dim(P_M) = n - c$ where c is the number of *connected components* of M.

Let $k \geq 0$ be an integer and let $P \subseteq \mathbb{R}^n$ be a polytope. We define $kP := \{kp | p \in P\}$ and the function

$$L_P(k) := \#(kP \cap \mathbb{Z}^n).$$

Note that $L_P(0) = 1$. It is well-known [13] that for integral polytopes, which is the case of matroid basis polytopes, the function L_P extends to a

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polynomial on $\mathbb{R}$, called the *Ehrhart polynomial* of P. The *Ehrhart series* of a polytope P is the infinite series

$$Ehr_P(z) = \sum_{k\geq 0} L_P(k) z^k.$$

A classical result about Ehrhart series states that if $P \subseteq \mathbb{R}^n$ is an integral convex polytope of dimension n then its Ehrhart series is a rational function,

$$Ehr_P(z) = \frac{h^*_P(z)}{(1-t)^{n+1}} = \frac{h^*_0 + h^*_1 t + \cdots + h^*_{n-1} t^{n-1} + h^*_n t^n}{(1-t)^{n+1}}.$$

The coefficients of h^*_P are the entries of the h^*-*vector* of P. The function $L_P(t)$ can be expressed as

$$L_P(t) = \sum_{j=0}^{n} h^*_j \binom{t+n-j}{n}. \tag{1}$$

In this paper we investigate the function $L_{P_M}(t)$ when M is a *lattice path matroid*. After introducing the necessary properties and in particular the subclass of snakes in Section 2, in Section 3 we provide a combinatorial characterization of the points in kP_M (Theorem 3.3) and the integer points in kP_M (Corollaries 3.5 and 3.6). As an application, we obtain an explicit formula for $L_{P_M}(t)$ for an infinite family of snakes (Theorem 3.7) and a matrix formula for $L_{P_M}(t)$ for snakes in general (Theorem 3.9).

We then study in Section 4 the *distributive lattice* structure associated to P_M. We prove that there exists a bijective affine transformation π taking P_M to a full-dimensional *distributive integer* polytope Q_M with $L_{P_M}(t) = L_{Q_M}(t)$ (Theorem 4.1). We use this to make a connection between a natural distributive lattice associated to Q_M (and thus to P_M) and their corresponding *chain partitioned posets* (Theorem 4.5). As an application, we present a characterization of snakes via *order polytopes* (Theorem 4.6). As a consequence we prove unimodality of the h^*-vector for two infinite families of snakes. This provides new evidence for a challenging conjecture of De Loera, Haws, and Köppe [10] (Theorem 4.8) which was only known to hold for the class of rank two uniform matroids and for a finite list of examples.

2. Lattice path matroids

In this section we address the class of lattice path matroids first introduced by Bonin, de Mier, and Noy [5]. Many different aspects of lattice path matroids have been studied: excluded minor results [4], algebraic geometry notions [11, 22, 23], the Tutte polynomial [6, 17, 18], and results around the matroid base polytope [1, 2, 7, 8].

A path in the plane is a *lattice path*, if it starts at the origin and only does steps of the form $+(1,0)$ and $+(0,1)$ and ends at a point (m,r). One way to encode a lattice path P is therefore simply to identify it with a vector $\mathrm{st}(P) = (P_1, \ldots, P_{r+m})$, where $P_i \in \{0,1\}$ corresponds to the y-coordinate

of the ith step of P for all $1 \leq i \leq r+m$. The vector $\mathrm{st}(P)$ is the *step vector* of P. We will often identify P with its step vector without explicitly mentioning it. Note that since $\mathrm{st}(P) \in \{0,1\}^{r+m}$ the path P can also be identified with a subset of $\{1,\ldots,r+m\}$ or cardinality r. Let L, U be two lattice paths from $(0,0)$ to (m,r), such that L never goes above U. The *lattice path matroid (LPM)* associated to U and L is the matroid $M[U,L]$ on the ground set $\{1,\ldots,m+r\}$ whose base set corresponds to all lattice paths from $(0,0)$ to (m,r) never going below L and never going above U. In [5, Theorem 3.3] it was proved, that $M[U,L]$ is indeed a matroid. See Figure 1 for an illustration.

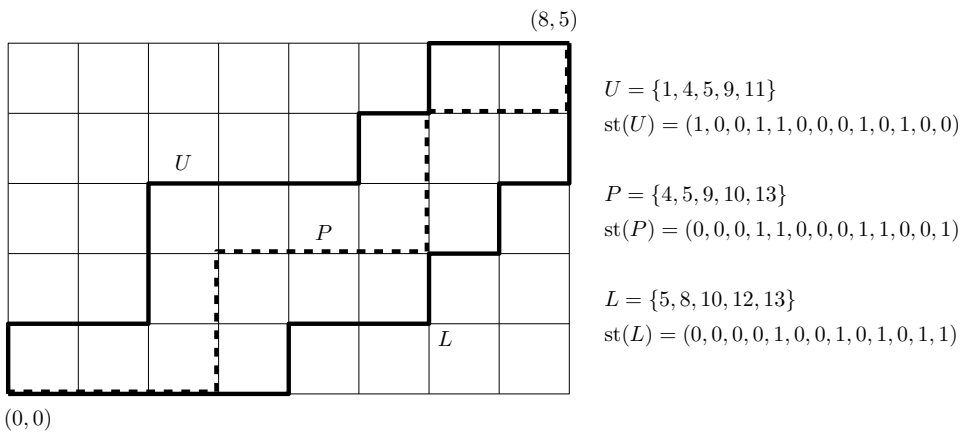

FIGURE 1. Left: Lattice paths U and L from $(0,0)$ to $(8,5)$ and a path P staying between U and L in the diagram of $M[U,L]$. Right: Representations of U, L, and P as subsets of $\{1,\ldots,13\}$ and as step vectors.

It is known [5] that the class of LPMs is closed under deletion, contraction, and duality. Indeed, to see the latter, for an LPM M, a base in the the dual matroid M^* consists of the 0-steps of a lattice path in the diagram of M. Thus, reflecting the diagram of M along the diagonal $x = y$ yields a diagram for M^* and shows that M^* is an LPM as well. See Figure 2.

LPMs are also closed under direct sum. The direct sum in terms of diagrams is illustrated in Figure 3. In particular, we shall later use the fact ([5, Theorem 3.6]) that the LPM $M[U,L]$ is connected if and only if paths U and L meet only at $(0,0)$ and (m,r).

It is known that if $M = M_1 \oplus \cdots \oplus M_n$ is the *direct sum* of matroids then $P_M = P_{M_1} \times \cdots \times P_{M_n}$ is the *Cartesian product* of the corresponding basis polytopes. Since the Ehrhart polynomial of the Cartesian product of two integral polytopes is just the product of their Ehrhart polynomials, it will be enough to work with *connected* matroids, that is, matroids that are not the direct sum of two non-trivial matroids. We thus often suppose that $\dim(P_M) = n-1$ where n is the number of elements of M.

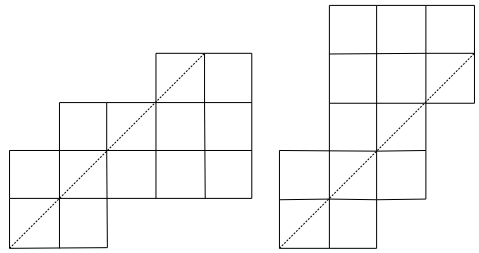

FIGURE 2. Presentations of an LPM and its dual.

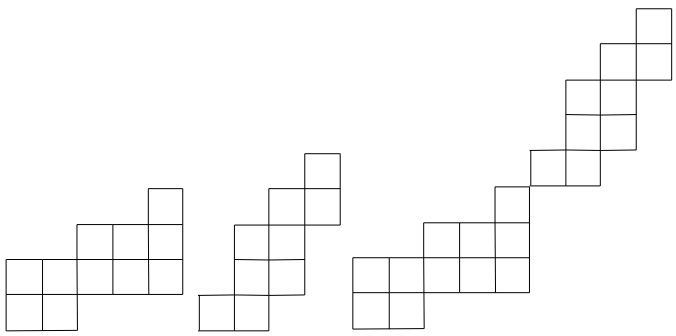

FIGURE 3. Diagrams of two LPMs and their direct sum.

An LPM is called *snake* if it has at least two elements, it is connected and its diagram has no interior lattice points, see Figure 4. Note that snakes have also been called border strips in [1, 2].

We represent a snake as $S(a_1, a_2, \ldots, a_n)$ if starting from the origin its diagram encloses $a_1 \geq 1$ squares to the right, then $a_2 \geq 2$ squares up, then $a_3 \geq 2$ squares to the right and so on up to $a_n \geq 2$, where the last square counted by each a_i coincides with the first square counted by a_{i+1} for all $i \leq n-1$. Moreover, unless $M = S(1) = U_{1,2}$, since Ehrhart polynomials of basis polytopes are invariant under matroid duality, we will suppose by duality of snakes that $a_1 \geq 2$, as well.

The base polytope P_M will be called *snake polytope* if M is a snake.

3. Integer points in lattice path matroid polytopes

We shall present a halfspace description of P_M when M is a connected LPM. In order to do this, we give an attractive geometric interpretation of the points in P_M in terms of polygonal paths. Let $M = M[U, L]$ be a rank r connected LPM with $r + m$ elements. Let l_i be the line defined by $x + y = i$ for each $i = 0, \ldots, r + m$ and denote by $R(M[U, L])$ the region bounded by U and L. Let $T_i = l_i \cap R(M[U, L])$ for each $i = 0, \ldots, r + m$, that is, T_i is the segment of l_i contained in R. We notice that the endpoints of T_i are

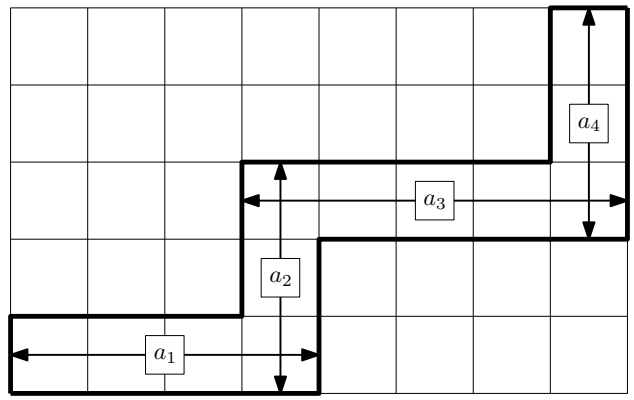

FIGURE 4. The diagram of the snake $S(a_1, a_2, a_3, a_4)$.

given by the intersection of l_i with U and L. Moreover, $T_0 = \{(0,0)\}$ and $T_{r+m} = \{(r, r+m)\}$.

We define a *generalized lattice path* P as a polygonal path formed by $r+m$ segments $S_{i+1}(P)$ joining (x_i, y_i) to (x_{i+1}, y_{i+1}) where $x_i, y_i \in T_i$, $x_i \leq x_{i+1}$ and $y_i \leq y_{i+1}$ for each $i = 0, \ldots, r+m-1$. Notice that a generalized lattice path is an ordinary lattice path if and only if all its coordinates (x_i, y_i) are integer points.

The points (x_i, y_i) will be called *bend points* (points where P may change slope). Let $\mathrm{st}(P) = (P_1, \ldots, P_{r+m})$ where $P_{i+1} = y_{i+1} - y_i$ for each $i = 0, \ldots, r+m-1$, i.e., $\mathrm{st}(P)$ stores the y-steps of the segments $S_i(P)$.

Example 3.1. *We construct the three generalized lattice paths A, B and C in the snake $S(1,2)$ given in Figure 5.*

A is formed by segments : $S_1(A) = \overline{(0,0)(\frac{1}{4}, \frac{3}{4})}$, $S_2(A) = \overline{(\frac{1}{4}, \frac{3}{4})(\frac{1}{2}, \frac{3}{2})}$ and $S_3(A) = \overline{(\frac{1}{2}, \frac{3}{2})(1,2)}$,

B is formed by segments : $S_1(B) = \overline{(0,0)(0,1)}$, $S_2(B) = \overline{(0,1)(1,1)}$ and $S_3(B) = \overline{(1,1)(1,2)}$ (B is an ordinary path corresponding to a base of $S(1,2)$) and

C is formed by segments : $S_1(C) = \overline{(0,0)(\frac{3}{4}, \frac{1}{4})}$, $S_2(C) = \overline{(\frac{3}{4}, \frac{1}{4})(\frac{3}{4}, \frac{5}{4})}$ and $S_3(C) = \overline{(\frac{3}{4}, \frac{5}{4})(1,2)}$.

We have that $\mathrm{st}(A) = (\frac{3}{4}, \frac{3}{4}, \frac{1}{2}), \mathrm{st}(B) = (1,0,1)$ and $\mathrm{st}(C) = (\frac{1}{4}, 1, \frac{3}{4})$.

Remark 3.2. *Let P be a generalized lattice path in the diagram of $M[U,L]$ of rank r and with $r+m$ elements. We have*

a) *P starts at $(x_0, y_0) = (0,0)$ and ends at $(x_{r+m}, y_{r+m}) = (r, r+m)$.*
b) *P is monotonously increasing since $x_i \leq x_{i+1}$ and $y_i \leq y_{i+1}$ for each $i = 0, \ldots, r+m-1$.*
c) *If P is a lattice path corresponding to a base in $M[U,L]$ then, P is a generalized lattice path where either $x_{i+1} = x_i + 1$ and $y_{i+1} = y_i$*

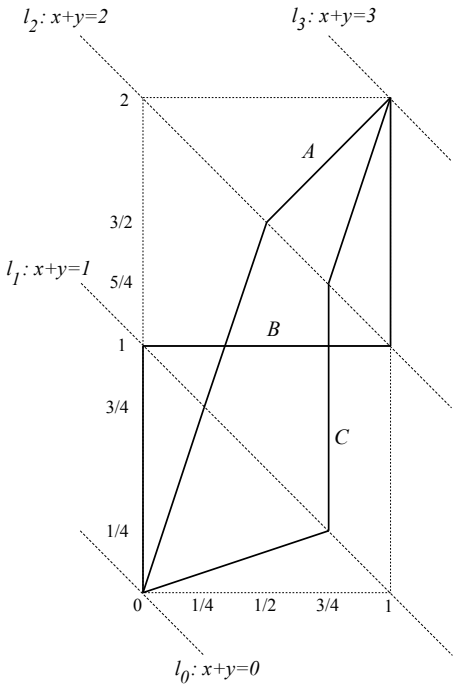

FIGURE 5. Three generalized lattice paths A, B and C in the snake $S(1,2)$

or $x_{i+1} = x_i$ *and* $y_{i+1} = y_i + 1$ *for each* $i = 0, \ldots, r+m-1$. *We thus have that* $\mathrm{st}(P) \in \{0,1\}^{r+m}$ *and the notion of step vector for generalized paths generalizes step vectors of ordinary lattice paths.*

d) *Since* $\mathrm{dist}(l_i, l_{i+1}) = 1$ *and* $x_{i+1} - x_i \geq 0$ *(resp.* $y_{i+1} - y_i \geq 0$*) then* $1 \geq y_{i+1} - y_i$ *(resp.* $1 \geq x_{i+1} - x_i$*).*

e) *Let* $\mathrm{st}(P) = (P_1, \ldots, P_{r+m})$. *Then, by definition, we clearly have that*

$$\sum_{j=1}^{i} L_j \leq \sum_{j=1}^{i} P_j \leq \sum_{j=1}^{i} U_j \text{ for all } i \in [r+m].$$

In particular, $\sum_{i=1}^{r+m} P_i = r$.

Let $\mathcal{C}_M$ be the family of step vectors of all the generalized lattice paths in $M[U,L]$.

Theorem 3.3. *Let* $M = M[U,L]$ *be a rank* r *LPM with* $r+m$ *elements and let* $\mathrm{st}(L) = (L_1, \ldots, L_{r+m})$ *and* $\mathrm{st}(U) = (U_1, \ldots, U_{r+m})$. *Then,* $\mathcal{C}_M$ *equals*

$$\left\{ p \in \mathbb{R}^{r+m} \mid 0 \leq p_i \leq 1 \text{ and } \sum_{j=1}^{i} L_j \leq \sum_{j=1}^{i} p_j \leq \sum_{j=1}^{i} U_j \text{ for all } i \in [r+m] \right\}.$$

Proof. Let $\mathrm{st}(P) = (P_1, \ldots, P_{r+m}) \in \mathcal{C}_M$. By definition $P_{i+1} = y_{i+1} - y_i$ for each $i = 0, \ldots, r+m-1$. Thus, $\mathrm{st}(P)$ satisfies the first set of inequalities by Remark 3.2 (d). By Remark 3.2 (e) we conclude that $\mathrm{st}(P) \in \mathbb{R}^{r+m}$ satisfies the remaining inequalities.

Let conversely $p \in \mathbb{R}^{r+m}$ such that $0 \leq p_i \leq 1$ and $\sum_{j=1}^{i} L_j \leq \sum_{j=1}^{i} p_j \leq \sum_{j=1}^{i} U_j$ for all $i \in [r+m]$. We consider the points

$$(x_0, y_0) = (0,0) \text{ and } (x_i, y_i) = (i - \sum_{j=1}^{i} p_j, \sum_{j=1}^{i} p_j) \text{ for each } i = 1, \ldots, r+m.$$

We clearly have that $y_{i+1} \geq y_i$ for all i since $p_i \geq 0$. Moreover,

$$x_{i+1} - x_i = (i+1) - \sum_{j=1}^{i+1} p_j - (i - \sum_{j=1}^{i} p_j) = 1 - p_{i+1}$$

but $1 - p_{i+1} \geq 0$ since $p_i \leq 1$.

Now, $(x_i, y_i) \in T_i$, indeed, $x_i + y_i = i - \sum_{j=1}^{i} p_j + \sum_{j=1}^{i} p_j = i$ and thus (x_i, y_i) belongs to line l_i. Moreover, (x_i, y_i) belongs to $R(M[U,L])$ since $\sum_{j=1}^{i} L_j \leq \sum_{j=1}^{i} p_j \leq \sum_{j=1}^{i} U_j$ and thus $\sum_{j=1}^{i}(1 - L_j) \geq \sum_{j=1}^{i}(1 - p_j) \geq \sum_{j=1}^{i}(1 - U_j)$. Therefore, the points (x_i, y_i) form the generalized lattice path C with $\mathrm{st}(C) = (p_1, \ldots, p_{r+m})$.

□

Theorem 3.4. *Let $M = M[U,L]$ be a rank r LPM on $r+m$ elements. Then, $P_M = \mathcal{C}_M$.*

Proof. We first prove that $P_M \subseteq \mathcal{C}_M$. By Remark 3.2 any base of M corresponds to a generalized lattice path. Therefore, by Theorem 3.3, any vertex of P_M belongs to $\mathcal{C}_M$ and since $\mathcal{C}_M$ is convex (it has a halfspace intersection description) then $P_M \subseteq \mathcal{C}_M$.

We now prove that $P_M \supseteq \mathcal{C}_M$. We show that every element in $\mathcal{C}_M$ is a convex combination of step vectors corresponding to ordinary lattice paths. We proceed by induction on the number n of elements of M. If $n = 1$, then there are only ordinary paths, so we are done. Now suppose $n > 1$. If $M = M_1 \oplus M_2$ is disconnected, we have $P_M = P_{M_1} \times P_{M_2}$ and by induction $P_{M_1} \supseteq \mathcal{C}_{M_1}$ and $P_{M_2} \supseteq \mathcal{C}_{M_2}$. This gives $P_M \supseteq \mathcal{C}_{M_1} \times \mathcal{C}_{M_2}$ where the latter consists of step vectors of concatenated generalized lattice paths in M_1 and M_2. Thus, $\mathcal{C}_{M_1} \times \mathcal{C}_{M_2} = \mathcal{C}_M$

Suppose now that M is connected. Note that $\mathcal{C}_M$ is contained in the $(n-1)$-dimensional subspace H defined by the equality

$$\sum_{j=1}^{r+m} L_j = \sum_{j=1}^{r+m} p_j = \sum_{j=1}^{r+m} U_j = r.$$

Let C be a point on the boundary of $\mathcal{C}_M$ with respect to H. Hence, C satisfies the equality in one of the inequalities of the halfspace description of $\mathcal{C}_M$ of Theorem 3.3. If $C_i = 0$ for some $i \in [n]$, then we can consider $M \setminus i$, which corresponds to all lattice paths of M with a 0 in the ith coordinate. By induction C is in the convex hull of these vectors. The dual argument works if $C_i = 1$. If $\sum_{j=1}^{i} L_j = \sum_{j=1}^{i} C_j$, we know that $\sum_{j=1}^{i} C_j < \sum_{j=1}^{i} U_j$ since the case of both equalities cannot happen in a connected M. Thus, C coincides with L at a point $(x, y) \in \mathbb{Z}^2$ which is not in U. We consider the lattice path matroid M' with lower path L and upper path U', where U' arises from U by going right from the point $(x', y) \in U$ until reaching (x, y) and then up until reaching the point $(x, y') \in U$. Now, M' contains C and is not connected. By applying the above argument for disconnected LPMs we get $C \in P_{M'}$, but clearly we have $P_{M'} \subseteq P_M$. If C coincides with U at a point which is not in L the analogous argument works. We thus have shown that all points on the boundary of $\mathcal{C}_M$ are in P_M, that is, $P_M \supseteq \mathcal{C}_M$. □

The above geometric description of the points in P_M seems to be new as far as we are aware. The equality

$$P_M = \left\{ p \in \mathbb{R}^n \mid \forall i \in [n] : 0 \leq p_i \leq 1; \sum_{j=1}^{i} L_j \leq \sum_{j=1}^{i} p_j \leq \sum_{j=1}^{i} U_j \right\}$$

was already stated in [2, Lemma 3.8] but it seems that the given proof contains a wrong argument. Indeed, in the proof a vector $B := (a - a_i X)/(1 - a_i)$ is defined, where a is a vector satisfying the inequalities describing $\mathcal{C}_M$ without any coordinate in $\{0, 1\}$, a_i a smallest entry of a, and X_i a $0,1$-step vector of an ordinary lattice path such that $X_i = 1$. It is claimed that B also satisfies the inequalities of $\mathcal{C}_M$, in particular verifying that $0 \leq B_i \leq 1$ for all i. However, if $a_j = 1 - \frac{a_i}{2}$ and $X_j = 0$, then $B_j = a_j/(1 - a_i) = (1 - \frac{a_i}{2})/(1 - a_i) > 1$.

Let $\mathcal{C}_M^k$ be the family of step vectors of all the generalized lattice paths P of $M[U, L]$ such that all the bend points (x, y) of P satisfy $kx, ky \in \mathbb{Z}$. The following two corollaries can be easily deduced from Theorems 3.3 and 3.4.

Corollary 3.5. *Let M be an LPM on n elements and let $k \in \mathbb{N}$. Then, a point $p \in \mathbb{R}^n$ is in $kP_M \cap \mathbb{Z}^n$ if and only if p corresponds to a generalized lattice path in $\mathcal{C}_M^k$.*

Corollary 3.6. *Let $M = M[U, L]$ be a rank r LPM on $r + m$ elements. Then, a point $p \in \mathbb{R}^{r+m}$ is in the interior of P_M if and only if p corresponds to a generalized lattice path P of M such that*

- *$P \cap U = P \cap L = L \cap U$,*
- *P is strictly monotone, that is, $x_i < x_{i+1}$ and $y_i < y_{i+1}$ for all i.*

3.1. **Application: a formula for $L_{P_{S(a,b)}}(t)$.** In [10], $L_{P_M}(t)$ is explicitly calculated for 28 selected matroids M and in [16] for all uniform matroids. The following result provides an explicit formula for $L_{P_{S(a,b)}}(t)$.

Theorem 3.7. *Let $a, b \geq 2$ be integers. Then, the Ehrhart polynomial of $P_{S(a,b)}$ is given by*

$$1 + \frac{1}{(a-1)!(b-1)!} \sum_{i=1}^{a+b-1} \left(\sum_{j=i-1}^{a+b-2} (-1)^{j-i+1} \cdot \frac{B_{j-i+1} \cdot \sigma_{a+b-2-j}}{j+1} \binom{j+1}{i} \right) t^i.$$

where σ_ℓ is the ℓth symmetric function on the numbers

$$1, 2, \ldots, a-1, 1, 2, \ldots, b-1$$

and B_m is the mth Bernoulli number.

In order to prove this, we will state a couple of general results for counting the number of integer points in $kP_{S(a_1,\ldots,a_n)}$. For this purpose, we will use the description given by Corollary 3.6, so we shall focus on counting the number of generalized lattice paths whose coordinates are integer multiples of $\frac{1}{k}$. Later on we will see how these results simplify in the case $n = 2$ and yield the formula in Theorem 3.7.

We begin with the following simple combinatorial lemma. We refer the reader to Figure 6 for a geometric interpretation that relates the lemma to generalized lattice paths.

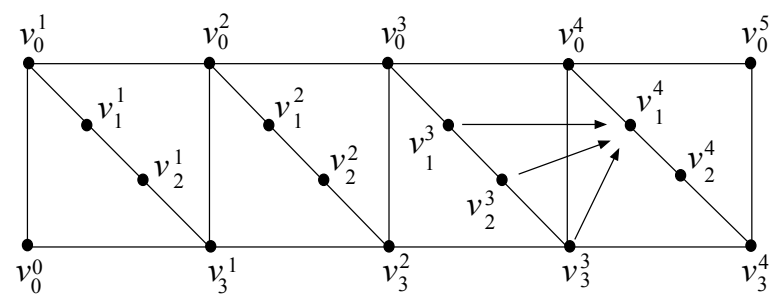

FIGURE 6. General construction.

Lemma 3.8. *Let k and a be positive integers. Let $u^1 = (u_0^1, \ldots, u_k^1)$ be a vector of positive integers. For $i = 1, \ldots, a-1$ define recursively the entries of vectors u^i as follows:*

$$u_j^{i+1} = \sum_{\ell=j}^{k} u_\ell^i, \qquad \text{for } j \in \{0, 1, \ldots, k\}.$$

Then $u^a = A(k,a)u^1$ where $A(k,a)$ is the $(k+1) \times (k+1)$ matrix with entries A_{ij} given by:

$$A_{ij} = \binom{a-2+j-i}{j-i}.$$

Proof. Consider the $(k+1)\times(k+1)$ matrix J with entries J_{ij} given by

$$J_{ij} = \begin{cases} 1 & \text{for } j \geq i \\ 0 & \text{for } j < i. \end{cases}$$

Using this matrix the recursion can be stated as $u^{i+1} = Ju^i$. Therefore, $u^a = J^{a-1}u^1$. An easy induction argument shows that $J^{a-1} = A(k,a)$. □

Suppose now that we want to count the number of generalized lattice paths in the snake $S(a_1, a_2, \ldots, a_n)$ whose coordinates are integer multiples of $\frac{1}{k}$. A standard technique is to count the number of paths recursively starting in the lower left corner and then writing at each bend point p the number of possible ways to get to that point. The number in p equals to the sum of the numbers in the possible points that precede p.

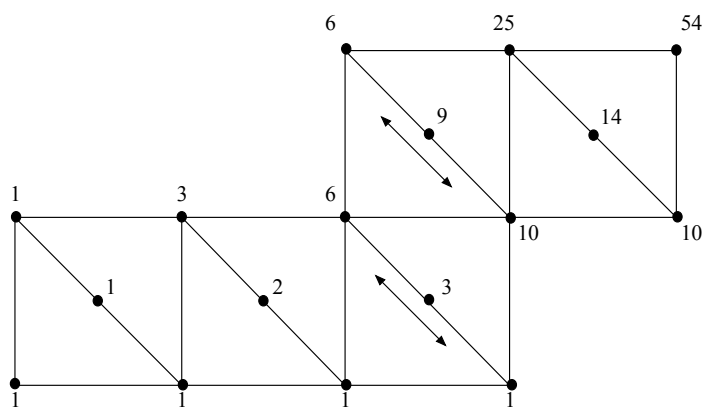

FIGURE 7. Counting bends horizontally and vertically.

As long as the snake is horizontal, the process above yields the recursion in Lemma 3.8: we add the numbers in points before and below p. However, whenever the snake bends the numbers in the diagonal play an "inverse role" and we have to invert the recursion accordingly, see Figure 7. If we proceed inductively on the number of bends in the snake we obtain the following result.

Theorem 3.9. *Let $k \geq 1$ be an integer. Then, the number of integer points in $kP_{S(a_1,\ldots,a_n)}$ is*

$$u^T \left(\prod_{j=1}^{n} A(k,a_i)R \right) u,$$

where $u = (1,1,\ldots,1)$ is the vector with $k+1$ ones, R is the matrix that inverts the coordinates of a vector and the matrices $A(k,a_i)$ are defined as in Lemma 3.8.

When $n = 2$ we have a snake with just one bend, say $S(a, b)$. This allows the formula above to be simplified to a polynomial in k.

Proof of Theorem 3.7. Theorem 3.9 states that the number of lattice points in the dilation $kP_{S(a,b)}$ is equal to

$$u^T A(k, b) R A(k, a) u.$$

First, the jth entry of $A(k, a)u$ is:

$$\sum_{i=0}^{k+1-j} \binom{a-2+i}{i} = \binom{a+k-j}{k+1-j}.$$

When we invert the coordinates, we get the vector

$$\left(\binom{a-1}{0}, \binom{a}{1}, \ldots, \binom{a+k-1}{k} \right).$$

Now we have to multiply from the left by $u^T A(k, b)$ or, equivalently, multiply by $A(k, b)$ and sum the coordinates of the obtained vector. After multiplying we can arrange the sum of entries as follows:

$$\begin{aligned}
\sum_{j=0}^{k} \sum_{i=0}^{j} \binom{a-1+j}{j} \binom{b-2+i}{i} &= \sum_{j=0}^{k} \binom{a-1+j}{j} \sum_{i=0}^{j} \binom{b-2+i}{i} \\
&= \sum_{j=0}^{k} \binom{a-1+j}{j} \binom{b-1+j}{j} \\
&= \sum_{j=0}^{k} \binom{a-1+j}{a-1} \binom{b-1+j}{b-1} \\
&= \frac{1}{(a-1)!(b-1)!} \sum_{j=0}^{k} \left((a-1+j) \cdots (j+1) \right) \left((b-1+1) \ldots (j+1) \right) \\
&= \frac{H(0) + H(1) + \ldots + H(k)}{(a-1)!(b-1)!}.
\end{aligned}$$

In the last line H is the polynomial

$$H(t) = (t+1)(t+2) \cdots (t+a-1)(t+1)(t+2) \cdots (t+b-1).$$

The proof ends by expanding the polynomial using symmetric functions on the multiset $\{1, 2, \ldots, a-1, 1, 2, \ldots, b-1\}$, grouping terms with the same t exponent, using standard formulas for the sums of first powers and regrouping as a polynomial in k.

□

4. Distributive lattice structure

In this section we will study a lattice structure induced by lattice path matroids. For more on posets and lattices we refer the reader to [9].

4.1. **Distributive polytopes.** A polytope $P \subseteq \mathbb{R}^n$ is called *distributive* if for all $x, y \in P$ also their componentwise maximum and minimum $\max(x, y)$ and $\min(x, y)$ are in P.

Distributive polytopes are those polytopes whose point set forms a distributive lattice as a sublattice of the componentwise ordering of $\mathbb{R}^n$. The latter is a distributive lattice itself with join and meet operations componentwise maximum and minimum, respectively. Distributive polytopes have been characterized combinatorially and geometrically in [15].

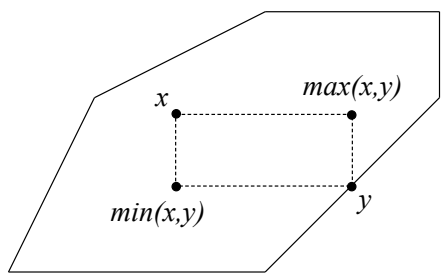

Figure 8. A distributive polytope in $\mathbb{R}^2$

The following relates P_M with distributive polytopes when M is an LPM.

Theorem 4.1. *Let $M = M[U, L]$ be a rank r connected LPM on $r + m$ elements. Then, there exists a bijective affine transformation taking $P_M \subset \mathbb{R}^{r+m}$ into a full-dimensional distributive integer polytope $Q_M \subset \mathbb{R}^{r+m-1}$ consisting of all $q \in \mathbb{R}^{r+m-1}$ such that*

$$0 \leq (-1)^{L_{i+1}}(q_{i+1} - q_i) \leq 1 \qquad \text{for all } i \in [r + m - 2]$$

and

$$0 \leq q_i \leq \sum_{j=1}^{i} (U_j - L_j) \qquad \text{for all } i \in [r + m - 1].$$

Moreover, $L_{P_M}(t) = L_{Q_M}(t)$.

Proof. Let $\mathrm{st}(L) = (L_1, \ldots, L_{r+m})$ and $\mathrm{st}(U) = (U_1, \ldots, U_{r+m})$. Then, by Theorems 4.6 and 3.4, we have that

$$P_M = \left\{ p \in \mathbb{R}^{r+m} \mid 0 \leq p_i \leq 1 \text{ and } \sum_{j=1}^{i} L_j \leq \sum_{j=1}^{i} p_j \leq \sum_{j=1}^{i} U_j \text{ for all } i \in [r + m] \right\}.$$

Let

$$\begin{array}{rlcl} \pi : & P_M \subset \mathbb{R}^{r+m} & \longrightarrow & \mathbb{R}^{r+m-1} \\ & p = (p_1, \ldots, p_{r+m}) & \mapsto & (p_1 - L_1, \ldots, \sum_{j=1}^{r+m-1}(p_j - L_j)) \end{array}$$

We thus have that π is an affine mapping consisting of a translation by $-\mathrm{st}(L)$ and of the linear map using the above halfplane description of P_M. Clearly, π is injective. Let p be a point in P_M and let P the corresponding generalized path of M with $\mathrm{st}(P) = p$. Let $\pi(p) = \pi(p_1, \ldots, p_{r+m}) = (q_1, \ldots, q_{r+m-1})$. By Remark 3.2 (e) we have both

$$\sum_{j=1}^{i} L_j \leq \sum_{j=1}^{i} P_j \text{ for all } i \in [r+m] \text{ and thus } q_i = \sum_{j=1}^{i}(P_i - L_j) \geq 0$$

and

$$\sum_{j=1}^{i} P_j \leq \sum_{j=1}^{i} U_j \text{ for all } i \in [r+m] \text{ and thus } q_i = \sum_{j=1}^{i}(P_i - L_j) \leq \sum_{j=1}^{i}(U_i - L_j).$$

Now,

$$q_{i+1} - q_i = \sum_{j=1}^{i+1}(p_j - L_j) - \sum_{j=1}^{i}(p_j - L_j) = p_{i+1} - L_{i+1}.$$

Since $L_{i+1} = 0$ or 1 and $0 \leq p_{i+1} \leq 1$ then we clearly have that $-1 \leq p_{i+1} - L_{i+1} \leq 0$ if $L_{i+1} = 1$ and $0 \leq p_{i+1} - L_{i+1} \leq 1$ if $L_{i+1} = 1$.

Therefore, $\pi(P_M)$ is a polytope contained in a polytope Q_M having the following description

$$Q_M = \{q \in \mathbb{R}^{r+m-1} \mid 0 \leq (-1)^{L_{i+1}}(q_{i+1} - q_i) \leq 1 \text{ for all } i \in [r+m-2] \text{ and}$$
$$0 \leq q_i \leq \sum_{j=1}^{i}(U_j - L_j) \text{ for all } i \in [r+m-1]\}.$$

Conversely, it is easy to see that $Q_M \subseteq \pi(P_M)$ by constructing for a given $q \in Q_M$ a preimage p in P_M under π by setting $p_1 = q_1$ and $p_i = q_i - p_{i-1} + \sum_{j=1}^{i} L_j$ for $1 < i < r+m$ and $p_{r+m} = r - \sum_{j=1}^{r+m-1} p_j$.

By using the above description or the characterization in [15], it is straightforward to verify that Q_P is closed under componentwise maximum and minimum. Therefore, Q_P is a distributive polytope.

Furthermore, since $\dim(P_M) = r+m-1$ then Q_M is full-dimensional. Let k be a positive integer. It is also immediate that π sends points in $\frac{1}{k}\mathbb{Z}^{r+m}$ to $\frac{1}{k}\mathbb{Z}^{r+m-1}$. Indeed, if π would send a point $p \in P \setminus \frac{1}{k}\mathbb{Z}^{r+m}$ to $\frac{1}{k}\mathbb{Z}^{r+m-1}$, then it would exist a minimal index i such that $p_i \notin \frac{1}{k}\mathbb{Z}$ and since $\sum_{j=1}^{i-1} p_j \in \frac{1}{k}\mathbb{Z}$ we would have $\sum_{j=1}^{i-1} p_j + p_i \notin \frac{1}{k}\mathbb{Z}$ which would be a contradiction. Therefore, $L_{P_M}(t) = L_{Q_M}(t)$. $\square$

The above theorem can be extended to a disconnected LPM M as follows. Let $M = M_1 \oplus \cdots \oplus M_c$ where M_i is a connected LPM on $e_i = r_i + m_i$ elements with r_i and m_i the number of lines (rank) and columns in the presentation of $M_i, i = 1, \ldots, c$. Thus, $M = M[U, L]$ is an LPM with c connected components and its representation consists by identifying the top-right corner of M_i with the bottom-left corner of M_{i+1} for each $i = 1, \ldots, c-1$, see Figure 3. We obtain that $L = L_1, \ldots, L_c$ and $U = U_1, \ldots, U_c$ and thus $M[U, L]$ is of rank $r = \sum_{i=1}^{c} r_i$ having $n = \sum_{j=1}^{c}(r_j + m_j)$ elements.

We clearly have that $\sum_{j=1}^{i} L_i < \sum_{j=1}^{i} U_i$ for all $i \in [n]$ except at the values $i = r_1 + m_1, r_1 + m_1 + r_2 + m_2, \ldots, \sum_{j=1}^{c}(r_j + m_j)$, that is, except at the c points where the paths U and L meet (other than $(0, 0)$).

Theorem 4.1 can be generalized to M by considering the map

$$\begin{array}{rlcl} \psi: & P_M \subset \mathbb{R}^{r+n} & \longrightarrow & \mathbb{R}^{r+n-c} \\ & p = (p_1, \ldots, p_{r+n}) & \mapsto & \psi(p) = (\psi_1, \ldots, \psi_{r+n-c}) \end{array}$$

where ψ_i is the ith nonzero coordinate of $q = (q_1, \ldots, q_{r+n})$ where $q_i = \sum_{j=1}^{i}(p_j - L_j)$ for all $i \in [n]$ with $\sum_{j=1}^{i} L_i < \sum_{j=1}^{i} U_i$ and zero otherwise (that is, $q_i = 0$ when $i = r_1 + m_1, r_1 + m_1 + r_2 + m_2, \ldots, \sum_{j=1}^{c}(r_j + m_j)$).

Notice that if M is connected then U and L meet only at one point (other than $(0, 0)$) and ψ becomes the map π given in Theorem 4.1.

Example 4.2. *Let us consider the snake $S(1, 2)$. Since $S(1, 2)$ consists of 3 elements and it is connected then* $\dim(P_{S(1,2)}) = 3 - 1 = 2$. *We have*

$$P_{S(1,2)} = \{p \in \mathbb{R}^3 \mid 0 \leq p_i \leq 1 \text{ and } 0 \leq p_1 \leq 1; \\ 1 \leq p_1 + p_2 \leq 2; 2 \leq p_1 + p_2 + p_3 \leq 2\}.$$

We notice that the vertices of $P_{S(1,2)}$ correspond to the three bases $u = \mathrm{st}(U) = (1, 1, 0), b = \mathrm{st}(B) = (1, 0, 1)$ and $l = \mathrm{st}(L) = (0, 1, 1)$, see Figure 9.

Since $\pi(\mathrm{st}(U)) = (1, 1)$ and $\pi(\mathrm{st}(L)) = (0, 0)$ then

$$Q_{S(1,2)} = \{q \in \mathbb{R}^2 \mid 0 \leq q_1 - q_2 \leq 1 \text{ and } 0 \leq q_1, q_2 \leq 1\}.$$

$Q_{S(1,2)}$ is illustrated in Figure 10. We can check that $\pi(a) = (\frac{3}{4} - 0, \frac{6}{4} - 1) = (\frac{3}{4}, \frac{1}{2}), \pi(b) = (1 - 0, 1 - 1) = (1, 0)$ and $\pi(c) = (\frac{1}{4} - 0, \frac{5}{4} - 1) = (\frac{1}{4}, \frac{1}{4})$.

Example 4.3. *Let us consider the snake $S(2, 2)$ ("L inversed"). Notice that $S(2, 2)$ is given by $U_{2,4}$ from which the base $\{1, 2\}$ is deleted. It is known that $P_{U_{2,4}}$ is the octahedron and thus $P_{S(2,2)}$ is the pyramid obtained from removing the north vertex of the octahedron.*

Since $\mathrm{st}(U) = (1, 0, 1, 0)$ and $\mathrm{st}(L) = (0, 0, 1, 1)$ then $\pi(\mathrm{st}(U)) = (1 - 0, 1 - 0, 2 - 1) = (1, 1, 1)$ and $\pi(\mathrm{st}(L)) = (0 - 0, 0 - 0, 0 - 0) = (0, 0, 0)$ and

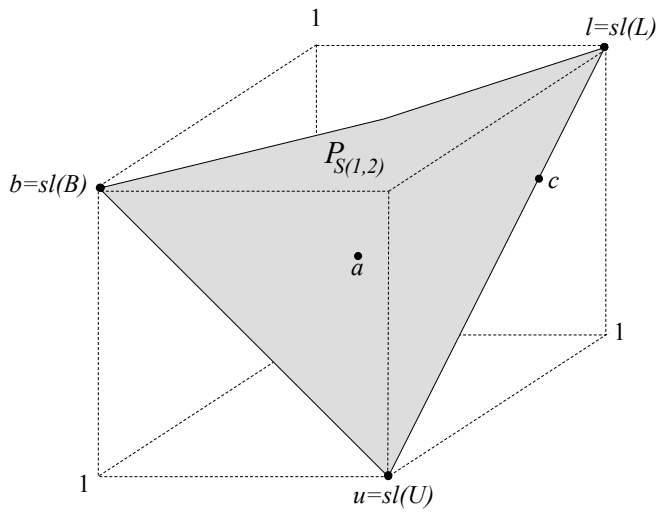

FIGURE 9. $P_{S(1,2)}$ where the three points a, b, c correspond to the step vectors of the generalized paths A, B, C given in Example 3.1.

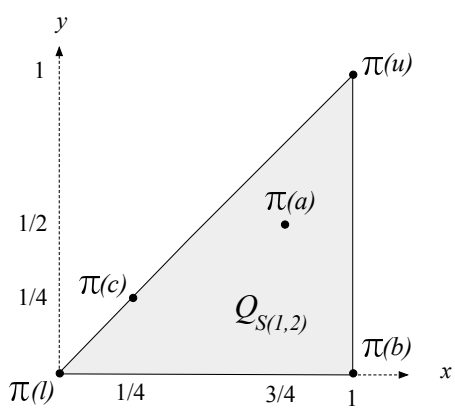

FIGURE 10. Polytope $Q_{S(1,2)}$.

thus

$$Q_{S(2,2)} = \{q \in \mathbb{R}^3 \mid 0 \leq q_2 - q_1 \leq 1, 0 \leq q_2 - q_3 \leq 1 \text{ and } 0 \leq q_1, q_2, q_3 \leq 1\}.$$

See Figure 11 for an illustration.

We are able to determine a distributive lattice structure in P_M by using the above application π, i.e., for $p, p' \in P_M$ we have $p \leq p'$ if and only if $\pi(p) \leq \pi(p')$ with respect to the componentwise ordering in Q_M. In particular, since the set $\frac{1}{t}\mathbb{Z}^{r+m-1}$ is closed under componentwise minimum

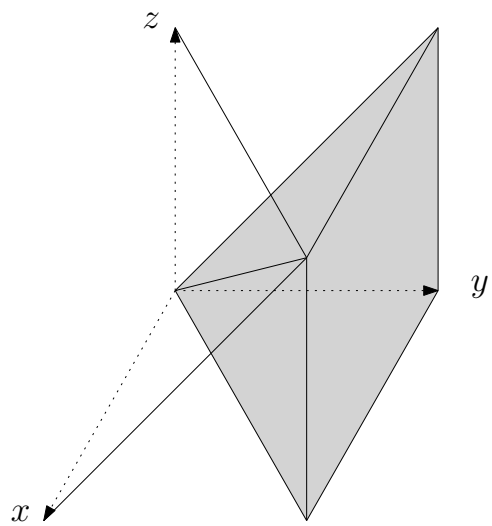

FIGURE 11. The distributive polytope $Q_{S(2,2)}$.

and maximum then π restricts to a distributive lattice on $tQ_M \cap \mathbb{Z}^{r+m-1}$ which carries over to $tP_M \cap \mathbb{Z}^{r+m}$. This leads to:

Corollary 4.4. *Let $M = M[U, L]$ be a rank r connected LPM on $r + m$ elements and let k be a positive integer. Then, both the points in P_M and the points in $kP_M \cap \mathbb{Z}^{r+m}$ carry a distributive lattice structure defined by*

$$p \leq p' \text{ if and only if } \sum_{j=1}^{i} p_j \leq \sum_{j=1}^{i} p'_j \text{ for all } i \in [r+m].$$

Moreover, in $\mathcal{C}_M$ and $\mathcal{C}_M^k$ this order corresponds to one generalized lattice path lying above another generalized lattice path.

4.2. **Chain partitioned poset.** We study further the distributive lattices given in Corollary 4.4. Let us quickly recall some definitions and notions needed for the rest of the section.

Given a poset X, an *order ideal* $I \subseteq X$ is a set such that $x \in I$ and $y \leq x$ implies $y \in I$. The poset $\mathcal{I}(X)$ of all order ideals of X (ordered by containment) is a distributive lattice. An element ℓ of a lattice L is *join-irreducible* if it cannot be written as the join of two other elements, that is, if $\ell = \ell' \vee \ell''$ then $\ell = \ell'$ or $\ell = \ell''$. The induced subposet (not sublattice) of L consisting of all join-irreducible elements is denoted by $\mathcal{J}(L)$

The Fundamental Theorem of Finite Distributive Lattices (FTFDL) [3] states:

> Up to isomorphism, the finite distributive lattices are exactly the lattices $\mathcal{I}(P)$ where P is a finite poset. Moreover, L is isomorphic to $\mathcal{I}(\mathcal{J}(L))$ for every lattice L and P isomorphic to $\mathcal{J}(\mathcal{I}(P))$ for every poset P.

We say that a distributive lattice L is *embedded* in $\mathbb{Z}^n$ if

- the affine hull of L is $\mathbb{Z}^n$ (full-dimensional),
- L is a sublattice of the componentwise order of $\mathbb{Z}^n$ (sublattice),
- the minimum of L is $\mathbf{0} \in \mathbb{Z}^n$ (normalized),
- if $\ell \prec \ell'$ in L then $\ell - \ell' = e_i$ for some unit vector e_i (cover-preserving).

Dilworth's Theorem [12] generalizes the FTFDL to a bijection between embedded distributive sublattices and chain partitioned posets. More precisely, given a chain partitioned poset $(X, C_1, \ldots, C_n)$ we associate an embedded distributive lattice by mapping ideals I of X to $\mathbb{Z}^n$ via

$$\phi(I)_i := |I \cap C_i| \text{ for all } i \in [n].$$

Conversely, given a join-irreducible element ℓ of the embedded lattice L, it is covering a unique ℓ' and $\ell - \ell' = e_i$ for some unit vector e_i. We put ℓ into the chain C_i. The FTFDL corresponds to embeddings into $\{0,1\}^n$ and posets with the trivial singleton-chain partition.

By using Theorem 3.3, the distributive lattice defined in Corollary 4.4 can be viewed in $\mathcal{C}_M$ as follows: for two generalized lattice paths P and Q, we have $P \geq Q$ if for every line l_i in the diagram the y-coordinate of $P \cap \ell$ is larger or equal than the y-coordinate of $Q \cap \ell$. Equivalently, the area below P contains Q.

Let us now describe the distributive lattice structure on $kP_M \cap \mathbb{Z}^{r+m}$ corresponding to elements of $\mathcal{C}_M^k$ (by Corollary 3.5).

Given the digram representing M it is easy to construct the chain partitioned poset representing the lattice on these paths, see Figure 12 for an illustration:

- insert lines T_i in the diagram,
- in each line T_i add between any two consecutive grid points $k-1$ equidistant points,
- connect two of these new points with a line if and only if their difference is $(1,0)$ or $(0,1)$.
- remove the points lying on the lower path L,
- rotate the drawing by 45 degree clockwise.

The resulting diagram is the Hasse diagram of a poset, that we denote by X_M^k. The chain partition that we consider simply puts all points on a given line l_i into a chain C_i of X_M^k.

Let L_M^k be the embedded distributed lattice obtained from the chain partitioned poset $(X_M^k, C_1, \ldots, C_n)$ via the Dilworth's bijection as described above.

Theorem 4.5. *Let M be a connected rank r LPM on $r+m$ elements. Then, $L_M^k = kQ_M \cap \mathbb{Z}^{r+m-1}$.*

Proof. A generalized lattice path with all its bends on coordinates (x, y) such that $kx, ky, x+y \in \mathbb{N}$ corresponds to choosing the ky^{th} element in

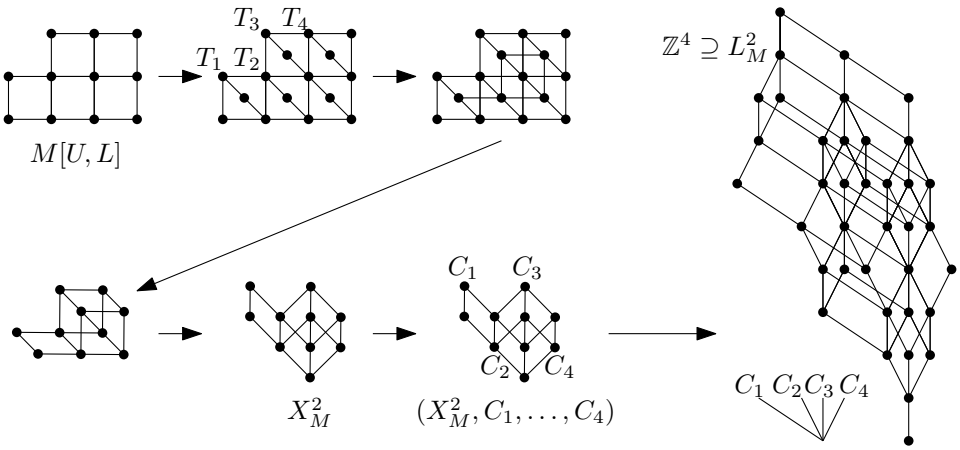

FIGURE 12. Constructing X_M^2, its chain partition, and the embedded $L_M^2 \subseteq \mathbb{Z}^4$ from a diagram representing M.

chain C_{x+y} of $(X_M^k, C_1, \ldots, C_n)$ for all its bends. Since generalized lattice paths are weakly monotone this choice indeed corresponds to an ideal in X_M^k. Conversely, any ideal in X_M^k can be viewed as an element of $\mathcal{C}_M^k$. Thus, the Dilworth mapping satisfies $\phi(X_M^k, C_1, \ldots, C_n) = \mathcal{C}_M^k$. Now, by Corollary 4.4 the ordering on $\mathcal{C}_M^k$ corresponds to the distributive lattice structure on $kQ_M \cap \mathbb{Z}^{r+m-1}$ whose embedding clearly corresponds to the chain partition $C_1, \ldots, C_n$. □

4.3. **Application: Ehrhart polynomial for snakes.** After having understood the combinatorics of the embedded distributive lattice L_M^k let us combine these results with the polyhedral structure in the case of snakes.

In this section, let X be a poset on $\{1, \ldots, n\}$ such that this labeling is *natural*, i.e., if $i <_X j$ then $i < j$. The *order polytope* $\mathcal{O}(X)$ of X is defined as the set of those $x \in \mathbb{R}^n$ such that

$$0 \leq x_i \leq 1, \text{ for all } i \in X \tag{2}$$

$$x_i \geq x_j, \text{ if } i \leq j \text{ in } X \tag{3}$$

Note that $\mathcal{O}(X)$ is a convex polytope since it is defined by linear inequalities and is bounded because of (2).

Let $\chi(I) \in \mathbb{R}^n$ denote the characteristic vector of an ideal I, i.e.,

$$\chi(I)_i = \begin{cases} 1 & \text{if } i \in I, \\ 0 & \text{if } i \notin I. \end{cases}$$

It is known [26, Corollary 1.3] that the vertices of $\mathcal{O}(X)$ are the characteristic vectors $\chi(I)$ as I runs through all order ideals I in X. In particular, the number of vertices of $\mathcal{O}(X)$ is the number of ideals of X.

For integers $a_1, \ldots, a_k \geq 2$ denote by $Z(a_1, \ldots, a_k)$ the *zig-zag-chain* poset on $\sum_{i=1}^{k} a_i - k + 1$ elements arising from k disjoint incomparable chains $C_1, \ldots, C_k$ of lengths $a_1, \ldots a_k$ by identifying the bottom elements of C_i and C_{i+1} for odd $1 \leq i < k$ and the top elements of C_i and C_{i+1} for even $1 \leq i < k$, see Figure 13.

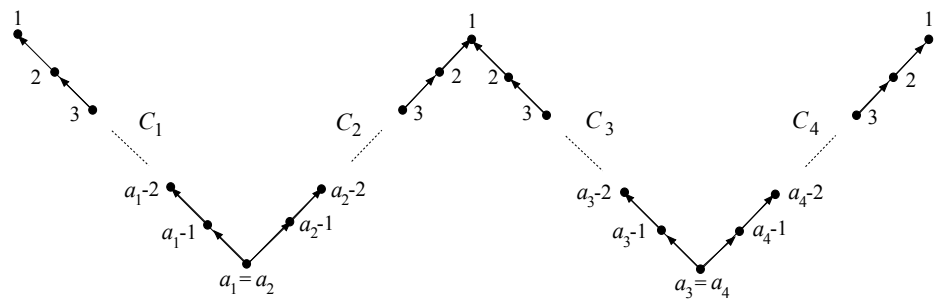

FIGURE 13. Zig-zag-chain $Z(a_1, a_2, a_3, a_4)$.

The following result relates snakes polytopes with order polytopes.

Theorem 4.6. *Let $a_1, \ldots a_k \geq 2$ be integers. Then a connected LPM M is the snake $S(a_1, \ldots a_k)$ if and only if Q_M is the order polytope of the poset $Z(a_1, \ldots, a_k)$.*

Proof. Let M be a snake of rank r with $r + m$ elements. By Theorem 4.1, we have that Q_M is a full $r + m$-dimensional distributive $(0, 1)$-polytope consisting of $q \in \mathbb{R}^{r+m-1}$ such that

$$0 \leq (-1)^{L_{i+1}}(q_{i+1} - q_i) \leq 1 \text{ for all } i \in [r + m - 2] \tag{4}$$

and

$$0 \leq q_i \leq \sum_{j=1}^{i} (U_j - L_j) \text{ for all } i \in [r + m - 1]. \tag{5}$$

Now since $L_i = 0$ or 1 for all i, we can write (4) as

$$q_{i+1} \leq q_i \text{ if } L_{i+1} = 1 \text{ and } q_i \leq q_{i+1} \text{ if } L_{i+1} = 0 \text{ for each } i \in [r+m-2]. \tag{6}$$

Moreover, since M is a snake then $\sum_{j=1}^{i} (U_j - L_j \leq 1$ for all $i \in [r+m-1]$, and thus, from (5), we get that

$$0 \leq q_i \leq 1 \text{ for all } i \in [r + m - 1]. \tag{7}$$

As $\mathbb{R}^X$ can be identified with $\mathbb{R}^n$ (X on $[n]$) then, by (2) and (3), $\mathcal{O}(X)$ is the polytope consisting of those points in $q \in \mathbb{R}^n$ such that

$$0 \leq q_i \leq 1 \text{ and } q_i \leq q_j \text{ if } i \geq_X j \text{ for all } i, j \in X. \tag{8}$$

Therefore, $\mathcal{O}(Z(a_1, \ldots, a_n))$ is of dimension $\sum_{i=1}^{k} a_i - k + 1$ which is exactly the number of elements of $S(a_1, \ldots a_k)$ and, in this case, it can be verified that inequalities in (8) are given by (6) and (7). □

Theorem 4.6 is useful in order to study Ehrhart polynomials of snake polytopes. Indeed, volumes and Ehrhart polynomials of order polytopes have been already studied. In [26, Corollary 4.2], it was proved that

$$\mathrm{vol}(\mathcal{O}(X)) = \frac{e(X)}{n!}$$

where $e(X)$ is the number of linear extensions of X and n the size of X. Here a permutation σ of X is in the set of linear extensions $\mathcal{L}(X)$ if $i <_X j$ implies $\sigma(i) < \sigma(j)$.

Given the poset X, we define the function $\Omega_X(k)$ as the number of order preserving mappings η from X to the total order on $[k]$ (i.e., if $x \leq y$ in X then $\eta(x) \leq \eta(y)$). In [25], it was proved that Ω_X is a polynomial (called the *order-polynomial* of X). Moreover, it was shown in [25, Theorem 2] that

$$\Omega_X(t+1) = \sum_{s=0}^{n-1} \omega_s \binom{n+t-s}{n} \tag{9}$$

where the sequence ω_s denotes the number of linear extensions of X such that exactly s consecutive pairs in the linear extension are not ordered as the natural order on $[n]$.

In [26, Theorem 4.1] it was proved that

$$\Omega_X(t+1) = L_{\mathcal{O}(X)}(t). \tag{10}$$

Remark 4.7. *In view of* (1) *the sequence* ω *in* (9) *corresponds to the* h^**-vector of* $\mathcal{O}(X)$*. Sometimes* ω *is defined in a different (but equivalent) way. For instance, in* [21] *it is defined setting* $\mathcal{L}'(X)$ *to be the set of permutations* σ *of* X *such that* $i <_X j$ *implies* $\sigma^{-1}(i) < \sigma^{-1}(j)$*, i.e., the inverse of a linear extension in our sense, and denoting by* ω_s *elements* $\sigma \in \mathcal{L}'(X)$ *such that exactly* s *consecutive pairs* $(i, i+1)$ *have* $\sigma(i) > \sigma(i+1)$*.*

Recall that the *rank* of a poset X is the length of its largest chain and that X is *graded* if all maximal chains have the same length. A vector $(c_0, \ldots, c_d)$ is *unimodal* if there exists an index $p, 0 \leq p \leq d$, such that $c_{i-1} \leq c_i$ for $i \leq p$ and $c_j \geq c_{j+1}$ for $j \geq p$.

Theorem 4.8. *Let* $a, b \geq 2$ *be integers. The* h^**-vectors of the snake polytopes* $P_{S(a,\ldots,a)}$ *and* $P_{S(a,b)}$ *are unimodal.*

Proof. By Theorem 4.1 we have that $P_{S(a,\ldots,a)}$ and $P_{S(a,b)}$ have the same Ehrhart polynomial as $Q_{S(a,\ldots,a)}$ and $Q_{S(a,b)}$. By Theorem 4.6 both these have the same Ehrhart polynomial as $\mathcal{O}(Z(a,\ldots,a))$ and $\mathcal{O}(Z(a,b))$. Now, Remark 4.7 the h^*-vectors of the latter two polytopes coincide with the sequence ω in (9) associated to $Z(a,\ldots,a)$ and $Z(a,b)$, respectively. In [21], it was proved that if X is a graded poset then the sequence ω is unimodal. Since the zig-zag poset $Z(a,\ldots,a)$ is graded, the latter implies that the h^*-vector of $P_{S(a,\ldots,a)}$ is unimodal.

Now, observe that all linear extensions of $Z(a,b)$ begin with the unique minimal element, which thus can be removed from $Z(a,b)$ without affecting ω. The resulting poset is a disjoint union of two chains. It can be easily shown by hand, that ω is unimodal in this case, but for shortness let us just refer to the stronger result of [24] showing unimodality of ω for any disjoint union of chains. Thus, implying the unimodularity of the h^*-vector of $P_{S(a,b)}$. □

Theorem 4.8 supports the more general conjecture [10, Conjecture 2] asserting that the h^*-vector of any matroid basis polytope is unimodal. Let us also note that we have shown that for matroids whose polytope is equivalent to an order polytope, this conjecture is a weakening of a famous conjecture of Neggers [19] which implies unimodality of h^* for order polytopes. Note that since the latter conjecture has also be confirmed for all naturally labeled posets on at most 7 elements, we get unimodality of h^* for snakes $S(a_1,\ldots,a_k)$ with $a_1+\ldots+a_k-(k-1)\leq 7$.

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