Stability analysis of coupled torsional vibration and pressure in oilwell drillstring system

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To address security issues in oilwell drillstring system, the drilling operation handling which is in generally not autonomous but ensured by an operator may be drill bit destructive or fatal for the machine. To control of stick-slip phenomenon, the drillstring control at the right speed taking only the drillstring vibration is not sufficient as the mud dynamics and the pressure change around the drill pipes cannot be neglected. A coupled torsional vibration and pressure model is presented, and the well-posedness problem is addressed. As a Partial Differential Equation-Ordinary Differential Equa-tion (PDE-ODE) coupled system, and in order to maintain a non destructive downhole pressure, we investigate the control stability with and without the damping term in the wave PDE. In terms of, the torsional variable, the downhole pressure, and the annulus pressure, the coupled system equilibrium is shown to be exponentially stable.

Torsional vibrations; drilling system; neutral delay; mud pressure; Partial Differential Equation, Ordinary Differential Equation; exponential stability

1. Introduction

In recent years an extensive literature was devoted to modelling and control of drilling systems (Bailey & Finnie, 1960; Fridman, Mondié, & Saldivar, 2010; Liu, Vlajic, Long, Meng, and Balachandran, 2014a; Navarro & Corts, 2007; Rouchon, 1998; Sagert, Di Meglio, Krstic, and Rouchon, 2013; Saldivar, Boussaada, Mounier, Iulian, & Niculescu, 2015; Saldivar, Mondie, Loiseau, & Rasvan, 2011; Toumi, Beji, Mlayeh, & Abichou, 2016a). Despite the work of Bailey and Finnie of Shell Development Company which is the first developed experimental and analytical study on axial and torsional drilling vibrations (Bailey & Finnie, 1960), nowadays many problems remains open for drilling systems (more details in Jansen, 1993). In fact, the petroleum and gas extraction industry consists in the creation of a borehole several hundred metres deep in the ground until an oil and gas reservoir is reached. The main components of drillstring are: drill collars, drill pipe, and the rock-cutting tool referred to as drill-bit rotates around its vertical axis. The rotary table gives the necessary torque to put the system into a rotary motion at the top of the drillstring. There exits three main types of instabilities (more details in Jansen, 1993): axial (bit-bouncing phenomenon) leading to pressure oscillations in the mud, lateral (whirling phenomenon) due to an imbalanced drillstring, and torsional vibrations (stick-slip oscillations).

The torsional dynamic of the drillstring is modelled by a damped wave equation (Saldivar et al., 2011; Sagert et al., 2013). The stick-slip phenomenon is a common type of instability for drilling, which is an undesirable limit cycle of the drill string velocity yielding potentially significant damages on oil production facilities. Many authors applied different strategies to suppress the torsional vibrations. In Sagert et al. (2013), the authors proposed a control law to avoid undesirable torsional oscillations of the drillstring in oilwell drilling systems using flatness approaches and backstepping techniques. We even refer to Saldivar et al. (2011) in which the authors proposed the energy function to find a control law that ensures the energy dissipation during the drilling operation. The range of stability for the coupled axial-torsional motions in the presence of state-dependent delay as well as to investigate the effectiveness of a feedback controller to suppress torsional vibrations can be found in Liu, Vlajic, Long, Meng, & Balachandran (2014b). In Toumi, Mlayeh, Beji, and Abichou (2016b), the authors introduced a control law to avoid the torsional vibrations of the drillstring in oilwell drilling systems using Lyapunov theory and backstepping techniques, leading to stability results. Also, an original model considering time-delay and stick-slip effects has been presented in Liu, Vlajic, Long, and Meng (2013).

During drilling operations, the torsional dynamics of the drillstring depend on a system of fluid mud which should be injected for the operation. It is called Managed Pressure Drilling (MPD) (Stamnes, Zhou, Kaasa, & Aamo, 2008). The principal reason for pressure control is above the pore pressure of the reservoir or the

collapse pressure of the bore hole, and below the fracturing pressure of the bore hole. Numerous researchers were interested in stabilising the MPD with different methods (Kaasa, Stamnes, Aamo, & Imsland, 2012; Zoghlami, Latrach, & Beji, 2015). In Zoghlami et al. (2015) the authors proposed a finite dimensional model to all fluid phenomena between the main pump till the fluid outlet through the annulus. The derivation of an MPD is modelled by the next fundamental equations (Merrit, 1967): fluid viscosity, equation of state, conservation of mass, conservation of momentum, and conservation of energy. Hence, the MPD is governed by five ordinary differential equations (Toumi et al., 2016a).

The key contribution in this work is to model and to study the MPD coupled with the torsional drilling system in oil-gas industry. First, we prove the well-posedness of the controlled coupled torsional vibration and pressure proposed model. In a next step, we show the stability of the coupled system. To achieve our objectives, we use the operator theory (Pazy, 2012), the backstepping techniques and the Lyapunov theory (Kokotovic, 1992; Skjetne & Fossen, 2004).

The paper is structured as follows: in Section 2, we recall the PDE as damped wave equation including the adequate boundary conditions that permits to describe the torsional vibration problem. Based on the mass balance method and the momentum balance equation, we present an MPD model that describes the flow behaviour in the drillstring and annulus. In Section 3, the controlled coupled torsional vibration and pressure is proved to be well-posed. The control objectives which is defined as a stabilising problem of the controlled coupled torsional-vibration and pressure at the equilibrium is presented in Section 4. In Section 5, using d'Alembert transformation the drillstring distributed parameter model is transformed to a neutral type retarded equation. The control laws obtained in this section are shown to have a key effect in the reduction of stick-slip oscillations at the bottom extremity. We provide simulation results and some concluding remarks in Section 6.

2. Managing coupled torsional vibration and pressure

2.1 Torsional vibration characteristics

The torsional vibration phenomenon, also known as the drillstring stick-slip oscillations, constitutes the major source of failures in drilling operations. It is characterised by two phases: stick phases, during which the rotation stops completely, and slip phases, during which the angular velocity of the tool increases up to two times the nominal angular velocity. These both phenomena occur when a section of the rotating drillstring is momentarily caught by friction against the borehole, and then releases.

The rotary drilling system is known in oilwell drilling processes, and the dynamic of the torsional variable $\vartheta(t, \varsigma)$ along the drill pipe is described by Sagert et al. (2013) and Saldivar et al. (2011):

$$GJ\frac{\partial^2\vartheta}{\partial\varsigma^2}(t,\varsigma) - I\frac{\partial^2\vartheta}{\partial t^2}(t,\varsigma) - \rho\frac{\partial\vartheta}{\partial t}(t,\varsigma) = 0,$$
$$\varsigma \in (0, L), t \in (0, +\infty) \quad (1)$$

An appropriate choice of boundary conditions allows characterisation of torsional waves propagation along the drillstring. This is introduced in the work of Saldivar et al. (2011).

- At the top ($\varsigma = 0$)

$$GJ\frac{\partial\vartheta}{\partial\varsigma}(t,0) = c_a\left(\frac{\partial\vartheta}{\partial t}(t,0) - \Omega(t)\right) \quad (2)$$

- At the bottom ($\varsigma = L$)

$$GJ\frac{\partial\vartheta}{\partial\varsigma}(t,L) + I_b\frac{\partial^2\vartheta}{\partial t^2}(t,L) = -T\left(\frac{\partial\vartheta}{\partial t}(t,L)\right) \quad (3)$$

where L is the drillstring length, c_a the sliding torque coefficient, I the inertia at the bottom, G the shear modulus, I_b inertia of the drill pipe, J the geometrical moment of inertia, ρ the drillstring damping, and Ω the control input which is also the angular velocity due to the rotary table. The extremity ($\varsigma = L$) is subject to a torque on the bit (Navarro and Corts, 2007; Saldivar et al., 2011)

$$T\left(\frac{\partial\vartheta}{\partial t}(t,L)\right) = c_b\frac{\partial\vartheta}{\partial t}(t,L) + W_{ob}R_b\mu\left(\frac{\partial\vartheta}{\partial t}(t,L)\right)\operatorname{sgn}\left(\frac{\partial\vartheta}{\partial t}(t,L)\right)$$

which is a function of the bit velocity such that $R_b > 0$ is the bit radius and $W_{ob} > 0$ the weight on the bit. The terms $c_b\frac{\partial\vartheta}{\partial t}(t,L)$ and $W_{ob}R_b\mu(\frac{\partial\vartheta}{\partial t}(t,L))\operatorname{sgn}(\frac{\partial\vartheta}{\partial t}(t,L))$ represent the viscous damping torque at the bit and the dry friction torque modelling the bit-rock contact, respectively. The bit dry friction coefficient is governed as (Saldivar et al., 2011)

$$\mu\left(\frac{\partial\vartheta}{\partial t}(t,L)\right) = \mu_{cb} + (\mu_{sb} - \mu_{cb})e^{-\frac{\gamma_b}{\upsilon_b}\left|\frac{\partial\vartheta}{\partial t}(t,L)\right|}$$

such that $\mu_{sb} \geq \mu_{cb} \in (0, 1)$ are the static and Coulomb friction coefficients, $0 < \gamma_b < 1$ is a constant defining the velocity decrease rate, and the constant velocity $\upsilon_b > 0$ is

introduced in order to have appropriate units. Using the following change of variable (Toumi et al., 2016b):

$$u(t,x) = \vartheta\left(L\sqrt{\frac{I}{GJ}}\,t, L(1-x)\right),\ x \in (0,1),$$

we obtain

$$\partial_{tt}u(t,x) = \partial_{xx}u(t,x) - \lambda\partial_t u(t,x),\ x \in (0,1) \quad (4)$$

$$\partial_t u(t,1) = L\sqrt{\frac{I}{GJ}}\Omega(t) - \frac{\sqrt{I\,GJ}}{c_a}u_x(t,1) \quad (5)$$

$$\partial_{tt}u(t,0) = a\partial_x u(t,0) + aF(\partial_t u(t,0)) \quad (6)$$

where $\lambda = \rho L\sqrt{\frac{1}{IGJ}}, a = \frac{LI}{I_b}$,

$$\begin{aligned} F\big(\partial_t u(t,0)\big) &= -\frac{L}{GJ}T\left(\frac{1}{L}\sqrt{\frac{GJ}{I}}\partial_t u(t,0)\right), \\ &= -c_b\sqrt{\frac{1}{GJI}}\partial_t u(t,0) - \sqrt{\frac{1}{GJI}}W_{ob}R_b \\ &\quad \times\mu\left(\frac{1}{L}\sqrt{\frac{GJ}{I}}\partial_t u(t,0)\right)\ \mathrm{sgn}(\partial_t u(t,0)) \end{aligned}$$

in which

$$\begin{aligned} &\mu\left(\frac{1}{L}\sqrt{\frac{GJ}{I}}\partial_t u(t,0)\right) \\ &\quad = \mu_{cb} + (\mu_{sb} - \mu_{cb})e^{-\frac{\gamma_b}{v_b}\left|\frac{1}{L}\sqrt{\frac{GJ}{I}}\partial_t u(t,0)\right|} \end{aligned}$$

2.2 Mud and slider dynamics

During drilling opertaions, down hole cuttings need to be transported out of the bore hole. This done by using a mud circulation system. On board the rig, tanks filled with drilling mud feed the principal mud pump which pumps the drilling fluid from the top drive and into the drillstring. Then, Drill-mud is pumped down the inside of drill pipe through the drillstring and fate through the choke. The drill mud transports cuttings out of the well bore and helps maintain the desired pressure in the well bore (see Figure 1).

The dynamic of the drillstring stability depends on a system of fluid which should be injected to bring cuttings out of the well bore (more discussion about mud in Skalle (2011)) and amplify the torsional vibrations phenomena. In drilling process, drilling mud is pumped down the drillstring and flows through the drill bit in the bottom of the well. Therefore, the mud flows up the annulus carrying cuttings out of the well. Then, the pressure of fluid in

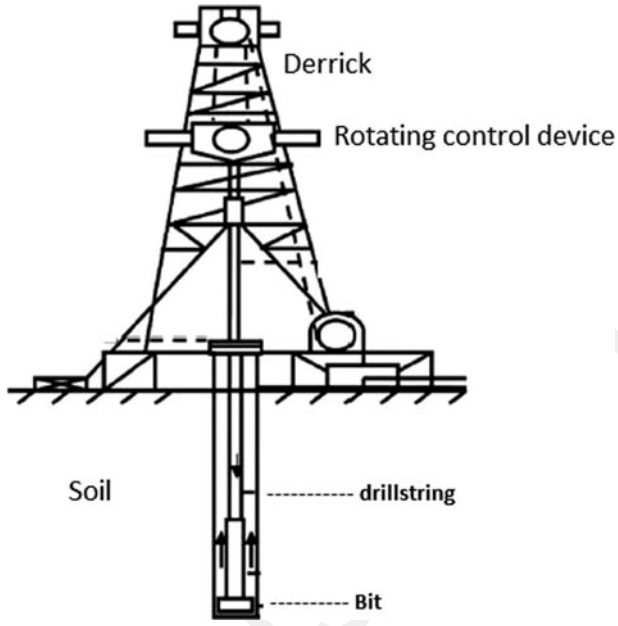

Figure 1. Drill mud through the drilling system.

the downhole can be so important and damage whole the system. To suppress fracturing collapse or influx of fluids surrounding the well, it is important to integrate the dynamic of the pressure in this study: PDE-ODE control problem. Hence, the model of the mud part is described by a finite dimensional system (Toumi et al., 2016a)

$$\begin{aligned} M\dot{q}_{\mathrm{bit}}(t) &= P_1(t) - P_3(t) \\ &\quad -R(q_{\mathrm{bit}}) + g(\rho_1 L - \rho_3 y(t)) \\ \frac{V_1}{\beta_1}\dot{P}_1(t) &= q_1(t) - q_{\mathrm{bit}}(t) \\ (V_0 + sy(t))\dot{P}_3(t) &= \beta_3\left[q_{\mathrm{bit}}(t) + q_2(t) - q_3(t) - \frac{dV_3}{dt}\right] \\ \dot{y}(t) &= v(t) \\ \dot{v}(t) &= \tau_3(t) \end{aligned}$$

in which M is the integrated density per cross section, $y(t) \in [0, L]$ is the penetration depth, $V_3 = V_0 + sy(t)$ is the crown's volume such that $\frac{dV_3}{dt} = sv(t)$ in which v is the penetration rate, q_1 is the pump flow at the surface, q_2 describes the amount of flow due to bit-rock interaction, s is the annular surface, q_3 is the flow out of the crown, V_1 is the inside the drillstring volume, β_1 and β_3 are the effective bulk moduli, q_{bit} is the flow rate from the tool, P_1 is the pump pressure, P_3 is the pressure when $y = L$, ρ_1 is the drilling mud density in the drillstring, and ρ_3 is the annulus fluid density. The total pressure drop due to friction on the bit is represented by $R(q_{\mathrm{bit}})$ (Mahdianfar, Pavlov, & Aamo, 2013). Let us introduce $q_1 = \tau_2$ as a control input in flow. Hence, (τ_2, τ_3) is the stabilising control vector of the mud pressure and the slider position and velocity.

From the next variable change $z(t) = \frac{1}{V_0 + sy(t)}$, the system given below is written as

$$\begin{aligned}
\dot{q}_{\text{bit}}(t) &= c_1 P_1(t) - c_1 P_3(t) - c_1 R(q_{\text{bit}}) + h(z) \\
\dot{P}_1(t) &= c\tau_2(t) - c q_{\text{bit}}(t) \\
\dot{P}_3(t) &= \beta_3 \left[q_{\text{bit}}(t) + q_2(t) - q_3(t) \right] z(t) - \beta_3 z(t) s \upsilon(t) \\
\dot{z}(t) &= -s z^2(t) \upsilon(t) \\
\dot{\upsilon}(t) &= \tau_3(t)
\end{aligned}$$

where $c_1 = \frac{1}{M}$, $c = \frac{\beta_1}{V_1}$, $h(z) = c_1 g(\rho_1 L - \frac{\rho_3}{s}(\frac{1}{z(t)} - V_0))$.

2.3 Coupled torsional vibration and pressure

The flow q_2 in rotary drilling system is written in this form $q_2 = \rho_3 s r \partial_t \vartheta(L\sqrt{\frac{I}{GJ}}\, t, L) = \epsilon \partial_t u(t, 0)$, where r is the bit annular radius and $\epsilon = \frac{\rho_3 s r}{L}\sqrt{\frac{GJ}{I}}$. Then, we obtain the following coupled PDE-ODE equations:

$$\begin{aligned}
\partial_{tt} u(t, x) &= \partial_{xx} u(t, x) - \lambda \partial_t u(t, x) \\
\partial_t u(t, 1) &= L\sqrt{\frac{I}{GJ}}\Omega(t) - \frac{\sqrt{I\,GJ}}{c_a} u_x(t, 1) \\
\partial_{tt} u(t, 0) &= a \partial_x u(t, 0) + aF(\partial_t u(t, 0)) \\
\dot{q}_{bit}(t) &= c_1 P_1(t) - c_1 P_3(t) - c_1 R(q_{\text{bit}}) + h(z) \\
\dot{P}_1(t) &= c\tau_2(t) - c q_{\text{bit}}(t) \\
\dot{P}_3(t) &= \beta_3 (q_{\text{bit}}(t) - q_3) z(t) \\
&\quad + \beta_3 \epsilon \partial_t u(t, 0) z(t) - \beta_3 z(t) s \upsilon(t) \\
\dot{z}(t) &= -s z^2(t) \upsilon(t) \\
\dot{\upsilon}(t) &= \tau_3(t).
\end{aligned}$$

The main purpose contribution is the study of both torsional and drilling mud occuring along a rotary oil-well drilling system. The model considered consits of a damped wave equation (PDE) coupled to an ordinary differential equation (ODE). Then, we are going to study the well-posedness and the stability of the PDE-ODE coupled equations obtained in which (Ω, τ_2, τ_3) as control inputs from, the injected fluid dynamic τ_2, the table rotary speed Ω and the slider velocity τ_3.

3. Well-posedness problem

The aim of this section is to prove the existence and uniqueness of the controlled coupled torsional-vibration and pressure using Lumer-Phillips' theorem and operator theory (see Coron, 2007; Pazy, 2012). Let $T > 0$, the solution of the Cauchy problem is written in this form

$$\partial_{tt} u(t, x) = \partial_{xx} u(t, x) - \lambda \partial_t u(t, x) \tag{7}$$

$$\partial_t u(t, 1) = L\sqrt{\frac{I}{GJ}}\Omega(t) - \frac{\sqrt{I\,GJ}}{c_a} u_x(t, 1) \tag{8}$$

$$\partial_{tt} u(t, 0) = a \partial_x u(t, 0) + aF(\partial_t u(t, 0)) \tag{9}$$

$$\begin{aligned}
\dot{q}_{\text{bit}}(t) &= c_1 P_1(t) - c_1 P_3(t) - c_1 R(q_{\text{bit}}) + h(z) \\
\dot{P}_1(t) &= c\tau_2(t) - c q_{\text{bit}}(t) \\
\dot{P}_3(t) &= \beta_3 (q_{\text{bit}}(t) - q_3) z(t) \\
&\quad + \beta_3 \epsilon \partial_t u(t, 0) z(t) - \beta_3 z(t) s \upsilon(t) \\
\dot{z}(t) &= -s z^2(t) \upsilon(t) \\
\dot{\upsilon}(t) &= \tau_3(t)
\end{aligned} \tag{10}$$

$$\begin{aligned}
&u(0, x) = u^0(x), \quad u_t(0, x) = u^1(x), q_{\text{bit}}(0) = q^0_{bit}, \\
&P_1(0) = P_1^0, P_3(0) = P_3^0, z(0) = z^0, \upsilon(0) = \upsilon^0
\end{aligned} \tag{11}$$

where $x \in (0, 1)$, $t \in (0, T)$, $u^0 \in K := \{u \in H^1(0, 1), u^0(0) = 0\}, u^1 \in L^2(0, 1)$. $q^0_{\text{bit}}, P_1^0, P_3^0, z^0, \upsilon^0$ are the imposed values to the solution at $t = 0$. Let us consider the vector space K which is equipped with the scalar product

$$< u^1(t, x), u^2(t, x) >_K = \int_0^1 u_x^1(t, x) u_x^2(t, x) dx \tag{12}$$

It is obvious that K is a Hilbert space. Let us introduce $Z = (u(t, x), u_t(t, x), u(t, 1), u_t(t, 0), q_{\text{bit}}(t), P_1(t), P_3(t), z(t), \nu(t))^T$. Equations (7)–(11) can be compactly written as

$$\dot{Z}(t) = AZ(t) + H(Z(t)) + f(t) \tag{13}$$

$$Z(0) = Z_0 \tag{14}$$

where

$$A = \begin{pmatrix}
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\partial_{xx} & -\lambda & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
k_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
k_2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & c_1 & -c_1 & 0 & 0 \\
0 & 0 & 0 & 0 & -c & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -\beta_3 q_3 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix},$$

$$H(Z(t)) = \begin{pmatrix} 0 \\ 0 \\ 0 \\ aF(u_t(t,0)) \\ -c_1R(q_{\text{bit}}) + h(z) \\ 0 \\ \beta_3 q_{\text{bit}} z - \beta_3 szv + \beta_3 \epsilon z u_t(t,0) \\ -sz^2 v \\ 0 \end{pmatrix}, \text{ and}$$

$$f(t) = \begin{pmatrix} 0 \\ 0 \\ L\sqrt{\frac{I}{GJ}}\Omega(t) \\ 0 \\ 0 \\ c\tau_2 \\ 0 \\ 0 \\ \tau_3 \end{pmatrix}$$

for which $k_1 = \frac{GJ}{L} < \delta_1'(x), . >$, $k_2 = -a < \delta_0'(x), . >$ such that δ denotes the Dirac function, $\langle \delta_1'(x), u(t,x)\rangle = -u_x(t,1)$ and $\langle \delta_0'(x), u(t,x)\rangle = -u_x(t,0)$.

Our purpose in this section is to prove the existence and uniqueness of system (13)–(14). First, let us consider the problem (13)–(14) with $H(Z) = 0$ and $f(t) = 0$, we have the next theorem.

Theorem 3.1: *The operator A generates a C_0 semigroup e^{At}, $t \geq 0$ of contractions on H.*

Proof: Let $X = \{Z : u \in K, u_t \in L^2([0,1]), u(t,1) \in \mathbb{R}, u_t(t,0) \in \mathbb{R}, q_{bit} \in \mathbb{R}, P_1 \in \mathbb{R}, P_3 \in \mathbb{R}, z \in \mathbb{R}, v \in \mathbb{R}\}$.

To show that A generates a C_0 semigroup of contractions on X, it is equivalent to prove that the operator A is dissipative and $(I - A)$ is surjective. Let us introduce

$$Z^1 = (u^1(t,x), u_t^1(t,x), u^1(t,1), u_t^1(t,0), q_{\text{bit}}^1(t), \times P_1^1(t), P_3^1(t), z^1(t), v^1(t))^T$$

and

$$Z^2 = (u^2(t,x), u_t^2(t,x), u^2(t,1), u_t^2(t,0), q_{\text{bit}}^2(t), \times P_1^2(t), P_3^2(t), z^2(t), v^2(t))^T$$

The vector space X is equipped with the inner product

$$\begin{aligned} \langle Z^1, Z^2\rangle_X = {} & \langle u^1, u^2\rangle_K + \langle u_t^1, u_t^2\rangle_{L^2[0,1]} \\ & + \langle u^1(t,1), u^2(t,1)\rangle_{\mathbb{R}} + \langle u_t^1(t,0), u_t^2(t,0)\rangle_{\mathbb{R}} \\ & + \langle q_{\text{bit}}^1, q_{\text{bit}}^2(t)\rangle_{\mathbb{R}} + \langle P_1^1(t), P_1^2(t)\rangle_{\mathbb{R}} \\ & + \langle P_3^1(t), P_3^2(t)\rangle_{\mathbb{R}} + \langle z_1^1(t), z_1^2(t)\rangle_{\mathbb{R}} \end{aligned}$$

We denote by $\|.\|$ the norm in X associated with this scalar product.

Let A: $D(A) \subset X \to X$ be the linear operator defined by

$D(A) = \{Z \in X, u \in H^2(0,1), u_t \in K, u_x(t,1) = u_x(t,0) = 0, u(t,1) \in \mathbb{R}, u_t(t,0) \in \mathbb{R}, q_{\text{bit}} \in \mathbb{R}, P_1 \in \mathbb{R}, P_3 \in \mathbb{R}, z \in \mathbb{R}, v \in \mathbb{R}\}$.

We have

$$A \begin{pmatrix} u(t,x) \\ u_t(t,x) \\ u(t,1) \\ u_t(t,0) \\ q_{\text{bit}}^1(t) \\ P_1^1(t) \\ P_3^1(t) \\ z^1(t) \\ v^1(t) \end{pmatrix} = \begin{pmatrix} u_t(t,x) \\ \partial_{xx}u(t,x) - \lambda u_t(t,x) \\ -\frac{GJ}{L}u_x(t,1) \\ au_x(t,0) \\ c_1P_1(t) - c_1P_3 \\ -cq_{\text{bit}} \\ -\beta_3 q_3(t) z(t) \\ 0 \\ 0 \end{pmatrix}, \forall Z \in D(A)$$

The operator A satisfies

$$\begin{aligned} < AZ, Z >_X = {} & -\lambda \int_0^1 u_t^2 dx + (c_1 - c)P_1(t)q_{\text{bit}} \\ & - c_1P_3q_{\text{bit}} - \beta_3 q_3(t)z(t)P_3 \end{aligned}$$

Physically $c_1 \leq c$, then $< AZ, Z >_X \leq 0$. This implies that A is dissipative.

It is obviously to verify that $\forall\, m = (m_1, m_2, m_3, m_4, m_5, m_6, m_7, m_8, m_9) \in X$, there exists $w = (w_1, w_2, w_3, w_4, w_5, w_6, w_7, w_8, w_9) \in D(A)$ such that $w - Aw = m$. Then, $D(A)$ is dense in X and A is closed. Hence, using the Lumer-Phillips theorem, A is the infinitesimal generator of a strongly continuous group of isometries $S(t)$, $t \in [0, +\infty)$, on X. ■

Remark 3.1: In this case, for every $Z_0 \in D(A)$, the cauchy problem

$$\dot{Z}(t) = AZ(t),\ Z(0) = Z_0$$

has a unique solution

$$Z \in C^1([0,T], X) \bigcap C^0([0,T], D(A))$$

Now, we are going to prove the existence and uniqueness of system (13)–(14) such that $H(Z)$ and $f(t)$ are different from zero.

Theorem 3.2: *Let $f \in L^1([0,T], X)$ and $Z_0 \in D(A)$, then the problem $\dot{Z}(t) = AZ(t) + H(Z) + f(t)$ has a unique solution*

$$Z \in C^1([0,T], X) \bigcap C^0([0,T], D(A))$$

given by:

$$Z(t) = S(t)Z(0) + \int_0^t S(t-s)(H(Z(s)) + f(s))ds$$

To prove Theorem 3.2, we need the following lemmas:

Lemma 3.1: *The nonlinear operator $H(Z)$ is dissipative and locally Lipschitz.*

Proof: As at first, it is obvious that $H(Z)$ is locally Lipschitz. Second, we will prove that $H(Z)$ is dissipative. We have

$$\begin{aligned}\langle H(Z(t))), Z(t)\rangle_X &= aF(u_t(t,0))u_t(t,0) - c_1R(q_{\text{bit}})q_{\text{bit}} \\ &\quad +h(z)q_{\text{bit}} + \beta_3 q_{\text{bit}} z P_3 - \beta_3 s z v P_3 \\ &\quad +\beta_3 \epsilon z u_t(t,0) P_3 - s z^3 v \\ &= F(u_t(t,0))u_t(t,0) - q_{\text{bit}}(c_1R(q_{\text{bit}}) \\ &\quad -h(z) - \beta_3 z P_3) - s z^3 v \\ &\quad -\beta_3 z P_3 (sv - \epsilon u_t(t,0)) \end{aligned} \tag{15}$$

Recall that the nonlinear function $F(\partial_t u(t,0))$ is written in this form

$$\begin{aligned} F\big(\partial_t u(t,0)\big) &= -\frac{L}{GJ} T\left(\frac{1}{L}\sqrt{\frac{GJ}{I}}\partial_t u(t,0)\right), \\ &= -c_b\sqrt{\frac{1}{GJI}}\partial_t u(t,0) \\ &\quad -\sqrt{\frac{1}{GJI}} W_{ob} R_b \mu(\partial_t u(t,0)) \\ &\quad \times \operatorname{sgn}\left(\frac{1}{L}\sqrt{\frac{GJ}{I}}\partial_t u(t,0)\right) \end{aligned}$$

where μ is given by

$$\begin{aligned} &\mu\left(\frac{1}{L}\sqrt{\frac{GJ}{I}}\partial_t u(t,0)\right) \\ &\quad = \mu_{cb} + (\mu_{sb} - \mu_{cb}) e^{-\frac{\gamma_b}{v_b}\left|\frac{1}{L}\sqrt{\frac{GJ}{I}}\partial_t u(t,0)\right|} \end{aligned}$$

Since $\mu_{sb} \geq \mu_{cb}$

$$\begin{aligned} &\langle F(\partial_t u(t,0)), \partial_t u(t,0)\rangle_{\mathbb{R}} = F(u_t(t,0))u_t(t,0) \\ &= -c_b\sqrt{\frac{1}{GJI}}(\partial_t u(t,0))^2 \\ &\quad -\frac{L}{GJ} W_{ob} R_b \left(\mu_{cb} + (\mu_{sb} - \mu_{cb}) e^{-\frac{\gamma_b}{v_b}\left|\frac{1}{L}\sqrt{\frac{GJ}{I}}\partial_t u(t,0)\right|}\right) \\ &\quad \times \operatorname{sgn}\left(\frac{1}{L}\sqrt{\frac{GJ}{I}}\partial_t u(t,0)\right)\partial_t u(t,0) \end{aligned}$$

Hence, $\langle F(\partial_t u(t,0)), \partial_t u(t,0)\rangle_{\mathbb{R}} \leq 0$.

As $v \in [v_{\min}, v_{\max}]$, choosing v such that $\frac{sv_{\min}}{\epsilon} \geq u_t(t,0)$ guarantees the negativity of the last term in (15). Further, let us consider the pressure at the annulus such that $P_3 \leq \frac{1}{\beta_3 z}[c_1R(q_{\text{bit}}) - h(z)]$. Accordingly to these assumptions, for a minimum slider velocity and a known behaviour of the annulus pressure P_3 with respect to the flow q_{bit} and the slider position $z(t)$, we may guarantee that the operator $H(Z)$ is dissipative. Moreover, $H(Z)$ is locally Lipschitz. ■

Lemma 3.2: *For any function $f \in L^1([0,\ T],\ X)$, and any initial condition $Z_0 \in D(A)$, the problem* (13)–(14) *has at most one solution in $C^1([0,\ T], X) \bigcap C^0([0,\ T], D(A))$.*

Proof: Suppose Z_1 and Z_2 are two solutions of (13)–(14) in the class $C^1([0,\ T], X) \bigcap C^0([0,\ T], D(A))$. Then the difference $Z = Z_1 - Z_2$ is an element of $C^1([0,\ T], X) \bigcap C^0([0,\ T], D(A))$ which satisfies the next system

$$\begin{cases} \dot{Z}(t) = AZ(t) + H(Z(t)) \\ Z(0) = 0 \end{cases} \tag{16}$$

Since A and $H(Z)$ are dissipative, we get

$$\langle \dot{Z}(t), Z(t)\rangle_X = \langle AZ(t), Z(t)\rangle_X + \langle H(Z(t)), Z(t)\rangle_X \leq 0$$

Then

$$\frac{1}{2}\frac{d}{dt} \parallel Z(t) \parallel_X \leq 0 \Rightarrow \parallel Z(t) \parallel_X = 0$$

Hence, problem (16) has a unique solution $Z(t) = 0$ for every $Z_0 \in D(A)$, thus proves that $Z_1 = Z_2$ and shows that (13)–(14) has a solution in $C^1([0,\ T], X) \bigcap C^0([0,\ T], D(A))$, then this one is unique. ■

Proof of Theorem 3.2: By applying the two Lemmas given above and from results given in Theorem 4.2 in Slemrod (1989), Nguyen and Egeland (2004), Pazy (2012), Coron (2007), it is easy to prove that our system (13)–(14) has a unique solution. ■

4. Stability analysis of the controlled coupled system

Let $\tau_1(t) = \frac{c_d L}{GJ}\big(\Omega(t) - \frac{1}{L}\sqrt{\frac{GJ}{I}}\partial_t u(t,1)\big)$. In order to make easy the stability study of the coupled torsional vibration and pressure system, we linearise the boundary condition (6) towards the next reference trajectory (as presented in Sagert et al., 2013)

$$\bar{u}(t,x) = \frac{\lambda w_r}{2}x^2 - F(w_r)x + w_r t + u_0$$

where $w_r = \partial_t \bar{u}(t,x)$. Then, we find the next system:

$$\partial_{tt} u(t,x) = \partial_{xx} u(t,x) - \lambda \partial_t u(t,x) \tag{17}$$

$$\partial_x u(t,1) = \tau_1(t) \tag{18}$$

$$\partial_{tt}u(t,0) = a\partial_x u(t,0) + ab\partial_t u(t,0) \tag{19}$$

$$\begin{aligned}
\dot{q}_{\text{bit}}(t) &= c_1P_1(t) - c_1P_3(t) - c_1R(q_{\text{bit}}) + h(z)\\
\dot{P}_1(t) &= c\tau_2(t) - cq_{\text{bit}}(t)\\
\dot{P}_3(t) &= \beta_3(q_{\text{bit}}(t) - q_3)z(t) + \beta_3\epsilon\partial_t u(t,0)z(t)\\
&\quad -\beta_3 z(t)sv(t)\\
\dot{z}(t) &= -sz^2(t)v(t)\\
\dot{v}(t) &= \tau_3(t)
\end{aligned} \tag{20}$$

such that $b = \frac{\partial F}{\partial w}(w_r)$ and $w(t) = \partial_t u(t,1)$.

In this subsection, we establish the stability of the controlled coupled torsional vibration and pressure model.

Theorem 4.1: *Consider system* (17)–(20)*, and the following three control laws*

$$\begin{aligned}
\tau_1(t) &= \frac{1}{\partial_t u(t,1)}\Big[(1-a)\partial_t u(t,0)\partial_x u(t,0) - ab\partial_t u(t,0)^2\\
&\quad - P_3\left(\beta_3(q_{\text{bit}}(t) - q_3(t))z(t) + \epsilon\beta_3\partial_t u(t,0)z(t)\right)\\
&\quad - q_{\text{bit}}(-c_1P_3(t) - c_1R(q_{\text{bit}}) + h(z))\Big]\\
\tau_2(t) &= \frac{c-c_1}{c}q_{\text{bit}}(t)\\
\tau_3(t) &= \beta_3P_3(t)z(t)s + sz^2(t)\left(z(t) - \frac{\rho_3}{\rho_3V_0 + \rho_1sL}\right)
\end{aligned}$$

Then system (17)–(20) *is stable at the equilibrium* $(0,0,0,0,\frac{\rho_3}{\rho_3V_0+\rho_1sL},0)$.

Proof: In order to prove the controlled coupled system stability, let us introduce the next Lyapunov function:

$$\begin{aligned}
L = \frac{1}{2}\Bigg[&\int_0^1 \left((\partial_x u)^2 + (\partial_t u)^2\right)dx + (\partial_t u(t,0))^2 + c_q q_{\text{bit}}^2\\
&+c_pP_1^2 + c_{p3}P_3^2 + c_z\left(z - \frac{\rho_3}{\rho_3V_0 + \rho_1sL}\right)^2 + c_v v^2\Bigg]
\end{aligned}$$

The constants c_q, c_p, c_{p3}, c_z, and c_v are introduced for the homogeneity of units and in the following we assume that $c_q = c_p = c_{p3} = c_z = c_v = 1$. Hence, differentiating L with respect to time and using the boundary conditions (18)–(19), we find

$$\begin{aligned}
\dot{L}(t) &= \int_0^1 \partial_{tx}u\partial_x u + \partial_t u\partial_{tt}u dx + \partial_t u(t,0)\partial_{tt}u(t,0)\\
&\quad + \dot{q}_{\text{bit}}q_{\text{bit}} + \dot{P}_1P_1 + \dot{P}_3P_3\\
&\quad + \dot{z}\left(z - \frac{\rho_3}{\rho_3V_0 + \rho_1sL}\right) + \dot{v}v\\
&= \int_0^1 \partial_{tx}u\partial_x u + \partial_t u(\partial_{xx}u - \lambda\partial_t u)dx\\
&\quad + \partial_t u(t,0)\partial_{tt}u(t,0)\\
&\quad + \dot{q}_{\text{bit}}q_{\text{bit}} + \dot{P}_1P_1 + \dot{P}_3P_3\\
&\quad + \dot{z}\left(z - \frac{\rho_3}{\rho_3V_0 + \rho_1sL}\right) + \dot{v}v\\
&= -\lambda\int_0^1 (\partial_t u)^2dx + \partial_t u(t,1)\tau_1(t)\\
&\quad - (1-a)\partial_t u(t,0)\partial_x u(t,0) + ab\partial_t u(t,0)^2\\
&\quad + q_{\text{bit}}(t)(c_1P_1(t) - c_1P_3(t) - c_1R(q_{\text{bit}})\\
&\quad + h(z)) + P_1(t)(c\tau_2(t) - cq_{\text{bit}}(t))\\
&\quad + P_3(t)(\beta_3(q_{\text{bit}}(t) - q_3(t))z(t)\\
&\quad + \epsilon\beta_3\partial_t u(t,0)z(t) - \beta_3z(t)sv(t))\\
&\quad - sz^2(t)\left(z(t) - \frac{\rho_3}{\rho_3V_0+\rho_1sL}\right)v(t)) + v(t)\tau_3(t)\\
&= -\lambda\int_0^1 (\partial_t u)^2dx + \partial_t u(t,1)\tau_1(t)\\
&\quad - (1-a)\partial_t u(t,0)\partial_x u(t,0) + ab\partial_t u(t,0)^2\\
&\quad + q_{\text{bit}}(t)(-c_1P_3(t) - c_1R(q_{\text{bit}}) + h(z))\\
&\quad + P_1(t)(c_1q_{\text{bit}}(t) + c\tau_2(t) - cq_{\text{bit}}(t))\\
&\quad + P_3(t)(\beta_3(q_{\text{bit}}(t) - q_3(t))z(t) + \epsilon\beta_3\partial_t u(t,0)z(t))\\
&\quad + v(t)\Bigg(-\beta_3P_3z(t)s - sz^2(t)\left(z(t) - \frac{\rho_3}{\rho_3V_0+\rho_1sL}\right)\\
&\quad + \tau_3(t)\Bigg)
\end{aligned}$$

In order to ensure the stability of system (17)–(20), the following three control laws are chosen:

$$\begin{aligned}
\tau_1(t) &= \frac{1}{\partial_t u(t,1)}[(1-a)\partial_t u(t,0)\partial_x u(t,0) - ab\partial_t u(t,0)^2\\
&\quad - P_3(\beta_3(q_{\text{bit}}(t) - q_3)z(t) + \epsilon\beta_3\partial_t u(t,0)z(t))\\
&\quad - q_{\text{bit}}(-c_1P_3(t) - c_1R(q_{\text{bit}}) + h(z))]\\
\tau_2(t) &= \frac{(c-c_1)}{c}q_{\text{bit}}(t)\\
\tau_3(t) &= \beta_3P_3z(t)s + sz^2(t)\left(z(t) - \frac{\rho_3}{\rho_3V_0 + \rho_1sL}\right)
\end{aligned}$$

Then, we find $\dot{L}(t) \le -\lambda\int_0^1(\partial_t u)^2dx$. Hence $\dot{L}(t) \le 0$. This establish the stability of the controlled system (17)–(20). ■

5. Neutral-type, delay-coupled torsional vibration and pressure

This section shows the importance of the backstepping techniques, and the Lyapunov theory, providing a useful analysis for the stability of the oilwell drilling system

at the bottom extremity. We assume that the damping λ (given in (17)) is zero. We use the d'Alembert's transformation to the distributed parameter model (17)–(19) to a difference equation model (neutral-type delay system). $u(t, x) = \eta(\sigma) + \nu(\gamma)$ is the general solution of the unidimensional wave equation (17) when $\lambda = 0$, with $\sigma = t + x$, $\gamma = t - x$. Defining

$$\dot{\alpha}(t) = \partial_t u(t, 0) = \dot{\eta}(t) + \dot{\nu}(t), \tag{21}$$

as the angular velocity at the drill pipe bottom extremity. Introducing this into the boundary conditions (18)–(19), this yields

$$\dot{\eta}(t+1) - \dot{\nu}(t-1) = \tau_4(t) \tag{22}$$

$$\ddot{\eta}(t) + \ddot{\nu}(t) = a(\dot{\eta}(t) - \dot{\nu}(t)) + ab(\dot{\eta}(t) + \dot{\nu}(t) \tag{23}$$

We can write (21) in this form

$$\dot{\eta}(t) = \dot{\alpha}(t) - \dot{\nu}(t) \tag{24}$$

Plugging (24) into (23), we get

$$\dot{\nu}(t) = \frac{1+b}{2}\dot{\alpha}(t) - \frac{1}{2a}\ddot{\alpha}(t) \tag{25}$$

Plugging (25) into (24), we get

$$\dot{\eta}(t) = \frac{1-b}{2}\dot{\alpha}(t) + \frac{1}{2a}\ddot{\alpha}(t) \tag{26}$$

Introducing both the η and ν expressions in (22), we get a neutral-type, time-delay drillstring model:

$$\frac{1-b}{2}\dot{\alpha}(t+1) + \frac{1}{2a}\ddot{\alpha}(t+1) - \frac{1+b}{2}\dot{\alpha}(t-1) + \frac{1}{2a}\ddot{\alpha}(t-1) = \tau_4(t)$$

Then, we may define the following

$$\ddot{\alpha}(t) = a(b-1)\dot{\alpha}(t) + a(1+b)\dot{\alpha}(t-2) - \ddot{\alpha}(t-2) + 2a\tau_4(t-1)$$

Now, the coupled PDE-ODE stability is analysed through

$$\begin{aligned}
\ddot{\alpha}(t) &= a(b-1)\dot{\alpha}(t) + a(1+b)\dot{\alpha}(t-2) \\
&\quad - \ddot{\alpha}(t-2) + 2a\tau_4(t-1) \\
\dot{q}_{\text{bit}}(t) &= c_1P_1(t) - c_1P_3(t) - c_1R(q_{\text{bit}}) + h(z) \\
\dot{P}_1(t) &= c\tau_5(t) - cq_{\text{bit}}(t) \\
\dot{P}_3(t) &= \beta_3(q_{\text{bit}}(t) - q_3(t))z(t) \\
&\quad + \epsilon\beta_3\dot{B}(t)z(t) - \beta_3 z(t)s\upsilon(t) \\
\dot{z}(t) &= -sz^2(t)\upsilon(t) \\
\dot{\upsilon}(t) &= \tau_6(t).
\end{aligned}$$

Let us introduce $x_1 = \alpha$, $x_2 = \dot{\alpha}$. Then, we have:

$$\dot{x}_1(t) = x_2(t) \tag{27}$$

$$\dot{x}_2(t) = a(b-1)x_2(t) + a(1+b)x_2(t-2) - \dot{x}_2(t-2) + 2a\tau_4(t-1) \tag{28}$$

$$\dot{q}_{bit}(t) = c_1P_1(t) - c_1P_3(t) - c_1R(q_{\text{bit}}) + h(z) \tag{29}$$

$$\dot{P}_1(t) = c\tau_5(t) - cq_{\text{bit}}(t) \tag{30}$$

$$\dot{P}_3(t) = \beta_3(q_{\text{bit}}(t) - q_3(t))z(t) + \epsilon\beta_3 x_2(t)z(t) - \beta_3 z(t)s\upsilon(t) \tag{31}$$

$$\dot{z}(t) = -sz^2(t)\upsilon(t) \tag{32}$$

$$\dot{\upsilon}(t) = \tau_6(t) \tag{33}$$

It is easy to verify that $(0, 0, 0, 0, 0, \frac{\rho_3}{\rho_3 V_0 + \rho_1 sL}, 0)$ is an equilibrium of (27)–(33).

In order to avoid a large increase of pressure which may lead to a plugging phenomenon, the coupled torsional vibration and pressure stabilising problem is summarized in controlling simultaneously the downhole pressure, the slider's linear velocity, and the bottomhole angular velocity.

Theorem 5.1: *Let consider system* (27)–(33) *and the three control laws given by:*

$$\tau_4(t-1) = \frac{1}{2a}\big[-x_1(t) - \epsilon\beta_3 P_3(t)z(t) + a\zeta + \dot{\zeta} - abx_2(t) - a(1+b)x_2(t-2) + \dot{x}_2(t-2)\big] \tag{34}$$

$$\tau_5(t) = -\frac{1}{c}(x_2(t) - \zeta_1(q_{\text{bit}}, P_3, z)) + \frac{c-c_1}{c}q_{\text{bit}}(t) + \frac{1}{c}\dot{\zeta}_1(q_{\text{bit}}, P_3, z).$$

$$\tau_6(t) = sz^2(t)\left(z(t) - \frac{\rho_3}{\rho_3 V_0 + \rho_1 sL}\right) + \zeta_2 - \upsilon(t) + \dot{\zeta}_2$$

in which for $x = 0$, *under*

$$u(t, 0) = u(0, 0) - x_1(0) - \epsilon\beta_3 P_3(t)z(t) + B,$$

$$\zeta(x_1, q_{\text{bit}}, P_3, z, \upsilon) = \frac{1}{(x_1(t) + \epsilon\beta_3 P_3(t)z(t))}\big[-s_0x_1^2(t) - s_1P_3^2(t) - P_3(t)(\beta_3(q_{\text{bit}}(t) - q_3(t))z(t) - \beta_3 z(t)s\upsilon(t))\big]$$

$$\zeta_1(q_{\text{bit}}, P_3, z) = P_3(t) - \left(\frac{1}{c_1}q_{\text{bit}}(t) - R(q_{\text{bit}})\right) - \frac{h(z)}{c_1}$$

$\zeta_2(z) = \frac{(z(t)-\frac{\rho_3}{\rho_3 V_0+\rho_1 sL})}{sz^2(t)}$ *such that* $\zeta(0, 0, 0, \frac{\rho_3}{\rho_3 V_0+\rho_1 sL}, 0) = 0$, $\zeta_1(0, 0, \frac{\rho_3}{\rho_3 V_0+\rho_1 sL}) = 0$, $\zeta_2(\frac{\rho_3}{\rho_3 V_0+\rho_1 sL}) = 0$, *B is a constant different from zero, and* s_i *is a constant positif for* $i = 0, 1$. *Then* (27)–(33) *is exponentially stable.*

Proof: To prove this theorem, we use the backstepping techniques. As at first, let us consider the next sub-system

$$\dot{x}_1(t) = x_2(t) \tag{35}$$

$$\dot{P}_3(t) = \beta_3(q_{\text{bit}}(t) - q_3(t))z(t) + \epsilon\beta_3 x_2(t)z(t) - \beta_3 z(t)s\upsilon(t) \tag{36}$$

$$\dot{x}_2(t) = a(b-1)x_2(t) + a(1+b)x_2(t-2) - \dot{x}_2(t-2) + 2a\tau_4(t-1) \tag{37}$$

We define $\zeta(x_1, q_{\text{bit}}, P_3, z, v)$ as a virtual control law for system (35)–(36). We introduce the Lyapunov function $W = \frac{1}{2}(x_1^2 + P_3^2)$. We differentiate W with respect to time,

$$\begin{aligned}\dot{W} &= \dot{x_1}x_1 + \dot{P_3}P_3 \\ &\leq -s_0x_1^2 - s_1P_3^2 + s_0x_1^2 + (x_1 + \epsilon\beta_3P_3z)x_2 + s_1P_3^2 \\ &\quad + P_3(\beta_3(q_{\text{bit}} - q_3)z - \beta_3 zs\upsilon)\end{aligned}$$

Hence, we select the virtual control law

$$\begin{aligned}\zeta = &\frac{1}{(x_1(t) + \epsilon\beta_3P_3(t)z(t))} \\ &\times\big[-s_0x_1^2(t) - s_1P_3^2(t) - P_3(t)(\beta_3(q_{\text{bit}}(t) \\ &- q_3(t))z(t) - \beta_3 z(t)s\upsilon(t))\big]\end{aligned}$$

Now, introducing the virtual state variable $\chi = x_2 - \zeta$ in (35)–(37), we get

$$\begin{aligned}\dot{x}_1 &= (\chi + \zeta) \\ \dot{P}_3(t) &= \beta_3(q_{\text{bit}}(t) - q_3(t))z(t) \\ &\quad + \epsilon\beta_3(\chi + \zeta)z(t) - \beta_3 z(t)s\upsilon(t) \\ \dot{\chi}(t) &= -\dot{\zeta} + a(b-1)(\chi + \zeta) + a(1+b)x_2(t-2) \\ &\quad - \dot{x}_2(t-2) + 2a\tau_4(t-1)\end{aligned}$$

Here, we propose the next Lyapunov function $W_1 = W + \frac{1}{2}\chi^2$ where the time derivative is as:

$$\begin{aligned}\dot{W}_1 &= \dot{W} + \chi\dot{\chi} \\ &\leq -\min(s_0, s_1)S - a\chi^2 + \chi\big(x_1 + \epsilon\beta_3P_3z \\ &\quad - \dot{\zeta} + ab\chi + a(b-1)\zeta + a(1+b)x_2(t-2) \\ &\quad - \dot{x}_2(t-2) + 2a\tau_4(t-1)\big)\end{aligned}$$

We select the actual control law given by

$$\begin{aligned}\tau_4(t-1) = \frac{1}{2a}\big[&-\beta_3\epsilon P_3z - a(b-1)\zeta - ab(x_2 - \zeta) \\ &+ \dot{\zeta} - a(1+b)x_2(t-2) + \dot{x}_2(t-2) - x_1\big]\end{aligned}$$

Then we get $\dot{W}_1 \leq -\min(s_0, s_1, a)W_1$.

Second, we consider the second sub-system

$$\dot{q}_{\text{bit}}(t) = c_1P_1(t) - c_1P_3(t) - c_1R(q_{\text{bit}}) + h(z) \tag{38}$$

$$\dot{P}_1(t) = c\tau_5(t) - cq_{\text{bit}}(t) \tag{39}$$

As aforementioned, we consider the next equation

$$\dot{q}_{\text{bit}}(t) = c_1P_1(t) - c_1P_3(t) - c_1R(q_{\text{bit}}) + h(z)$$

and introduce a virtual control law which might $\zeta_1(q_{\text{bit}}, P_3, z) = P_3 - (\frac{1}{c_1}q_{\text{bit}} - R(q_{\text{bit}})) - \frac{h(z)}{c_1}$ in which $\zeta_1(0, 0, \frac{\rho_3}{\rho_3 V_0+\rho_1 sL}) = 0$. Then we get $\dot{q}_{\text{bit}} = -q_{\text{bit}}$. We introduce the next Lyapunov function $W_2 = \frac{1}{2}q_{\text{bit}}^2$, then $\dot{W}_2 = -2W_2$. This proves that (38) is exponentially stable at the equilibrium $q_{\text{bit}} = 0$.

Now, let us consider the next virtual state variable $\chi_1(t) = P_1(t) - \zeta_1(q_{\text{bit}}, P_3, z)$. Then we obtain

$$\begin{aligned}\dot{q}_{\text{bit}}(t) &= c_1\chi_1(t) - q_{\text{bit}}(t) \\ \dot{\chi}_1(t) &= -\dot{\zeta}_1(q_{\text{bit}}, P_3, z) + c\tau_5(t) - cq_{\text{bit}}(t)\end{aligned}$$

We choose the following Lyapunov function

$W_3 = W_2 + \frac{1}{2}\chi_1^2 = \frac{1}{2}(q_{\text{bit}}^2 + \chi_1^2)$. The time derivative of W_3 is given by

$$\begin{aligned}\dot{W}_3 &= \dot{q}_{\text{bit}}q_{\text{bit}} + \dot{\chi}_1\chi_1 \\ &= -q_{\text{bit}}^2 - \chi_1^2 + \chi_1(\chi_1 + (c_1 - c)q_{\text{bit}} \\ &\quad - \dot{\zeta}_1(q_{\text{bit}}, P_3, z) + c\tau_5)\end{aligned}$$

Hence, tacking

$$\tau_5(t) = -\frac{1}{c}\chi_1(t) + \frac{c - c_1}{c}q_{\text{bit}}(t) + \frac{1}{c}\dot{\zeta}_1(q_{\text{bit}}, P_3, z)$$

This proves that $\dot{W}_3 = -2W_3$. Then system (38)–(39) is exponentially stable at the equilibrium $(q_{\text{bit}}, P_1) = (0, 0)$.

Finally, let us consider the last sub-system

$$\dot{z}(t) = -sz^2(t)\upsilon(t) \tag{40}$$

$$\dot{\upsilon}(t) = \tau_6(t) \tag{41}$$

We define a virtual control law for equation system (40), say $\zeta_2(z) = \frac{(z(t)-\frac{\rho_3}{\rho_3 V_0+\rho_1 sL})}{sz^2(t)}$ in which $\zeta_2(\frac{\rho_3}{\rho_3 V_0+\rho_1 sL}) = 0$.

We introduce $\chi_2 = \upsilon - \zeta_2$ be a virtual state variable. Then we find

$$\dot{z}(t) = -sz^2(t)\chi_2(t) - sz^2(t)\zeta_2(z)$$
$$\dot{\chi}_2(t) = \tau_6(t) - \dot{\zeta}_2(z)$$

We consider the next Lyapunov function

$$W_4(P_3, z) = \frac{1}{2}\left(\chi_2^2 + \left(z - \frac{\rho_3}{\rho_3 V_0 + \rho_1 sL}\right)^2\right)$$

We differentiate W_4 with respect to time. Then we get

$$\dot{W}_4 = \dot{z}\left(z - \frac{\rho_3}{\rho_3 V_0 + \rho_1 sL}\right) + \dot{\chi}_2\chi_2$$

$$= -sz^2\left(z - \frac{\rho_3}{\rho_3 V_0 + \rho_1 sL}\right)\zeta_2(z) + \chi_2(-\dot{\zeta}_2 + \tau_6)$$
$$\leq -\left(z - \frac{\rho_3}{\rho_3 V_0 + \rho_1 sL}\right)^2 - \chi_2^2$$
$$+ \chi_2\left(-sz^2(z - \frac{\rho_3}{\rho_3 V_0 + \rho_1 sL}) + \chi_2 - \dot{\zeta}_2 + \tau_6\right)$$

Now, we select the actual control law

$$\tau_6 = sz^2\left(z - \frac{\rho_3}{\rho_3 V_0 + \rho_1 sL}\right) - \chi_2 + \dot{\zeta}_2$$
$$= sz^2\left(z - \frac{\rho_3}{\rho_3 V_0 + \rho_1 sL}\right) + \zeta_2 - \upsilon + \dot{\zeta}_2$$

Then $\dot{W}_4 \leq -2W_4$.

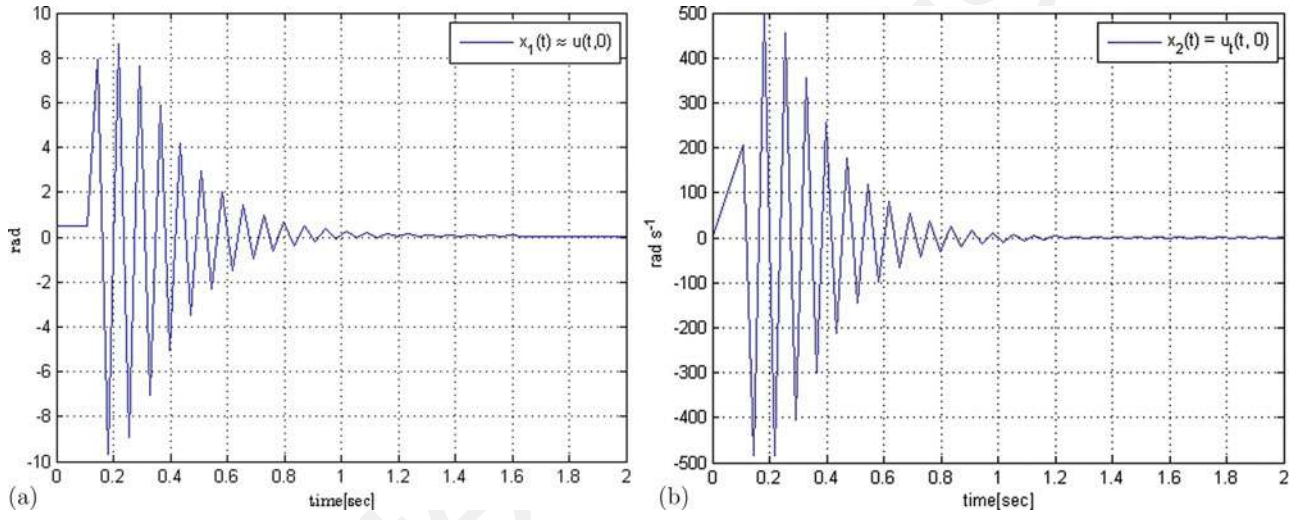

Figure 2. Velocity in the bottom extremity of the drillstring system. (a) Time evolution of the state $x_1(t) \approx u(t, 0)$. (b) Time evolution of the state $x_2(t) = \partial_t u(t, 0)$.

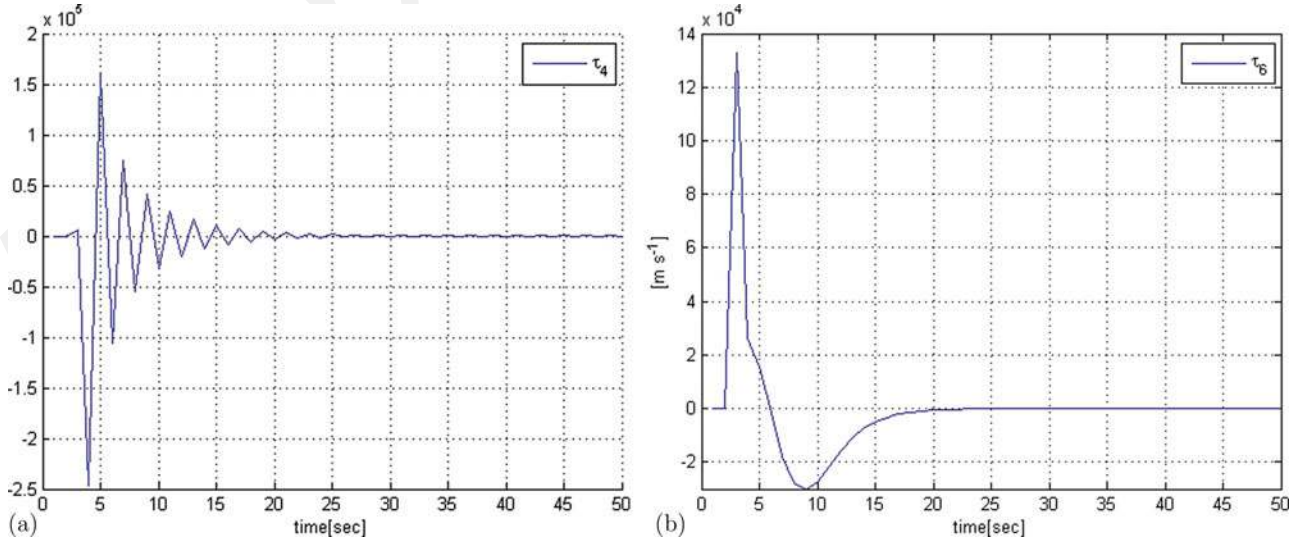

Figure 3. Control laws. (a) Schema of the control law $\tau_4(t-1)$. (b) Schema of the control law $\tau_6(t)$.

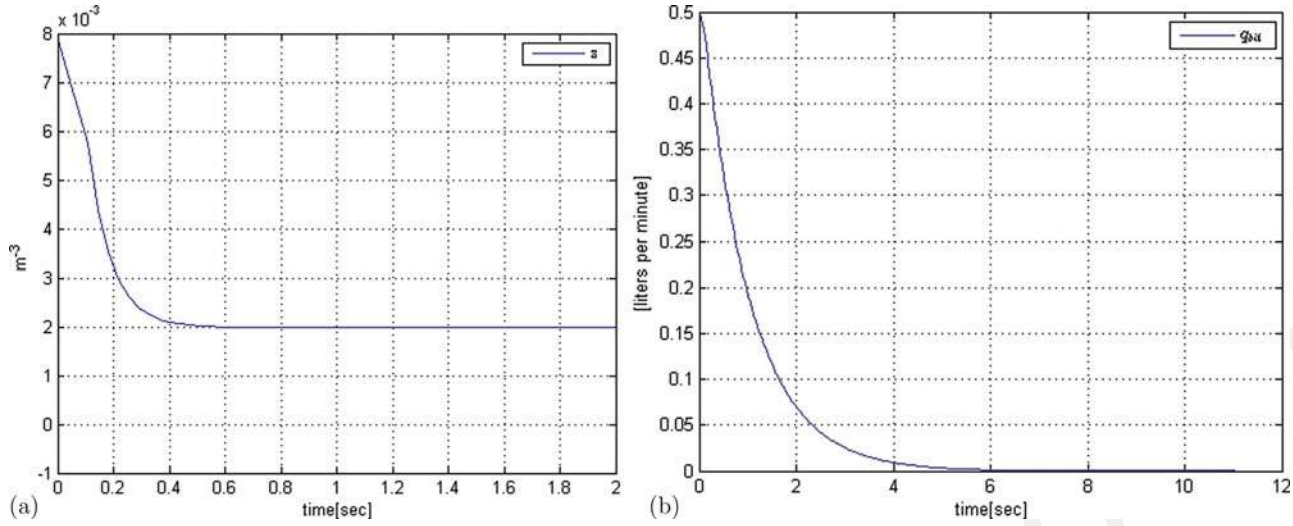

Figure 4. Time evolution of the states. (a) Time evolution of the state $z(t)$. (b) Time evolution of the state $q_{\text{bit}}(t)$.

Table 1. Different physical parameters.

Variable	Value	Description
L	2000 m	Drillstring length
I	0.095 kg.m	Inertia per unit length
I_b	311 kg.m^2	Inertia at the drillstring bottom
J	1.19.10^{-5} m^4	Geometrical moment of inertia
c_a	2000 Nm.s.rad^{-1}	Sliding torque coefficient
G	79.3.10^9 $N.m^{-2}$	Shear modulus
$\rho_1 = \rho_3$	1250 kg.m^{-3}	Density of the drilling
M	8300 kg.m^{-4}	Integrated density
$\beta_1 = \beta_3$	24, 750 bar	Effective bulk modulus,
V_0	110 m^3	Volume of the drillstring
g	9.81 m.s^{-2}	Gravity
s	$\pi \times (0.25)^2$ m^2	Surface
c_d	0.61	

Here, let us introduce the global Lyapunov function

$$L_1 = \frac{1}{2}\left(x_1^2 + P_3^2 + (x_2 - \zeta)^2 + q_{\text{bit}}^2 + (P_1 - \zeta_1)^2 + (\upsilon - \zeta_2)^2 + \left(z - \frac{\rho_3}{\rho_3 V_0 + \rho_1 s L}\right)^2\right)$$

Then

$$\dot{L}_1 \leq -2(W_1 + W_3 + W_4) \leq -2L_1$$

This proves that system (27)–(33) is exponentially stable at the equilibrium $(0, 0, 0, 0, 0, \frac{\rho_3}{\rho_3 V_0 + \rho_1 s L}, 0)$. ■

6. Numerical simulation

Now, we are able to test the effectiveness of the proposed control schemes for the neutral-type, delay-coupled torsional vibration and pressure. The numerical values corresponding to the torsional were taken from the work of Sagert et al. (2013), Saldivar et al. (2015), Challamel (2000), and the ones corresponding to the mud part from that of Mahdianfar et al. (2013). The next physical parameters are used in simulation:

The signification of each system physical parameter is addressed in Table 1. In Toumi et al. (2016a), the flow rate due to the mud leaving through an open annulus is given by (the balance momentum Bernoulli's law):

$$q_3 = c_d s \sqrt{\frac{2}{\rho_3}\left(P_{dh} - P_3 + \frac{\rho_3 g}{s}\left(\frac{1}{z} - V_0\right)\right)}$$

The bit displacement and characteristics permit to construct the bottomhole pressure P_{dh} from the annulus pressure P_3 (Zoghlami et al., 2015)

$$P_{dh} = P_3 - \frac{\rho_3 g}{s}\left(\frac{1}{z} - V_0\right) + F_a q_{\text{bit}}^2$$

with F_a denotes the friction factor in the annulus.

According to practical experience, two strategies to suppress the torsional vibration phenomenon are: the reduction of the weight on the bit and the increase of the velocity at the surface. From Figures 2–4, all the simulations imply an adequate convergence of the system variables to the equilibrium. As an analysis example, a limit cycle stability is reduced by means of the control law $\tau_4(t-1)$ in (34) depending on the angular velocity

at the bottom extremity (Figure 2). Also, under the proposed control inputs $\tau_{4,6}$ (Figure 3), the controlled coupled system avoids an excessive increase (saturation) in pressures. One notes that $z(t)$ converges to the equilibrium $\frac{\rho_3}{\rho_3 V_0 + \rho_1 sL} = 2.10^{-3}\mathrm{m}^{-3}$ (Figure 4(a)).

7. Conclusion

In this paper, by coupling the mud flow dynamics and the torsional vibration in oilwell drilling operation, we have shown that the mud evacuation control is crucial. This is well known as a MPD which is commonly studied in the literature but with respect to the torsional variables, and no interaction phenomenon with fluid was investigated. We use the D'Alembert transformation to obtain a coupled neutral-type delay system that includes the pressure behaviour in order to simplify the analysis and the simulations of the drillstring. From the Lyapunov theory and the backstepping approach, control inputs are designed in rotary's velocity, the injected fluid flow, and the force acting on the drillstring's displacement. In the future work, automatic drilling applications performances are to be studied, and some variables such that the downhole pressure need to be estimated.

Disclosure statement

No potential conflict of interest was reported by the authors.

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