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ON SPECTRAL PROPERTIES OF THE BLOCH-TORREY OPERATOR IN TWO DIMENSIONS

DENIS S. GREBENKOV* AND BERNARD HELFFER†

Abstract. We investigate a two-dimensional Schrödinger operator, $-h^2\Delta + iV(x)$, with a purely complex potential $iV(x)$. A rigorous definition of this non-selfadjoint operator is provided for bounded and unbounded domains with common boundary conditions (Dirichlet, Neumann, Robin and transmission). We propose a general perturbative approach to construct its quasimodes in the semi-classical limit. An alternative WKB construction is also discussed. These approaches are local and thus valid for both bounded and unbounded domains, allowing one to compute the approximate eigenvalues to any order in the small h limit. The general results are further illustrated on the particular case of the Bloch-Torrey operator, $-h^2\Delta + ix_1$, for which a four-term asymptotics is explicitly computed. Its high accuracy is confirmed by a numerical computation of the eigenvalues and eigenfunctions of this operator for a disk and circular annuli. The localization of eigenfunctions near the specific boundary points is revealed. Some applications in the field of diffusion nuclear magnetic resonance are discussed.

Key words. Transmission boundary condition, spectral theory, Bloch-Torrey equation, semi-classical analysis, WKB

AMS subject classifications. 35P10, 47A10, 47A75

1. Introduction.

In a previous paper [17], we have analyzed in collaboration with R. Henry one-dimensional models associated with the complex Airy operator $-\frac{d^2}{dx^2} + igx$ on the line, with $g \in \mathbb{R}$. We revisited the Dirichlet and Neumann realization of this operator in $\mathbb{R}^+$ and the main novelty was to consider a transmission problem at 0. In higher dimensions, an extension of the complex Airy operator is the differential operator that we call the Bloch-Torrey operator or simply the BT-operator

$$-D\Delta + igx_1\,,$$

where $\Delta = \partial^2/\partial x_1^2 + \ldots + \partial^2/\partial x_n^2$ is the Laplace operator in $\mathbb{R}^n$, and D and g are real parameters. More generally, we will study the spectral properties of some realizations of the differential Schrödinger operator

$$\mathcal{A}_h^{\#} = -h^2\Delta + i\,V(x)\,, \tag{1.1}$$

in an open set Ω, where h is a real parameter and $V(x)$ a real-valued potential with controlled behavior at ∞, and the superscript # distinguishes Dirichlet (D), Neumann (N), Robin (R), or transmission (T) conditions. More precisely we discuss

1. the case of a bounded open set Ω with Dirichlet, Neumann or Robin boundary condition;
2. the case of a complement $\Omega := \complement\overline{\Omega}_-$ of a bounded set Ω_- with Dirichlet, Neumann or Robin boundary condition;
3. the case of two components $\Omega_- \cup \Omega_+$, with $\Omega_- \subset \overline{\Omega}_- \subset \Omega$ and $\Omega_+ = \Omega\backslash\overline{\Omega}_-$, with Ω bounded and transmission conditions at the interface between Ω_- and Ω_+;

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4. the case of two components $\Omega_- \cup \complement\,\overline{\Omega}_-$, with Ω_- bounded and transmission conditions at the boundary;
5. the case of two unbounded components Ω_- and Ω_+ separated by a hypersurface with transmission conditions.

In all cases, we assume that the boundary is C^∞ to avoid technical difficulties related to irregular boundaries (see [18]). Roughly speaking (see the next section for a precise definition), the state u (in the first two items) or the pair (u_-, u_+) in the last items should satisfy some boundary or transmission condition at the interface. In this paper, we consider the following situations:

- the Dirichlet condition: $u_{|\partial\Omega} = 0$;
- the Neumann condition: $\partial_\nu u_{|\partial\Omega} = 0$, where $\partial_\nu = \nu \cdot \nabla$, with ν being the outwards pointing normal;
- the Robin condition: $h^2 \partial_\nu u_{|\partial\Omega} = -\mathcal{K} u_{|\partial\Omega}$, where $\mathcal{K} \geq 0$ denotes the Robin parameter;
- the transmission condition:

$$h^2 \partial_\nu u_{+\,|\partial\Omega_-} = h^2 \partial_\nu u_{-\,|\partial\Omega_-} = \mathcal{K}(u_{+\,|\partial\Omega_-} - u_{-\,|\partial\Omega_-})\,,$$

  where $\mathcal{K} \geq 0$ denotes the transmission parameter, and the normal ν is directed outwards Ω_-.

From now on $\Omega^\#$ denotes Ω if $\# \in \{D, N, R\}$ and Ω_- if $\# = T$. $L^2_\#$ will denote $L^2(\Omega)$ if $\# \in \{D, N, R\}$ and $L^2(\Omega_-) \times L^2(\Omega_+)$ if $\# = T$.

In [17], we have analyzed in detail various realizations of the complex Airy (or Bloch-Torrey) operator $A_0^\# := -\frac{d^2}{d\tau^2} + i\tau$ in the four cases corresponding to Dirichlet, Neumann, and Robin on the half-line $\mathbb{R}^+$ or for the transmission problem on the whole line $\mathbb{R}$ (in what follows, $\mathbb{R}^\#$ will denote $\mathbb{R}^+$ if $\# \in \{D, N, R\}$ and $\mathbb{R}$ if $\# = T$). The boundary conditions read respectively:

- $u(0) = 0$;
- $u'(0) = 0$;
- $u'(0) = \kappa\, u(0)$;
- $u'_-(0) = u'_+(0) = \kappa\,(u_+(0) - u_-(0))$

(with $\kappa \geq 0$ in the last items). For all these cases, we have proven the existence of a discrete spectrum and the completeness of the corresponding generalized eigenfunctions. Moreover, there is no Jordan block (for the fourth case, this statement was proven only for κ small enough).

In this article, we start the analysis of the spectral properties of the BT operator in dimensions 2 or higher that are relevant for applications in superconductivity theory [2, 5, 6, 7], in fluid dynamics [30], in control theory [10], and in diffusion magnetic resonance imaging [12, 16] (and references therein). We will mainly focus on

- definition of the operator,
- construction of approximate eigenvalues in some asymptotic regimes,
- localization of quasimode states near certain boundary points,
- numerical simulations.

In particular, we will discuss the semi-classical asymptotics $h \to 0$, the large domain limit, the asymptotics when $g \to 0$ or $+\infty$, the asymptotics when the transmission or Robin parameter tends to 0 . Some other important questions remain unsolved like the existence of eigenvalues close to the approximate eigenvalues (a problem which is only solved in particular situations). We hope to contribute to this point in the future.

When $g = 0$, the BT-operator is reduced to the Laplace operator for which the answers are well known. In particular, the spectrum is discrete in the case of bounded domains and equals $[0, +\infty)$ when one or both components are unbounded. In the case $g \neq 0$, we show that if there is at least one boundary point at which the normal vector to the boundary is parallel to the coordinate x_1, then there exist approximate eigenvalues of the BT-operator *suggesting* the existence of eigenvalues while the associated eigenfunctions are localized near this point. This localization property has been already discussed in physics literature for bounded domains [35], for which the existence of eigenvalues is trivial. Since our asymptotic constructions are local and thus hold for unbounded domains, the localization behavior can be conjectured for exterior problems involving the BT-operator.

Some of these questions have been already analyzed by Y. Almog (see [2] and references therein for earlier contributions), R. Henry [25, 26] and Almog-Henry [8] but they were mainly devoted to the case of a Dirichlet realization in bounded domains in $\mathbb{R}^2$ or particular unbounded domains like $\mathbb{R}^2$ and $\mathbb{R}^2_+$, these two last cases playing an important role in the local analysis of the global problem.
Different realizations of the operator $\mathcal{A}_h$ in Ω are denoted by $\mathcal{A}_h^D$, $\mathcal{A}_h^N$, $\mathcal{A}_h^R$ and $\mathcal{A}_h^T$. These realizations will be properly defined in Section 2 under the condition that, when Ω is unbounded, there exists $C > 0$ such that

$$|\nabla V(x)| \leq C\sqrt{1 + V(x)^2}\,. \tag{1.2}$$

Our main construction is local and summarized in the following

THEOREM 1. *Let $\Omega \subset \mathbb{R}^2$ as above, $V \in \mathcal{C}^\infty(\overline{\Omega}; \mathbb{R})$ and $x^0 \in \partial\Omega^\#$ such that*[1]

$$\nabla V(x^0) \neq 0\,, \quad \nabla V(x^0) \wedge \nu(x^0) = 0\,, \tag{1.3}$$

where $\nu(x^0)$ denotes the outward normal on $\partial\Omega$ at x^0.
Let us also assume that, in the local curvilinear coordinates, the second derivative of the restriction of V to the boundary at x^0 (denoted as $2\,v_{20}$) satisfies

$$v_{20} \neq 0\,.$$

For the Robin and transmission cases, we also assume that for some $\kappa > 0$

$$\mathcal{K} = h^{\frac{4}{3}}\kappa\,. \tag{1.4}$$

If $\mu_0^\#$ is a simple eigenvalue of the realization "#" of the complex Airy operator $-\frac{d^2}{dx^2} + ix$ in $L^2_\#$, and μ_2 is an eigenvalue of the Davies operator $-\frac{d^2}{dy^2} + iy^2$ on $L^2(\mathbb{R})$, then there exists an approximate pair $(\lambda_h^\#, u_h^\#)$ with $u_h^\#$ in the domain of $\mathcal{A}_h^\#$, such that

$$\lambda_h^\# = i\,V(x^0) + h^{\frac{2}{3}} \sum_{j \in \mathbb{N}} \lambda_{2j}^\#\, h^{\frac{j}{3}} + \mathcal{O}(h^\infty)\,, \tag{1.5}$$

$$(\mathcal{A}_h^\# - \lambda_h^\#)\, u_h^\# = \mathcal{O}(h^\infty) \ \text{ in } L^2_\#(\Omega)\,, \quad ||u_h^\#||_{L^2} \sim 1\,, \tag{1.6}$$

[1] As noticed in [8], a point satisfying the second condition in (1.3) always exists when $\partial\Omega^\#$ is bounded.

where

$$\lambda_0^{\#} = \mu_0^{\#}\,|\,v_{01}|^{\frac{2}{3}} \exp\left(i\frac{\pi}{3}\text{sign}\ v_{01}\right)\ , \quad \lambda_2 = \mu_2 |v_{20}|^{\frac{1}{2}} \exp\left(i\frac{\pi}{4}\text{sign}\, v_{20}\right)\ , \tag{1.7}$$

with $v_{01} := \nu \cdot \nabla V(x^0)\,.$

In addition, we will compute $\lambda_4^{\#}$ explicitly (see the Appendix) in the four types of boundary conditions and also describe an alternative WKB construction to have a better understanding of the structure of the presumably corresponding eigenfunctions. We will also discuss a physically interesting case when κ in (1.4) depends on h and tends to 0.

The proof of this theorem provides a general scheme for quasimode construction in an arbitrary planar domain with smooth boundary $\partial\Omega$. In particular, this construction allow us to retrieve and further generalize the asymptotic expansion of eigenvalues obtained by de Swiet and Sen for the Bloch-Torrey operator in the case of a disk [35]. The generalization is applicable for any smooth boundary, with Neumann, Dirichlet, Robin, or transmission boundary condition. Moreover, since the analysis is local, the construction is applicable to both bounded and unbounded components.

The paper is organized as follows. In Sec. 2, we provide rigorous definitions and basic properties of the BT-operator in bounded and unbounded domains, with Dirichlet, Neumann, Robin, and Transmission conditions. Section 3 recalls former semi-classical results for a general operator $-h^2\Delta + iV(x)$. In Sec. 4, we provide preliminaries for semi-classical quasimode constructions in the two-dimensional case. The construction scheme is detailed in Sec. 5. In particular, the four-terms asymptotics of the approximate eigenvalues is obtained and we prove the main theorem. In Sec. 6 we consider other scaling regimes for the Robin or transmission parameter. In Sec. 7 we propose an alternative construction for the first approximate eigenvalue using WKB quasi-mode states. In Sec. 8, we illustrate general results for simple domains such as disk and annulus. Sec. 9 describes numerical results in order to check the accuracy of the derived four-terms asymptotics of eigenvalues of the BT-operator in simple domains such as a disk, an annulus, and the union of disk and annulus with transmission boundary condition. We also illustrate the localization of eigenfunctions near circular boundaries of these domains. Since a direct numerical computation for unbounded domains (e.g., an exterior of the disk) was not possible, we approach this problem by considering an annulus with a fixed inner circle and a moving away outer circle. We check that the localization of some eigenfunctions near the inner circle makes them independent of the outer circle. We therefore conjecture that the BT-operator has some discrete spectrum for the exterior of the disk. More generally, this property is conjectured to hold for any domain in $\mathbb{R}^n$ (bounded or not) with smooth boundary which has points whose normal is parallel to the gradient direction. Finally, we briefly discuss in Sec. 10 an application of the obtained results in the field of diffusion nuclear magnetic resonance.

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2. Definition of the various realizations of the Bloch-Torrey operator.

2.1. The case of a bounded open set Ω. This is the simplest case. For the analysis of the Dirichlet (resp. Neumann) realization $\mathcal{A}_h^D$ (resp. $\mathcal{A}_h^N$) of the BT-operator, the term $V(x)$ is simply a bounded non self-adjoint perturbation of the

Dirichlet (resp. Neumann) Laplacian.
We have for three boundary conditions:

- For the Neumann case, the form domain $\mathcal{V}$ is $H^1(\Omega)$ and (if Ω is regular) the domain of the operator is $\{u \in H^2(\Omega)\,,\, \partial_\nu u_{/\partial\Omega} = 0\}$. The quadratic form reads

$$\mathcal{V} \ni u \mapsto q_V(u) := h^2\, ||\nabla u||^2_\Omega + i \int_\Omega V(x)\, |u(x)|^2\, dx\,. \tag{2.1}$$

- For the Dirichlet case, the form domain is $H^1_0(\Omega)$ and (if Ω is regular) the domain of the operator is $H^2(\Omega) \cap H^1_0(\Omega)$. The quadratic form is given by (2.1).
- For the Robin case (which is a generalization of the Neumann case), the form domain is $H^1(\Omega)$ and (if Ω is regular) the domain of the operator $\mathcal{A}^R_h$ is $\{u \in H^2(\Omega)\,,\, -h^2 \partial_\nu u_{/\partial\Omega} = \mathcal{K} u_{/\partial\Omega}\}$, where $\mathcal{K}$ denotes the Robin coefficient, and ν is pointing outwards. The quadratic form reads

$$u \mapsto q_V(u) := h^2\, ||\nabla u||^2_\Omega + i \int_\Omega V(x) |u(x)|^2\, dx + \mathcal{K} \int_{\partial\Omega} |u|^2 ds\,. \tag{2.2}$$

  The Neumann case is retrieved for $\mathcal{K} = 0$.

For bounded domains, there are standard theorems, coming back to Agmon [1], permitting to prove the non-emptiness of the spectrum and moreover the completeness of the "generalized" eigenfunctions[2]. In the case $V(x) = g x_1$ (here we can think of $g \in \mathbb{C}$), the limit $g \longrightarrow 0$ can be treated by regular perturbation theory. In particular, Kato's theory [29] can be applied, the spectrum being close (modulo $\mathcal{O}(g)$) to the real axis. It is interesting to determine the variation of the lowest real part of an eigenvalue.

For the Dirichlet problem, the Feynman-Hellmann formula gives the coefficient in front of g as $i \int_\Omega x_1 |u_0(x)|^2\, dx$, where u_0 is the first $L^2(\Omega)$-normalized eigenfunction of the Dirichlet Laplacian. In fact, using the standard Kato's procedure we can look for an approximate eigenpair (λ, u) in the form:

$$u = u_0 + i\, g\, u_1 + g^2\, u_2 + \dots \tag{2.3}$$

and

$$\lambda = \lambda_0 + i\, g\, \lambda_1 + g^2\, \lambda_2 + \dots \tag{2.4}$$

Developing in powers of g, we get for the coefficient in front of g:

$$(-\Delta - \lambda_0) u_1 = -x_1 u_0 + \lambda_1 u_0\,, \tag{2.5}$$

and λ_1 is chosen in order to solve (2.5)

$$\lambda_1 = \int_\Omega x_1 |u_0(x)|^2\, dx\,. \tag{2.6}$$

We then take

$$u_1 = -(-\Delta - \lambda_0)^{(-1,reg)}\, ((x_1 - \lambda_1) u_0)\,, \tag{2.7}$$

[2] By this we mean elements in the kernel of $(\mathcal{A}^\#_h - \lambda)^k$ for some $k \geq 1$.

where $(-\Delta - \lambda_0)^{(-1,reg)}$ is the regularized resolvent, defined on the vector space generated by u_0 as

$$(-\Delta - \lambda_0)^{(-1,reg)} u_0 = 0\,,$$

and as the resolvent on the orthogonal space to u_0.
To look at the coefficient in front of g^2, we write

$$(-\Delta - \lambda_0) u_2 = (x_1 - \lambda_1)\, u_1 + \lambda_2 u_0\,, \tag{2.8}$$

and get

$$\lambda_2 = -\int_\Omega (x_1 - \lambda_1) u_1(x) u_0(x)\, dx\,,$$

from which

$$\lambda_2 = \big\langle (-\Delta - \lambda_0)^{(-1,reg)}((x_1 - \lambda_0)u_0) \,|\, ((x_1 - \lambda_0)u_0) \big\rangle_{L^2(\Omega)} > 0\,.$$

The effect of the perturbation is thus to shift the real part of the "first" eigenvalue on the right.

The limit $g \to +\infty$ for a fixed domain, or the limit of increasing domains (i.e. the domain obtained by dilation by a factor $R \to +\infty$) for a fixed g can be reduced by rescaling to a semi-classical limit $h \to 0$ of the operator $\mathcal{A}_h$ with a fixed potential $V(x)$. In this way, the BT-operator appears as a particular case (with $V(x) = x_1$) of a more general problem. We can mention (and will discuss) several recent papers, mainly devoted to the Dirichlet case, including: Almog [2], Henry [25] (Chapter 4), Beauchard-Helffer-Henry-Robbiano [10] (analysis of the 1D problem), Henry [26], Almog-Henry [8] and in the physics literature [35, 12] (and references therein).

2.2. The case of a bounded set in $\mathbb{R}^n$ and its complementary set with transmission condition at the boundary. We consider $\Omega_- \cup \complement\,\overline{\Omega}_-$, with Ω_- bounded in $\mathbb{R}^n$ and $\partial\Omega_-$ connected. In this case the definition of the operator is similar to what was done for the one-dimensional case in [17]. However, we start with a simpler case when $\Omega_- \subset \overline{\Omega}_- \subset \Omega$ with Ω bounded and $\Omega_+ = \Omega \setminus \overline{\Omega}_-$ (with Neumann boundary condition imposed on the exterior boundary $\partial\Omega$). After that, we explain how to treat the unbounded case with $\Omega = \mathbb{R}^n$ and $\Omega_+ = \complement\,\overline{\Omega}_-$.

2.2.1. Transmission property in the bounded case. To treat the difficulties one by one, we start with the situation when $\Omega_- \subset \overline{\Omega}_- \subset \Omega$, $\Omega_+ := \Omega \setminus \overline{\Omega}_-$, and Ω bounded and connected (e.g., a disk inside a larger disk).
We first introduce the variational problem, with the Hilbert space

$$\mathcal{H} = L^2(\Omega_-) \times L^2(\Omega_+)$$

and the form domain

$$\mathcal{V} := H^1(\Omega_-) \times H^1(\Omega_+)\,.$$

The quadratic form reads on $\mathcal{V}$

(2.9)
$$u = (u_-, u_+) \mapsto q_V(u) := h^2 ||\nabla u_-||^2_{\Omega_-} + h^2 ||\nabla u_+||^2_{\Omega_+} + \mathcal{K}\, ||(u_- - u_+)||^2_{L^2(\partial\Omega_-)} + i \int_{\Omega_-} V(x)|u_-(x)|^2\, dx + i \int_{\Omega_+} V(x)|u_+(x)|^2\, dx\ ,$$

where $\mathcal{K}$ is a positive parameter of the transmission problem, and $h > 0$ is a semi-classical parameter whose role will be explained later and which can be thought of as equal to one in this section. The dependence of $\mathcal{K}$ on $h > 0$ will be discussed later. We denote by $\mathfrak{a}_V$ the associated sesquilinear form:

$$\mathfrak{a}_V(u, u) = q_V(u)\,.$$

The potential $V(x)$ is assumed to be real (and we are particularly interested in the example $V(x) = gx_1$). In this case, one gets continuity and coercivity of the associated sesquilinear form on $\mathcal{V}$. This is true for any $\mathcal{K}$ without assumption on its sign. The trace of u_- and u_+ on $\partial\Omega_-$ is indeed well defined for $(u_-, u_+) \in \mathcal{V}$.

Applying Lax-Milgram's theorem, we first get that (u_-, u_+) should satisfy $\Delta u_- \in L^2(\Omega_-)$ and $\Delta u_+ \in L^2(\Omega_+)$. Together with $(u_-, u_+) \in \mathcal{V}$ this permits to define the Neumann condition (via the Green formula) for both u_- and u_+ in $H^{-\frac{1}{2}}(\partial\Omega_-)$, and in addition for u_+ in $H^{-\frac{1}{2}}(\partial\Omega)$. Indeed, to define $\partial_\nu u_-$ as a linear form on $H^{\frac{1}{2}}(\partial\Omega_-)$, we use that for any $v \in H^1(\Omega_-)$,

$$-\int_{\Omega_-} \Delta u_-\, v\, dx = \int_{\Omega_-} \nabla u_- \cdot \nabla v\, dx + \int_{\partial\Omega_-} \partial_\nu u_-\, v\, d\sigma\,, \tag{2.10}$$

and the existence of a continuous right inverse for the trace from $H^{\frac{1}{2}}(\partial\Omega_-)$ into $H^1(\Omega_-)$. Here the normal ν is oriented outwards Ω_- and when u_- is more regular ($u_- \in H^2(\Omega_-)$), we have $\partial_\nu u_- = \nu \cdot \nabla u_-$. In a second step we get the Neumann condition for u_+ on $\partial\Omega$,

$$\partial_\nu u_+ = 0 \text{ on } \partial\Omega\,, \tag{2.11}$$

and the transmission condition on $\partial\Omega_-$

$$\begin{aligned} \partial_\nu u_- &= \partial_\nu u_+ \\ h^2 \partial_\nu u_- &= \mathcal{K}\left(u_+ - u_-\right) \end{aligned} \qquad \text{on } \partial\Omega_-\,, \tag{2.12}$$

which is satisfied in $H^{-\frac{1}{2}}(\partial\Omega_-)$. We keep here the previous convention about the outwards direction of ν on $\partial\Omega_-$.
Finally, we observe that the first traces of u_- and u_+ on $\partial\Omega_-$ belong to $H^{\frac{1}{2}}(\partial\Omega_-)$. Hence by (2.12), the second traces of u_- and u_+ are in $H^{\frac{1}{2}}(\partial\Omega_-)$. But now the regularity of the Neumann problem in Ω_- and Ω_+ implies that

$$(u_-, u_+) \in H^2(\Omega_-) \times H^2(\Omega_+)\,.$$

Here we have assumed that all the boundaries are regular.

Remark 2. *One can actually consider a more general problem in which the two diffusion coefficients D_- and D_+ in Ω_- and Ω_+ are different. The transmission condition reads*

$$D_-\partial_\nu u_- = D_+\partial_\nu u_+ = \mathcal{K}(u_+ - u_-) \quad \text{on } \partial\Omega_- \,.$$

If we take $D_- = D_+ = D = h^2$, we recover the preceding case. In the limit $D_+ \to \infty$, we can consider the particular case where u_+ is identically 0 and we recover the Robin condition on the boundary $\partial\Omega_-$ of the domain Ω_-.

2.2.2. The unbounded case with bounded transmission boundary. In the case $\Omega_+ = \complement\overline{\Omega}_-$ (i.e., $\Omega = \mathbb{R}^n$), we have to treat the transmission problem through $\partial\Omega_-$ with the operator $-h^2\Delta + iV(x)$ on $L^2(\Omega_-) \times L^2(\Omega_+)$. Nothing changes at the level of the transmission property because $\partial\Omega_-$ is bounded. However, the variational space has to be changed in order to get the continuity of the sesquilinear form. Here we have to account for the unboundedness of V in Ω_+. For this purpose, we introduce

$$\mathcal{V} := \{(u_-, u_+) \in \mathcal{H}\,,\ |V|^{\frac{1}{2}} u_+ \in L^2(\Omega_+)\}\,. \tag{2.13}$$

If V has constant sign outside a compact, there is no problem to get the coercivity by looking separately at $\operatorname{Re}\mathfrak{a}_V(u,u)$ and $\operatorname{Im}\mathfrak{a}_V(u,u)$. When V does not have this property (as it is in the case $V(x) = x_1$), one cannot apply Lax-Milgram's theorem in its standard form. We will instead use the generalized Lax-Milgram Theorem as presented in [4] (see also [17]).

Theorem 3. *Let $\mathcal{V}$ denote a Hilbert space and let $\mathfrak{a}$ be a continuous sesquilinear form on $\mathcal{V} \times \mathcal{V}$. If $\mathfrak{a}$ satisfies, for some $\Phi_1, \Phi_2 \in \mathcal{L}(\mathcal{V})$, and some $\alpha > 0$,*

$$|\mathfrak{a}(u,u)| + |\mathfrak{a}(u, \Phi_1(u))| \geq \alpha \|u\|^2_{\mathcal{V}}\,, \quad \forall u \in \mathcal{V}\,, \tag{2.14}$$

$$|\mathfrak{a}(u,u)| + |\mathfrak{a}(\Phi_2(u), u)| \geq \alpha \|u\|^2_{\mathcal{V}}\,, \quad \forall u \in \mathcal{V}\,, \tag{2.15}$$

then $A \in \mathcal{L}(\mathcal{V})$ defined by

$$\mathfrak{a}(u,v) = \langle Au\,,\, v\rangle_{\mathcal{V}}\,, \forall u \in \mathcal{V}\,, \quad \forall v \in \mathcal{V}\,, \tag{2.16}$$

is a continuous isomorphism from $\mathcal{V}$ onto $\mathcal{V}$.

We now consider two Hilbert spaces $\mathcal{V}$ and $\mathcal{H}$ such that $\mathcal{V} \subset \mathcal{H}$ (with continuous injection and dense image). Let $\mathcal{A}$ be defined by

$$D(\mathcal{A}) = \{u \in \mathcal{V} \mid v \mapsto \mathfrak{a}(u,v) \text{ is continuous on } \mathcal{V} \text{ in the norm of } \mathcal{H}\} \tag{2.17}$$

and

$$\mathfrak{a}(u,v) = \langle \mathcal{A}u\,,\, v\rangle_{\mathcal{H}} \quad \forall u \in D(\mathcal{A}) \text{ and } \forall v \in \mathcal{V}\,. \tag{2.18}$$

Then we have

Theorem 4. *Let $\mathfrak{a}$ be a continuous sesquilinear form satisfying (2.14) and (2.15). Assume further that Φ_1 and Φ_2 extend into continuous linear maps in $\mathcal{L}(\mathcal{H})$. Let $\mathcal{A}$ be defined by (2.17)-(2.18). Then*

1. *$\mathcal{A}$ is bijective from $D(\mathcal{A})$ onto $\mathcal{H}$.*
2. *$D(\mathcal{A})$ is dense in both $\mathcal{V}$ and $\mathcal{H}$.*

3. *$\mathcal{A}$ is closed.*

EXAMPLE 5. *For $V(x) = x_1$, we can use on $\mathcal{V}$ the multiplier*

$$\Phi_1(u_-, u_+) = \left(u_- \, , \, \frac{x_1}{\sqrt{1+x_1^2}} \, u_+ \right).$$

We first observe that, for some $C > 0$,

$$Re\, \mathfrak{a}_V(u,u) \geq \frac{1}{C} \left(||\nabla u_-||^2 + ||\nabla u_+||^2 \right) - C \left(||u_-||^2 + ||u_+||^2 \right) .$$

To obtain the generalized coercivity, we now look at $Im\, \mathfrak{a}_V(u, \Phi_1(u))$ and get, for some $\hat{C} > 0$,

$$Im\, \mathfrak{a}_V(u, \phi_1(u)) \geq \int_{\Omega_+} |V(x)||u_+|^2 \, dx - \hat{C} \left(||u||^2 + ||\nabla u||^2 \right).$$

Note that this works (see [4]) for general potentials $V(x)$ satisfying (1.2).

Note also that the domain of the operator $\mathcal{A}^T$ associated with the sesquilinear form is described as follows

$$\begin{aligned} D \quad := \{ & u \in \mathcal{V} \, , \, (-h^2\Delta + iV)u_- \in L^2(\Omega_-) \, , \, (-h^2\Delta + iV)u_+ \in L^2(\Omega_+) \\ & \text{and transmission condition on } \partial\Omega_- \} \, . \end{aligned} \tag{2.19}$$

It is clear that this implies $u_- \in H^2(\Omega_-)$. The question of showing that $u_+ \in H^2(\Omega_+)$ is *a priori* unclear. By using the local regularity, we can show that for any χ in $C_0^\infty(\overline{\Omega_+})$,

$$(-h^2\Delta + iV)(\chi u) \in L^2(\mathbb{R}^n) \, ,$$

and consequently $\chi u \in H^2(\mathbb{R}^n)$.
In order to show that $u_+ \in H^2(\Omega_+)$, one needs to introduce other techniques and additional assumptions. For example, using the pseudodifferential calculus, it is possible to prove (see [32]), that $u_+ \in H^2(\Omega_+)$ and $Vu_+ \in L^2(\Omega_+)$ under the stronger condition that for any $\alpha \in \mathbb{N}^n$, there exists C_α such that

$$|D_x^\alpha V(x)| \leq C_\alpha \sqrt{1 + V(x)^2} \, , \quad \forall x \in \mathbb{R}^n \, . \tag{2.20}$$

REMARK 6 (No compactness of the resolvent). *There is no compact resolvent in this problem. We note indeed that the pairs (u_-, u_+) with $u_- = 0$ and $u_+ \in C_0^\infty(\Omega_+)$ belong to the domain of the operator. It is easy to construct a sequence of L^2 normalized $u_+^{(k)}$ in $C_0^\infty(\Omega_+)$ which is bounded in $H^2(\Omega_+)$, with support in $(-R, +R) \times \mathbb{R}^{n-1}$, and weakly convergent to 0 in $L^2(\Omega_+)$. This implies that the resolvent cannot be compact.*

REMARK 7. *The noncompactness of the resolvent does not exclude the existence of eigenvalues. Actually, when $\mathcal{K} = 0$, the spectral problem is decoupled into two independent problems: the Neumann problem in Ω_- which gives eigenvalues (the potential ix_1 in Ω_- is just a bounded perturbation, as discussed in Sec. 2.2.1) and the Neumann problem for the exterior problem in Ω_+ with $-\Delta + igx_1$ for which the question of existence of eigenvalues is more subtle if we think of the model of the half-space analyzed in Almog [2] or [25]. We will see that in the semi-classical limit (or equivalently $g \to +\infty$) the points of $\partial\Omega_-$ at which the normal vector to $\partial\Omega_-$ is parallel to $(1, 0, \ldots, 0)$, play a particular role.*

2.2.3. The case of two unbounded components in $\mathbb{R}^2$ separated by a curve. The case of two half-spaces is of course the simplest because we can come back to the one-dimensional problem using the partial Fourier transform. The analysis of the resolvent should however be detailed (see Henry [25] who treats the model of the half-space for the BT operator with Neumann or Dirichlet conditions). In fact, we consider the quadratic form

$$\begin{aligned} q(u) = h^2 \int_{x_1<0} |\nabla u_-(x)|^2\, dx + i \int_{x_1<0} \ell(x)|u_-(x)|^2\, dx \\ + h^2 \int_{x_1>0} |\nabla u_+(x)|^2\, dx + i \int_{x_1>0} \ell(x)|u_+(x)|^2\, dx \\ + \mathcal{K} \int |u_-(0,x_2) - u_+(0,x_2)|^2\, dx_2\,, \end{aligned}$$

where $x \mapsto \ell(x)$ is a nonzero linear form on $\mathbb{R}^2$:

$$\ell(x) = \alpha x_1 + \beta x_2\,.$$

Here, we can also apply the general Lax-Milgram theorem in order to define a closed operator associated to this quadratic form. The extension to a more general curve should be possible under the condition that the curve admits two asymptotes at infinity.

In this section, we have described how to associate to a given sesquilinear form $\mathfrak{a}$ defined on a form domain $\mathcal{V}$ an unbounded closed operator $\mathcal{A}$ in some Hilbert space $\mathcal{H}$. We will add the superscript $\#$ with $\# \in \{D, N, R, T\}$ in order to treat simultaneously the different cases. The space $\mathcal{H}^\#$ will be $L^2(\Omega)$ when $\# \in \{D, N, R\}$ and will be $L^2(\Omega_-) \times L^2(\Omega_+)$ in the case with transmission $\# = T$. $\mathcal{V}^\#$ will be respectively $H^1_0(\Omega)$, $H^1(\Omega)$, $H^1(\Omega)$, and $H^1(\Omega_-) \times H^1(\Omega_+)$. The corresponding operators are denoted $\mathcal{A}^\#_h$ with $\# \in \{D, N, R, T\}$.

3. Former semi-classical results. In order to treat simultaneously various problems we introduce $\Omega^\#$ with $\# \in \{D, N, R, T\}$ and $\Omega^D = \Omega$, $\Omega^N = \Omega$, $\Omega^R = \Omega$ and $\Omega^T = \Omega_-$.
R. Henry [26] (see also [8]) looked at the Dirichlet realization of the differential operator

$$\mathcal{A}^D_h := -h^2\Delta + i\,V(x)\,, \tag{3.1}$$

in a fixed bounded domain Ω, where V is a real potential and h a semi-classical parameter that goes to 0.
Setting $V(x) = x_1$, one gets a problem considered by de Swiet and Sen [35] in the simple case of a disk but these authors mentioned a possible extension of their computations to more general cases.

For a bounded regular open set, R. Henry in [26] (completed by Almog-Henry [8], see below) proved the following

THEOREM 8. *Let $V \in \mathcal{C}^\infty(\overline{\Omega};\mathbb{R})$ be such that, for every $x \in \overline{\Omega}$,*

$$\nabla V(x) \neq 0\,. \tag{3.2}$$

Then, we have

$$\varliminf_{h\to 0} \frac{1}{h^{2/3}} \inf\big\{Re\,\sigma(\mathcal{A}^D_h)\big\} \geq \frac{|a_1|}{2} J_m^{2/3}\,, \tag{3.3}$$

where $\mathcal{A}_h^D$ is the operator defined by (3.1) with the Dirichlet condition, $a_1 < 0$ is the rightmost zero of the Airy function Ai , and

$$J_m = \min_{x \in \partial\Omega_\perp} |\nabla V(x)| \,, \tag{3.4}$$

where

$$\partial\Omega_\perp = \{x \in \partial\Omega \,,\, \nabla V(x) \wedge \nu(x) = 0\} \,.$$

This result is essentially a reformulation of the result stated by Y. Almog in [2].

Remark 9. *The theorem holds in particular when $V(x) = x_1$ in the case of the disk (two points) and in the case of an annulus (four points). Note that in this application $J_m = 1$.*

A similar result can be proved for the Neumann case.

Remark 10. *To our knowledge, the equivalent theorems in the Robin case and the transmission case are open. We hope to come back to this point in a future work.*

A more detailed information is available in dimension 1 (see [10]) and in higher dimension [8] under some additional assumption on $\partial\Omega_\perp$. The authors in [8] prove the existence of an approximate eigenvalue. Our main goal is to propose a more general construction which will work in particular for the case with transmission condition.

Remark 11 (Computation of the Hessian). *For a planar domain, let us denote by $(x_1(s), x_2(s))$ the parameterization of the boundary by the arc length s starting from some point, $t(s) = (x_1'(s), x_2'(s))$ is the normalized oriented tangent, and $\nu(s)$ is the outwards normal to the boundary at s. Now we compute at $s = 0$ (corresponding to a point $x^0 = x(0) \in \partial\Omega_\perp^\#$, where $\nabla V \cdot t(0) = 0$),*

$$\begin{aligned}\left(\frac{d^2}{ds^2} V(x_1(s), x_2(s))\right)_{s=0} &= \langle t(0) | \mathrm{Hess V}(x_1(0), x_2(0))\, | t(0) \rangle \\ &\quad - \mathfrak{c}(0)\, (\nabla V(x_1(0), x_2(0)) \cdot \nu(0)) \,,\end{aligned}$$

where we used $t'(s) = -\mathfrak{c}(s)\nu(s)$, $\mathfrak{c}(s)$ representing the curvature of the boundary at the point $x(s)$.

Example 12. *When $V(x_1, x_2) = x_1$, we get*

$$\left(\frac{d^2}{ds^2} V(x_1(s), x_2(s))\right)_{s=0} = -\mathfrak{c}(0)(e_1 \cdot \nu(0)) \,,$$

with $e_1 = (1, 0)$.
In the case of the disk of radius 1, we get

$$\left(\frac{d^2}{ds^2} V(x_1(s), x_2(s))\right)_{s=0} \big(e_1 \cdot \nu(0)\big) = -1 \,, \tag{3.5}$$

for $(x_1, x_2) = (\pm 1, 0)$.

Let us now introduce a stronger assumption for $\# \in \{N, D\}$.

Assumption 13. *At each point x of $\partial\Omega_\perp^\#$, the Hessian of $V_{/\partial\Omega}$ is*
- *positive definite if $\partial_\nu V < 0$,*
- *negative definite if $\partial_\nu V > 0$,*

with ν being the outwards normal and $\partial_\nu V := \nu \cdot \nabla V$.

Under this additional assumption[3], the authors in [8] (Theorem 1.1) prove the equality in (3.3) by proving the existence of an eigenvalue near each previously constructed approximate eigenvalue, and get a three-terms asymptotics.

Remark 14. *Note that this additional assumption is verified for all points of $\partial\Omega_\perp$ when $V(x) = x_1$ and Ω is the disk. In fact, for this model, there are two points $(-1, 0)$ and $(1, 0)$, and formula (3.5) gives the solution.*

Y. Almog and R. Henry considered in [2, 26, 8] the Dirichlet case but, as noted by these authors in [8], one can similarly consider the Neumann case.

Without Assumption 13, there is indeed a difficulty for proving the existence of an eigenvalue close to the approximate eigenvalue. This is for example the case for the model operator

$$-h^2 \frac{d^2}{dx^2} - h^2 \frac{d^2}{dy^2} + i(y - x^2)\,,$$

on the half space. The operator is indeed not sectorial, and Lemma 4.2 in [8] is not proved in this case. The definition of the closed operator is questionable. One cannot use the technique given in a previous section because the condition (1.2) is not satisfied. The argument used by R. Henry in [25] for the analysis of the Dirichlet BT-operator in a half space $\mathbb{R}^2_+$ (based on [31] (Theorem X.49) and [28]) can be extended to this case.

This problem occurs for the transmission problem in which the model could be related to

$$-h^2 \frac{d^2}{dx^2} - h^2 \frac{d^2}{dy^2} + i(y + x^2)\,,$$

on the whole space $\mathbb{R}^2$ with transmission on $y = 0$. This case will not be treated in this paper.

On the growth of semi-groups. In the case of Dirichlet and Neumann realizations, one can study the decay of the semi-group $\exp(-t\mathcal{A}_h^\#)$ relying on the previous results and additional controls of the resolvent (see [25], [8]). When the domain is bounded, the potential is a bounded perturbation of self-adjoint operators. In this case, the control of the resolvent when $\mathrm{Im}\lambda$ tends to $\pm\infty$ is straightforward, with the decay as $\mathcal{O}(1/|\mathrm{Im}\lambda|)$. Applying the Gearhardt-Prüss theorem (see for example in [19]), the decay is

$$\mathcal{O}_\epsilon\Big(\exp\big(-t(1-\epsilon)\inf_{\lambda\in\sigma(\mathcal{A}_h^\#)}\{\mathrm{Re}\lambda\}\big)\Big) \quad \forall\epsilon > 0\,,$$

where $\sigma(\mathcal{A})$ denotes the spectrum of $\mathcal{A}$. In this case, $\sigma(\mathcal{A}_h^\#)$ is not empty and the set of generalized eigenfunctions is complete (see [1]).

In the unbounded case, the situation is much more delicate. The spectrum $\sigma(\mathcal{A}_h^\#)$ can be empty and one has to control the resolvent as $|\mathrm{Im}\lambda| \to +\infty$. The behavior of the associate semi-group can be super-exponential when $\sigma(\mathcal{A}_h^\#)$ is empty. Moreover, it is not granted that $\inf_{\lambda\in\sigma(\mathcal{A}_h^\#)}\{\mathrm{Re}\lambda\}$ gives the decay rate of the semi-group.

[3] We actually need this assumption only for the points x of $\partial\Omega_\perp$ such that $|\nabla V(x)| = J_m$.

4. Quasimode constructions – Preliminaries. Let us present in more detail the situation considered in Theorem 1.

4.1. Local coordinates. Choosing the origin at a point x^0 at which $\nabla V(x_0) \wedge \nu(x_0) = 0$, we replace the Cartesian coordinates (x_1, x_2) by the standard local variables (s, ρ), where ρ is the signed distance to the boundary, and s is the arc length starting from x^0. Hence

- In the case of one component, $\rho = 0$ defines the boundary $\partial\Omega$ and Ω is locally defined by $\rho > 0$.
- In the case of two components, $\rho = 0$ defines $\partial\Omega_-$, while $\rho < 0$ and $\rho > 0$ correspond, in the neighborhood of $\partial\Omega_-$, respectively to Ω_- and Ω_+.

In the (s, ρ) coordinates, the operator reads

$$\mathcal{A}_h = -h^2 a^{-1}\partial_s(a^{-1}\partial_s) - h^2 a^{-1}\partial_\rho(a\partial_\rho) + i\,\widetilde{V}(s,\rho)\,, \tag{4.1}$$

with

$$\widetilde{V}(s,\rho) := V(x_1(s,\rho), x_2(s,\rho))\,,$$

where

$$a(s,\rho) = 1 - \mathfrak{c}(s)\,\rho\,, \tag{4.2}$$

$\mathfrak{c}(s)$ representing the curvature of the boundary at $x(s, 0)$.
For future computation, we also rewrite (4.1) as

$$\mathcal{A}_h = -h^2 a^{-2}\partial_s^2 + h^2 a^{-3}\partial_s a\,\partial_s - h^2\partial_\rho^2 - h^2 a^{-1}\partial_\rho a\,\partial_\rho + i\,\widetilde{V}(s,\rho)\,. \tag{4.3}$$

The boundary conditions read

- Dirichlet condition

$$u(s,0) = 0\,, \tag{4.4}$$

- Neumann condition

$$\partial_\rho u(s,0) = 0\,, \tag{4.5}$$

- Robin condition with parameter $\mathcal{K}$

$$h^2\partial_\rho u(s,0) = \mathcal{K} u(s,0)\,, \tag{4.6}$$

- Transmission condition with parameter $\mathcal{K}$

$$\begin{cases} \partial_\rho u_+(s,0) = \partial_\rho u_-(s,0)\,, \\ h^2\partial_\rho u_+(s,0) = \mathcal{K}\big(u_+(s,0) - u_-(s,0)\big)\,. \end{cases} \tag{4.7}$$

In the last two cases, the link between $\mathcal{K}$ and h will be given later in (4.30).

We omit the tilde of $\tilde{V}$ in what follows.

We recall that the origin of the coordinates is at a point x^0 such that

$$\nabla V(x_0) \neq 0 \quad \text{and} \quad \nabla V(x_0) \wedge \vec{\nu}(x^0) = 0\,.$$

Hence we have

$$\frac{\partial V}{\partial s}(0,0) = 0\,, \tag{4.8}$$

and

$$\frac{\partial V}{\partial \rho}(0,0) \neq 0\,. \tag{4.9}$$

We also assume in our theorem that

$$\frac{\partial^2 V}{\partial s^2}(0,0) \neq 0\,. \tag{4.10}$$

Hence we have the following Taylor expansion

$$V(s,\rho) \sim \sum_{j,k} v_{jk} s^j \rho^k\,, \tag{4.11}$$

where

$$v_{jk} = \frac{1}{j!\;k!}\;\left(\frac{\partial^{j+k}}{\partial s^j \partial \rho^k} V(s,\rho)\right)_{s=\rho=0}\,, \tag{4.12}$$

with

$$v_{00} = V(0,0)\,, \quad v_{10} = 0\,, \quad v_{01} \neq 0\,, \quad v_{20} \neq 0\,, \tag{4.13}$$

corresponding to the assumptions of Theorem 1.

4.2. The blowing up argument. Approximating the potential V near x^0 by the first terms of its Taylor expansion $v_{00} + v_{01}\rho + v_{20}s^2$, a basic model reads

$$-h^2 \frac{d^2}{ds^2} - h^2 \frac{d^2}{d\rho^2} + i\,(v_{01}\rho + v_{20}s^2) \quad \text{on the half space } \{\rho > 0\},$$

in the case when $\# \in \{D, N, R\}$, and on $\mathbb{R}^2$ when $\# = T$, which is reduced by a natural scaling

$$(s,\rho) = (h^{\frac{1}{2}}\sigma, h^{\frac{2}{3}}\tau) \tag{4.14}$$

to

$$h\left(-\frac{d^2}{d\sigma^2} + i v_{20}\sigma^2\right) + h^{\frac{2}{3}}\left(-\frac{d^2}{d\tau^2} + i v_{01}\tau\right),$$

whose definition and spectrum can be obtained by separation of variables in the four cases.

4.2.1. Expansions. In the new variables (σ,τ) introduced in (4.14), the expansion is

$$\widehat{V}_h(\sigma,\tau) := V(h^{\frac{1}{2}}\sigma, h^{\frac{2}{3}}\tau) \sim \sum_{m\geq 0} h^{\frac{m}{6}} \left(\sum_{3k+4p=m} v_{kp}\sigma^k\tau^p\right)\,. \tag{4.15}$$

In particular, the first terms are

$$\widehat{V}_h(\sigma,\tau) = v_{00} + h^{\frac{2}{3}}\, v_{01}\tau + hv_{20}\sigma^2 + h^{\frac{7}{6}} v_{11}\sigma\tau + h^{\frac{8}{6}} v_{02}\tau^2 + h^{\frac{3}{2}} v_{30}\sigma^3 + \mathcal{O}(h^{\frac{5}{3}})\,. \tag{4.16}$$

Similarly, we consider the dilation of $a(s,\rho)$

$$\widehat{a}_h(\sigma,\tau) := a(h^{\frac{1}{2}}\sigma, h^{\frac{2}{3}}\tau) = 1 - h^{\frac{2}{3}}\tau\ \mathfrak{c}(h^{\frac{1}{2}}\sigma), \tag{4.17}$$

which can be expanded in the form

$$\widehat{a}_h(\sigma,\tau) \sim 1 - h^{\frac{2}{3}}\tau \left(\sum_{\ell} \frac{1}{\ell!}\mathfrak{c}^{(\ell)}(0)\sigma^\ell h^{\frac{\ell}{2}} \right)\,. \tag{4.18}$$

In the (σ,τ) coordinates, we get

$$\widehat{\mathcal{A}}_h = -h\widehat{a}_h^{-2}\partial_\sigma^2 + h^{\frac{3}{2}}\widehat{a}_h^{-3}\widehat{(\partial_s a)}_h\,\partial_\sigma - h^{\frac{2}{3}}\partial_\tau^2 - h^{\frac{4}{3}}\widehat{a}_h^{-1}\widehat{(\partial_\rho a)}_h\,\partial_\tau + i\,\widehat{V}_h(\sigma,\tau)\,. \tag{4.19}$$

We note that

$$\widehat{(\partial_s a)}_h(\sigma,\tau) = -h^{\frac{2}{3}}\mathfrak{c}'(h^{\frac{1}{2}}\sigma) \quad \text{and} \quad \widehat{(\partial_\rho a)}_h(\sigma,\tau) = -\mathfrak{c}(h^{\frac{1}{2}}\sigma)\,.$$

We rewrite $\widehat{\mathcal{A}}_h$ by expanding in powers of $h^{\frac{1}{6}}$:

$$\widehat{\mathcal{A}}_h \sim i\,v_{00} + h^{\frac{2}{3}} \sum_{j\geq 0} h^{\frac{j}{6}}\mathcal{L}_j(\sigma,\tau,\partial_\sigma,\partial_\tau)\,, \tag{4.20}$$

where the first terms are given by

$$\begin{aligned}
\mathcal{L}_0 &= -\partial_\tau^2 + i\ v_{01}\,\tau\,,\\
\mathcal{L}_1 &= 0\,,\\
\mathcal{L}_2 &= -\partial_\sigma^2 + i\,v_{20}\,\sigma^2\,,\\
\mathcal{L}_3 &= i\,v_{11}\,\sigma\tau\,,\\
\mathcal{L}_4 &= \mathfrak{c}(0)\,\partial_\tau + i\,v_{02}\,\tau^2\,,\\
\mathcal{L}_5 &= -\mathfrak{c}'(0)\partial_\sigma + i\,v_{03}\,\sigma^3\,.
\end{aligned} \tag{4.21}$$

For any $j \geq 0$, each $\mathcal{L}_j$ is a differential operator of order ≤ 2 with polynomial coefficients of degree which can be controlled as a function of j. In particular these operators preserve the vector space $\mathcal{S}(\mathbb{R}_\sigma) \otimes \mathcal{S}^\#$. The Fréchet space $\mathcal{S}^\#$ denotes $\mathcal{S}(\overline{\mathbb{R}_+})$ in the case when $\# \in \{D,N,R\}$ and $\mathcal{S}(\overline{\mathbb{R}_-}) \times \mathcal{S}(\overline{\mathbb{R}_+})$ when $\# = T$.

4.2.2. Parity. Note also that we have

Lemma 15.

$$(\check{\mathcal{L}_j f}) = (-1)^j \mathcal{L}_j \check{f}\,, \tag{4.22}$$

where $\check{f}(\tau,\sigma) = f(\tau,-\sigma)$.

Proof

This is a consequence of

$$\mathcal{L}_j(\sigma,\tau,\partial_\sigma,\partial_\tau) = (-1)^j\,\mathcal{L}_j(-\sigma,\tau,-\partial_\sigma,\partial_\tau) \tag{4.23}$$

that can be seen by observing that

$$
\begin{aligned}
h^{-\frac{2}{3}}(\widehat{\mathcal{A}}_h - i v_{00}) &= h^{\frac{1}{3}}\widehat{a}_h^{-2}\partial_\sigma^2 + h^{\frac{5}{6}}\widehat{a}_h^{-3}\widehat{(\partial_s a)}_h\,\partial_\sigma - \partial_\tau^2 \\
&\quad - h^{\frac{2}{3}}\widehat{a}_h^{-1}\widehat{(\partial_\rho a)}_h\,\partial_\tau + i\,h^{-\frac{2}{3}}(\widehat{V}_h(\sigma,\tau) - v_{00}) \\
&\sim \sum_{j\geq 0} h^{\frac{j}{6}}\mathcal{L}_j(\sigma,\tau,\partial_\sigma,\partial_\tau)\,.
\end{aligned}
\tag{4.24}
$$

We will see that each term in the right hand side of (4.24) satisfies (4.22). First, denoting $\hat{h} = h^{\frac{1}{6}}$, we can rewrite

$$
\widehat{a}_h(\sigma,\tau) \sim 1 - \hat{h}^4\tau\left(\sum_{\ell\geq 0}\frac{1}{\ell!}\mathfrak{c}^{(\ell)}(0)\sigma^\ell\hat{h}^{3\ell}\right)\,,
\tag{4.25}
$$

and expanding in powers of $\hat{h}$, we see that the coefficient in front of $\hat{h}^\ell$ has the parity of ℓ in σ. The same is true for $\widehat{a}_h(\sigma,\tau)^{-2}$. Hence the coefficient in front of $h^{\frac{j}{6}}$ in $h^{\frac{1}{3}}\widehat{a}_h(\sigma,\tau)^{-2}\partial_\sigma^2$ satisfies (4.23).

We now look at $h^{\frac{5}{6}}\widehat{a}_h^{-3}\widehat{(\partial_s a)}_h$ and write

$$
h^{\frac{5}{6}}\widehat{(\partial_s a)}_h(\sigma,\tau)\partial_\sigma = -\hat{h}^9\mathfrak{c}'(\hat{h}^3\sigma)\partial_\sigma\,.
$$

It is clear from this formula that the second term in the right hand side of (4.24) satisfies (4.23).

The third term $-\partial_\tau^2$ clearly satisfies (4.23). For the forth term $-h^{\frac{2}{3}}\widehat{a}_h^{-1}\widehat{(\partial_\rho a)}_h\,\partial_\tau$, it is enough to use the previous expansions and to observe that

$$
\widehat{(\partial_\rho a)}_h(\sigma,\tau) = -\mathfrak{c}(\hat{h}^3\sigma)\,.
$$

Finally, we consider

$$
i\,h^{-\frac{2}{3}}(\widehat{V}_h(\sigma,\tau) - v_{00}) \sim i\sum_{m\geq 4}\hat{h}^{m-4}\left(\sum_{3k+4p=m} v_{kp}\,\sigma^k\,\tau^p\right)\,,
$$

and we observe that k and m should have the same parity.

This lemma will be useful for explaining cancellations in the expansion of the quasi-mode.

4.2.3. Boundary or transmission conditions. In these local coordinates, the boundary conditions read

- the Dirichlet condition

$$
u(\sigma,0) = 0\,,
\tag{4.26}
$$

- the Neumann condition

$$
\partial_\tau u(\sigma,0) = 0\,,
\tag{4.27}
$$

- the Robin condition

$$
\partial_\tau u(\sigma,0) = \mathcal{K}h^{-\frac{4}{3}}u(\sigma,0)\,,
\tag{4.28}
$$

- the transmission condition

$$\partial_\tau u_-(\sigma,0) = \partial_\tau u_+(\sigma,0)\,, \quad \partial_\tau u_+(\sigma,0) = \mathcal{K}\, h^{-\frac{4}{3}}\big(u_+(\sigma,0) - u_-(\sigma,0)\big)\,. \tag{4.29}$$

Depending on the physical problem, the Robin or Transmission parameter $\mathcal{K}$ can exhibit different scaling with h. Here we assume the scaling

$$\mathcal{K} = \kappa\, h^{\frac{4}{3}}\,, \tag{4.30}$$

so that the Robin or transmission conditions in the variables (σ,τ) are independent of h and read

$$\partial_\tau u(\sigma,0) = \kappa\, u(\sigma,0)\,, \tag{4.31}$$

and

$$\partial_\tau u_-(\sigma,0) = \partial_\tau u_+(\sigma,0)\,, \quad \partial_\tau u_+(\sigma,0) = \kappa\big(u_+(\sigma,0) - u_-(\sigma,0)\big)\,. \tag{4.32}$$

In Sec. 4.3, we justify this scaling by considering the transmission problem in dilated domains, while other scalings are discussed in Sec. 6. We denote by $\mathcal{L}_0^{\#}$ the realization of $\mathcal{L}_0$ with $\# = D, N, R, T$ for Dirichlet, Neumann, Robin, or Transmission condition. We recall that the Hilbert space $L^2_{\#}$ denotes $L^2(\mathbb{R}_+)$ in the case when $\# \in \{D, N, R\}$, and $L^2(\mathbb{R}_-) \times L^2(\mathbb{R}_+)$ when $\# = T$. For the complex harmonic oscillator $\mathcal{L}_2$ we consider (with the same notation) the self-adjoint realization on $L^2(\mathbb{R}_\sigma)$.

4.3. Comparison with the large domain limit. We assume that $0 \in \Omega_-$ and we dilate Ω_- and Ω by the map $(x_1, x_2) \mapsto (Sx_1, Sx_2)$ ($S > 0$ supposed to be large) and get Ω_-^S and Ω^S.
It remains to check how the transmission problem for Ω^S with $V(x) = x_1$ is modified by dilation. If we start from the form

$$u \mapsto \int_{\Omega^S} |\nabla u|^2 dx + i \int_{\Omega^S} x_1 |u(x)|^2\, dx + \kappa_S \int_{\partial\Omega_-^S} |u_+ - u_-|^2\, ds_S\,,$$

with a transmission coefficient κ_S, we get by the change of coordinates $x = Sy$, for $v(y) = u(Sy)$,

$$\int_\Omega |\nabla_y v|^2 dy + i\, S^3 \int_\Omega y_1 |v(y)|^2\, dy + \kappa_S\, S \int_{\partial\Omega_-} |v_+ - v_-|^2\, ds\,.$$

Dividing by S^3, we get

$$\frac{1}{S^3} \int_\Omega |\nabla_y v|^2 dy + i \int_\Omega y_1 |v(y)|^2\, dy + \kappa_S\, S^{-2} \int_{\partial\Omega_-} |v_+ - v_-|^2\, ds\,.$$

In order to treat this problem as semi-classical, we set

$$h^2 = \frac{1}{S^3}\,, \qquad \mathcal{K} = \kappa_S\, S^{-2}\,,$$

Hence we get

$$\kappa = \kappa_S\,,$$

and our assumption (4.30) on $\mathcal{K}$ corresponds to what we get by rescaling from the problem in Ω_S with κ_S independent of R.

For this application, Theorem 1 gives the following

THEOREM 16. *For $S > 0$, let $V_S(x) = S\,V(S^{-1}x)$, with the potential V defined on Ω satisfying the conditions of Theorem 1, and κ_S is independent of S. Then, with the notation of Theorem 1, one can construct a quasimode $\lambda_S^{\#}$ of the $\#$ realization of the operator $-\Delta + iV_S$ in $\Omega_S^{\#}$ such that*

$$\lambda_S^{\#} = i\,SV(x^0) + \sum_{j\in\mathbb{N}} \lambda_{2j}^{\#}\, S^{-\frac{j}{2}} + \mathcal{O}(S^{-\infty})\,, \tag{4.33}$$

as $S \to +\infty$.

This theorem can also be applied to $V(x) = x_1$, in which case V_S is independent of S.

REMARK 17. *More generally, one can consider*

$$V_S(x) = S^m\, V(S^{-1}x)\,,$$

with $m > -2$. In this case, we get $\kappa = \kappa_S\, S^{1-m}$. If κ is independent of S or tends to 0 as $S \to +\infty$, one can apply the semi-classical analysis of the previous sections.

5. The quasimode construction. Proof of the main theorem.

5.1. The form of the quasimode. In what follows, we assume in the Robin or transmission cases that κ is independent of h (see (4.30)). We now look for a quasimode $u_h^{app,\#}$ that we write in the (σ,τ) variables in the form:

$$u_h^{app,\#} \sim d(h) \left(\sum_{j\geq 0} h^{\frac{j}{6}} u_j^{\#}(\sigma,\tau) \right)\,, \tag{5.1}$$

associated with an approximate eigenvalue

$$\lambda_h^{app,\#} \sim i\,v_{00} + h^{\frac{2}{3}} \sum_{j\geq 0} h^{\frac{j}{6}} \lambda_j^{\#}\,. \tag{5.2}$$

Here $d(h) \sim d_0 h^{-\frac{7}{12}}$ with $d_0 \neq 0$ chosen such that, coming back to the initial coordinates, the L^2-norm of the trial state equals 1.

Note that the $u_j^{\#}$ are in the domain of $\mathcal{L}_j^{\#}$ if we take the condition $\#$ (with $\# \in \{N, D, R, T\}$).
Note also that we do not assume *a priori* that the $\lambda_j^{\#}$ for j odd are 0 as claimed in our theorem.
As will be seen in the proof, we can choose

$$u_j^{\#}(\sigma,\tau) = \phi_j^{\#}(\sigma)\psi_0^{\#}(\tau)\,, \quad j = 0,1,2\,, \tag{5.3}$$

and

$$u_j^{\#}(\sigma,\tau) = \phi_j^{\#}(\sigma)\psi_0^{\#}(\tau) + \sum_{\ell=1}^{N_j} \phi_{j,\ell}^{\#}(\sigma)\psi_{j,\ell}^{\#}(\tau)\,, \quad j \geq 3\,, \tag{5.4}$$

with $\phi_{j,\ell}^{\#} \in \mathcal{S}(\mathbb{R})$ and $\psi_{j,\ell} \in \mathcal{S}^{\#}$ to be specified below.
Moreover, we have

$$\mathcal{L}_0^{\#}\psi_0^{\#} = \lambda_0^{\#}\psi_0^{\#}\,, \tag{5.5}$$

$$\mathcal{L}_2 \phi_0^{\#}(\sigma) = \lambda_2^{\#} \phi_0^{\#}(\sigma)\,, \tag{5.6}$$

with

$$\langle \psi_{j,\ell}^{\#}\,,\, \bar{\psi}_0^{\#}\,\rangle_{L^2_{\#}} = 0\,, \tag{5.7}$$

and

$$\langle \psi_0^{\#}\,,\, \bar{\psi}_0^{\#}\rangle_{L^2_{\#}} \neq 0\,. \tag{5.8}$$

The construction will consist in expanding $(\widehat{\mathcal{A}_h} - \lambda_h^{app,\#})u_h^{app,\#}$ in powers of $h^{\frac{1}{6}}$ and finding the conditions of cancellation for each coefficient of this expansion.

If we succeed in this construction and come back to the initial coordinates, using a Borel procedure to sum the formal expansions and multiplying by a cutoff function in the neighborhood of a point x^0 of $\partial\Omega^{\#}$, we obtain an approximate spectral pair localized near x^0 (i.e. $\mathcal{O}(h^\infty)$ outside any neighborhood of x^0).
The Borel procedure consists in choosing a cutoff function θ (with $\theta = 1$ in a small neighborhood of 0 and a sequence H_n such that $\beta \mapsto \sum_j \beta^j \lambda_j \theta(\beta/H_j)$ converges in $C^\infty([0,\beta_0])$ for some $\beta_0 > 0$. We then define

$$\lambda_h^{\#} = i\, v_{00} + h^{\frac{2}{3}} \sum_{j\geq 0} \beta^j \lambda_j \theta(\beta/H_j)\,,$$

with $\beta = h^{\frac{1}{6}}$.
This $\lambda_h^{\#}$ is not unique but the difference between two different choices is $\mathcal{O}(h^\infty)$. A similar procedure can be used to define a quasimode state $u_h^{\#}$ strongly localized near x^0.

Remark 18. *We emphasize that the above construction is not sufficient (the problem being non self-adjoint) for proving the existence of an eigenvalue with this expansion. The construction is true for any regular domain (exterior or interior) under the conditions* (4.8)-(4.10)*. When $V(x) = x_1$, we recover in this way the condition that the curvature does not vanish at x^0. We recall that this construction can be done near each point where $\nabla V(x^0) \wedge \nu(x^0) = 0$. The candidates for the spectrum are determined by ordering different quasimodes and comparing their real parts. We guess that the true eigenfunctions will have the same localization properties as the constructed quasimode states.*

5.2. Term $j = 0$. Identifying the powers in front of $h^{\frac{1}{6}}$, after division by $d(h)$, one gets the first equation corresponding to $j = 0$.
We consider four boundary conditions.

Neumann and Dirichlet cases. For the Neumann boundary condition, one has

$$\mathcal{L}_0^N\, u_0^N = \lambda_0^N u_0^N\,, \quad \partial_\tau u_0^N(\sigma, 0) = 0\,, \tag{5.9}$$

and we look for a solution in the form

$$u_0^N(\sigma,\tau) = \phi_0^N(\sigma)\psi_0^N(\tau)\,. \tag{5.10}$$

At this step, we only look for a pair (λ_0^N, ψ_0^N) with ψ_0^N non identically 0 such that

$$(-\partial_\tau^2 + i\, v_{01}\tau)\, \psi_0^N(\tau) = \lambda_0^N \psi_0^N(\tau) \text{ in } \mathbb{R}^+\,, \quad (\psi_0^N)'(0) = 0\,. \tag{5.11}$$

We recall from (4.13) that $v_{01} \neq 0$ so we have the standard spectral problem for the complex Airy operator in the half line with Neumann condition at 0. The spectral theory of this operator is recalled in [17]. The spectrum consists of an infinite sequence of eigenvalues $(\lambda^{N,(n)})_{n\geq 1}$ (ordered by increasing real part) that can be expressed through the zeros a_n' $(n \geq 1)$ of the derivative of the Airy function $\mathrm{Ai}'(z)$:

$$\lambda^{N,(n)} = -a_n' \,|\, v_{01}|^{\frac{2}{3}} \exp\left(\frac{i\pi}{3} \operatorname{sign} v_{01}\right)\,. \tag{5.12}$$

Different choices of n will determine the asymptotic expansion of different approximate eigenvalues of the original problem. If we are interested in controlling the decay of the associated semi-group, we choose $\lambda_0^N = \lambda^{N,(1)}$ which corresponds to the eigenvalue with the smallest real part.

One can similarly treat the Dirichlet problem (like in [8]). In this case, one has

$$\mathcal{L}_0^D\, u_0^D = \lambda_0^D u_0^D \text{ in } \mathbb{R}_+\,, \quad u_0^D(\sigma, 0) = 0\,, \tag{5.13}$$

and we look for a solution in the form

$$u_0^D(\sigma,\tau) = \phi_0^D(\sigma)\psi_0^D(\tau)\,, \tag{5.14}$$

where $\psi_0^D(\tau)$ satisfies

$$\mathcal{L}_0^D\, \psi_0^D = \lambda_0^D \psi_0^D \text{ in } \mathbb{R}_+\,, \quad \psi_0^D(0) = 0\,. \tag{5.15}$$

The spectral theory of this operator is also recalled in [17]. The spectrum consists of an infinite sequence of eigenvalues $(\lambda^{D,(n)})_{n\geq 1}$ (ordered by increasing real part) that can be expressed through the zeros a_n $(n \geq 1)$ of the Airy function $\mathrm{Ai}(z)$:

$$\lambda^{D,(n)} = -a_n \,|\, v_{01}|^{\frac{2}{3}} \exp\left(\frac{i\pi}{3} \operatorname{sign} v_{01}\right)\,. \tag{5.16}$$

One can show (see [25] for a proof by analytic dilation) that

$$\int_0^{+\infty} \psi_0^N(\tau)^2\, d\tau \neq 0 \quad \text{and} \quad \int_0^{+\infty} \psi_0^D(\tau)^2\, d\tau \neq 0\,. \tag{5.17}$$

This is also a consequence of the completeness of the eigenfunctions of the complex Airy operator in the half-line with Neumann or Dirichlet boundary condition. This property is true for any eigenvalue $\lambda_0^\#$ of $\mathcal{L}_0^\#$.

For $n \geq 1$, the eigenfunctions $\psi_0^N = \psi^{N,(n)}$ and $\psi_0^D = \psi^{D,(n)}$ are specifically translated and complex dilated Airy functions:

$$\psi^{N,(n)}(\tau) = c_n^N\, \mathrm{Ai}\left(a_n' + \tau\,|\, v_{01}|^{\frac{1}{3}} \exp\left(\frac{i\pi}{6} \operatorname{sign} v_{01}\right)\right) \quad \text{for } \tau \geq 0\,, \tag{5.18}$$

$$\psi^{D,(n)}(\tau) = c_n^D\, \mathrm{Ai}\left(a_n + \tau\,|\, v_{01}|^{\frac{1}{3}} \exp\left(\frac{i\pi}{6} \operatorname{sign} v_{01}\right)\right) \quad \text{for } \tau \geq 0\,, \tag{5.19}$$

where the normalization constants c_n^N and c_n^D can be fixed by choosing the following normalization that we keep throughout the paper:

$$\int_0^\infty \psi_0^{\#}(\tau)^2 d\tau = 1\,. \tag{}$$

These coefficients are computed explicitly in Appendix A (see (A.24), (A.20)) .

Robin case. For the Robin boundary condition, one has

$$\mathcal{L}_0^R\, u_0^R = \lambda_0^R u_0^R\,, \quad \partial_\tau u_0^R(\sigma,0) = \kappa\, u_0^R(\sigma,0)\,, \tag{5.20}$$

and we look for a solution in the form

$$u_0^R(\sigma,\tau) = \phi_0^R(\sigma)\psi_0^R(\tau)\,, \tag{5.21}$$

where the function $\psi_0^R(\tau)$ satisfies

$$(-\partial_\tau^2 + iv_{01}\tau)\psi_0^R(\tau) = \lambda_0^R\psi_0^R(\tau) \ \text{ in } \mathbb{R}_+, \quad (\psi_0^R)'(0) = \kappa\,\psi_0^R(0)\,. \tag{5.22}$$

This one-dimensional problem was studied in [17]. In particular, the spectrum consists of an infinite sequence of eigenvalues $(\lambda^{R,(n)})_{n\geq 1}$ (ordered by increasing real part) that can be expressed as

$$\lambda^{R,(n)}(\kappa) = -a_n^R(\kappa)\,|\,v_{01}|^{\frac{2}{3}}\,\exp\left(\frac{\pi i}{3}\,\mathrm{sign}\,v_{01}\right), \tag{5.23}$$

where $a_n^R(\kappa)$ is a solution of the equation

$$\exp\left(\frac{\pi i}{6}\,\mathrm{sign}\,v_{01}\right)\mathrm{Ai}'\big(a_n^R(\kappa)\big) - \frac{\kappa}{|v_{01}|^{\frac{1}{3}}}\,\mathrm{Ai}\big(a_n^R(\kappa)\big) = 0\,, \tag{5.24}$$

and $\kappa \geq 0$ denotes the Robin parameter.[4]
Except for the case of small κ, in which the eigenvalues are close to the eigenvalues of the Neumann problem, it does not seem easy to localize all the solutions of (5.24) in general. Note that from (5.24), we deduce that

$$(\lambda^{R,(n)})'(0) = -(a_n^R)'(0)\,|\,v_{01}|^{\frac{2}{3}}\,\exp\left(\frac{\pi i}{3}\,\mathrm{sign}\,v_{01}\right), \tag{5.25}$$

where

$$(a_n^R)'(0) = \frac{1}{a_n^R(0)|v_{01}|^{\frac{1}{3}}}\,\exp\left(-\frac{\pi i}{6}\,\mathrm{sign}\,v_{01}\right) \neq 0\,. \tag{5.26}$$

Nevertheless it is proven in [17] that the zeros of the function in (5.24) are simple and that there is no Jordan block. So as can be deduced from the next lemma, any eigenfunction satisfies $\int \psi_0^R(\tau)^2\,d\tau \neq 0$. We consequently fix the normalization of ψ_0^R by imposing

$$\int_0^\infty \psi_0^R(\tau)^2\,d\tau = 1\,. \tag{5.27}$$

[4] In [17], we discussed the complex Airy operator with $v_{01} = -1$, see Eq. (3.25).

For $n \geq 1$, the associated eigenfunction $\psi_0^R = \psi^{R,(n)}$ reads

$$\psi^{R,(n)}(\tau) = c_n^R \,\mathrm{Ai}\Big(a_n^R(\kappa) + \tau |v_{01}|^{\frac{1}{3}} \exp\Big(\frac{\pi i}{6} \,\mathrm{sign}\, v_{01}\Big)\Big) \quad (\tau \geq 0), \tag{5.28}$$

where c_n^R is the normalization constant given by (A.28).

5.2.1. Transmission case. In the transmission case, one gets, with $\psi_0^T = (\psi_0^-, \psi_0^+)$,

$$\begin{aligned} &\mathcal{L}_0 \psi_0^- = \lambda_0^T \psi_0^- \text{ in } \mathbb{R}_-\,, \quad \mathcal{L}_0 \psi_0^+ = \lambda_0^T \psi_0^+ \text{ in } \mathbb{R}_+\,, \\ &\partial_\tau \psi_0^-(\sigma, 0) = \partial_\tau \psi_0^+(\sigma, 0)\,, \quad \partial_\tau \psi_0^+(\sigma, 0) = \kappa \left(\psi_0^+(\sigma, 0) - \psi_0^-(\sigma, 0)\right). \end{aligned} \tag{5.29}$$

The existence of λ_0^T has been proved in [17]. In addition, the eigenvalue (of the smallest real part) is simple (no Jordan block) for $\kappa \geq 0$ small. We can use the explicit computations in [17] or the following abstract lemma by Aslayan-Davies for a closed operator A [9]:

Lemma 19. *If f and f^* are the normalized eigenvectors of A and A^* associated with the eigenvalues λ and $\bar{\lambda}$ respectively, and if the the spectral projector P has rank 1, then $\langle f\,,\, f^* \rangle \neq 0$ and*

$$||P|| = \frac{1}{|\langle f\,,\, f^* \rangle|}\,.$$

The proof that P has rank 1 for the case $V(x) = x_1$ is given in [17] but only for $\kappa \geq 0$. In general, we make the assumption

Assumption 20. *$\lambda_0^T(\kappa)$ is simple (no Jordan block).*

Under this assumption, we have

$$\int_{-\infty}^{\infty} \psi_0^T(\tau)^2 \, d\tau := \int_{-\infty}^{0} \psi_0^-(\tau)^2 \, d\tau + \int_0^{+\infty} \psi_0^+(\tau)^2 \, d\tau \neq 0\,. \tag{5.30}$$

The explicit form of the eigenfunctions $\psi^{T,(n)}$ $(n \geq 1)$ can be obtained from the analysis provided in [16, 17]:

$$\begin{aligned} \psi^{+,(n)}(\tau) &= -c_n^T \,\bar{\delta}\, \mathrm{Ai}'\big(a_n^-(\kappa)\big)\, \mathrm{Ai}\left(a_n^+(\kappa) + \tau \,|\, v_{01}|^{\frac{1}{3}} \delta\right), \\ \psi^{-,(n)}(\tau) &= c_n^T \,\delta\, \mathrm{Ai}'\big(a_n^+(\kappa)\big)\, \mathrm{Ai}\left(a_1^-(\kappa) - \tau \,|\, v_{01}|^{\frac{1}{3}} \bar{\delta}\right), \end{aligned} \tag{5.31}$$

where c_n^T is a normalization constant (to satisfy (5.35)), $\delta = \exp\big(\frac{i\pi}{6} \,\mathrm{sign}\, v_{01}\big)$, and

$$a_n^{\pm}(\kappa) = \hat{\lambda}^{T,(n)}\big(\kappa / |\, v_{01}|^{\frac{1}{3}}\big) \,\exp\left(\pm \frac{2\pi i}{3} \,\mathrm{sign}\, v_{01}\right), \tag{5.32}$$

where the $\hat{\lambda}^{T,(n)}(\check{\kappa})$ are the eigenvalues of the complex Airy operator $-\frac{d^2}{dx^2} + ix$ on the line with transmission condition at 0, with coefficient

$$\check{\kappa} = \kappa / |v_{01}|^{\frac{1}{3}}\,.$$

They are defined implicitly as complex-valued solutions (enumerated by the index $n = 1, 2, \ldots$) of the equation [16, 17]

$$2\pi \,\mathrm{Ai}'\big(e^{2\pi i/3} \hat{\lambda}^{T,(n)}(\check{\kappa})\big)\, \mathrm{Ai}'\big(e^{-2\pi i/3} \hat{\lambda}^{T,(n)}(\check{\kappa})\big) = -\check{\kappa}\,. \tag{5.33}$$

The eigenvalues $\hat{\lambda}^{T,(n)}(\check{\kappa})$ are ordered according to their increasing real parts:

$$\mathrm{Re}\{\hat{\lambda}^{T,(1)}(\check{\kappa})\} \leq \mathrm{Re}\{\hat{\lambda}^{T,(2)}(\check{\kappa})\} \leq \dots$$

Note that $\psi^{-,(n)}(0_-) \neq \psi^{+,(n)}(0_+)$. The associated eigenvalue is

$$\lambda^{T,(n)}(\kappa) = \hat{\lambda}^{T,(n)}\big(\kappa/|v_0|^{\frac{1}{3}}\big)\,|\,v_{01}|^{\frac{2}{3}}\,. \tag{5.34}$$

In what follows, $(\lambda_0^T(\kappa), \psi_0^T)$ denotes an eigenpair $(\lambda^{T,(n)}(\kappa), \psi^{T,(n)})$ corresponding to a particular choice of $n \geq 1$.

Summary at this stage. For $\# \in \{D, N, R, T\}$, we have constructed $u_0^\#$ in the form (5.3). At this step $\phi_0^\#(\sigma)$ remains "free" except that it should not be identically 0. We have chosen $\lambda_0^\#$ as an eigenvalue of $\mathcal{L}_0^\#$ (assuming that it is simple, with no Jordan block) and $\psi_0^\#$ is the associated eigenfunction of $\mathcal{L}_0^\#$, which belongs to $\mathcal{S}^\#$ and permits, according to Lemma 19, to have the normalization

$$\int_{\mathbb{R}^\#} \psi_0^\#(\tau)^2 d\tau = 1\,. \tag{5.35}$$

From now on, we do not mention (except for explicit computations) the reference to Dirichlet, Neumann, Robin or Transmission condition when the construction is independent of the considered case.

5.3. Term $j = 1$. The second equation (corresponding to $j = 1$) reads

$$(\mathcal{L}_0^\# - \lambda_0)\, u_1^\# = \lambda_1\, u_0^\#\,. \tag{5.36}$$

We omit sometimes the superscript # for simplicity.
The guess is that $\lambda_1 = 0$. To see if it is a necessary condition, one can take the scalar product (in the τ variable) with $\bar{\psi}_0$ (to be understood as the element in $\mathrm{Ker}(\mathcal{L}_0^* - \bar{\lambda}_0)$). We take the convention that the scalar product is antilinear in the second argument. This leads to

$$\left(\int \psi_0^2(\tau) d\tau\right)\, \lambda_1\, \phi_0(\sigma) = 0\,,$$

the integral being on $\mathbb{R}^+$ for Dirichlet, Neumann or Robin, and on $\mathbb{R}$ in the transmission case. From Eq. (5.35), we get then

$$\lambda_1 \phi_0(\sigma) = 0\,,$$

and by the previous condition on $\phi_0(\sigma)$

$$\lambda_1 = 0\,. \tag{5.37}$$

Hence, coming back to (5.36), we choose

$$u_1^\#(\sigma, \tau) = \phi_1^\#(\sigma)\psi_0^\#(\tau)\,, \tag{5.38}$$

where $\phi_1^\#$ remains free at this step.

5.4. Term $j = 2$. The third condition (corresponding to $j = 2$) reads

$$(\mathcal{L}_0^{\#} - \lambda_0)\, u_2 + \mathcal{L}_2\, u_0 = \lambda_2\, u_0\,. \tag{5.39}$$

To find a necessary condition, we take the scalar product (in the τ variable) with $\bar{\psi}_0$. In this way we get (having in mind (5.35))

$$\langle \mathcal{L}_2\, u_0\,,\, \bar{\psi}_0 \rangle = \lambda_2\, \phi_0(\sigma)\,.$$

Computing the left hand side, we get

$$(-\partial_\sigma^2 + i\, v_{20}\, \sigma^2)\, \phi_0(\sigma) = \lambda_2\, \phi_0(\sigma)\,.$$

From Assumption (4.13), we know that $v_{20} \neq 0$. Hence we are dealing with an effective complex harmonic oscillator whose spectral analysis has been done in detail (see Davies [11] or the book by Helffer [19]). The eigenvalues can be explicitly computed (by analytic dilation) and there is no Jordan block. Moreover the system of corresponding eigenfunctions is complete. This implies that (λ_2, ϕ_0) should be a spectral pair for $(-\partial_\sigma^2 + i\, v_{20}\, \sigma^2)$.
The eigenpairs of the quantum harmonic oscillator are well known:

$$\lambda_2^{(k)} = \gamma(2k-1), \quad \phi_0^{(k)}(\sigma) = \frac{\gamma^{\frac{1}{4}}\, e^{-\gamma\sigma^2/2}\, H_{k-1}(\gamma^{\frac{1}{2}}\sigma)}{\pi^{\frac{1}{4}}\sqrt{2^{k-1}\,(k-1)!}} \quad (k = 1, 2, \ldots), \tag{5.40}$$

where $\gamma^s = |v_{20}|^{\frac{s}{2}} \exp\big(\frac{\pi s i}{4} \operatorname{sign} v_{20}\big)$ (for $s = \frac{1}{4}, \frac{1}{2}, 1$), $H_k(z)$ are Hermite polynomials, and the prefactor ensures that

$$\int_{-\infty}^{\infty} \phi_0^{(k)}(\sigma)^2\, d\sigma = 1\,.$$

The eigenvalue with the smallest real part corresponds to $k = 1$ for which

$$\phi_0^{(1)}(\sigma) = c_{\phi_0} \exp\left(-\lambda_2 \frac{\sigma^2}{2}\right)\,, \tag{5.41}$$

while the corresponding eigenvalue is

$$\lambda_2^{(1)} = |v_{20}|^{\frac{1}{2}} \exp\left(\frac{i\pi}{4} \operatorname{sign}\, v_{20}\right)\,, \tag{5.42}$$

and c_{ϕ_0} ensures the normalization of $\phi_0^{(1)}(\sigma)$:

$$c_{\phi_0} = |v_{20}|^{\frac{1}{8}}\, \pi^{-\frac{1}{4}} \exp\left(\frac{i\pi}{16} \operatorname{sign}\, v_{20}\right)\,. \tag{5.43}$$

We do not need actually the specific expression of $\phi_0^{\#} = \phi_0$ and it is enough to know that $\phi_0^{\#} \in \mathcal{S}(\mathbb{R})$.

Coming back to the solution of (5.39), which simply reads

$$(\mathcal{L}_0 - \lambda_0)\, u_2 = 0\,, \tag{5.44}$$

we consequently look for $u_2^{\#}(\sigma,\tau)$ in the form

$$u_2^{\#}(\sigma,\tau) = \phi_2^{\#}(\sigma)\,\psi_0^{\#}(\tau)\,, \tag{5.45}$$

where $\phi_2^{\#}(\sigma)$ is free at this stage.

Summary at this stage. We note that the construction is conform with the general form introduced in (5.3). At this stage, $(\lambda_0^{\#},\psi_0^{\#})$ is a spectral pair for $\mathcal{L}_0^{\#}$, $\lambda_1^{\#}=0$, $u_1^{\#}(\sigma,\tau)=\phi_1^{\#}(\sigma)\psi_0^{\#}(\tau)$ (with $\phi_1^{\#}$ free), $(\lambda_2^{\#},\phi_0^{\#})$ is a spectral pair for $\mathcal{L}_2$ (actually independent of #).

5.5. Term $j=3$. The fourth equation corresponds to $j=3$ and reads

$$(\mathcal{L}_0-\lambda_0)u_3+(\mathcal{L}_2-\lambda_2)u_1+\mathcal{L}_3u_0=\lambda_3u_0\,. \tag{5.46}$$

Taking the scalar product (in $L^2\hat{\otimes}L^2_{\#}:=L^2(\mathbb{R}_\sigma\times\mathbb{R}^+_\tau)$ for Dirichlet, Neumann and Robin, and in $L^2\hat{\otimes}L^2_{\#}:=L^2(\mathbb{R}_\sigma\times\mathbb{R}^-_\tau)\times L^2(\mathbb{R}_\sigma\times\mathbb{R}^+_\tau)$ for the transmission case) with $\bar{u}_0$ and having in mind our normalizations of ψ_0 and ϕ_0, we obtain

$$\langle\mathcal{L}_3u_0\,,\,\bar{u}_0\,\rangle=\lambda_3\,,$$

so λ_3 is determined by

$$\lambda_3=i\,v_{11}\left(\int\sigma\phi_0(\sigma)^2d\sigma\right)\left(\int\tau\psi_0^{\#}(\tau)^2d\tau\right)\,. \tag{5.47}$$

Note that whatever the parity of ϕ_0, ϕ_0^2 is even, so $\int\sigma\phi_0(\sigma)^2d\sigma=0$. Hence,

$$\lambda_3=0\,. \tag{5.48}$$

We come back to (5.46), but now take the scalar product with $\bar{\psi}_0$ in the τ variable. So we get

$$\langle(\mathcal{L}_2-\lambda_2)u_1+(\mathcal{L}_3-\lambda_3)u_0\,,\,\bar{\psi}_0\rangle=0\,.$$

Taking into account (4.13) and the form of u_0 and u_1, this reads

$$(\mathcal{L}_2-\lambda_2)\,\phi_1=-i\,v_{11}\,\sigma\,\left(\int\tau\psi_0(\tau)^2d\tau\right)\,\phi_0\,. \tag{5.49}$$

The right hand side is in the image of the realization of $(\mathcal{L}_2-\lambda_2)$. There is a unique ϕ_1 solution of (5.49) satisfying

$$\int_{\mathbb{R}}\phi_1(\sigma)\phi_0(\sigma)\,d\sigma=0\,. \tag{5.50}$$

Remark 21. *Note that $\phi_0\phi_1$ is odd.*

We can now solve (5.46). We observe that

$$(\mathcal{L}_2-\lambda_2)u_1+(\mathcal{L}_3-\lambda_3)u_0=((\mathcal{L}_2-\lambda_2)\phi_1)\psi_0+(\mathcal{L}_3-\lambda_3)u_0\,.$$

According to what we have done already, (5.46) has the form

$$((\mathcal{L}_0-\lambda_0)u_3)\,(\sigma,\tau)=g_3(\tau)f_3(\sigma)\,,$$

where

$$g_3(\tau) = (\tau - c_3)\, \psi_0(\tau)$$

is orthogonal to $\bar{\psi}_0$, i.e.

$$c_3 = \int \tau \psi_0(\tau)^2 \, d\tau \,,$$

and

$$f_3(\sigma) = i\, v_{11}\, \sigma \phi_0(\sigma) \,.$$

Remark 22. *Note that $\phi_0 f_3$ is odd.*

We then write for $j = 3$ the expression (5.4), with $N_3 = 1$,

$$u_3(\sigma, \tau) = \phi_3(\sigma)\psi_0(\tau) + \phi_{3,1}(\sigma)\psi_{3,1}(\tau) \,, \tag{5.51}$$

where $\psi_{3,1}$ is determined as the unique solution of the problem

$$(\mathcal{L}_0^{\#} - \lambda_0^{\#})\, \psi_{3,1} = g_3 \,, \tag{5.52}$$

which is orthogonal to $\bar{\psi}_0$, and

$$\phi_{3,1}(\sigma) = f_3(\sigma) \,. \tag{5.53}$$

Remark 23. *Note that $\phi_0 \phi_{3,1}$ is odd.*

Summary at this stage. We note that the construction is conform with the general form introduced in (5.3)-(5.4). At this stage, $\phi_3^{\#}$ is introduced, $\lambda_3^{\#} = 0$ and $\phi_1^{\#}$ are determined but $\phi_2^{\#}$ and $\phi_3^{\#}$ remain free. Note that $N_3 = 1$ in (5.4), $\phi_{3,1}^{\#}$ is determined in $\mathcal{S}(\mathbb{R})$ and $\psi_{3,1}^{\#}$ is determined in $\mathcal{S}^{\#}$.

5.6. Term $j = 4$. The fifth condition corresponds to $j = 4$ and reads

$$(\mathcal{L}_0 - \lambda_0)u_4 + (\mathcal{L}_2 - \lambda_2)u_2 + (\mathcal{L}_3 - \lambda_3)u_1 + \mathcal{L}_4 u_0 = \lambda_4 u_0 \,. \tag{5.54}$$

We follow the same procedure as in the preceding step. λ_4 is determined by integrating (5.54) after multiplication by u_0:

$$\begin{aligned} \lambda_4 &= \langle (\mathcal{L}_3 - \lambda_3)u_1 + \mathcal{L}_4 u_0 \,,\, \bar{u}_0 \rangle \\ &= i\, v_{11} \left(\int \sigma \phi_1(\sigma)\phi_0(\sigma) d\sigma \right) \left(\int \tau \psi_0(\tau)^2 d\tau \right) \\ &\quad + \mathfrak{c}(0) \int \psi_0'(\tau)\psi_0(\tau) d\tau + i\, v_{02} \int \tau^2 \psi_0(\tau)^2 \, d\tau \,. \end{aligned} \tag{5.55}$$

ϕ_2 is determined by integrating (5.54) in the τ variable over $\mathbb{R}^{\#}$ after multiplication by ψ_0. We get

$$(\mathcal{L}_2 - \lambda_2)\phi_2 = \langle (\mathcal{L}_3 - \lambda_3)u_1 \,,\, \bar{\psi}_0 \rangle_{L^2_\tau} + \langle \mathcal{L}_4 u_1 \,,\, \bar{\psi}_0 \rangle_{L^2_\tau} - \lambda_4 := f_4 \,, \tag{5.56}$$

where our choice of λ_4 implies the orthogonality of f_4 to $\bar{\phi}_0$ in $L^2_\#$.
There exists consequently a unique ϕ_2 solution of (5.56) that is orthogonal to $\bar{\phi}_0$.

We then proceed like in the fourth step, observing that u_4 should satisfy, for some $N_4 \geq 1$,

$$(\mathcal{L}_0 - \lambda_0)u_4 = \sum_{\ell=1}^{N_4} f_{4,\ell}(\sigma)\, g_{4,\ell}(\tau)\,, \tag{5.57}$$

with $f_{4,\ell}$ in $\mathcal{S}(\mathbb{R})$, $g_{4,\ell}$ in $\mathcal{S}^\#$ and orthogonal to $\bar{\psi}_0$ in $L^2_\#$. The expression in the right hand side is deduced from our previous computations of u_0, u_2 and u_3 and λ_4 .
We then look for a solution u_4 in the form

$$u_4(\sigma,\tau) = \phi_4(\sigma)\,\psi_0(\tau) + \sum_{\ell=1}^{N_4} \phi_{4,\ell}(\sigma)\,\psi_{4,\ell}(\tau)\,, \tag{5.58}$$

which is obtained by solving for each ℓ

$$(\mathcal{L}_0^\# - \lambda_0^\#)\psi_{4,\ell} = g_{4,\ell}\,, \qquad \int \psi_{4,\ell}(\tau)\,\psi_0(\tau)\,d\tau = 0\,, \tag{5.59}$$

with the suitable boundary (or transmission) condition at 0 and taking

$$\phi_{4,\ell} = f_{4,\ell}\,.$$

Although not needed, we make explicit the computation of the right hand side in (5.57). Using our choice of λ_4 and ϕ_2, we obtain

$$\begin{aligned}
&-(\mathcal{L}_2 - \lambda_2)u_2 - (\mathcal{L}_3 - \lambda_3)u_1 - \mathcal{L}_4 u_0 + \lambda_4 u_0 \\
&= (-(\mathcal{L}_2 - \lambda_2)\phi_2)\psi_0 - ((\mathcal{L}_3 - \lambda_3)\phi_1\psi_0 - \mathcal{L}_4\phi_0\psi_0 + \lambda_4\phi_0\psi_0 \\
&= g_{4,1}(\sigma)(\tau - c_3)\psi_0(\tau) + g_{4,2}(\sigma)(\partial_\tau\psi_0 - c_4\psi_0) + g_{4,3}(\sigma)(\tau^2 - c_5)\psi_0\,,
\end{aligned}$$

with $c_4 = \int(\partial_\tau\psi_0)(\tau)\psi_0(\tau)d\tau$ and $c_5 = \int \tau^2\psi_0(\tau)d\tau$.
Moreover the $g_{4,\ell}$ are even with respect with σ.
Hence we can take $N_4 = 3$ and

$$\begin{aligned}
g_{4,1}(\tau) &:= (\tau - c_3)\psi_0(\tau)\,, \\
g_{4,2}(\tau) &:= (\partial_\tau\psi_0 - c_4\psi_0)\,, \\
g_{4,3}(\tau) &:= (\tau^2 - c_5)\psi_0(\tau)\,.
\end{aligned} \tag{5.60}$$

We do not provide explicit formula for the corresponding $\psi_{4,\ell}$.

Summary at this stage. At the end of this step we have determined the $\lambda_j^\#$ for $j \leq 4$, the $\psi_{j,\ell}^\#$ and $\phi_{4,\ell}^\#$ for $3 \leq j \leq 4$ and the $\phi_j^\#(\sigma)$ for $j \leq 2$. Like in [21], this construction can be continued to any order. This achieves the proof of Theorem 1.

5.7. Term $j = 5$ and vanishing of the odd terms. We first focus on the sixth step corresponding to the computation of λ_5. The sixth condition corresponds to $j = 5$ and reads

$$(\mathcal{L}_0 - \lambda_0)u_5 + (\mathcal{L}_2 - \lambda_2)u_3 + (\mathcal{L}_3 - \lambda_3)u_2 + (\mathcal{L}_4 - \lambda_4)u_1 + \mathcal{L}_5 u_0 = \lambda_5 u_0\,. \tag{5.61}$$

λ_5 is determined by integrating (5.61) after multiplication by $\bar{u}_0$. By our preceding constructions and (4.22), we see that

$$\sigma \mapsto u_0(\sigma)\,\big((\mathcal{L}_2-\lambda_2)\,u_3+(\mathcal{L}_3-\lambda_3)\,u_2+(\mathcal{L}_4-\lambda_4)\,u_1+\mathcal{L}_5\,u_0\big)(\sigma)$$

is odd. This immediately leads to $\lambda_5 = 0$.

With some extra work consisting in examining the symmetry properties with respect to σ and using Sec. 4.2.2, we obtain

PROPOSITION 24. *In the formal expansion, $\lambda_j = 0$ if j is odd.*

5.8. Four-terms asymptotics. Gathering (4.20), (5.12) and (5.55), the four-terms asymptotics of approximate eigenvalues reads for $n, k = 1, 2, \ldots$

$$\begin{aligned}\lambda_h^{app,\#} := \lambda_h^{\#,(n,k)} &= i\,v_{00} + h^{\frac{2}{3}}|\,v_{01}|^{\frac{2}{3}}\mu_n^{\#}\exp\left(\frac{i\pi}{3}\mathrm{sign}\ v_{01}\right)\\ &\quad + h(2k-1)|v_{20}|^{\frac{1}{2}}\exp\left(\frac{i\pi}{4}\mathrm{sign}\,v_{20}\right) + h^{\frac{4}{3}}\lambda_4^{\#,(n)} + \mathcal{O}(h^{\frac{5}{3}})\,,\end{aligned} \tag{5.62}$$

where $\mu_n^D = -a_n$, $\mu_n^N = -a_n'$, $\mu_n^R = -a_n^R(\kappa)$ (defined by (5.24)), and $\mu_n^T = -a_n^+(\kappa)$ (defined by (5.32)), while $\lambda_4^{\#,(n)}$ is explicitly computed in Appendix A (see (A.23), (A.27), (A.31), and (A.39) for Dirichlet, Neumann, Robin, and Transmission cases), and the involved coefficients v_{jk} of the potential $V(s,\rho)$ are defined in (4.12).

REMARK 25. *In the above construction, if we take $\phi_j^{\#} = 0$ for $j \geq 3$, we get an eigenpair $(\lambda_h^{app,\#}, u_h^{app,\#})$ with*

$$u_h^{app,\#} = u_0^{\#} + h^{\frac{1}{6}}u_1^{\#} + h^{\frac{1}{3}}u_2^{\#}\,,$$

such that

$$(\mathcal{A}_h^{\#} - \lambda_h^{app,\#})u_h^{app,\#} = \mathcal{O}(h^{\frac{3}{2}})\,. \tag{5.63}$$

To get in (5.63) the remainder $\mathcal{O}(h^{\frac{5}{3}})$, one should continue the construction for two more steps.

REMARK 26. *Note that the leading terms in the eigenvalue expansion do not contain the curvature which appears only in λ_4 (see Eq. (A.27)) and is thus of order $h^{\frac{4}{3}}$.*

6. Other scalings in the Robin or transmission problems. The scaling (4.30) of the transmission parameter $\mathcal{K}$ with h was appropriate to keep the Robin or transmission condition for the rescaled problem. In biophysical applications, the transmission condition reads

$$D\partial_\nu u_+ = D\partial_\nu u_- = \mathcal{K}\,\big(u_+ - u_-\big), \tag{6.1}$$

where D is the bulk diffusion coefficient, while the transmission parameter $\mathcal{K}$ represents the permeability of a membrane which is set by the membrane properties and thus does not necessarily scale with h. Similarly, in the Robin boundary condition,

$$-\,D\partial_\nu u_- = \mathcal{K}\,u_-, \tag{6.2}$$

which accounts for partial reflections on the boundary, $\mathcal{K}$ represents partial reactivity or surface relaxivity which are set by properties of the boundary.

We consider two practically relevant situations for the BT-operator $-D\Delta + i\,g\,x_1$:

- When $D \to 0$ with fixed g, one can identify $h^2 = D$ and $V(x) = g\,x_1$ so that the rescaled transmission condition in (4.29) gives $\mathcal{K}h^{-\frac{4}{3}}$ which tends to $+\infty$ as $h \to 0$ if $\mathcal{K}$ is fixed. In this limit, the transmission condition is formally reduced to the continuity condition at the boundary: $u_+(\sigma, 0) = u_-(\sigma, 0)$, together with the flux continuity in the first relation of (4.29). In other words, the interface between two subdomains is removed. The construction of the previous section seems difficult to control in this asymptotics and the mathematical proof of the heuristics should follow other ways.
- When $g \to \pm\infty$ with fixed D, one can divide the BT-operator and (6.1) by g and then identify $h^2 = D/g$ and $V(x) = x_1$. In this situation, the rescaled transmission condition in (4.29) gives a parameter $\kappa = (\mathcal{K}/D)h^{\frac{2}{3}}$ which tends to 0 as $h \to 0$. In this limit, the transmission condition is reduced to two Neumann boundary conditions on both sides of the interface: $\partial_\tau u_+(\sigma, 0) = \partial_\tau u_-(\sigma, 0) = 0$.

We now discuss how the eigenvalue asymptotic expansion obtained for rescaled $\mathcal{K}$ can be modified for the second situation. The constructions of the previous section can be adapted and controlled with respect to κ for κ small enough. As observed along the construction, one can start with (5.62) and then expand the factor $\mu_n^{\#}(\kappa)$ into Taylor series that results in the quasi-mode in the Robin or Transmission case:

THEOREM 27. *With the notation of Theorem 1 except that in* (1.4) *we assume*

$$\kappa = \hat{\kappa}\, h^{\frac{2}{3}}\,, \tag{6.3}$$

we have for $\# \in \{R, T\}$, $n, k = 1, 2, \ldots$

$$\begin{aligned} \lambda_h^{\#,(n,k)} &= i\,v_{00} - h^{\frac{2}{3}}|\,v_{01}|^{\frac{2}{3}} a_n' \exp\left(\frac{i\pi}{3}\text{sign } v_{01}\right) \\ &\quad + h\,(2k-1)\,|v_{20}|^{\frac{1}{2}} \exp\left(\frac{i\pi}{4}\text{sign}\, v_{20}\right) \\ &\quad + h^{\frac{4}{3}}\left(\lambda_4^{N,(n)} - \hat{\kappa}\,\frac{|v_{01}|^{\frac{1}{3}}}{a_n'} \exp\left(\frac{i\pi}{6}\,\text{sign}\, v_{01}\right)\right) + O(h^{\frac{5}{3}})\,, \end{aligned} \tag{6.4}$$

where $\lambda_4^{N,(n)}$ *is explicitly given in* (A.27)*, and the involved coefficients* v_{jk} *of the potential* $V(s, \rho)$ *are defined in (4.12).*

Here, we have used that $\lambda_4^{\#,(n)}(\kappa) = \lambda_4^{N,(n)}$ for $\kappa = 0$ (see Remark 33). The coefficient in front of $\hat{\kappa}$ involves $(\mu_n^{\#})'(0)$ that was computed explicitly by differentiating the relation determining $\mu_n^{\#}(\kappa)$ with respect to κ. For the Robin case, we used (5.26) to get

$$(\mu_n^R)'(0) = -(a_n^R)'(0) = -\frac{1}{a_n'\,|v_{01}|^{\frac{1}{3}}} \exp\left(-\frac{i\pi}{6}\text{sign } v_{01}\right), \tag{6.5}$$

with $a_n^R(0) = a_n'$.

Similarly, differentiating (5.33) with respect to κ and using (5.32), we got (see Appendix A.3)

$$(\mu_n^T)'(0) = -(a_n^+)'(0) = -\frac{1}{a_n' \, |v_{01}|^{\frac{1}{3}}} \, \exp\left(-\frac{i\pi}{6} \operatorname{sign} v_{01}\right), \tag{6.6}$$

with $a_n^+(0) = a_n'$. The effect of Robin or transmission condition appears only in the coefficient of $h^{\frac{4}{3}}$.

In order to control the construction with respect to κ, it is enough to get an expression of the kernel of the regularized resolvent for $z = \lambda_0^{\#}$. Let us treat the Robin case and assume $v_{01} = -1$.

As proven in [17], the kernel of the resolvent is given by

$$\mathcal{G}^{-,R}(x,y\,;\lambda) = \mathcal{G}_0^-(x,y\,;\lambda) + \mathcal{G}_1^{-,R}(x,y\,;\kappa,\lambda) \quad \text{for } (x,y) \in \mathbb{R}_+^2,$$

where

$$\mathcal{G}_0^-(x,y\,;\lambda) = \begin{cases} 2\pi \mathrm{Ai}(e^{i\alpha} w_x)\mathrm{Ai}(e^{-i\alpha} w_y) & (x<y), \\ 2\pi \mathrm{Ai}(e^{-i\alpha} w_x)\mathrm{Ai}(e^{i\alpha} w_y) & (x>y), \end{cases} \tag{6.7}$$

and

$$\begin{aligned} \mathcal{G}_1^{-,R}(x,y\,;\kappa,\lambda) = -2\pi & \frac{ie^{i\alpha}\mathrm{Ai}'(e^{i\alpha}\lambda) - \kappa \mathrm{Ai}(e^{i\alpha}\lambda)}{ie^{-i\alpha}\mathrm{Ai}'(e^{-i\alpha}\lambda) - \kappa \mathrm{Ai}(e^{-i\alpha}\lambda)} \\ & \times \mathrm{Ai}\big(e^{-i\alpha}(ix+\lambda)\big) \, \mathrm{Ai}\big(e^{-i\alpha}(iy+\lambda)\big). \end{aligned} \tag{6.8}$$

The kernel $\mathcal{G}_0^-(x,y\,;\lambda)$ is holomorphic in λ and independent of κ. Setting $\kappa = 0$, one retrieves the resolvent for the Neumann case. Its poles are determined as (complex-valued) solutions of the equation

$$f^R(\kappa,\lambda) := ie^{-i\alpha}\mathrm{Ai}'(e^{-i\alpha}\lambda) - \kappa \mathrm{Ai}(e^{-i\alpha}\lambda) = 0\,. \tag{6.9}$$

For $\kappa = 0$, we recover the equation determining the poles of the Neumann problem:

$$f^N(\lambda) := ie^{-i\alpha}\mathrm{Ai}'(e^{-i\alpha}\lambda) = 0\,.$$

We look at the first pole and observe that

$$(\partial_\lambda f^R)(0, \lambda^{R,(1)}(0)) = (\partial_\lambda f^R)(0, \lambda^{N,(1)}) = (f^N)'(\lambda^{N,(1)}) \neq 0\,. \tag{6.10}$$

This evidently remains true for κ small enough:

$$(\partial_\lambda f^R)(\kappa, \lambda^{R,(1)}(\kappa)) \neq 0\,. \tag{6.11}$$

As done in [17], we can compute the distribution kernel of the projector associated with

$$\lambda_0(\kappa) := \lambda^{R,(1)}(\kappa)\,.$$

We get

$$\begin{aligned} \Pi_1^R(x,y;\kappa) = -2\pi & \frac{ie^{i\alpha}\mathrm{Ai}'(e^{i\alpha}\lambda_0(\kappa)) - \kappa \mathrm{Ai}(e^{i\alpha}\lambda_0(\kappa))}{(\partial_\lambda f^R)(\kappa, \lambda_0(\kappa))} \\ & \times \mathrm{Ai}\big(e^{-i\alpha}(ix+\lambda_0(\kappa))\big) \, \mathrm{Ai}\big(e^{-i\alpha}(iy+\lambda_0(\kappa))\big). \end{aligned} \tag{6.12}$$

This kernel is regular with respect to κ.
The distribution kernel of the regularized resolvent at $\lambda_0(\kappa)$ is obtained as

$$\begin{aligned}\mathcal{G}^{R,reg}(x,y;\kappa,\lambda_0(\kappa)) &:= \mathcal{G}_0^-(x,y;\kappa,\lambda_0(\kappa)) \\ &\quad + \lim_{\lambda\to\lambda_0}\left(\mathcal{G}_1^{-,R}(x,y\,;\kappa,\lambda) - (\lambda_0-\lambda)^{-1}\Pi_1^R(x,y;\kappa)\right).\end{aligned}$$

It remains to compute the second term of the right hand side. Writing $\mathcal{G}_1^{-,R}(x,y;\kappa,\lambda)$ in the form

$$\mathcal{G}_1^{-,R}(x,y,\kappa,\lambda) = \frac{\Phi(x,y;\kappa,\lambda)}{\lambda-\lambda_0(\kappa)},$$

we observe that $\Phi(x,y;\kappa,\lambda)$ is regular in κ,λ and we get

$$\mathcal{G}^{R,reg}(x,y;\kappa,\lambda_0(\kappa)) := \mathcal{G}_0^-(x,y;\kappa,\lambda_0(\kappa)) + \partial_\lambda\Phi(x,y;\kappa,\lambda_0(\kappa))\,.$$

It is regular in κ and we recover for $\kappa=0$ the regularized resolvent of the Neumann problem at $\lambda=\lambda^{N,(1)}$.

With this regularity with respect to κ, we can control all the constructions for $j=0,\dots,4$ (and actually any j) and in particular solve (5.52) for κ small and similarly (5.59), with a complete expansion in powers of κ at the origin.

Remark 28. *Similarly, one can treat the transmission case.*

7. WKB construction. In this section, we propose an alternative analysis based on the WKB method. This construction is restricted to quasimodes with $k=1$ in (5.40) but it gives a quasimode state that is closer to the eigenfunction than that obtained by the earlier perturbative approach. Here we follow the constructions of [21, 22] developed for a Robin problem.

We start from

$$\mathcal{A}_h = -h^2a^{-2}\partial_s^2 + h^2a^{-3}(\partial_s a)\,\partial_s - h^2\partial_\rho^2 - h^2a^{-1}(\partial_\rho a)\,\partial_\rho + i\,\widetilde{V}(s,\rho)\,. \tag{7.1}$$

Here, instead of what was done in (4.14), we only dilate in the ρ variable:

$$\rho = h^{\frac{2}{3}}\tau\,.$$

In the (s,τ) coordinates, we get

$$\widehat{\mathcal{A}}_h = -h^2\check{a}_h^{-2}\partial_s^2 + h^2\check{a}_h^{-3}(\partial_s\check{a}_h)\,\partial_s - h^{\frac{2}{3}}\partial_\tau^2 - h^{\frac{4}{3}}\check{a}_h^{-1}(\check{\partial_\rho a})\,\partial_\tau + i\,\check{V}_h(s,\tau)\,, \tag{7.2}$$

with

$$\begin{aligned}\check{V}_h(s,\tau) &= \widetilde{V}(s,h^{\frac{2}{3}}\tau)\,,\\ \check{a}_h(s,\tau) &= 1-\tau h^{\frac{2}{3}}\mathfrak{c}\,(s)\,,\\ \partial_s\check{a}_h(s,\tau) &= -\tau h^{\frac{2}{3}}\mathfrak{c}'\,(s)\,,\\ \check{\partial_\rho a} &= -\mathfrak{c}(s)\,,\\ \check{a}_h(s,\tau)^2 &= 1-2\tau h^{\frac{2}{3}}\mathfrak{c}\,(s)+\tau^2h^{\frac{4}{3}}\mathfrak{c}\,(s)^2\,,\\ \check{a}_h(s,\tau)^{-2} &= 1+2\tau h^{\frac{2}{3}}\mathfrak{c}\,(s)+3\tau^2h^{\frac{4}{3}}\mathfrak{c}\,(s)^2+\mathcal{O}(h^2)\,.\end{aligned} \tag{7.3}$$

We consider the Taylor expansion of $\check{V}_h$:

$$\check{V}_h(s,\tau) \sim \sum_{j\in\mathbb{N}} v_j(s) h^{\frac{2j}{3}} \tau^j \,, \tag{7.4}$$

with

$$v_j(s) = \frac{1}{j!}(\partial_\rho^j \widetilde{V})(s,0)\,. \tag{7.5}$$

We look for a trial state in the form

$$u_h^{\#,wkb} := d(h) b_h(s,\tau) \exp\left(-\frac{\theta(s,h)}{h}\right)\,, \tag{7.6}$$

with

$$\theta(s,h) = \theta_0(s) + h^{\frac{2}{3}}\theta_1(s)\,, \tag{7.7}$$

and

$$b_h(s,\tau) \sim \sum_{j\in\mathbb{N}} b_j(s,\tau) h^{\frac{j}{3}}\,. \tag{7.8}$$

Here $d(h)$ is a normalizing constant such that, when coming back to the initial coordinates, the L^2 norm of $u_h^{\#,wkb}$ is 1. In the initial coordinates, we should actually consider $u_h^{\#,wkb}(s, h^{-\frac{2}{3}}\rho)$ multiplied by a suitable cut-off function in the neighborhood of the point x^0 of $\partial\Omega^\perp$.
This gives an operator acting on b_h

$$\begin{aligned}\widehat{\mathcal{A}}_{h,\theta} &:= \exp\left(\frac{\theta(s,h)}{h}\right) \widehat{\mathcal{A}}_h \exp\left(-\frac{\theta(s,h)}{h}\right)\\ &= -\check{a}_h^{-2}(h\partial_s - \theta'(s,h))^2 + h\check{a}_h^{-3}(\partial_s \check{a}_h)\,(h\partial_s - \theta'(s,h))\\ &\quad - h^{\frac{2}{3}}\partial_\tau^2 - h^{\frac{4}{3}}\check{a}_h^{-1}(\check{\partial_\rho a})\,\partial_\tau + i\,\check{V}_h(s,\tau)\,.\end{aligned} \tag{7.9}$$

We rewrite this operator in the form

$$\widehat{\mathcal{A}}_{h,\theta} \sim \sum_{j\geq 0} \Lambda_j h^{\frac{j}{3}}\,, \tag{7.10}$$

with

$$\begin{aligned}&\Lambda_0 := iv_0(s) - \theta_0'(s)^2\,,\\ &\Lambda_1 := 0\,,\\ &\Lambda_2 := -\partial_\tau^2 + (iv_1(s) - 2\mathfrak{c}(s)\theta_0'(s)^2)\tau - 2\theta_0'(s)\theta_1'(s)\,,\\ &\Lambda_3 := 2\theta_0'(s)\partial_s\,,\\ &\Lambda_4 := \mathfrak{c}(s)\partial_\tau + (iv_2(s) - \mathfrak{c}(s)^2\theta_0'(s)^2)\tau^2 + 4\mathfrak{c}(s)^2\theta_0'(s)\theta_1'(s)\tau - \theta_1'(s)^2\,.\end{aligned} \tag{7.11}$$

We recall that $v_0'(0) = 0$, $v_1(0) \neq 0$.
We look for a quasimode in the form

$$\lambda_h^{\#,wkb} \sim iv_0(0) + h^{\frac{2}{3}} \sum_{j\in\mathbb{N}} \mu_j h^{\frac{j}{3}}\,. \tag{7.12}$$

The construction should be local in the s-variable near 0 and global in the τ variable in $\mathbb{R}^{\#}$.

Expanding $(\widehat{\mathcal{A}}_{h,\theta} - \lambda_h)b_h$ in powers of $h^{\frac{1}{3}}$ and looking at the coefficient in front of h^0, we get

$$(\Lambda_0 - iv_0(0))b_0 = 0$$

as a necessary condition. Hence we choose θ_0 as a solution of

$$i(v_0(s) - v_0(0)) - \theta_0'(s)^2 = 0\,, \tag{7.13}$$

which is usually called the (first) eikonal equation.
We take the solution such that

$$\operatorname{Re}\theta_0(s) \geq 0\,, \quad \theta_0(0) = 0\,, \tag{7.14}$$

and we note that

$$\theta_0'(0) = 0\,. \tag{7.15}$$

With this choice of θ_0, we note that

$$\Lambda_2 = -\partial_\tau^2 + i\big(v_1(s) - 2\mathfrak{c}(s)[v_0(s) - v_0(0)]\big)\tau - 2\theta_0'(s)\theta_1'(s)\,, \tag{7.16}$$

with

$$\hat{v}_1(s) := v_1(s) - 2\mathfrak{c}(s)[v_0(s) - v_0(0)] \tag{7.17}$$

being real.
As operator on $L^2_{\#}$, with the corresponding boundary or transmission condition $\# \in \{D, N, R, T\}$, it satisfies

$$\Lambda_2^{\#,*} = \overline{\Lambda_2^{\#}}\,.$$

The coefficient in front of $h^{\frac{1}{3}}$ vanishes and we continue with imposing the cancellation of the coefficient in front of $h^{\frac{2}{3}}$ which reads

$$(\Lambda_0 - iv_0(0))b_2 + \Lambda_2 b_0 = \mu_0 b_0\,,$$

or, taking account of our choice of θ_0,

$$-2\theta_0'(s)\theta_1'(s)b_0(s,\tau) + (-\partial_\tau^2 + i\hat{v}_1(s)\tau)b_0(s,\tau) - \mu_0 b_0(s,\tau) = 0\,. \tag{7.18}$$

Considering this equation at $s = 0$, we get as a necessary condition

$$(-\partial_\tau^2 + iv_1(0)\tau)\, b_0(0,\tau) = \mu_0\, b_0(0,\tau)\,. \tag{7.19}$$

If we impose a choice such that $b_0(0,\tau)$ is not identically 0, we get that μ_0 should be an eigenvalue of (the suitable realization of) $-\partial_\tau^2 + iv_1(0)\tau$, i.e. $\mathcal{L}_0^{\#}$. We take some simple eigenvalue μ_0 and define $\mu_0(s)$ as the eigenvalue of the operator

$$-\partial_\tau^2 + i\hat{v}_1(s)\tau \tag{7.20}$$

such that $\mu_0(0) = \mu_0$. If $f_0(s,\tau)$ denotes the corresponding eigenfunction normalized as

$$\int f_0(s,\tau)^2 d\tau = 1\,, \tag{7.21}$$

we can look for

$$b_0(s,\tau) = c_0(s) f_0(s,\tau)\,. \tag{7.22}$$

We now come back to (7.18), which reads, assuming $c_0(s) \neq 0$,

$$-2\theta_0'(s)\theta_1'(s) + (\mu_0(s) - \mu_0) = 0\,. \tag{7.23}$$

This equation can be seen as the second eikonal equation. It has a unique regular solution θ_1 if we add the condition

$$\theta_1(0) = 0\,. \tag{7.24}$$

The first transport equation is obtained when looking at the coefficient in front of h which reads

$$(\Lambda_0 - iv_0(0))b_3 + (\Lambda_2 - \mu_0)b_1 + \Lambda_3 b_0 = \mu_1 b_0\,,$$

or

$$(-\partial_\tau^2 + i\hat{v}_1(s)\,\tau - \mu_0(s))b_1(s,\tau) + 2\theta_0'(s)\partial_s b_0(s,\tau) + \theta_0''(s)b_0(s,\tau) - \mu_1 b_0(s,\tau) = 0\,. \tag{7.25}$$

We assume

$$b_1(s,\tau) = c_1(s) f_0(s,\tau) + \hat{b}_1(s,\tau)\,, \quad \text{with} \int f_0(s,\tau)\hat{b}_1(s,\tau)d\tau = 0\,. \tag{7.26}$$

Multiplying it by $f_0(s,\tau)$ and integrating with respect to τ, we get

$$2\theta_0'(s)\int \partial_s b_0(s,\tau) f_0(s,\tau)d\tau + \theta_0''(s)c_0(s) = \mu_1 c_0(s)\,, \tag{7.27}$$

which leads to

$$2\theta_0'(s)c_0'(s) + \theta_0''(s)c_0(s) = \mu_1 c_0(s)\,, \tag{7.28}$$

where we have used in the last line (7.21). Taking $s = 0$ and assuming $c_0(0) \neq 0$, one gets

$$\theta_0''(0) = \mu_1\,, \tag{7.29}$$

which is also sufficient for solving (7.28). We have determined at this stage $c_0(s)$ assuming for normalization

$$c_0(0) = 1\,. \tag{7.30}$$

Coming back to (7.25), we have to solve, for each s in a neighborhood of 0

$$\big(-\partial_\tau^2 + i\hat{v}_1(s)\tau - \mu_0(s)\big)\hat{b}_1(s,\tau) = g_1(s,\tau)\,, \tag{7.31}$$

with $g_1(s,\tau)$ satisfying $\int f_0(s,\tau)g_1(s,\tau)d\tau = 0$.
At this stage, the function c_1 is free.
We continue, one step more, in order to see if the proposed approach is general.
The second transport equation is obtained when looking at the coefficient in front of $h^{\frac{4}{3}}$, which reads

$$(\Lambda_0 - iv_0(0))b_4 + (\Lambda_2 - \mu_0)b_2 + (\Lambda_3 - \mu_1)b_1 + \Lambda_4 b_0 = \mu_2 b_0\,,$$

or

$$\begin{aligned}&(-\partial_\tau^2 + i\hat{v}_1(s)\tau - \mu_0(s))b_2(s,\tau) + 2\theta_0'(s)\partial_s b_1(s,\tau) + \theta_0''(s)b_1(s,\tau)\\ &- \mu_1 b_1(s,\tau) - \mu_2 b_0(s,\tau) + (iv_2(s)\tau^2 - \theta_1'(s)^2)b_0(s,\tau) - 3\tau^2\mathfrak{c}(s)^2\theta_0'(s)^2\\ &+ 4\tau\mathfrak{c}(s)^2\theta_0'(s)\theta_1'(s)b_0 + \mathfrak{c}(s)\partial_\tau b_0 = 0\,.\end{aligned} \tag{7.32}$$

We look for b_2 in the form

$$b_2(s,\tau) = c_2(s)f_0(s,\tau) + \hat{b}_2(s,\tau)\,, \quad \text{with} \int f_0(s,\tau)\hat{b}_1(s,\tau)d\tau = 0\,. \tag{7.33}$$

We then proceed as before. If we write

$$\begin{aligned}g_2(s,\tau) = &-2\theta_0'(s)\partial_s b_1(s,\tau)\\ &- \theta_0''(s)b_1(s,\tau) + \mu_1 b_1(s,\tau) + \mu_2 b_0(s,\tau)(s) + (\theta_1'(s)^2 - iv_2\tau^2)b_0(s,\tau)\\ &- 3\tau^2\mathfrak{c}(s)^2\theta_0'(s)^2 b_0 + 4\tau\mathfrak{c}(s)^2\theta_0'(s)\theta_1'(s)b_0 + \mathfrak{c}(s)\partial_\tau b_0\,,\end{aligned}$$

the orthogonality condition reads

$$\begin{aligned}0 &= \int g_2(s,\tau)f_0(s,\tau)\,d\tau\\ &= -2\theta_0'(s)c_1'(s) + (\mu_1 - \theta_0''(s))c_1(s) - 2\theta_0'(s)\int \partial_s\hat{b}_1(s,\tau)f_0(s,\tau)d\tau\\ &\quad + \left(\mu_2 + \theta_1'(s)^2 - iv_2\int \tau^2 f_0(s,\tau)^2 d\tau\right)c_0(s)\\ &\quad + \int \left(-3\tau^2\mathfrak{c}(s)^2\theta_0'(s)^2 b_0 + 4\tau\mathfrak{c}(s)^2\theta_0'(s)\theta_1'(s)b_0\, f_0(s,\tau)d\tau\right)\\ &\quad + \int \mathfrak{c}(s)\partial_\tau b_0\, f_0(s,\tau)\,d\tau\,.\end{aligned}$$

Observing that

$$\begin{aligned}&\int \left(-3\tau^2\mathfrak{c}(s)^2\theta_0'(s)^2 b_0(s,\tau) + 4\tau\mathfrak{c}(s)^2\theta_0'(s)\theta_1'(s)b_0(s,\tau)\, f_0(s,\tau) + \mathfrak{c}(s)\partial_\tau b_0\, f_0(s,\tau)\right)d\tau\\ &= \mathfrak{c}(0)\left(\int \partial_\tau f_0(0,\tau)f_0(0,\tau)\,d\tau\right)c_0(0)\,,\end{aligned}$$

for $s = 0$, this determines μ_2 as a necessary condition at $s = 0$ which reads

$$\mu_2 = iv_2(0)\int \tau^2 f_0(0,\tau)^2 d\tau - \theta_1'(0)^2 - \mathfrak{c}(0)\int \partial_\tau f_0(0,\tau)f_0(0,\tau)\,d\tau\,. \tag{7.34}$$

Note that in the case when $\# \in \{D, N, R\}$, we get

$$\int \partial_\tau f_0(0,\tau)f_0(0,\tau)\,d\tau = \frac{1}{2}f_0(0,0)^2\,.$$

We can then determine c_1 if we add the condition $c_1(0) = 0$.
Since g_2 is orthogonal to $\bar{f}_0$, we can find $\hat{b}_2$, while c_2 remains free for the next step.

Hence, we have obtained the following theorem

THEOREM 29. *Under the assumptions of Theorem 1, if $\mu_0^{\#}$ is a simple eigenvalue of the realization "#" of the complex Airy operator $-\frac{d^2}{dx^2} + ix$ in $L^2_{\#}$, and $\tilde{\mu}_1$ is the eigenvalue of the Davies operator $-\frac{d^2}{dy^2} + iy^2$ on $L^2(\mathbb{R})$ with the smallest real part, then there exists an approximate pair $(\lambda_h^{\#,wkb}, u_h^{\#,wkb})$ with $u_h^{\#,wkb}$ in the domain of $\mathcal{A}_h^{\#}$, such that (7.6), (7.7) and (7.8) are satisfied and*

$$\exp\left(\frac{\theta}{h}\right)(\mathcal{A}_h^{\#} - \lambda_h^{\#})\, u_h^{\#,wkb} = \mathcal{O}(h^{\infty}) \ \text{ in } L^2_{\#}(\Omega)\,,\ ||u_h^{\#,wkb}||_{L^2} \sim 1\,, \tag{7.35}$$

where

$$\lambda_0^{\#} = \mu_0^{\#}\,|\,v_{01}|^{\frac{2}{3}} \exp\left(i\frac{\pi}{3}\text{sign } v_{01}\right)\,, \quad \lambda_2 = \tilde{\mu}_1 |v_{20}|^{\frac{1}{2}} \exp\left(i\frac{\pi}{4}\text{sign}\, v_{20}\right)\,, \tag{7.36}$$

with $v_{01} := \nu \cdot \nabla V(x^0)$.

REMARK 30. *In this approach, we understand more directly why no odd power of $h^{\frac{1}{6}}$ appears for λ_h. Note that $\mu_j = \lambda_{2j}$.*

8. Examples. In this Section, we illustrate the above general results for the potential $V(x) = x_1$ and some simple domains.

8.1. Disk. Let $\Omega = \{(x_1, x_2) \in \mathbb{R}^2 \ : \ |x| < R_0\}$ be the disk of radius R_0. In this case, $\Omega_{\perp} = \{(R_0, 0), (-R_0, 0)\}$. The local parameterization around the point $(R_0, 0)$ reads in polar coordinates (r, θ) as $\rho = R_0 - r$, $s = R_0\theta$, so that

$$V(x) = x_1(s, \rho) = (R_0 - \rho)\cos(s/R_0), \tag{8.1}$$

$\mathfrak{c}(0) = 1/R_0$, and we get

$$v_{00} = R_0, \quad v_{01} = -1, \quad v_{20} = -\frac{1}{2R_0}, \quad v_{11} = v_{02} = 0\,. \tag{8.2}$$

Using Eqs. (A.27), (A.23), (A.31) or (A.39) for $\lambda_4^{\#,(n)}$, one can write explicitly the four-term expansion for four types of boundary condition:

• Dirichlet case,

$$\lambda_h^{D,(n,k)} = iR_0 - h^{\frac{2}{3}} a_n e^{-i\pi/3} + h(2k-1)\frac{e^{-i\pi/4}}{\sqrt{2R_0}} + \mathcal{O}(h^{\frac{5}{3}})\,. \tag{8.3}$$

• Neumann case

$$\lambda_h^{N,(n,k)} = iR_0 - h^{\frac{2}{3}} a'_n e^{-i\pi/3} + h(2k-1)\frac{e^{-i\pi/4}}{\sqrt{2R_0}} + h^{\frac{4}{3}}\frac{e^{-\pi i/6}}{2R_0\, a'_n} + \mathcal{O}(h^{\frac{5}{3}})\,. \tag{8.4}$$

• Robin case

$$\begin{aligned}\lambda_h^{R,(n,k)} &= iR_0 - h^{\frac{2}{3}} a_n^R(\kappa) e^{-i\pi/3} + h(2k-1)\frac{e^{-i\pi/4}}{\sqrt{2R_0}} \\ &\quad + h^{\frac{4}{3}}\frac{i}{2R_0(\kappa^2 - a_n^R(\kappa)e^{-\pi i/3})} + \mathcal{O}(h^{\frac{5}{3}})\,.\end{aligned} \tag{8.5}$$

When $\kappa = 0$, $a_n^R(0) = a_n'$, and one retrieves the expansion (8.4) for Neumann case.

• Transmission case,

$$\lambda_h^{T,(n,k)} = iR_0 - h^{\frac{2}{3}} a_n^+(\kappa) e^{-i\pi/3} + h(2k-1)\frac{e^{-i\pi/4}}{\sqrt{2R_0}} + h^{\frac{4}{3}} \frac{e^{-i\pi/6}}{2R_0\, a_n^+(\kappa)} + \mathcal{O}(h^{\frac{5}{3}})\,. \tag{8.6}$$

When $\kappa = 0$, one has $a_n^+(0) = a_n'$ and thus retrieves the expansion (8.4) for Neumann case.

We recall that the indices $n = 1, 2, \dots$ and $k = 1, 2, \dots$ enumerate eigenvalues of the operators $\mathcal{L}_0^{\#}$ and $\mathcal{L}_2^{\#}$ that were used in the asymptotic expansion. The approximate eigenvalue with the smallest real part corresponds to $n = k = 1$.

The three-terms version of the Neumann expansion (8.4) was first derived by de Swiet and Sen [35] (note that we consider the eigenvalues of the operator $-h^2\Delta + ix_1$ while de Swiet and Sen looked at the complex conjugate operator).

Remark 31. *At the other point $(-R_0, 0)$, the parameterization is simply*

$$V(x) = -(R_0 - \rho)\cos(s/R_0)$$

that alters the signs of the all involved coefficients v_{jk}. As a consequence, the asymptotics is obtained as the complex conjugate of $\lambda_h^{\#,(n,k)}$.

In the WKB approach, one needs to compute the functions $\theta_0(s)$ and $\theta_1(s)$ that determine the asymptotic decay of the quasimode state in the tangential direction. We only consider the Neumann boundary condition while the computation for other cases is similar. From (7.5) and (7.17), we have for the potential in (8.1):

$$v_0(s) = R_0\cos(s/R_0), \qquad v_1(s) = -\cos(s/R_0), \qquad \hat{v}_1(s) = 2 - 3\cos(s/R_0).$$

In what follows, we consider $s > 0$ though the results will be the same for $s < 0$ due to the symmetry. From Eqs. (7.13, 7.14), we first obtain

$$\theta_0(s) = \int_0^s \sqrt{-iR_0(1-\cos(s'/R_0))}\, ds' = e^{-\pi i/4}\,(2R_0)^{\frac{3}{2}}\big(1 - \cos(s/(2R_0))\big)\,. \tag{8.7}$$

For Neumann boundary condition, $\mu_0 = -a_1' e^{-\pi i/3}$ (here $v_1(0) = -1$) and the eigenvalue of the operator in (7.20) reads

$$\mu_0(s) = -a_1'\,|2 - 3\cos(s/R_0)|^{\frac{2}{3}} \exp\left(\frac{\pi i}{3}\mathrm{sign}\,(2 - 3\cos(s/R_0))\right).$$

Since $\hat{v}_1(s)$ was assumed to be nonzero, we restrict the analysis to $|s/R_0| < \arccos(2/3)$ for which $2 - 3\cos x$ does not vanish (and remains negative) so that

$$\mu_0(s) = -a_1'\,\big(3\cos(s/R_0) - 2\big)^{\frac{2}{3}} \exp\left(-\frac{\pi i}{3}\right). \tag{8.8}$$

From (7.23), one gets then

$$\begin{aligned}\theta_1(s) &= \int_0^s \frac{-a_1'\, e^{-\pi i/3}\left[(3\cos(s'/R_0) - 2)^{\frac{2}{3}} - 1\right]}{2e^{-\pi i/4} R_0^{\frac{1}{2}}\sqrt{1 - \cos(s'/R_0)}}\, ds' \\ &= \frac{1}{2}|a_1'|\, e^{-\pi i/12} R_0^{\frac{1}{2}} \int_0^{s/R_0} \frac{(3\cos x - 2)^{\frac{2}{3}} - 1}{\sqrt{1 - \cos x}}\, dx\,.\end{aligned} \tag{8.9}$$

8.2. Annulus. For an annulus $\Omega = \{(x_1, x_2) \in \mathbb{R}^2 \ : \ R_1 < |x| < R_2\}$ between two circles of radii R_1 and R_2, there are four points in $\Omega_\perp$: $(\pm R_1, 0)$ and $(\pm R_2, 0)$. In order to determine the candidate for an eigenvalue with the smallest real part (in short the "first eigenvalue"), one needs to compare the asymptotics of the quasimodes associated with these points and identify those with the smallest real part. Of course, this analysis depends on the imposed boundary conditions. We consider four combinations: NN (Neumann condition on both circles), ND (Neumann condition on the inner circle and Dirichlet on the outer circle), DN (Dirichlet condition on the inner circle and Neumann on the outer circle), and DD (Dirichlet condition on both circles). Since the leading contribution is proportional $|a_1| \approx 2.3381$ for the Dirichlet case and to $|a_1'| \approx 1.0188$ for the Neumann case, the asymptotics for the circle with Neumann boundary condition always contributes to the first eigenvalue. In turn, when the same boundary condition is imposed on the two circles, the first eigenvalue expansion corresponds to the outer circle of larger radius because the real part of the next-order term (of order h) is always positive and scales as $1/\sqrt{R_0}$. As a consequence, the first eigenvalue asymptotics is given by (8.4) with $R_0 = R_2$ for cases NN and DN, and by (8.3) with $R_0 = R_2$ for the case DD. Only in the case ND, the first eigenvalue asymptotics is determined by the points $(\pm R_1, 0)$ on the inner circle. In this case, the potential reads in local coordinates around $(R_1, 0)$ as $V(s, \rho) = (R_1 + \rho)\cos(s/R_1)$ so that the only change with respect to the above results is $v_{01} = 1$ (instead of $v_{01} = -1$) and $\mathfrak{c}(0) = -1/R_1$ (instead of $\mathfrak{c}(0) = 1/R_1$) so that Eq. (8.4) becomes

$$\lambda_{app}^{ND,(n,k)} = iR_1 + h^{\frac{2}{3}}|a_n'|e^{i\pi/3} + h(2k-1)\frac{e^{-i\pi/4}}{\sqrt{2R_1}} + h^{\frac{4}{3}}\frac{e^{\pi i/6}}{2|a_n'|R_1} + \mathcal{O}(h^{\frac{5}{3}})\,. \tag{8.10}$$

Remark 32. *When the outer radius R_2 of an annulus goes to infinity, the above problem should progressively*[5] *become an exterior problem in the complement of a disk: $\Omega = \{(x_1, x_2) \in \mathbb{R}^2 \ : \ |x| > R_1\}$. Due to the local character of the asymptotic analysis, the expansion (8.10) is independent of the outer radius R_2 and holds even for the unbounded case. This argument suggests the non-emptiness of the spectrum for unbounded domains. This conjecture is confirmed by numerical results in Sec. 9.*

8.3. Domain with transmission condition. Finally, we consider the union of two subdomains, the disk $\Omega_- = \{(x_1, x_2) \in \mathbb{R}^2 \ : \ |x| < R_1\}$ and the annulus $\Omega_+ = \{(x_1, x_2) \in \mathbb{R}^2 \ : \ R_1 < |x| < R_2\}$ separated by a circle on which the transmission boundary condition is imposed. A Dirichlet, Neumann or Robin boundary condition can be imposed at the outer boundary (circle of radius R_2). As for the annulus, there are four points in $\Omega_\perp$: $(\pm R_1, 0)$ and $(\pm R_2, 0)$. Here we focus only on the asymptotic behavior at points $(\pm R_1, 0)$ for the transmission boundary condition (the behavior at the points $(\pm R_2, 0)$ was described in Sec. 8.1). We consider the case described in Theorem 27 when the transmission parameter κ scales with h according to (6.3). As discussed in Sec. 6, this situation is relevant for diffusion MRI applications. The case with fixed κ can be treated similarly.

As stated in Theorem 27, the asymptotic expansion is obtained by starting from the "basic" expansion (with $\kappa = 0$) of either of two problems with Neumann boundary condition corresponding to the two subdomains Ω_- and Ω_+.
If we start from the expansion for the disk, one has $V(x) = (R_1 - \rho)\cos(s/R_1)$, and

[5] We do not have a mathematical proof, the statement remains conjectural.

the asymptotic expansion (6.4) at the point $(R_1, 0)$ reads

$$\begin{aligned}\lambda_h^{\#,(n,k)} &= i\,R_1 - h^{\frac{2}{3}} a_n' e^{-\pi i/3} + h\,(2k-1)\,\frac{e^{-\pi i/4}}{\sqrt{2R_1}} \\ &\quad - h^{\frac{4}{3}} \frac{e^{-\pi i/6}}{a_n'}\left(\hat{\kappa} - \frac{1}{2R_1}\right) + O(h^{\frac{5}{3}})\,.\end{aligned} \tag{8.11}$$

In turn, if we start from the expansion for the inner boundary of the annulus, one has $V(x) = (R_1 + \rho)\cos(s/R_1)$, and the asymptotic expansion (6.4) at the point $(R_1, 0)$ reads

$$\begin{aligned}\lambda_h^{\#,(n,k)} &= i\,R_1 - h^{\frac{2}{3}} a_n' e^{\pi i/3} + h\,(2k-1)\,\frac{e^{-\pi i/4}}{\sqrt{2R_1}} \\ &\quad - h^{\frac{4}{3}} \frac{e^{\pi i/6}}{a_n'}\left(\hat{\kappa} + \frac{1}{2R_1}\right) + O(h^{\frac{5}{3}})\,.\end{aligned} \tag{8.12}$$

These two expressions are different, in particular, their imaginary parts differ already in the order $h^{\frac{2}{3}}$. In turn, the real parts differ at the term of order $h^{\frac{4}{3}}$ that contains two contributions: from the curvature of the boundary, and from the transmission. While the curvature changes its sign on both sides of the boundary, the contribution due to the transmission remains the same. As a consequence, the real part of (8.12) is larger than the real part of (8.11). One can thus expect the existence of two distinct eigenstates living on both sides of the boundary, as confirmed numerically in the next section. For $k = 1$, the eigenstate associated with the eigenvalue with the smallest real part is mainly localized in the disk side of the boundary.

9. Numerical results. This section presents some numerical results to illustrate our analysis. The claims of this section are supported by numerical evidence but should not be considered as rigorous statements, in contrast to previous sections.

The numerical analysis will be limited to bounded domains, for which the BT-operator has compact resolvent and hence discrete spectrum (see Sec. 2). In order to compute numerically its eigenvalues and eigenfunctions, one needs to approximate the BT-operator in a matrix form. For this purpose, one can either (i) discretize the domain by a square lattice and replace the Laplace operator by finite differences (finite difference method); (ii) discretize the domain by a mesh and use a weak formulation of the eigenvalue problem (finite elements method); or (iii) project the BT-operator onto an appropriate complete basis of functions. We choose the last option and project the BT-operator onto the Laplacian eigenfunctions which for rotation-invariant domains (such as disk, annuli, circular layers) are known explicitly [15]. In this basis, the Laplace operator $-\Delta$ is represented by a diagonal matrix Λ. Moreover, the matrix representation of the potential $V(x) = x_1$ was computed analytically, i.e., the elements of the corresponding matrix $\mathcal{B}$ are known explicitly [12, 13, 14]. As a consequence, the computation is reduced to finding the Laplacian eigenvalues for these rotation-invariant domains, constructing the matrices Λ and $\mathcal{B}$ through explicit formulas, and then diagonalizing numerically the truncated matrix $h^2\Lambda + i\mathcal{B}$ which is an approximate representation of the BT-operator $-h^2\Delta + ix_1$. This numerical procedure yields the eigenvalues $\lambda_h^{(m)}$ of the truncated matrix $h^2\Lambda + i\mathcal{B}$, while the associated eigenvectors allow one to construct the eigenfunctions $u_h^{(m)}$. All eigenvalues are ordered according to their increasing real parts:

$$\mathrm{Re}\{\lambda_h^{(1)}\} \leq \mathrm{Re}\{\lambda_h^{(2)}\} \leq \dots \tag{9.1}$$

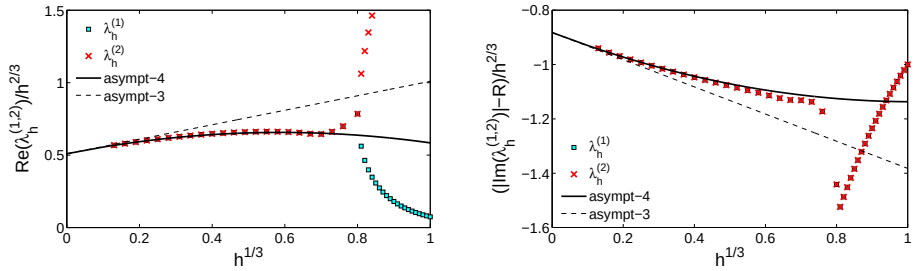

FIG. 1. *The rescaled eigenvalues $\lambda_h^{(1)}$ and $\lambda_h^{(2)}$ of the BT-operator in the unit disk with Neumann boundary condition. Symbols (squares and crosses) show the numerical results of the diagonalization of the matrix $h^2\Lambda + i\mathcal{B}$ (truncated to the size 2803×2803), solid line presents the four-terms asymptotics (8.4) for $\lambda_h^{N,(1,1)}$ while the dashed line shows its three-terms versions (without $h^{\frac{4}{3}}$ term).*

Note that, for a bounded domain, the potential ix is a bounded perturbation of the unbounded Laplace operator $-h^2\Delta$, if $h \neq 0$. To preserve this property after truncation of the matrix $h^2\Lambda + i\mathcal{B}$, the truncation size should be chosen such that $h^2\mu^{(M)} \gg 1$, where $\mu^{(M)}$ is the largest element of the matrix Λ. Due to the Weyl's law, $M \sim \frac{|\Omega|}{4\pi}\mu^{(M)}$ so that the truncation size M should satisfy:

$$h^2 M \gg \frac{|\Omega|}{4\pi}, \tag{9.2}$$

where $|\Omega|$ is the surface area of Ω. For larger domains, either larger truncation sizes are needed (that can be computationally limiting), or h should be limited to larger values. In practice, we use M around 3000 to access h up to 0.01. We have checked that the truncation size does not affect the computed eigenvalues.

9.1. Eigenvalues. For large h, one can divide the BT operator by h^2, $-\Delta + ix_1/h^2$, to get a small bounded perturbation of the Laplace operator. In particular, the eigenvalues of the operator $-h^2\Delta + ix_1$ behave asymptotically as $h^2\mu^{(m)}$, where $\mu^{(m)}$ are the eigenvalues of the Laplace. In this Section, we focus on the more complicated semi-classical limit $h \to 0$ which is the main topic of the paper.

9.1.1. Disk. In order to check the accuracy of the asymptotic expansion of eigenvalues, we first consider the BT-operator in the unit disk: $\Omega = \{(x_1, x_2) \in \mathbb{R}^2 \ : \ |x| < R_0\}$, with $R_0 = 1$. We will present *rescaled* eigenvalues, $(\lambda_h^{(m)} - iR)/h^{\frac{2}{3}}$, for which the constant imaginary offset iR is subtracted and the difference $\lambda_h^{(m)} - iR$ is divided by $h^{\frac{2}{3}}$ in order to emphasize the asymptotic behavior. Note also that, according to Remark 31, the asymptotic expansions for the approximate eigenvalues corresponding to the points $(-R, 0)$ and $(R, 0)$ are the complex conjugates to each other. In order to facilitate their comparison and check this property for numerically computed eigenvalues, we will plot the absolute value of the imaginary part.

Figure 1 shows the first two eigenvalues $\lambda_h^{(1)}$ and $\lambda_h^{(2)}$. For $h^{\frac{1}{3}} \lesssim 0.8$, these eigenvalues turn out to be the complex conjugate to each other, as expected from their asymptotic expansions (the difference $\lambda_h^{(1)} - \bar{\lambda}_h^{(2)}$ being negligible within numerical precision). In turn, the eigenvalues $\lambda_h^{(1)}$ and $\lambda_h^{(2)}$ become real and split for $h^{\frac{1}{3}} \gtrsim 0.8$. The splitting is expected because these eigenvalues behave differently in the large h limit. This numerical observation suggests the existence of branch points in the spectrum (similar features were earlier reported for the complex Airy operator on the

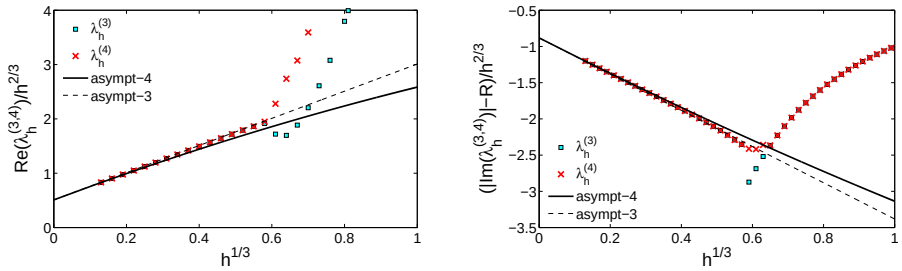

FIG. 2. *The rescaled eigenvalues $\lambda_h^{(3)}$ and $\lambda_h^{(4)}$ of the BT-operator in the unit disk with Neumann boundary condition. Symbols (squares and crosses) show the numerical results of the diagonalization of the matrix $h^2\Lambda + i\mathcal{B}$ (truncated to the size 2803×2803), solid line presents the four-terms asymptotics (8.4) for $\lambda_h^{N,(1,3)}$ while the dashed line shows its three-terms versions (without $h^{\frac{4}{3}}$ term).*

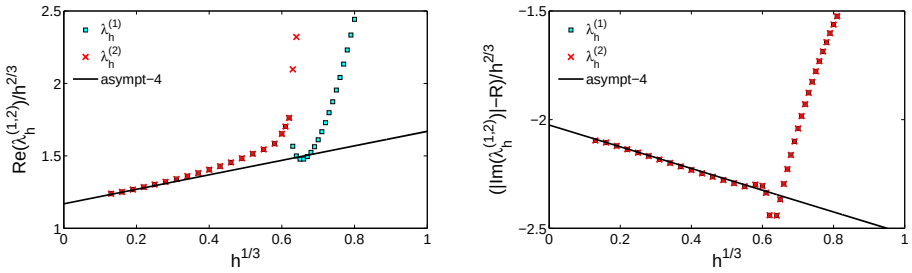

FIG. 3. *The rescaled eigenvalues $\lambda_h^{(1)}$ and $\lambda_h^{(2)}$ of the BT-operator in the unit disk with Dirichlet boundary condition. Symbols (squares and crosses) show the numerical results of the diagonalization of the matrix $h^2\Lambda + i\mathcal{B}$ (truncated to the size 2731×2731), while solid line shows the four-terms asymptotic expansion (8.3) for $\lambda_h^{D,(1,1)}$.*

one-dimensional interval with Neumann boundary condition, see [34]). For comparison, the four-terms asymptotics (8.4) for $\lambda_h^{N,(1,1)}$ and its three-terms version (without term $h^{\frac{4}{3}}$) are shown by solid and dashed lines, respectively. These expansions start to be applicable for $h^{\frac{1}{3}} \lesssim 0.7$, while their accuracy increases as h decreases.

Figure 2 shows the next eigenvalues $\lambda_h^{(3)}$ and $\lambda_h^{(4)}$, the four-terms asymptotics (8.4) for $\lambda_h^{N,(1,3)}$ and its three-terms version. These eigenvalues are the complex conjugates to each other for $h^{\frac{1}{3}} \lesssim 0.57$ while become real and split for larger h. One can see that the four-terms asymptotics is less accurate for these eigenvalues than for those from Fig. 1. A small deviation can probably be attributed to higher-order terms (it is worth noting that contributions from the $h^{\frac{4}{3}}$ and $h^{\frac{5}{3}}$ terms can be comparable for the considered values of h).

For comparison, Figure 3 shows the first rescaled eigenvalues $\lambda_h^{(1)}$ and $\lambda_h^{(2)}$ of the BT-operator in the unit disk with Dirichlet boundary condition. As earlier for the Neumann case, these eigenvalues are complex conjugate to each other for $h^{\frac{1}{3}} \lesssim 0.6$ while become real and split for larger h. One can see that the asymptotics (8.3) for $\lambda_h^{D,(1,1)}$ captures the behavior for the imaginary part very accurately. In turn, the behavior of the real part is less accurate, probably due to higher-order corrections.

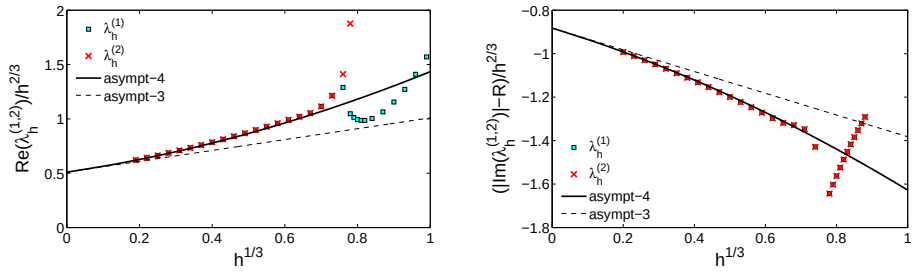

FIG. 4. *The rescaled eigenvalues $\lambda_h^{(1)}$ and $\lambda_h^{(2)}$ of the BT-operator in the unit disk with Robin boundary condition (with $\hat\kappa = 1$ and $\kappa = \hat\kappa h^{\frac{2}{3}}$). Symbols (squares and crosses) show the numerical results of the diagonalization of the matrix $h^2\Lambda + i\mathcal{B}$ (truncated to the size 2803×2803), while solid and dashed lines show the four-terms asymptotic expansion (8.3) for $\lambda_h^{R,(1,1)}$ and its three-term version (without term $h^{\frac{4}{3}}$).*

Finally, Figure 4 illustrates the case with Robin boundary condition, with $\hat\kappa = 1$ while κ scaling as $\hat\kappa h^{\frac{2}{3}}$. The four-term expansion (6.4) accurately captures their asymptotic behavior.

9.1.2. Annulus. Due to its local character, the quasimodes construction is expected to be applicable to the exterior problem, i.e., in the complement of a disk of radius R_1, $\Omega = \{(x_1, x_2) \in \mathbb{R}^2 \ : \ |x| > R_1\}$. Since we cannot numerically solve this problem for unbounded domains, we consider a circular annulus $\Omega = \{x \in \mathbb{R}^2 \ : \ R_1 < |x| < R_2\}$ with a fixed inner radius $R_1 = 1$ and then increase the outer radius R_2. In the limit $h \to 0$, the eigenfunctions are expected to be localized around the four points $(\pm R_1, 0)$, $(\pm R_2, 0)$ from the set $\Omega_\perp$, with corresponding asymptotic expansions for eigenvalues.

Figure 5 illustrates the discussion in Sec. 8.2 about different asymptotics of the first eigenvalue $\lambda_h^{(1)}$ for four combinations of Neumann/Dirichlet boundary conditions on inner and outer circles. In particular, one observes the same asymptotic expansion (8.4) with $R = R_2$ for NN and DN cases because the first eigenvalue is determined by the local behavior near the point $(R_2, 0)$ which is independent of the boundary condition on the inner circle as $h \to 0$. The expansion (8.3) with $R = R_2$ for the Dirichlet condition appears only for the case DD. Finally, the case ND is described by the local behavior at the inner circle by the expansion (8.10) with $R = R_1$. In what follows, we focus on this case in order to illustrate that the local behavior at the inner boundary is not affected by the position of the outer circle as $h \to 0$.

For the case ND, Fig. 6 shows the first rescaled eigenvalue $\lambda_h^{(1)}$ that corresponds to an eigenfunction which, for small h, is localized near the inner circle. As a consequence, the asymptotic behavior of $\lambda_h^{(1)}$ as $h \to 0$ is expected to be independent of the outer boundary. This is indeed confirmed because the numerical results for three annuli with $R_2 = 1.5$, $R_2 = 2$ and $R_3 = 3$ are indistinguishable for $h^{\frac{1}{3}}$ smaller than 0.5. For comparison, we also plot the four-terms asymptotics (8.10) that we derived for the exterior of the disk of radius $R_1 = 1$. One can see that the inclusion of the term $h^{\frac{4}{3}}$ improves the quality of the expansion (as compared to its reduced three-terms version without $h^{\frac{4}{3}}$ term).

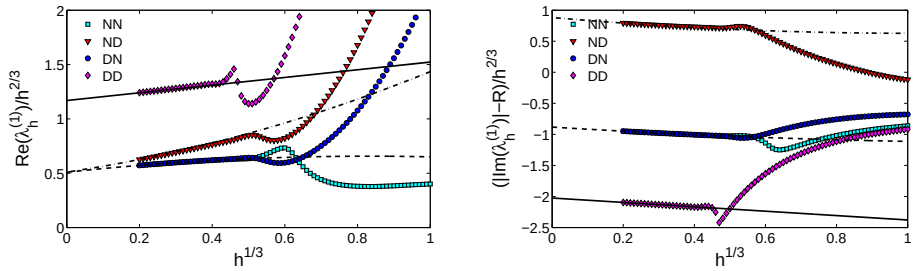

FIG. 5. *The rescaled eigenvalue $\lambda_h^{(1)}$ of the BT-operator in the annulus with four combinations of Neumann/Dirichlet boundary conditions at the inner and outer circles of radii $R_1 = 1$ and $R_2 = 2$: NN (squares), ND (triangles), DN (circles), and DD (diamonds), obtained by the diagonalization of the truncated matrix $h^2\Lambda + i\mathcal{B}$. The solid line presents the expansion (8.3) with $R = R_2$ for Dirichlet condition, the dashed line shows the expansion (8.4) with $R = R_2$ for Neumann condition, and the dash-dotted line shows the expansion (8.10) with $R = R_1$ for Neumann condition.*

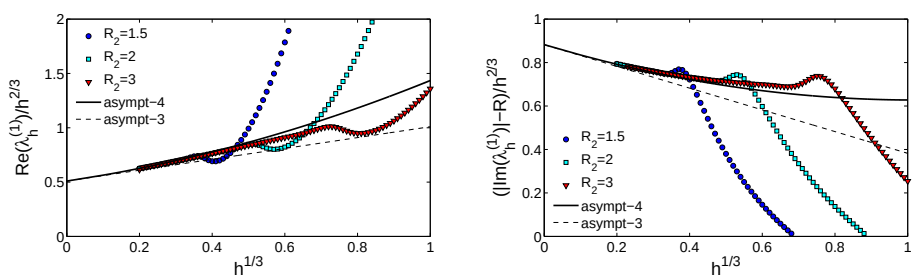

FIG. 6. *The rescaled eigenvalue $\lambda_h^{(1)}$ of the BT-operator in the annulus with Neumann boundary condition at the inner circle of radius $R_1 = 1$ and Dirichlet boundary condition at the outer circle of radius R_2, with $R_2 = 1.5$ (circles), $R_2 = 2$ (squares) and $R_3 = 3$ (triangles), obtained by the diagonalization of the matrix $h^2\Lambda + i\mathcal{B}$ (truncated to sizes 1531×1531 for $R_2 = 1.5$, 2334×2334 for $R_2 = 2$, and 2391×2391 for $R_2 = 3$). Solid line presents the four-terms expansion (8.10) for $\lambda_h^{ND,(1,1)}$, while dashed line shows its reduced three-terms version (without $h^{\frac{4}{3}}$ term).*

9.1.3. Domain with transmission condition. Finally, we consider the BT-operator in the union of two subdomains, the disk $\Omega_- = \{(x_1, x_2) \in \mathbb{R}^2 \ : \ |x| < R_1\}$ and the annulus $\Omega_+ = \{(x_1, x_2) \in \mathbb{R}^2 \ : \ R_1 < |x| < R_2\}$ separated by the circle of radius R_1 on which the transmission boundary condition is imposed. We impose the Dirichlet boundary condition at the outer boundary of the domain (at the circle of radius R_2) to ensure that first eigenfunctions are localized near points $(\pm R_1, 0)$ with transmission boundary condition.

Figure 7 shows the rescaled eigenvalues $\lambda_h^{(1)}$ and $\lambda_h^{(2)}$ of the BT-operator with a fixed $\hat{\kappa} = 1$ and κ scaling as $\hat{\kappa} h^{\frac{2}{3}}$. As in earlier examples, the first two eigenvalues are complex conjugate to each other for small h but they split at larger h. One can see that the asymptotic relation (8.11) with $n = k = 1$ accurately describes the behavior of the these eigenvalues for small h.

Figure 8 shows the first rescaled eigenvalue $\lambda_h^{(1)}$ for several values of $\hat{\kappa}$ (with κ scaling as $\hat{\kappa} h^{\frac{2}{3}}$). In the special case $\hat{\kappa} = 0$, the two subdomains are separated from each other by Neumann boundary condition, and the spectrum of the BT operator

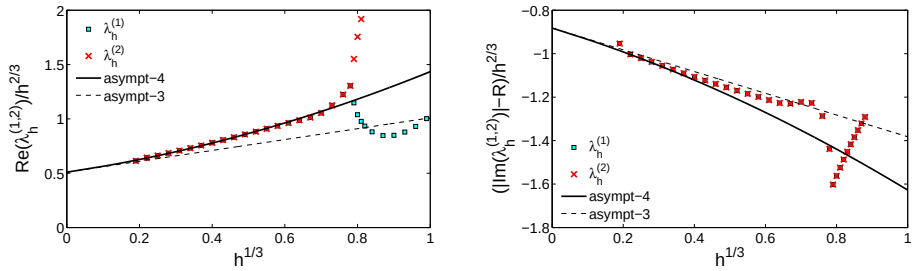

FIG. 7. *The rescaled eigenvalues $\lambda_h^{(1)}$ and $\lambda_h^{(2)}$ of the BT-operator in the union of the disk and annulus with transmission condition at the inner boundary of radius $R_1 = 1$ (with $\hat{\kappa} = 1$) and Dirichlet condition at the outer boundary of radius $R_2 = 2$. Symbols (squares and crosses) show the numerical results of the diagonalization of the matrix $h^2\Lambda + i\mathcal{B}$ (truncated to the size 3197×3197), solid line presents the four-terms expansion (8.11) for $\lambda_h^{T,(1,1)}$, while dashed line shows its reduced three-terms version (without $h^{\frac{4}{3}}$ term).*

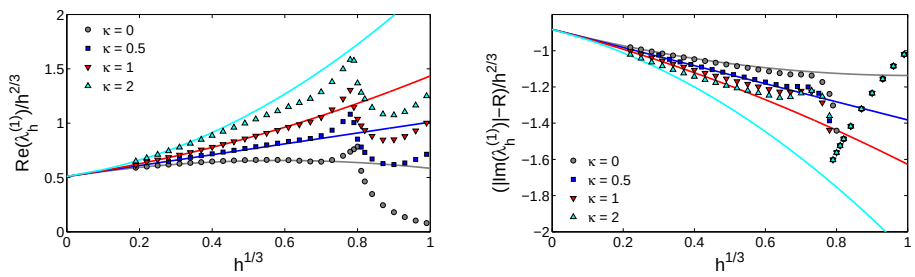

FIG. 8. *The rescaled eigenvalue $\lambda_h^{(1)}$ of the BT-operator in the union of the disk and annulus with transmission condition at the inner boundary of radius $R_1 = 1$ (with several values of $\hat{\kappa}$: 0, 0.5, 1, 2) and Dirichlet condition at the outer boundary of radius $R_2 = 2$. Symbols (circles, squares, triangles) show the numerical results of the diagonalization of the truncated matrix $h^2\Lambda + i\mathcal{B}$, solid lines present the four-terms expansion (8.11) for $\lambda_h^{T,(1,1)}$.*

is obtained from its spectra for each subdomain. As a consequence, we plot in this case the first rescaled eigenvalue for the BT operator in the unit disk with Neumann boundary condition (as in Fig. 1). One can see that the expansion (8.11) accurately captures the asymptotic behavior. We recall that the transmission parameter $\hat{\kappa}$ appears only in the fourth term of order $h^{\frac{4}{3}}$. Note also that this term vanishes in the case $\hat{\kappa} = 1/2$ as two contributions in (8.11) compensate each other.

9.2. Eigenfunctions. For the annulus with Neumann boundary condition at the inner circle of radius $R_1 = 1$ and Dirichlet boundary condition at the outer circle of radius $R_2 = 2$, Fig. 9(top) shows two eigenfunctions of the BT operator with $h = 0.1$ (corresponding to $h^{\frac{1}{3}} \approx 0.4642$). One can already recognize the localization of the first eigenfunction $u_h^{(1)}$ at the inner boundary, while the eigenfunction $u_h^{(3)}$ tends to localize near the outer boundary. Their pairs $u_h^{(2)}$ and $u_h^{(4)}$ (not shown) exhibit the same behavior near the opposite points $(-R_1, 0)$ and $(-R_2, 0)$, respectively. Since $h = 0.1$ is not small enough, the localization becomes less and less marked for other eigenfunctions which progressively spread over the whole annulus (not shown). For comparison, we also plot in Fig. 9(bottom) the eigenfunctions $u_h^{(1)}$ and $u_h^{(3)}$ for a thicker annulus of outer radius $R_2 = 3$. One can see that these eigenfunctions look

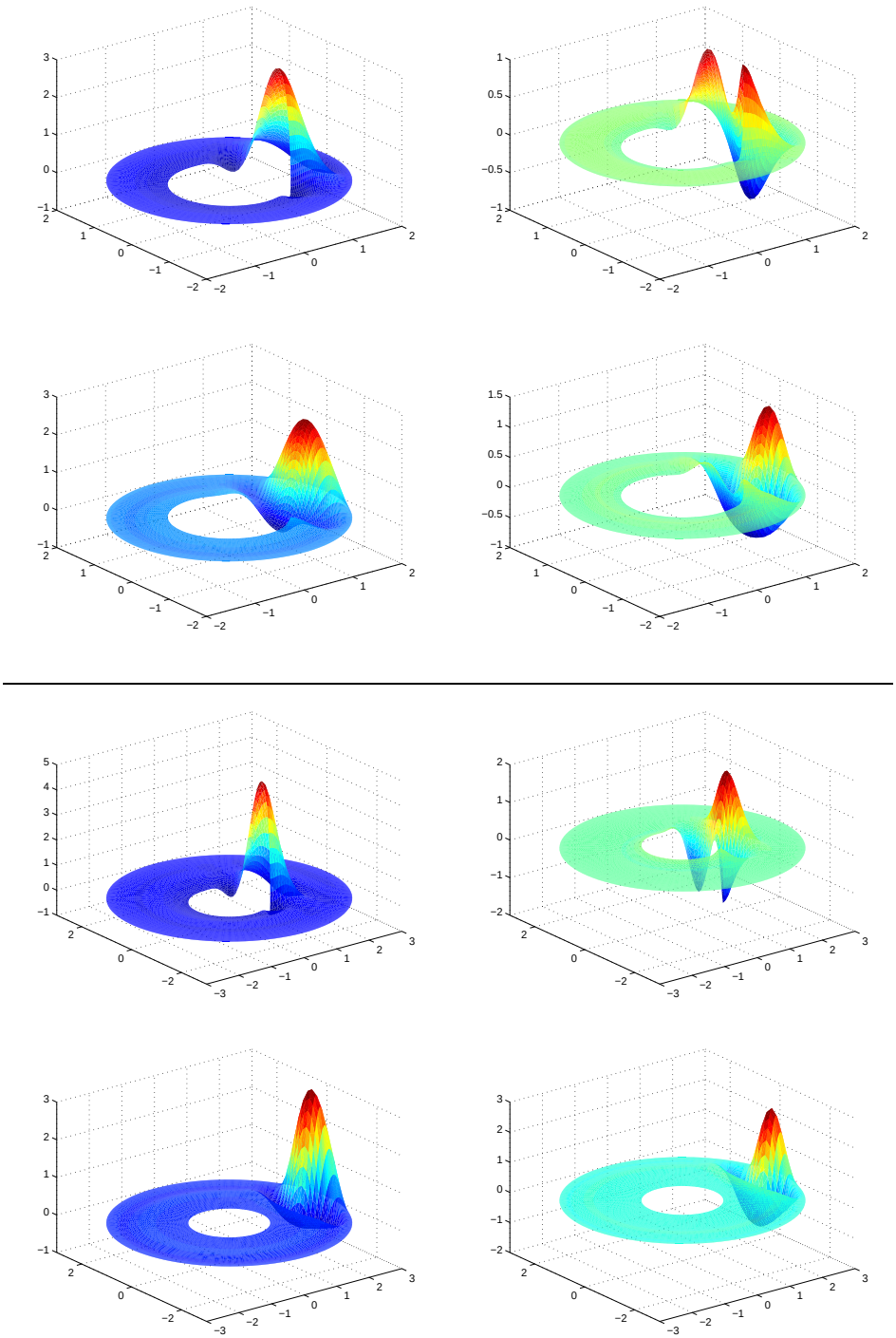

FIG. 9. *Real (left) and imaginary (right) parts of the eigenfunctions* $u_h^{(1)}$ *(top) and* $u_h^{(3)}$ *(bottom) at* $h = 0.1$ *for the annulus with Neumann boundary condition at the inner circle of radius* $R_1 = 1$ *and Dirichlet boundary condition at the outer circle of radius* $R_2 = 2$ *(four plots above horizontal line) or* $R_2 = 3$ *(four plots below horizontal line). Numerical computation is based on the truncated matrix representation of sizes* 2334×2334 *and* 2391×2391*, respectively.*

very similar to that of the annulus with $R_2 = 2$.

For smaller $h = 0.01$ (corresponding to $h^{\frac{1}{3}} \approx 0.2154$), the localization of eigenfunctions is much more pronounced. Figure 10 shows four eigenfunctions for the annulus of radii $R_1 = 1$ (Neumann condition) and $R_2 = 2$ (Dirichlet condition). One can see that the eigenfunctions $u_h^{(1)}$, $u_h^{(3)}$, and $u_h^{(7)}$ are localized near the inner circle while $u_h^{(5)}$ is localized near the outer circle. When the outer circle is moved away, the former eigenfunctions remain almost unchanged, suggesting that they would exist even in the limiting domain with $R_2 = \infty$, i.e., in the complement of the unit disk. In turn, the eigenfunctions that are localized near the outer boundary (such as $u_h^{(5)}$) will be eliminated. In spite of this numerical evidence, the existence of eigenfunctions of the BT operator for unbounded domains remains conjectural.

Figure 11 shows the eigenfunctions $u_h^{(1)}$ and $u_h^{(3)}$ at $h = 0.01$ for the union of the disk and annulus with transmission condition at the inner boundary of radius $R_1 = 1$ (with $\hat{\kappa} = 1$ and $\kappa = \hat{\kappa} h^{\frac{2}{3}}$) and Dirichlet condition at the outer boundary of radius $R_2 = 2$. Both eigenfunctions are localized near the inner boundary. Moreover, a careful inspection of this figure shows that $u_h^{(1)}$ is mainly supported by the disk and vanishes rapidly on the other side of the inner circle (i.e., in the annulus side), while $u_h^{(3)}$ exhibits the opposite (i.e., it is localized in the annulus). This is a new feature of localization as compared to the one-dimensional case studied in [16, 17] because the curvature has the opposite signs on two sides of the boundary.

Finally, we check the accuracy of the WKB approximation of the first eigenfunction $u_h^{(1)}$ for the unit disk with Neumann boundary condition. To make the illustration easier, we plot in Figure 12 the absolute value of $u_h^{(1)}$ at $h = 0.01$, normalized by its maximum, along the boundary (on the circle of radius $R_0 = 1$), near the localization point $s = 0$. One can see that the WKB approximation, $\exp(-(\theta_0(s) + h^{\frac{2}{3}}\theta_1(s))/h)$, obtained with $\theta_0(s)$ and $\theta_1(s)$ given by (8.7) and (8.9), accurately captures the behavior over the range of s between -0.3 and 0.3. Note that its reduced version, $\exp(-\theta_0(s)/h)$, is also accurate.

10. Application to diffusion NMR. In this section, we briefly discuss (with no pretention to mathematical rigor) a possible application of the proposed spectral analysis of the Bloch-Torrey operator to diffusion NMR [12]. In this field, the BT-operator governs the evolution of the transverse nuclear magnetization which satisfies the Bloch-Torrey equation

$$\frac{\partial}{\partial t} m(x,t) = \big[D\Delta - i\gamma g x_1\big] m(x,t), \tag{10.1}$$

subject to the uniform initial condition $m(x,0) = 1$. Here D is the diffusion coefficient, g the magnetic field gradient, γ the gyromagnetic ratio, and the gradient is considered to be constant in time. For a bounded domain, the long-time asymptotic behavior of the solution is determined by the first eigenvalue $\lambda^{(1)}$ of the BT-operator (with the smallest real part):

$$m(x,t) \simeq C u^{(1)}(x) \exp(-\omega t) \quad (t \to \infty), \tag{10.2}$$

where

$$\omega = \gamma g \lambda_h^{(1)}, \qquad h^2 = D/(\gamma g). \tag{10.3}$$

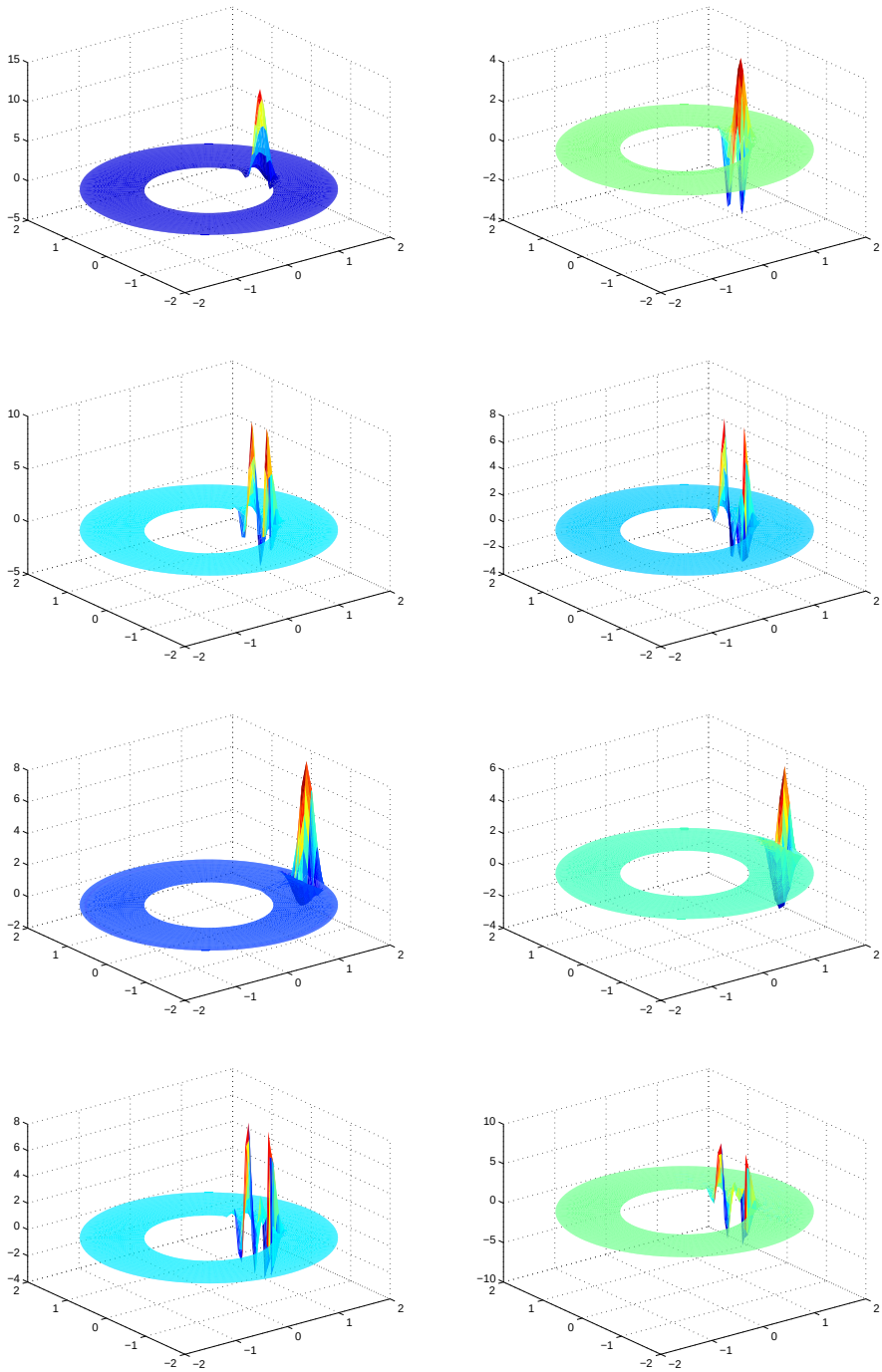

FIG. 10. *Real (left) and imaginary (right) parts of the eigenfunctions* $u_h^{(1)}$ *(top),* $u_h^{(3)}$, $u_h^{(5)}$ *and* $u_h^{(7)}$ *(bottom) at* $h = 0.01$ *for the annulus with Neumann boundary condition on the inner circle of radius* $R_1 = 1$ *and Dirichlet boundary condition on the outer circle of radius* $R_2 = 2$ *(numerical computation based on the truncated matrix representation of size* 2334×2334*).*

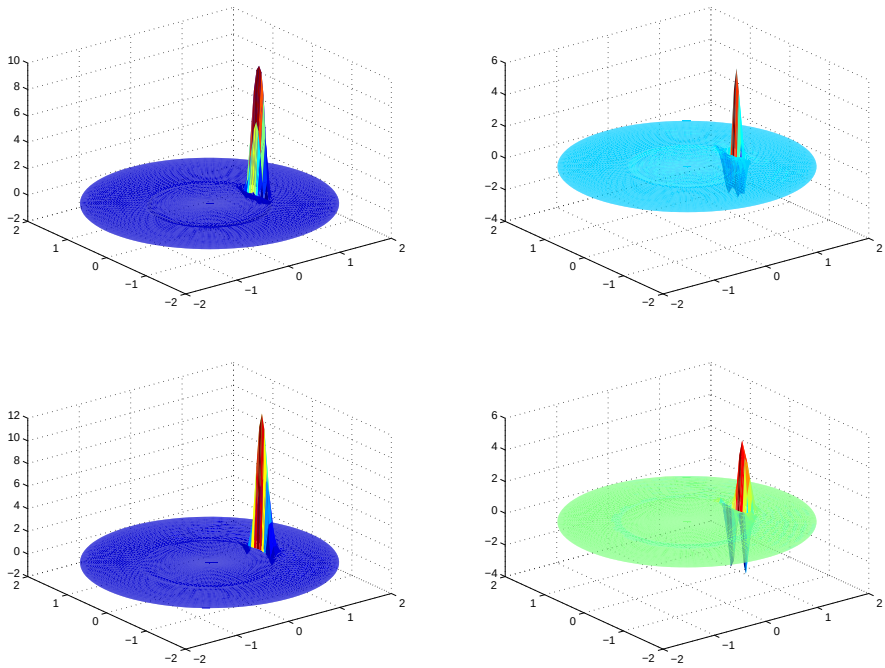

FIG. 11. *Real (left) and imaginary (right) parts of the eigenfunctions* $u_h^{(1)}$ *(top) and* $u_h^{(3)}$ *(bottom) at* $h = 0.01$ *for the union of the disk and annulus with a transmission boundary condition (with* $\hat{\kappa} = 1$ *and* $\kappa = \hat{\kappa} h^{\frac{2}{3}}$*) at the inner circle of radius* $R_1 = 1$ *and Dirichlet boundary condition at the outer circle of radius* $R_2 = 2$ *(numerical computation based on the truncated matrix representation of size* 3197×3197*).*

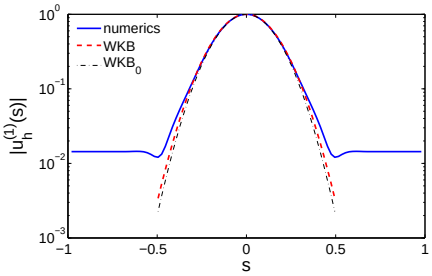

FIG. 12. *The absolute value of the first eigenfunction* $u_h^{(1)}(r, s)$ *(solid line) at* $h = 0.01$ *and* $r = 1$ *for the unit disk with Neumann boundary condition, near the boundary point* $s = 0$*. For convenience,* $u_h^{(1)}(r, s)$ *is normalized by its maximum at* $s = 0$*. For comparison, the absolute value of the WKB approximation,* $\exp(-(\theta_0(s) + h^{\frac{2}{3}}\theta_1(s))/h)$ *and of its reduced version,* $\exp(-\theta_0(s)/h)$*, are shown by dashed and dash-dotted lines, respectively.*

Admitting[6] that the formal asymptotic expansion (5.42) with $n = k = 1$ is the asymptotics of the eigenvalue $\lambda_h^{(1)}$ with the smallest real part, we obtain in the limit

[6]This has not be proven mathematically.

of large g

$$\begin{aligned}\omega = i\,\gamma g v_{00} + D^{\frac{1}{3}}(\gamma g)^{\frac{2}{3}}\mu_0^{\#}|v_{01}|^{\frac{2}{3}}\exp\left(\frac{i\pi}{3}\text{sign } v_{01}\right)\\ + D^{\frac{1}{2}}(\gamma g)^{\frac{1}{2}}|v_{20}|^{\frac{1}{2}}\exp\left(\frac{i\pi}{4}\text{sign}\, v_{20}\right) + D^{\frac{2}{3}}(\gamma g)^{\frac{1}{3}}\lambda_4^{\#,(1)} + \mathcal{O}(g^{\frac{1}{6}})\,,\end{aligned} \tag{10.4}$$

where the coefficients v_{jk} are defined by the local parameterization $V(x) = x_1$ of the boundary near a point from $\Omega_\perp$. The real part of ω determines the decay rate of the transverse magnetization and the related macroscopic signal.

The leading term of order $(\gamma g)^{\frac{2}{3}}$ was predicted for impermeable one-dimensional domains (with Neumann boundary condition) by Stoller *et al.* [34] and experimentally confirmed by Hürlimann *et al.* [27]. The next-order correction was obtained by de Swiet and Sen [35] for an impermeable disk. In the present paper, we generalized these results to arbitrary planar domains with smooth boundary and to various boundary conditions (Neumann, Dirichlet, Robin, transmission) and provided a general technique for getting higher-order corrections (in particular, we derived the last term). Moreover, we argued (without rigorous proof) that these asymptotic relations should also hold for unbounded domains.

Appendix A. Explicit computation of λ_4.

A.1. Evaluation of the integral with ϕ_1. In order to compute λ_4 from (5.55), we first evaluate the integral

$$\eta = \int_{-\infty}^{\infty} \sigma\,\phi_1(\sigma)\,\phi_0(\sigma)\,d\sigma. \tag{A.1}$$

We recall that $\phi_1(\sigma)$ satisfies

$$(\mathcal{L}_2 - \lambda_2)\,\phi_1 = c_{11}\,\sigma\,\phi_0\,, \tag{A.2}$$

with

$$c_{11} := -i\,v_{11}\int \tau\psi_0^{\#}(\tau)^2 d\tau\,. \tag{A.3}$$

As a solution of (A.2), we search for some eigenpair $\{\lambda_2, \phi_0\} = \{\lambda_2^{(k)}, \phi_0^{(k)}\}$, with some fixed $k \geq 1$, where $\lambda_2^{(k)}$ and $\phi_0^{(k)}$ are the eigenvalues and eigenfunctions of the quantum harmonic oscillator given explicitly in (5.40). Since $\phi_0^{(k)}$ are expressed through the Hermite polynomials H_k, one can use their recurrence relation, $H_{k+1}(x) = 2xH_k(x) - 2kH_{k-1}(x)$, to express

$$\sigma\,\phi_0^{(k)} = \frac{\sqrt{k}\,\phi_0^{(k+1)} + \sqrt{k-1}\,\phi_0^{(k-1)}}{(2\gamma)^{\frac{1}{2}}}. \tag{A.4}$$

It is therefore natural to search for the solution of (A.2) in the form

$$\phi_1(\sigma) = C_1\,\phi_0^{(k+1)}(\sigma) + C_2\,\phi_0^{(k-1)}(\sigma)\,. \tag{A.5}$$

The coefficients C_1 and C_2 are determined by substituting this expression into (A.2):

$$\begin{aligned}(\mathcal{L}_2 - \lambda_2)\,\phi_1 &= C_1\big(\lambda_2^{(k+1)} - \lambda_2^{(k)}\big)\phi_0^{(k+1)} + C_2\big(\lambda_2^{(k-1)} - \lambda_2^{(k)}\big)\phi_0^{(k-1)}\\ &= c_{11}\,\frac{\sqrt{k}\,\phi_0^{(k+1)} + \sqrt{k-1}\,\phi_0^{(k-1)}}{(2\gamma)^{\frac{1}{2}}},\end{aligned} \tag{A.6}$$

from which $C_1 = c_{11}\sqrt{k}/(2\gamma)^{\frac{3}{2}}$ and $C_2 = -c_{11}\sqrt{k-1}/(2\gamma)^{\frac{3}{2}}$, where we used $\lambda_2^{(k)} = \gamma(2k-1)$, with $\gamma = |v_{20}|^{\frac{1}{2}} \exp\left(\frac{\pi i}{4} \operatorname{sign} v_{20}\right)$. We get then

$$\phi_1(\sigma) = \frac{c_{11}}{(2\gamma)^{\frac{3}{2}}} \left(\sqrt{k}\, \phi_0^{(k+1)}(\sigma) - \sqrt{k-1}\, \phi_0^{(k-1)}(\sigma) \right). \tag{A.7}$$

Substituting this expression into (A.1), one gets

$$\eta = \frac{c_{11}}{4\gamma^2} = -\frac{v_{11}}{4v_{20}} \int \tau \psi_0^{\#}(\tau)^2 d\tau\,, \tag{A.8}$$

independently of n. We conclude from (5.55) that

$$\lambda_4^{\#} = -i\frac{v_{11}^2 [I_1^{\#}]^2}{4v_{20}} + \frac{\mathfrak{c}(0)}{2} \int \partial_\tau [\psi_0^{\#}(\tau)]^2 + i v_{02} I_2^{\#}, \tag{A.9}$$

where

$$I_1^{\#} = \int \tau\, \psi_0^{\#}(\tau)^2\, d\tau\,, \qquad I_2^{\#} = \int \tau^2\, \psi_0^{\#}(\tau)^2\, d\tau\,. \tag{A.10}$$

A.2. Evaluation of the integrals with $\psi_0^{\#}$. In order to compute these integrals, we consider the function $\Psi(x) = \mathrm{Ai}(\alpha + \beta x)$ that satisfies the Airy equation

$$(-\partial_x^2 + \beta^3 x + \beta^2 \alpha)\Psi(x) = 0\,. \tag{A.11}$$

Multiplying this equation by $\Psi'(x)$, $\Psi(x)$, $x\Psi'(x)$, $x\Psi(x)$, or $x^2\Psi'(x)$ and integrating from 0 to infinity, one gets the following five relations:

1.

$$-\int_0^\infty \Psi''(x)\Psi'(x)\, dx + \int_0^\infty (\beta^3 x + \beta^2\alpha)\Psi(x)\Psi'(x)\, dx = 0\,,$$

which leads to the determination of $\int_0^{+\infty} \Psi(x)^2 dx$ by the formula

$$\Psi'(0)^2 - \beta^2\alpha\Psi(0)^2 - \beta^3 \int_0^\infty \Psi(x)^2\, dx = 0\,. \tag{A.12}$$

2.

$$-\int_0^\infty \Psi''(x)\Psi(x)\, dx + \int_0^\infty (\beta^3 x + \beta^2\alpha)\Psi(x)^2 dx = 0\,.$$

Here we remark that

$$\int_0^\infty \Psi''(x)\Psi(x)\, dx = \Psi'(0)\Psi(0) - \int_0^\infty \Psi'(x)^2\, dx$$

and get

$$-\Psi'(0)\Psi(0) + \int_0^\infty \Psi'(x)^2\, dx + \int_0^\infty (\beta^3 x + \beta^2\alpha)\Psi(x)^2 dx = 0\,. \tag{A.13}$$

3.

$$-\int_0^\infty \Psi''(x)x\Psi'(x)\,dx + \int_0^\infty (\beta^3 x + \beta^2\alpha)x\Psi(x)\Psi' dx = 0 \quad \Longrightarrow$$

$$\frac{1}{2}\int_0^\infty \Psi'(x)^2\,dx - \frac{1}{2}\int_0^\infty (2\beta^3 x + \beta^2\alpha)\Psi(x)^2 dx = 0\,. \tag{A.14}$$

4.

$$-\int_0^\infty \Psi''(x)x\Psi(x)\,dx + \int_0^\infty (\beta^3 x + \beta^2\alpha)x\Psi(x)^2 dx = 0 \quad \Longrightarrow$$

$$\int_0^\infty \Psi'(x)(x\Psi(x))'\,dx + \int_0^\infty (\beta^3 x + \beta^2\alpha)x\Psi(x)^2 dx = 0 \quad \Longrightarrow$$

$$\int_0^\infty x\Psi'(x)^2\,dx - \frac{1}{2}\Psi(0)^2 + \int_0^\infty (\beta^3 x + \beta^2\alpha)x\Psi(x)^2 dx = 0\,. \tag{A.15}$$

5.

$$-\int_0^\infty \Psi''(x)x^2\Psi'(x)\,dx + \int_0^\infty (\beta^3 x + \beta^2\alpha)x^2\Psi(x)\Psi' dx = 0 \quad \Longrightarrow$$

$$\int_0^\infty x\Psi'(x)^2\,dx + \int_0^\infty (\beta^3 x + \beta^2\alpha)x^2\Psi(x)\Psi' dx = 0\,. \tag{A.16}$$

So we get a linear system of five equations satisfied by $\int \Psi^2 dx$, $\int x\Psi^2 dx$, $\int x^2\Psi^2 dx$, $\int \Psi'(x)^2 dx$ and $\int x\Psi'(x)^2 dx$. Solving this system, we obtain

$$\begin{aligned}\int_0^\infty \Psi^2(x)dx &= \beta^{-3}[\Psi'(0)]^2 - \alpha\beta^{-1}[\Psi(0)]^2 \\ &= \frac{[\mathrm{Ai}'(\alpha)]^2 - \alpha[\mathrm{Ai}(\alpha)]^2}{\beta},\end{aligned} \tag{A.17}$$

$$\begin{aligned}\int_0^\infty x\Psi^2(x)dx &= \frac{1}{3\beta^3}\left(-\Psi'(0)\Psi(0) - 2\alpha\beta^2\int_0^\infty \Psi^2(x)dx\right) \\ &= -\frac{\mathrm{Ai}(\alpha)\mathrm{Ai}'(\alpha) + 2\alpha[\mathrm{Ai}'(\alpha)]^2 - 2\alpha^2[\mathrm{Ai}(\alpha)]^2}{3\beta^2}\,,\end{aligned} \tag{A.18}$$

$$\begin{aligned}\int_0^\infty x^2\Psi^2(x)dx &= \frac{1}{5\beta^3}\left([\Psi(0)]^2 - 4\alpha\beta^2\int_0^\infty x\ \Psi^2(x)dx\right) \\ &= \frac{[\mathrm{Ai}(\alpha)]^2 + \frac{4}{3}\alpha\big(\mathrm{Ai}(\alpha)\mathrm{Ai}'(\alpha) + 2\alpha[\mathrm{Ai}'(\alpha)]^2 - 2\alpha^2[\mathrm{Ai}(\alpha)]^2\big)}{5\beta^3}\,,\end{aligned} \tag{A.19}$$

where we assume that the parameter β is such that $|\arg(\beta)| < \pi/3$ so that $\Psi(+\infty) = \Psi'(+\infty) = 0$ (otherwise the integrals could diverge). These relations allow one to compute the normalization constant $c_n^{\#}$ of quasimodes and the contribution $\lambda_4^{\#}$ to the eigenvalue. We consider successively Dirichlet, Neumann, Robin, and Transmission cases.

Dirichlet case. The function $\psi_0^D(\tau)$ from (5.19) corresponds to $\alpha = a_n$ and $\beta = |v_{01}|^{\frac{1}{3}} \exp\left(\frac{i\pi}{6}\text{sign } v_{01}\right)$ so that $\text{Ai}(\alpha) = 0$. The normalization constant c_n^D in (5.18) is then

$$(c_n^D)^{-2} = \frac{[\text{Ai}'(a_n)]^2}{\beta}\,. \tag{A.20}$$

Using (A.17), one gets

$$I_1^D = \int_0^\infty \tau[\psi_0^D(\tau)]^2 d\tau = -\frac{2a_n}{3\beta}\,, \tag{A.21}$$

$$I_2^D = \int_0^\infty \tau^2[\psi_0^D(\tau)]^2 d\tau = \frac{8a_n^2}{15\beta^2}\,. \tag{A.22}$$

Using (5.42) and (A.8), we obtain

$$\begin{aligned} \lambda_4^{D,(n)} &= i\frac{v_{11}^2 a_n^2}{9v_{20}\beta^2} - \frac{\mathfrak{c}(0)}{2}[\psi_0^D(0)]^2 + iv_{02}\frac{8a_n^2}{15\beta^2} \\ &= \frac{ia_n^2}{|v_{01}|^{\frac{2}{3}}\exp\left(\frac{i\pi}{3}\text{sign } v_{01}\right)}\left(\frac{1}{9}\,\frac{v_{11}^2}{v_{20}} + \frac{8}{15}v_{02}\right), \end{aligned} \tag{A.23}$$

where we used $\psi_0^D(0) = 0$.

Neumann case. The function $\psi_0^N(\tau)$ from (5.18) corresponds to $\alpha = a_n'$ and $\beta = |v_{01}|^{\frac{1}{3}} \exp\left(\frac{i\pi}{6}\text{sign } v_{01}\right)$ so that $\text{Ai}'(\alpha) = 0$. The normalization constant c_n^N in (5.18) is then

$$(c_n^N)^{-2} = \frac{[\text{Ai}'(\alpha)]^2 - \alpha[\text{Ai}(\alpha)]^2}{\beta} = -\frac{a_n'[\text{Ai}(a_n')]^2}{\beta}. \tag{A.24}$$

Using (A.17), one gets

$$I_1^N = \int_0^\infty \tau[\psi_0^N(\tau)]^2 d\tau = -\frac{2a_n'}{3\beta}\,, \tag{A.25}$$

$$I_2^N = \int_0^\infty \tau^2[\psi_0^N(\tau)]^2 d\tau = \frac{8(a_n')^3 - 3}{15a_n'\beta^2}\,, \tag{A.26}$$

from which

$$\lambda_4^{N,(n)} = \frac{i}{|v_{01}|^{\frac{2}{3}}\exp\left(\frac{i\pi}{3}\text{sign } v_{01}\right)}\left(-\frac{(a_n')^2}{18}\,\frac{v_{11}^2}{v_{20}} + \frac{1}{2a_n'}\mathfrak{c}(0)\,v_{01} + \frac{8(a_n')^3 - 3}{15a_n'}v_{02}\right). \tag{A.27}$$

Robin case. The function $\psi_0^R(\tau)$ from (5.28) corresponds to $\beta = |v_{01}|^{\frac{1}{3}}\delta$ and $\alpha = a_n^R(\kappa)$ so that $\mathrm{Ai}'(\alpha) = \hat{\kappa}\,\mathrm{Ai}(\alpha)$, with $\hat{\kappa} = \kappa/(\delta\,|v_{01}|^{\frac{1}{3}})$ and $\delta = \exp\left(\frac{i\pi}{6}\,\mathrm{sign}\ v_{01}\right)$. The normalization constant c_n^R in (5.28) is then

$$(c_n^R)^{-2} = \frac{[\mathrm{Ai}(a_n^R(\kappa))]^2}{\beta}\left[\hat{\kappa}^2 - a_n^R(\kappa)\right] = [\mathrm{Ai}(a_n^R(\kappa))]^2\ \frac{\kappa^2+\lambda_0^R}{iv_{01}}\,, \tag{A.28}$$

where we used (5.23) for λ_0^R.

Using (A.17), one gets

$$\begin{aligned} I_1^R = \int_0^\infty \tau[\psi_0^R(\tau)]^2 d\tau &= -\frac{\hat{\kappa} + 2\hat{\kappa}^2 a_n^R(\kappa) - 2[a_n^R(\kappa)]^2}{3\beta[\hat{\kappa}^2 - a_n^R(\kappa)]} \\ &= \frac{2\lambda_0^R}{3iv_{01}} - \frac{\kappa}{3(\kappa^2+\lambda_0^R)}\,, \end{aligned} \tag{A.29}$$

$$\begin{aligned} I_2^R = \int_0^\infty \tau^2[\psi_0^R(\tau)]^2 d\tau &= \frac{1 + \frac{4}{3}a_n^R(\kappa)\left[\hat{\kappa} + 2\hat{\kappa}^2 a_n^R(\kappa) - 2[a_n^R(\kappa)]^2\right]}{5\beta^2[\hat{\kappa}^2 - a_n^R(\kappa)]} \\ &= \frac{1}{5(\kappa^2+\lambda_0^R)} - \frac{8[\lambda_0^R]^2}{15\,v_{01}^2} - \frac{4\kappa\lambda_0^R}{15\,iv_{01}(\kappa^2+\lambda_0^R)}\,. \end{aligned} \tag{A.30}$$

Using (5.42) and (A.8), we obtain

$$\begin{aligned} \lambda_4^{R,(n)} &= -i\frac{v_{11}^2[I_1^R]^2}{4v_{20}} - \frac{\mathfrak{c}(0)}{2}[\psi_0^R(0)]^2 + iv_{02}I_2^R \\ &= -i\frac{v_{11}^2[I_1^R]^2}{4v_{20}} - \frac{\mathfrak{c}(0)}{2}\frac{iv_{01}}{\kappa^2+\lambda_0^R} + iv_{02}I_2^R\,. \end{aligned} \tag{A.31}$$

Remark 33. *It is clear from the computation that $\lambda_4^{R,(n)}$ belongs to C^∞ in a neighborhood of 0. In particular, we recover*

$$\lambda_4^{R,(n)}(0) = \lambda_4^{N,(n)}\,. \tag{A.32}$$

Transmission case. In order to compute the above integrals for the transmission case, we note that (5.33) can be written as

$$\mathrm{Ai}'(a_n^+)\,\mathrm{Ai}'(a_n^-) = -\frac{\kappa}{2\pi|\,v_{01}|^{\frac{1}{3}}}\,, \tag{A.33}$$

while the Wronskian for Airy functions yields another relation:

$$\bar{\delta}\mathrm{Ai}'(a_n^-)\mathrm{Ai}(a_n^+) + \delta\mathrm{Ai}'(a_n^+)\mathrm{Ai}(a_n^-) = -\frac{1}{2\pi}\,, \tag{A.34}$$

where $\delta = \exp\left(\frac{\pi i}{6}\,\mathrm{sign}\ v_{01}\right)$, and $a_n^\pm = a_n^\pm(\kappa)$ are given by (5.32). From (5.31), we then obtain

$$(c_n^T)^{-2} = \frac{a_n^+\bar{\delta}}{2\pi|\,v_{01}|^{\frac{1}{3}}}\left(\bar{\delta}\mathrm{Ai}'(a_n^-)\mathrm{Ai}(a_n^+) - \delta\mathrm{Ai}'(a_n^+)\mathrm{Ai}(a_n^-)\right). \tag{A.35}$$

Using (A.17), we get

$$I_1^T = \int\limits_{-\infty}^{\infty} \tau \psi_0^T(\tau)^2 d\tau = \frac{(c_n^T)^2}{3|\,v_{01}|^{\frac{2}{3}}} \Bigg(\frac{\kappa \delta^3}{4\pi^2 |\,v_{01}|^{\frac{1}{3}}} + \frac{(a_n^+)^2 \delta^4}{\pi} \big(\bar{\delta} \mathrm{Ai}'(a_n^-) \mathrm{Ai}(a_n^+) - \delta \mathrm{Ai}'(a_n^+) \mathrm{Ai}(a_n^-) \big) \Bigg)$$

$$= (c_n^T)^2 \frac{\kappa i}{12\pi^2\, v_{01}} - \frac{2a_n^+}{3\delta |\,v_{01}|^{\frac{1}{3}}}\,, \tag{A.36}$$

$$I_2^T = \int\limits_{-\infty}^{\infty} \tau^2 \psi_0^T(\tau)^2 d\tau = \frac{(c_n^T)^2}{5|\,v_{01}|} \Bigg(\frac{\kappa a_n^+ \bar{\delta}^4}{3\pi^2 |\,v_{01}|^{\frac{1}{3}}} - \frac{8[a_n^+]^3 - 3}{6\pi} \delta^3 \big(\bar{\delta} \mathrm{Ai}'(a_n^-) \mathrm{Ai}(a_n^+) - \delta \mathrm{Ai}'(a_n^+) \mathrm{Ai}(a_n^-) \big) \Bigg)$$

$$= (c_n^T)^2 \, \frac{\kappa a_n^+ \bar{\delta}^4}{15\pi^2 |\,v_{01}|^{\frac{4}{3}}} + \frac{8[a_n^+]^3 - 3}{15 a_n^+ \delta^2 |\,v_{01}|^{\frac{2}{3}}}\,. \tag{A.37}$$

Finally, we compute the coefficient in front of $\frac{1}{2}\mathfrak{c}(0)$ in (A.9):

$$I_0^T := \int \partial_\tau [\psi_0^T(\tau)]^2 = [\psi_0^-(0)]^2 - [\psi_0^+(0)]^2 = \frac{|\,v_{01}|^{\frac{1}{3}} \exp\big(\frac{i\pi}{6} \operatorname{sign}\, v_{01}\big)}{a_n^+}\,. \tag{A.38}$$

We conclude that

$$\lambda_4^{T,(n)} = -i \frac{v_{11}^2 [I_1^T]^2}{4 v_{20}} + \mathfrak{c}(0) \frac{|\,v_{01}|^{\frac{1}{3}} \exp\big(\frac{i\pi}{6} \operatorname{sign}\, v_{01}\big)}{2a_n^+} + i\, v_{02}\, I_2^T\,. \tag{A.39}$$

Remark 34. *It is clear from the computation that $\lambda_4^{T,(n)}(\kappa)$ belongs to C^∞ in a neighborhood of* 0. *In particular, we recover*

$$\lambda_4^{T,(n)}(0) = \lambda_4^{N,(n)}\,. \tag{A.40}$$

A.3. Evaluation of the derivative $(\mu_n^T)'(0)$. The asymptotic relation (6.4) involves the derivative of $\mu_n^{\#}(\kappa)$ with respect to κ at $\kappa = 0$. In this subsection, we provide its explicit computation for the transmission case. According to (5.32), we have

$$\mu_n^T(\kappa) = -a_n^+(\kappa) = -\lambda_n^T(\kappa/|v_{01}|^{\frac{1}{3}}) \exp\left(\frac{2\pi i}{3} \operatorname{sign} v_{01} \right)\,, \tag{A.41}$$

where λ_n^T satisfies (5.33).
The derivative with respect to κ at $\kappa = 0$ reads

$$(\mu_n^T)'(0) = \left(\frac{\partial}{\partial \kappa} \mu_n^T(\kappa) \right)_{\kappa=0} = -(\lambda_n^T)'(0) \, \frac{1}{|v_{01}|^{\frac{1}{3}}} \exp\left(\frac{2\pi i}{3} \operatorname{sign} v_{01} \right)\,. \tag{A.42}$$

In turn, $(\lambda_n^T)'(0)$ can be obtained by differentiating (5.33) with respect to κ

$$2\pi \frac{(\lambda_n^T)'(0)}{|v_{01}|^{\frac{1}{3}}} \Big[e^{-i\alpha}\, \lambda_n^T(0)\, \mathrm{Ai}'(e^{-i\alpha} \lambda_n^T(0))\, \mathrm{Ai}(e^{i\alpha} \lambda_n^T(0)) + e^{i\alpha}\, \lambda_n^T(0)\, \mathrm{Ai}'(e^{i\alpha} \lambda_n^T(0))\, \mathrm{Ai}(e^{-i\alpha} \lambda_n^T(0)) \Big] = -\frac{1}{|v_{01}|^{\frac{1}{3}}}\,, \tag{A.43}$$

where we used the Airy equation: $\mathrm{Ai}''(z) = z\mathrm{Ai}(z)$, and a shortcut notation $\alpha = 2\pi/3$. At $\kappa = 0$, (5.33) admits two solutions, $\lambda_n^T(0) = e^{i\alpha}\, a_n'$ and $\lambda_n^T(0) = e^{-i\alpha}\, a_n'$, that correspond to $v_{01} < 0$ and $v_{01} > 0$, respectively.

When $v_{01} < 0$, the first term in (A.43) vanishes (as $\mathrm{Ai}'(e^{-i\alpha}\lambda_n^T(0)) = 0$), while the second term can be expressed by using the Wronskian,

$$e^{-i\alpha}\mathrm{Ai}'(e^{-i\alpha}z)\mathrm{Ai}(e^{i\alpha}z) - e^{i\alpha}\mathrm{Ai}'(e^{i\alpha}z)\mathrm{Ai}(e^{-i\alpha}z) = \frac{i}{2\pi} \qquad \forall\, z \in \mathbb{C}\,. \tag{A.44}$$

We get then

$$(\lambda_n^T)'(0) = \frac{i}{\lambda_n^T(0)} = \frac{i}{a_n' e^{i\alpha}}\,.$$

In turn, when $v_{01} > 0$, the second term in (A.43) vanishes, while the first term yields

$$(\lambda_n^T)'(0) = \frac{-i}{\lambda_n^T(0)} = \frac{-i}{a_n' e^{-i\alpha}}\,.$$

Combining these relations, we obtain

$$(\mu_n^T)'(0) = -\frac{1}{a_n'\,|v_{01}|^{\frac{1}{3}}}\,\exp\left(-\frac{\pi i}{6}\,\mathrm{sign}\, v_{01}\right). \tag{A.45}$$

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