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1 **ON SPECTRAL PROPERTIES OF THE BLOCH-TORREY**  
2 **OPERATOR IN TWO DIMENSIONS**

3 DENIS S. GREBENKOV\* AND BERNARD HELFFER†

4 **Abstract.** We investigate a two-dimensional Schrödinger operator,  $-h^2\Delta + iV(x)$ , with a  
5 purely complex potential  $iV(x)$ . A rigorous definition of this non-selfadjoint operator is provided for  
6 bounded and unbounded domains with common boundary conditions (Dirichlet, Neumann, Robin  
7 and transmission). We propose a general perturbative approach to construct its quasimodes in the  
8 semi-classical limit. An alternative WKB construction is also discussed. These approaches are local  
9 and thus valid for both bounded and unbounded domains, allowing one to compute the approximate  
10 eigenvalues to any order in the small  $h$  limit. The general results are further illustrated on the  
11 particular case of the Bloch-Torrey operator,  $-h^2\Delta + ix_1$ , for which a four-term asymptotics is  
12 explicitly computed. Its high accuracy is confirmed by a numerical computation of the eigenvalues  
13 and eigenfunctions of this operator for a disk and circular annuli. The localization of eigenfunctions  
14 near the specific boundary points is revealed. Some applications in the field of diffusion nuclear  
15 magnetic resonance are discussed.

16 **Key words.** Transmission boundary condition, spectral theory, Bloch-Torrey equation, semi-  
17 classical analysis, WKB

18 **AMS subject classifications.** 35P10, 47A10, 47A75

19 **1. Introduction.** In a previous paper [17], we have analyzed in collaboration  
20 with R. Henry one-dimensional models associated with the complex Airy operator  
21  $-\frac{d^2}{dx^2} + igx$  on the line, with  $g \in \mathbb{R}$ . We revisited the Dirichlet and Neumann real-  
22 ization of this operator in  $\mathbb{R}^+$  and the main novelty was to consider a transmission  
23 problem at 0. In higher dimensions, an extension of the complex Airy operator is the  
24 differential operator that we call the Bloch-Torrey operator or simply the BT-operator

25 
$$-D\Delta + igx_1,$$

26 where  $\Delta = \partial^2/\partial x_1^2 + \dots + \partial^2/\partial x_n^2$  is the Laplace operator in  $\mathbb{R}^n$ , and  $D$  and  $g$  are real  
27 parameters. More generally, we will study the spectral properties of some realizations  
28 of the differential Schrödinger operator

29 (1.1) 
$$\mathcal{A}_h^\# = -h^2\Delta + iV(x),$$

30 in an open set  $\Omega$ , where  $h$  is a real parameter and  $V(x)$  a real-valued potential with  
31 controlled behavior at  $\infty$ , and the superscript  $\#$  distinguishes Dirichlet (D), Neumann  
32 (N), Robin (R), or transmission (T) conditions. More precisely we discuss

- 33 1. the case of a bounded open set  $\Omega$  with Dirichlet, Neumann or Robin boundary  
34 condition;  
35 2. the case of a complement  $\Omega := \mathbb{C}\overline{\Omega}_-$  of a bounded set  $\Omega_-$  with Dirichlet,  
36 Neumann or Robin boundary condition;  
37 3. the case of two components  $\Omega_- \cup \Omega_+$ , with  $\Omega_- \subset \overline{\Omega}_- \subset \Omega$  and  $\Omega_+ = \Omega \setminus \overline{\Omega}_-$ ,  
38 with  $\Omega$  bounded and transmission conditions at the interface between  $\Omega_-$  and  
39  $\Omega_+$ ;

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\*Laboratoire de Physique de la Matière Condensée, CNRS–Ecole Polytechnique, University Paris-Saclay, 91128 Palaiseau, France ([denis.grebenkov@polytechnique.edu](mailto:denis.grebenkov@polytechnique.edu), <http://pmc.polytechnique.fr/pagesperso/dg/>).

†Laboratoire de Mathématiques Jean Leray, Université de Nantes 2 rue de la Houssinière 44322 Nantes, France, and Laboratoire de Mathématiques, Université Paris-Sud, CNRS, Univ. Paris Saclay, France ([bernard.helffer@math.u-psud.fr](mailto:bernard.helffer@math.u-psud.fr)).

- 40 4. the case of two components  $\Omega_- \cup \mathbb{C}\overline{\Omega_-}$ , with  $\Omega_-$  bounded and transmission  
 41 conditions at the boundary;  
 42 5. the case of two unbounded components  $\Omega_-$  and  $\Omega_+$  separated by a hypersur-  
 43 face with transmission conditions.

44 In all cases, we assume that the boundary is  $C^\infty$  to avoid technical difficulties related  
 45 to irregular boundaries (see [18]). Roughly speaking (see the next section for a precise  
 46 definition), the state  $u$  (in the first two items) or the pair  $(u_-, u_+)$  in the last items  
 47 should satisfy some boundary or transmission condition at the interface. In this paper,  
 48 we consider the following situations:

- 49 • the Dirichlet condition:  $u|_{\partial\Omega} = 0$ ;
- 50 • the Neumann condition:  $\partial_\nu u|_{\partial\Omega} = 0$ , where  $\partial_\nu = \nu \cdot \nabla$ , with  $\nu$  being the  
 51 outwards pointing normal;
- 52 • the Robin condition:  $h^2 \partial_\nu u|_{\partial\Omega} = -\mathcal{K}u|_{\partial\Omega}$ , where  $\mathcal{K} \geq 0$  denotes the Robin  
 53 parameter;
- the transmission condition:

$$h^2 \partial_\nu u_+|_{\partial\Omega_-} = h^2 \partial_\nu u_-|_{\partial\Omega_-} = \mathcal{K}(u_+|_{\partial\Omega_-} - u_-|_{\partial\Omega_-}),$$

54 where  $\mathcal{K} \geq 0$  denotes the transmission parameter, and the normal  $\nu$  is directed  
 55 outwards  $\Omega_-$ .

56 From now on  $\Omega^\#$  denotes  $\Omega$  if  $\# \in \{D, N, R\}$  and  $\Omega_-$  if  $\# = T$ .  $L^2_\#$  will denote  
 57  $L^2(\Omega)$  if  $\# \in \{D, N, R\}$  and  $L^2(\Omega_-) \times L^2(\Omega_+)$  if  $\# = T$ .

58

59 In [17], we have analyzed in detail various realizations of the complex Airy (or  
 60 Bloch-Torrey) operator  $A_0^\# := -\frac{d^2}{dx^2} + i\tau$  in the four cases corresponding to Dirichlet,  
 61 Neumann, and Robin on the half-line  $\mathbb{R}^+$  or for the transmission problem on the whole  
 62 line  $\mathbb{R}$  (in what follows,  $\mathbb{R}^\#$  will denote  $\mathbb{R}^+$  if  $\# \in \{D, N, R\}$  and  $\mathbb{R}$  if  $\# = T$ ). The  
 63 boundary conditions read respectively:

- 64 •  $u(0) = 0$ ;
- 65 •  $u'(0) = 0$ ;
- 66 •  $u'(0) = \kappa u(0)$ ;
- 67 •  $u'_-(0) = u'_+(0) = \kappa(u_+(0) - u_-(0))$

68 (with  $\kappa \geq 0$  in the last items). For all these cases, we have proven the existence of a  
 69 discrete spectrum and the completeness of the corresponding generalized eigenfunc-  
 70 tions. Moreover, there is no Jordan block (for the fourth case, this statement was  
 71 proven only for  $\kappa$  small enough).

72

73 In this article, we start the analysis of the spectral properties of the BT operator  
 74 in dimensions 2 or higher that are relevant for applications in superconductivity theory  
 75 [2, 5, 6, 7], in fluid dynamics [30], in control theory [10], and in diffusion magnetic  
 76 resonance imaging [12, 16] (and references therein). We will mainly focus on

- 77 • definition of the operator,
- 78 • construction of approximate eigenvalues in some asymptotic regimes,
- 79 • localization of quasimode states near certain boundary points,
- 80 • numerical simulations.

81 In particular, we will discuss the semi-classical asymptotics  $h \rightarrow 0$ , the large domain  
 82 limit, the asymptotics when  $g \rightarrow 0$  or  $+\infty$ , the asymptotics when the transmission or  
 83 Robin parameter tends to 0. Some other important questions remain unsolved like  
 84 the existence of eigenvalues close to the approximate eigenvalues (a problem which is  
 85 only solved in particular situations). We hope to contribute to this point in the future.

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When  $g = 0$ , the BT-operator is reduced to the Laplace operator for which the answers are well known. In particular, the spectrum is discrete in the case of bounded domains and equals  $[0, +\infty)$  when one or both components are unbounded. In the case  $g \neq 0$ , we show that if there is at least one boundary point at which the normal vector to the boundary is parallel to the coordinate  $x_1$ , then there exist approximate eigenvalues of the BT-operator *suggesting* the existence of eigenvalues while the associated eigenfunctions are localized near this point. This localization property has been already discussed in physics literature for bounded domains [35], for which the existence of eigenvalues is trivial. Since our asymptotic constructions are local and thus hold for unbounded domains, the localization behavior can be conjectured for exterior problems involving the BT-operator.

Some of these questions have been already analyzed by Y. Almog (see [2] and references therein for earlier contributions), R. Henry [25, 26] and Almog-Henry [8] but they were mainly devoted to the case of a Dirichlet realization in bounded domains in  $\mathbb{R}^2$  or particular unbounded domains like  $\mathbb{R}^2$  and  $\mathbb{R}_+^2$ , these two last cases playing an important role in the local analysis of the global problem.

Different realizations of the operator  $\mathcal{A}_h$  in  $\Omega$  are denoted by  $\mathcal{A}_h^D$ ,  $\mathcal{A}_h^N$ ,  $\mathcal{A}_h^R$  and  $\mathcal{A}_h^T$ . These realizations will be properly defined in Section 2 under the condition that, when  $\Omega$  is unbounded, there exists  $C > 0$  such that

$$(1.2) \quad |\nabla V(x)| \leq C\sqrt{1 + V(x)^2}.$$

Our main construction is local and summarized in the following

THEOREM 1. *Let  $\Omega \subset \mathbb{R}^2$  as above,  $V \in \mathcal{C}^\infty(\overline{\Omega}; \mathbb{R})$  and  $x^0 \in \partial\Omega^\#$  such that<sup>1</sup>*

$$(1.3) \quad \nabla V(x^0) \neq 0, \quad \nabla V(x^0) \wedge \nu(x^0) = 0,$$

where  $\nu(x^0)$  denotes the outward normal on  $\partial\Omega$  at  $x^0$ .

Let us also assume that, in the local curvilinear coordinates, the second derivative of the restriction of  $V$  to the boundary at  $x^0$  (denoted as  $2v_{20}$ ) satisfies

$$(1.4) \quad v_{20} \neq 0.$$

For the Robin and transmission cases, we also assume that for some  $\kappa > 0$

$$(1.4) \quad \mathcal{K} = h^{\frac{4}{3}}\kappa.$$

If  $\mu_0^\#$  is a simple eigenvalue of the realization “#” of the complex Airy operator  $-\frac{d^2}{dx^2} + ix$  in  $L_\#^2$ , and  $\mu_2$  is an eigenvalue of the Davies operator  $-\frac{d^2}{dy^2} + iy^2$  on  $L^2(\mathbb{R})$ , then there exists an approximate pair  $(\lambda_h^\#, u_h^\#)$  with  $u_h^\#$  in the domain of  $\mathcal{A}_h^\#$ , such that

$$(1.5) \quad \lambda_h^\# = iV(x^0) + h^{\frac{2}{3}} \sum_{j \in \mathbb{N}} \lambda_{2j}^\# h^{\frac{j}{3}} + \mathcal{O}(h^\infty),$$

121

$$(1.6) \quad (\mathcal{A}_h^\# - \lambda_h^\#) u_h^\# = \mathcal{O}(h^\infty) \text{ in } L_\#^2(\Omega), \quad \|u_h^\#\|_{L^2} \sim 1,$$

<sup>1</sup> As noticed in [8], a point satisfying the second condition in (1.3) always exists when  $\partial\Omega^\#$  is bounded.

123 *where*

124 (1.7)  $\lambda_0^\# = \mu_0^\# |v_{01}|^{\frac{2}{3}} \exp\left(i\frac{\pi}{3}\text{sign } v_{01}\right), \quad \lambda_2 = \mu_2 |v_{20}|^{\frac{1}{2}} \exp\left(i\frac{\pi}{4}\text{sign } v_{20}\right),$

125 *with*  $v_{01} := \nu \cdot \nabla V(x^0)$ .

126 In addition, we will compute  $\lambda_4^\#$  explicitly (see the Appendix) in the four types of  
 127 boundary conditions and also describe an alternative WKB construction to have a  
 128 better understanding of the structure of the presumably corresponding eigenfunctions.  
 129 We will also discuss a physically interesting case when  $\kappa$  in (1.4) depends on  $h$  and  
 130 tends to 0.

131 The proof of this theorem provides a general scheme for quasimode construction in  
 132 an arbitrary planar domain with smooth boundary  $\partial\Omega$ . In particular, this construction  
 133 allow us to retrieve and further generalize the asymptotic expansion of eigenvalues  
 134 obtained by de Swiet and Sen for the Bloch-Torrey operator in the case of a disk [35].  
 135 The generalization is applicable for any smooth boundary, with Neumann, Dirichlet,  
 136 Robin, or transmission boundary condition. Moreover, since the analysis is local, the  
 137 construction is applicable to both bounded and unbounded components.

138 The paper is organized as follows. In Sec. 2, we provide rigorous definitions  
 139 and basic properties of the BT-operator in bounded and unbounded domains, with  
 140 Dirichlet, Neumann, Robin, and Transmission conditions. Section 3 recalls former  
 141 semi-classical results for a general operator  $-h^2\Delta + iV(x)$ . In Sec. 4, we provide pre-  
 142 liminaries for semi-classical quasimode constructions in the two-dimensional case. The  
 143 construction scheme is detailed in Sec. 5. In particular, the four-terms asymptotics of  
 144 the approximate eigenvalues is obtained and we prove the main theorem. In Sec. 6 we  
 145 consider other scaling regimes for the Robin or transmission parameter. In Sec. 7 we  
 146 propose an alternative construction for the first approximate eigenvalue using WKB  
 147 quasi-mode states. In Sec. 8, we illustrate general results for simple domains such as  
 148 disk and annulus. Sec. 9 describes numerical results in order to check the accuracy  
 149 of the derived four-terms asymptotics of eigenvalues of the BT-operator in simple do-  
 150 mains such as a disk, an annulus, and the union of disk and annulus with transmission  
 151 boundary condition. We also illustrate the localization of eigenfunctions near circular  
 152 boundaries of these domains. Since a direct numerical computation for unbounded  
 153 domains (e.g., an exterior of the disk) was not possible, we approach this problem by  
 154 considering an annulus with a fixed inner circle and a moving away outer circle. We  
 155 check that the localization of some eigenfunctions near the inner circle makes them  
 156 independent of the outer circle. We therefore conjecture that the BT-operator has  
 157 some discrete spectrum for the exterior of the disk. More generally, this property is  
 158 conjectured to hold for any domain in  $\mathbb{R}^n$  (bounded or not) with smooth boundary  
 159 which has points whose normal is parallel to the gradient direction. Finally, we briefly  
 160 discuss in Sec. 10 an application of the obtained results in the field of diffusion nuclear  
 161 magnetic resonance.

162 **Acknowledgments.**

163 We thank Raphael Henry who collaborated with us in [17] and in the preliminary  
 164 discussions for the present paper. The second author would also like to thank Yaniv  
 165 Almog and Didier Robert for useful discussions.

166 **2. Definition of the various realizations of the Bloch-Torrey operator.**

167 **2.1. The case of a bounded open set  $\Omega$ .** This is the simplest case. For the  
 168 analysis of the Dirichlet (resp. Neumann) realization  $\mathcal{A}_h^D$  (resp.  $\mathcal{A}_h^N$ ) of the BT-  
 169 operator, the term  $V(x)$  is simply a bounded non self-adjoint perturbation of the

170 Dirichlet (resp. Neumann) Laplacian.

171 We have for three boundary conditions:

- 172 • For the Neumann case, the form domain  $\mathcal{V}$  is  $H^1(\Omega)$  and (if  $\Omega$  is regular) the  
 173 domain of the operator is  $\{u \in H^2(\Omega), \partial_\nu u|_{\partial\Omega} = 0\}$ . The quadratic form  
 174 reads

$$175 \quad (2.1) \quad \mathcal{V} \ni u \mapsto q_V(u) := h^2 \|\nabla u\|_\Omega^2 + i \int_\Omega V(x) |u(x)|^2 dx.$$

- 176 • For the Dirichlet case, the form domain is  $H_0^1(\Omega)$  and (if  $\Omega$  is regular) the  
 177 domain of the operator is  $H^2(\Omega) \cap H_0^1(\Omega)$ . The quadratic form is given by  
 178 (2.1).

- 179 • For the Robin case (which is a generalization of the Neumann case), the form  
 180 domain is  $H^1(\Omega)$  and (if  $\Omega$  is regular) the domain of the operator  $\mathcal{A}_h^R$  is  
 181  $\{u \in H^2(\Omega), -h^2 \partial_\nu u|_{\partial\Omega} = \mathcal{K}u|_{\partial\Omega}\}$ , where  $\mathcal{K}$  denotes the Robin coefficient,  
 182 and  $\nu$  is pointing outwards. The quadratic form reads

$$183 \quad (2.2) \quad u \mapsto q_V(u) := h^2 \|\nabla u\|_\Omega^2 + i \int_\Omega V(x) |u(x)|^2 dx + \mathcal{K} \int_{\partial\Omega} |u|^2 ds.$$

184 The Neumann case is retrieved for  $\mathcal{K} = 0$ .

185 For bounded domains, there are standard theorems, coming back to Agmon [1], per-  
 186 mitting to prove the non-emptiness of the spectrum and moreover the completeness  
 187 of the “generalized” eigenfunctions<sup>2</sup>. In the case  $V(x) = gx_1$  (here we can think of  
 188  $g \in \mathbb{C}$ ), the limit  $g \rightarrow 0$  can be treated by regular perturbation theory. In particular,  
 189 Kato’s theory [29] can be applied, the spectrum being close (modulo  $\mathcal{O}(g)$ ) to the  
 190 real axis. It is interesting to determine the variation of the lowest real part of an  
 191 eigenvalue.

192 For the Dirichlet problem, the Feynman-Hellmann formula gives the coefficient in  
 193 front of  $g$  as  $i \int_\Omega x_1 |u_0(x)|^2 dx$ , where  $u_0$  is the first  $L^2(\Omega)$ -normalized eigenfunction  
 194 of the Dirichlet Laplacian. In fact, using the standard Kato’s procedure we can look  
 195 for an approximate eigenpair  $(\lambda, u)$  in the form:

$$196 \quad (2.3) \quad u = u_0 + i g u_1 + g^2 u_2 + \dots$$

197 and

$$198 \quad (2.4) \quad \lambda = \lambda_0 + i g \lambda_1 + g^2 \lambda_2 + \dots$$

199 Developing in powers of  $g$ , we get for the coefficient in front of  $g$ :

$$200 \quad (2.5) \quad (-\Delta - \lambda_0)u_1 = -x_1 u_0 + \lambda_1 u_0,$$

201 and  $\lambda_1$  is chosen in order to solve (2.5)

$$202 \quad (2.6) \quad \lambda_1 = \int_\Omega x_1 |u_0(x)|^2 dx.$$

203 We then take

$$204 \quad (2.7) \quad u_1 = -(-\Delta - \lambda_0)^{(-1, reg)} ((x_1 - \lambda_1)u_0),$$

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<sup>2</sup> By this we mean elements in the kernel of  $(\mathcal{A}_h^\# - \lambda)^k$  for some  $k \geq 1$ .

205 where  $(-\Delta - \lambda_0)^{(-1, reg)}$  is the regularized resolvent, defined on the vector space  
 206 generated by  $u_0$  as

$$207 \quad (-\Delta - \lambda_0)^{(-1, reg)} u_0 = 0,$$

208 and as the resolvent on the orthogonal space to  $u_0$ .  
 209 To look at the coefficient in front of  $g^2$ , we write

$$210 \quad (2.8) \quad (-\Delta - \lambda_0) u_2 = (x_1 - \lambda_1) u_1 + \lambda_2 u_0,$$

211 and get

$$212 \quad \lambda_2 = - \int_{\Omega} (x_1 - \lambda_1) u_1(x) u_0(x) dx,$$

213 from which

$$214 \quad \lambda_2 = \langle (-\Delta - \lambda_0)^{(-1, reg)}((x_1 - \lambda_0)u_0) | ((x_1 - \lambda_0)u_0) \rangle_{L^2(\Omega)} > 0.$$

215 The effect of the perturbation is thus to shift the real part of the “first” eigenvalue  
 216 on the right.

217

218 The limit  $g \rightarrow +\infty$  for a fixed domain, or the limit of increasing domains (i.e.  
 219 the domain obtained by dilation by a factor  $R \rightarrow +\infty$ ) for a fixed  $g$  can be reduced  
 220 by rescaling to a semi-classical limit  $h \rightarrow 0$  of the operator  $\mathcal{A}_h$  with a fixed potential  
 221  $V(x)$ . In this way, the BT-operator appears as a particular case (with  $V(x) = x_1$ )  
 222 of a more general problem. We can mention (and will discuss) several recent pa-  
 223 pers, mainly devoted to the Dirichlet case, including: Almgren [2], Henry [25] (Chapter  
 224 4), Beauchard-Helffer-Henry-Robbiano [10] (analysis of the 1D problem), Henry [26],  
 225 Almgren-Henry [8] and in the physics literature [35, 12] (and references therein).

226 **2.2. The case of a bounded set in  $\mathbb{R}^n$  and its complementary set with**  
 227 **transmission condition at the boundary.** We consider  $\Omega_- \cup \mathbb{C}\overline{\Omega_-}$ , with  $\Omega_-$   
 228 bounded in  $\mathbb{R}^n$  and  $\partial\Omega_-$  connected. In this case the definition of the operator is  
 229 similar to what was done for the one-dimensional case in [17]. However, we start with  
 230 a simpler case when  $\Omega_- \subset \overline{\Omega_-} \subset \Omega$  with  $\Omega$  bounded and  $\Omega_+ = \Omega \setminus \overline{\Omega_-}$  (with Neumann  
 231 boundary condition imposed on the exterior boundary  $\partial\Omega$ ). After that, we explain  
 232 how to treat the unbounded case with  $\Omega = \mathbb{R}^n$  and  $\Omega_+ = \mathbb{C}\overline{\Omega_-}$ .

233 **2.2.1. Transmission property in the bounded case.** To treat the difficulties  
 234 one by one, we start with the situation when  $\Omega_- \subset \overline{\Omega_-} \subset \Omega$ ,  $\Omega_+ := \Omega \setminus \overline{\Omega_-}$ , and  $\Omega$   
 235 bounded and connected (e.g., a disk inside a larger disk).  
 236 We first introduce the variational problem, with the Hilbert space

$$237 \quad \mathcal{H} = L^2(\Omega_-) \times L^2(\Omega_+)$$

238 and the form domain

$$239 \quad \mathcal{V} := H^1(\Omega_-) \times H^1(\Omega_+).$$

240 The quadratic form reads on  $\mathcal{V}$

(2.9)

$$241 \quad u = (u_-, u_+) \mapsto q_V(u) := h^2 \|\nabla u_-\|_{\Omega_-}^2 + h^2 \|\nabla u_+\|_{\Omega_+}^2 + \mathcal{K} \|(u_- - u_+)\|_{L^2(\partial\Omega_-)}^2 \\ + i \int_{\Omega_-} V(x) |u_-(x)|^2 dx + i \int_{\Omega_+} V(x) |u_+(x)|^2 dx ,$$

242 where  $\mathcal{K}$  is a positive parameter of the transmission problem, and  $h > 0$  is a semi-  
243 classical parameter whose role will be explained later and which can be thought of as  
244 equal to one in this section. The dependence of  $\mathcal{K}$  on  $h > 0$  will be discussed later.  
245 We denote by  $\mathbf{a}_V$  the associated sesquilinear form:

$$246 \quad \mathbf{a}_V(u, u) = q_V(u) .$$

247 The potential  $V(x)$  is assumed to be real (and we are particularly interested in the  
248 example  $V(x) = gx_1$ ). In this case, one gets continuity and coercivity of the associ-  
249 ated sesquilinear form on  $\mathcal{V}$ . This is true for any  $\mathcal{K}$  without assumption on its sign.  
250 The trace of  $u_-$  and  $u_+$  on  $\partial\Omega_-$  is indeed well defined for  $(u_-, u_+) \in \mathcal{V}$ .

251

252 Applying Lax-Milgram's theorem, we first get that  $(u_-, u_+)$  should satisfy  $\Delta u_- \in$   
253  $L^2(\Omega_-)$  and  $\Delta u_+ \in L^2(\Omega_+)$ . Together with  $(u_-, u_+) \in \mathcal{V}$  this permits to define the  
254 Neumann condition (via the Green formula) for both  $u_-$  and  $u_+$  in  $H^{-\frac{1}{2}}(\partial\Omega_-)$ , and  
255 in addition for  $u_+$  in  $H^{-\frac{1}{2}}(\partial\Omega)$ . Indeed, to define  $\partial_\nu u_-$  as a linear form on  $H^{\frac{1}{2}}(\partial\Omega_-)$ ,  
256 we use that for any  $v \in H^1(\Omega_-)$ ,

$$257 \quad (2.10) \quad - \int_{\Omega_-} \Delta u_- v dx = \int_{\Omega_-} \nabla u_- \cdot \nabla v dx + \int_{\partial\Omega_-} \partial_\nu u_- v d\sigma ,$$

258 and the existence of a continuous right inverse for the trace from  $H^{\frac{1}{2}}(\partial\Omega_-)$  into  
259  $H^1(\Omega_-)$ . Here the normal  $\nu$  is oriented outwards  $\Omega_-$  and when  $u_-$  is more regular  
260 ( $u_- \in H^2(\Omega_-)$ ), we have  $\partial_\nu u_- = \nu \cdot \nabla u_-$ . In a second step we get the Neumann  
261 condition for  $u_+$  on  $\partial\Omega$ ,

$$262 \quad (2.11) \quad \partial_\nu u_+ = 0 \text{ on } \partial\Omega ,$$

263 and the transmission condition on  $\partial\Omega_-$

$$264 \quad (2.12) \quad \begin{aligned} \partial_\nu u_- &= \partial_\nu u_+ \\ h^2 \partial_\nu u_- &= \mathcal{K} (u_+ - u_-) \end{aligned} \quad \text{on } \partial\Omega_- ,$$

265 which is satisfied in  $H^{-\frac{1}{2}}(\partial\Omega_-)$ . We keep here the previous convention about the  
266 outwards direction of  $\nu$  on  $\partial\Omega_-$ .

267 Finally, we observe that the first traces of  $u_-$  and  $u_+$  on  $\partial\Omega_-$  belong to  $H^{\frac{1}{2}}(\partial\Omega_-)$ .  
268 Hence by (2.12), the second traces of  $u_-$  and  $u_+$  are in  $H^{\frac{1}{2}}(\partial\Omega_-)$ . But now the  
269 regularity of the Neumann problem in  $\Omega_-$  and  $\Omega_+$  implies that

$$270 \quad (u_-, u_+) \in H^2(\Omega_-) \times H^2(\Omega_+) .$$

271 Here we have assumed that all the boundaries are regular.

272 **REMARK 2.** *One can actually consider a more general problem in which the two*  
 273 *diffusion coefficients  $D_-$  and  $D_+$  in  $\Omega_-$  and  $\Omega_+$  are different. The transmission*  
 274 *condition reads*

$$275 \quad D_- \partial_\nu u_- = D_+ \partial_\nu u_+ = \mathcal{K}(u_+ - u_-) \quad \text{on } \partial\Omega_-.$$

276 *If we take  $D_- = D_+ = D = h^2$ , we recover the preceding case. In the limit  $D_+ \rightarrow \infty$ ,*  
 277 *we can consider the particular case where  $u_+$  is identically 0 and we recover the Robin*  
 278 *condition on the boundary  $\partial\Omega_-$  of the domain  $\Omega_-$ .*

279 **2.2.2. The unbounded case with bounded transmission boundary.** In  
 280 the case  $\Omega_+ = \mathbb{C}\overline{\Omega_-}$  (i.e.,  $\Omega = \mathbb{R}^n$ ), we have to treat the transmission problem through  
 281  $\partial\Omega_-$  with the operator  $-h^2\Delta + iV(x)$  on  $L^2(\Omega_-) \times L^2(\Omega_+)$ . Nothing changes at the  
 282 level of the transmission property because  $\partial\Omega_-$  is bounded. However, the variational  
 283 space has to be changed in order to get the continuity of the sesquilinear form. Here  
 284 we have to account for the unboundedness of  $V$  in  $\Omega_+$ . For this purpose, we introduce

$$285 \quad (2.13) \quad \mathcal{V} := \{(u_-, u_+) \in \mathcal{H}, |V|^{\frac{1}{2}} u_+ \in L^2(\Omega_+)\}.$$

286 If  $V$  has constant sign outside a compact, there is no problem to get the coercivity  
 287 by looking separately at  $\operatorname{Re} \mathfrak{a}_V(u, u)$  and  $\operatorname{Im} \mathfrak{a}_V(u, u)$ . When  $V$  does not have this  
 288 property (as it is in the case  $V(x) = x_1$ ), one cannot apply Lax-Milgram's theorem  
 289 in its standard form. We will instead use the generalized Lax-Milgram Theorem as  
 290 presented in [4] (see also [17]).

291 **THEOREM 3.** *Let  $\mathcal{V}$  denote a Hilbert space and let  $\mathfrak{a}$  be a continuous sesquilinear*  
 292 *form on  $\mathcal{V} \times \mathcal{V}$ . If  $\mathfrak{a}$  satisfies, for some  $\Phi_1, \Phi_2 \in \mathcal{L}(\mathcal{V})$ , and some  $\alpha > 0$ ,*

$$293 \quad (2.14) \quad |\mathfrak{a}(u, u)| + |\mathfrak{a}(u, \Phi_1(u))| \geq \alpha \|u\|_{\mathcal{V}}^2, \quad \forall u \in \mathcal{V},$$

294

$$295 \quad (2.15) \quad |\mathfrak{a}(u, u)| + |\mathfrak{a}(\Phi_2(u), u)| \geq \alpha \|u\|_{\mathcal{V}}^2, \quad \forall u \in \mathcal{V},$$

296 *then  $A \in \mathcal{L}(\mathcal{V})$  defined by*

$$297 \quad (2.16) \quad \mathfrak{a}(u, v) = \langle Au, v \rangle_{\mathcal{V}}, \quad \forall u \in \mathcal{V}, \quad \forall v \in \mathcal{V},$$

298 *is a continuous isomorphism from  $\mathcal{V}$  onto  $\mathcal{V}$ .*

299 We now consider two Hilbert spaces  $\mathcal{V}$  and  $\mathcal{H}$  such that  $\mathcal{V} \subset \mathcal{H}$  (with continuous  
 300 injection and dense image). Let  $\mathcal{A}$  be defined by

$$301 \quad (2.17) \quad D(\mathcal{A}) = \{u \in \mathcal{V} \mid v \mapsto \mathfrak{a}(u, v) \text{ is continuous on } \mathcal{V} \text{ in the norm of } \mathcal{H}\}$$

302 and

$$303 \quad (2.18) \quad \mathfrak{a}(u, v) = \langle Au, v \rangle_{\mathcal{H}} \quad \forall u \in D(\mathcal{A}) \text{ and } \forall v \in \mathcal{V}.$$

304 Then we have

305 **THEOREM 4.** *Let  $\mathfrak{a}$  be a continuous sesquilinear form satisfying (2.14) and (2.15).*  
 306 *Assume further that  $\Phi_1$  and  $\Phi_2$  extend into continuous linear maps in  $\mathcal{L}(\mathcal{H})$ . Let  $\mathcal{A}$*   
 307 *be defined by (2.17)-(2.18). Then*

- 308 1.  $\mathcal{A}$  is bijective from  $D(\mathcal{A})$  onto  $\mathcal{H}$ .
- 309 2.  $D(\mathcal{A})$  is dense in both  $\mathcal{V}$  and  $\mathcal{H}$ .

310 3.  $\mathcal{A}$  is closed.

311 EXAMPLE 5. For  $V(x) = x_1$ , we can use on  $\mathcal{V}$  the multiplier

$$312 \quad \Phi_1(u_-, u_+) = \left( u_-, \frac{x_1}{\sqrt{1+x_1^2}} u_+ \right).$$

313 We first observe that, for some  $C > 0$ ,

$$314 \quad \operatorname{Re} \mathbf{a}_V(u, u) \geq \frac{1}{C} (\|\nabla u_-\|^2 + \|\nabla u_+\|^2) - C (\|u_-\|^2 + \|u_+\|^2).$$

315 To obtain the generalized coercivity, we now look at  $\operatorname{Im} \mathbf{a}_V(u, \Phi_1(u))$  and get, for some  
316  $\hat{C} > 0$ ,

$$317 \quad \operatorname{Im} \mathbf{a}_V(u, \phi_1(u)) \geq \int_{\Omega_+} |V(x)| |u_+|^2 dx - \hat{C} (\|u\|^2 + \|\nabla u\|^2).$$

318 Note that this works (see [4]) for general potentials  $V(x)$  satisfying (1.2).

319 Note also that the domain of the operator  $\mathcal{A}^T$  associated with the sesquilinear  
320 form is described as follows

$$321 \quad (2.19) \quad D := \{u \in \mathcal{V}, (-h^2\Delta + iV)u_- \in L^2(\Omega_-), (-h^2\Delta + iV)u_+ \in L^2(\Omega_+) \\ \text{and transmission condition on } \partial\Omega_-\}.$$

322 It is clear that this implies  $u_- \in H^2(\Omega_-)$ . The question of showing that  $u_+ \in H^2(\Omega_+)$   
323 is *a priori* unclear. By using the local regularity, we can show that for any  $\chi$  in  
324  $C_0^\infty(\overline{\Omega_+})$ ,

$$325 \quad (-h^2\Delta + iV)(\chi u) \in L^2(\mathbb{R}^n),$$

326 and consequently  $\chi u \in H^2(\mathbb{R}^n)$ .

327 In order to show that  $u_+ \in H^2(\Omega_+)$ , one needs to introduce other techniques and  
328 additional assumptions. For example, using the pseudodifferential calculus, it is possi-  
329 ble to prove (see [32]), that  $u_+ \in H^2(\Omega_+)$  and  $Vu_+ \in L^2(\Omega_+)$  under the stronger  
330 condition that for any  $\alpha \in \mathbb{N}^n$ , there exists  $C_\alpha$  such that

$$331 \quad (2.20) \quad |D_x^\alpha V(x)| \leq C_\alpha \sqrt{1+V(x)^2}, \quad \forall x \in \mathbb{R}^n.$$

332 REMARK 6 (No compactness of the resolvent). *There is no compact resolvent in*  
333 *this problem. We note indeed that the pairs  $(u_-, u_+)$  with  $u_- = 0$  and  $u_+ \in C_0^\infty(\Omega_+)$*   
334 *belong to the domain of the operator. It is easy to construct a sequence of  $L^2$  normal-*  
335 *ized  $u_+^{(k)}$  in  $C_0^\infty(\Omega_+)$  which is bounded in  $H^2(\Omega_+)$ , with support in  $(-R, +R) \times \mathbb{R}^{n-1}$ ,*  
336 *and weakly convergent to 0 in  $L^2(\Omega_+)$ . This implies that the resolvent cannot be*  
337 *compact.*

338 REMARK 7. *The noncompactness of the resolvent does not exclude the existence*  
339 *of eigenvalues. Actually, when  $\mathcal{K} = 0$ , the spectral problem is decoupled into two inde-*  
340 *pendent problems: the Neumann problem in  $\Omega_-$  which gives eigenvalues (the potential*  
341  *$ix_1$  in  $\Omega_-$  is just a bounded perturbation, as discussed in Sec. 2.2.1) and the Neu-*  
342 *mann problem for the exterior problem in  $\Omega_+$  with  $-\Delta + igx_1$  for which the question*  
343 *of existence of eigenvalues is more subtle if we think of the model of the half-space*  
344 *analyzed in Almog [2] or [25]. We will see that in the semi-classical limit (or equiv-*  
345 *alently  $g \rightarrow +\infty$ ) the points of  $\partial\Omega_-$  at which the normal vector to  $\partial\Omega_-$  is parallel to*  
346  *$(1, 0, \dots, 0)$ , play a particular role.*

347 **2.2.3. The case of two unbounded components in  $\mathbb{R}^2$  separated by a**  
 348 **curve.** The case of two half-spaces is of course the simplest because we can come back  
 349 to the one-dimensional problem using the partial Fourier transform. The analysis of  
 350 the resolvent should however be detailed (see Henry [25] who treats the model of the  
 351 half-space for the BT operator with Neumann or Dirichlet conditions). In fact, we  
 352 consider the quadratic form

$$\begin{aligned}
 q(u) &= h^2 \int_{x_1 < 0} |\nabla u_-(x)|^2 dx + i \int_{x_1 < 0} \ell(x) |u_-(x)|^2 dx \\
 &+ h^2 \int_{x_1 > 0} |\nabla u_+(x)|^2 dx + i \int_{x_1 > 0} \ell(x) |u_+(x)|^2 dx \\
 &+ \mathcal{K} \int |u_-(0, x_2) - u_+(0, x_2)|^2 dx_2,
 \end{aligned}$$

354 where  $x \mapsto \ell(x)$  is a nonzero linear form on  $\mathbb{R}^2$ :

$$355 \quad \ell(x) = \alpha x_1 + \beta x_2.$$

356 Here, we can also apply the general Lax-Milgram theorem in order to define a closed  
 357 operator associated to this quadratic form. The extension to a more general curve  
 358 should be possible under the condition that the curve admits two asymptotes at in-  
 359 finity.

360  
 361 In this section, we have described how to associate to a given sesquilinear form  $\mathfrak{a}$   
 362 defined on a form domain  $\mathcal{V}$  an unbounded closed operator  $\mathcal{A}$  in some Hilbert space  $\mathcal{H}$ .  
 363 We will add the superscript  $\#$  with  $\# \in \{D, N, R, T\}$  in order to treat simultaneously  
 364 the different cases. The space  $\mathcal{H}^\#$  will be  $L^2(\Omega)$  when  $\# \in \{D, N, R\}$  and will be  
 365  $L^2(\Omega_-) \times L^2(\Omega_+)$  in the case with transmission  $\# = T$ .  $\mathcal{V}^\#$  will be respectively  
 366  $H_0^1(\Omega)$ ,  $H^1(\Omega)$ ,  $H^1(\Omega)$ , and  $H^1(\Omega_-) \times H^1(\Omega_+)$ . The corresponding operators are  
 367 denoted  $\mathcal{A}_h^\#$  with  $\# \in \{D, N, R, T\}$ .

368 **3. Former semi-classical results.** In order to treat simultaneously various  
 369 problems we introduce  $\Omega^\#$  with  $\# \in \{D, N, R, T\}$  and  $\Omega^D = \Omega$ ,  $\Omega^N = \Omega$ ,  $\Omega^R = \Omega$   
 370 and  $\Omega^T = \Omega_-$ .

371 R. Henry [26] (see also [8]) looked at the Dirichlet realization of the differential oper-  
 372 ator

$$373 \quad (3.1) \quad \mathcal{A}_h^D := -h^2 \Delta + iV(x),$$

374 in a fixed bounded domain  $\Omega$ , where  $V$  is a real potential and  $h$  a semi-classical pa-  
 375 rameter that goes to 0.

376 Setting  $V(x) = x_1$ , one gets a problem considered by de Swiet and Sen [35] in the  
 377 simple case of a disk but these authors mentioned a possible extension of their com-  
 378 putations to more general cases.

379 For a bounded regular open set, R. Henry in [26] (completed by Almog-Henry [8],  
 380 see below) proved the following

381 **THEOREM 8.** *Let  $V \in C^\infty(\overline{\Omega}; \mathbb{R})$  be such that, for every  $x \in \overline{\Omega}$ ,*

$$382 \quad (3.2) \quad \nabla V(x) \neq 0.$$

383 *Then, we have*

$$384 \quad (3.3) \quad \liminf_{h \rightarrow 0} \frac{1}{h^{2/3}} \inf \{ \operatorname{Re} \sigma(\mathcal{A}_h^D) \} \geq \frac{|a_1|}{2} J_m^{2/3},$$

385 where  $\mathcal{A}_h^D$  is the operator defined by (3.1) with the Dirichlet condition,  $a_1 < 0$  is the  
 386 rightmost zero of the Airy function  $Ai$ , and

387 (3.4) 
$$J_m = \min_{x \in \partial\Omega_\perp} |\nabla V(x)|,$$

where

$$\partial\Omega_\perp = \{x \in \partial\Omega, \nabla V(x) \wedge \nu(x) = 0\}.$$

388 This result is essentially a reformulation of the result stated by Y. Almog in [2].

389 **REMARK 9.** *The theorem holds in particular when  $V(x) = x_1$  in the case of the*  
 390 *disk (two points) and in the case of an annulus (four points). Note that in this*  
 391 *application  $J_m = 1$ .*

392 A similar result can be proved for the Neumann case.

393 **REMARK 10.** *To our knowledge, the equivalent theorems in the Robin case and*  
 394 *the transmission case are open. We hope to come back to this point in a future work.*

395 A more detailed information is available in dimension 1 (see [10]) and in higher di-  
 396 mension [8] under some additional assumption on  $\partial\Omega_\perp$ . The authors in [8] prove the  
 397 existence of an approximate eigenvalue. Our main goal is to propose a more general  
 398 construction which will work in particular for the case with transmission condition.

399 **REMARK 11 (Computation of the Hessian).** *For a planar domain, let us denote*  
 400 *by  $(x_1(s), x_2(s))$  the parameterization of the boundary by the arc length  $s$  starting from*  
 401 *some point,  $t(s) = (x'_1(s), x'_2(s))$  is the normalized oriented tangent, and  $\nu(s)$  is the*  
 402 *outwards normal to the boundary at  $s$ . Now we compute at  $s = 0$  (corresponding to a*  
 403 *point  $x^0 = x(0) \in \partial\Omega_\perp^\#$ , where  $\nabla V \cdot t(0) = 0$ ),*

404 
$$\left( \frac{d^2}{ds^2} V(x_1(s), x_2(s)) \right)_{s=0} = \langle t(0) | \text{Hess} V(x_1(0), x_2(0)) | t(0) \rangle$$

$$- \mathfrak{c}(0) (\nabla V(x_1(0), x_2(0)) \cdot \nu(0)) ,$$

405 where we used  $t'(s) = -\mathfrak{c}(s)\nu(s)$ ,  $\mathfrak{c}(s)$  representing the curvature of the boundary at  
 406 the point  $x(s)$ .

407 **EXAMPLE 12.** *When  $V(x_1, x_2) = x_1$ , we get*

408 
$$\left( \frac{d^2}{ds^2} V(x_1(s), x_2(s)) \right)_{s=0} = -\mathfrak{c}(0)(e_1 \cdot \nu(0)) ,$$

409 with  $e_1 = (1, 0)$ .

410 *In the case of the disk of radius 1, we get*

411 (3.5) 
$$\left( \frac{d^2}{ds^2} V(x_1(s), x_2(s)) \right)_{s=0} (e_1 \cdot \nu(0)) = -1 ,$$

412 for  $(x_1, x_2) = (\pm 1, 0)$ .

413 Let us now introduce a stronger assumption for  $\# \in \{N, D\}$ .

414

415 **ASSUMPTION 13.** *At each point  $x$  of  $\partial\Omega_\perp^\#$ , the Hessian of  $V|_{\partial\Omega}$  is*

- 416 • *positive definite if  $\partial_\nu V < 0$ ,*
- 417 • *negative definite if  $\partial_\nu V > 0$ ,*

418 with  $\nu$  being the outwards normal and  $\partial_\nu V := \nu \cdot \nabla V$ .

419 Under this additional assumption<sup>3</sup>, the authors in [8] (Theorem 1.1) prove the  
420 equality in (3.3) by proving the existence of an eigenvalue near each previously con-  
421 structed approximate eigenvalue, and get a three-terms asymptotics.

422 **REMARK 14.** Note that this additional assumption is verified for all points of  $\partial\Omega_\perp$   
423 when  $V(x) = x_1$  and  $\Omega$  is the disk. In fact, for this model, there are two points  $(-1, 0)$   
424 and  $(1, 0)$ , and formula (3.5) gives the solution.

425 Y. Almog and R. Henry considered in [2, 26, 8] the Dirichlet case but, as noted  
426 by these authors in [8], one can similarly consider the Neumann case.

427 Without Assumption 13, there is indeed a difficulty for proving the existence of  
428 an eigenvalue close to the approximate eigenvalue. This is for example the case for  
429 the model operator

$$430 \quad -h^2 \frac{d^2}{dx^2} - h^2 \frac{d^2}{dy^2} + i(y - x^2),$$

431 on the half space. The operator is indeed not sectorial, and Lemma 4.2 in [8] is  
432 not proved in this case. The definition of the closed operator is questionable. One  
433 cannot use the technique given in a previous section because the condition (1.2) is not  
434 satisfied. The argument used by R. Henry in [25] for the analysis of the Dirichlet BT-  
435 operator in a half space  $\mathbb{R}_+^2$  (based on [31] (Theorem X.49) and [28]) can be extended  
436 to this case.

437 This problem occurs for the transmission problem in which the model could be related  
438 to

$$439 \quad -h^2 \frac{d^2}{dx^2} - h^2 \frac{d^2}{dy^2} + i(y + x^2),$$

440 on the whole space  $\mathbb{R}^2$  with transmission on  $y = 0$ . This case will not be treated in  
441 this paper.

442 **On the growth of semi-groups.** In the case of Dirichlet and Neumann realiza-  
443 tions, one can study the decay of the semi-group  $\exp(-t\mathcal{A}_h^\#)$  relying on the previous  
444 results and additional controls of the resolvent (see [25], [8]). When the domain is  
445 bounded, the potential is a bounded perturbation of self-adjoint operators. In this  
446 case, the control of the resolvent when  $\text{Im}\lambda$  tends to  $\pm\infty$  is straightforward, with  
447 the decay as  $\mathcal{O}(1/|\text{Im}\lambda|)$ . Applying the Gearhardt-Prüss theorem (see for example in  
448 [19]), the decay is

$$449 \quad \mathcal{O}_\epsilon \left( \exp(-t(1 - \epsilon) \inf_{\lambda \in \sigma(\mathcal{A}_h^\#)} \{\text{Re}\lambda\}) \right) \quad \forall \epsilon > 0,$$

450 where  $\sigma(\mathcal{A})$  denotes the spectrum of  $\mathcal{A}$ . In this case,  $\sigma(\mathcal{A}_h^\#)$  is not empty and the set  
451 of generalized eigenfunctions is complete (see [1]).

452 In the unbounded case, the situation is much more delicate. The spectrum  $\sigma(\mathcal{A}_h^\#)$   
453 can be empty and one has to control the resolvent as  $|\text{Im}\lambda| \rightarrow +\infty$ . The behavior of  
454 the associate semi-group can be super-exponential when  $\sigma(\mathcal{A}_h^\#)$  is empty. Moreover,  
455 it is not granted that  $\inf_{\lambda \in \sigma(\mathcal{A}_h^\#)} \{\text{Re}\lambda\}$  gives the decay rate of the semi-group.  
456

---

<sup>3</sup> We actually need this assumption only for the points  $x$  of  $\partial\Omega_\perp$  such that  $|\nabla V(x)| = J_m$ .

457 **4. Quasimode constructions – Preliminaries.** Let us present in more detail  
458 the situation considered in Theorem 1.

459 **4.1. Local coordinates.** Choosing the origin at a point  $x^0$  at which  $\nabla V(x_0) \wedge$   
460  $\nu(x_0) = 0$ , we replace the Cartesian coordinates  $(x_1, x_2)$  by the standard local vari-  
461 ables  $(s, \rho)$ , where  $\rho$  is the signed distance to the boundary, and  $s$  is the arc length  
462 starting from  $x^0$ . Hence

- 463 • In the case of one component,  $\rho = 0$  defines the boundary  $\partial\Omega$  and  $\Omega$  is locally  
464 defined by  $\rho > 0$ .
- 465 • In the case of two components,  $\rho = 0$  defines  $\partial\Omega_-$ , while  $\rho < 0$  and  $\rho > 0$   
466 correspond, in the neighborhood of  $\partial\Omega_-$ , respectively to  $\Omega_-$  and  $\Omega_+$ .

467 In the  $(s, \rho)$  coordinates, the operator reads

$$468 \quad (4.1) \quad \mathcal{A}_h = -h^2 a^{-1} \partial_s (a^{-1} \partial_s) - h^2 a^{-1} \partial_\rho (a \partial_\rho) + i \tilde{V}(s, \rho),$$

469 with

$$470 \quad \tilde{V}(s, \rho) := V(x_1(s, \rho), x_2(s, \rho)),$$

471 where

$$472 \quad (4.2) \quad a(s, \rho) = 1 - \mathbf{c}(s) \rho,$$

473  $\mathbf{c}(s)$  representing the curvature of the boundary at  $x(s, 0)$ .

474 For future computation, we also rewrite (4.1) as

$$475 \quad (4.3) \quad \mathcal{A}_h = -h^2 a^{-2} \partial_s^2 + h^2 a^{-3} \partial_s a \partial_s - h^2 \partial_\rho^2 - h^2 a^{-1} \partial_\rho a \partial_\rho + i \tilde{V}(s, \rho).$$

476 The boundary conditions read

- 477 • Dirichlet condition

$$478 \quad (4.4) \quad u(s, 0) = 0,$$

- 479 • Neumann condition

$$480 \quad (4.5) \quad \partial_\rho u(s, 0) = 0,$$

- 481 • Robin condition with parameter  $\mathcal{K}$

$$482 \quad (4.6) \quad h^2 \partial_\rho u(s, 0) = \mathcal{K} u(s, 0),$$

- 483 • Transmission condition with parameter  $\mathcal{K}$

$$484 \quad (4.7) \quad \begin{cases} \partial_\rho u_+(s, 0) = \partial_\rho u_-(s, 0), \\ h^2 \partial_\rho u_+(s, 0) = \mathcal{K} (u_+(s, 0) - u_-(s, 0)). \end{cases}$$

485 In the last two cases, the link between  $\mathcal{K}$  and  $h$  will be given later in (4.30).

486 We omit the tilde of  $\tilde{V}$  in what follows.

487

488 We recall that the origin of the coordinates is at a point  $x^0$  such that

$$489 \quad \nabla V(x_0) \neq 0 \quad \text{and} \quad \nabla V(x_0) \wedge \vec{\nu}(x^0) = 0.$$

490 Hence we have

$$491 \quad (4.8) \quad \frac{\partial V}{\partial s}(0, 0) = 0,$$

492 and

$$493 \quad (4.9) \quad \frac{\partial V}{\partial \rho}(0, 0) \neq 0.$$

494 We also assume in our theorem that

$$495 \quad (4.10) \quad \frac{\partial^2 V}{\partial s^2}(0, 0) \neq 0.$$

496 Hence we have the following Taylor expansion

$$497 \quad (4.11) \quad V(s, \rho) \sim \sum_{j,k} v_{jk} s^j \rho^k,$$

498 where

$$499 \quad (4.12) \quad v_{jk} = \frac{1}{j! k!} \left( \frac{\partial^{j+k}}{\partial s^j \partial \rho^k} V(s, \rho) \right)_{s=\rho=0},$$

500 with

$$501 \quad (4.13) \quad v_{00} = V(0, 0), \quad v_{10} = 0, \quad v_{01} \neq 0, \quad v_{20} \neq 0,$$

502 corresponding to the assumptions of Theorem 1.

503 **4.2. The blowing up argument.** Approximating the potential  $V$  near  $x^0$  by  
504 the first terms of its Taylor expansion  $v_{00} + v_{01}\rho + v_{20}s^2$ , a basic model reads

$$505 \quad -h^2 \frac{d^2}{ds^2} - h^2 \frac{d^2}{d\rho^2} + i(v_{01}\rho + v_{20}s^2) \quad \text{on the half space } \{\rho > 0\},$$

506 in the case when  $\# \in \{D, N, R\}$ , and on  $\mathbb{R}^2$  when  $\# = T$ , which is reduced by a  
507 natural scaling

$$508 \quad (4.14) \quad (s, \rho) = (h^{\frac{1}{2}}\sigma, h^{\frac{2}{3}}\tau)$$

509 to

$$510 \quad h \left( -\frac{d^2}{d\sigma^2} + iv_{20}\sigma^2 \right) + h^{\frac{2}{3}} \left( -\frac{d^2}{d\tau^2} + iv_{01}\tau \right),$$

511 whose definition and spectrum can be obtained by separation of variables in the four  
512 cases.

513

514 **4.2.1. Expansions.** In the new variables  $(\sigma, \tau)$  introduced in (4.14), the expansion is  
515

$$516 \quad (4.15) \quad \widehat{V}_h(\sigma, \tau) := V(h^{\frac{1}{2}}\sigma, h^{\frac{2}{3}}\tau) \sim \sum_{m \geq 0} h^{\frac{m}{6}} \left( \sum_{3k+4p=m} v_{kp} \sigma^k \tau^p \right).$$

517 In particular, the first terms are

$$518 \quad (4.16) \quad \widehat{V}_h(\sigma, \tau) = v_{00} + h^{\frac{2}{3}} v_{01} \tau + h v_{20} \sigma^2 + h^{\frac{7}{6}} v_{11} \sigma \tau + h^{\frac{8}{6}} v_{02} \tau^2 + h^{\frac{3}{2}} v_{30} \sigma^3 + \mathcal{O}(h^{\frac{5}{3}}).$$

519

520 Similarly, we consider the dilation of  $a(s, \rho)$

$$521 \quad (4.17) \quad \widehat{a}_h(\sigma, \tau) := a(h^{\frac{1}{2}} \sigma, h^{\frac{2}{3}} \tau) = 1 - h^{\frac{2}{3}} \tau \mathbf{c}(h^{\frac{1}{2}} \sigma),$$

522 which can be expanded in the form

$$523 \quad (4.18) \quad \widehat{a}_h(\sigma, \tau) \sim 1 - h^{\frac{2}{3}} \tau \left( \sum_{\ell} \frac{1}{\ell!} \mathbf{c}^{(\ell)}(0) \sigma^{\ell} h^{\frac{\ell}{2}} \right).$$

524 In the  $(\sigma, \tau)$  coordinates, we get

$$525 \quad (4.19) \quad \widehat{\mathcal{A}}_h = -h \widehat{a}_h^{-2} \partial_{\sigma}^2 + h^{\frac{3}{2}} \widehat{a}_h^{-3} (\widehat{\partial_s a})_h \partial_{\sigma} - h^{\frac{2}{3}} \partial_{\tau}^2 - h^{\frac{4}{3}} \widehat{a}_h^{-1} (\widehat{\partial_{\rho} a})_h \partial_{\tau} + i \widehat{V}_h(\sigma, \tau).$$

526 We note that

$$527 \quad (\widehat{\partial_s a})_h(\sigma, \tau) = -h^{\frac{2}{3}} \mathbf{c}'(h^{\frac{1}{2}} \sigma) \quad \text{and} \quad (\widehat{\partial_{\rho} a})_h(\sigma, \tau) = -\mathbf{c}(h^{\frac{1}{2}} \sigma).$$

528 We rewrite  $\widehat{\mathcal{A}}_h$  by expanding in powers of  $h^{\frac{1}{6}}$ :

$$529 \quad (4.20) \quad \widehat{\mathcal{A}}_h \sim i v_{00} + h^{\frac{2}{3}} \sum_{j \geq 0} h^{\frac{j}{6}} \mathcal{L}_j(\sigma, \tau, \partial_{\sigma}, \partial_{\tau}),$$

530 where the first terms are given by

$$531 \quad (4.21) \quad \begin{aligned} \mathcal{L}_0 &= -\partial_{\tau}^2 + i v_{01} \tau, \\ \mathcal{L}_1 &= 0, \\ \mathcal{L}_2 &= -\partial_{\sigma}^2 + i v_{20} \sigma^2, \\ \mathcal{L}_3 &= i v_{11} \sigma \tau, \\ \mathcal{L}_4 &= \mathbf{c}(0) \partial_{\tau} + i v_{02} \tau^2, \\ \mathcal{L}_5 &= -\mathbf{c}'(0) \partial_{\sigma} + i v_{03} \sigma^3. \end{aligned}$$

532 For any  $j \geq 0$ , each  $\mathcal{L}_j$  is a differential operator of order  $\leq 2$  with polynomial co-  
533 efficients of degree which can be controlled as a function of  $j$ . In particular these  
534 operators preserve the vector space  $\mathcal{S}(\mathbb{R}_{\sigma}) \otimes \mathcal{S}^{\#}$ . The Fréchet space  $\mathcal{S}^{\#}$  denotes  
535  $\mathcal{S}(\overline{\mathbb{R}_{+}})$  in the case when  $\# \in \{D, N, R\}$  and  $\mathcal{S}(\overline{\mathbb{R}_{-}}) \times \mathcal{S}(\overline{\mathbb{R}_{+}})$  when  $\# = T$ .

536 **4.2.2. Parity.** Note also that we have

LEMMA 15.

$$537 \quad (4.22) \quad (\mathcal{L}_j \check{f}) = (-1)^j \mathcal{L}_j \check{f},$$

538 where  $\check{f}(\tau, \sigma) = f(\tau, -\sigma)$ .

539 **Proof**

540 This is a consequence of

$$541 \quad (4.23) \quad \mathcal{L}_j(\sigma, \tau, \partial_{\sigma}, \partial_{\tau}) = (-1)^j \mathcal{L}_j(-\sigma, \tau, -\partial_{\sigma}, \partial_{\tau})$$

542 that can be seen by observing that

$$\begin{aligned}
(4.24) \quad h^{-\frac{2}{3}}(\widehat{\mathcal{A}}_h - iv_{00}) &= h^{\frac{1}{3}}\widehat{a}_h^{-2}\partial_\sigma^2 + h^{\frac{5}{6}}\widehat{a}_h^{-3}(\widehat{\partial_s a})_h \partial_\sigma - \partial_\tau^2 \\
&\quad - h^{\frac{2}{3}}\widehat{a}_h^{-1}(\widehat{\partial_\rho a})_h \partial_\tau + i h^{-\frac{2}{3}}(\widehat{V}_h(\sigma, \tau) - v_{00}) \\
&\sim \sum_{j \geq 0} h^{\frac{j}{6}} \mathcal{L}_j(\sigma, \tau, \partial_\sigma, \partial_\tau).
\end{aligned}$$

544 We will see that each term in the right hand side of (4.24) satisfies (4.22).

545 First, denoting  $\hat{h} = h^{\frac{1}{6}}$ , we can rewrite

$$(4.25) \quad \widehat{a}_h(\sigma, \tau) \sim 1 - \hat{h}^4 \tau \left( \sum_{\ell \geq 0} \frac{1}{\ell!} \mathbf{c}^{(\ell)}(0) \sigma^\ell \hat{h}^{3\ell} \right),$$

547 and expanding in powers of  $\hat{h}$ , we see that the coefficient in front of  $\hat{h}^\ell$  has the parity  
548 of  $\ell$  in  $\sigma$ . The same is true for  $\widehat{a}_h(\sigma, \tau)^{-2}$ . Hence the coefficient in front of  $h^{\frac{j}{6}}$   
549  $h^{\frac{1}{3}}\widehat{a}_h(\sigma, \tau)^{-2}\partial_\sigma^2$  satisfies (4.23).

550 We now look at  $h^{\frac{5}{6}}\widehat{a}_h^{-3}(\widehat{\partial_s a})_h$  and write

$$(4.26) \quad h^{\frac{5}{6}}(\widehat{\partial_s a})_h(\sigma, \tau) \partial_\sigma = -\hat{h}^9 \mathbf{c}'(\hat{h}^3 \sigma) \partial_\sigma.$$

552 It is clear from this formula that the second term in the right hand side of (4.24)  
553 satisfies (4.23).

554 The third term  $-\partial_\tau^2$  clearly satisfies (4.23). For the fourth term  $-h^{\frac{2}{3}}\widehat{a}_h^{-1}(\widehat{\partial_\rho a})_h \partial_\tau$ , it  
555 is enough to use the previous expansions and to observe that

$$(4.27) \quad (\widehat{\partial_\rho a})_h(\sigma, \tau) = -\mathbf{c}(\hat{h}^3 \sigma).$$

557 Finally, we consider

$$(4.28) \quad i h^{-\frac{2}{3}}(\widehat{V}_h(\sigma, \tau) - v_{00}) \sim i \sum_{m \geq 4} \hat{h}^{m-4} \left( \sum_{3k+4p=m} v_{kp} \sigma^k \tau^p \right),$$

559 and we observe that  $k$  and  $m$  should have the same parity.

560 This lemma will be useful for explaining cancellations in the expansion of the quasi-  
561 mode.

562 **4.2.3. Boundary or transmission conditions.** In these local coordinates, the  
563 boundary conditions read

- 564 • the Dirichlet condition

$$(4.26) \quad u(\sigma, 0) = 0,$$

- 565 • the Neumann condition

$$(4.27) \quad \partial_\tau u(\sigma, 0) = 0,$$

- 566 • the Robin condition

$$(4.28) \quad \partial_\tau u(\sigma, 0) = \mathcal{K} h^{-\frac{4}{3}} u(\sigma, 0),$$

- the transmission condition

$$(4.29) \quad \partial_\tau u_-(\sigma, 0) = \partial_\tau u_+(\sigma, 0), \quad \partial_\tau u_+(\sigma, 0) = \mathcal{K} h^{-\frac{4}{3}} (u_+(\sigma, 0) - u_-(\sigma, 0)).$$

Depending on the physical problem, the Robin or Transmission parameter  $\mathcal{K}$  can exhibit different scaling with  $h$ . Here we assume the scaling

$$(4.30) \quad \mathcal{K} = \kappa h^{\frac{4}{3}},$$

so that the Robin or transmission conditions in the variables  $(\sigma, \tau)$  are independent of  $h$  and read

$$(4.31) \quad \partial_\tau u(\sigma, 0) = \kappa u(\sigma, 0),$$

and

$$(4.32) \quad \partial_\tau u_-(\sigma, 0) = \partial_\tau u_+(\sigma, 0), \quad \partial_\tau u_+(\sigma, 0) = \kappa (u_+(\sigma, 0) - u_-(\sigma, 0)).$$

In Sec. 4.3, we justify this scaling by considering the transmission problem in dilated domains, while other scalings are discussed in Sec. 6. We denote by  $\mathcal{L}_0^\#$  the realization of  $\mathcal{L}_0$  with  $\# = D, N, R, T$  for Dirichlet, Neumann, Robin, or Transmission condition. We recall that the Hilbert space  $L_\#^2$  denotes  $L^2(\mathbb{R}_+)$  in the case when  $\# \in \{D, N, R\}$ , and  $L^2(\mathbb{R}_-) \times L^2(\mathbb{R}_+)$  when  $\# = T$ . For the complex harmonic oscillator  $\mathcal{L}_2$  we consider (with the same notation) the self-adjoint realization on  $L^2(\mathbb{R}_\sigma)$ .

**4.3. Comparison with the large domain limit.** We assume that  $0 \in \Omega_-$  and we dilate  $\Omega_-$  and  $\Omega$  by the map  $(x_1, x_2) \mapsto (Sx_1, Sx_2)$  ( $S > 0$  supposed to be large) and get  $\Omega_-^S$  and  $\Omega^S$ .

It remains to check how the transmission problem for  $\Omega^S$  with  $V(x) = x_1$  is modified by dilation. If we start from the form

$$u \mapsto \int_{\Omega^S} |\nabla u|^2 dx + i \int_{\Omega^S} x_1 |u(x)|^2 dx + \kappa_S \int_{\partial\Omega_-^S} |u_+ - u_-|^2 ds_S,$$

with a transmission coefficient  $\kappa_S$ , we get by the change of coordinates  $x = Sy$ , for  $v(y) = u(Sy)$ ,

$$\int_{\Omega} |\nabla_y v|^2 dy + i S^3 \int_{\Omega} y_1 |v(y)|^2 dy + \kappa_S S \int_{\partial\Omega_-} |v_+ - v_-|^2 ds.$$

Dividing by  $S^3$ , we get

$$\frac{1}{S^3} \int_{\Omega} |\nabla_y v|^2 dy + i \int_{\Omega} y_1 |v(y)|^2 dy + \kappa_S S^{-2} \int_{\partial\Omega_-} |v_+ - v_-|^2 ds.$$

In order to treat this problem as semi-classical, we set

$$h^2 = \frac{1}{S^3}, \quad \mathcal{K} = \kappa_S S^{-2},$$

Hence we get

$$\kappa = \kappa_S,$$

and our assumption (4.30) on  $\mathcal{K}$  corresponds to what we get by rescaling from the problem in  $\Omega_S$  with  $\kappa_S$  independent of  $R$ .

For this application, Theorem 1 gives the following

605 **THEOREM 16.** For  $S > 0$ , let  $V_S(x) = SV(S^{-1}x)$ , with the potential  $V$  defined  
 606 on  $\Omega$  satisfying the conditions of Theorem 1, and  $\kappa_S$  is independent of  $S$ . Then, with  
 607 the notation of Theorem 1, one can construct a quasimode  $\lambda_S^\#$  of the  $\#$  realization of  
 608 the operator  $-\Delta + iV_S$  in  $\Omega_S^\#$  such that

$$609 \quad (4.33) \quad \lambda_S^\# = iSV(x^0) + \sum_{j \in \mathbb{N}} \lambda_{2j}^\# S^{-\frac{j}{2}} + \mathcal{O}(S^{-\infty}),$$

610 as  $S \rightarrow +\infty$ .

611 This theorem can also be applied to  $V(x) = x_1$ , in which case  $V_S$  is independent  
 612 of  $S$ .

613 **REMARK 17.** More generally, one can consider

$$614 \quad V_S(x) = S^m V(S^{-1}x),$$

615 with  $m > -2$ . In this case, we get  $\kappa = \kappa_S S^{1-m}$ . If  $\kappa$  is independent of  $S$  or tends to  
 616 0 as  $S \rightarrow +\infty$ , one can apply the semi-classical analysis of the previous sections.

## 617 **5. The quasimode construction. Proof of the main theorem.**

618 **5.1. The form of the quasimode.** In what follows, we assume in the Robin  
 619 or transmission cases that  $\kappa$  is independent of  $h$  (see (4.30)). We now look for a  
 620 quasimode  $u_h^{app,\#}$  that we write in the  $(\sigma, \tau)$  variables in the form:

$$621 \quad (5.1) \quad u_h^{app,\#} \sim d(h) \left( \sum_{j \geq 0} h^{\frac{j}{6}} u_j^\#(\sigma, \tau) \right),$$

622 associated with an approximate eigenvalue

$$623 \quad (5.2) \quad \lambda_h^{app,\#} \sim i v_{00} + h^{\frac{2}{3}} \sum_{j \geq 0} h^{\frac{j}{6}} \lambda_j^\#.$$

624 Here  $d(h) \sim d_0 h^{-\frac{7}{12}}$  with  $d_0 \neq 0$  chosen such that, coming back to the initial coordi-  
 625 nates, the  $L^2$ -norm of the trial state equals 1.

626

627 Note that the  $u_j^\#$  are in the domain of  $\mathcal{L}_j^\#$  if we take the condition  $\#$  (with  
 628  $\# \in \{N, D, R, T\}$ ).

629 Note also that we do not assume *a priori* that the  $\lambda_j^\#$  for  $j$  odd are 0 as claimed in  
 630 our theorem.

631 As will be seen in the proof, we can choose

$$632 \quad (5.3) \quad u_j^\#(\sigma, \tau) = \phi_j^\#(\sigma) \psi_0^\#(\tau), \quad j = 0, 1, 2,$$

633 and

$$634 \quad (5.4) \quad u_j^\#(\sigma, \tau) = \phi_j^\#(\sigma) \psi_0^\#(\tau) + \sum_{\ell=1}^{N_j} \phi_{j,\ell}^\#(\sigma) \psi_{j,\ell}^\#(\tau), \quad j \geq 3,$$

635 with  $\phi_{j,\ell}^\# \in \mathcal{S}(\mathbb{R})$  and  $\psi_{j,\ell}^\# \in \mathcal{S}^\#$  to be specified below.

636 Moreover, we have

$$637 \quad (5.5) \quad \mathcal{L}_0^\# \psi_0^\# = \lambda_0^\# \psi_0^\#,$$

638

$$639 \quad (5.6) \quad \mathcal{L}_2 \phi_0^\#(\sigma) = \lambda_2^\# \phi_0^\#(\sigma),$$

640 with

$$641 \quad (5.7) \quad \langle \psi_{j,\ell}^\#, \bar{\psi}_0^\# \rangle_{L^2_\#} = 0,$$

642 and

$$643 \quad (5.8) \quad \langle \psi_0^\#, \bar{\psi}_0^\# \rangle_{L^2_\#} \neq 0.$$

644 The construction will consist in expanding  $(\widehat{\mathcal{A}}_h - \lambda_h^{app,\#})u_h^{app,\#}$  in powers of  $h^{\frac{1}{6}}$  and  
645 finding the conditions of cancellation for each coefficient of this expansion.

646

647 If we succeed in this construction and come back to the initial coordinates, using  
648 a Borel procedure to sum the formal expansions and multiplying by a cutoff function  
649 in the neighborhood of a point  $x^0$  of  $\partial\Omega^\#$ , we obtain an approximate spectral pair  
650 localized near  $x^0$  (i.e.  $\mathcal{O}(h^\infty)$  outside any neighborhood of  $x^0$ ).

651 The Borel procedure consists in choosing a cutoff function  $\theta$  (with  $\theta = 1$  in a small  
652 neighborhood of 0 and a sequence  $H_n$  such that  $\beta \mapsto \sum_j \beta^j \lambda_j \theta(\beta/H_j)$  converges in  
653  $C^\infty([0, \beta_0])$  for some  $\beta_0 > 0$ ). We then define

$$654 \quad \lambda_h^\# = i v_{00} + h^{\frac{2}{3}} \sum_{j \geq 0} \beta^j \lambda_j \theta(\beta/H_j),$$

655 with  $\beta = h^{\frac{1}{6}}$ .

656 This  $\lambda_h^\#$  is not unique but the difference between two different choices is  $\mathcal{O}(h^\infty)$ . A  
657 similar procedure can be used to define a quasimode state  $u_h^\#$  strongly localized near  
658  $x^0$ .

659

660 **REMARK 18.** *We emphasize that the above construction is not sufficient (the prob-*  
661 *lem being non self-adjoint) for proving the existence of an eigenvalue with this expan-*  
662 *sion. The construction is true for any regular domain (exterior or interior) under the*  
663 *conditions (4.8)-(4.10). When  $V(x) = x_1$ , we recover in this way the condition that*  
664 *the curvature does not vanish at  $x^0$ . We recall that this construction can be done near*  
665 *each point where  $\nabla V(x^0) \wedge \nu(x^0) = 0$ . The candidates for the spectrum are deter-*  
666 *mined by ordering different quasimodes and comparing their real parts. We guess that*  
667 *the true eigenfunctions will have the same localization properties as the constructed*  
668 *quasimode states.*

669 **5.2. Term  $j = 0$ .** Identifying the powers in front of  $h^{\frac{1}{6}}$ , after division by  $d(h)$ ,  
670 one gets the first equation corresponding to  $j = 0$ .

671 We consider four boundary conditions.

672

673 **Neumann and Dirichlet cases.** For the Neumann boundary condition, one  
674 has

$$675 \quad (5.9) \quad \mathcal{L}_0^N u_0^N = \lambda_0^N u_0^N, \quad \partial_\tau u_0^N(\sigma, 0) = 0,$$

676 and we look for a solution in the form

$$677 \quad (5.10) \quad u_0^N(\sigma, \tau) = \phi_0^N(\sigma) \psi_0^N(\tau).$$

678 At this step, we only look for a pair  $(\lambda_0^N, \psi_0^N)$  with  $\psi_0^N$  non identically 0 such that

$$679 \quad (5.11) \quad (-\partial_\tau^2 + i v_{01} \tau) \psi_0^N(\tau) = \lambda_0^N \psi_0^N(\tau) \text{ in } \mathbb{R}^+, \quad (\psi_0^N)'(0) = 0.$$

680 We recall from (4.13) that  $v_{01} \neq 0$  so we have the standard spectral problem for the  
681 complex Airy operator in the half line with Neumann condition at 0. The spectral  
682 theory of this operator is recalled in [17]. The spectrum consists of an infinite sequence  
683 of eigenvalues  $(\lambda^{N,(n)})_{n \geq 1}$  (ordered by increasing real part) that can be expressed  
684 through the zeros  $a'_n$  ( $n \geq 1$ ) of the derivative of the Airy function  $\text{Ai}'(z)$ :

$$685 \quad (5.12) \quad \lambda^{N,(n)} = -a'_n |v_{01}|^{\frac{2}{3}} \exp\left(\frac{i\pi}{3} \text{sign } v_{01}\right).$$

686 Different choices of  $n$  will determine the asymptotic expansion of different approximate  
687 eigenvalues of the original problem. If we are interested in controlling the decay of the  
688 associated semi-group, we choose  $\lambda_0^N = \lambda^{N,(1)}$  which corresponds to the eigenvalue  
689 with the smallest real part.

690 One can similarly treat the Dirichlet problem (like in [8]). In this case, one has

$$691 \quad (5.13) \quad \mathcal{L}_0^D u_0^D = \lambda_0^D u_0^D \text{ in } \mathbb{R}_+, \quad u_0^D(\sigma, 0) = 0,$$

692 and we look for a solution in the form

$$693 \quad (5.14) \quad u_0^D(\sigma, \tau) = \phi_0^D(\sigma) \psi_0^D(\tau),$$

694 where  $\psi_0^D(\tau)$  satisfies

$$695 \quad (5.15) \quad \mathcal{L}_0^D \psi_0^D = \lambda_0^D \psi_0^D \text{ in } \mathbb{R}_+, \quad \psi_0^D(0) = 0.$$

696 The spectral theory of this operator is also recalled in [17]. The spectrum consists of  
697 an infinite sequence of eigenvalues  $(\lambda^{D,(n)})_{n \geq 1}$  (ordered by increasing real part) that  
698 can be expressed through the zeros  $a_n$  ( $n \geq 1$ ) of the Airy function  $\text{Ai}(z)$ :

$$699 \quad (5.16) \quad \lambda^{D,(n)} = -a_n |v_{01}|^{\frac{2}{3}} \exp\left(\frac{i\pi}{3} \text{sign } v_{01}\right).$$

700 One can show (see [25] for a proof by analytic dilation) that

$$701 \quad (5.17) \quad \int_0^{+\infty} \psi_0^N(\tau)^2 d\tau \neq 0 \quad \text{and} \quad \int_0^{+\infty} \psi_0^D(\tau)^2 d\tau \neq 0.$$

702 This is also a consequence of the completeness of the eigenfunctions of the complex  
703 Airy operator in the half-line with Neumann or Dirichlet boundary condition. This  
704 property is true for any eigenvalue  $\lambda_0^\#$  of  $\mathcal{L}_0^\#$ .

705

706 For  $n \geq 1$ , the eigenfunctions  $\psi_0^N = \psi^{N,(n)}$  and  $\psi_0^D = \psi^{D,(n)}$  are specifically  
707 translated and complex dilated Airy functions:

$$708 \quad (5.18) \quad \psi^{N,(n)}(\tau) = c_n^N \text{Ai}\left(a'_n + \tau |v_{01}|^{\frac{1}{3}} \exp\left(\frac{i\pi}{6} \text{sign } v_{01}\right)\right) \quad \text{for } \tau \geq 0,$$

$$709 \quad (5.19) \quad \psi^{D,(n)}(\tau) = c_n^D \text{Ai}\left(a_n + \tau |v_{01}|^{\frac{1}{3}} \exp\left(\frac{i\pi}{6} \text{sign } v_{01}\right)\right) \quad \text{for } \tau \geq 0,$$

710 where the normalization constants  $c_n^N$  and  $c_n^D$  can be fixed by choosing the following  
711 normalization that we keep throughout the paper:

$$712 \quad \int_0^\infty \psi_0^\#(\tau)^2 d\tau = 1.$$

713 These coefficients are computed explicitly in Appendix A (see (A.24), (A.20)).

714 **Robin case.** For the Robin boundary condition, one has

$$715 \quad (5.20) \quad \mathcal{L}_0^R u_0^R = \lambda_0^R u_0^R, \quad \partial_\tau u_0^R(\sigma, 0) = \kappa u_0^R(\sigma, 0),$$

716 and we look for a solution in the form

$$717 \quad (5.21) \quad u_0^R(\sigma, \tau) = \phi_0^R(\sigma) \psi_0^R(\tau),$$

718 where the function  $\psi_0^R(\tau)$  satisfies

$$719 \quad (5.22) \quad (-\partial_\tau^2 + iv_{01}\tau)\psi_0^R(\tau) = \lambda_0^R \psi_0^R(\tau) \text{ in } \mathbb{R}_+, \quad (\psi_0^R)'(0) = \kappa \psi_0^R(0).$$

720 This one-dimensional problem was studied in [17]. In particular, the spectrum consists  
721 of an infinite sequence of eigenvalues  $(\lambda^{R,(n)})_{n \geq 1}$  (ordered by increasing real part) that  
722 can be expressed as

$$723 \quad (5.23) \quad \lambda^{R,(n)}(\kappa) = -a_n^R(\kappa) |v_{01}|^{\frac{2}{3}} \exp\left(\frac{\pi i}{3} \text{sign } v_{01}\right),$$

724 where  $a_n^R(\kappa)$  is a solution of the equation

$$725 \quad (5.24) \quad \exp\left(\frac{\pi i}{6} \text{sign } v_{01}\right) \text{Ai}'(a_n^R(\kappa)) - \frac{\kappa}{|v_{01}|^{\frac{1}{3}}} \text{Ai}(a_n^R(\kappa)) = 0,$$

726 and  $\kappa \geq 0$  denotes the Robin parameter.<sup>4</sup>

727 Except for the case of small  $\kappa$ , in which the eigenvalues are close to the eigenvalues  
728 of the Neumann problem, it does not seem easy to localize all the solutions of (5.24)  
729 in general. Note that from (5.24), we deduce that

$$730 \quad (5.25) \quad (\lambda^{R,(n)})'(0) = -(a_n^R)'(0) |v_{01}|^{\frac{2}{3}} \exp\left(\frac{\pi i}{3} \text{sign } v_{01}\right),$$

731 where

$$732 \quad (5.26) \quad (a_n^R)'(0) = \frac{1}{a_n^R(0) |v_{01}|^{\frac{1}{3}}} \exp\left(-\frac{\pi i}{6} \text{sign } v_{01}\right) \neq 0.$$

733 Nevertheless it is proven in [17] that the zeros of the function in (5.24) are simple  
734 and that there is no Jordan block. So as can be deduced from the next lemma, any  
735 eigenfunction satisfies  $\int \psi_0^R(\tau)^2 d\tau \neq 0$ . We consequently fix the normalization of  $\psi_0^R$   
736 by imposing

$$737 \quad (5.27) \quad \int_0^\infty \psi_0^R(\tau)^2 d\tau = 1.$$

---

<sup>4</sup> In [17], we discussed the complex Airy operator with  $v_{01} = -1$ , see Eq. (3.25).

738 For  $n \geq 1$ , the associated eigenfunction  $\psi_0^R = \psi^{R,(n)}$  reads

$$739 \quad (5.28) \quad \psi^{R,(n)}(\tau) = c_n^R \operatorname{Ai} \left( a_n^R(\kappa) + \tau |v_{01}|^{\frac{1}{3}} \exp \left( \frac{\pi i}{6} \operatorname{sign} v_{01} \right) \right) \quad (\tau \geq 0),$$

740 where  $c_n^R$  is the normalization constant given by (A.28).

741 **5.2.1. Transmission case.** In the transmission case, one gets, with  $\psi_0^T =$   
742  $(\psi_0^-, \psi_0^+)$ ,

$$743 \quad (5.29) \quad \begin{aligned} \mathcal{L}_0 \psi_0^- &= \lambda_0^T \psi_0^- \text{ in } \mathbb{R}_-, & \mathcal{L}_0 \psi_0^+ &= \lambda_0^T \psi_0^+ \text{ in } \mathbb{R}_+, \\ \partial_\tau \psi_0^-(\sigma, 0) &= \partial_\tau \psi_0^+(\sigma, 0), & \partial_\tau \psi_0^+(\sigma, 0) &= \kappa (\psi_0^+(\sigma, 0) - \psi_0^-(\sigma, 0)). \end{aligned}$$

744 The existence of  $\lambda_0^T$  has been proved in [17]. In addition, the eigenvalue (of the  
745 smallest real part) is simple (no Jordan block) for  $\kappa \geq 0$  small. We can use the  
746 explicit computations in [17] or the following abstract lemma by Aslayan-Davies for  
747 a closed operator  $A$  [9]:

748 **LEMMA 19.** *If  $f$  and  $f^*$  are the normalized eigenvectors of  $A$  and  $A^*$  associated*  
749 *with the eigenvalues  $\lambda$  and  $\bar{\lambda}$  respectively, and if the spectral projector  $P$  has rank*  
750 *1, then  $\langle f, f^* \rangle \neq 0$  and*

$$751 \quad \|P\| = \frac{1}{|\langle f, f^* \rangle|}.$$

752 The proof that  $P$  has rank 1 for the case  $V(x) = x_1$  is given in [17] but only for  $\kappa \geq 0$ .  
753 In general, we make the assumption

754 **ASSUMPTION 20.**  $\lambda_0^T(\kappa)$  is simple (no Jordan block).

755 Under this assumption, we have

$$756 \quad (5.30) \quad \int_{-\infty}^{\infty} \psi_0^T(\tau)^2 d\tau := \int_{-\infty}^0 \psi_0^-(\tau)^2 d\tau + \int_0^{+\infty} \psi_0^+(\tau)^2 d\tau \neq 0.$$

757 The explicit form of the eigenfunctions  $\psi^{T,(n)}$  ( $n \geq 1$ ) can be obtained from the  
758 analysis provided in [16, 17]:

$$759 \quad (5.31) \quad \begin{aligned} \psi^{+,(n)}(\tau) &= -c_n^T \bar{\delta} \operatorname{Ai}'(a_n^-(\kappa)) \operatorname{Ai} \left( a_n^+(\kappa) + \tau |v_{01}|^{\frac{1}{3}} \delta \right), \\ \psi^{-,(n)}(\tau) &= c_n^T \delta \operatorname{Ai}'(a_n^+(\kappa)) \operatorname{Ai} \left( a_n^-(\kappa) - \tau |v_{01}|^{\frac{1}{3}} \bar{\delta} \right), \end{aligned}$$

760 where  $c_n^T$  is a normalization constant (to satisfy (5.35)),  $\delta = \exp \left( \frac{i\pi}{6} \operatorname{sign} v_{01} \right)$ , and

$$761 \quad (5.32) \quad a_n^\pm(\kappa) = \hat{\lambda}^{T,(n)}(\kappa / |v_{01}|^{\frac{1}{3}}) \exp \left( \pm \frac{2\pi i}{3} \operatorname{sign} v_{01} \right),$$

where the  $\hat{\lambda}^{T,(n)}(\tilde{\kappa})$  are the eigenvalues of the complex Airy operator  $-\frac{d^2}{dx^2} + ix$  on  
the line with transmission condition at 0, with coefficient

$$\tilde{\kappa} = \kappa / |v_{01}|^{\frac{1}{3}}.$$

762 They are defined implicitly as complex-valued solutions (enumerated by the index  
763  $n = 1, 2, \dots$ ) of the equation [16, 17]

$$764 \quad (5.33) \quad 2\pi \operatorname{Ai}' \left( e^{2\pi i/3} \hat{\lambda}^{T,(n)}(\tilde{\kappa}) \right) \operatorname{Ai}' \left( e^{-2\pi i/3} \hat{\lambda}^{T,(n)}(\tilde{\kappa}) \right) = -\tilde{\kappa}.$$

765 The eigenvalues  $\hat{\lambda}^{T,(n)}(\tilde{\kappa})$  are ordered according to their increasing real parts:

$$766 \quad \operatorname{Re}\{\hat{\lambda}^{T,(1)}(\tilde{\kappa})\} \leq \operatorname{Re}\{\hat{\lambda}^{T,(2)}(\tilde{\kappa})\} \leq \dots$$

767 Note that  $\psi^{-,(n)}(0_-) \neq \psi^{+,(n)}(0_+)$ . The associated eigenvalue is

$$769 \quad (5.34) \quad \lambda^{T,(n)}(\kappa) = \hat{\lambda}^{T,(n)}(\kappa/|v_0|^{\frac{1}{3}}) |v_{01}|^{\frac{2}{3}}.$$

770 In what follows,  $(\lambda_0^T(\kappa), \psi_0^T)$  denotes an eigenpair  $(\lambda^{T,(n)}(\kappa), \psi^{T,(n)})$  corresponding to  
771 a particular choice of  $n \geq 1$ .

772 **Summary at this stage.** For  $\# \in \{D, N, R, T\}$ , we have constructed  $u_0^\#$  in the  
773 form (5.3). At this step  $\phi_0^\#(\sigma)$  remains “free” except that it should not be identically  
774 0. We have chosen  $\lambda_0^\#$  as an eigenvalue of  $\mathcal{L}_0^\#$  (assuming that it is simple, with no  
775 Jordan block) and  $\psi_0^\#$  is the associated eigenfunction of  $\mathcal{L}_0^\#$ , which belongs to  $\mathcal{S}^\#$   
776 and permits, according to Lemma 19, to have the normalization  
777

$$778 \quad (5.35) \quad \int_{\mathbb{R}^\#} \psi_0^\#(\tau)^2 d\tau = 1.$$

779 From now on, we do not mention (except for explicit computations) the reference  
780 to Dirichlet, Neumann, Robin or Transmission condition when the construction is  
781 independent of the considered case.

782 **5.3. Term  $j = 1$ .** The second equation (corresponding to  $j = 1$ ) reads

$$783 \quad (5.36) \quad (\mathcal{L}_0^\# - \lambda_0) u_1^\# = \lambda_1 u_0^\#.$$

784 We omit sometimes the superscript  $\#$  for simplicity.

785 The guess is that  $\lambda_1 = 0$ . To see if it is a necessary condition, one can take the scalar  
786 product (in the  $\tau$  variable) with  $\bar{\psi}_0$  (to be understood as the element in  $\operatorname{Ker}(\mathcal{L}_0^* - \bar{\lambda}_0)$ ).

787 We take the convention that the scalar product is antilinear in the second argument.

788 This leads to

$$789 \quad \left( \int \psi_0^2(\tau) d\tau \right) \lambda_1 \phi_0(\sigma) = 0,$$

790 the integral being on  $\mathbb{R}^+$  for Dirichlet, Neumann or Robin, and on  $\mathbb{R}$  in the transmis-  
791 sion case. From Eq. (5.35), we get then

$$792 \quad \lambda_1 \phi_0(\sigma) = 0,$$

793 and by the previous condition on  $\phi_0(\sigma)$

$$794 \quad (5.37) \quad \lambda_1 = 0.$$

795 Hence, coming back to (5.36), we choose

$$796 \quad (5.38) \quad u_1^\#(\sigma, \tau) = \phi_1^\#(\sigma) \psi_0^\#(\tau),$$

797 where  $\phi_1^\#$  remains free at this step.

798

799 **5.4. Term  $j = 2$ .** The third condition (corresponding to  $j = 2$ ) reads

$$800 \quad (5.39) \quad (\mathcal{L}_0^\# - \lambda_0) u_2 + \mathcal{L}_2 u_0 = \lambda_2 u_0.$$

801 To find a necessary condition, we take the scalar product (in the  $\tau$  variable) with  $\bar{\psi}_0$ .  
802 In this way we get (having in mind (5.35))

$$803 \quad \langle \mathcal{L}_2 u_0, \bar{\psi}_0 \rangle = \lambda_2 \phi_0(\sigma).$$

804 Computing the left hand side, we get

$$805 \quad (-\partial_\sigma^2 + i v_{20} \sigma^2) \phi_0(\sigma) = \lambda_2 \phi_0(\sigma).$$

806 From Assumption (4.13), we know that  $v_{20} \neq 0$ . Hence we are dealing with an effective complex harmonic oscillator whose spectral analysis has been done in detail (see  
807 Davies [11] or the book by Helffer [19]). The eigenvalues can be explicitly computed  
808 (by analytic dilation) and there is no Jordan block. Moreover the system of corresponding  
809 eigenfunctions is complete. This implies that  $(\lambda_2, \phi_0)$  should be a spectral  
810 pair for  $(-\partial_\sigma^2 + i v_{20} \sigma^2)$ .  
811 The eigenpairs of the quantum harmonic oscillator are well known:

$$813 \quad (5.40) \quad \lambda_2^{(k)} = \gamma(2k - 1), \quad \phi_0^{(k)}(\sigma) = \frac{\gamma^{\frac{1}{4}} e^{-\gamma\sigma^2/2} H_{k-1}(\gamma^{\frac{1}{2}}\sigma)}{\pi^{\frac{1}{4}} \sqrt{2^{k-1} (k-1)!}} \quad (k = 1, 2, \dots),$$

814 where  $\gamma^s = |v_{20}|^{\frac{s}{2}} \exp(\frac{\pi s i}{4} \text{sign } v_{20})$  (for  $s = \frac{1}{4}, \frac{1}{2}, 1$ ),  $H_k(z)$  are Hermite polynomials,  
815 and the prefactor ensures that

$$816 \quad \int_{-\infty}^{\infty} \phi_0^{(k)}(\sigma)^2 d\sigma = 1.$$

817 The eigenvalue with the smallest real part corresponds to  $k = 1$  for which

$$818 \quad (5.41) \quad \phi_0^{(1)}(\sigma) = c_{\phi_0} \exp\left(-\lambda_2 \frac{\sigma^2}{2}\right),$$

819 while the corresponding eigenvalue is

$$820 \quad (5.42) \quad \lambda_2^{(1)} = |v_{20}|^{\frac{1}{2}} \exp\left(\frac{i\pi}{4} \text{sign } v_{20}\right),$$

821 and  $c_{\phi_0}$  ensures the normalization of  $\phi_0^{(1)}(\sigma)$ :

$$822 \quad (5.43) \quad c_{\phi_0} = |v_{20}|^{\frac{1}{8}} \pi^{-\frac{1}{4}} \exp\left(\frac{i\pi}{16} \text{sign } v_{20}\right).$$

823 We do not need actually the specific expression of  $\phi_0^\# = \phi_0$  and it is enough to know  
824 that  $\phi_0^\# \in \mathcal{S}(\mathbb{R})$ .

825

826 Coming back to the solution of (5.39), which simply reads

$$827 \quad (5.44) \quad (\mathcal{L}_0 - \lambda_0) u_2 = 0,$$

828 we consequently look for  $u_2^\#(\sigma, \tau)$  in the form

$$829 \quad (5.45) \quad u_2^\#(\sigma, \tau) = \phi_2^\#(\sigma) \psi_0^\#(\tau),$$

830 where  $\phi_2^\#(\sigma)$  is free at this stage.

831

832 **Summary at this stage.** We note that the construction is conform with the  
 833 general form introduced in (5.3). At this stage,  $(\lambda_0^\#, \psi_0^\#)$  is a spectral pair for  $\mathcal{L}_0^\#$ ,  
 834  $\lambda_1^\# = 0$ ,  $u_1^\#(\sigma, \tau) = \phi_1^\#(\sigma) \psi_0^\#(\tau)$  (with  $\phi_1^\#$  free),  $(\lambda_2^\#, \phi_0^\#)$  is a spectral pair for  $\mathcal{L}_2$   
 835 (actually independent of  $\#$ ).

836 **5.5. Term  $j = 3$ .** The fourth equation corresponds to  $j = 3$  and reads

$$837 \quad (5.46) \quad (\mathcal{L}_0 - \lambda_0)u_3 + (\mathcal{L}_2 - \lambda_2)u_1 + \mathcal{L}_3 u_0 = \lambda_3 u_0.$$

838 Taking the scalar product (in  $L^2 \hat{\otimes} L^2_\# := L^2(\mathbb{R}_\sigma \times \mathbb{R}_\tau^+)$  for Dirichlet, Neumann and  
 839 Robin, and in  $L^2 \hat{\otimes} L^2_\# := L^2(\mathbb{R}_\sigma \times \mathbb{R}_\tau^-) \times L^2(\mathbb{R}_\sigma \times \mathbb{R}_\tau^+)$  for the transmission case) with  
 840  $\bar{u}_0$  and having in mind our normalizations of  $\psi_0$  and  $\phi_0$ , we obtain

$$841 \quad \langle \mathcal{L}_3 u_0, \bar{u}_0 \rangle = \lambda_3,$$

842 so  $\lambda_3$  is determined by

$$843 \quad (5.47) \quad \lambda_3 = i v_{11} \left( \int \sigma \phi_0(\sigma)^2 d\sigma \right) \left( \int \tau \psi_0^\#(\tau)^2 d\tau \right).$$

844 Note that whatever the parity of  $\phi_0$ ,  $\phi_0^2$  is even, so  $\int \sigma \phi_0(\sigma)^2 d\sigma = 0$ . Hence,

$$845 \quad (5.48) \quad \lambda_3 = 0.$$

846 We come back to (5.46), but now take the scalar product with  $\bar{\psi}_0$  in the  $\tau$  variable.  
 847 So we get

$$848 \quad \langle (\mathcal{L}_2 - \lambda_2)u_1 + (\mathcal{L}_3 - \lambda_3)u_0, \bar{\psi}_0 \rangle = 0.$$

849 Taking into account (4.13) and the form of  $u_0$  and  $u_1$ , this reads

$$850 \quad (5.49) \quad (\mathcal{L}_2 - \lambda_2) \phi_1 = -i v_{11} \sigma \left( \int \tau \psi_0(\tau)^2 d\tau \right) \phi_0.$$

851 The right hand side is in the image of the realization of  $(\mathcal{L}_2 - \lambda_2)$ . There is a unique  
 852  $\phi_1$  solution of (5.49) satisfying

$$853 \quad (5.50) \quad \int_{\mathbb{R}} \phi_1(\sigma) \phi_0(\sigma) d\sigma = 0.$$

854 **REMARK 21.** Note that  $\phi_0 \phi_1$  is odd.

855 We can now solve (5.46). We observe that

$$856 \quad (\mathcal{L}_2 - \lambda_2)u_1 + (\mathcal{L}_3 - \lambda_3)u_0 = ((\mathcal{L}_2 - \lambda_2)\phi_1)\psi_0 + (\mathcal{L}_3 - \lambda_3)u_0.$$

857 According to what we have done already, (5.46) has the form

$$858 \quad ((\mathcal{L}_0 - \lambda_0)u_3)(\sigma, \tau) = g_3(\tau) f_3(\sigma),$$

859 where

$$860 \quad g_3(\tau) = (\tau - c_3) \psi_0(\tau)$$

861 is orthogonal to  $\bar{\psi}_0$ , i.e.

$$862 \quad c_3 = \int \tau \psi_0(\tau)^2 d\tau,$$

863 and

$$864 \quad f_3(\sigma) = i v_{11} \sigma \phi_0(\sigma).$$

865 **REMARK 22.** Note that  $\phi_0 f_3$  is odd.

866 We then write for  $j = 3$  the expression (5.4), with  $N_3 = 1$ ,

$$867 \quad (5.51) \quad u_3(\sigma, \tau) = \phi_3(\sigma) \psi_0(\tau) + \phi_{3,1}(\sigma) \psi_{3,1}(\tau),$$

868 where  $\psi_{3,1}$  is determined as the unique solution of the problem

$$869 \quad (5.52) \quad (\mathcal{L}_0^\# - \lambda_0^\#) \psi_{3,1} = g_3,$$

870 which is orthogonal to  $\bar{\psi}_0$ , and

$$871 \quad (5.53) \quad \phi_{3,1}(\sigma) = f_3(\sigma).$$

872

873 **REMARK 23.** Note that  $\phi_0 \phi_{3,1}$  is odd.

874 **Summary at this stage.** We note that the construction is conform with the  
875 general form introduced in (5.3)-(5.4). At this stage,  $\phi_3^\#$  is introduced,  $\lambda_3^\# = 0$  and  
876  $\phi_1^\#$  are determined but  $\phi_2^\#$  and  $\phi_3^\#$  remain free. Note that  $N_3 = 1$  in (5.4),  $\phi_{3,1}^\#$  is  
877 determined in  $\mathcal{S}(\mathbb{R})$  and  $\psi_{3,1}^\#$  is determined in  $\mathcal{S}^\#$ .

878 **5.6. Term  $j = 4$ .** The fifth condition corresponds to  $j = 4$  and reads

$$879 \quad (5.54) \quad (\mathcal{L}_0 - \lambda_0)u_4 + (\mathcal{L}_2 - \lambda_2)u_2 + (\mathcal{L}_3 - \lambda_3)u_1 + \mathcal{L}_4 u_0 = \lambda_4 u_0.$$

880 We follow the same procedure as in the preceding step.  $\lambda_4$  is determined by integrating  
881 (5.54) after multiplication by  $u_0$  :

$$882 \quad (5.55) \quad \begin{aligned} \lambda_4 &= \langle (\mathcal{L}_3 - \lambda_3)u_1 + \mathcal{L}_4 u_0, \bar{u}_0 \rangle \\ &= i v_{11} \left( \int \sigma \phi_1(\sigma) \phi_0(\sigma) d\sigma \right) \left( \int \tau \psi_0(\tau)^2 d\tau \right) \\ &\quad + \epsilon(0) \int \psi_0'(\tau) \psi_0(\tau) d\tau + i v_{02} \int \tau^2 \psi_0(\tau)^2 d\tau. \end{aligned}$$

883  $\phi_2$  is determined by integrating (5.54) in the  $\tau$  variable over  $\mathbb{R}^\#$  after multiplication  
884 by  $\psi_0$ . We get

$$885 \quad (5.56) \quad (\mathcal{L}_2 - \lambda_2)\phi_2 = \langle (\mathcal{L}_3 - \lambda_3)u_1, \bar{\psi}_0 \rangle_{L^2_\tau} + \langle \mathcal{L}_4 u_1, \bar{\psi}_0 \rangle_{L^2_\tau} - \lambda_4 := f_4,$$

886 where our choice of  $\lambda_4$  implies the orthogonality of  $f_4$  to  $\bar{\phi}_0$  in  $L_{\#}^2$ .  
 887 There exists consequently a unique  $\phi_2$  solution of (5.56) that is orthogonal to  $\bar{\phi}_0$ .  
 888

889 We then proceed like in the fourth step, observing that  $u_4$  should satisfy, for some  
 890  $N_4 \geq 1$ ,

$$891 \quad (5.57) \quad (\mathcal{L}_0 - \lambda_0)u_4 = \sum_{\ell=1}^{N_4} f_{4,\ell}(\sigma) g_{4,\ell}(\tau),$$

892 with  $f_{4,\ell}$  in  $\mathcal{S}(\mathbb{R})$ ,  $g_{4,\ell}$  in  $\mathcal{S}^{\#}$  and orthogonal to  $\bar{\psi}_0$  in  $L_{\#}^2$ . The expression in the right  
 893 hand side is deduced from our previous computations of  $u_0$ ,  $u_2$  and  $u_3$  and  $\lambda_4$ .  
 894 We then look for a solution  $u_4$  in the form

$$895 \quad (5.58) \quad u_4(\sigma, \tau) = \phi_4(\sigma) \psi_0(\tau) + \sum_{\ell=1}^{N_4} \phi_{4,\ell}(\sigma) \psi_{4,\ell}(\tau),$$

896 which is obtained by solving for each  $\ell$

$$897 \quad (5.59) \quad (\mathcal{L}_0^{\#} - \lambda_0^{\#})\psi_{4,\ell} = g_{4,\ell}, \quad \int \psi_{4,\ell}(\tau) \psi_0(\tau) d\tau = 0,$$

898 with the suitable boundary (or transmission) condition at 0 and taking

$$899 \quad \phi_{4,\ell} = f_{4,\ell}.$$

900 Although not needed, we make explicit the computation of the right hand side in  
 901 (5.57). Using our choice of  $\lambda_4$  and  $\phi_2$ , we obtain

$$\begin{aligned} & -(\mathcal{L}_2 - \lambda_2)u_2 - (\mathcal{L}_3 - \lambda_3)u_1 - \mathcal{L}_4 u_0 + \lambda_4 u_0 \\ 903 \quad & = -(\mathcal{L}_2 - \lambda_2)\phi_2 \psi_0 - ((\mathcal{L}_3 - \lambda_3)\phi_1 \psi_0 - \mathcal{L}_4 \phi_0 \psi_0 + \lambda_4 \phi_0 \psi_0) \\ & = g_{4,1}(\sigma)(\tau - c_3)\psi_0(\tau) + g_{4,2}(\sigma)(\partial_{\tau} \psi_0 - c_4 \psi_0) + g_{4,3}(\sigma)(\tau^2 - c_5)\psi_0, \end{aligned}$$

904 with  $c_4 = \int (\partial_{\tau} \psi_0)(\tau) \psi_0(\tau) d\tau$  and  $c_5 = \int \tau^2 \psi_0(\tau) d\tau$ .

905 Moreover the  $g_{4,\ell}$  are even with respect with  $\sigma$ .

906 Hence we can take  $N_4 = 3$  and

$$\begin{aligned} & g_{4,1}(\tau) := (\tau - c_3)\psi_0(\tau), \\ 907 \quad (5.60) \quad & g_{4,2}(\tau) := (\partial_{\tau} \psi_0 - c_4 \psi_0), \\ & g_{4,3}(\tau) := (\tau^2 - c_5)\psi_0(\tau). \end{aligned}$$

908 We do not provide explicit formula for the corresponding  $\psi_{4,\ell}$ .

909 **Summary at this stage.** At the end of this step we have determined the  $\lambda_j^{\#}$   
 910 for  $j \leq 4$ , the  $\psi_{j,\ell}^{\#}$  and  $\phi_{4,\ell}^{\#}$  for  $3 \leq j \leq 4$  and the  $\phi_j^{\#}(\sigma)$  for  $j \leq 2$ . Like in [21], this  
 912 construction can be continued to any order. This achieves the proof of Theorem 1.

913 **5.7. Term  $j = 5$  and vanishing of the odd terms.** We first focus on the  
 914 sixth step corresponding to the computation of  $\lambda_5$ . The sixth condition corresponds  
 915 to  $j = 5$  and reads

$$916 \quad (5.61) \quad (\mathcal{L}_0 - \lambda_0)u_5 + (\mathcal{L}_2 - \lambda_2)u_3 + (\mathcal{L}_3 - \lambda_3)u_2 + (\mathcal{L}_4 - \lambda_4)u_1 + \mathcal{L}_5 u_0 = \lambda_5 u_0.$$

917  $\lambda_5$  is determined by integrating (5.61) after multiplication by  $\bar{u}_0$ . By our preceding  
918 constructions and (4.22), we see that

$$919 \quad \sigma \mapsto u_0(\sigma) \left( (\mathcal{L}_2 - \lambda_2) u_3 + (\mathcal{L}_3 - \lambda_3) u_2 + (\mathcal{L}_4 - \lambda_4) u_1 + \mathcal{L}_5 u_0 \right) (\sigma)$$

920 is odd. This immediately leads to  $\lambda_5 = 0$ .

921

922 With some extra work consisting in examining the symmetry properties with  
923 respect to  $\sigma$  and using Sec. 4.2.2, we obtain

924 PROPOSITION 24. *In the formal expansion,  $\lambda_j = 0$  if  $j$  is odd.*

925 **5.8. Four-terms asymptotics.** Gathering (4.20), (5.12) and (5.55), the four-  
926 terms asymptotics of approximate eigenvalues reads for  $n, k = 1, 2, \dots$

$$927 \quad (5.62) \quad \lambda_h^{app, \#} := \lambda_h^{\#, (n, k)} = i v_{00} + h^{\frac{2}{3}} |v_{01}|^{\frac{2}{3}} \mu_n^{\#} \exp\left(\frac{i\pi}{3} \text{sign } v_{01}\right) \\ + h(2k-1) |v_{20}|^{\frac{1}{2}} \exp\left(\frac{i\pi}{4} \text{sign } v_{20}\right) + h^{\frac{4}{3}} \lambda_4^{\#, (n)} + \mathcal{O}(h^{\frac{5}{3}}),$$

928 where  $\mu_n^D = -a_n$ ,  $\mu_n^N = -a'_n$ ,  $\mu_n^R = -a_n^R(\kappa)$  (defined by (5.24)), and  $\mu_n^T = -a_n^+(\kappa)$   
929 (defined by (5.32)), while  $\lambda_4^{\#, (n)}$  is explicitly computed in Appendix A (see (A.23),  
930 (A.27), (A.31), and (A.39) for Dirichlet, Neumann, Robin, and Transmission cases),  
931 and the involved coefficients  $v_{jk}$  of the potential  $V(s, \rho)$  are defined in (4.12).

932

933 REMARK 25. *In the above construction, if we take  $\phi_j^{\#} = 0$  for  $j \geq 3$ , we get an*  
934 *eigenpair  $(\lambda_h^{app, \#}, u_h^{app, \#})$  with*

$$935 \quad u_h^{app, \#} = u_0^{\#} + h^{\frac{1}{6}} u_1^{\#} + h^{\frac{1}{3}} u_2^{\#},$$

936 *such that*

$$937 \quad (5.63) \quad (\mathcal{A}_h^{\#} - \lambda_h^{app, \#}) u_h^{app, \#} = \mathcal{O}(h^{\frac{3}{2}}).$$

938 *To get in (5.63) the remainder  $\mathcal{O}(h^{\frac{5}{3}})$ , one should continue the construction for two*  
939 *more steps.*

940 REMARK 26. *Note that the leading terms in the eigenvalue expansion do not con-*  
941 *tain the curvature which appears only in  $\lambda_4$  (see Eq. (A.27)) and is thus of order*  
942  *$h^{\frac{4}{3}}$ .*

943 **6. Other scalings in the Robin or transmission problems.** The scaling  
944 (4.30) of the transmission parameter  $\mathcal{K}$  with  $h$  was appropriate to keep the Robin  
945 or transmission condition for the rescaled problem. In biophysical applications, the  
946 transmission condition reads

$$947 \quad (6.1) \quad D\partial_\nu u_+ = D\partial_\nu u_- = \mathcal{K}(u_+ - u_-),$$

948 where  $D$  is the bulk diffusion coefficient, while the transmission parameter  $\mathcal{K}$  repre-  
949 sents the permeability of a membrane which is set by the membrane properties and  
950 thus does not necessarily scale with  $h$ . Similarly, in the Robin boundary condition,

$$951 \quad (6.2) \quad -D\partial_\nu u_- = \mathcal{K} u_-,$$

952 which accounts for partial reflections on the boundary,  $\mathcal{K}$  represents partial reactivity  
 953 or surface relaxivity which are set by properties of the boundary.

954

955 We consider two practically relevant situations for the BT-operator  
 956  $-D\Delta + igx_1$ :

- 957 • When  $D \rightarrow 0$  with fixed  $g$ , one can identify  $h^2 = D$  and  $V(x) = gx_1$  so that  
 958 the rescaled transmission condition in (4.29) gives  $\mathcal{K}h^{-\frac{4}{3}}$  which tends to  $+\infty$   
 959 as  $h \rightarrow 0$  if  $\mathcal{K}$  is fixed. In this limit, the transmission condition is formally  
 960 reduced to the continuity condition at the boundary:  $u_+(\sigma, 0) = u_-(\sigma, 0)$ ,  
 961 together with the flux continuity in the first relation of (4.29). In other  
 962 words, the interface between two subdomains is removed. The construction  
 963 of the previous section seems difficult to control in this asymptotics and the  
 964 mathematical proof of the heuristics should follow other ways.
- 965 • When  $g \rightarrow \pm\infty$  with fixed  $D$ , one can divide the BT-operator and (6.1)  
 966 by  $g$  and then identify  $h^2 = D/g$  and  $V(x) = x_1$ . In this situation, the  
 967 rescaled transmission condition in (4.29) gives a parameter  $\kappa = (\mathcal{K}/D)h^{\frac{2}{3}}$   
 968 which tends to 0 as  $h \rightarrow 0$ . In this limit, the transmission condition is  
 969 reduced to two Neumann boundary conditions on both sides of the interface:  
 970  $\partial_\tau u_+(\sigma, 0) = \partial_\tau u_-(\sigma, 0) = 0$ .

971 We now discuss how the eigenvalue asymptotic expansion obtained for rescaled  
 972  $\mathcal{K}$  can be modified for the second situation. The constructions of the previous section  
 973 can be adapted and controlled with respect to  $\kappa$  for  $\kappa$  small enough. As observed  
 974 along the construction, one can start with (5.62) and then expand the factor  $\mu_n^\#(\kappa)$   
 975 into Taylor series that results in the quasi-mode in the Robin or Transmission case:

976 THEOREM 27. *With the notation of Theorem 1 except that in (1.4) we assume*

$$977 \quad (6.3) \quad \kappa = \hat{\kappa} h^{\frac{2}{3}},$$

978 *we have for  $\# \in \{R, T\}$ ,  $n, k = 1, 2, \dots$*

$$979 \quad (6.4) \quad \begin{aligned} \lambda_h^{\#, (n, k)} &= i v_{00} - h^{\frac{2}{3}} |v_{01}|^{\frac{2}{3}} a'_n \exp\left(\frac{i\pi}{3} \text{sign } v_{01}\right) \\ &+ h(2k-1) |v_{20}|^{\frac{1}{2}} \exp\left(\frac{i\pi}{4} \text{sign } v_{20}\right) \\ &+ h^{\frac{4}{3}} \left( \lambda_4^{N, (n)} - \hat{\kappa} \frac{|v_{01}|^{\frac{1}{3}}}{a'_n} \exp\left(\frac{i\pi}{6} \text{sign } v_{01}\right) \right) + O(h^{\frac{5}{3}}), \end{aligned}$$

980 *where  $\lambda_4^{N, (n)}$  is explicitly given in (A.27), and the involved coefficients  $v_{jk}$  of the*  
 981 *potential  $V(s, \rho)$  are defined in (4.12).*

982 Here, we have used that  $\lambda_4^{\#, (n)}(\kappa) = \lambda_4^{N, (n)}$  for  $\kappa = 0$  (see Remark 33). The coefficient  
 983 in front of  $\hat{\kappa}$  involves  $(\mu_n^\#)'(0)$  that was computed explicitly by differentiating the  
 984 relation determining  $\mu_n^\#(\kappa)$  with respect to  $\kappa$ . For the Robin case, we used (5.26) to  
 985 get

$$986 \quad (6.5) \quad (\mu_n^R)'(0) = -(a_n^R)'(0) = -\frac{1}{a'_n |v_{01}|^{\frac{1}{3}}} \exp\left(-\frac{i\pi}{6} \text{sign } v_{01}\right),$$

987 with  $a_n^R(0) = a'_n$ .

988 Similarly, differentiating (5.33) with respect to  $\kappa$  and using (5.32), we got (see Ap-  
989 pendix A.3)

$$990 \quad (6.6) \quad (\mu_n^T)'(0) = -(a_n^+)'(0) = -\frac{1}{a_n' |v_{01}|^{\frac{1}{3}}} \exp\left(-\frac{i\pi}{6} \operatorname{sign} v_{01}\right),$$

991 with  $a_n^+(0) = a_n'$ . The effect of Robin or transmission condition appears only in the  
992 coefficient of  $h^{\frac{1}{3}}$ .

993 In order to control the construction with respect to  $\kappa$ , it is enough to get an  
994 expression of the kernel of the regularized resolvent for  $z = \lambda_0^\#$ . Let us treat the  
995 Robin case and assume  $v_{01} = -1$ .

996 As proven in [17], the kernel of the resolvent is given by

$$997 \quad \mathcal{G}^{-,R}(x, y; \lambda) = \mathcal{G}_0^-(x, y; \lambda) + \mathcal{G}_1^{-,R}(x, y; \kappa, \lambda) \quad \text{for } (x, y) \in \mathbb{R}_+^2,$$

998 where

$$999 \quad (6.7) \quad \mathcal{G}_0^-(x, y; \lambda) = \begin{cases} 2\pi \operatorname{Ai}(e^{i\alpha} w_x) \operatorname{Ai}(e^{-i\alpha} w_y) & (x < y), \\ 2\pi \operatorname{Ai}(e^{-i\alpha} w_x) \operatorname{Ai}(e^{i\alpha} w_y) & (x > y), \end{cases}$$

1000 and

$$1001 \quad (6.8) \quad \mathcal{G}_1^{-,R}(x, y; \kappa, \lambda) = -2\pi \frac{ie^{i\alpha} \operatorname{Ai}'(e^{i\alpha} \lambda) - \kappa \operatorname{Ai}(e^{i\alpha} \lambda)}{ie^{-i\alpha} \operatorname{Ai}'(e^{-i\alpha} \lambda) - \kappa \operatorname{Ai}(e^{-i\alpha} \lambda)} \\ \times \operatorname{Ai}(e^{-i\alpha}(ix + \lambda)) \operatorname{Ai}(e^{-i\alpha}(iy + \lambda)).$$

1002 The kernel  $\mathcal{G}_0^-(x, y; \lambda)$  is holomorphic in  $\lambda$  and independent of  $\kappa$ . Setting  $\kappa = 0$ , one  
1003 retrieves the resolvent for the Neumann case. Its poles are determined as (complex-  
1004 valued) solutions of the equation

$$1005 \quad (6.9) \quad f^R(\kappa, \lambda) := ie^{-i\alpha} \operatorname{Ai}'(e^{-i\alpha} \lambda) - \kappa \operatorname{Ai}(e^{-i\alpha} \lambda) = 0.$$

1006 For  $\kappa = 0$ , we recover the equation determining the poles of the Neumann problem:

$$1007 \quad f^N(\lambda) := ie^{-i\alpha} \operatorname{Ai}'(e^{-i\alpha} \lambda) = 0.$$

1008 We look at the first pole and observe that

$$1009 \quad (6.10) \quad (\partial_\lambda f^R)(0, \lambda^{R,(1)}(0)) = (\partial_\lambda f^R)(0, \lambda^{N,(1)}) = (f^N)'(\lambda^{N,(1)}) \neq 0.$$

1010 This evidently remains true for  $\kappa$  small enough:

$$1011 \quad (6.11) \quad (\partial_\lambda f^R)(\kappa, \lambda^{R,(1)}(\kappa)) \neq 0.$$

1012 As done in [17], we can compute the distribution kernel of the projector associated  
1013 with

$$1014 \quad \lambda_0(\kappa) := \lambda^{R,(1)}(\kappa).$$

1015 We get

$$1016 \quad (6.12) \quad \Pi_1^R(x, y; \kappa) = -2\pi \frac{ie^{i\alpha} \operatorname{Ai}'(e^{i\alpha} \lambda_0(\kappa)) - \kappa \operatorname{Ai}(e^{i\alpha} \lambda_0(\kappa))}{(\partial_\lambda f^R)(\kappa, \lambda_0(\kappa))} \\ \times \operatorname{Ai}(e^{-i\alpha}(ix + \lambda_0(\kappa))) \operatorname{Ai}(e^{-i\alpha}(iy + \lambda_0(\kappa))).$$

1017 This kernel is regular with respect to  $\kappa$ .  
 1018 The distribution kernel of the regularized resolvent at  $\lambda_0(\kappa)$  is obtained as

$$1019 \quad \mathcal{G}^{R,reg}(x, y; \kappa, \lambda_0(\kappa)) := \mathcal{G}_0^-(x, y; \kappa, \lambda_0(\kappa)) \\ + \lim_{\lambda \rightarrow \lambda_0} \left( \mathcal{G}_1^{-,R}(x, y; \kappa, \lambda) - (\lambda_0 - \lambda)^{-1} \Pi_1^R(x, y; \kappa) \right).$$

1020 It remains to compute the second term of the right hand side. Writing  $\mathcal{G}_1^{-,R}(x, y; \kappa, \lambda)$   
 1021 in the form

$$1022 \quad \mathcal{G}_1^{-,R}(x, y; \kappa, \lambda) = \frac{\Phi(x, y; \kappa, \lambda)}{\lambda - \lambda_0(\kappa)},$$

1023 we observe that  $\Phi(x, y; \kappa, \lambda)$  is regular in  $\kappa, \lambda$  and we get

$$1024 \quad \mathcal{G}^{R,reg}(x, y; \kappa, \lambda_0(\kappa)) := \mathcal{G}_0^-(x, y; \kappa, \lambda_0(\kappa)) + \partial_\lambda \Phi(x, y; \kappa, \lambda_0(\kappa)).$$

1025 It is regular in  $\kappa$  and we recover for  $\kappa = 0$  the regularized resolvent of the Neumann  
 1026 problem at  $\lambda = \lambda^{N,(1)}$ .

1027

1028 With this regularity with respect to  $\kappa$ , we can control all the constructions for  
 1029  $j = 0, \dots, 4$  (and actually any  $j$ ) and in particular solve (5.52) for  $\kappa$  small and simi-  
 1030 larly (5.59), with a complete expansion in powers of  $\kappa$  at the origin.

1031

1032 **REMARK 28.** *Similarly, one can treat the transmission case.*

1033 **7. WKB construction.** In this section, we propose an alternative analysis  
 1034 based on the WKB method. This construction is restricted to quasimodes with  $k = 1$   
 1035 in (5.40) but it gives a quasimode state that is closer to the eigenfunction than that  
 1036 obtained by the earlier perturbative approach. Here we follow the constructions of  
 1037 [21, 22] developed for a Robin problem.

1038 We start from

$$1039 \quad (7.1) \quad \mathcal{A}_h = -h^2 a^{-2} \partial_s^2 + h^2 a^{-3} (\partial_s a) \partial_s - h^2 \partial_\rho^2 - h^2 a^{-1} (\partial_\rho a) \partial_\rho + i \tilde{V}(s, \rho).$$

1040 Here, instead of what was done in (4.14), we only dilate in the  $\rho$  variable:

$$1041 \quad \rho = h^{\frac{2}{3}} \tau.$$

1042 In the  $(s, \tau)$  coordinates, we get

$$1043 \quad (7.2) \quad \widehat{\mathcal{A}}_h = -h^2 \check{a}_h^{-2} \partial_s^2 + h^2 \check{a}_h^{-3} (\partial_s \check{a}_h) \partial_s - h^{\frac{2}{3}} \partial_\tau^2 - h^{\frac{4}{3}} \check{a}_h^{-1} (\partial_\tau \check{a}_h) \partial_\tau + i \check{V}_h(s, \tau),$$

1044 with

$$1045 \quad (7.3) \quad \begin{aligned} \check{V}_h(s, \tau) &= \tilde{V}(s, h^{\frac{2}{3}} \tau), \\ \check{a}_h(s, \tau) &= 1 - \tau h^{\frac{2}{3}} \mathbf{c}(s), \\ \partial_s \check{a}_h(s, \tau) &= -\tau h^{\frac{2}{3}} \mathbf{c}'(s), \\ \partial_\tau \check{a}_h &= -\mathbf{c}(s), \\ \check{a}_h(s, \tau)^2 &= 1 - 2\tau h^{\frac{2}{3}} \mathbf{c}(s) + \tau^2 h^{\frac{4}{3}} \mathbf{c}(s)^2, \\ \check{a}_h(s, \tau)^{-2} &= 1 + 2\tau h^{\frac{2}{3}} \mathbf{c}(s) + 3\tau^2 h^{\frac{4}{3}} \mathbf{c}(s)^2 + \mathcal{O}(h^2). \end{aligned}$$

1046 We consider the Taylor expansion of  $\tilde{V}_h$ :

$$1047 \quad (7.4) \quad \tilde{V}_h(s, \tau) \sim \sum_{j \in \mathbb{N}} v_j(s) h^{\frac{2j}{3}} \tau^j,$$

1048 with

$$1049 \quad (7.5) \quad v_j(s) = \frac{1}{j!} (\partial_\rho^j \tilde{V})(s, 0).$$

1050 We look for a trial state in the form

$$1051 \quad (7.6) \quad u_h^{\#, wkb} := d(h) b_h(s, \tau) \exp\left(-\frac{\theta(s, h)}{h}\right),$$

1052 with

$$1053 \quad (7.7) \quad \theta(s, h) = \theta_0(s) + h^{\frac{2}{3}} \theta_1(s),$$

1054 and

$$1055 \quad (7.8) \quad b_h(s, \tau) \sim \sum_{j \in \mathbb{N}} b_j(s, \tau) h^{\frac{j}{3}}.$$

1056 Here  $d(h)$  is a normalizing constant such that, when coming back to the initial co-  
 1057 ordinates, the  $L^2$  norm of  $u_h^{\#, wkb}$  is 1. In the initial coordinates, we should actually  
 1058 consider  $u_h^{\#, wkb}(s, h^{-\frac{2}{3}}\rho)$  multiplied by a suitable cut-off function in the neighborhood  
 1059 of the point  $x^0$  of  $\partial\Omega^\perp$ .

1060 This gives an operator acting on  $b_h$

$$1061 \quad (7.9) \quad \begin{aligned} \widehat{\mathcal{A}}_{h, \theta} &:= \exp\left(\frac{\theta(s, h)}{h}\right) \widehat{\mathcal{A}}_h \exp\left(-\frac{\theta(s, h)}{h}\right) \\ &= -\check{a}_h^{-2} (h\partial_s - \theta'(s, h))^2 + h\check{a}_h^{-3} (\partial_s \check{a}_h) (h\partial_s - \theta'(s, h)) \\ &\quad - h^{\frac{2}{3}} \partial_\tau^2 - h^{\frac{4}{3}} \check{a}_h^{-1} (\partial_\rho a) \partial_\tau + i \tilde{V}_h(s, \tau). \end{aligned}$$

1062 We rewrite this operator in the form

$$1063 \quad (7.10) \quad \widehat{\mathcal{A}}_{h, \theta} \sim \sum_{j \geq 0} \Lambda_j h^{\frac{j}{3}},$$

1064 with

$$1065 \quad (7.11) \quad \begin{aligned} \Lambda_0 &:= iv_0(s) - \theta'_0(s)^2, \\ \Lambda_1 &:= 0, \\ \Lambda_2 &:= -\partial_\tau^2 + (iv_1(s) - 2\mathbf{c}(s)\theta'_0(s)^2)\tau - 2\theta'_0(s)\theta'_1(s), \\ \Lambda_3 &:= 2\theta'_0(s)\partial_s, \\ \Lambda_4 &:= \mathbf{c}(s)\partial_\tau + (iv_2(s) - \mathbf{c}(s)^2\theta'_0(s)^2)\tau^2 + 4\mathbf{c}(s)^2\theta'_0(s)\theta'_1(s)\tau - \theta'_1(s)^2. \end{aligned}$$

1066 We recall that  $v'_0(0) = 0$ ,  $v_1(0) \neq 0$ .

1067 We look for a quasimode in the form

$$1068 \quad (7.12) \quad \lambda_h^{\#, wkb} \sim iv_0(0) + h^{\frac{2}{3}} \sum_{j \in \mathbb{N}} \mu_j h^{\frac{j}{3}}.$$

1069 The construction should be local in the  $s$ -variable near 0 and global in the  $\tau$  variable  
1070 in  $\mathbb{R}^\#$ .

1071

1072 Expanding  $(\widehat{\mathcal{A}}_{h,\theta} - \lambda_h)b_h$  in powers of  $h^{\frac{1}{3}}$  and looking at the coefficient in front  
1073 of  $h^0$ , we get

$$1074 \quad (\Lambda_0 - iv_0(0))b_0 = 0$$

1075 as a necessary condition. Hence we choose  $\theta_0$  as a solution of

$$1076 \quad (7.13) \quad i(v_0(s) - v_0(0)) - \theta'_0(s)^2 = 0,$$

1077 which is usually called the (first) eikonal equation.

1078 We take the solution such that

$$1079 \quad (7.14) \quad \operatorname{Re} \theta_0(s) \geq 0, \quad \theta_0(0) = 0,$$

1080 and we note that

$$1081 \quad (7.15) \quad \theta'_0(0) = 0.$$

1082 With this choice of  $\theta_0$ , we note that

$$1083 \quad (7.16) \quad \Lambda_2 = -\partial_\tau^2 + i(v_1(s) - 2c(s)[v_0(s) - v_0(0)])\tau - 2\theta'_0(s)\theta'_1(s),$$

1084 with

$$1085 \quad (7.17) \quad \hat{v}_1(s) := v_1(s) - 2c(s)[v_0(s) - v_0(0)]$$

1086 being real.

1087 As operator on  $L^2_\#$ , with the corresponding boundary or transmission condition  $\# \in$   
1088  $\{D, N, R, T\}$ , it satisfies

$$1089 \quad \Lambda_2^{\#,*} = \overline{\Lambda_2^\#}.$$

1090 The coefficient in front of  $h^{\frac{1}{3}}$  vanishes and we continue with imposing the cancellation  
1091 of the coefficient in front of  $h^{\frac{2}{3}}$  which reads

$$1092 \quad (\Lambda_0 - iv_0(0))b_2 + \Lambda_2 b_0 = \mu_0 b_0,$$

1093 or, taking account of our choice of  $\theta_0$ ,

$$1094 \quad (7.18) \quad -2\theta'_0(s)\theta'_1(s)b_0(s, \tau) + (-\partial_\tau^2 + i\hat{v}_1(s)\tau)b_0(s, \tau) - \mu_0 b_0(s, \tau) = 0.$$

1095 Considering this equation at  $s = 0$ , we get as a necessary condition

$$1096 \quad (7.19) \quad (-\partial_\tau^2 + iv_1(0)\tau)b_0(0, \tau) = \mu_0 b_0(0, \tau).$$

1097 If we impose a choice such that  $b_0(0, \tau)$  is not identically 0, we get that  $\mu_0$  should be  
1098 an eigenvalue of (the suitable realization of)  $-\partial_\tau^2 + iv_1(0)\tau$ , i.e.  $\mathcal{L}_0^\#$ . We take some  
1099 simple eigenvalue  $\mu_0$  and define  $\mu_0(s)$  as the eigenvalue of the operator

$$1100 \quad (7.20) \quad -\partial_\tau^2 + i\hat{v}_1(s)\tau$$

1101 such that  $\mu_0(0) = \mu_0$ . If  $f_0(s, \tau)$  denotes the corresponding eigenfunction normalized  
1102 as

$$1103 \quad (7.21) \quad \int f_0(s, \tau)^2 d\tau = 1,$$

1104 we can look for

$$1105 \quad (7.22) \quad b_0(s, \tau) = c_0(s) f_0(s, \tau).$$

1106 We now come back to (7.18), which reads, assuming  $c_0(s) \neq 0$ ,

$$1107 \quad (7.23) \quad -2\theta'_0(s)\theta'_1(s) + (\mu_0(s) - \mu_0) = 0.$$

1108 This equation can be seen as the second eikonal equation. It has a unique regular  
1109 solution  $\theta_1$  if we add the condition

$$1110 \quad (7.24) \quad \theta_1(0) = 0.$$

1111 The first transport equation is obtained when looking at the coefficient in front  
1112 of  $h$  which reads

$$1113 \quad (\Lambda_0 - iv_0(0))b_3 + (\Lambda_2 - \mu_0)b_1 + \Lambda_3b_0 = \mu_1b_0,$$

1114 or

$$1115 \quad (7.25) \quad (-\partial_\tau^2 + i\hat{v}_1(s)\tau - \mu_0(s))b_1(s, \tau) + 2\theta'_0(s)\partial_s b_0(s, \tau) + \theta''_0(s)b_0(s, \tau) - \mu_1b_0(s, \tau) = 0.$$

1116 We assume

$$1117 \quad (7.26) \quad b_1(s, \tau) = c_1(s)f_0(s, \tau) + \hat{b}_1(s, \tau), \quad \text{with} \quad \int f_0(s, \tau)\hat{b}_1(s, \tau)d\tau = 0.$$

1118 Multiplying it by  $f_0(s, \tau)$  and integrating with respect to  $\tau$ , we get

$$1119 \quad (7.27) \quad 2\theta'_0(s) \int \partial_s b_0(s, \tau) f_0(s, \tau) d\tau + \theta''_0(s) c_0(s) = \mu_1 c_0(s),$$

1120 which leads to

$$1121 \quad (7.28) \quad 2\theta'_0(s)c'_0(s) + \theta''_0(s)c_0(s) = \mu_1c_0(s),$$

1122 where we have used in the last line (7.21). Taking  $s = 0$  and assuming  $c_0(0) \neq 0$ , one  
1123 gets

$$1124 \quad (7.29) \quad \theta''_0(0) = \mu_1,$$

1125 which is also sufficient for solving (7.28). We have determined at this stage  $c_0(s)$   
1126 assuming for normalization

$$1127 \quad (7.30) \quad c_0(0) = 1.$$

1128 Coming back to (7.25), we have to solve, for each  $s$  in a neighborhood of 0

$$1129 \quad (7.31) \quad (-\partial_\tau^2 + i\hat{v}_1(s)\tau - \mu_0(s))\hat{b}_1(s, \tau) = g_1(s, \tau),$$

1130 with  $g_1(s, \tau)$  satisfying  $\int f_0(s, \tau)g_1(s, \tau)d\tau = 0$ .

1131 At this stage, the function  $c_1$  is free.

1132 We continue, one step more, in order to see if the proposed approach is general.

1133 The second transport equation is obtained when looking at the coefficient in front of  
1134  $h^{\frac{4}{3}}$ , which reads

$$1135 \quad (\Lambda_0 - iv_0(0))b_4 + (\Lambda_2 - \mu_0)b_2 + (\Lambda_3 - \mu_1)b_1 + \Lambda_4b_0 = \mu_2b_0,$$

1136 or

$$1137 \quad (7.32) \quad \begin{aligned} & (-\partial_\tau^2 + i\hat{v}_1(s)\tau - \mu_0(s))b_2(s, \tau) + 2\theta'_0(s)\partial_s b_1(s, \tau) + \theta''_0(s)b_1(s, \tau) \\ & - \mu_1b_1(s, \tau) - \mu_2b_0(s, \tau) + (iv_2(s)\tau^2 - \theta'_1(s)^2)b_0(s, \tau) - 3\tau^2\mathfrak{c}(s)^2\theta'_0(s)^2 \\ & + 4\tau\mathfrak{c}(s)^2\theta'_0(s)\theta'_1(s)b_0 + \mathfrak{c}(s)\partial_\tau b_0 = 0. \end{aligned}$$

1138 We look for  $b_2$  in the form

$$1139 \quad (7.33) \quad b_2(s, \tau) = c_2(s)f_0(s, \tau) + \hat{b}_2(s, \tau), \quad \text{with} \quad \int f_0(s, \tau)\hat{b}_1(s, \tau)d\tau = 0.$$

1140 We then proceed as before. If we write

$$1141 \quad \begin{aligned} g_2(s, \tau) &= -2\theta'_0(s)\partial_s b_1(s, \tau) \\ & - \theta''_0(s)b_1(s, \tau) + \mu_1b_1(s, \tau) + \mu_2b_0(s, \tau)(s) + (\theta'_1(s)^2 - iv_2\tau^2)b_0(s, \tau) \\ & - 3\tau^2\mathfrak{c}(s)^2\theta'_0(s)^2b_0 + 4\tau\mathfrak{c}(s)^2\theta'_0(s)\theta'_1(s)b_0 + \mathfrak{c}(s)\partial_\tau b_0, \end{aligned}$$

1142 the orthogonality condition reads

$$1143 \quad \begin{aligned} 0 &= \int g_2(s, \tau)f_0(s, \tau) d\tau \\ &= -2\theta'_0(s)c'_1(s) + (\mu_1 - \theta''_0(s))c_1(s) - 2\theta'_0(s) \int \partial_s \hat{b}_1(s, \tau)f_0(s, \tau)d\tau \\ & + \left( \mu_2 + \theta'_1(s)^2 - iv_2 \int \tau^2 f_0(s, \tau)^2 d\tau \right) c_0(s) \\ & + \int (-3\tau^2\mathfrak{c}(s)^2\theta'_0(s)^2b_0 + 4\tau\mathfrak{c}(s)^2\theta'_0(s)\theta'_1(s)b_0) f_0(s, \tau)d\tau \\ & + \int \mathfrak{c}(s)\partial_\tau b_0 f_0(s, \tau) d\tau. \end{aligned}$$

1144 Observing that

$$1145 \quad \begin{aligned} & \int (-3\tau^2\mathfrak{c}(s)^2\theta'_0(s)^2b_0(s, \tau) + 4\tau\mathfrak{c}(s)^2\theta'_0(s)\theta'_1(s)b_0(s, \tau) f_0(s, \tau) + \mathfrak{c}(s)\partial_\tau b_0 f_0(s, \tau))d\tau \\ & = \mathfrak{c}(0) \left( \int \partial_\tau f_0(0, \tau)f_0(0, \tau) d\tau \right) c_0(0), \end{aligned}$$

1146 for  $s = 0$ , this determines  $\mu_2$  as a necessary condition at  $s = 0$  which reads

$$1147 \quad (7.34) \quad \mu_2 = iv_2(0) \int \tau^2 f_0(0, \tau)^2 d\tau - \theta'_1(0)^2 - \mathfrak{c}(0) \int \partial_\tau f_0(0, \tau)f_0(0, \tau) d\tau.$$

1148 Note that in the case when  $\# \in \{D, N, R\}$ , we get

$$1149 \quad \int \partial_\tau f_0(0, \tau)f_0(0, \tau) d\tau = \frac{1}{2}f_0(0, 0)^2.$$

1150 We can then determine  $c_1$  if we add the condition  $c_1(0) = 0$ .  
 1151 Since  $g_2$  is orthogonal to  $\bar{f}_0$ , we can find  $\hat{b}_2$ , while  $c_2$  remains free for the next step.  
 1152

1153 Hence, we have obtained the following theorem

1154 **THEOREM 29.** *Under the assumptions of Theorem 1, if  $\mu_0^\#$  is a simple eigenvalue*  
 1155 *of the realization “#” of the complex Airy operator  $-\frac{d^2}{dx^2} + ix$  in  $L^2_\#$ , and  $\tilde{\mu}_1$  is the*  
 1156 *eigenvalue of the Davies operator  $-\frac{d^2}{dy^2} + iy^2$  on  $L^2(\mathbb{R})$  with the smallest real part,*  
 1157 *then there exists an approximate pair  $(\lambda_h^\#, u_h^\#)$  with  $u_h^\#$  in the domain of*  
 1158  *$\mathcal{A}_h^\#$ , such that (7.6), (7.7) and (7.8) are satisfied and*

$$1159 \quad (7.35) \quad \exp\left(\frac{\theta}{h}\right) (\mathcal{A}_h^\# - \lambda_h^\#) u_h^\# = \mathcal{O}(h^\infty) \text{ in } L^2_\#(\Omega), \quad \|u_h^\#\|_{L^2} \sim 1,$$

1160 where

$$1161 \quad (7.36) \quad \lambda_0^\# = \mu_0^\# |v_{01}|^{\frac{2}{3}} \exp\left(i\frac{\pi}{3} \text{sign } v_{01}\right), \quad \lambda_2 = \tilde{\mu}_1 |v_{20}|^{\frac{1}{2}} \exp\left(i\frac{\pi}{4} \text{sign } v_{20}\right),$$

1162 with  $v_{01} := \nu \cdot \nabla V(x^0)$ .

1163 **REMARK 30.** *In this approach, we understand more directly why no odd power of*  
 1164  *$h^{\frac{1}{6}}$  appears for  $\lambda_h$ . Note that  $\mu_j = \lambda_{2j}$ .*

1165 **8. Examples.** In this Section, we illustrate the above general results for the  
 1166 potential  $V(x) = x_1$  and some simple domains.

1167 **8.1. Disk.** Let  $\Omega = \{(x_1, x_2) \in \mathbb{R}^2 : |x| < R_0\}$  be the disk of radius  $R_0$ . In this  
 1168 case,  $\Omega_\perp = \{(R_0, 0), (-R_0, 0)\}$ . The local parameterization around the point  $(R_0, 0)$   
 1169 reads in polar coordinates  $(r, \theta)$  as  $\rho = R_0 - r$ ,  $s = R_0\theta$ , so that

$$1170 \quad (8.1) \quad V(x) = x_1(s, \rho) = (R_0 - \rho) \cos(s/R_0),$$

1171  $\mathfrak{c}(0) = 1/R_0$ , and we get

$$1172 \quad (8.2) \quad v_{00} = R_0, \quad v_{01} = -1, \quad v_{20} = -\frac{1}{2R_0}, \quad v_{11} = v_{02} = 0.$$

1173 Using Eqs. (A.27), (A.23), (A.31) or (A.39) for  $\lambda_4^{\#, (n)}$ , one can write explicitly the  
 1174 four-term expansion for four types of boundary condition:

1175 • Dirichlet case,

$$1176 \quad (8.3) \quad \lambda_h^{D, (n, k)} = iR_0 - h^{\frac{2}{3}} a_n e^{-i\pi/3} + h(2k-1) \frac{e^{-i\pi/4}}{\sqrt{2R_0}} + \mathcal{O}(h^{\frac{5}{3}}).$$

1177

1178 • Neumann case

$$1179 \quad (8.4) \quad \lambda_h^{N, (n, k)} = iR_0 - h^{\frac{2}{3}} a_n' e^{-i\pi/3} + h(2k-1) \frac{e^{-i\pi/4}}{\sqrt{2R_0}} + h^{\frac{4}{3}} \frac{e^{-\pi i/6}}{2R_0 a_n'} + \mathcal{O}(h^{\frac{5}{3}}).$$

1180

1181 • Robin case

$$1182 \quad (8.5) \quad \lambda_h^{R, (n, k)} = iR_0 - h^{\frac{2}{3}} a_n^R(\kappa) e^{-i\pi/3} + h(2k-1) \frac{e^{-i\pi/4}}{\sqrt{2R_0}} \\ + h^{\frac{4}{3}} \frac{i}{2R_0(\kappa^2 - a_n^R(\kappa) e^{-\pi i/3})} + \mathcal{O}(h^{\frac{5}{3}}).$$

1183 When  $\kappa = 0$ ,  $a_n^R(0) = a'_n$ , and one retrieves the expansion (8.4) for Neumann case.  
 1184 • Transmission case,

$$1185 \quad (8.6) \quad \lambda_h^{T,(n,k)} = iR_0 - h^{\frac{2}{3}} a_n^+(\kappa) e^{-i\pi/3} + h(2k-1) \frac{e^{-i\pi/4}}{\sqrt{2R_0}} + h^{\frac{4}{3}} \frac{e^{-i\pi/6}}{2R_0 a_n^+(\kappa)} + \mathcal{O}(h^{\frac{5}{3}}).$$

1186 When  $\kappa = 0$ , one has  $a_n^+(0) = a'_n$  and thus retrieves the expansion (8.4) for Neumann  
 1187 case.

1188 We recall that the indices  $n = 1, 2, \dots$  and  $k = 1, 2, \dots$  enumerate eigenvalues of  
 1189 the operators  $\mathcal{L}_0^\#$  and  $\mathcal{L}_2^\#$  that were used in the asymptotic expansion. The approxi-  
 1190 mate eigenvalue with the smallest real part corresponds to  $n = k = 1$ .

1191 The three-terms version of the Neumann expansion (8.4) was first derived by de  
 1192 Swiet and Sen [35] (note that we consider the eigenvalues of the operator  $-h^2\Delta + ix_1$   
 1193 while de Swiet and Sen looked at the complex conjugate operator).

1194 **REMARK 31.** *At the other point  $(-R_0, 0)$ , the parameterization is simply*

$$1195 \quad V(x) = -(R_0 - \rho) \cos(s/R_0)$$

1196 *that alters the signs of the all involved coefficients  $v_{jk}$ . As a consequence, the asymp-*  
 1197 *totics is obtained as the complex conjugate of  $\lambda_h^{\#, (n,k)}$ .*

1198 In the WKB approach, one needs to compute the functions  $\theta_0(s)$  and  $\theta_1(s)$  that  
 1199 determine the asymptotic decay of the quasimode state in the tangential direction.  
 1200 We only consider the Neumann boundary condition while the computation for other  
 1201 cases is similar. From (7.5) and (7.17), we have for the potential in (8.1):

$$1202 \quad v_0(s) = R_0 \cos(s/R_0), \quad v_1(s) = -\cos(s/R_0), \quad \hat{v}_1(s) = 2 - 3 \cos(s/R_0).$$

1203 In what follows, we consider  $s > 0$  though the results will be the same for  $s < 0$  due  
 1204 to the symmetry. From Eqs. (7.13, 7.14), we first obtain

$$1205 \quad (8.7) \quad \theta_0(s) = \int_0^s \sqrt{-iR_0(1 - \cos(s'/R_0))} ds' = e^{-\pi i/4} (2R_0)^{\frac{3}{2}} (1 - \cos(s/(2R_0))).$$

1206 For Neumann boundary condition,  $\mu_0 = -a'_1 e^{-\pi i/3}$  (here  $v_1(0) = -1$ ) and the eigen-  
 1207 value of the operator in (7.20) reads

$$1208 \quad \mu_0(s) = -a'_1 |2 - 3 \cos(s/R_0)|^{\frac{2}{3}} \exp\left(\frac{\pi i}{3} \text{sign}(2 - 3 \cos(s/R_0))\right).$$

1209 Since  $\hat{v}_1(s)$  was assumed to be nonzero, we restrict the analysis to  $|s/R_0| < \arccos(2/3)$   
 1210 for which  $2 - 3 \cos x$  does not vanish (and remains negative) so that

$$1211 \quad (8.8) \quad \mu_0(s) = -a'_1 (3 \cos(s/R_0) - 2)^{\frac{2}{3}} \exp\left(-\frac{\pi i}{3}\right).$$

1212 From (7.23), one gets then

$$1213 \quad (8.9) \quad \begin{aligned} \theta_1(s) &= \int_0^s \frac{-a'_1 e^{-\pi i/3} \left[ (3 \cos(s'/R_0) - 2)^{\frac{2}{3}} - 1 \right]}{2e^{-\pi i/4} R_0^{\frac{1}{2}} \sqrt{1 - \cos(s'/R_0)}} ds' \\ &= \frac{1}{2} |a'_1| e^{-\pi i/12} R_0^{\frac{1}{2}} \int_0^{s/R_0} \frac{(3 \cos x - 2)^{\frac{2}{3}} - 1}{\sqrt{1 - \cos x}} dx. \end{aligned}$$

1214 **8.2. Annulus.** For an annulus  $\Omega = \{(x_1, x_2) \in \mathbb{R}^2 : R_1 < |x| < R_2\}$  between  
 1215 two circles of radii  $R_1$  and  $R_2$ , there are four points in  $\Omega_\perp$ :  $(\pm R_1, 0)$  and  $(\pm R_2, 0)$ .  
 1216 In order to determine the candidate for an eigenvalue with the smallest real part (in  
 1217 short the “first eigenvalue”), one needs to compare the asymptotics of the quasimodes  
 1218 associated with these points and identify those with the smallest real part. Of course,  
 1219 this analysis depends on the imposed boundary conditions. We consider four combi-  
 1220 nations: NN (Neumann condition on both circles), ND (Neumann condition on the  
 1221 inner circle and Dirichlet on the outer circle), DN (Dirichlet condition on the inner  
 1222 circle and Neumann on the outer circle), and DD (Dirichlet condition on both circles).  
 1223 Since the leading contribution is proportional  $|a_1| \approx 2.3381$  for the Dirichlet case and  
 1224 to  $|a'_1| \approx 1.0188$  for the Neumann case, the asymptotics for the circle with Neumann  
 1225 boundary condition always contributes to the first eigenvalue. In turn, when the same  
 1226 boundary condition is imposed on the two circles, the first eigenvalue expansion cor-  
 1227 responds to the outer circle of larger radius because the real part of the next-order  
 1228 term (of order  $h$ ) is always positive and scales as  $1/\sqrt{R_0}$ . As a consequence, the first  
 1229 eigenvalue asymptotics is given by (8.4) with  $R_0 = R_2$  for cases NN and DN, and  
 1230 by (8.3) with  $R_0 = R_2$  for the case DD. Only in the case ND, the first eigenvalue  
 1231 asymptotics is determined by the points  $(\pm R_1, 0)$  on the inner circle. In this case, the  
 1232 potential reads in local coordinates around  $(R_1, 0)$  as  $V(s, \rho) = (R_1 + \rho) \cos(s/R_1)$  so  
 1233 that the only change with respect to the above results is  $v_{01} = 1$  (instead of  $v_{01} = -1$ )  
 1234 and  $\mathfrak{c}(0) = -1/R_1$  (instead of  $\mathfrak{c}(0) = 1/R_1$ ) so that Eq. (8.4) becomes

$$1235 \quad (8.10) \quad \lambda_{app}^{ND, (n, k)} = iR_1 + h^{\frac{2}{3}} |a'_n| e^{i\pi/3} + h(2k - 1) \frac{e^{-i\pi/4}}{\sqrt{2R_1}} + h^{\frac{4}{3}} \frac{e^{\pi i/6}}{2|a'_n|R_1} + \mathcal{O}(h^{\frac{5}{3}}).$$

1236 **REMARK 32.** *When the outer radius  $R_2$  of an annulus goes to infinity, the above*  
 1237 *problem should progressively<sup>5</sup> become an exterior problem in the complement of a*  
 1238 *disk:  $\Omega = \{(x_1, x_2) \in \mathbb{R}^2 : |x| > R_1\}$ . Due to the local character of the asymptotic*  
 1239 *analysis, the expansion (8.10) is independent of the outer radius  $R_2$  and holds even*  
 1240 *for the unbounded case. This argument suggests the non-emptiness of the spectrum*  
 1241 *for unbounded domains. This conjecture is confirmed by numerical results in Sec. 9.*

1242 **8.3. Domain with transmission condition.** Finally, we consider the union  
 1243 of two subdomains, the disk  $\Omega_- = \{(x_1, x_2) \in \mathbb{R}^2 : |x| < R_1\}$  and the annulus  $\Omega_+ =$   
 1244  $\{(x_1, x_2) \in \mathbb{R}^2 : R_1 < |x| < R_2\}$  separated by a circle on which the transmission  
 1245 boundary condition is imposed. A Dirichlet, Neumann or Robin boundary condition  
 1246 can be imposed at the outer boundary (circle of radius  $R_2$ ). As for the annulus, there  
 1247 are four points in  $\Omega_\perp$ :  $(\pm R_1, 0)$  and  $(\pm R_2, 0)$ . Here we focus only on the asymptotic  
 1248 behavior at points  $(\pm R_1, 0)$  for the transmission boundary condition (the behavior  
 1249 at the points  $(\pm R_2, 0)$  was described in Sec. 8.1). We consider the case described in  
 1250 Theorem 27 when the transmission parameter  $\kappa$  scales with  $h$  according to (6.3). As  
 1251 discussed in Sec. 6, this situation is relevant for diffusion MRI applications. The case  
 1252 with fixed  $\kappa$  can be treated similarly.

1253 As stated in Theorem 27, the asymptotic expansion is obtained by starting from  
 1254 the “basic” expansion (with  $\kappa = 0$ ) of either of two problems with Neumann boundary  
 1255 condition corresponding to the two subdomains  $\Omega_-$  and  $\Omega_+$ .  
 1256 If we start from the expansion for the disk, one has  $V(x) = (R_1 - \rho) \cos(s/R_1)$ , and

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<sup>5</sup>We do not have a mathematical proof, the statement remains conjectural.

1257 the asymptotic expansion (6.4) at the point  $(R_1, 0)$  reads

$$1258 \quad (8.11) \quad \lambda_h^{\#, (n, k)} = i R_1 - h^{\frac{2}{3}} a'_n e^{-\pi i/3} + h(2k-1) \frac{e^{-\pi i/4}}{\sqrt{2R_1}} \\ - h^{\frac{4}{3}} \frac{e^{-\pi i/6}}{a'_n} \left( \hat{\kappa} - \frac{1}{2R_1} \right) + O(h^{\frac{5}{3}}).$$

1259 In turn, if we start from the expansion for the inner boundary of the annulus, one has  
1260  $V(x) = (R_1 + \rho) \cos(s/R_1)$ , and the asymptotic expansion (6.4) at the point  $(R_1, 0)$   
1261 reads

$$1262 \quad (8.12) \quad \lambda_h^{\#, (n, k)} = i R_1 - h^{\frac{2}{3}} a'_n e^{\pi i/3} + h(2k-1) \frac{e^{-\pi i/4}}{\sqrt{2R_1}} \\ - h^{\frac{4}{3}} \frac{e^{\pi i/6}}{a'_n} \left( \hat{\kappa} + \frac{1}{2R_1} \right) + O(h^{\frac{5}{3}}).$$

1263 These two expressions are different, in particular, their imaginary parts differ already  
1264 in the order  $h^{\frac{2}{3}}$ . In turn, the real parts differ at the term of order  $h^{\frac{4}{3}}$  that contains  
1265 two contributions: from the curvature of the boundary, and from the transmission.  
1266 While the curvature changes its sign on both sides of the boundary, the contribution  
1267 due to the transmission remains the same. As a consequence, the real part of (8.12) is  
1268 larger than the real part of (8.11). One can thus expect the existence of two distinct  
1269 eigenstates living on both sides of the boundary, as confirmed numerically in the next  
1270 section. For  $k = 1$ , the eigenstate associated with the eigenvalue with the smallest  
1271 real part is mainly localized in the disk side of the boundary.

1272 **9. Numerical results.** This section presents some numerical results to illustrate  
1273 our analysis. The claims of this section are supported by numerical evidence but  
1274 should not be considered as rigorous statements, in contrast to previous sections.

1275 The numerical analysis will be limited to bounded domains, for which the BT-  
1276 operator has compact resolvent and hence discrete spectrum (see Sec. 2). In order  
1277 to compute numerically its eigenvalues and eigenfunctions, one needs to approximate  
1278 the BT-operator in a matrix form. For this purpose, one can either (i) discretize the  
1279 domain by a square lattice and replace the Laplace operator by finite differences (finite  
1280 difference method); (ii) discretize the domain by a mesh and use a weak formulation of  
1281 the eigenvalue problem (finite elements method); or (iii) project the BT-operator onto  
1282 an appropriate complete basis of functions. We choose the last option and project the  
1283 BT-operator onto the Laplacian eigenfunctions which for rotation-invariant domains  
1284 (such as disk, annuli, circular layers) are known explicitly [15]. In this basis, the  
1285 Laplace operator  $-\Delta$  is represented by a diagonal matrix  $\Lambda$ . Moreover, the matrix  
1286 representation of the potential  $V(x) = x_1$  was computed analytically, i.e., the elements  
1287 of the corresponding matrix  $\mathcal{B}$  are known explicitly [12, 13, 14]. As a consequence,  
1288 the computation is reduced to finding the Laplacian eigenvalues for these rotation-  
1289 invariant domains, constructing the matrices  $\Lambda$  and  $\mathcal{B}$  through explicit formulas, and  
1290 then diagonalizing numerically the truncated matrix  $h^2\Lambda + i\mathcal{B}$  which is an approximate  
1291 representation of the BT-operator  $-h^2\Delta + ix_1$ . This numerical procedure yields the  
1292 eigenvalues  $\lambda_h^{(m)}$  of the truncated matrix  $h^2\Lambda + i\mathcal{B}$ , while the associated eigenvectors  
1293 allow one to construct the eigenfunctions  $u_h^{(m)}$ . All eigenvalues are ordered according  
1294 to their increasing real parts:

$$1295 \quad (9.1) \quad \operatorname{Re}\{\lambda_h^{(1)}\} \leq \operatorname{Re}\{\lambda_h^{(2)}\} \leq \dots$$

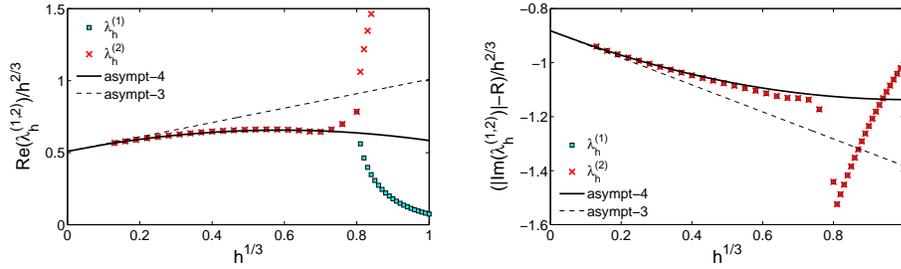


FIG. 1. The rescaled eigenvalues  $\lambda_h^{(1)}$  and  $\lambda_h^{(2)}$  of the BT-operator in the unit disk with Neumann boundary condition. Symbols (squares and crosses) show the numerical results of the diagonalization of the matrix  $h^2\Lambda + i\mathcal{B}$  (truncated to the size  $2803 \times 2803$ ), solid line presents the four-terms asymptotics (8.4) for  $\lambda_h^{N,(1,1)}$  while the dashed line shows its three-terms versions (without  $h^{\frac{4}{3}}$  term).

1296 Note that, for a bounded domain, the potential  $ix$  is a bounded perturbation of  
 1297 the unbounded Laplace operator  $-h^2\Delta$ , if  $h \neq 0$ . To preserve this property after  
 1298 truncation of the matrix  $h^2\Lambda + i\mathcal{B}$ , the truncation size should be chosen such that  
 1299  $h^2\mu^{(M)} \gg 1$ , where  $\mu^{(M)}$  is the largest element of the matrix  $\Lambda$ . Due to the Weyl's  
 1300 law,  $M \sim \frac{|\Omega|}{4\pi}\mu^{(M)}$  so that the truncation size  $M$  should satisfy:

$$1301 \quad (9.2) \quad h^2M \gg \frac{|\Omega|}{4\pi},$$

1302 where  $|\Omega|$  is the surface area of  $\Omega$ . For larger domains, either larger truncation sizes  
 1303 are needed (that can be computationally limiting), or  $h$  should be limited to larger  
 1304 values. In practice, we use  $M$  around 3000 to access  $h$  up to 0.01. We have checked  
 1305 that the truncation size does not affect the computed eigenvalues.

1306 **9.1. Eigenvalues.** For large  $h$ , one can divide the BT operator by  $h^2$ ,  $-\Delta +$   
 1307  $ix_1/h^2$ , to get a small bounded perturbation of the Laplace operator. In particular, the  
 1308 eigenvalues of the operator  $-h^2\Delta + ix_1$  behave asymptotically as  $h^2\mu^{(m)}$ , where  $\mu^{(m)}$   
 1309 are the eigenvalues of the Laplace. In this Section, we focus on the more complicated  
 1310 semi-classical limit  $h \rightarrow 0$  which is the main topic of the paper.

1311 **9.1.1. Disk.** In order to check the accuracy of the asymptotic expansion of eigen-  
 1312 values, we first consider the BT-operator in the unit disk:  $\Omega = \{(x_1, x_2) \in \mathbb{R}^2 : |x| <$   
 1313  $R_0\}$ , with  $R_0 = 1$ . We will present *rescaled* eigenvalues,  $(\lambda_h^{(m)} - iR)/h^{\frac{2}{3}}$ , for which  
 1314 the constant imaginary offset  $iR$  is subtracted and the difference  $\lambda_h^{(m)} - iR$  is divided  
 1315 by  $h^{\frac{2}{3}}$  in order to emphasize the asymptotic behavior. Note also that, according to  
 1316 Remark 31, the asymptotic expansions for the approximate eigenvalues correspond-  
 1317 ing to the points  $(-R, 0)$  and  $(R, 0)$  are the complex conjugates to each other. In  
 1318 order to facilitate their comparison and check this property for numerically computed  
 1319 eigenvalues, we will plot the absolute value of the imaginary part.

1320 Figure 1 shows the first two eigenvalues  $\lambda_h^{(1)}$  and  $\lambda_h^{(2)}$ . For  $h^{\frac{1}{3}} \lesssim 0.8$ , these eigen-  
 1321 values turn out to be the complex conjugate to each other, as expected from their  
 1322 asymptotic expansions (the difference  $\lambda_h^{(1)} - \bar{\lambda}_h^{(2)}$  being negligible within numerical  
 1323 precision). In turn, the eigenvalues  $\lambda_h^{(1)}$  and  $\lambda_h^{(2)}$  become real and split for  $h^{\frac{1}{3}} \gtrsim 0.8$ .  
 1324 The splitting is expected because these eigenvalues behave differently in the large  
 1325  $h$  limit. This numerical observation suggests the existence of branch points in the  
 1326 spectrum (similar features were earlier reported for the complex Airy operator on the

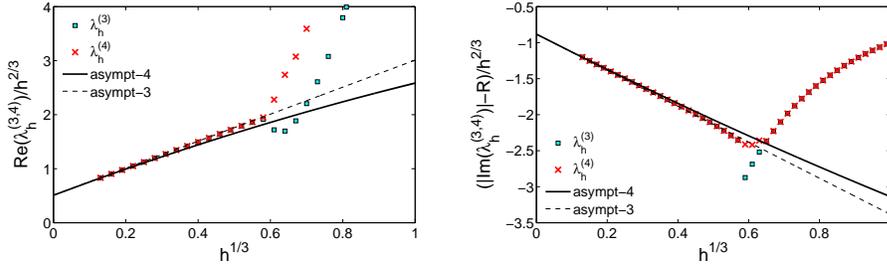


FIG. 2. The rescaled eigenvalues  $\lambda_h^{(3)}$  and  $\lambda_h^{(4)}$  of the BT-operator in the unit disk with Neumann boundary condition. Symbols (squares and crosses) show the numerical results of the diagonalization of the matrix  $h^2\Lambda + i\mathcal{B}$  (truncated to the size  $2803 \times 2803$ ), solid line presents the four-terms asymptotics (8.4) for  $\lambda_h^{N,(1,3)}$  while the dashed line shows its three-terms versions (without  $h^{\frac{4}{3}}$  term).

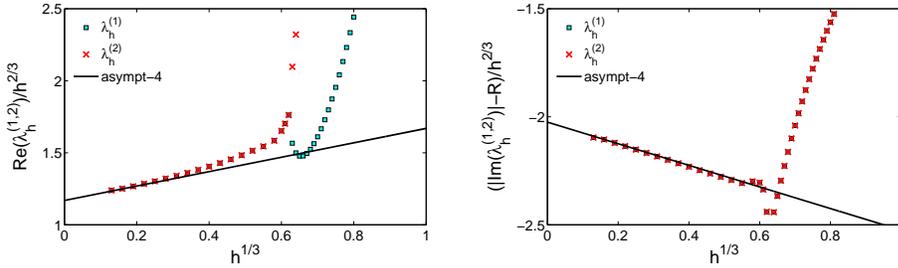


FIG. 3. The rescaled eigenvalues  $\lambda_h^{(1)}$  and  $\lambda_h^{(2)}$  of the BT-operator in the unit disk with Dirichlet boundary condition. Symbols (squares and crosses) show the numerical results of the diagonalization of the matrix  $h^2\Lambda + i\mathcal{B}$  (truncated to the size  $2731 \times 2731$ ), while solid line shows the four-terms asymptotic expansion (8.3) for  $\lambda_h^{D,(1,1)}$ .

1327 one-dimensional interval with Neumann boundary condition, see [34]). For compari-  
 1328 son, the four-terms asymptotics (8.4) for  $\lambda_h^{N,(1,1)}$  and its three-terms (without  
 1329 term  $h^{\frac{4}{3}}$ ) are shown by solid and dashed lines, respectively. These expansions start  
 1330 to be applicable for  $h^{\frac{1}{3}} \lesssim 0.7$ , while their accuracy increases as  $h$  decreases.

1331  
 1332 Figure 2 shows the next eigenvalues  $\lambda_h^{(3)}$  and  $\lambda_h^{(4)}$ , the four-terms asymptotics  
 1333 (8.4) for  $\lambda_h^{N,(1,3)}$  and its three-terms version. These eigenvalues are the complex con-  
 1334 jugates to each other for  $h^{\frac{1}{3}} \lesssim 0.57$  while become real and split for larger  $h$ . One  
 1335 can see that the four-terms asymptotics is less accurate for these eigenvalues than for  
 1336 those from Fig. 1. A small deviation can probably be attributed to higher-order terms  
 1337 (it is worth noting that contributions from the  $h^{\frac{4}{3}}$  and  $h^{\frac{5}{3}}$  terms can be comparable  
 1338 for the considered values of  $h$ ).

1339  
 1340 For comparison, Figure 3 shows the first rescaled eigenvalues  $\lambda_h^{(1)}$  and  $\lambda_h^{(2)}$  of the  
 1341 BT-operator in the unit disk with Dirichlet boundary condition. As earlier for the  
 1342 Neumann case, these eigenvalues are complex conjugate to each other for  $h^{\frac{1}{3}} \lesssim 0.6$   
 1343 while become real and split for larger  $h$ . One can see that the asymptotics (8.3) for  
 1344  $\lambda_h^{D,(1,1)}$  captures the behavior for the imaginary part very accurately. In turn, the  
 1345 behavior of the real part is less accurate, probably due to higher-order corrections.

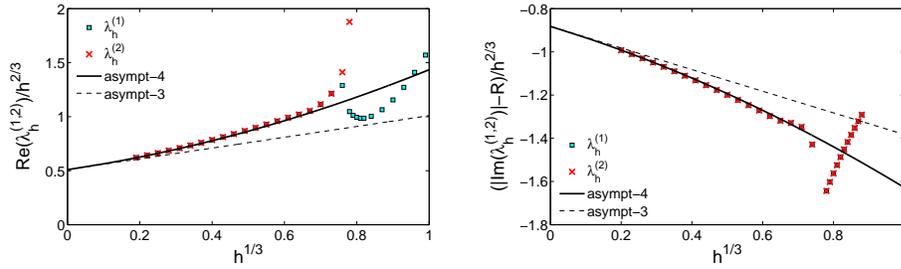


FIG. 4. The rescaled eigenvalues  $\lambda_h^{(1)}$  and  $\lambda_h^{(2)}$  of the BT-operator in the unit disk with Robin boundary condition (with  $\hat{\kappa} = 1$  and  $\kappa = \hat{\kappa}h^{\frac{2}{3}}$ ). Symbols (squares and crosses) show the numerical results of the diagonalization of the matrix  $h^2\Lambda + i\mathcal{B}$  (truncated to the size  $2803 \times 2803$ ), while solid and dashed lines show the four-terms asymptotic expansion (8.3) for  $\lambda_h^{R,(1,1)}$  and its three-term version (without term  $h^{\frac{4}{3}}$ ).

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Finally, Figure 4 illustrates the case with Robin boundary condition, with  $\hat{\kappa} = 1$  while  $\kappa$  scaling as  $\hat{\kappa}h^{\frac{2}{3}}$ . The four-term expansion (6.4) accurately captures their asymptotic behavior.

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**9.1.2. Annulus.** Due to its local character, the quasimodes construction is expected to be applicable to the exterior problem, i.e., in the complement of a disk of radius  $R_1$ ,  $\Omega = \{(x_1, x_2) \in \mathbb{R}^2 : |x| > R_1\}$ . Since we cannot numerically solve this problem for unbounded domains, we consider a circular annulus  $\Omega = \{x \in \mathbb{R}^2 : R_1 < |x| < R_2\}$  with a fixed inner radius  $R_1 = 1$  and then increase the outer radius  $R_2$ . In the limit  $h \rightarrow 0$ , the eigenfunctions are expected to be localized around the four points  $(\pm R_1, 0)$ ,  $(\pm R_2, 0)$  from the set  $\Omega_\perp$ , with corresponding asymptotic expansions for eigenvalues.

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Figure 5 illustrates the discussion in Sec. 8.2 about different asymptotics of the first eigenvalue  $\lambda_h^{(1)}$  for four combinations of Neumann/Dirichlet boundary conditions on inner and outer circles. In particular, one observes the same asymptotic expansion (8.4) with  $R = R_2$  for NN and DN cases because the first eigenvalue is determined by the local behavior near the point  $(R_2, 0)$  which is independent of the boundary condition on the inner circle as  $h \rightarrow 0$ . The expansion (8.3) with  $R = R_2$  for the Dirichlet condition appears only for the case DD. Finally, the case ND is described by the local behavior at the inner circle by the expansion (8.10) with  $R = R_1$ . In what follows, we focus on this case in order to illustrate that the local behavior at the inner boundary is not affected by the position of the outer circle as  $h \rightarrow 0$ .

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For the case ND, Fig. 6 shows the first rescaled eigenvalue  $\lambda_h^{(1)}$  that corresponds to an eigenfunction which, for small  $h$ , is localized near the inner circle. As a consequence, the asymptotic behavior of  $\lambda_h^{(1)}$  as  $h \rightarrow 0$  is expected to be independent of the outer boundary. This is indeed confirmed because the numerical results for three annuli with  $R_2 = 1.5$ ,  $R_2 = 2$  and  $R_3 = 3$  are indistinguishable for  $h^{\frac{1}{3}}$  smaller than 0.5. For comparison, we also plot the four-terms asymptotics (8.10) that we derived for the exterior of the disk of radius  $R_1 = 1$ . One can see that the inclusion of the term  $h^{\frac{4}{3}}$  improves the quality of the expansion (as compared to its reduced three-terms version without  $h^{\frac{4}{3}}$  term).

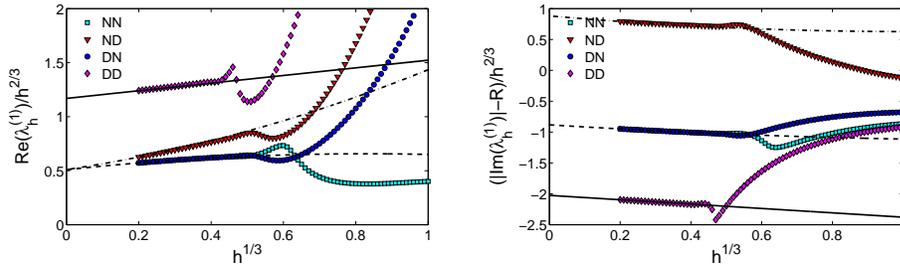


FIG. 5. The rescaled eigenvalue  $\lambda_h^{(1)}$  of the BT-operator in the annulus with four combinations of Neumann/Dirichlet boundary conditions at the inner and outer circles of radii  $R_1 = 1$  and  $R_2 = 2$ : NN (squares), ND (triangles), DN (circles), and DD (diamonds), obtained by the diagonalization of the truncated matrix  $h^2\Lambda + i\mathcal{B}$ . The solid line presents the expansion (8.3) with  $R = R_2$  for Dirichlet condition, the dashed line shows the expansion (8.4) with  $R = R_2$  for Neumann condition, and the dash-dotted line shows the expansion (8.10) with  $R = R_1$  for Neumann condition.

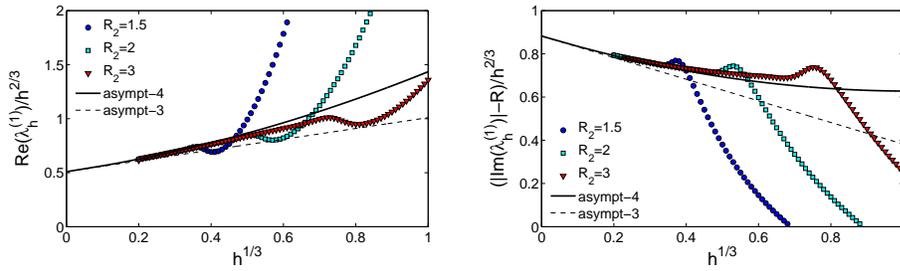


FIG. 6. The rescaled eigenvalue  $\lambda_h^{(1)}$  of the BT-operator in the annulus with Neumann boundary condition at the inner circle of radius  $R_1 = 1$  and Dirichlet boundary condition at the outer circle of radius  $R_2$ , with  $R_2 = 1.5$  (circles),  $R_2 = 2$  (squares) and  $R_2 = 3$  (triangles), obtained by the diagonalization of the matrix  $h^2\Lambda + i\mathcal{B}$  (truncated to sizes  $1531 \times 1531$  for  $R_2 = 1.5$ ,  $2334 \times 2334$  for  $R_2 = 2$ , and  $2391 \times 2391$  for  $R_2 = 3$ ). Solid line presents the four-terms expansion (8.10) for  $\lambda_h^{ND,(1,1)}$ , while dashed line shows its reduced three-terms version (without  $h^{\frac{4}{3}}$  term).

1377 **9.1.3. Domain with transmission condition.** Finally, we consider the BT-  
 1378 operator in the union of two subdomains, the disk  $\Omega_- = \{(x_1, x_2) \in \mathbb{R}^2 : |x| < R_1\}$   
 1379 and the annulus  $\Omega_+ = \{(x_1, x_2) \in \mathbb{R}^2 : R_1 < |x| < R_2\}$  separated by the circle of  
 1380 radius  $R_1$  on which the transmission boundary condition is imposed. We impose the  
 1381 Dirichlet boundary condition at the outer boundary of the domain (at the circle of  
 1382 radius  $R_2$ ) to ensure that first eigenfunctions are localized near points  $(\pm R_1, 0)$  with  
 1383 transmission boundary condition.

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1385 Figure 7 shows the rescaled eigenvalues  $\lambda_h^{(1)}$  and  $\lambda_h^{(2)}$  of the BT-operator with a  
 1386 fixed  $\hat{\kappa} = 1$  and  $\kappa$  scaling as  $\hat{\kappa}h^{\frac{2}{3}}$ . As in earlier examples, the first two eigenvalues are  
 1387 complex conjugate to each other for small  $h$  but they split at larger  $h$ . One can see  
 1388 that the asymptotic relation (8.11) with  $n = k = 1$  accurately describes the behavior  
 1389 of the these eigenvalues for small  $h$ .

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1391 Figure 8 shows the first rescaled eigenvalue  $\lambda_h^{(1)}$  for several values of  $\hat{\kappa}$  (with  $\kappa$   
 1392 scaling as  $\hat{\kappa}h^{\frac{2}{3}}$ ). In the special case  $\hat{\kappa} = 0$ , the two subdomains are separated from  
 1393 each other by Neumann boundary condition, and the spectrum of the BT operator

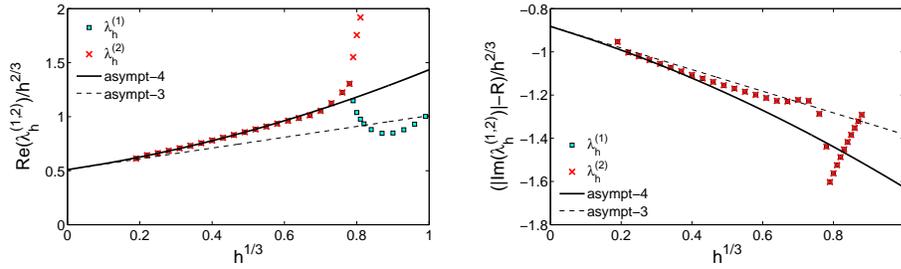


FIG. 7. The rescaled eigenvalues  $\lambda_h^{(1)}$  and  $\lambda_h^{(2)}$  of the BT-operator in the union of the disk and annulus with transmission condition at the inner boundary of radius  $R_1 = 1$  (with  $\hat{\kappa} = 1$ ) and Dirichlet condition at the outer boundary of radius  $R_2 = 2$ . Symbols (squares and crosses) show the numerical results of the diagonalization of the matrix  $h^2\Lambda + i\mathcal{B}$  (truncated to the size  $3197 \times 3197$ ), solid line presents the four-terms expansion (8.11) for  $\lambda_h^{T,(1,1)}$ , while dashed line shows its reduced three-terms version (without  $h^{\frac{4}{3}}$  term).

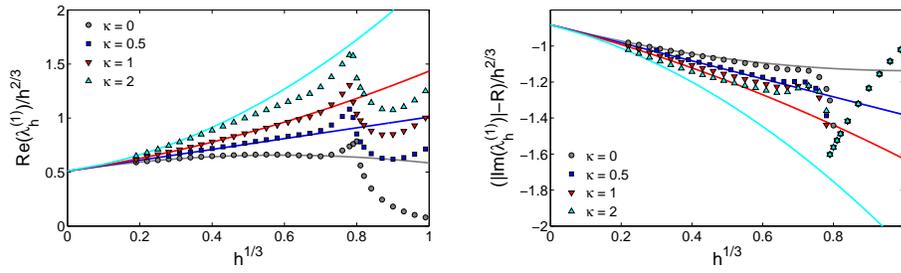


FIG. 8. The rescaled eigenvalue  $\lambda_h^{(1)}$  of the BT-operator in the union of the disk and annulus with transmission condition at the inner boundary of radius  $R_1 = 1$  (with several values of  $\hat{\kappa}$ : 0, 0.5, 1, 2) and Dirichlet condition at the outer boundary of radius  $R_2 = 2$ . Symbols (circles, squares, triangles) show the numerical results of the diagonalization of the truncated matrix  $h^2\Lambda + i\mathcal{B}$ , solid lines present the four-terms expansion (8.11) for  $\lambda_h^{T,(1,1)}$ .

1394 is obtained from its spectra for each subdomain. As a consequence, we plot in this  
 1395 case the first rescaled eigenvalue for the BT operator in the unit disk with Neumann  
 1396 boundary condition (as in Fig. 1). One can see that the expansion (8.11) accurately  
 1397 captures the asymptotic behavior. We recall that the transmission parameter  $\hat{\kappa}$  ap-  
 1398 pears only in the fourth term of order  $h^{\frac{4}{3}}$ . Note also that this term vanishes in the  
 1399 case  $\hat{\kappa} = 1/2$  as two contributions in (8.11) compensate each other.

1400 **9.2. Eigenfunctions.** For the annulus with Neumann boundary condition at  
 1401 the inner circle of radius  $R_1 = 1$  and Dirichlet boundary condition at the outer circle  
 1402 of radius  $R_2 = 2$ , Fig. 9(top) shows two eigenfunctions of the BT operator with  
 1403  $h = 0.1$  (corresponding to  $h^{\frac{1}{3}} \approx 0.4642$ ). One can already recognize the localization  
 1404 of the first eigenfunction  $u_h^{(1)}$  at the inner boundary, while the eigenfunction  $u_h^{(3)}$   
 1405 tends to localize near the outer boundary. Their pairs  $u_h^{(2)}$  and  $u_h^{(4)}$  (not shown) exhibit  
 1406 the same behavior near the opposite points  $(-R_1, 0)$  and  $(-R_2, 0)$ , respectively. Since  
 1407  $h = 0.1$  is not small enough, the localization becomes less and less marked for other  
 1408 eigenfunctions which progressively spread over the whole annulus (not shown). For  
 1409 comparison, we also plot in Fig. 9(bottom) the eigenfunctions  $u_h^{(1)}$  and  $u_h^{(3)}$  for a  
 1410 thicker annulus of outer radius  $R_2 = 3$ . One can see that these eigenfunctions look

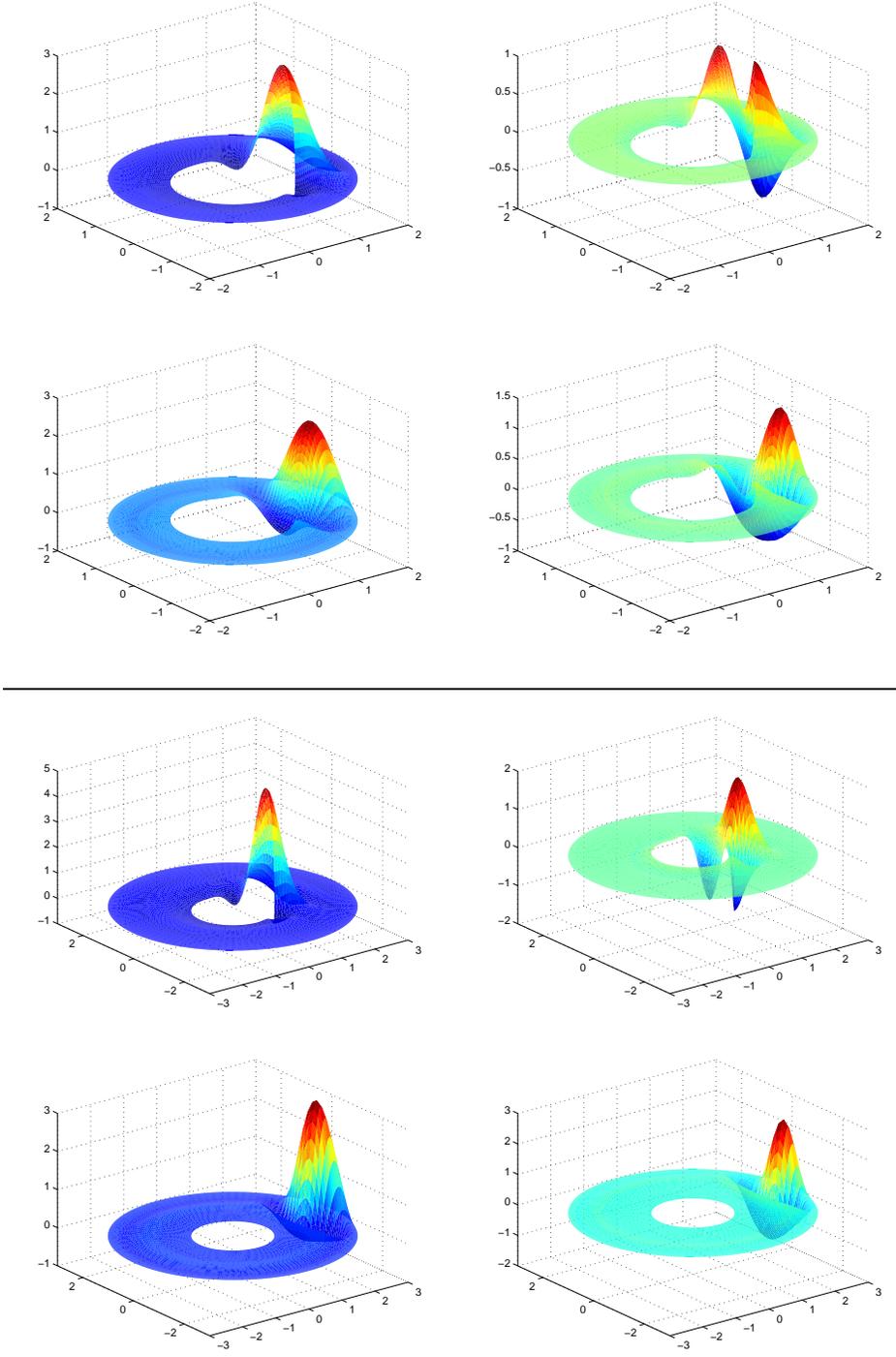


FIG. 9. Real (left) and imaginary (right) parts of the eigenfunctions  $u_h^{(1)}$  (top) and  $u_h^{(3)}$  (bottom) at  $h = 0.1$  for the annulus with Neumann boundary condition at the inner circle of radius  $R_1 = 1$  and Dirichlet boundary condition at the outer circle of radius  $R_2 = 2$  (four plots above horizontal line) or  $R_2 = 3$  (four plots below horizontal line). Numerical computation is based on the truncated matrix representation of sizes  $2334 \times 2334$  and  $2391 \times 2391$ , respectively.

1411 very similar to that of the annulus with  $R_2 = 2$ .

1412 For smaller  $h = 0.01$  (corresponding to  $h^{\frac{1}{3}} \approx 0.2154$ ), the localization of eigen-  
 1413 functions is much more pronounced. Figure 10 shows four eigenfunctions for the  
 1414 annulus of radii  $R_1 = 1$  (Neumann condition) and  $R_2 = 2$  (Dirichlet condition). One  
 1415 can see that the eigenfunctions  $u_h^{(1)}$ ,  $u_h^{(3)}$ , and  $u_h^{(7)}$  are localized near the inner circle  
 1416 while  $u_h^{(5)}$  is localized near the outer circle. When the outer circle is moved away,  
 1417 the former eigenfunctions remain almost unchanged, suggesting that they would exist  
 1418 even in the limiting domain with  $R_2 = \infty$ , i.e., in the complement of the unit disk.  
 1419 In turn, the eigenfunctions that are localized near the outer boundary (such as  $u_h^{(5)}$ )  
 1420 will be eliminated. In spite of this numerical evidence, the existence of eigenfunctions  
 1421 of the BT operator for unbounded domains remains conjectural.

1422 Figure 11 shows the eigenfunctions  $u_h^{(1)}$  and  $u_h^{(3)}$  at  $h = 0.01$  for the union of the  
 1423 disk and annulus with transmission condition at the inner boundary of radius  $R_1 = 1$   
 1424 (with  $\hat{\kappa} = 1$  and  $\kappa = \hat{\kappa}h^{\frac{2}{3}}$ ) and Dirichlet condition at the outer boundary of radius  
 1425  $R_2 = 2$ . Both eigenfunctions are localized near the inner boundary. Moreover, a  
 1426 careful inspection of this figure shows that  $u_h^{(1)}$  is mainly supported by the disk and  
 1427 vanishes rapidly on the other side of the inner circle (i.e., in the annulus side), while  
 1428  $u_h^{(3)}$  exhibits the opposite (i.e., it is localized in the annulus). This is a new feature of  
 1429 localization as compared to the one-dimensional case studied in [16, 17] because the  
 1430 curvature has the opposite signs on two sides of the boundary.

1431 Finally, we check the accuracy of the WKB approximation of the first eigenfunc-  
 1432 tion  $u_h^{(1)}$  for the unit disk with Neumann boundary condition. To make the illustration  
 1433 easier, we plot in Figure 12 the absolute value of  $u_h^{(1)}$  at  $h = 0.01$ , normalized by its  
 1434 maximum, along the boundary (on the circle of radius  $R_0 = 1$ ), near the localization  
 1435 point  $s = 0$ . One can see that the WKB approximation,  $\exp(-(\theta_0(s) + h^{\frac{2}{3}}\theta_1(s))/h)$ ,  
 1436 obtained with  $\theta_0(s)$  and  $\theta_1(s)$  given by (8.7) and (8.9), accurately captures the be-  
 1437 havior over the range of  $s$  between  $-0.3$  and  $0.3$ . Note that its reduced version,  
 1438  $\exp(-\theta_0(s)/h)$ , is also accurate.

1439 **10. Application to diffusion NMR.** In this section, we briefly discuss (with  
 1440 no pretention to mathematical rigor) a possible application of the proposed spectral  
 1441 analysis of the Bloch-Torrey operator to diffusion NMR [12]. In this field, the BT-  
 1442 operator governs the evolution of the transverse nuclear magnetization which satisfies  
 1443 the Bloch-Torrey equation

$$1444 \quad (10.1) \quad \frac{\partial}{\partial t} m(x, t) = [D\Delta - i\gamma g x_1] m(x, t),$$

1445 subject to the uniform initial condition  $m(x, 0) = 1$ . Here  $D$  is the diffusion coefficient,  
 1446  $g$  the magnetic field gradient,  $\gamma$  the gyromagnetic ratio, and the gradient is considered  
 1447 to be constant in time. For a bounded domain, the long-time asymptotic behavior of  
 1448 the solution is determined by the first eigenvalue  $\lambda^{(1)}$  of the BT-operator (with the  
 1449 smallest real part):

$$1450 \quad (10.2) \quad m(x, t) \simeq C u^{(1)}(x) \exp(-\omega t) \quad (t \rightarrow \infty),$$

1451 where

$$1452 \quad (10.3) \quad \omega = \gamma g \lambda_h^{(1)}, \quad h^2 = D/(\gamma g).$$

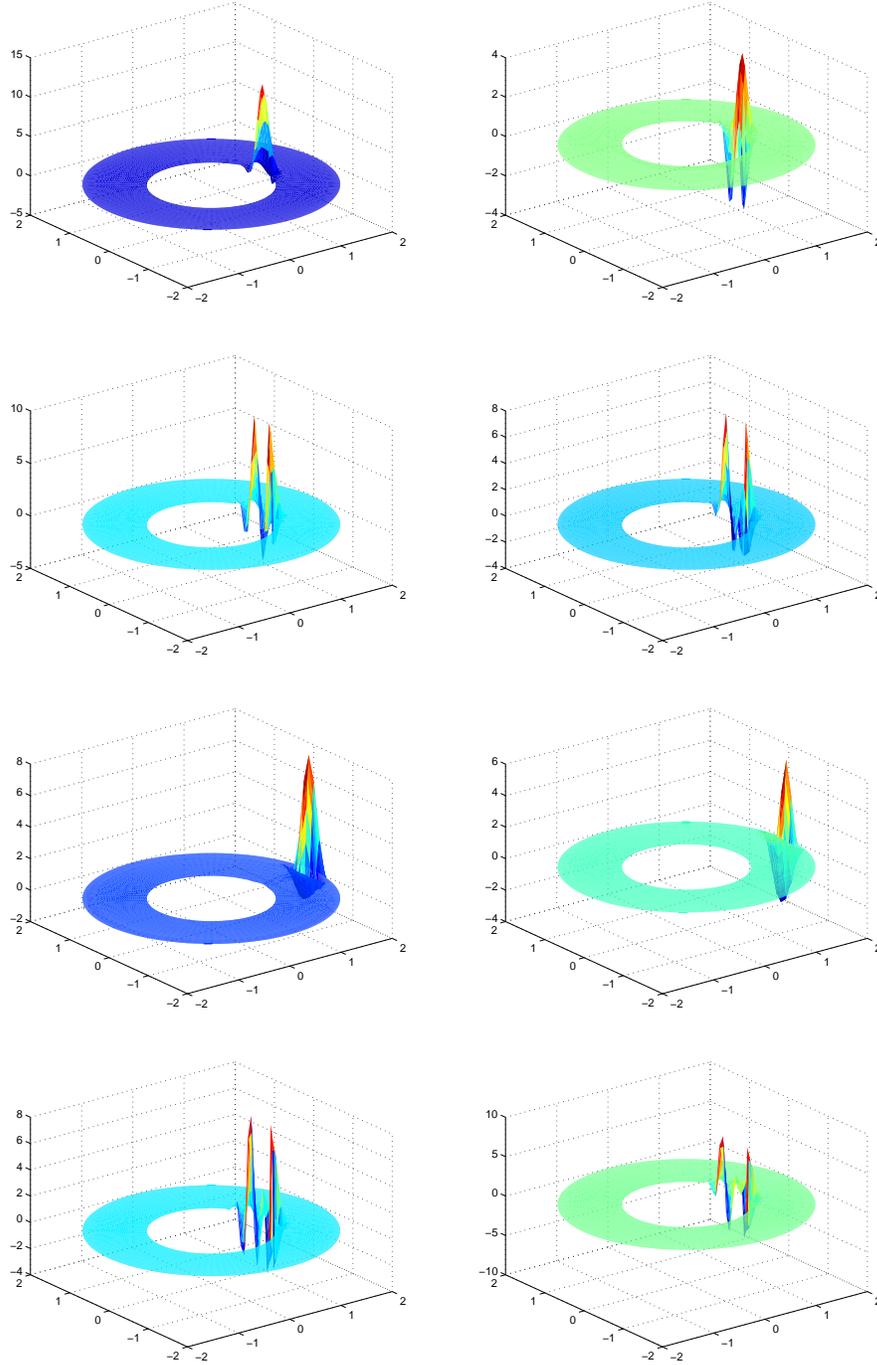


FIG. 10. Real (left) and imaginary (right) parts of the eigenfunctions  $u_h^{(1)}$  (top),  $u_h^{(3)}$ ,  $u_h^{(5)}$  and  $u_h^{(7)}$  (bottom) at  $h = 0.01$  for the annulus with Neumann boundary condition on the inner circle of radius  $R_1 = 1$  and Dirichlet boundary condition on the outer circle of radius  $R_2 = 2$  (numerical computation based on the truncated matrix representation of size  $2334 \times 2334$ ).

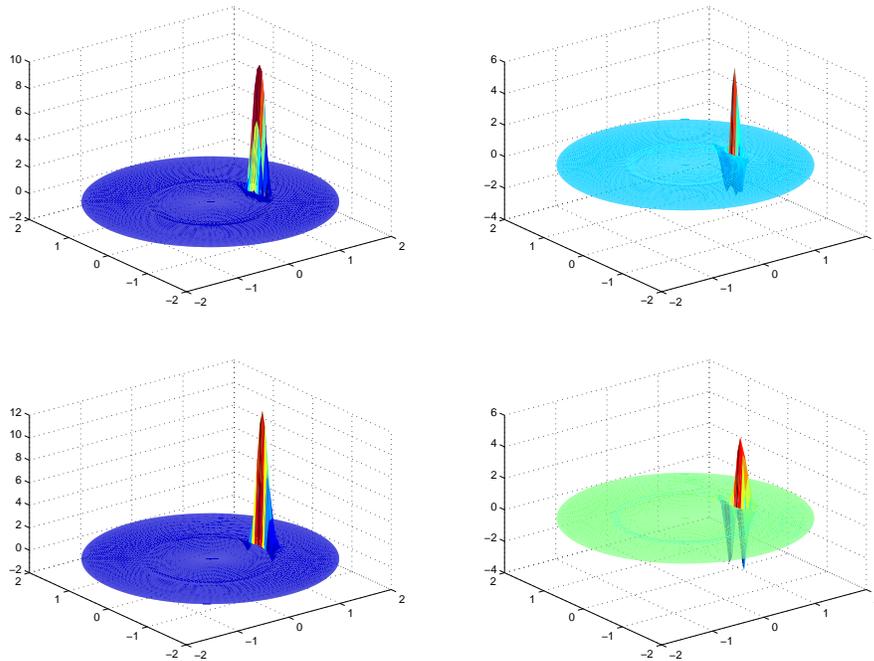


FIG. 11. Real (left) and imaginary (right) parts of the eigenfunctions  $u_h^{(1)}$  (top) and  $u_h^{(3)}$  (bottom) at  $h = 0.01$  for the union of the disk and annulus with a transmission boundary condition (with  $\hat{\kappa} = 1$  and  $\kappa = \hat{\kappa}h^{\frac{2}{3}}$ ) at the inner circle of radius  $R_1 = 1$  and Dirichlet boundary condition at the outer circle of radius  $R_2 = 2$  (numerical computation based on the truncated matrix representation of size  $3197 \times 3197$ ).

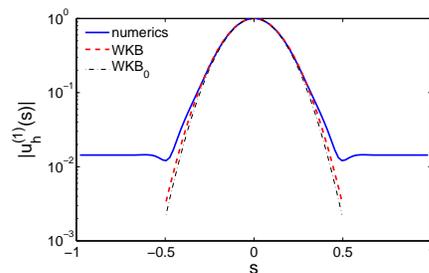


FIG. 12. The absolute value of the first eigenfunction  $u_h^{(1)}(r, s)$  (solid line) at  $h = 0.01$  and  $r = 1$  for the unit disk with Neumann boundary condition, near the boundary point  $s = 0$ . For convenience,  $u_h^{(1)}(r, s)$  is normalized by its maximum at  $s = 0$ . For comparison, the absolute value of the WKB approximation,  $\exp(-(\theta_0(s) + h^{\frac{2}{3}}\theta_1(s))/h)$  and of its reduced version,  $\exp(-\theta_0(s)/h)$ , are shown by dashed and dash-dotted lines, respectively.

1453 Admitting<sup>6</sup> that the formal asymptotic expansion (5.42) with  $n = k = 1$  is the  
 1454 asymptotics of the eigenvalue  $\lambda_h^{(1)}$  with the smallest real part, we obtain in the limit

<sup>6</sup>This has not been proven mathematically.

1455 of large  $g$

$$1456 \quad (10.4) \quad \begin{aligned} \omega &= i \gamma g v_{00} + D^{\frac{1}{3}} (\gamma g)^{\frac{2}{3}} \mu_0^\# |v_{01}|^{\frac{2}{3}} \exp\left(\frac{i\pi}{3} \text{sign } v_{01}\right) \\ &+ D^{\frac{1}{2}} (\gamma g)^{\frac{1}{2}} |v_{20}|^{\frac{1}{2}} \exp\left(\frac{i\pi}{4} \text{sign } v_{20}\right) + D^{\frac{2}{3}} (\gamma g)^{\frac{1}{3}} \lambda_4^{\#, (1)} + \mathcal{O}(g^{\frac{1}{6}}), \end{aligned}$$

1457 where the coefficients  $v_{jk}$  are defined by the local parameterization  $V(x) = x_1$  of the  
1458 boundary near a point from  $\Omega_\perp$ . The real part of  $\omega$  determines the decay rate of the  
1459 transverse magnetization and the related macroscopic signal.

1460 The leading term of order  $(\gamma g)^{\frac{2}{3}}$  was predicted for impermeable one-dimensional  
1461 domains (with Neumann boundary condition) by Stoller *et al.* [34] and experimen-  
1462 tally confirmed by Hürlimann *et al.* [27]. The next-order correction was obtained by  
1463 de Swiet and Sen [35] for an impermeable disk. In the present paper, we general-  
1464 ized these results to arbitrary planar domains with smooth boundary and to various  
1465 boundary conditions (Neumann, Dirichlet, Robin, transmission) and provided a gen-  
1466 eral technique for getting higher-order corrections (in particular, we derived the last  
1467 term). Moreover, we argued (without rigorous proof) that these asymptotic relations  
1468 should also hold for unbounded domains.

#### 1469 Appendix A. Explicit computation of $\lambda_4$ .

1470 **A.1. Evaluation of the integral with  $\phi_1$ .** In order to compute  $\lambda_4$  from (5.55),  
1471 we first evaluate the integral

$$1472 \quad (A.1) \quad \eta = \int_{-\infty}^{\infty} \sigma \phi_1(\sigma) \phi_0(\sigma) d\sigma.$$

1473 We recall that  $\phi_1(\sigma)$  satisfies

$$1474 \quad (A.2) \quad (\mathcal{L}_2 - \lambda_2) \phi_1 = c_{11} \sigma \phi_0,$$

1475 with

$$1476 \quad (A.3) \quad c_{11} := -i v_{11} \int \tau \psi_0^\#(\tau)^2 d\tau.$$

1477 As a solution of (A.2), we search for some eigenpair  $\{\lambda_2, \phi_0\} = \{\lambda_2^{(k)}, \phi_0^{(k)}\}$ , with  
1478 some fixed  $k \geq 1$ , where  $\lambda_2^{(k)}$  and  $\phi_0^{(k)}$  are the eigenvalues and eigenfunctions of  
1479 the quantum harmonic oscillator given explicitly in (5.40). Since  $\phi_0^{(k)}$  are expressed  
1480 through the Hermite polynomials  $H_k$ , one can use their recurrence relation,  $H_{k+1}(x) =$   
1481  $2xH_k(x) - 2kH_{k-1}(x)$ , to express

$$1482 \quad (A.4) \quad \sigma \phi_0^{(k)} = \frac{\sqrt{k} \phi_0^{(k+1)} + \sqrt{k-1} \phi_0^{(k-1)}}{(2\gamma)^{\frac{1}{2}}}.$$

1483 It is therefore natural to search for the solution of (A.2) in the form

$$1484 \quad (A.5) \quad \phi_1(\sigma) = C_1 \phi_0^{(k+1)}(\sigma) + C_2 \phi_0^{(k-1)}(\sigma).$$

1485 The coefficients  $C_1$  and  $C_2$  are determined by substituting this expression into (A.2):

$$1486 \quad (A.6) \quad \begin{aligned} (\mathcal{L}_2 - \lambda_2) \phi_1 &= C_1 (\lambda_2^{(k+1)} - \lambda_2^{(k)}) \phi_0^{(k+1)} + C_2 (\lambda_2^{(k-1)} - \lambda_2^{(k)}) \phi_0^{(k-1)} \\ &= c_{11} \frac{\sqrt{k} \phi_0^{(k+1)} + \sqrt{k-1} \phi_0^{(k-1)}}{(2\gamma)^{\frac{1}{2}}}, \end{aligned}$$

1487 from which  $C_1 = c_{11}\sqrt{k}/(2\gamma)^{\frac{3}{2}}$  and  $C_2 = -c_{11}\sqrt{k-1}/(2\gamma)^{\frac{3}{2}}$ , where we used  $\lambda_2^{(k)} =$   
 1488  $\gamma(2k-1)$ , with  $\gamma = |v_{20}|^{\frac{1}{2}} \exp(\frac{\pi i}{4} \text{sign } v_{20})$ . We get then

$$1489 \quad (\text{A.7}) \quad \phi_1(\sigma) = \frac{c_{11}}{(2\gamma)^{\frac{3}{2}}} \left( \sqrt{k} \phi_0^{(k+1)}(\sigma) - \sqrt{k-1} \phi_0^{(k-1)}(\sigma) \right).$$

1490 Substituting this expression into (A.1), one gets

$$1491 \quad (\text{A.8}) \quad \eta = \frac{c_{11}}{4\gamma^2} = -\frac{v_{11}}{4v_{20}} \int \tau \psi_0^\#(\tau)^2 d\tau,$$

1492 independently of  $n$ . We conclude from (5.55) that

$$1493 \quad (\text{A.9}) \quad \lambda_4^\# = -i \frac{v_{11}^2 [I_1^\#]^2}{4v_{20}} + \frac{c(0)}{2} \int \partial_\tau [\psi_0^\#(\tau)]^2 + iv_{02} I_2^\#,$$

1494 where

$$1495 \quad (\text{A.10}) \quad I_1^\# = \int \tau \psi_0^\#(\tau)^2 d\tau, \quad I_2^\# = \int \tau^2 \psi_0^\#(\tau)^2 d\tau.$$

1496 **A.2. Evaluation of the integrals with  $\psi_0^\#$ .** In order to compute these inte-  
 1497 grals, we consider the function  $\Psi(x) = \text{Ai}(\alpha + \beta x)$  that satisfies the Airy equation

$$1498 \quad (\text{A.11}) \quad (-\partial_x^2 + \beta^3 x + \beta^2 \alpha) \Psi(x) = 0.$$

1499 Multiplying this equation by  $\Psi'(x)$ ,  $\Psi(x)$ ,  $x\Psi'(x)$ ,  $x\Psi(x)$ , or  $x^2\Psi'(x)$  and integrating  
 1500 from 0 to infinity, one gets the following five relations:

1.

$$1501 \quad - \int_0^\infty \Psi''(x) \Psi'(x) dx + \int_0^\infty (\beta^3 x + \beta^2 \alpha) \Psi(x) \Psi'(x) dx = 0,$$

1502 which leads to the determination of  $\int_0^{+\infty} \Psi(x)^2 dx$  by the formula

$$1503 \quad (\text{A.12}) \quad \Psi'(0)^2 - \beta^2 \alpha \Psi(0)^2 - \beta^3 \int_0^\infty \Psi(x)^2 dx = 0.$$

2.

$$1504 \quad - \int_0^\infty \Psi''(x) \Psi(x) dx + \int_0^\infty (\beta^3 x + \beta^2 \alpha) \Psi(x)^2 dx = 0.$$

1505 Here we remark that

$$1506 \quad \int_0^\infty \Psi''(x) \Psi(x) dx = \Psi'(0) \Psi(0) - \int_0^\infty \Psi'(x)^2 dx$$

1507 and get

$$1508 \quad (\text{A.13}) \quad - \Psi'(0) \Psi(0) + \int_0^\infty \Psi'(x)^2 dx + \int_0^\infty (\beta^3 x + \beta^2 \alpha) \Psi(x)^2 dx = 0.$$

3.

$$\begin{aligned}
1509 \quad & - \int_0^{\infty} \Psi''(x)x\Psi'(x) dx + \int_0^{\infty} (\beta^3x + \beta^2\alpha)x\Psi(x)\Psi' dx = 0 \implies \\
1510 \quad & \\
1511 \quad (A.14) \quad & \frac{1}{2} \int_0^{\infty} \Psi'(x)^2 dx - \frac{1}{2} \int_0^{\infty} (2\beta^3x + \beta^2\alpha)\Psi(x)^2 dx = 0.
\end{aligned}$$

4.

$$\begin{aligned}
1512 \quad & - \int_0^{\infty} \Psi''(x)x\Psi(x) dx + \int_0^{\infty} (\beta^3x + \beta^2\alpha)x\Psi(x)^2 dx = 0 \implies \\
1513 \quad & \\
1514 \quad & \int_0^{\infty} \Psi'(x)(x\Psi(x))' dx + \int_0^{\infty} (\beta^3x + \beta^2\alpha)x\Psi(x)^2 dx = 0 \implies \\
1515 \quad & \\
1516 \quad (A.15) \quad & \int_0^{\infty} x\Psi'(x)^2 dx - \frac{1}{2}\Psi(0)^2 + \int_0^{\infty} (\beta^3x + \beta^2\alpha)x\Psi(x)^2 dx = 0.
\end{aligned}$$

5.

$$\begin{aligned}
1517 \quad & - \int_0^{\infty} \Psi''(x)x^2\Psi'(x) dx + \int_0^{\infty} (\beta^3x + \beta^2\alpha)x^2\Psi(x)\Psi' dx = 0 \implies \\
1518 \quad & \\
1519 \quad (A.16) \quad & \int_0^{\infty} x\Psi'(x)^2 dx + \int_0^{\infty} (\beta^3x + \beta^2\alpha)x^2\Psi(x)\Psi' dx = 0.
\end{aligned}$$

1520 So we get a linear system of five equations satisfied by  $\int \Psi^2 dx$ ,  $\int x\Psi^2 dx$ ,  $\int x^2\Psi^2 dx$ ,  
1521  $\int \Psi'(x)^2 dx$  and  $\int x\Psi'(x)^2 dx$ . Solving this system, we obtain

$$\begin{aligned}
1522 \quad & \int_0^{\infty} \Psi^2(x) dx = \beta^{-3}[\Psi'(0)]^2 - \alpha\beta^{-1}[\Psi(0)]^2 \\
1523 \quad (A.17) \quad & = \frac{[\text{Ai}'(\alpha)]^2 - \alpha[\text{Ai}(\alpha)]^2}{\beta},
\end{aligned}$$

$$\begin{aligned}
1524 \quad & \int_0^{\infty} x\Psi^2(x) dx = \frac{1}{3\beta^3} \left( -\Psi'(0)\Psi(0) - 2\alpha\beta^2 \int_0^{\infty} \Psi^2(x) dx \right) \\
1525 \quad (A.18) \quad & = -\frac{\text{Ai}(\alpha)\text{Ai}'(\alpha) + 2\alpha[\text{Ai}'(\alpha)]^2 - 2\alpha^2[\text{Ai}(\alpha)]^2}{3\beta^2},
\end{aligned}$$

$$\begin{aligned}
1526 \quad & \int_0^{\infty} x^2\Psi^2(x) dx = \frac{1}{5\beta^3} \left( [\Psi(0)]^2 - 4\alpha\beta^2 \int_0^{\infty} x\Psi^2(x) dx \right) \\
1527 \quad (A.19) \quad & = \frac{[\text{Ai}(\alpha)]^2 + \frac{4}{3}\alpha(\text{Ai}(\alpha)\text{Ai}'(\alpha) + 2\alpha[\text{Ai}'(\alpha)]^2 - 2\alpha^2[\text{Ai}(\alpha)]^2)}{5\beta^3},
\end{aligned}$$

1528 where we assume that the parameter  $\beta$  is such that  $|\arg(\beta)| < \pi/3$  so that  $\Psi(+\infty) =$   
 1529  $\Psi'(+\infty) = 0$  (otherwise the integrals could diverge). These relations allow one to  
 1530 compute the normalization constant  $c_n^\#$  of quasimodes and the contribution  $\lambda_4^\#$  to the  
 1531 eigenvalue. We consider successively Dirichlet, Neumann, Robin, and Transmission  
 1532 cases.

1533 **Dirichlet case.** The function  $\psi_0^D(\tau)$  from (5.19) corresponds to  $\alpha = a_n$  and  
 1534  $\beta = |v_{01}|^{\frac{1}{3}} \exp(\frac{i\pi}{6} \text{sign } v_{01})$  so that  $\text{Ai}(\alpha) = 0$ . The normalization constant  $c_n^D$  in  
 1535 (5.18) is then

$$1536 \quad (\text{A.20}) \quad (c_n^D)^{-2} = \frac{[\text{Ai}'(a_n)]^2}{\beta}.$$

1537 Using (A.17), one gets

$$1538 \quad (\text{A.21}) \quad I_1^D = \int_0^\infty \tau [\psi_0^D(\tau)]^2 d\tau = -\frac{2a_n}{3\beta},$$

$$1539 \quad (\text{A.22}) \quad I_2^D = \int_0^\infty \tau^2 [\psi_0^D(\tau)]^2 d\tau = \frac{8a_n^2}{15\beta^2}.$$

1540 Using (5.42) and (A.8), we obtain

$$1541 \quad \lambda_4^{D,(n)} = i \frac{v_{11}^2 a_n^2}{9v_{20}\beta^2} - \frac{\mathfrak{c}(0)}{2} [\psi_0^D(0)]^2 + iv_{02} \frac{8a_n^2}{15\beta^2}$$

$$1542 \quad (\text{A.23}) \quad = \frac{ia_n^2}{|v_{01}|^{\frac{2}{3}} \exp(\frac{i\pi}{3} \text{sign } v_{01})} \left( \frac{1}{9} \frac{v_{11}^2}{v_{20}} + \frac{8}{15} v_{02} \right),$$

1543 where we used  $\psi_0^D(0) = 0$ .

1544 **Neumann case.** The function  $\psi_0^N(\tau)$  from (5.18) corresponds to  $\alpha = a'_n$  and  
 1545  $\beta = |v_{01}|^{\frac{1}{3}} \exp(\frac{i\pi}{6} \text{sign } v_{01})$  so that  $\text{Ai}'(\alpha) = 0$ . The normalization constant  $c_n^N$  in  
 1546 (5.18) is then

$$1547 \quad (\text{A.24}) \quad (c_n^N)^{-2} = \frac{[\text{Ai}'(\alpha)]^2 - \alpha [\text{Ai}(\alpha)]^2}{\beta} = -\frac{a'_n [\text{Ai}(a'_n)]^2}{\beta}.$$

1548 Using (A.17), one gets

$$1549 \quad (\text{A.25}) \quad I_1^N = \int_0^\infty \tau [\psi_0^N(\tau)]^2 d\tau = -\frac{2a'_n}{3\beta},$$

$$1550 \quad (\text{A.26}) \quad I_2^N = \int_0^\infty \tau^2 [\psi_0^N(\tau)]^2 d\tau = \frac{8(a'_n)^3 - 3}{15a'_n\beta^2},$$

1551 from which

$$1552 \quad (\text{A.27}) \quad \lambda_4^{N,(n)} = \frac{i}{|v_{01}|^{\frac{2}{3}} \exp(\frac{i\pi}{3} \text{sign } v_{01})} \left( -\frac{(a'_n)^2}{18} \frac{v_{11}^2}{v_{20}} + \frac{1}{2a'_n} \mathfrak{c}(0) v_{01} + \frac{8(a'_n)^3 - 3}{15a'_n} v_{02} \right).$$

1553 **Robin case.** The function  $\psi_0^R(\tau)$  from (5.28) corresponds to  $\beta = |v_{01}|^{\frac{1}{3}}\delta$  and  
 1554  $\alpha = a_n^R(\kappa)$  so that  $\text{Ai}'(\alpha) = \hat{\kappa} \text{Ai}(\alpha)$ , with  $\hat{\kappa} = \kappa/(\delta|v_{01}|^{\frac{1}{3}})$  and  $\delta = \exp(\frac{i\pi}{6} \text{sign } v_{01})$ .  
 1555 The normalization constant  $c_n^R$  in (5.28) is then

$$1556 \quad (\text{A.28}) \quad (c_n^R)^{-2} = \frac{[\text{Ai}(a_n^R(\kappa))]^2}{\beta} [\hat{\kappa}^2 - a_n^R(\kappa)] = [\text{Ai}(a_n^R(\kappa))]^2 \frac{\kappa^2 + \lambda_0^R}{iv_{01}},$$

1557 where we used (5.23) for  $\lambda_0^R$ .

1558 Using (A.17), one gets

$$1559 \quad I_1^R = \int_0^\infty \tau [\psi_0^R(\tau)]^2 d\tau = -\frac{\hat{\kappa} + 2\hat{\kappa}^2 a_n^R(\kappa) - 2[a_n^R(\kappa)]^2}{3\beta[\hat{\kappa}^2 - a_n^R(\kappa)]}$$

$$1560 \quad (\text{A.29}) \quad = \frac{2\lambda_0^R}{3iv_{01}} - \frac{\kappa}{3(\kappa^2 + \lambda_0^R)},$$

$$1561 \quad I_2^R = \int_0^\infty \tau^2 [\psi_0^R(\tau)]^2 d\tau = \frac{1 + \frac{4}{3}a_n^R(\kappa)[\hat{\kappa} + 2\hat{\kappa}^2 a_n^R(\kappa) - 2[a_n^R(\kappa)]^2]}{5\beta^2[\hat{\kappa}^2 - a_n^R(\kappa)]}$$

$$1562 \quad (\text{A.30}) \quad = \frac{1}{5(\kappa^2 + \lambda_0^R)} - \frac{8[\lambda_0^R]^2}{15v_{01}^2} - \frac{4\kappa\lambda_0^R}{15iv_{01}(\kappa^2 + \lambda_0^R)}.$$

1563 Using (5.42) and (A.8), we obtain

$$1564 \quad \lambda_4^{R,(n)} = -i \frac{v_{11}^2 [I_1^R]^2}{4v_{20}} - \frac{\mathfrak{c}(0)}{2} [\psi_0^R(0)]^2 + iv_{02} I_2^R$$

$$1565 \quad (\text{A.31}) \quad = -i \frac{v_{11}^2 [I_1^R]^2}{4v_{20}} - \frac{\mathfrak{c}(0)}{2} \frac{iv_{01}}{\kappa^2 + \lambda_0^R} + iv_{02} I_2^R.$$

1566 **REMARK 33.** It is clear from the computation that  $\lambda_4^{R,(n)}$  belongs to  $C^\infty$  in a  
 1567 neighborhood of 0. In particular, we recover

$$1568 \quad (\text{A.32}) \quad \lambda_4^{R,(n)}(0) = \lambda_4^{N,(n)}.$$

1569 **Transmission case.** In order to compute the above integrals for the transmission  
 1570 case, we note that (5.33) can be written as

$$1571 \quad (\text{A.33}) \quad \text{Ai}'(a_n^+) \text{Ai}'(a_n^-) = -\frac{\kappa}{2\pi|v_{01}|^{\frac{1}{3}}},$$

1572 while the Wronskian for Airy functions yields another relation:

$$1573 \quad (\text{A.34}) \quad \bar{\delta} \text{Ai}'(a_n^-) \text{Ai}(a_n^+) + \delta \text{Ai}'(a_n^+) \text{Ai}(a_n^-) = -\frac{1}{2\pi},$$

1574 where  $\delta = \exp(\frac{\pi i}{6} \text{sign } v_{01})$ , and  $a_n^\pm = a_n^\pm(\kappa)$  are given by (5.32).

1575 From (5.31), we then obtain

$$1576 \quad (\text{A.35}) \quad (c_n^T)^{-2} = \frac{a_n^+ \bar{\delta}}{2\pi|v_{01}|^{\frac{1}{3}}} \left( \bar{\delta} \text{Ai}'(a_n^-) \text{Ai}(a_n^+) - \delta \text{Ai}'(a_n^+) \text{Ai}(a_n^-) \right).$$

1577 Using (A.17), we get

$$\begin{aligned}
1578 \quad I_1^T &= \int_{-\infty}^{\infty} \tau \psi_0^T(\tau)^2 d\tau = \frac{(c_n^T)^2}{3|v_{01}|^{\frac{2}{3}}} \left( \frac{\kappa \delta^3}{4\pi^2 |v_{01}|^{\frac{1}{3}}} \right. \\
1579 \quad &\quad \left. + \frac{(a_n^+)^2 \delta^4}{\pi} (\bar{\delta} \text{Ai}'(a_n^-) \text{Ai}(a_n^+) - \delta \text{Ai}'(a_n^+) \text{Ai}(a_n^-)) \right) \\
1580 \quad (\text{A.36}) \quad &= (c_n^T)^2 \frac{\kappa i}{12\pi^2 v_{01}} - \frac{2a_n^+}{3\delta |v_{01}|^{\frac{1}{3}}},
\end{aligned}$$

$$\begin{aligned}
1581 \quad I_2^T &= \int_{-\infty}^{\infty} \tau^2 \psi_0^T(\tau)^2 d\tau = \frac{(c_n^T)^2}{5|v_{01}|} \left( \frac{\kappa a_n^+ \bar{\delta}^4}{3\pi^2 |v_{01}|^{\frac{1}{3}}} \right. \\
1582 \quad &\quad \left. - \frac{8[a_n^+]^3 - 3}{6\pi} \delta^3 (\bar{\delta} \text{Ai}'(a_n^-) \text{Ai}(a_n^+) - \delta \text{Ai}'(a_n^+) \text{Ai}(a_n^-)) \right) \\
1583 \quad (\text{A.37}) \quad &= (c_n^T)^2 \frac{\kappa a_n^+ \bar{\delta}^4}{15\pi^2 |v_{01}|^{\frac{4}{3}}} + \frac{8[a_n^+]^3 - 3}{15a_n^+ \delta^2 |v_{01}|^{\frac{2}{3}}}.
\end{aligned}$$

1584 Finally, we compute the coefficient in front of  $\frac{1}{2}\mathfrak{c}(0)$  in (A.9):

$$1585 \quad (\text{A.38}) \quad I_0^T := \int \partial_\tau [\psi_0^T(\tau)]^2 = [\psi_0^-(0)]^2 - [\psi_0^+(0)]^2 = \frac{|v_{01}|^{\frac{1}{3}} \exp\left(\frac{i\pi}{6} \text{sign } v_{01}\right)}{a_n^+}.$$

1586 We conclude that

$$1587 \quad (\text{A.39}) \quad \lambda_4^{T,(n)} = -i \frac{v_{11}^2 [I_1^T]^2}{4v_{20}} + \mathfrak{c}(0) \frac{|v_{01}|^{\frac{1}{3}} \exp\left(\frac{i\pi}{6} \text{sign } v_{01}\right)}{2a_n^+} + i v_{02} I_2^T.$$

1588 **REMARK 34.** *It is clear from the computation that  $\lambda_4^{T,(n)}(\kappa)$  belongs to  $C^\infty$  in a*  
1589 *neighborhood of 0. In particular, we recover*

$$1590 \quad (\text{A.40}) \quad \lambda_4^{T,(n)}(0) = \lambda_4^{N,(n)}.$$

1591 **A.3. Evaluation of the derivative  $(\mu_n^T)'(0)$ .** The asymptotic relation (6.4)  
1592 involves the derivative of  $\mu_n^\#(\kappa)$  with respect to  $\kappa$  at  $\kappa = 0$ . In this subsection, we  
1593 provide its explicit computation for the transmission case. According to (5.32), we  
1594 have

$$1595 \quad (\text{A.41}) \quad \mu_n^T(\kappa) = -a_n^+(\kappa) = -\lambda_n^T(\kappa/|v_{01}|^{\frac{1}{3}}) \exp\left(\frac{2\pi i}{3} \text{sign } v_{01}\right),$$

1596 where  $\lambda_n^T$  satisfies (5.33).

1597 The derivative with respect to  $\kappa$  at  $\kappa = 0$  reads

$$1598 \quad (\text{A.42}) \quad (\mu_n^T)'(0) = \left( \frac{\partial}{\partial \kappa} \mu_n^T(\kappa) \right)_{\kappa=0} = -(\lambda_n^T)'(0) \frac{1}{|v_{01}|^{\frac{1}{3}}} \exp\left(\frac{2\pi i}{3} \text{sign } v_{01}\right).$$

1599 In turn,  $(\lambda_n^T)'(0)$  can be obtained by differentiating (5.33) with respect to  $\kappa$

$$\begin{aligned}
1600 \quad (\text{A.43}) \quad & 2\pi \frac{(\lambda_n^T)'(0)}{|v_{01}|^{\frac{1}{3}}} \left[ e^{-i\alpha} \lambda_n^T(0) \text{Ai}'(e^{-i\alpha} \lambda_n^T(0)) \text{Ai}(e^{i\alpha} \lambda_n^T(0)) \right. \\
& \quad \left. + e^{i\alpha} \lambda_n^T(0) \text{Ai}'(e^{i\alpha} \lambda_n^T(0)) \text{Ai}(e^{-i\alpha} \lambda_n^T(0)) \right] = -\frac{1}{|v_{01}|^{\frac{1}{3}}},
\end{aligned}$$

1601 where we used the Airy equation:  $\text{Ai}''(z) = z\text{Ai}(z)$ , and a shortcut notation  $\alpha = 2\pi/3$ .  
 1602 At  $\kappa = 0$ , (5.33) admits two solutions,  $\lambda_n^T(0) = e^{i\alpha} a'_n$  and  $\lambda_n^T(0) = e^{-i\alpha} a'_n$ , that  
 1603 correspond to  $v_{01} < 0$  and  $v_{01} > 0$ , respectively.

1604 When  $v_{01} < 0$ , the first term in (A.43) vanishes (as  $\text{Ai}'(e^{-i\alpha}\lambda_n^T(0)) = 0$ ), while  
 1605 the second term can be expressed by using the Wronskian,

$$1606 \quad (\text{A.44}) \quad e^{-i\alpha}\text{Ai}'(e^{-i\alpha}z)\text{Ai}(e^{i\alpha}z) - e^{i\alpha}\text{Ai}'(e^{i\alpha}z)\text{Ai}(e^{-i\alpha}z) = \frac{i}{2\pi} \quad \forall z \in \mathbb{C}.$$

1607 We get then

$$1608 \quad (\lambda_n^T)'(0) = \frac{i}{\lambda_n^T(0)} = \frac{i}{a'_n e^{i\alpha}}.$$

1609 In turn, when  $v_{01} > 0$ , the second term in (A.43) vanishes, while the first term yields

$$1610 \quad (\lambda_n^T)'(0) = \frac{-i}{\lambda_n^T(0)} = \frac{-i}{a'_n e^{-i\alpha}}.$$

1611 Combining these relations, we obtain

$$1612 \quad (\text{A.45}) \quad (\mu_n^T)'(0) = -\frac{1}{a'_n |v_{01}|^{\frac{1}{3}}} \exp\left(-\frac{\pi i}{6} \text{sign } v_{01}\right).$$

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