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► **To cite this version:**

Denis S Grebenkov, Bernard Helffer, Raphael Henry. THE COMPLEX AIRY OPERATOR ON THE LINE WITH A SEMI-PERMEABLE BARRIER . SIAM Journal on Mathematical Analysis, 2017, 49 (3). hal-01456380

HAL Id: hal-01456380

<https://hal.science/hal-01456380>

Submitted on 4 Feb 2017

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1 **THE COMPLEX AIRY OPERATOR ON THE LINE WITH A**
2 **SEMI-PERMEABLE BARRIER**

3 DENIS S. GREBENKOV*, BERNARD HELFFER†, AND RAPHAEL HENRY‡

4 **Abstract.** We consider a suitable extension of the complex Airy operator, $-d^2/dx^2 + ix$, on the
5 real line with a transmission boundary condition at the origin. We provide a rigorous definition of
6 this operator and study its spectral properties. In particular, we show that the spectrum is discrete,
7 the space generated by the generalized eigenfunctions is dense in L^2 (completeness), and we analyze
8 the decay of the associated semi-group. We also present explicit formulas for the integral kernel of
9 the resolvent in terms of Airy functions, investigate its poles, and derive the resolvent estimates.

10 **Key words.** Airy operator, transmission boundary condition, spectral theory, Bloch-Torrey
11 equation

12 **AMS subject classifications.** 35P10, 47A10, 47A75

13 **1. Introduction.** The transmission boundary condition which is considered in
14 this article appears in various exchange problems such as molecular diffusion across
15 semi-permeable membranes [37, 34, 33], heat transfer in composite materials [11,
16 18, 8], or transverse magnetization evolution in nuclear magnetic resonance (NMR)
17 experiments [20]. In the simplest setting of the latter case, one considers the local
18 transverse magnetization $G(x, y; t)$ produced by the nuclei that started from a fixed
19 initial point y and diffused in a constant magnetic field gradient g up to time t . This
20 magnetization is also called the propagator or the Green function of the Bloch-Torrey
21 equation [39]:

22 (1.1)
$$\frac{\partial}{\partial t} G(x, y; t) = (D\Delta - i\gamma g x_1) G(x, y; t),$$

23 with the initial condition

24 (1.2)
$$G(x, y; t = 0) = \delta(x - y),$$

25 where $\delta(x)$ is the Dirac distribution, D the intrinsic diffusion coefficient, $\Delta = \partial^2/\partial x_1^2 +$
26 $\dots + \partial^2/\partial x_d^2$ the Laplace operator in \mathbb{R}^d , γ the gyromagnetic ratio, and x_1 the co-
27 ordinate in a prescribed direction. From an experimental point of view, the local
28 transverse magnetization is too weak to be detected but the macroscopic signal pro-
29 duced by all the nuclei in a sample can be measured. In other words, one can access
30 the double integral of $G(x, y; t) \rho(y)$ over the starting and arrival points y and x ,
31 where $\rho(y)$ is the initial density of the nuclei in the sample. The microstructure
32 of the sample, which can eventually be introduced through boundary conditions to
33 (1.1), affects the motion of the nuclei, the magnetization $G(x, y; t)$, and the resulting
34 macroscopic signal. Measuring the signal at various times t and magnetic field gradi-
35 ents g , one aims at inferring structural properties of the sample [19]. Although this

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36 non-invasive experimental technique has found numerous applications in material sci-
 37 ences and medicine, mathematical aspects of this formidable inverse problem remain
 38 poorly understood. Even the forward problem of relating a given microstructure to
 39 the macroscopic signal is challenging because of the non-selfadjoint character of the
 40 Bloch-Torrey operator $D\Delta - i\gamma gx_1$ in (1.1). In particular, the spectral properties of
 41 this operator were rigorously established only on the line \mathbb{R} (no boundary condition)
 42 and on the half-axis \mathbb{R}_+ with Dirichlet or Neumann boundary conditions (see Sec. 3).

43 Throughout this paper, we focus on the one-dimensional situation
 44 ($d = 1$), in which the operator

$$45 \quad D_x^2 + ix = -\frac{d^2}{dx^2} + ix$$

46 is called the complex Airy operator and appears in many contexts: mathematical
 47 physics, fluid dynamics, time dependent Ginzburg-Landau problems and also as an
 48 interesting toy model in spectral theory (see [3]). We will consider a suitable exten-
 49 sion \mathcal{A}_1^+ of this differential operator and its associated evolution operator $e^{-t\mathcal{A}_1^+}$. The
 50 Green function $G(x, y; t)$ is the distribution kernel of $e^{-t\mathcal{A}_1^+}$. A separate article will
 51 address this operator in higher dimensions [24].

52
 53 For the problem on the line \mathbb{R} , an intriguing property is that this non self-adjoint
 54 operator, which has compact resolvent, has empty spectrum (see Section 3.1). How-
 55 ever, the situation is completely different on the half-line \mathbb{R}_+ . The eigenvalue problem

$$56 \quad (D_x^2 + ix)u = \lambda u,$$

57 for a spectral pair (u, λ) with $u \in H^2(\mathbb{R}_+)$ and $xu \in L^2(\mathbb{R}^+)$ has been thoroughly
 58 analyzed for both Dirichlet ($u(0) = 0$) and Neumann ($u'(0) = 0$) boundary condi-
 59 tions. The spectrum consists of an infinite sequence of eigenvalues of multiplicity one
 60 explicitly related to the zeros of the Airy function (see [36, 26]). The space gener-
 61 ated by the eigenfunctions is dense in $L^2(\mathbb{R}_+)$ (completeness property) but there is
 62 no Riesz basis of eigenfunctions.¹ Finally, the decay of the associated semi-group has
 63 been analyzed in detail. The physical consequences of these spectral properties for
 64 NMR experiments have been first revealed by Stoller, Happer and Dyson [36] and
 65 then thoroughly discussed in [15, 19, 22].

66
 67 In this article, we consider another problem for the complex Airy operator on the
 68 line but with a transmission condition at 0 which reads [22]:

$$69 \quad (1.3) \quad \begin{cases} u'(0_+) &= u'(0_-), \\ u'(0) &= \kappa(u(0_+) - u(0_-)), \end{cases}$$

70 where $\kappa \geq 0$ is a real parameter. In physical terms, the transmission condition ac-
 71 counts for the diffusive exchange between two media \mathbb{R}_- and \mathbb{R}_+ across the barrier
 72 at 0, while κ is defined as the ratio between the barrier permeability and the bulk
 73 diffusion coefficient. This situation is particularly relevant for biological samples and
 74 applications [19, 21, 22]. The case $\kappa = 0$ corresponds to two independent Neumann
 75 problems on \mathbb{R}_- and \mathbb{R}_+ for the complex Airy operator. When κ tends to $+\infty$, the

¹ We recall that a collection of vectors (x_k) in a Hilbert space \mathcal{H} is called Riesz basis if it is an image of an orthonormal basis in \mathcal{H} under some isomorphism.

76 second relation in (1.3) becomes the continuity condition, $u(0_+) = u(0_-)$, and the
 77 barrier disappears. As a consequence, the problem tends (at least formally) to the
 78 standard problem for the complex Airy operator on the line.

79 The main purpose of this paper is to define the complex Airy operator with trans-
 80 mission (Section 4) and then to analyze its spectral properties. Before starting the
 81 analysis of the complex Airy operator with transmission, we first recall in Section 2
 82 the spectral properties of the one-dimensional Laplacian with the transmission condi-
 83 tion, and summarize in Section 3 the known properties of the complex Airy operator.
 84 New properties are also established concerning the Robin boundary condition and
 85 the behavior of the resolvent for real λ going to $+\infty$. In Section 4 we will show that
 86 the complex Airy operator $\mathcal{A}_1^+ = D_x^2 + ix$ on the line \mathbb{R} with a transmission prop-
 87 erty (1.3) is well defined by an appropriate sesquilinear form and an extension of the
 88 Lax-Milgram theorem. Section 5 focuses on the exponential decay of the associated
 89 semi-group. In Section 6, we present explicit formulas for the integral kernel of the
 90 resolvent and investigate its poles. In Section 7, the resolvent estimates as $|\operatorname{Im} \lambda| \rightarrow 0$
 91 are discussed. Finally, the proof of completeness is reported in Section 8. In five
 92 Appendices, we recall the basic properties of Airy functions (Appendix A), determine
 93 the asymptotic behavior of the resolvent as $\lambda \rightarrow +\infty$ for extensions of the complex
 94 Airy operator on the line (Appendix B) and in the semi-axis (Appendix C), give
 95 the statement of the needed Phragmen-Lindelöf theorem (Appendix D) and finally
 96 describe the numerical method for computing the eigenvalues (Appendix E).

97 We summarize our main results in the following:

98 **THEOREM 1.** *The semigroup $\exp(-t\mathcal{A}_1^+)$ is contracting. The operator \mathcal{A}_1^+ has a*
 99 *discrete spectrum $\{\lambda_n(\kappa)\}$. The eigenvalues $\lambda_n(\kappa)$ are determined as (complex-valued)*
 100 *solutions of the equation*

101 (1.4)
$$2\pi \operatorname{Ai}'(e^{2\pi i/3}\lambda) \operatorname{Ai}'(e^{-2\pi i/3}\lambda) + \kappa = 0,$$

102 where $\operatorname{Ai}'(z)$ is the derivative of the Airy function.

103 For all $\kappa \geq 0$, there exists N such that, for all $n \geq N$, there exists a unique eigenvalue
 104 of \mathcal{A}_1^+ in the ball $B(\lambda_n^\pm, 2\kappa|\lambda_n^\pm|^{-1})$, where $\lambda_n^\pm = e^{\pm 2\pi i/3} a'_n$, and a'_n are the zeros of
 105 $\operatorname{Ai}'(z)$.

106 Finally, for any $\kappa \geq 0$ the space generated by the generalized eigenfunctions of the
 107 complex Airy operator with transmission is dense in $L^2(\mathbb{R}_-) \times L^2(\mathbb{R}_+)$.

108 **REMARK 2.** *Numerical computations suggest that all the spectral projections have*
 109 *rank one (no Jordan block) but we shall only prove in Proposition 36 that there are*
 110 *at most a finite number of eigenvalues with nontrivial Jordan blocks. It will be shown*
 111 *in [6] that the eigenvalues are actually simple. Hence one can replace “generalized*
 112 *eigenfunctions” by “eigenfunctions” in the Theorem 1.*

113 **2. The free Laplacian with a semi-permeable barrier.** As an enlightening
 114 exercise, let us consider in this section the case of the free one-dimensional Laplacian
 115 $-\frac{d^2}{dx^2}$ on $\mathbb{R} \setminus \{0\}$ with the transmission condition (1.3) at $x = 0$. We work in the
 116 Hilbert space

117
$$\mathcal{H} := L^2_- \times L^2_+,$$

118 where $L^2_- := L^2(\mathbb{R}_-)$ and $L^2_+ := L^2(\mathbb{R}_+)$.

119 An element $u \in L^2_- \times L^2_+$ will be denoted by $u = (u_-, u_+)$ and we shall use the
 120 notation $H^s_- = H^s(\mathbb{R}_-)$, $H^s_+ = H^s(\mathbb{R}_+)$ for $s \geq 0$.

121 So (1.3) reads

$$122 \quad (2.1) \quad \begin{cases} u'_+(0) &= u'_-(0), \\ u'_+(0) &= \kappa(u_+(0) - u_-(0)). \end{cases}$$

123 In order to define appropriately the corresponding operator, we start by consid-
124 ering a sesquilinear form defined on the domain

$$125 \quad V = H_-^1 \times H_+^1.$$

126 The space V is endowed with the Hilbert norm $\|\cdot\|_V$ defined for all $u = (u_-, u_+)$ in
127 V by

$$128 \quad \|u\|_V^2 = \|u_-\|_{H_-^1}^2 + \|u_+\|_{H_+^1}^2.$$

129 We then define a Hermitian sesquilinear form a_ν acting on $V \times V$ by the formula

$$130 \quad a_\nu(u, v) = \int_{-\infty}^0 \left(u'_-(x) \bar{v}'_-(x) + \nu u_-(x) \bar{v}_-(x) \right) dx \\ 131 \quad + \int_0^{+\infty} \left(u'_+(x) \bar{v}'_+(x) + \nu u_+(x) \bar{v}_+(x) \right) dx \\ 132 \quad + \kappa (u_+(0) - u_-(0)) \overline{(v_+(0) - v_-(0))},$$

133 for all pairs $u = (u_-, u_+)$ and $v = (v_-, v_+)$ in V . For $z \in \mathbb{C}$, \bar{z} denotes the complex
134 conjugate of z . The parameter $\nu \geq 0$ will be determined later to ensure the coercivity
135 of a_ν .

136 **LEMMA 3.** *The sesquilinear form a_ν is continuous on V .*

137 **Proof**

138 We want to show that, for any $\nu \geq 0$, there exists a positive constant c such that, for
139 all $(u, v) \in V \times V$,

$$140 \quad (2.2) \quad |a_\nu(u, v)| \leq c \|u\|_V \|v\|_V.$$

141 We have, for some $c_0 > 0$,

$$142 \quad \left| \int_{-\infty}^0 \left(u'_-(x) \bar{v}'_-(x) + \nu u_-(x) \bar{v}_-(x) \right) dx \right. \\ 143 \quad \left. + \int_0^{+\infty} \left(u'_+(x) \bar{v}'_+(x) + \nu u_+(x) \bar{v}_+(x) \right) dx \right| \leq c_0 \|u\|_V \|v\|_V.$$

144 On the other hand,

$$145 \quad (2.4) \quad |u_+(0)|^2 = - \int_0^{+\infty} (u_+ \bar{u}_+)'(x) dx \leq 2 \|u\|_{L^2} \|u'\|_{L^2},$$

146 and similarly for $|u_-(0)|^2$, $|v_+(0)|^2$ and $|v_-(0)|^2$. Thus there exists $c_1 > 0$ such that,
147 for all $(u, v) \in V \times V$,

$$148 \quad \left| \kappa (u_-(0) - u_+(0)) \overline{(v_-(0) - v_+(0))} \right| \leq c_1 \|u\|_V \|v\|_V,$$

149 and (2.3) follows with $c = c_0 + c_1$. \square

150

151 The coercivity of the sesquilinear form a_ν for ν large enough is proved in the
152 following lemma. It allows us to define a closed operator associated with a_ν by using
153 the Lax-Milgram theorem.

154 LEMMA 4. *There exist $\nu_0 > 0$ and $\alpha > 0$ such that, for all $\nu \geq \nu_0$,*

$$155 \quad (2.5) \quad \forall u \in V, \quad a_\nu(u, u) \geq \alpha \|u\|_V^2.$$

156 **Proof**

157 The proof is elementary for $\kappa \geq 0$. For completeness, we provide below the proof for
 158 the case $\kappa < 0$ (in which an additional difficulty occurs), but we will keep considering
 159 the physically relevant case $\kappa \geq 0$ throughout the paper.
 160 Using the estimate (2.4) as well as the Young inequality

$$161 \quad \forall e, f, \delta > 0, \quad ef \leq \frac{1}{2}(\delta e^2 + \delta^{-1} f^2),$$

162 we get that, for all $\varepsilon > 0$, there exists $C(\varepsilon) > 0$ such that, for all $u \in V$,

$$163 \quad (2.6) \quad |u_-(0) - u_+(0)|^2 \leq \varepsilon \left(\int_{-\infty}^0 |u'_-(x)|^2 dx + \int_0^{+\infty} |u'_+(x)|^2 dx \right) + C(\varepsilon) \|u\|_{L^2}^2.$$

164 Thus for all $u \in V$ we have

$$165 \quad a_\nu(u, u) \geq (1 - |\kappa|\varepsilon) \left(\int_{-\infty}^0 |u'_-(x)|^2 dx + \int_0^{+\infty} |u'_+(x)|^2 dx \right) \\ 166 \quad (2.7) \quad + (\nu - |\kappa|C(\varepsilon)) \|u\|_{L^2}^2.$$

167 Choosing $\varepsilon < |\kappa|^{-1}$ and $\nu > |\kappa|C(\varepsilon)$, we get (2.5). □

168
 169 The sesquilinear form a_ν being symmetric, continuous and coercive in the sense of
 170 (2.5) on $V \times V$, we can use the Lax-Milgram theorem [26] to define a closed, densely
 171 defined selfadjoint operator S_ν associated with a_ν . Then we set $\mathcal{T}_0 = S_\nu - \nu$. By
 172 construction, the domain of S_ν and \mathcal{T}_0 is

$$173 \quad \mathcal{D}(\mathcal{T}_0) = \{u \in V : v \mapsto a_\nu(u, v) \text{ can be extended continuously} \\ 174 \quad (2.8) \quad \text{on } L_-^2 \times L_+^2\},$$

175 and the operator \mathcal{T}_0 satisfies, for all $(u, v) \in \mathcal{D}(\mathcal{T}_0) \times V$,

$$176 \quad a_\nu(u, v) = \langle \mathcal{T}_0 u, v \rangle + \nu \langle u, v \rangle.$$

177 Now we look for an explicit description of the domain (2.8). The antilinear form
 178 $a(u, \cdot)$ can be extended continuously on $L_-^2 \times L_+^2$ if and only if there exists $w_u =$
 179 $(w_u^-, w_u^+) \in L_-^2 \times L_+^2$ such that

$$180 \quad \forall v \in V, \quad a_\nu(u, v) = \langle w_u, v \rangle.$$

181 According to the expression (2.2), we have necessarily

$$182 \quad w_u = (-u''_- + \nu u_-, -u''_+ + \nu u_+) \in L_-^2 \times L_+^2,$$

183 where u''_- and u''_+ are *a priori* defined in the sense of distributions respectively in
 184 $\mathcal{D}'(\mathbb{R}_-)$ and $\mathcal{D}'(\mathbb{R}_+)$. Moreover (u_-, u_+) has to satisfy conditions (1.3). Consequently
 185 we have

$$186 \quad \mathcal{D}(\mathcal{T}_0) = \left\{ u = (u_-, u_+) \in H_-^1 \times H_+^1 : (u''_-, u''_+) \in L_-^2 \times L_+^2 \right. \\ 187 \quad \left. \text{and } u \text{ satisfies conditions (1.3)} \right\}.$$

188 Finally we have introduced a closed, densely defined selfadjoint operator \mathcal{T}_0 acting by

$$189 \quad \mathcal{T}_0 u = -u''$$

190 on $(-\infty, 0) \cup (0, +\infty)$, with domain

$$191 \quad \mathcal{D}(\mathcal{T}_0) = \{u \in H_-^2 \times H_+^2 : u \text{ satisfies conditions (1.3)}\}.$$

192 Note that at the end \mathcal{T}_0 is independent of the ν chosen for its construction.

193 We observe also that because of the transmission conditions (1.3), the operator \mathcal{T}_0
194 might not be positive when $\kappa < 0$, hence there can be a negative spectrum, as can be
195 seen in the following statement.

196 **PROPOSITION 5.** *For all $\kappa \in \mathbb{R}$, the essential spectrum of \mathcal{T}_0 is*

$$197 \quad (2.9) \quad \sigma_{ess}(\mathcal{T}_0) = [0, +\infty).$$

198 *Moreover, if $\kappa \geq 0$ the operator \mathcal{T}_0 has empty discrete spectrum and*

$$199 \quad (2.10) \quad \sigma(\mathcal{T}_0) = \sigma_{ess}(\mathcal{T}_0) = [0, +\infty).$$

200 *On the other hand, if $\kappa < 0$ there exists a unique negative eigenvalue $-4\kappa^2$, which is*
201 *simple, and*

$$202 \quad (2.11) \quad \sigma(\mathcal{T}_0) = \{-4\kappa^2\} \cup [0, +\infty).$$

203 **Proof**

204 Let us first prove that $[0, +\infty) \subset \sigma_{ess}(\mathcal{T}_0)$. This can be achieved by a standard
205 singular sequence construction.

206 Let $(a_j)_{j \in \mathbb{N}}$ be a positive increasing sequence such that, for all $j \in \mathbb{N}$, $a_{j+1} - a_j >$
207 $2j + 1$. Let $\chi_j \in \mathcal{C}_0^\infty(\mathbb{R})$ ($j \in \mathbb{N}$) such that

$$208 \quad \text{Supp } \chi_j \subset (a_j - j, a_j + j), \|\chi_j\|_{L_+^2} = 1 \text{ and } \sup |\chi_j^{(p)}| \leq \frac{C}{j^p}, \quad p = 1, 2,$$

209 for some C independent of j . Then, for all $r \geq 0$, the sequence $u_j^r(x) = (0, \chi_j(x)e^{irx})$
210 is a singular sequence for \mathcal{T}_0 corresponding to $z = r^2$ in the sense of [17, Definition
211 IX.1.2]. Hence according to [17, Theorem IX.1.3], we have $[0, +\infty) \subset \sigma_{ess}(\mathcal{T}_0)$.

212

213 Now let us prove that $(\mathcal{T}_0 - \mu)$ is invertible for all $\mu \in (-\infty, 0)$ if $\kappa \geq 0$, and for
214 all $\mu \in (-\infty, 0) \setminus \{-4\kappa^2\}$ if $\kappa < 0$.

215 Let $\mu < 0$ and $f = (f_-, f_+) \in L_-^2 \times L_+^2$. We are going to determine explicitly the
216 solutions $u = (u_-, u_+)$ to the equation

$$217 \quad (2.12) \quad \mathcal{T}_0 u = \mu u + f.$$

218 Any solution of the equation $-u_\pm'' = \mu u_\pm + f_\pm$ has the form
219 (2.13)

$$219 \quad u_\pm(x) = \frac{1}{2\sqrt{-\mu}} \int_0^x f_\pm(y) (e^{-\sqrt{-\mu}(x-y)} - e^{\sqrt{-\mu}(x-y)}) dy + A_\pm e^{\sqrt{-\mu}x} + B_\pm e^{-\sqrt{-\mu}x},$$

220 for some $A_\pm, B_\pm \in \mathbb{R}$.

221 We shall now determine A_+ , A_- , B_+ and B_- such that (u_-, u_+) belongs to the
222 domain $\mathcal{D}(\mathcal{T}_0)$. The conditions (1.3) yield

$$223 \quad \begin{cases} A_+ - B_+ &= A_- - B_-, \\ \sqrt{-\mu}(A_+ - B_+) &= -\kappa(A_- + B_- - A_+ - B_+). \end{cases}$$

224 Moreover, the decay conditions at $\pm\infty$ imposed by $u_{\pm} \in H_{\pm}^2$ lead to the following
 225 values for A_+ and B_- :

$$226 \quad (2.14) \quad A_+ = \frac{1}{2\sqrt{-\mu}} \int_0^{+\infty} f_+(y) e^{-\sqrt{-\mu}y} dy, \quad B_- = \frac{1}{2\sqrt{-\mu}} \int_{-\infty}^0 f_-(y) e^{\sqrt{-\mu}y} dy.$$

227 The remaining constants A_- and B_+ have to satisfy the system

$$228 \quad (2.15) \quad \begin{cases} A_- + B_+ &= A_+ + B_-, \\ -\kappa A_- + (\sqrt{-\mu} + \kappa) B_+ &= (\sqrt{-\mu} - \kappa) A_+ + \kappa B_-. \end{cases}$$

229 We then notice that the equation (2.12) has a unique solution $u = (u_-, u_+)$ if and
 230 only if $\kappa \geq 0$ or $\mu \neq -4\kappa^2$.

231 Finally in the case $\kappa < 0$ and $\mu = -4\kappa^2$, the homogeneous equation associated with
 232 (2.12) (*i.e* with $f \equiv 0$) has a one-dimensional space of solutions, namely

$$233 \quad u(x) = (-Ke^{-2\kappa x}, Ke^{2\kappa x})$$

234 with $K \in \mathbb{R}$. Consequently if $\kappa < 0$, the eigenvalue $\mu = -4\kappa^2$ is simple, and the
 235 desired statement is proved. \square

236

237 The expression (2.14) along with the system (2.15) yield the values of A_- and
 238 B_+ when $\mu \notin \sigma(\mathcal{T}_0)$:

$$239 \quad A_- = \frac{2\kappa}{2\sqrt{-\mu}(\sqrt{-\mu} + 2\kappa)} \int_0^{+\infty} f_+(y) e^{-\sqrt{-\mu}y} dy \\ + \frac{1}{2(\sqrt{-\mu} + 2\kappa)} \int_{-\infty}^0 f_-(y) e^{\sqrt{-\mu}y} dy$$

240 and

$$241 \quad B_+ = \frac{1}{2(\sqrt{-\mu} + 2\kappa)} \int_0^{+\infty} f_+(y) e^{-\sqrt{-\mu}y} dy \\ + \frac{2\kappa}{2\sqrt{-\mu}(\sqrt{-\mu} + 2\kappa)} \int_{-\infty}^0 f_-(y) e^{\sqrt{-\mu}y} dy.$$

242 Using (2.13), we are then able to obtain the expression of the integral kernel of $(\mathcal{T}_0 -$
 243 $\mu)^{-1}$. More precisely we have, for all $f = (f_-, f_+) \in L_-^2 \times L_+^2$,

$$244 \quad (\mathcal{T}_0 - \mu)^{-1} = \begin{pmatrix} \mathcal{R}_{\mu}^{-+} & \mathcal{R}_{\mu}^{-+} \\ \mathcal{R}_{\mu}^{+-} & \mathcal{R}_{\mu}^{++} \end{pmatrix},$$

245 where for $\varepsilon, \sigma \in \{-, +\}$ the operator $\mathcal{R}_{\mu}^{\varepsilon\sigma} : \mathbb{R}^{\sigma} \rightarrow \mathbb{R}^{\varepsilon}$ is an integral operator whose
 246 kernel (still denoted $\mathcal{R}_{\mu}^{\varepsilon\sigma}$) is given for all $(x, y) \in \mathbb{R}^{\varepsilon} \times \mathbb{R}^{\sigma}$ by

$$247 \quad (2.16) \quad \mathcal{R}_{\mu}^{\varepsilon,\sigma}(x, y) = \frac{1}{2\sqrt{-\mu}} e^{-\sqrt{-\mu}|x-y|} + \varepsilon\sigma \frac{1}{2(\sqrt{-\mu} + 2\kappa)} e^{-\sqrt{-\mu}(|x|+|y|)}.$$

248 Noticing that the first term in the right-hand side of (2.16) is the integral kernel of
 249 the Laplacian on \mathbb{R} , and that the second term is the kernel of a rank one operator,

250 we finally get the following expression of $(\mathcal{T}_0 - \mu)^{-1}$ as a rank one perturbation of the
251 Laplacian:

$$252 \quad (\mathcal{T}_0 - \mu)^{-1} = (-\Delta - \mu)^{-1} \\ + \frac{1}{2(\sqrt{-\mu} + 2\kappa)} \begin{pmatrix} \langle \cdot, \ell_\mu \rangle_- (\ell_\mu)_- & -\langle \cdot, \ell_\mu \rangle_+ (\ell_\mu)_- \\ -\langle \cdot, \ell_\mu \rangle_- (\ell_\mu)_+ & \langle \cdot, \ell_\mu \rangle_+ (\ell_\mu)_+ \end{pmatrix},$$

253 where $\ell_\mu(x) = e^{-\sqrt{-\mu}|x|}$ and $\langle \cdot, \cdot \rangle_\pm$ denotes the L^2 scalar product on \mathbb{R}^\pm .
254 Here the operator $(-\Delta - \mu)^{-1}$ denotes the operator acting on $L^2_- \times L^2_+$ like the
255 resolvent of the Laplacian on $L^2(\mathbb{R})$:

$$256 \quad (-\Delta - \mu)^{-1}(u_-, u_+) := (-\Delta - \mu)^{-1}(u_- \mathbf{1}_{(-\infty, 0)} + u_+ \mathbf{1}_{(0, +\infty)}),$$

257 composed with the map $L^2(\mathbb{R}) \ni v \mapsto (v|_{\mathbb{R}_-}, v|_{\mathbb{R}_+}) \in L^2_- \times L^2_+$.
258

259 **3. Reminder on the complex Airy operator.** Here we recall relatively basic
260 facts coming from [32, 3, 10, 26, 25, 28, 29] and discuss new questions concerning
261 estimates on the resolvent and the Robin boundary condition. Complements will also
262 be given in Appendices A, B and C.

263 **3.1. The complex Airy operator on the line.** The complex Airy operator
264 on the line can be defined as the closed extension \mathcal{A}^+ of the differential operator
265 $\mathcal{A}_0^+ := D_x^2 + ix$ on $C_0^\infty(\mathbb{R})$. We observe that $\mathcal{A}^+ = (\mathcal{A}_0^-)^*$ with $\mathcal{A}_0^- := D_x^2 - ix$ and
266 that its domain is

$$267 \quad \mathcal{D}(\mathcal{A}^+) = \{u \in H^2(\mathbb{R}), xu \in L^2(\mathbb{R})\}.$$

268 In particular, \mathcal{A}^+ has a compact resolvent. It is also easy to see that $-\mathcal{A}^+$ is the
269 generator of a semi-group S_t of contraction,

$$270 \quad (3.1) \quad S_t = \exp(-t\mathcal{A}^+).$$

271 Hence the results of the theory of semi-groups can be applied (see for example [12]).
272 In particular, we have, for $\operatorname{Re} \lambda < 0$,

$$273 \quad (3.2) \quad \|(\mathcal{A}^+ - \lambda)^{-1}\| \leq \frac{1}{|\operatorname{Re} \lambda|}.$$

274 A very special property of this operator is that, for any $a \in \mathbb{R}$,

$$275 \quad (3.3) \quad T_a \mathcal{A}^+ = (\mathcal{A}^+ - ia) T_a,$$

276 where T_a is the translation operator: $(T_a u)(x) = u(x - a)$.

277 As an immediate consequence, we obtain that the spectrum is empty and that the
278 resolvent of \mathcal{A}^+ ,

$$279 \quad \mathcal{G}_0^+(\lambda) = (\mathcal{A}^+ - \lambda)^{-1},$$

280 which is defined for any $\lambda \in \mathbb{C}$, satisfies

$$281 \quad (3.4) \quad \|(\mathcal{A}^+ - \lambda)^{-1}\| = \|(\mathcal{A}^+ - \operatorname{Re} \lambda)^{-1}\|.$$

282 The most interesting property is the control of the resolvent for $\operatorname{Re} \lambda \geq 0$.

283 PROPOSITION 6 (W. Bordeaux-Montrieux [10]).
 284 As $\operatorname{Re} \lambda \rightarrow +\infty$, we have

$$285 \quad (3.5) \quad \|\mathcal{G}_0^+(\lambda)\| \sim \sqrt{\frac{\pi}{2}} (\operatorname{Re} \lambda)^{-\frac{1}{4}} \exp\left(\frac{4}{3} (\operatorname{Re} \lambda)^{\frac{3}{2}}\right),$$

286 where $f(\lambda) \sim g(\lambda)$ means that the ratio $f(\lambda)/g(\lambda)$ tends to 1 in the limit $\lambda \rightarrow +\infty$.

287 This improves a previous result (see Appendix B) by J. Martinet [32] (see also in
 288 [26, 25]) who also proved²

PROPOSITION 7.

$$289 \quad (3.6) \quad \|\mathcal{G}_0^+(\lambda)\|_{HS} = \|\mathcal{G}_0^+(\operatorname{Re} \lambda)\|_{HS},$$

290 and

$$291 \quad (3.7) \quad \|\mathcal{G}_0^+(\lambda)\|_{HS} \sim \sqrt{\frac{\pi}{2}} (\operatorname{Re} \lambda)^{-\frac{1}{4}} \exp\left(\frac{4}{3} (\operatorname{Re} \lambda)^{\frac{3}{2}}\right) \quad \text{as } \operatorname{Re} \lambda \rightarrow +\infty,$$

292 where $\|\cdot\|_{HS}$ is the Hilbert-Schmidt norm. This is consistent with the well-known
 293 translation invariance properties of the operator \mathcal{A}^+ , see [26]. The comparison be-
 294 tween the HS -norm and the norm in $\mathcal{L}(L^2(\mathbb{R}))$ immediately implies that Proposition
 295 7 gives the upper bound in Proposition 6.

296 **3.2. The complex Airy operator on the half-line: Dirichlet case.** It is
 297 not difficult to define the Dirichlet realization $\mathcal{A}^{\pm, D}$ of $D_x^2 \pm ix$ on \mathbb{R}_+ (the analysis on
 298 the negative semi-axis is similar). One can use for example the Lax-Milgram theorem
 299 and take as form domain

$$300 \quad V^D := \{u \in H_0^1(\mathbb{R}_+), x^{\frac{1}{2}}u \in L_+^2\}.$$

301 It can also be shown that the domain is

$$302 \quad \mathcal{D}^D := \{u \in V^D, u \in H_+^2\}.$$

303 This implies

304 PROPOSITION 8. *The resolvent $\mathcal{G}^{\pm, D}(\lambda) := (\mathcal{A}^{\pm, D} - \lambda)^{-1}$ is in the Schatten class*
 305 *C^p for any $p > \frac{3}{2}$ (see [16] for definition), where $\mathcal{A}^{\pm, D}$ is the Dirichlet realization of*
 306 *$D_x^2 \pm ix$, as emphasized by the superscript D .*

307 More precisely we provide the distribution kernel $\mathcal{G}^{-, D}(x, y; \lambda)$ of the resolvent for
 308 the complex Airy operator $D_x^2 - ix$ on the positive semi-axis with Dirichlet bound-
 309 ary condition at the origin (the results for $\mathcal{G}^{+, D}(x, y; \lambda)$ are similar). Matching the
 310 boundary conditions, one gets

$$311 \quad (3.8) \quad \mathcal{G}^{-, D}(x, y; \lambda) = \begin{cases} 2\pi \frac{\operatorname{Ai}(e^{-i\alpha}w_y)}{\operatorname{Ai}(e^{-i\alpha}w_0)} [\operatorname{Ai}(e^{i\alpha}w_x)\operatorname{Ai}(e^{-i\alpha}w_0) \\ \quad - \operatorname{Ai}(e^{-i\alpha}w_x)\operatorname{Ai}(e^{i\alpha}w_0)] & (0 < x < y), \\ 2\pi \frac{\operatorname{Ai}(e^{-i\alpha}w_x)}{\operatorname{Ai}(e^{-i\alpha}w_0)} [\operatorname{Ai}(e^{i\alpha}w_y)\operatorname{Ai}(e^{-i\alpha}w_0) \\ \quad - \operatorname{Ai}(e^{-i\alpha}w_y)\operatorname{Ai}(e^{i\alpha}w_0)] & (x > y), \end{cases}$$

²The coefficient was wrong in [32] and is corrected here, see Appendix B.

312 where $\text{Ai}(z)$ is the Airy function,

$$313 \quad w_x = ix + \lambda,$$

314 and

$$315 \quad \alpha = 2\pi/3.$$

316 The above expression can also be written as

$$317 \quad (3.9) \quad \mathcal{G}^{-,D}(x, y; \lambda) = \mathcal{G}_0^-(x, y; \lambda) + \mathcal{G}_1^{-,D}(x, y; \lambda),$$

318 where $\mathcal{G}_0^-(x, y; \lambda)$ is the resolvent for the complex Airy operator $D_x^2 - ix$ on the whole
319 line,

$$320 \quad (3.10) \quad \mathcal{G}_0^-(x, y; \lambda) = \begin{cases} 2\pi \text{Ai}(e^{i\alpha} w_x) \text{Ai}(e^{-i\alpha} w_y) & (x < y), \\ 2\pi \text{Ai}(e^{-i\alpha} w_x) \text{Ai}(e^{i\alpha} w_y) & (x > y), \end{cases}$$

321 and

$$322 \quad (3.11) \quad \mathcal{G}_1^{-,D}(x, y; \lambda) = -2\pi \frac{\text{Ai}(e^{i\alpha} \lambda)}{\text{Ai}(e^{-i\alpha} \lambda)} \text{Ai}(e^{-i\alpha}(ix + \lambda)) \text{Ai}(e^{-i\alpha}(iy + \lambda)).$$

323 The resolvent is compact. The poles of the resolvent are determined by the zeros of
324 $\text{Ai}(e^{-i\alpha} \lambda)$, i.e., $\lambda_n = e^{i\alpha} a_n$, where the a_n are zeros of the Airy function: $\text{Ai}(a_n) = 0$.
325 The eigenvalues have multiplicity 1 (no Jordan block). See Appendix A.

326

327 As a consequence of the analysis of the numerical range of the operator, we have

PROPOSITION 9.

$$328 \quad (3.12) \quad \|\mathcal{G}^{\pm, D}(\lambda)\| \leq \frac{1}{|\text{Re } \lambda|}, \quad \text{if } \text{Re } \lambda < 0;$$

329 and

$$330 \quad (3.13) \quad \|\mathcal{G}^{\pm, D}(\lambda)\| \leq \frac{1}{|\text{Im } \lambda|}, \quad \text{if } \mp \text{Im } \lambda > 0.$$

331 This proposition together with the Phragmen-Lindelöf principle (Theorem 54) and
332 Proposition 8 implies (see [2] or [16])

333 PROPOSITION 10. *The space generated by the eigenfunctions of the Dirichlet re-*
334 *alization $\mathcal{A}^{\pm, D}$ of $D_x^2 \pm ix$ is dense in L_+^2 .*

335 It is proven in [28] that there is no Riesz basis of eigenfunctions.

336 At the boundary of the numerical range of the operator, it is interesting to analyze
337 the behavior of the resolvent. Numerical computations lead to the observation that

$$338 \quad (3.14) \quad \lim_{\lambda \rightarrow +\infty} \|\mathcal{G}^{\pm, D}(\lambda)\|_{\mathcal{L}(L_+^2)} = 0.$$

339 As a new result, we will prove

340 PROPOSITION 11. *When λ tends to $+\infty$, we have*

$$341 \quad (3.15) \quad \|\mathcal{G}^{\pm, D}(\lambda)\|_{HS} \approx \lambda^{-\frac{1}{4}} (\log \lambda)^{\frac{1}{2}}.$$

342 The convention “ $A(\lambda) \approx B(\lambda)$ as $\lambda \rightarrow +\infty$ ” means that there exist C and λ_0 such
 343 that

$$344 \quad \frac{1}{C} \leq \frac{|A(\lambda)|}{|B(\lambda)|} \leq C, \quad \forall \lambda \geq \lambda_0,$$

345 or, in other words, $A = \mathcal{O}(|B|)$ and $B = \mathcal{O}(|A|)$.

346 The proof of this proposition will be given in Appendix C.

347 Note that, as $\|\mathcal{G}^{\pm,D}(\lambda)\|_{\mathcal{L}(L^2)} \leq \|\mathcal{G}^{\pm,D}(\lambda)\|_{HS}$, the estimate (3.15) implies (3.14).

348

349 **3.3. The complex Airy operator on the half-line: Neumann case.** Sim-
 350 ilarly, we can look at the Neumann realization $\mathcal{A}^{\pm,N}$ of $D_x^2 \pm ix$ on \mathbb{R}_+ (the analysis
 351 on the negative semi-axis is similar).

352 One can use for example the Lax-Milgram theorem and take as form domain

$$353 \quad V^N = \{u \in H_+^1, x^{\frac{1}{2}}u \in L_+^2\}.$$

354 We recall that the Neumann condition appears when writing the domain of the oper-
 355 ator $\mathcal{A}^{\pm,N}$.

356 As in the Dirichlet case (Proposition 8), this implies

357 PROPOSITION 12. *The resolvent $\mathcal{G}^{\pm,N}(\lambda) := (\mathcal{A}^{\pm,N} - \lambda)^{-1}$ is in the Schatten class*
 358 *\mathcal{C}^p for any $p > \frac{3}{2}$.*

359 More explicitly, the resolvent of $\mathcal{A}^{-,N}$ is obtained as

$$360 \quad \mathcal{G}^{-,N}(x, y; \lambda) = \mathcal{G}_0^-(x, y; \lambda) + \mathcal{G}_1^{-,N}(x, y; \lambda) \quad \text{for } (x, y) \in \mathbb{R}_+^2,$$

361 where $\mathcal{G}_0^-(x, y; \lambda)$ is given by (3.10) and $\mathcal{G}_1^{-,N}(x, y; \lambda)$ is

$$362 \quad (3.16) \quad \mathcal{G}_1^{-,N}(x, y; \lambda) = -2\pi \frac{e^{i\alpha} \text{Ai}'(e^{i\alpha}\lambda)}{e^{-i\alpha} \text{Ai}'(e^{-i\alpha}\lambda)} \text{Ai}(e^{-i\alpha}(ix + \lambda)) \text{Ai}(e^{-i\alpha}(iy + \lambda)).$$

363 The poles of the resolvent are determined by zeros of $\text{Ai}'(e^{-i\alpha}\lambda)$, i.e., $\lambda_n = e^{i\alpha} a'_n$,
 364 where a'_n are zeros of the derivative of the Airy function: $\text{Ai}'(a'_n) = 0$. The eigenvalues
 365 have multiplicity 1 (no Jordan block). See Appendix A.

366 As a consequence of the analysis of the numerical range of the operator, we have

PROPOSITION 13.

$$367 \quad (3.17) \quad \|\mathcal{G}^{\pm,N}(\lambda)\| \leq \frac{1}{|\text{Re } \lambda|}, \quad \text{if } \text{Re } \lambda < 0;$$

368 and

$$369 \quad (3.18) \quad \|\mathcal{G}^{\pm,N}(\lambda)\| \leq \frac{1}{|\text{Im } \lambda|}, \quad \text{if } \mp \text{Im } \lambda > 0.$$

370 This proposition together with Proposition 12 and the Phragmen-Lindelöf principle
 371 implies the completeness of the eigenfunctions:

372

373 PROPOSITION 14. *The space generated by the eigenfunctions of the Neumann re-*
 374 *alization $\mathcal{A}^{\pm,N}$ of $D_x^2 \pm ix$ is dense in L_+^2 .*

375 At the boundary of the numerical range of the operator, we have

376 PROPOSITION 15. *When λ tends to $+\infty$, we have*

$$377 \quad (3.19) \quad \|\mathcal{G}^{\pm, N}(\lambda)\|_{HS} \approx \lambda^{-\frac{1}{4}} (\log \lambda)^{\frac{1}{2}}.$$

378 **Proof**

379 Using the Wronskian (A.3) for Airy functions, we have

$$380 \quad (3.20) \quad \mathcal{G}^{-, D}(x, y; \lambda) - \mathcal{G}^{-, N}(x, y; \lambda) = -ie^{i\alpha} \frac{\text{Ai}(e^{-i\alpha} w_x) \text{Ai}(e^{-i\alpha} w_y)}{\text{Ai}(e^{-i\alpha} \lambda) \text{Ai}'(e^{-i\alpha} \lambda)}.$$

381 Hence

$$382 \quad \|\mathcal{G}^{-, D}(x, y; \lambda) - \mathcal{G}^{-, N}(x, y; \lambda)\|_{HS}^2 = \frac{(\int_0^{+\infty} |\text{Ai}(e^{-i\alpha} w_x)|^2 dx)^2}{|\text{Ai}(e^{-i\alpha} \lambda)|^2 |\text{Ai}'(e^{-i\alpha} \lambda)|^2}.$$

383 We will show in (8.10) that there exists $C > 0$ such that

$$384 \quad \int_0^{+\infty} |\text{Ai}(e^{-i\alpha} w_x)|^2 dx \leq C \lambda^{-\frac{1}{2}} \exp\left(\frac{4}{3} \lambda^{\frac{3}{2}}\right).$$

385 On the other hand, using (A.5) and (A.6), we obtain, for $\lambda \geq \lambda_0$

$$386 \quad |\text{Ai}(e^{-i\alpha} \lambda) \text{Ai}'(e^{-i\alpha} \lambda)| \geq \frac{1}{4\pi} \exp\left(\frac{4}{3} \lambda^{\frac{3}{2}}\right)$$

387 (this argument will also be used in the proof of (8.7)). We have consequently obtained
388 that there exist $C > 0$ and $\lambda_0 > 0$ such that, for $\lambda \geq \lambda_0$,

$$389 \quad (3.21) \quad \|\mathcal{G}^{-, D}(\lambda) - \mathcal{G}^{-, N}(\lambda)\|_{HS} \leq C |\lambda|^{-\frac{1}{4}}.$$

390 The proof of the proposition follows from Proposition 11.

391 **3.4. The complex Airy operator on the half-line: Robin case.** For com-
392 pleteness, we provide new results for the complex Airy operator on the half-line with
393 the Robin boundary condition that naturally extends both Dirichlet and Neumann
394 cases:

$$395 \quad (3.22) \quad \left[\frac{\partial}{\partial x} \mathcal{G}^{-, R}(x, y; \lambda, \kappa) - \kappa \mathcal{G}^{-, R}(x, y; \lambda, \kappa) \right]_{x=0} = 0,$$

396 with a positive parameter κ . The operator is associated with the sesquilinear form
397 defined on $H_+^1 \times H_+^1$ by

$$398 \quad (3.23) \quad a^{-, R}(u, v) = \int_0^{+\infty} u'(x) \bar{v}'(x) dx - i \int_0^{+\infty} x u(x) \bar{v}(x) dx + \kappa u(0) \bar{v}(0).$$

399 The distribution kernel of the resolvent is obtained as

$$400 \quad \mathcal{G}^{-, R}(x, y; \lambda) = \mathcal{G}_0^{-}(x, y; \lambda) + \mathcal{G}_1^{-, R}(x, y; \lambda, \kappa) \quad \text{for } (x, y) \in \mathbb{R}_+^2,$$

401 where

$$402 \quad (3.24) \quad \mathcal{G}_1^{-, R}(x, y; \lambda, \kappa) = -2\pi \frac{ie^{i\alpha} \text{Ai}'(e^{i\alpha} \lambda) - \kappa \text{Ai}(e^{i\alpha} \lambda)}{ie^{-i\alpha} \text{Ai}'(e^{-i\alpha} \lambda) - \kappa \text{Ai}(e^{-i\alpha} \lambda)} \\ \times \text{Ai}(e^{-i\alpha}(ix + \lambda)) \text{Ai}(e^{-i\alpha}(iy + \lambda)).$$

403 Setting $\kappa = 0$, one retrieves (3.16) for the Neumann case, while the limit $\kappa \rightarrow +\infty$
 404 yields (3.11) for the Dirichlet case, as expected. As previously, the resolvent is compact
 405 and actually in the Schatten class \mathcal{C}^p for any $p > \frac{3}{2}$ (see Proposition 8). Its poles are
 406 determined as (complex-valued) solutions of the equation

$$407 \quad (3.25) \quad f^R(\kappa, \lambda) := ie^{-i\alpha} \text{Ai}'(e^{-i\alpha} \lambda) - \kappa \text{Ai}(e^{-i\alpha} \lambda) = 0.$$

408 Except for the case of small κ , in which the eigenvalues might be localized close to
 409 the eigenvalues of the Neumann problem (see Section 4 for an analogous case), it does
 410 not seem easy to localize all the solutions of (3.25) in general. Nevertheless one can
 411 prove that the zeros of $f^R(\kappa, \cdot)$ are simple. If indeed λ is a common zero of f^R and
 412 $(f^R)'$, then either $\lambda + \kappa^2 = 0$, or $e^{-i\alpha} \lambda$ is a common zero of Ai and Ai' . The second
 413 option is excluded by the properties of the Airy function, whereas the first option is
 414 excluded for $\kappa \geq 0$ because the spectrum is contained in the positive half-plane.

415 As a consequence of the analysis of the numerical range of the operator, we have

PROPOSITION 16.

$$416 \quad (3.26) \quad \|\mathcal{G}^{\pm, R}(\lambda, \kappa)\| \leq \frac{1}{|\text{Re } \lambda|}, \quad \text{if } \text{Re } \lambda < 0;$$

417 and

$$418 \quad (3.27) \quad \|\mathcal{G}^{\pm, R}(\lambda, \kappa)\| \leq \frac{1}{|\text{Im } \lambda|}, \quad \text{if } \mp \text{Im } \lambda > 0.$$

419 This proposition together with the Phragmen-Lindelöf principle (Theorem 54) and
 420 the fact that the resolvent is in the Schatten class \mathcal{C}^p , for any $p > \frac{3}{2}$, implies

421 PROPOSITION 17. *For any $\kappa \geq 0$, the space generated by the eigenfunctions of*
 422 *the Robin realization $\mathcal{A}^{\pm, R}$ of $D_x^2 \pm ix$ is dense in L_+^2 .*

423 At the boundary of the numerical range of the operator, it is interesting to analyze
 424 the behavior of the resolvent. Equivalently to Propositions 11 or 15, we have

425 PROPOSITION 18. *When λ tends to $+\infty$, we have*

$$426 \quad (3.28) \quad \|\mathcal{G}^{\pm, R}(\lambda, \kappa)\|_{HS} \approx \lambda^{-\frac{1}{4}} (\log \lambda)^{\frac{1}{2}}.$$

427 **Proof**

428 The proof is obtained by using Proposition 15 and computing, using (A.3),

$$429 \quad \|\mathcal{G}^{-, N}(\lambda) - \mathcal{G}^{-, R}(\kappa, \lambda)\|_{HS}^2 = \left(\int_0^{+\infty} |\text{Ai}(e^{-i\alpha} w_x)|^2 dx \right)^2 \\
 430 \quad \times \frac{\kappa}{2\pi} \frac{1}{|ie^{-i\alpha} \text{Ai}'(e^{-i\alpha} \lambda) - \kappa \text{Ai}(e^{-i\alpha} \lambda)|^2 |\text{Ai}'(e^{-i\alpha} \lambda)|^2}.$$

431 As in the proof of Proposition 15, we show that for any $\kappa_0 > 0$, there exist $C > 0$ and
 432 λ_0 such that, for $\lambda \geq \lambda_0$ and $\kappa \in [0, \kappa_0]$,

$$433 \quad \|\mathcal{G}^{-, N}(\lambda) - \mathcal{G}^{-, R}(\lambda, \kappa)\|_{HS} \leq C |\kappa| \lambda^{-\frac{3}{4}}.$$

434 **4. The complex Airy operator on the line with a semi-permeable barrier: definition and properties.** In comparison with Section 2, we now replace
 435 the differential operator $-\frac{d^2}{dx^2}$ by $\mathcal{A}_1^+ = -\frac{d^2}{dx^2} + ix$ but keep the same transmission
 436

437 condition. To give a precise mathematical definition of the associated closed operator,
438 we consider the sesquilinear form a_ν defined for $u = (u_-, u_+)$ and $v = (v_-, v_+)$ by

$$\begin{aligned}
439 \quad a_\nu(u, v) &= \int_{-\infty}^0 \left(u'_-(x) \bar{v}'_-(x) + i x u_-(x) \bar{v}_-(x) + \nu u_-(x) \bar{v}_-(x) \right) dx \\
440 \quad &+ \int_0^{+\infty} \left(u'_+(x) \bar{v}'_+(x) + i x u_+(x) \bar{v}_+(x) + \nu u_+(x) \bar{v}_+(x) \right) dx \\
441 \quad (4.1) \quad &+ \kappa (u_+(0) - u_-(0)) \overline{(v_+(0) - v_-(0))},
\end{aligned}$$

442 where the form domain \mathcal{V} is

$$443 \quad \mathcal{V} := \left\{ u = (u_-, u_+) \in H_-^1 \times H_+^1 : |x|^{\frac{1}{2}} u \in L_-^2 \times L_+^2 \right\}.$$

444 The space \mathcal{V} is endowed with the Hilbert norm

$$445 \quad \|u\|_{\mathcal{V}} := \left(\|u_-\|_{H_-^1}^2 + \|u_+\|_{H_+^1}^2 + \| |x|^{\frac{1}{2}} u \|_{L_-^2 \times L_+^2}^2 \right)^{\frac{1}{2}}.$$

446 We first observe

447 **LEMMA 19.** *For any $\nu \geq 0$, the sesquilinear form a_ν is continuous on \mathcal{V} .*

448 **Proof**

449 The proof is similar to that of Lemma 3, the additional term

$$450 \quad i \left(\int_{-\infty}^0 x u_-(x) \bar{v}_-(x) dx + \int_0^{+\infty} x u_+(x) \bar{v}_+(x) dx \right)$$

451 being obviously bounded by $\|u\|_{\mathcal{V}} \|v\|_{\mathcal{V}}$. □

452

453 Let us notice that, if u and v belong to $H_-^2 \times H_+^2$ and satisfy the boundary
454 conditions (1.3), then an integration by parts yields

$$\begin{aligned}
455 \quad a_\nu(u, v) &= \int_{-\infty}^0 \left(-u''_-(x) + i x u_-(x) + \nu u_-(x) \right) \bar{v}_-(x) dx \\
456 \quad &+ \int_0^{+\infty} \left(-u''_+(x) + i x u_+(x) + \nu u_+(x) \right) \bar{v}_+(x) dx \\
457 \quad &+ (u'_+(0) + \kappa(u_-(0) - u_+(0))) \overline{(v_-(0) - v_+(0))} \\
458 \quad &= \left\langle \left(-\frac{d^2}{dx^2} + i x + \nu \right) u, v \right\rangle_{L_-^2 \times L_+^2}.
\end{aligned}$$

459 Hence the operator associated with the form a_ν , once defined appropriately, will act
460 as $-\frac{d^2}{dx^2} + i x + \nu$ on $C_0^\infty(\mathbb{R} \setminus \{0\})$.

461

462 As the imaginary part of the potential $i x$ changes sign, it is not straightforward
463 to determine whether the sesquilinear form a_ν is coercive, i.e., whether there exists
464 ν_0 such that for $\nu \geq \nu_0$ the following estimate holds:

$$465 \quad (4.2) \quad \exists \alpha > 0, \quad \forall u \in \mathcal{V}, \quad |a_\nu(u, u)| \geq \alpha \|u\|_{\mathcal{V}}^2.$$

466 Let us show that it is indeed not true. Consider for instance the sequence

$$467 \quad u_n(x) = (\chi(x+n), \chi(x-n)), \quad n \geq 1,$$

468 where $\chi \in C_0^\infty(-1, 1)$ is an even function such that $\chi(x) = 1$ for $x \in [-1/2, 1/2]$.
 469 Then $\|u'_n\|_{L^2(-\infty, 0)}$ and $\|u'_n\|_{L^2(0, +\infty)}$ are bounded, and

$$470 \quad \int_{\mathbb{R}} x |u_n(x)|^2 dx = 0,$$

471 since $x \mapsto x|u_n(x)|^2$ is odd, whereas $\| |x|^{\frac{1}{2}} u_n \|_{L^2} \rightarrow +\infty$ as $n \rightarrow +\infty$. Consequently

$$472 \quad \frac{|a_\nu(u_n, u_n)|}{\|u_n\|_{\mathcal{V}}^2} \rightarrow 0 \text{ as } n \rightarrow +\infty,$$

473 and (4.2) does not hold.

474

475 Due to the lack of coercivity, the standard version of the Lax-Milgram theorem
 476 does not apply. We shall instead use the following generalization introduced in [4].

477 **THEOREM 20.** *Let $\mathcal{V} \subset \mathcal{H}$ be two Hilbert spaces such that \mathcal{V} is continuously em-
 478 bedded in \mathcal{H} and \mathcal{V} is dense in \mathcal{H} . Let a be a continuous sesquilinear form on $\mathcal{V} \times \mathcal{V}$,
 479 and assume that there exist $\alpha > 0$ and two bounded linear operators Φ_1 and Φ_2 on \mathcal{V}
 480 such that, for all $u \in \mathcal{V}$,*

$$481 \quad (4.3) \quad \begin{cases} |a(u, u)| + |a(u, \Phi_1 u)| & \geq \alpha \|u\|_{\mathcal{V}}^2, \\ |a(u, u)| + |a(\Phi_2 u, u)| & \geq \alpha \|u\|_{\mathcal{V}}^2. \end{cases}$$

482 Assume further that Φ_1 extends to a bounded linear operator on \mathcal{H} .

483 Then there exists a closed, densely-defined operator S on \mathcal{H} with domain

$$484 \quad \mathcal{D}(S) = \{u \in \mathcal{V} : v \mapsto a(u, v) \text{ can be extended continuously on } \mathcal{H}\},$$

485 such that, for all $u \in \mathcal{D}(S)$ and $v \in \mathcal{V}$,

$$486 \quad a(u, v) = \langle Su, v \rangle_{\mathcal{H}}.$$

487 Now we want to find two operators Φ_1 and Φ_2 on \mathcal{V} such that the estimates (4.3)

488 hold for the form a_ν defined by (4.1).

489 First we have, as in (2.7),

$$490 \quad \operatorname{Re} a_\nu(u, u) \geq (1 - |\kappa|\varepsilon) \left(\int_{-\infty}^0 |u'_-(x)|^2 dx + \int_0^{+\infty} |u'_+(x)|^2 dx \right) \\ 491 \quad + (\nu - |\kappa|C(\varepsilon)) \|u\|_{L^2}^2.$$

492 Thus by choosing ε and ν appropriately we get, for some $\alpha_1 > 0$,

$$493 \quad (4.4) \quad |a_\nu(u, u)| \geq \alpha_1 \left(\int_{-\infty}^0 |u'_-(x)|^2 dx + \int_0^{+\infty} |u'_+(x)|^2 dx + \|u\|_{L^2}^2 \right).$$

494 It remains to estimate the term $\| |x|^{\frac{1}{2}} u \|_{L^2}$ appearing in the norm $\|u\|_{\mathcal{V}}$. For this
 495 purpose, we introduce the operator

$$496 \quad \rho : (u_-, u_+) \mapsto (-u_-, u_+),$$

497 which corresponds to the multiplication operator by the function $\operatorname{sign} x$.

498 It is clear that ρ maps \mathcal{H} onto \mathcal{H} and \mathcal{V} onto \mathcal{V} . Then we have

$$499 \quad (4.5) \quad \operatorname{Im} a_\nu(u, \rho u) = \| |x|^{\frac{1}{2}} u \|_{L^2}^2.$$

500 Thus using (4.4), there exists α_0 such that, for all $u \in \mathcal{V}$,

$$501 \quad |a_\nu(u, u)| + |a_\nu(u, \rho u)| \geq \alpha \|u\|_{\mathcal{V}}^2.$$

502 Similarly, for all $u \in \mathcal{V}$,

$$503 \quad |a_\nu(u, u)| + |a_\nu(\rho u, u)| \geq \alpha \|u\|_{\mathcal{V}}^2.$$

504 In other words, the estimate (4.3) holds, with $\Phi_1 = \Phi_2 = \rho$. Hence the assumptions
505 of Theorem 20 are satisfied, and we can define a closed operator $\mathcal{A}_1^+ := S - \nu$, which
506 is given by the identity

$$507 \quad \forall u \in \mathcal{D}(\mathcal{A}_1^+), \forall v \in \mathcal{V}, \quad a_\nu(u, v) = \langle \mathcal{A}_1^+ u + \nu u, v \rangle_{L_-^2 \times L_+^2}$$

508 on the domain

$$509 \quad \mathcal{D}(\mathcal{A}_1^+) = \mathcal{D}(S) = \left\{ u \in \mathcal{V} : v \mapsto a_\nu(u, v) \text{ can be extended continuously} \right. \\ 510 \quad \left. \text{on } L_-^2 \times L_+^2 \right\}.$$

511 Now we shall determine explicitly the domain $\mathcal{D}(\mathcal{A}_1^+)$.

512 Let $u \in \mathcal{V}$. The map $v \mapsto a_\nu(u, v)$ can be extended continuously on $L_-^2 \times L_+^2$ if
513 and only if there exists some $w_u = (w_u^-, w_u^+) \in L_-^2 \times L_+^2$ such that, for all $v \in \mathcal{V}$,
514 $a_\nu(u, v) = \langle w_u, v \rangle_{L^2}$. Then due to the definition of $a_\nu(u, v)$, we have necessarily

$$515 \quad w_u^- = -u''_- + i x u_- + \nu u_- \quad \text{and} \quad w_u^+ = -u''_+ + i x u_+ + \nu u_+$$

516 in the sense of distributions respectively in \mathbb{R}_- and \mathbb{R}_+ , and u satisfies the conditions
517 (1.3). Consequently, the domain of \mathcal{A}_1^+ can be rewritten as

$$518 \quad \mathcal{D}(\mathcal{A}_1^+) = \left\{ u \in \mathcal{V} : (-u''_- + i x u_-, -u''_+ + i x u_+) \in L_-^2 \times L_+^2 \right. \\ 519 \quad \left. \text{and } u \text{ satisfies conditions (1.3)} \right\}.$$

520 We now prove that $\mathcal{D}(\mathcal{A}_1^+) = \widehat{\mathcal{D}}$ where

$$521 \quad \widehat{\mathcal{D}} = \left\{ u \in \mathcal{V} : (u_-, u_+) \in H_-^2 \times H_+^2, (x u_-, x u_+) \in L_-^2 \times L_+^2 \right. \\ 522 \quad \left. \text{and } u \text{ satisfies conditions (1.3)} \right\}.$$

523 It remains to check that this implies $(u_-, u_+) \in H_-^2 \times H_+^2$. The only problem is at
524 $+\infty$. Let u_+ be as above and let χ be a nonnegative function equal to 1 on $[1, +\infty)$
525 and with support in $(\frac{1}{2}, +\infty)$. It is clear that the natural extension by 0 of χu_+ to \mathbb{R}
526 belongs to $L^2(\mathbb{R})$ and satisfies

$$527 \quad \left(-\frac{d^2}{dx^2} + i x \right) (\chi u_+) \in L^2(\mathbb{R}).$$

528 One can apply for χu_+ a standard result for the domain of the accretive maximal
529 extension of the complex Airy operator on \mathbb{R} (see for example [26]).

530

531 Finally, let us notice that the continuous embedding

$$532 \quad \mathcal{V} \hookrightarrow L^2(\mathbb{R}; |x| dx) \cap (H_-^1 \times H_+^1)$$

533 implies that \mathcal{A}_1^+ has a compact resolvent; hence its spectrum is discrete.

534

535 Moreover, from the characterization of the domain and its inclusion in $\widehat{\mathcal{D}}$, we
536 deduce the stronger

537 PROPOSITION 21. *There exists λ_0 ($\lambda_0 = 0$ for $\kappa > 0$) such that $(\mathcal{A}_1^+ - \lambda_0)^{-1}$*
 538 *belongs to the Schatten class \mathcal{C}^p for any $p > \frac{3}{2}$.*

539 Note that if it is true for some λ_0 it is true for any λ in the resolvent set.

540 REMARK 22. *The adjoint of \mathcal{A}_1^+ is the operator associated by the same construc-*
 541 *tion with $D_x^2 - ix$. $\mathcal{A}_1^- + \lambda$ being injective, this implies by a general criterion [26] that*
 542 *$\mathcal{A}_1^+ + \lambda$ is maximal accretive, hence generates a contraction semigroup.*

543 The following statement summarizes the previous discussion.

544 PROPOSITION 23. *The operator \mathcal{A}_1^+ acting as*

$$545 \quad u \mapsto \mathcal{A}_1^+ u = \left(-\frac{d^2}{dx^2} u_- + ixu_-, -\frac{d^2}{dx^2} u_+ + ixu_+ \right)$$

546 *on the domain*

$$547 \quad \mathcal{D}(\mathcal{A}_1^+) = \{u \in H_-^2 \times H_+^2 : xu \in L_-^2 \times L_+^2$$

548 (4.6) *and u satisfies conditions (1.3)*

549 *is a closed operator with compact resolvent.*

550 *There exists some positive λ such that the operator $\mathcal{A}_1^+ + \lambda$ is maximal accretive.*

551 REMARK 24. *We have*

$$552 \quad (4.7) \quad \Gamma \mathcal{A}_1^+ = \mathcal{A}_1^- \Gamma,$$

553 *where Γ denotes the complex conjugation:*

$$554 \quad \Gamma(u_-, u_+) = (\bar{u}_-, \bar{u}_+).$$

555 *This implies that the distribution kernel of the resolvent satisfies:*

$$556 \quad (4.8) \quad \mathcal{G}(x, y; \lambda) = \mathcal{G}(y, x; \lambda),$$

557 *for any λ in the resolvent set.*

558 REMARK 25 (PT-Symmetry). *If (λ, u) is an eigenpair, then $(\bar{\lambda}, \bar{u}(-x))$ is also an*
 559 *eigenpair. Let indeed $v(x) = \bar{u}(-x)$. This means $v_-(x) = \bar{u}_+(-x)$ and $v_+(x) =$*
 560 *$\bar{u}_-(-x)$. Hence we get that v satisfies (2.1) if u satisfies the same condition:*

$$561 \quad v'_-(0) = -\bar{u}'_+(0) = \kappa(\bar{u}_-(0) - \bar{u}_+(0)) = +\kappa(v_+(0) - v_-(0)).$$

562 *Similarly one can verify that*

$$563 \quad \begin{aligned} \left(-\frac{d^2}{dx^2} + ix \right) v_+(x) &= \overline{\left(-\frac{d^2}{dx^2} - ix \right) u_-(-x)} \\ &= \overline{\left(\left(-\frac{d^2}{dx^2} + ix \right) u_- \right) (-x)} \\ &= \bar{\lambda} v_+(x). \end{aligned}$$

564 **5. Exponential decay of the associated semi-group.** In order to control the
 565 decay of the associated semi-group, we follow what has been done for the Neumann
 566 or Dirichlet realization of the complex Airy operator on the half-line (see for example
 567 [26] or [28, 29]).

568 THEOREM 26. Assume $\kappa > 0$, then for any $\omega < \inf\{\operatorname{Re} \sigma(\mathcal{A}_1^+)\}$, there exists M_ω
 569 such that, for all $t \geq 0$,

$$570 \quad \|\exp(-t\mathcal{A}_1^+)\|_{\mathcal{L}(L^2_- \times L^2_+)} \leq M_\omega \exp(-\omega t),$$

571 where $\sigma(\mathcal{A}_1^+)$ is the spectrum of \mathcal{A}_1^+ .

572 To apply the quantitative Gearhart-Prüss theorem (see [26]) to the operator \mathcal{A}_1^+ , we
 573 should prove that

$$574 \quad \sup_{\operatorname{Re} z \leq \omega} \|(\mathcal{A}_1^+ - z)^{-1}\| \leq C_\omega,$$

575 for all $\omega < \inf \operatorname{Re} \sigma(\mathcal{A}_1^+) := \omega_1$.

576 First we have by accretivity (remember that $\kappa > 0$), for $\operatorname{Re} \lambda < 0$,

$$577 \quad (5.1) \quad \|(\mathcal{A}_1^+ - \lambda)^{-1}\| \leq \frac{1}{|\operatorname{Re} \lambda|}.$$

578 So it remains to analyze the resolvent in the set

$$579 \quad 0 \leq \operatorname{Re} \lambda \leq \omega_1 - \epsilon, \quad |\operatorname{Im} \lambda| \geq C_\epsilon > 0,$$

580 where $C_\epsilon > 0$ is sufficiently large. Let us prove the following lemma.

581 LEMMA 27. For any $\alpha > 0$, there exist $C_\alpha > 0$ and $D_\alpha > 0$ such that for any
 582 $\lambda \in \{\omega \in \mathbb{C} : \operatorname{Re} \omega \in [-\alpha, +\alpha] \text{ and } |\operatorname{Im} \omega| > D_\alpha\}$,

$$583 \quad (5.2) \quad \|(\mathcal{A}_1^\pm - \lambda)^{-1}\| \leq C_\alpha.$$

584 Proof

585 Without loss of generality, we treat the case when $\operatorname{Im} \lambda > 0$. As in [9], the main idea
 586 of the proof is to approximate $(\mathcal{A}_1^+ - \lambda)^{-1}$ by a sum of two operators: one of them
 587 is a good approximation when applied to functions supported near the transmission
 588 point, while the other one takes care of functions whose support lies far away from
 589 this point.

590 The first operator $\tilde{\mathcal{A}}$ is associated with the sesquilinear form \tilde{a} defined for $u =$
 591 (u_-, u_+) and $v = (v_-, v_+)$ by

$$592 \quad \tilde{a}(u, v) = \int_{-\operatorname{Im} \lambda/2}^0 \left(u'_-(x) \bar{v}'_-(x) + i x u_-(x) \bar{v}_-(x) + \lambda u_-(x) \bar{v}_-(x) \right) dx$$

$$593 \quad + \int_0^{\operatorname{Im} \lambda/2} \left(u'_+(x) \bar{v}'_+(x) + i x u_+(x) \bar{v}_+(x) + \lambda u_+(x) \bar{v}_+(x) \right) dx$$

$$594 \quad (5.3) \quad + \kappa (u_+(0) - u_-(0)) \overline{(v_+(0) - v_-(0))},$$

595 where u and v belong to the following space:

$$596 \quad \mathbb{H}_0^1(\mathcal{S}_\lambda, \mathbb{C}) := (H^1(\mathcal{S}_\lambda^-) \times H^1(\mathcal{S}_\lambda^+)) \cap \{u_-(-\operatorname{Im} \lambda/2) = 0, u_+(\operatorname{Im} \lambda/2) = 0\},$$

597 with $\mathcal{S}_\lambda^- := (-\operatorname{Im} \lambda/2, 0)$ and $\mathcal{S}_\lambda^+ := (0, +\operatorname{Im} \lambda/2)$.

598 The domain $\mathcal{D}(\tilde{\mathcal{A}})$ of $\tilde{\mathcal{A}}$ is the set of $u \in H^2(\mathcal{S}_\lambda^-) \times H^2(\mathcal{S}_\lambda^+)$ such that $u_-(-\operatorname{Im} \lambda/2) =$
 599 $0, u_+(\operatorname{Im} \lambda/2) = 0$ and u satisfies conditions (1.3). Denote the resolvent of $\tilde{\mathcal{A}}$ by
 600 $R_1(\lambda)$ in $\mathcal{L}(L^2(\mathcal{S}_\lambda^-, \mathbb{C}) \times L^2(\mathcal{S}_\lambda^+, \mathbb{C}))$ and observe also that $R_1(\lambda) \in \mathcal{L}(L^2(\mathcal{S}_\lambda^-, \mathbb{C}) \times$
 601 $L^2(\mathcal{S}_\lambda^+, \mathbb{C}), \mathbb{H}_0^1(\mathcal{S}_\lambda, \mathbb{C}))$.

602 We easily obtain (looking at the imaginary part of the sesquilinear form) that

$$603 \quad (5.4) \quad \|R_1(\lambda)\| \leq \frac{2}{\operatorname{Im} \lambda}.$$

604 Furthermore, we have, for $u = R_1(\lambda)f$ (with $u = (u_-, u_+)$, $f = (f_-, f_+)$)

$$\begin{aligned} 605 \quad \|D_x R_1(\lambda)f\|^2 &= \|D_x u\|^2 \\ &\leq \|(\mathcal{A}_1^+ - \lambda)u\| \|u\| + \operatorname{Re} \lambda \|u\|^2 \\ &\leq \|f\| \|R_1(\lambda)f\| + |\alpha| \|\mathcal{R}_1(\lambda)f\|^2 \\ &\leq \left(\frac{2}{|\operatorname{Im} \lambda|} + \frac{4|\alpha|}{|\operatorname{Im} \lambda|^2} \right) \|f\|^2. \end{aligned}$$

606 Hence there exists $C_0(\alpha)$ such that, for $\operatorname{Im} \lambda \geq 1$ and $\operatorname{Re} \lambda \in [-\alpha, +\alpha]$,

$$607 \quad (5.5) \quad \|D_x R_1(\lambda)\| \leq C_0(\alpha) |\operatorname{Im} \lambda|^{-\frac{1}{2}}.$$

608 Far from the transmission point 0, we approximate by the resolvent \mathcal{G}_0^+ of the complex
609 Airy operator \mathcal{A}^+ on the line. Denote this resolvent by $R_2(\lambda)$ when considered as
610 an operator in $\mathcal{L}(L_-^2 \times L_+^2)$. We recall from Section 3 that the norm $\|R_2(\lambda)\|$ is
611 independent of $\operatorname{Im} \lambda$. Since $R_2(\lambda)$ is an entire function of λ , we easily obtain a uniform
612 bound on $\|R_2(\lambda)\|$ for $\operatorname{Re} \lambda \in [-\alpha, +\alpha]$. Hence,

$$613 \quad (5.6) \quad \|R_2(\lambda)\| \leq C_1(\alpha).$$

614 As for the proof of (5.5), we then show

$$615 \quad (5.7) \quad \|D_x R_2(\lambda)\| \leq C(\alpha).$$

616 We now use a partition of unity in the x variable in order to construct an approx-
617 imate inverse $R^{\text{app}}(\lambda)$ for $\mathcal{A}_1^+ - \lambda$. We shall then prove that the difference between
618 the approximation and the exact resolvent is well controlled as $\operatorname{Im} \lambda \rightarrow +\infty$. For this
619 purpose, we define the following triple (ϕ_-, ψ, ϕ_+) of cutoff functions in $C^\infty(\mathbb{R}, [0, 1])$
620 satisfying

$$\begin{aligned} 621 \quad \phi_-(t) &= 1 \text{ on } (-\infty, -1/2], \quad \phi_-(t) = 0 \text{ on } [-1/4, +\infty) \\ \psi(t) &= 1 \text{ on } [-1/4, 1/4], \quad \psi(t) = 0 \text{ on } (-\infty, -1/2] \cup [1/2, +\infty), \\ \phi_+(t) &= 1 \text{ on } [1/2, +\infty), \quad \phi_+(t) = 0 \text{ on } (-\infty, 1/4], \\ \phi_-(t)^2 + \psi(t)^2 + \phi_+(t)^2 &= 1 \text{ on } \mathbb{R}, \end{aligned}$$

622 and then set

$$623 \quad \phi_{\pm, \lambda}(x) = \phi_{\pm} \left(\frac{x}{\operatorname{Im} \lambda} \right), \quad \psi_{\lambda}(x) = \psi \left(\frac{x}{\operatorname{Im} \lambda} \right).$$

624 The approximate inverse $R^{\text{app}}(\lambda)$ is then constructed as

$$625 \quad (5.8) \quad R^{\text{app}}(\lambda) = \phi_{-, \lambda} R_2(\lambda) \phi_{-, \lambda} + \psi_{\lambda} R_1(\lambda) \psi_{\lambda} + \phi_{+, \lambda} R_2(\lambda) \phi_{+, \lambda},$$

626 where $\phi_{\pm, \lambda}$ and ψ_{λ} denote the operators of multiplication by the functions $\phi_{\pm, \lambda}$ and
627 ψ_{λ} . Note that ψ_{λ} maps $L_-^2 \times L_+^2$ into $L^2(\mathcal{S}_\lambda^-) \times L^2(\mathcal{S}_\lambda^+)$. In addition,

$$\begin{aligned} 628 \quad \psi_{\lambda} &: \mathcal{D}(\check{\mathcal{A}}) \rightarrow \mathcal{D}(\mathcal{A}_1^+), \\ \phi_{\lambda} &: \mathcal{D}(\mathcal{A}^+) \rightarrow \mathcal{D}(\mathcal{A}_1^+), \end{aligned}$$

629 where we have defined $\phi_\lambda(u_-, u_+)$ as $(\phi_{-, \lambda} u_-, \phi_{+, \lambda} u_+)$.
 630 From (5.4) and (5.6) we get, for sufficiently large $\text{Im } \lambda$,

$$631 \quad (5.9) \quad \|R^{\text{app}}(\lambda)\| \leq C_3(\alpha).$$

632 Note that

$$633 \quad (5.10) \quad |\phi'_\lambda(x)| + |\psi'_\lambda(x)| \leq \frac{C}{|\text{Im } \lambda|}, \quad |\phi''_\lambda(x)| + |\psi''_\lambda(x)| \leq \frac{C}{|\text{Im } \lambda|^2}.$$

634 Next, we apply $\mathcal{A}_1^+ - \lambda$ to R^{app} to obtain that

$$635 \quad (5.11) \quad (\mathcal{A}_1^+ - \lambda)R^{\text{app}}(\lambda) = I + [\mathcal{A}_1^+, \psi_\lambda]R_1(\lambda)\psi_\lambda + [\mathcal{A}_1^+, \phi_\lambda]R_2(\lambda)\phi_\lambda,$$

636 where I is the identity operator on $L_-^2 \times L_+^2$, and

$$\begin{aligned} 637 \quad [\mathcal{A}_1^+, \phi_\lambda] &:= \mathcal{A}_1^+ \phi_\lambda - \phi_\lambda \mathcal{A}_1^+ \\ 638 \quad &= [D_x^2, \phi_\lambda] \\ 639 \quad (5.12) \quad &= -\frac{2i}{\text{Im } \lambda} \phi' \left(\frac{x}{\text{Im } \lambda} \right) D_x - \frac{1}{(\text{Im } \lambda)^2} \phi'' \left(\frac{x}{\text{Im } \lambda} \right). \end{aligned}$$

640 A similar relation holds for $[\mathcal{A}_1^+, \psi_\lambda]$. Here we have used (5.8), and the fact that

$$641 \quad (\mathcal{A}_1^+ - \lambda)R_1(\lambda)\psi_\lambda u = \psi_\lambda u, \quad (\mathcal{A}_1^+ - \lambda)R_2(\lambda)\phi_\lambda u = \phi_\lambda u, \quad \forall u \in L_-^2 \times L_+^2.$$

642 Using (5.4), (5.5), (5.7), and (5.12) we then easily obtain, for sufficiently large $\text{Im } \lambda$,

$$643 \quad (5.13) \quad \|[\mathcal{A}_1^+, \psi_\lambda]R_1(\lambda)\| + \|[\mathcal{A}_1^+, \phi_\lambda]R_2(\lambda)\| \leq \frac{C_4(\alpha)}{|\text{Im } \lambda|}.$$

644 Hence, if $|\text{Im } \lambda|$ is large enough then $I + [\mathcal{A}_1^+, \psi_\lambda]R_1(\lambda)\psi_\lambda + [\mathcal{A}_1^+, \phi_\lambda]R_2(\lambda)\phi_\lambda$ is in-
 645 vertible in $\mathcal{L}(L_-^2 \times L_+^2)$, and

$$646 \quad (5.14) \quad \left\| \left(I + [\mathcal{A}_1^+, \psi_\lambda]R_1(\lambda)\psi_\lambda + [\mathcal{A}_1^+, \phi_\lambda]R_2(\lambda)\phi_\lambda \right)^{-1} \right\| \leq C_5(\alpha).$$

647 Finally, since

$$648 \quad (\mathcal{A}_1^+ - \lambda)^{-1} = R^{\text{app}}(\lambda) \circ \left(I + [\mathcal{A}_1^+, \psi_\lambda]R_1(\lambda)\psi_\lambda + [\mathcal{A}_1^+, \phi_\lambda]R_2(\lambda)\phi_\lambda \right)^{-1},$$

649 we have

$$650 \quad \|(\mathcal{A}_1^+ - \lambda)^{-1}\| \leq \|R^{\text{app}}(\lambda)\| \left\| \left(I + [\mathcal{A}_1^+, \psi_\lambda]R_1(\lambda)\psi_\lambda + [\mathcal{A}_1^+, \phi_\lambda]R_2(\lambda)\phi_\lambda \right)^{-1} \right\|.$$

651 Using (5.9) and (5.14) we conclude that (5.2) is true. \square

652 **REMARK 28.** *One could alternatively use more directly the expression of the ker-*
 653 *nel $\mathcal{G}^+(x, y; \lambda)$ of $(\mathcal{A}_1^+ - \lambda)^{-1}$ in terms of Ai and Ai' , together with the asymptotic*
 654 *expansions of the Airy function, see Appendix A and the discussion at the beginning*
 655 *of Section 7.*

656 **6. Integral kernel of the resolvent and its poles.** Here we revisit some
 657 of the computations of [22, 23] with the aim to complete some formal proofs. We
 658 are looking for the distribution kernel $\mathcal{G}^-(x, y; \lambda)$ of the resolvent $(\mathcal{A}_1^- - \lambda)^{-1}$ which
 659 satisfies in the sense of distribution

$$660 \quad (6.1) \quad \left(-\lambda - ix - \frac{\partial^2}{\partial x^2}\right) \mathcal{G}^-(x, y; \lambda) = \delta(x - y),$$

661 as well as the boundary conditions

$$662 \quad (6.2) \quad \left[\frac{\partial}{\partial x} \mathcal{G}^-(x, y; \lambda)\right]_{x=0^+} = \left[\frac{\partial}{\partial x} \mathcal{G}^-(x, y; \lambda)\right]_{x=0^-} \\ = \kappa [\mathcal{G}^-(0^+, y; \lambda) - \mathcal{G}^-(0^-, y; \lambda)].$$

663 Sometimes, we will write $\mathcal{G}^-(x, y; \lambda, \kappa)$, in order to stress the dependence on κ .
 664 Note that one can easily come back to the kernel of the resolvent of \mathcal{A}_1^+ by using

$$665 \quad (6.3) \quad \mathcal{G}^+(x, y; \lambda) = \overline{\mathcal{G}^-(y, x; \bar{\lambda})}.$$

666 Using (4.8), we also get

$$667 \quad (6.4) \quad \mathcal{G}^+(x, y; \lambda) = \overline{\mathcal{G}^-(x, y; \bar{\lambda})}.$$

668 We search for the solution $\mathcal{G}^-(x, y; \lambda)$ in three subdomains: the negative semi-axis
 669 $(-\infty, 0)$, the interval $(0, y)$, and the positive semi-axis $(y, +\infty)$ (here we assumed
 670 that $y > 0$; the opposite case is similar). For each subdomain, the solution is a linear
 671 combination of two Airy functions:

$$672 \quad (6.5) \quad \mathcal{G}^-(x, y; \lambda) = \begin{cases} A^- \text{Ai}(e^{-i\alpha} w_x) + B^- \text{Ai}(e^{i\alpha} w_x) & (x < 0), \\ A^+ \text{Ai}(e^{-i\alpha} w_x) + B^+ \text{Ai}(e^{i\alpha} w_x) & (0 < x < y), \\ C^+ \text{Ai}(e^{-i\alpha} w_x) + D^+ \text{Ai}(e^{i\alpha} w_x) & (x > y), \end{cases}$$

673 with six unknown coefficients (which are functions of $y > 0$). We recall that

$$674 \quad \alpha = \frac{2\pi}{3}$$

675 and

$$676 \quad w_x = ix + \lambda.$$

677 The boundary conditions (6.2) read as

$$678 \quad (6.6) \quad B^- i e^{i\alpha} \text{Ai}'(e^{i\alpha} w_0) \\ = A^+ i e^{-i\alpha} \text{Ai}'(e^{-i\alpha} w_0) + B^+ i e^{i\alpha} \text{Ai}'(e^{i\alpha} w_0) \\ = \kappa [A^+ \text{Ai}(e^{-i\alpha} w_0) + B^+ \text{Ai}(e^{i\alpha} w_0) - B^- \text{Ai}(e^{i\alpha} w_0)],$$

679 where $w_0 = \lambda$ and we set $A^- = 0$ and $D^+ = 0$ to ensure the decay of $\mathcal{G}^-(x, y; \lambda)$ as
 680 $x \rightarrow -\infty$ and as $x \rightarrow +\infty$, respectively.

681 We now look at the condition at $x = y$ in order to have (6.1) satisfied in the distribu-
 682 tion sense. We write the continuity condition,

$$683 \quad A^+ \text{Ai}(e^{-i\alpha} w_y) + B^+ \text{Ai}(e^{i\alpha} w_y) = C^+ \text{Ai}(e^{-i\alpha} w_y),$$

684 and the discontinuity jump of the derivative,

$$685 \quad A^+ i e^{-i\alpha} \text{Ai}'(e^{-i\alpha} w_y) + B^+ i e^{i\alpha} \text{Ai}'(e^{i\alpha} w_y) = C^+ i e^{-i\alpha} \text{Ai}'(e^{-i\alpha} w_y) + 1.$$

686 This can be considered as a linear system for A^+ and B^+ . Using the Wronskian
687 (A.3), one expresses A^+ and B^+ in terms of C^+ :

$$688 \quad (6.7) \quad A^+ = C^+ - 2\pi \text{Ai}(e^{i\alpha} w_y), \quad B^+ = 2\pi \text{Ai}(e^{-i\alpha} w_y).$$

689 We can rewrite (6.6) in the form

$$690 \quad (6.8) \quad B^- = e^{-2i\alpha} \frac{\text{Ai}'(e^{-i\alpha} w_0)}{\text{Ai}'(e^{i\alpha} w_0)} A^+ + B^+,$$

691 and

$$692 \quad (6.9) \quad \begin{aligned} & A^+ i e^{-i\alpha} \text{Ai}'(e^{-i\alpha} w_0) + B^+ i e^{i\alpha} \text{Ai}'(e^{i\alpha} w_0) \\ & = \kappa A^+ \left[\text{Ai}(e^{-i\alpha} w_0) - e^{-2i\alpha} \text{Ai}(e^{i\alpha} w_0) \frac{\text{Ai}'(e^{-i\alpha} w_0)}{\text{Ai}'(e^{i\alpha} w_0)} \right]. \end{aligned}$$

693 Using again the Wronskian (A.3), we obtain

$$694 \quad A^+ \text{Ai}'(e^{-i\alpha} w_0) + B^+ e^{2i\alpha} \text{Ai}'(e^{i\alpha} w_0) = -\kappa A^+ \frac{1}{2\pi \text{Ai}'(e^{i\alpha} w_0)},$$

695 that is

$$696 \quad A^+ (f(\lambda) + \kappa) + B^+ (2\pi) e^{2i\alpha} (\text{Ai}'(e^{i\alpha} w_0))^2 = 0,$$

697 where

$$698 \quad (6.10) \quad f(\lambda) := 2\pi \text{Ai}'(e^{-i\alpha} \lambda) \text{Ai}'(e^{i\alpha} \lambda).$$

699 So we now get

$$700 \quad (6.11) \quad A^+ = -\frac{1}{f(\lambda) + \kappa} (2\pi)^2 e^{2i\alpha} (\text{Ai}'(e^{i\alpha} w_0))^2 \text{Ai}(e^{-i\alpha} w_y),$$

701

$$702 \quad (6.12) \quad B^- = 2\pi \text{Ai}(e^{-i\alpha} w_y) - 2\pi \frac{f(\lambda)}{f(\lambda) + \kappa} \text{Ai}(e^{-i\alpha} w_y),$$

703 and

$$704 \quad (6.13) \quad C^+ = 2\pi \text{Ai}(e^{i\alpha} w_y) - 4\pi^2 \frac{e^{2i\alpha} [\text{Ai}'(e^{i\alpha} \lambda)]^2}{f(\lambda) + \kappa} \text{Ai}(e^{-i\alpha} w_y).$$

705 Combining these expressions, one finally gets

$$706 \quad (6.14) \quad \mathcal{G}^-(x, y; \lambda, \kappa) = \mathcal{G}_0^-(x, y; \lambda) + \mathcal{G}_1(x, y; \lambda, \kappa),$$

707 where $\mathcal{G}_0^-(x, y; \lambda)$ is the distribution kernel of the resolvent of the operator $\mathcal{A}_0^* :=$
708 $-\frac{d^2}{dx^2} - ix$ on the line (given by Eq. (3.10)), whereas $\mathcal{G}_1(x, y; \lambda, \kappa)$ is given by the
709 following expressions

$$710 \quad (6.15) \quad \mathcal{G}_1(x, y; \lambda, \kappa) = \begin{cases} -4\pi^2 \frac{e^{2i\alpha} [\text{Ai}'(e^{i\alpha} \lambda)]^2}{f(\lambda) + \kappa} \text{Ai}(e^{-i\alpha} w_x) \text{Ai}(e^{-i\alpha} w_y), & (x > 0), \\ -2\pi \frac{f(\lambda)}{f(\lambda) + \kappa} \text{Ai}(e^{i\alpha} w_x) \text{Ai}(e^{-i\alpha} w_y), & (x < 0), \end{cases}$$

711 for $y > 0$, and

$$712 \quad (6.16) \quad \mathcal{G}_1(x, y; \lambda, \kappa) = \begin{cases} -2\pi \frac{f(\lambda)}{f(\lambda) + \kappa} \text{Ai}(e^{-i\alpha} w_x) \text{Ai}(e^{i\alpha} w_y), & (x > 0), \\ -4\pi^2 \frac{e^{-2i\alpha} [\text{Ai}'(e^{-i\alpha} \lambda)]^2}{f(\lambda) + \kappa} \text{Ai}(e^{i\alpha} w_x) \text{Ai}(e^{i\alpha} w_y), & (x < 0), \end{cases}$$

713 for $y < 0$. Hence the poles are determined by the equation

$$714 \quad (6.17) \quad f(\lambda) = -\kappa,$$

715 with f defined in (6.10).

716 **REMARK 29.** For $\kappa = 0$, one recovers the conjugated pairs associated with the
717 zeros a'_n of Ai' . We have indeed as poles

$$718 \quad (6.18) \quad \lambda_n^+ = e^{i\alpha} a'_n, \quad \lambda_n^- = e^{-i\alpha} a'_n,$$

719 where a'_n is the n -th zero (starting from the right) of Ai' . Note that $a'_n < 0$ so that
720 $\text{Re } \lambda_n^\pm > 0$, as expected.

721 In this case, the restriction of $\mathcal{G}_1(x, y; \lambda, 0)$ to \mathbb{R}_+^2 is the kernel of the resolvent of the
722 Neumann problem in \mathbb{R}_+ .

723 We also know that the eigenvalues for the Neumann problem are simple. Hence
724 by the local inversion theorem we get the existence of a solution close to each λ_n^\pm for
725 κ small enough (possibly depending on n) if we show that $f'(\lambda_n^\pm) \neq 0$. For λ_n^+ , we
726 have, using the Wronskian relation (A.3) and $\text{Ai}'(e^{-i\alpha} \lambda_n^+) = 0$,

$$727 \quad (6.19) \quad \begin{aligned} f'(\lambda_n^+) &= 2\pi e^{-i\alpha} \text{Ai}''(e^{-i\alpha} \lambda_n^+) \text{Ai}'(e^{i\alpha} \lambda_n^+) \\ &= 2\pi e^{-2i\alpha} \lambda_n^+ \text{Ai}(e^{-i\alpha} \lambda_n^+) \text{Ai}'(e^{i\alpha} \lambda_n^+) \\ &= -i \lambda_n^+. \end{aligned}$$

728 Similar computations hold for λ_n^- . We recall that

$$729 \quad \lambda_n^+ = \overline{\lambda_n^-}.$$

730 The above argument shows that $f'(\lambda_n) \neq 0$, with $\lambda_n = \lambda_n^+$ or $\lambda_n = \lambda_n^-$. Hence by
731 the holomorphic inversion theorem we get that, for any $n \in \mathbb{N}^*$ (with $\mathbb{N}^* = \mathbb{N} \setminus \{0\}$),
732 and any ϵ , there exists $h_n(\epsilon)$ such that for $|\kappa| \leq h_n(\epsilon)$, we have a unique solution
733 $\lambda_n(\kappa)$ of (6.17) such that $|\lambda_n(\kappa) - \lambda_n| \leq \epsilon$.

734

735 We would like to have a control of $h_n(\epsilon)$ with respect to n . What we should do
736 is inspired by the Taylor expansion given in [23] (Formula (33)) of $\lambda_n^\pm(\kappa)$ for fixed n :

$$737 \quad (6.20) \quad \lambda_n^\pm(\kappa) = \lambda_n^\pm + e^{\pm i\frac{\pi}{6}} \frac{1}{a'_n} \kappa + \mathcal{O}_n(\kappa^2).$$

738 Since $|\lambda_n|$ behaves as $n^{\frac{2}{3}}$ (see Appendix A), the guess is that $\lambda_{n+1}^\pm(\kappa) - \lambda_n^\pm(\kappa)$ behaves
739 as $n^{-\frac{1}{3}}$.

740 To justify this guess, one needs to control the derivative in a suitable neighborhood
741 of λ_n .

742 **PROPOSITION 30.** *There exist $\eta > 0$ and $h_\infty > 0$, such that, for all $n \in \mathbb{N}^*$, for*
743 *any κ such that $|\kappa| \leq h_\infty$ there exists a unique solution of (6.17) in $B(\lambda_n, \eta |\lambda_n|^{-1})$*
744 *with $\lambda_n = \lambda_n^\pm$.*

745 **Proof of the proposition**

746 Using the previous arguments, it is enough to establish the proposition for n large
 747 enough. Hence it remains to establish a local inversion theorem uniform with respect
 748 to n for $n \geq N$. For this purpose, we consider the holomorphic function

$$749 \quad B(0, \eta) \ni t \mapsto \phi_n(t) = f(\lambda_n + t\lambda_n^{-1}).$$

750 To have a local inversion theorem uniform with respect to n , we need to control $|\phi'_n(t)|$
 751 from below.

752

753 **LEMMA 31.** *For any $\eta > 0$, there exists N such that, $\forall n \geq N$,*

$$754 \quad (6.21) \quad |\phi'_n(t)| \geq \frac{1}{2}, \quad \forall t \in B(0, \eta).$$

755 **Proof of the lemma**

756 We have

$$757 \quad \phi'_n(t) = \lambda_n^{-1} f'(\lambda_n + t\lambda_n^{-1}),$$

758 and

$$759 \quad \phi'_n(0) = -i.$$

760 Hence it remains to control $\phi'_n(t) - \phi'_n(0)$ in $B(0, \eta)$. We treat the case $\lambda_n = \lambda_n^+$.

761 We recall that

$$\begin{aligned} f'(\lambda) &= 2\pi e^{-i\alpha} \text{Ai}''(e^{-i\alpha}\lambda) \text{Ai}'(e^{i\alpha}\lambda) + 2\pi e^{i\alpha} \text{Ai}'(e^{-i\alpha}\lambda) \text{Ai}''(e^{i\alpha}\lambda) \\ (6.22) \quad &= 2\pi\lambda (e^{-2i\alpha} \text{Ai}(e^{-i\alpha}\lambda) \text{Ai}'(e^{i\alpha}\lambda) + e^{2i\alpha} \text{Ai}'(e^{-i\alpha}\lambda) \text{Ai}(e^{i\alpha}\lambda)) \\ &= -i\lambda + 4\pi\lambda e^{2i\alpha} \text{Ai}'(e^{-i\alpha}\lambda) \text{Ai}(e^{i\alpha}\lambda). \end{aligned}$$

763 Hence we have

$$764 \quad (6.23) \quad \phi'_n(t) - \phi'_n(0) = 4\pi\lambda\lambda_n^{-1} e^{2i\alpha} \text{Ai}'(e^{-i\alpha}\lambda) \text{Ai}(e^{i\alpha}\lambda) - it\lambda_n^{-2},$$

765 with $\lambda = \lambda_n + t\lambda_n^{-1}$.

766

767 The last term in (6.22) tends to zero. It remains to control $\text{Ai}'(e^{-i\alpha}\lambda) \text{Ai}(e^{i\alpha}\lambda)$ in
 768 $B(\lambda_n, \eta|\lambda_n|^{-1})$ and to show that this expression tends to zero as $n \rightarrow +\infty$.

769 We have

$$770 \quad \text{Ai}'(e^{-i\alpha}\lambda) = e^{-i\alpha}(\lambda - \lambda_n) \text{Ai}''(e^{-i\alpha}\tilde{\lambda}) = e^{-2i\alpha}(\lambda - \lambda_n) \tilde{\lambda} \text{Ai}(e^{-i\alpha}\tilde{\lambda}),$$

771 with $\tilde{\lambda} \in B(\lambda_n, \eta|\lambda_n|^{-1})$.

772 Hence it remains to show that the product $|\text{Ai}(e^{-i\alpha}\tilde{\lambda}) \text{Ai}(e^{i\alpha}\lambda)|$ for λ and $\tilde{\lambda}$
 773 in $B(\lambda_n, \eta|\lambda_n|^{-1})$ tends to 0. For this purpose, we will use the known expansion
 774 for the Airy function (recalled in Appendix A) in the balls $B(e^{-i\alpha}\lambda_n, \eta|\lambda_n|^{-1})$ and
 775 $B(e^{i\alpha}\lambda_n, \eta|\lambda_n|^{-1})$.

776

777 (i) For the factor $|\text{Ai}(e^{-i\alpha}\tilde{\lambda})|$, we need the expansion of $\text{Ai}(z)$ for z in a neighbor-
 778 hood of a'_n of size $C|\lambda_n|^{-1}$. Using the asymptotic relation (A.7), we observe that

$$779 \quad \exp\left(\pm i \frac{2}{3} z^{\frac{3}{2}}\right) = \exp\left(\pm i \left(\frac{2}{3} (-a'_n)^{\frac{3}{2}} (1 + \mathcal{O}(1/|a'_n|^2))\right)\right) = \mathcal{O}(1).$$

780 Hence we get

$$781 \quad |\text{Ai}(e^{-i\tilde{\alpha}}\lambda)| \leq C |a'_n|^{-\frac{1}{4}} \quad \forall \tilde{\lambda} \in B(\lambda_n, \eta|\lambda_n|^{-1}).$$

782 (ii) For the factor $|\text{Ai}(e^{i\alpha}\lambda)|$, we use (A.5) to observe that

$$783 \quad \exp\left(-\frac{2}{3}(e^{i\alpha}\lambda)^{\frac{3}{2}}\right) = \exp\left(-i\frac{2}{3}(-a'_n)^{\frac{3}{2}}(1 + \mathcal{O}((-a'_n)^{-2}))\right),$$

784 and we get, for any $\lambda \in B(\lambda_n, \eta|\lambda_n|^{-1})$

$$785 \quad (6.24) \quad |\text{Ai}(e^{i\alpha}\lambda)| \leq C |a'_n|^{-\frac{1}{4}}.$$

786 This completes the proof of the lemma and of the proposition.

787 Actually, we have proved on the way the more precise

788 **PROPOSITION 32.** *For all $\eta > 0$ and $0 \leq \kappa < \frac{\eta}{2}$, there exists N such that, for all*
 789 *$n \geq N$, there exists a unique solution of (6.17) in $B(\lambda_n, \eta|\lambda_n|^{-1})$.*

790 Figure 1 illustrates Proposition 30. Solving Eq. (6.17) numerically, we find the
 791 first 100 zeros $\lambda_n(\kappa)$ with $\text{Im } \lambda_n(\kappa) > 0$. According to Proposition 30, these zeros are
 792 within distance $1/|\lambda_n|$ from the zeros $\lambda_n = \lambda_n(0) = e^{i\alpha}a'_n$ which are given explicitly
 793 through the zeros a'_n . Moreover, the second order term in (6.20) that was computed
 794 in [23], suggests that the rescaled distance

$$795 \quad (6.25) \quad \delta_n(\kappa) = |\lambda_n(\kappa) - \lambda_n| |\lambda_n| / \kappa,$$

796 behaves as

$$797 \quad (6.26) \quad \delta_n(\kappa) = 1 - c\kappa n^{-\frac{1}{3}} + o(n^{-\frac{1}{3}}),$$

798 with a nonzero constant c . Figure 1(top) shows that the distance $\delta_n(\kappa)$ remains below
 799 1 for three values of κ : 0.1, 1, and 10. The expected asymptotic behavior given in
 800 (6.26) is confirmed by Figure 1(bottom), from which the constant c is estimated to
 801 be around 0.31.

802 **REMARK 33.** *The local inversion theorem with control with respect to n permits*
 803 *to have the asymptotic behavior of the $\lambda_n(\kappa)$ uniformly for κ small:*

$$804 \quad (6.27) \quad \lambda_n^\pm(\kappa) = \lambda_n^\pm + e^{\pm i\frac{\pi}{6}} \frac{1}{a'_n} \kappa + \frac{1}{a'_n} \mathcal{O}(\kappa^2).$$

805 *An improvement of (6.27) (as formulated by (6.26)) results from a good estimate on*
 806 *$\phi_n''(t)$. Observing that $|\phi_n''(t)| \leq C|a'_n|^{-\frac{1}{2}}$ in the ball $B(0, \eta)$, we obtain*

$$807 \quad (6.28) \quad \lambda_n^\pm(\kappa) = \lambda_n^\pm + e^{\pm i\frac{\pi}{6}} \frac{1}{a'_n} \kappa + \frac{1}{|a'_n|^{\frac{3}{2}}} \mathcal{O}(\kappa^2).$$

808 *If one needs finer estimates, one can compute $\phi_n''(0)$ and estimate ϕ_n''' , and so on.*

809 *It would also be interesting to analyze the case $\kappa \rightarrow +\infty$. The limiting problem in*
 810 *this case is the realization of the complex Airy operator on the line which has empty*
 811 *spectrum. See [23] for a preliminary non rigorous analysis.*

812 In the remaining part of this section, we describe the distribution kernel of the
 813 projector Π_n^\pm associated with $\lambda_n^\pm(\kappa)$.

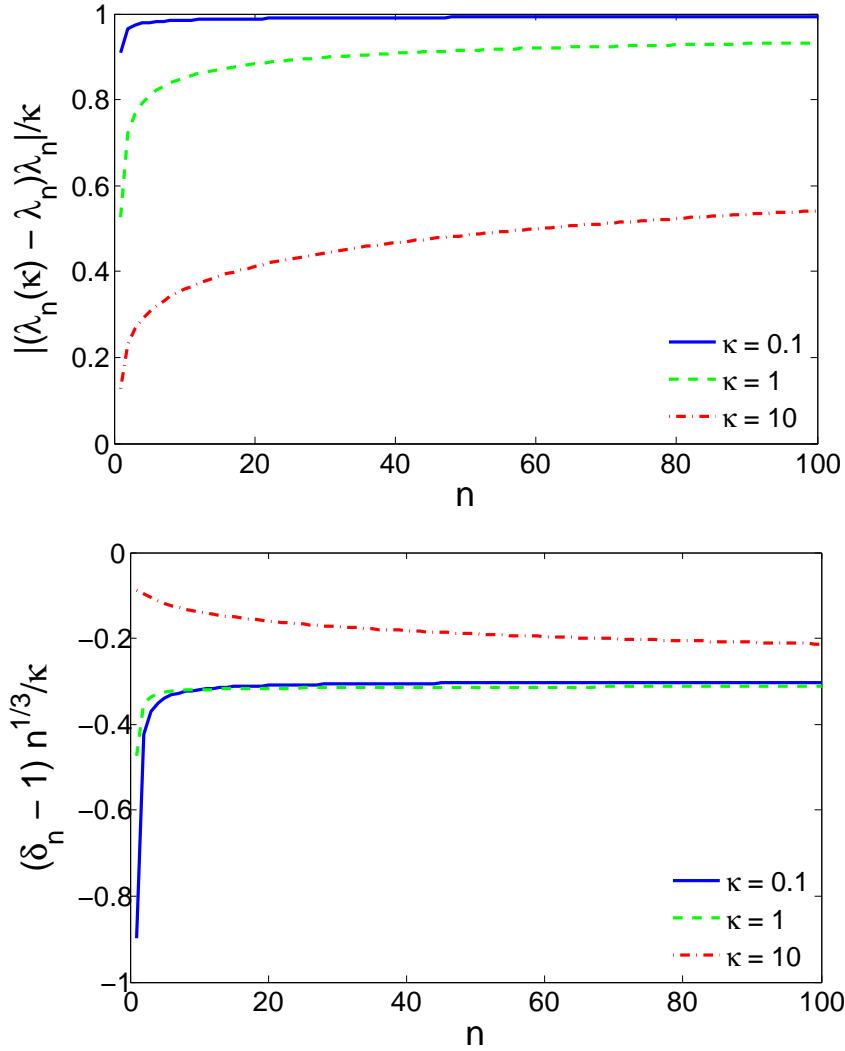


FIG. 1. Illustration of Proposition 30 by the numerical computation of the first 100 zeros $\lambda_n^+(\kappa)$ of (6.17). At the top, the rescaled distance $\delta_n(\kappa)$ from (6.25) between $\lambda_n^+(\kappa)$ and $\lambda_n^+ = \lambda_n^+(0)$. At the bottom, the asymptotic behavior of this distance.

814 PROPOSITION 34. There exists $\kappa_0 > 0$ such that, for any $\kappa \in [0, \kappa_0]$ and any
 815 $n \in \mathbb{N}^*$, the rank of Π_n^\pm is equal to one. Moreover, if ψ_n^\pm is an eigenfunction, then

$$816 \quad (6.29) \quad \int_{-\infty}^{+\infty} \psi_n^\pm(x)^2 dx \neq 0.$$

817 **Proof**

818 To write the projector Π_n^\pm associated with an eigenvalue λ_n^\pm we integrate the resolvent
 819 along a small contour γ_n^\pm around λ_n^\pm :

$$820 \quad (6.30) \quad \Pi_n^\pm = \frac{1}{2i\pi} \int_{\gamma_n^\pm} (\mathcal{A}_1^\pm - \lambda)^{-1} d\lambda.$$

821 If we consider the associated kernels, we get, using (6.14) and the fact that \mathcal{G}_0^- is
 822 holomorphic in λ :

$$823 \quad (6.31) \quad \Pi_n^\pm(x, y; \kappa) = \frac{1}{2i\pi} \int_{\gamma_n^\pm} \mathcal{G}_1(x, y; \lambda, \kappa) d\lambda.$$

824 The projector is given by the following expression (with $w_x^{\pm, n} = ix + \lambda_n^\pm$) for
 825 $y > 0$

$$826 \quad (6.32) \quad \Pi_n^\pm(x, y; \kappa) = \begin{cases} -4\pi^2 \frac{e^{2i\alpha} [\text{Ai}'(e^{i\alpha} \lambda_n^\pm)]^2}{f'(\lambda_n^\pm)} \text{Ai}(e^{-i\alpha} w_x^{\pm, n}) \text{Ai}(e^{-i\alpha} w_y^{\pm, n}) & (x > 0), \\ 2\pi \frac{\kappa}{f'(\lambda_n^\pm)} \text{Ai}(e^{i\alpha} w_x^{\pm, n}) \text{Ai}(e^{-i\alpha} w_y^{\pm, n}) & (x < 0), \end{cases}$$

827 and for $y < 0$
 (6.33)

$$828 \quad \Pi_n^\pm(x, y; \kappa) = \begin{cases} 2\pi \frac{\kappa}{f'(\lambda_n^\pm)} \text{Ai}(e^{-i\alpha} w_x^{\pm, n}) \text{Ai}(e^{i\alpha} w_y^{\pm, n}) & (x > 0), \\ -4\pi^2 \frac{e^{-2i\alpha} [\text{Ai}'(e^{-i\alpha} \lambda_n^\pm)]^2}{f'(\lambda_n^\pm)} \text{Ai}(e^{i\alpha} w_x^{\pm, n}) \text{Ai}(e^{i\alpha} w_y^{\pm, n}) & (x < 0). \end{cases}$$

829 Here we recall that we have established that for $|\kappa|$ small enough $f'(\lambda_n^\pm) \neq 0$. It
 830 remains to show that the rank of Π_n^\pm is one that will yield an expression for the eigen-
 831 function. It is clear from (6.32) and (6.33) that the rank of Π_n^\pm is at most two and that
 832 every function in the range of Π_n^\pm has the form $(c_- \text{Ai}(e^{i\alpha} w_x^{\pm, n}), c_+ \text{Ai}(e^{-i\alpha} w_x^{\pm, n}))$,
 833 where $c_-, c_+ \in \mathbb{R}$. It remains to establish the existence of a relation between c_- and
 834 c_+ . This is directly obtained by using the first part of the transmission condition. If
 835 $\kappa \neq 0$, the functions in the range of Π_n^\pm have the form

$$836 \quad c_n (\text{Ai}'(e^{-i\alpha} \lambda_n^\pm) \text{Ai}(e^{i\alpha} w_x^{\pm, n}), e^{2i\alpha} \text{Ai}'(e^{i\alpha} \lambda_n^\pm) \text{Ai}(e^{-i\alpha} w_x^{\pm, n})),$$

837 with $c_n \in \mathbb{C}$. Inequality (6.29) results from an abstract lemma in [7] once we have
 838 proved that the rank of the projector is one. We have indeed

$$839 \quad (6.34) \quad \|\Pi_n^\pm\| = \frac{1}{|\int_{-\infty}^{+\infty} \psi_n^\pm(x)^2 dx|}.$$

840 More generally, what we have proven can be formulated in this way:

841 PROPOSITION 35. *If $f(\lambda) + \kappa = 0$ and $f'(\lambda) \neq 0$, then the associated projector*
 842 *has rank 1 (no Jordan block).*

843 The condition of κ being small in Proposition 34 is only used for proving the property
 844 $f'(\lambda) \neq 0$. For the case of the Dirichlet or Neumann realization of the complex
 845 Airy operator in \mathbb{R}_+ , we refer to Section 3. The nonemptiness was obtained directly
 846 by using the properties of the Airy function. Note that our numerical solutions did
 847 not reveal projectors of rank higher than 1. We conjecture that the rank of these
 848 projectors is 1 for any $0 \leq \kappa < +\infty$ but we could only prove the weaker

849 PROPOSITION 36. *For any $\kappa \geq 0$, there is at most a finite number of eigenvalues*
 850 *with nontrivial Jordan blocks.*

851 **Proof**

852 We start from

$$853 \quad f(\lambda) := 2\pi \text{Ai}'(e^{i\alpha} \lambda) \text{Ai}'(e^{-i\alpha} \lambda),$$

854 and get by derivation

$$855 \quad (6.35) \quad \frac{1}{2\pi} f'(\lambda) = e^{i\alpha} \text{Ai}''(e^{i\alpha}\lambda) \text{Ai}'(e^{-i\alpha}\lambda) + e^{-i\alpha} \text{Ai}'(e^{i\alpha}\lambda) \text{Ai}''(e^{-i\alpha}\lambda).$$

856 What we have to prove is that $f'(\lambda)$ is different from 0 for a large solution λ of
857 $f(\lambda) = -\kappa$. We know already that $\text{Re } \lambda \geq 0$. We note that $f(0) > 0$. Hence 0 is not
858 a pole for $\kappa \geq 0$. More generally f is real and strictly positive on the real axis. Hence
859 $f(\lambda) + \kappa > 0$ on the real axis.

860 We can assume that $\text{Im } \lambda > 0$ (the other case can be treated similarly). Using the
861 equation satisfied by the Airy function, we get

$$862 \quad (6.36) \quad \frac{1}{2\pi\lambda} f'(\lambda) = e^{-i\alpha} \text{Ai}(e^{i\alpha}\lambda) \text{Ai}'(e^{-i\alpha}\lambda) + e^{i\alpha} \text{Ai}'(e^{i\alpha}\lambda) \text{Ai}(e^{-i\alpha}\lambda),$$

863 and by the Wronskian relation (A.3):

$$864 \quad (6.37) \quad e^{-i\alpha} \text{Ai}'(e^{-i\alpha}\lambda) \text{Ai}(e^{i\alpha}\lambda) - e^{i\alpha} \text{Ai}'(e^{i\alpha}\lambda) \text{Ai}(e^{-i\alpha}\lambda) = \frac{i}{2\pi}.$$

865 Suppose that $f(\lambda) = -\kappa$ and that $f'(\lambda) = 0$.

866 We have

$$867 \quad -e^{i\alpha} \text{Ai}'(e^{i\alpha}\lambda) \text{Ai}(e^{-i\alpha}\lambda) = e^{-i\alpha} \text{Ai}'(e^{-i\alpha}\lambda) \text{Ai}(e^{i\alpha}\lambda) = \frac{i}{4\pi}.$$

868 and get

$$869 \quad \kappa = -\frac{ie^{i\alpha}}{2} \frac{\text{Ai}'(e^{i\alpha}\lambda)}{\text{Ai}(e^{i\alpha}\lambda)} = \frac{ie^{-i\alpha}}{2} \frac{\text{Ai}'(e^{-i\alpha}\lambda)}{\text{Ai}(e^{-i\alpha}\lambda)}.$$

870 Using the last equality and the asymptotics (A.5), (A.6) for Ai and Ai' , we get as
871 $|\lambda| \rightarrow +\infty$ satisfying the previous condition

$$872 \quad \kappa \sim \frac{1}{2} |\lambda|^{\frac{1}{2}},$$

873 which cannot be true for λ large. This completes the proof of the proposition.

874 **7. Resolvent estimates as $|\text{Im } \lambda| \rightarrow +\infty$.** The resolvent estimates have been
875 already proved in Section 5 and were used in the analysis of the decay of the associated
876 semigroup. We propose here another approach which leads to more precise results.
877 We keep in mind (6.14) and the discussion in Section 5.

878 For $\lambda = \lambda_0 + i\eta$, we have

$$879 \quad \|\mathcal{G}_0^-(\cdot, \cdot; \lambda)\|_{L^2(\mathbb{R}^2)} = \|\mathcal{G}_0^-(\cdot, \cdot; \lambda_0)\|_{L^2(\mathbb{R}^2)}.$$

880 Hence the Hilbert-Schmidt norm of the resolvent $(\mathcal{A}^+ - \lambda)^{-1}$ does not depend on the
881 imaginary part of λ .

882 As a consequence, to recover Lemma 27 by this approach, it only remains to check
883 the following lemma

884 **LEMMA 37.** *For any λ_0 , there exist $C > 0$ and $\eta_0 > 0$ such that*

$$885 \quad (7.1) \quad \sup_{|\eta| > \eta_0} \|\mathcal{G}_1(\cdot, \cdot; \lambda_0 + i\eta)\|_{L^2(\mathbb{R}^2)} \leq C.$$

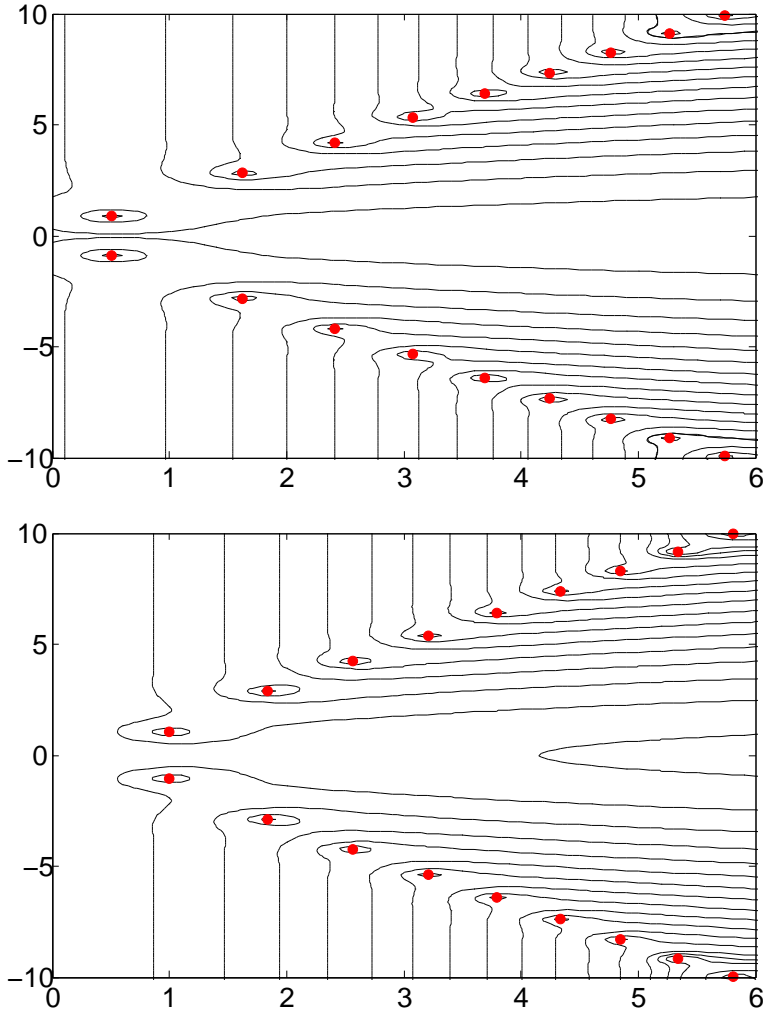


FIG. 2. Numerically computed pseudospectrum in the complex plane of the complex Airy operator with Neumann boundary conditions (top) and with the transmission boundary condition at the origin with $\kappa = 1$ (bottom). The red points show the poles $\lambda_n^\pm(\kappa)$ found by solving numerically Eq. (6.17) that corresponds to the original problem on \mathbb{R} . The presented picture corresponds to a zoom (eliminating numerical artefacts) in a computation done for a large interval $[-L, +L]$ with the transmission condition at the origin and Dirichlet boundary conditions at $\pm L$. The pseudospectrum was computed for $L^3 = 10^4$ by projecting the complex Airy operator onto the orthogonal basis of eigenfunctions of the corresponding Laplace operator and then diagonalizing the obtained truncated matrix representation (see Appendix E for details). We only keep a few lines of pseudospectra for the clarity of the picture. As predicted by the theory, the vertical lines are related to the pseudospectrum of the free complex Airy operator on the line.

886 The proof is included in the proof of the following improvement which is the main
 887 result of this section and is confirmed by the numerical computations. One indeed
 888 observes that the lines of the pseudospectrum are asymptotically vertical as $\text{Im } \lambda \rightarrow$
 889 $\pm\infty$ when $\text{Re } \lambda > 0$, see Figure 2.

890 PROPOSITION 38. For any $\lambda_0 > 0$,

891
$$\lim_{\eta \rightarrow \pm\infty} \|\mathcal{G}_1(\cdot, \cdot; \lambda_0 + i\eta)\|_{L^2(\mathbb{R}^2)} = 0.$$

892 Moreover, this convergence is uniform for λ_0 in a compact set.

893 **Proof**

894 We have

895
$$e^{i\alpha\lambda} = e^{i\alpha\lambda_0} - e^{i\pi/6}\eta$$

896 and

897
$$e^{-i\alpha\lambda} = e^{-i\alpha\lambda_0} + e^{-i\pi/6}\eta.$$

898 Then according to (A.6), one can easily check that the term $\text{Ai}'(e^{\pm i\alpha}\lambda)$ decays expo-
899 nentially as $\eta \rightarrow \mp\infty$ and grows exponentially as $\eta \rightarrow \pm\infty$. On the other hand, the
900 term $\text{Ai}'(e^{i\alpha}\lambda)$ decays exponentially as $\eta \rightarrow \pm\infty$.

901 More precisely, we have

902 (7.2)
$$\begin{aligned} |\text{Ai}'(e^{i\alpha}(\lambda_0 + i\eta))|^2 &\sim |c|^2 \eta^{\frac{1}{2}} \exp\left(\frac{2\sqrt{2}}{3}\eta^{\frac{3}{2}}\right), \text{ as } \eta \rightarrow +\infty; \\ &\sim |c|^2 (-\eta)^{\frac{1}{2}} \exp\left(-\frac{2\sqrt{2}}{3}\eta^{\frac{3}{2}}\right), \text{ as } \eta \rightarrow -\infty; \\ |\text{Ai}'(e^{-i\alpha}(\lambda_0 + i\eta))|^2 &\sim |c|^2 \eta^{\frac{1}{2}} \exp\left(-\frac{2\sqrt{2}}{3}\eta^{\frac{3}{2}}\right), \text{ as } \eta \rightarrow +\infty; \\ &\sim |c|^2 (-\eta)^{\frac{1}{2}} \exp\left(\frac{2\sqrt{2}}{3}\eta^{\frac{3}{2}}\right), \text{ as } \eta \rightarrow -\infty. \end{aligned}$$

903 As a consequence, the function $f(\lambda)$, which was defined in (6.10) by

904
$$f(\lambda) := 2\pi \text{Ai}'(e^{-i\alpha}\lambda) \text{Ai}'(e^{i\alpha}\lambda),$$

905 has the following asymptotic behavior as $\eta \rightarrow \mp\infty$:

906 (7.3)
$$f(\lambda_0 + i\eta) = 2\pi |c|^2 |\eta|^{\frac{1}{2}} (1 + o(1)).$$

907 **We treat the case $\eta > 0$** (the other case can be deduced by considering the
908 complex conjugate).

909 Coming back to the two formulas giving \mathcal{G}_1 in (6.15) and (6.16) and starting with the
910 first one, we have to analyze the L^2 norm over $\mathbb{R}_+ \times \mathbb{R}_+$ of

911
$$(x, y) \mapsto -4\pi^2 \frac{e^{2i\alpha} [\text{Ai}'(e^{i\alpha}\lambda)]^2}{f(\lambda) + \kappa} \text{Ai}(e^{-i\alpha}w_x) \text{Ai}(e^{-i\alpha}w_y).$$

912 This norm N_1 is given by

913
$$N_1 := 4\pi^2 |\text{Ai}'(e^{i\alpha}\lambda)|^2 |f(\lambda) + \kappa|^{-1} \|\text{Ai}(e^{-i\alpha}w_x)\|_{L^2(\mathbb{R}_+)}^2.$$

914 Hence we have to estimate $\int_0^{+\infty} |\text{Ai}(e^{-i\alpha}w_x)|^2 dx$. We observe that

915
$$e^{-i\alpha}w_x = e^{-i\frac{\pi}{6}}(x + \eta) + e^{-i\alpha}\lambda_0,$$

916 and that the argument of $e^{-i\alpha}w_x$ is very close to $-\frac{\pi}{6}$ as $\eta \rightarrow +\infty$ (uniformly for
917 $x > 0$). This is rather simple for $\eta > 0$ because x and η have the same sign. We can
918 use the asymptotics (A.5) (with $z = e^{-i\alpha}w_x$) in order to get

919 (7.4)
$$\int_0^{+\infty} |\text{Ai}(e^{-i\alpha}w_x)|^2 dx \leq C (|\eta|^2 + 1)^{-\frac{1}{2}} \exp\left(-\frac{2\sqrt{2}}{3}|\eta|^{\frac{3}{2}}\right).$$

920 Here we have used that, for $\beta > 0$,

$$921 \quad \int_{\eta}^{+\infty} \exp\left(-\beta y^{\frac{3}{2}}\right) dy = \frac{2}{3\beta} \exp\left(-\beta \eta^{\frac{3}{2}} (1 + \mathcal{O}(|\eta|^{-\frac{1}{2}}))\right).$$

922 The control of $|\text{Ai}'(e^{i\alpha}(\lambda_0 + i\eta))|^2$ given in (7.2) and (7.3) finally yields

$$923 \quad (7.5) \quad N_1 \lesssim (|\eta|^2 + 1)^{-\frac{1}{2}}.$$

924 By the notation \lesssim , we mean that there exists a constant C such that

$$925 \quad N_1 \leq C(|\eta|^2 + 1)^{-\frac{1}{2}}.$$

926 For the L^2 -norm of the second term (see (6.16)),

$$927 \quad N_2 := \left\| -2\pi \frac{f(\lambda)}{f(\lambda) + \kappa} \text{Ai}(e^{i\alpha} w_x) \text{Ai}(e^{-i\alpha} w_y) \right\|_{L^2(\mathbb{R}_- \times \mathbb{R}_+^*)},$$

928 we observe that

$$929 \quad N_2 \lesssim \|\text{Ai}(e^{i\alpha} w_x)\|_{L^2(\mathbb{R}_-)} \|\text{Ai}(e^{-i\alpha} w_x)\|_{L^2(\mathbb{R}_+)},$$

930 and having in mind (7.4), we have only to bound $\int_{-\infty}^0 |\text{Ai}(e^{i\alpha} w_x)|^2 dx$. We can no
931 more use the asymptotic for the Airy function as $(x + \eta)$ is small. We have indeed

$$932 \quad e^{i\alpha} w_x = -e^{i\frac{\pi}{6}}(x + \eta) + e^{i\alpha} \lambda_0.$$

933 We rewrite the integral as the sum

$$934 \quad \begin{aligned} \int_{-\infty}^0 |\text{Ai}(e^{i\alpha} w_x)|^2 dx &= \int_{-\infty}^{-\eta-C} |\text{Ai}(e^{i\alpha} w_x)|^2 dx \\ &+ \int_{-\eta-C}^{-\eta+C} |\text{Ai}(e^{i\alpha} w_x)|^2 dx + \int_{-\eta+C}^0 |\text{Ai}(e^{i\alpha} w_x)|^2 dx. \end{aligned}$$

935 The integral in the middle of the r.h.s. is bounded. The first one is also bounded
936 according to the behavior of the Airy function. So the dominant term is the third one

$$937 \quad \begin{aligned} \int_{-\eta+C}^0 |\text{Ai}(e^{i\alpha} w_x)|^2 dx &= \int_C^\eta |\text{Ai}(-e^{i\frac{\pi}{6}} x + e^{i\alpha} \lambda_0)|^2 dx \\ &\leq \tilde{C}(|\eta|^2 + 1)^{\frac{1}{4}} \exp\left(+\frac{2\sqrt{2}}{3} |\eta|^{\frac{3}{2}}\right). \end{aligned}$$

938 Combining with (7.4), the L^2 -norm of the second term decays as $\eta \rightarrow +\infty$:

$$939 \quad (7.6) \quad N_2 \lesssim (|\eta|^2 + 1)^{-\frac{1}{8}}.$$

940 This achieves the proof of the proposition, the uniformity for λ_0 in a compact being
941 controlled at each step of the proof.

942 **8. Proof of the completeness.** We have already recalled or established in Sec-
943 tion 3 (Propositions 10, 14, and 17) the results for the Dirichlet, Neumann or Robin
944 realization of the complex Airy operator in \mathbb{R}_+ . The aim of this section is to establish
945 the same result in the case with transmission. The new difficulty is that the operator
946 is no longer sectorial.

947

948 **8.1. Reduction to the case $\kappa = 0$.** We first reduce the analysis to the case
 949 $\kappa = 0$ by comparison of the two kernels. We have indeed

$$950 \quad (8.1) \quad \begin{aligned} \mathcal{G}^-(x, y; \lambda, \kappa) - \mathcal{G}^-(x, y; \lambda, 0) &= \mathcal{G}_1(x, y; \lambda, \kappa) - \mathcal{G}_1(x, y; \lambda, 0) \\ &= -\kappa(f(\lambda) + \kappa)^{-1} \mathcal{G}_1(x, y; \lambda, 0), \end{aligned}$$

951 where $\mathcal{G}^-(x, y; \lambda, \kappa)$ denotes the kernel of the resolvent for the transmission problem
 952 associated to $\kappa \geq 0$ and $D_x^2 - ix$.

953 We will also use the alternative equivalent relation:

$$954 \quad (8.2) \quad \mathcal{G}^-(x, y; \lambda, \kappa) = \mathcal{G}^-(x, y; \lambda, 0) f(\lambda) (f(\lambda) + \kappa)^{-1} + \kappa (f(\lambda) + \kappa)^{-1} \mathcal{G}_0^-(x, y; \lambda, 0).$$

955 **REMARK 39.** *This formula gives another way for proving that the operator with*
 956 *kernel $\mathcal{G}^\pm(x, y; \lambda, \kappa)$ is in a suitable Schatten class (see Proposition 21). It is indeed*
 957 *enough to have the result for $\kappa = 0$, that is to treat the Neumann case on the half line.*

958 Another application of this formula is

959 **PROPOSITION 40.** *There exists $M > 0$ such that for all $\lambda > 0$,*

$$960 \quad (8.3) \quad \|(\mathcal{A}_1^\pm - \lambda)^{-1}\|_{HS} \leq M(1 + \lambda)^{-\frac{1}{4}} (\log \lambda)^{\frac{1}{2}}.$$

961 **Proof**

962 Proposition 40 is a consequence of Proposition 15, and Formula (8.1).

963 **REMARK 41.** *Similar estimates are obtained in the case without boundary (typ-*
 964 *ically for a model like the Davies operator $D_x^2 + ix^2$) by Dencker-Sjöstrand-Zworski*
 965 *[14] or more recently by Sjöstrand [35].*

966 **8.2. Estimate for $f(\lambda)$.** We recall that $f(\lambda)$ was defined in (6.10) by

$$967 \quad f(\lambda) := 2\pi \text{Ai}'(e^{-i\alpha}\lambda) \text{Ai}'(e^{i\alpha}\lambda).$$

968 Recalling the asymptotic expansions (A.6) and (A.8) of Ai' , it is immediate to
 969 get

970 **LEMMA 42.** *The function $\lambda \mapsto f(\lambda)$ is an entire function of type $\frac{3}{2}$, i.e. there*
 971 *exists $D > 0$ such that*

$$972 \quad (8.4) \quad |f(\lambda)| \leq D \exp(D|\lambda|^{\frac{3}{2}}), \quad \forall \lambda \in \mathbb{C}.$$

973 Focusing now on the main purpose of this section, we get from (A.6) that for any
 974 $\epsilon > 0$ there exists $\lambda_1 > 0$ such that, for $\lambda \geq \lambda_1$,

$$975 \quad (8.5) \quad |\text{Ai}'(e^{i\alpha}\lambda)|^2 = |\text{Ai}'(e^{-i\alpha}\lambda)|^2 \geq \frac{1-\epsilon}{4\pi} \lambda^{\frac{1}{2}} \exp\left(\frac{4}{3}\lambda^{\frac{3}{2}}\right).$$

976 Here we have also used that

$$977 \quad (8.6) \quad \text{Ai}(z) = \overline{\text{Ai}(\bar{z})} \quad \text{and} \quad \text{Ai}'(z) = \overline{\text{Ai}'(\bar{z})}$$

978 (note that $\text{Ai}(x)$ is real for x real). Thus there exists $C_1 > 0$ such that, for $\lambda \geq 1$,

$$979 \quad (8.7) \quad \frac{1}{|f(\lambda)|} \leq \frac{C_1}{\lambda^{\frac{1}{2}}} \exp\left(-\frac{4}{3}\lambda^{\frac{3}{2}}\right).$$

980 **8.3. Estimate of the L^2 norm of $\mathcal{G}_1(\cdot, \cdot; \lambda, 0)$.** Having in mind (6.15)-(6.16)
 981 and noting that, for $\lambda > 0$,

$$982 \quad \frac{|\text{Ai}'(e^{i\alpha}\lambda)|^2}{|f(\lambda)|} = \frac{1}{2\pi},$$

983 it is enough to estimate

$$984 \quad (8.8) \quad \int_0^{+\infty} |\text{Ai}(e^{-i\alpha}(ix + \lambda))|^2 dx = I_0(\lambda) = \int_{-\infty}^0 |\text{Ai}(e^{i\alpha}(ix + \lambda))|^2 dx.$$

985 It is enough to observe from (3.10), (8.6) and the comparison of the domain of inte-
 986 gration in \mathbb{R}^2 , that

$$987 \quad (8.9) \quad 2I_0(\lambda)^2 \leq \|\mathcal{G}_0^-(\cdot, \cdot; \lambda)\|^2.$$

988 Applying (3.7), we get

$$989 \quad (8.10) \quad I_0(\lambda) \lesssim \lambda^{-\frac{1}{4}} \exp\left(\frac{4}{3}\lambda^{\frac{3}{2}}\right).$$

990 Hence, coming back to (8.1), we have obtained

991 **PROPOSITION 43.** *There exist κ_0, C and $\lambda_0 > 0$ such that, for all $\kappa \in [0, \kappa_0]$, for*
 992 *all $\lambda \geq \lambda_0$,*

$$993 \quad (8.11) \quad \|\mathcal{G}^-(\cdot, \cdot; \lambda, \kappa) - \mathcal{G}^-(\cdot, \cdot; \lambda, 0)\|_{L^2(\mathbb{R}^2)} \leq C\kappa |\lambda|^{-\frac{3}{4}}.$$

994 Hence we are reduced to the case $\kappa = 0$ which can be decoupled (see Remark 29) in
 995 two Neumann problems on \mathbb{R}_- and \mathbb{R}_+ .

996

997 Using (8.2) and the estimates established for $\mathcal{G}_0^-(\cdot, \cdot; \lambda, 0)$ (which depends only
 998 on $\text{Re } \lambda$) (see (3.7) or (3.5)), we have

999 **PROPOSITION 44.** *For all κ_0 , there exist C and $\lambda_0 > 0$ such that, for all $\kappa \in$
 1000 $[0, \kappa_0]$, for all real $\lambda \geq \lambda_0$, one has*

$$1001 \quad (8.12) \quad \|\mathcal{G}^-(\cdot, \cdot; \lambda, \kappa) - (f(\lambda)(f(\lambda) + \kappa)^{-1})\mathcal{G}^-(\cdot, \cdot; \lambda, 0)\|_{L^2(\mathbb{R}^2)} \leq C\kappa |\lambda|^{-\frac{3}{4}}.$$

1002 This immediately implies

1003 **PROPOSITION 45.** *For any $g = (g_-, g_+)$, $h = (h_-, h_+)$ in $L_-^2 \times L_+^2$, we have*

$$1004 \quad (8.13) \quad |\langle \mathcal{G}^-(\lambda, \kappa)g, h \rangle - (f(\lambda)(f(\lambda) + \kappa)^{-1})\langle \mathcal{G}^-(\lambda, 0)g, h \rangle| \leq C(g, h)\kappa |\lambda|^{-\frac{3}{4}},$$

1005 where $\langle \cdot, \cdot \rangle$ denotes the scalar product in the Hilbert space $L_-^2 \times L_+^2$.

1006 We now adapt the proof of the completeness from [2].

1007 If we denote by E the closed space generated by the generalized eigenfunctions of \mathcal{A}_1^- ,
 1008 the proof of [2] in the presentation of [28] consists in introducing

$$1009 \quad F(\lambda) = \langle \mathcal{G}^-(\lambda, \kappa)g, h \rangle,$$

1010 where

$$1011 \quad (8.14) \quad h \in E^\perp \quad \text{and} \quad g \in L_-^2 \times L_+^2.$$

1012 As a consequence of the assumption on h , one observes that $F(\lambda)$ is an entire function
 1013 and the problem is to show that F is identically 0. The completeness is obtained if
 1014 we prove this statement for any g and h satisfying the condition (8.14).

1015 Outside the numerical range of \mathcal{A}_1^- , i.e. in the negative half-plane, it is immediate to
 1016 see that $F(\lambda)$ tends to zero as $\operatorname{Re} \lambda \rightarrow -\infty$. If we show that $|F(\lambda)| \leq C(1 + |\lambda|)^M$ for
 1017 some $M > 0$ in the whole complex plane, we will get by Liouville's theorem that F
 1018 is a polynomial and, with the control in the left half-plane, we should get that F is
 1019 identically 0.

1020 Hence it remains to control $F(\lambda)$ in a neighborhood of the positive half-plane $\{\lambda \in$
 1021 $\mathbb{C}, \operatorname{Re} \lambda \geq 0\}$.

1022

1023 As in [2], we apply Phragmen-Lindelöf principle (see Appendix D). The natural
 1024 idea (suggested by the numerical picture) is to control the resolvent on the positive
 1025 real axis. We first recall some additional material present in Chapter 16 in [2].

1026 **THEOREM 46.** *Let $\phi(\lambda)$ be an entire complex-valued function of finite order ρ*
 1027 *(and $\phi(\lambda)$ is not identically equal to 0). Then for any $\epsilon > 0$ there exists an infinite*
 1028 *increasing sequence $(r_k)_{k \in \mathbb{N}}$ in \mathbb{R}^+ and tending to $+\infty$ such that*

$$1029 \quad \min_{|\lambda|=r_k} |\phi(\lambda)| > \exp(-r_k^{\rho+\epsilon}).$$

1030 For this theorem (Theorem 6.2 in [2]), S. Agmon refers to the book of E.C. Titchmarsh
 1031 [38] (p. 273).

1032 This theorem is used for proving an inequality of the type ρ with $\rho = 2$ in the
 1033 Hilbert-Schmidt case. We avoid an abstract lemma [2] (Lemma 16.3) but follow the
 1034 scheme of its proof for controlling directly the Hilbert-Schmidt norm of the resolvent
 1035 along an increasing sequence of circles.

1036 **PROPOSITION 47.** *For $\epsilon > 0$, there exists an infinite increasing sequence $(r_k)_{k \in \mathbb{N}}$*
 1037 *in \mathbb{R}^+ tending to $+\infty$ such that*

$$1038 \quad \max_{|\lambda|=r_k} \|\mathcal{G}^\pm(\cdot, \cdot; \lambda, \kappa)\|_{HS} \leq \exp(r_k^{\frac{3}{2}+\epsilon}).$$

1039 **Proof**

1040 We start from

$$1041 \quad (8.15) \quad \mathcal{G}^-(x, y; \lambda, \kappa) = \mathcal{G}^-(x, y; \lambda, 0) f(\lambda) (f(\lambda) + \kappa)^{-1} + \kappa (f(\lambda) + \kappa)^{-1} \mathcal{G}_0^-(x, y; \lambda, 0).$$

1042 We apply Theorem 46 with $\phi(\lambda) = f(\lambda) + \kappa$. It is proven in Lemma 42 that f is
 1043 of type $\frac{3}{2}$. Hence we get for $\epsilon > 0$ (arbitrary small) the existence of a sequence
 1044 $r_1 < r_2 < \dots < r_k < \dots$ such that

$$1045 \quad \max_{|\lambda|=r_k} \left| \frac{1}{f(\lambda) + \kappa} \right| \leq \exp(r_k^{\frac{3}{2}+\epsilon}).$$

1046 In view of (8.15), it remains to control the Hilbert-Schmidt norm of

$$1047 \quad \mathcal{G}^-(x, y; \lambda, 0) f(\lambda) + \kappa \mathcal{G}_0^-(x, y; \lambda, 0).$$

1048 Hence the remaining needed estimates only concern the case $\kappa = 0$. The estimate on
 1049 the Hilbert-Schmidt norm of \mathcal{G}_0^- is recalled in (3.7). It remains to get an estimate for
 1050 the entire function $\mathcal{G}^-(x, y; \lambda, 0) f(\lambda)$.

1051 Because $\kappa = 0$, this is reduced to the Neumann problem on the half-line for the
 1052 complex Airy operator $D_x^2 - ix$. For $y > 0$ and $x > 0$, $f(\lambda) \mathcal{G}_1^N(x, y; \lambda)$ is given by
 1053 the following expression

$$1054 \quad (8.16) \quad f(\lambda) \mathcal{G}_1^N(x, y; \lambda) = -4\pi^2 [e^{2i\alpha} \text{Ai}'(e^{i\alpha}\lambda)]^2 \text{Ai}(e^{-i\alpha}w_x) \text{Ai}(e^{-i\alpha}w_y).$$

1055 We only need the estimate for λ in a sector containing $\mathbb{R}_+ \times \mathbb{R}_+$. This is done in [28]
 1056 but we will give a direct proof below. In the other region, we can first control the
 1057 resolvent in $\mathcal{L}(L^2)$ and then use the resolvent identity

$$1058 \quad \mathcal{G}^{\pm, N}(\lambda) - \mathcal{G}^{\pm, N}(\lambda_0) = (\lambda - \lambda_0) \mathcal{G}^{\pm, N}(\lambda) \mathcal{G}^{\pm, N}(\lambda_0).$$

1059 This shows that in order to control the Hilbert-Schmidt norm of $\mathcal{G}^{\pm, N}(\lambda)$ for any λ ,
 1060 it is enough to control the Hilbert-Schmidt norm of $\mathcal{G}^{\pm, N}(\lambda_0)$ for some λ_0 , as well as
 1061 the $\mathcal{L}(L^2)$ norm of $\mathcal{G}^{\pm, N}(\lambda)$, the latter being easier to estimate.

1062

1063 More directly the control of the Hilbert-Schmidt norm is reduced to the existence
 1064 of a constant $C > 0$ such that

$$1065 \quad \int_0^{+\infty} |\text{Ai}(e^{-i\alpha}(ix + \lambda))|^2 dx \leq C \exp(C|\lambda|^{\frac{3}{2}}).$$

In this case, we have to control the resolvent in a neighborhood of the sector
 $\text{Im } \lambda \leq 0$, $\text{Re } \lambda \geq 0$, which corresponds to the numerical range of the operator.

As $x \rightarrow +\infty$, the dominant term in the argument of the Airy function is $e^{i(-\alpha + \frac{\pi}{2})x} = e^{-i\frac{\pi}{6}x}$. As expected we arrive in a zone of the complex plane where the Airy function is exponentially decreasing. It remains to estimate for which x we enter in this zone. We claim that there exists $C > 0$ such that if $x \geq C|\lambda|$ and $|\lambda| \geq 1$, then

$$|\text{Ai}(e^{-i\alpha}(ix + \lambda))| \leq C \exp(-C(x + |\lambda|)^{\frac{3}{2}}).$$

1066 In the remaining zone, we obtain an upper bound of the integral by $\mathcal{O}(\exp(C|\lambda|^{\frac{3}{2}}))$.

1067

1068 We will then use the Phragmen-Lindelöf principle (Theorem 54). For this purpose,
 1069 it remains to control the resolvent on the positive real line. It is enough to prove the
 1070 theorem for $g^+ = (0, g_+)$ and $g^- = (g_-, 0)$. In other words, it is enough to consider
 1071 F_+ (resp. F_-) associated with g^+ (resp. g^-).

1072 Let us treat the case of F_+ and use Formula (8.2) and Proposition 45:

$$1073 \quad (8.17) \quad |\langle \mathcal{G}^-(\lambda, \kappa)g^+, h \rangle - (f(\lambda)(f(\lambda) + \kappa)^{-1}) \langle \mathcal{G}^-(\lambda, 0)g_+, h_+ \rangle| \leq C(g, h)\kappa |\lambda|^{-\frac{3}{4}}.$$

1074 This estimate is true on the positive real axis. It remains to control the term
 1075 $|\langle \mathcal{G}^-(\lambda, 0)g^+, h \rangle|$. Along this positive real axis, we have by Proposition 15 the decay
 1076 of $F_+(\lambda)$. Using Phragmen-Lindelöf principle completes the proof.

1077 Note that for $F_-(\lambda)$, we have to use the symmetric (with respect to the real axis)
 1078 curve in $\text{Im } \lambda > 0$.

1079 In summary, we have obtained the following

1080 **PROPOSITION 48.** *For any $\kappa \geq 0$, the space generated by the generalized eigen-*
 1081 *functions of the complex Airy operator on the line with transmission is dense in*
 1082 $L_-^2 \times L_+^2$.

1083 **Appendices.**1084 **Appendix A. Basic properties of the Airy function.**

1085 In this Appendix, we summarize the basic properties of the Airy function $\text{Ai}(z)$
 1086 and its derivative $\text{Ai}'(z)$ that we used (see [1] for details).

1087 We recall that the Airy function is the unique solution of

$$1088 \quad (D_x^2 + x)u = 0$$

1089 on the line such that $u(x)$ tends to 0 as $x \rightarrow +\infty$ and

$$1090 \quad \text{Ai}(0) = \frac{1}{3^{\frac{2}{3}} \Gamma(\frac{2}{3})}.$$

1091 This Airy function extends into a holomorphic function in \mathbb{C} .

1092 The Airy function is positive decreasing on \mathbb{R}_+ but has an infinite number of zeros
 1093 in \mathbb{R}_- . We denote by a_n ($n \in \mathbb{N}$) the decreasing sequence of zeros of Ai . Similarly we
 1094 denote by a'_n the sequence of zeros of Ai' . They have the following asymptotics (see
 1095 for example [1]), as $n \rightarrow +\infty$,

$$1096 \quad (\text{A.1}) \quad a_n \underset{n \rightarrow +\infty}{\sim} - \left(\frac{3\pi}{2}(n - 1/4) \right)^{\frac{2}{3}},$$

1097 and

$$1098 \quad (\text{A.2}) \quad a'_n \underset{n \rightarrow +\infty}{\sim} - \left(\frac{3\pi}{2}(n - 3/4) \right)^{\frac{2}{3}}.$$

1099 The functions $\text{Ai}(e^{i\alpha}z)$ and $\text{Ai}(e^{-i\alpha}z)$ (with $\alpha = 2\pi/3$) are two independent
 1100 solutions of the differential equation

$$1101 \quad \left(-\frac{d^2}{dz^2} + z \right) w(z) = 0.$$

1102 Their Wronskian reads

$$1103 \quad (\text{A.3}) \quad e^{-i\alpha} \text{Ai}'(e^{-i\alpha}z) \text{Ai}(e^{i\alpha}z) - e^{i\alpha} \text{Ai}'(e^{i\alpha}z) \text{Ai}(e^{-i\alpha}z) = \frac{i}{2\pi} \quad \forall z \in \mathbb{C}.$$

1104 Note that these two functions are related to $\text{Ai}(z)$ by the identity

$$1105 \quad (\text{A.4}) \quad \text{Ai}(z) + e^{-i\alpha} \text{Ai}(e^{-i\alpha}z) + e^{i\alpha} \text{Ai}(e^{i\alpha}z) = 0 \quad \forall z \in \mathbb{C}.$$

1106 The Airy function and its derivative satisfy different asymptotic expansions de-
 1107 pending on their argument:

1108 (i) For $|\arg z| < \pi$,

$$1109 \quad (\text{A.5}) \quad \text{Ai}(z) = \frac{1}{2\sqrt{\pi}} z^{-\frac{1}{4}} \exp\left(-\frac{2}{3}z^{\frac{3}{2}}\right) (1 + \mathcal{O}(|z|^{-\frac{3}{2}})),$$

$$1110 \quad (\text{A.6}) \quad \text{Ai}'(z) = -\frac{1}{2\sqrt{\pi}} z^{\frac{1}{4}} \exp\left(-\frac{2}{3}z^{\frac{3}{2}}\right) (1 + \mathcal{O}(|z|^{-\frac{3}{2}}))$$

1111 (moreover \mathcal{O} is, for any $\epsilon > 0$, uniform when $|\arg z| \leq \pi - \epsilon$).

1112

1113 (ii) For $|\arg z| < \frac{2}{3}\pi$,

$$1114 \quad (\text{A.7}) \quad \text{Ai}(-z) = \frac{1}{\sqrt{\pi}} z^{-\frac{1}{4}} \left(\sin\left(\frac{2}{3}z^{\frac{3}{2}} + \frac{\pi}{4}\right) (1 + \mathcal{O}(|z|^{-\frac{3}{2}})) \right. \\ 1115 \quad \left. - \frac{5}{72} \left(\frac{2}{3}z^{\frac{3}{2}}\right)^{-1} \cos\left(\frac{2}{3}z^{\frac{3}{2}} + \frac{\pi}{4}\right) (1 + \mathcal{O}(|z|^{-\frac{3}{2}})) \right)$$

$$1116 \quad (\text{A.8}) \quad \text{Ai}'(-z) = -\frac{1}{\sqrt{\pi}} z^{\frac{1}{4}} \left(\cos\left(\frac{2}{3}z^{\frac{3}{2}} + \frac{\pi}{4}\right) (1 + \mathcal{O}(|z|^{-\frac{3}{2}})) \right. \\ 1117 \quad \left. + \frac{7}{72} \left(\frac{2}{3}z^{\frac{3}{2}}\right)^{-1} \sin\left(\frac{2}{3}z^{\frac{3}{2}} + \frac{\pi}{4}\right) (1 + \mathcal{O}(|z|^{-\frac{3}{2}})) \right)$$

1118 (moreover \mathcal{O} is for any $\epsilon > 0$, uniform in the sector $\{|\arg z| \leq \frac{2\pi}{3} - \epsilon\}$).

1119 **Appendix B. Analysis of the resolvent of \mathcal{A}^+ on the line for $\lambda > 0$ (after**
1120 **[32]).**

1121 On the line \mathbb{R} , \mathcal{A}^+ is the closure of the operator \mathcal{A}_0^+ defined on $C_0^\infty(\mathbb{R})$ by
1122 $\mathcal{A}_0^+ = D_x^2 + ix$. A detailed description of its properties can be found in [26]. In this
1123 appendix, we give the asymptotic control of the resolvent $(\mathcal{A}^+ - \lambda)^{-1}$ as $\lambda \rightarrow +\infty$.
1124 We successively discuss the control in $\mathcal{L}(L^2(\mathbb{R}))$ and in the Hilbert-Schmidt space
1125 $\mathcal{C}^2(L^2(\mathbb{R}))$. These two spaces are equipped with their canonical norms.

1126 **B.1. Control in $\mathcal{L}(L^2(\mathbb{R}))$.** Here we follow an idea present in the book of E. B.
1127 Davies [13] and used in J. Martinet's PHD [32] (see also [26]).

1128 PROPOSITION 49.

1129 For all $\lambda > \lambda_0$,

$$1130 \quad (\text{B.1}) \quad \|(\mathcal{A}^+ - \lambda)^{-1}\|_{\mathcal{L}(L^2(\mathbb{R}))} \leq \sqrt{2\pi} \lambda^{-\frac{1}{4}} \exp\left(\frac{4}{3}\lambda^{\frac{3}{2}}\right) (1 + o(1)).$$

1131 **Proof**

1132 The proof is obtained by considering \mathcal{A}^+ in the Fourier space, i.e.

$$1133 \quad (\text{B.2}) \quad \widehat{\mathcal{A}}^+ = \xi^2 - \frac{d}{d\xi}.$$

1134 The associated semi-group $T_t := \exp(-\widehat{\mathcal{A}}^+ t)$ is given by

$$1135 \quad (\text{B.3}) \quad T_t u(\xi) = \exp\left(-\xi^2 t - \xi t^2 - \frac{t^3}{3}\right) u(\xi - t), \quad \forall u \in \mathcal{S}(\mathbb{R}).$$

1136 T_t appears as the composition of a multiplication by $\exp(-\xi^2 t - \xi t^2 - \frac{t^3}{3})$ and a
1137 translation by t . Computing $\sup_{\xi} \{\exp(-\xi^2 t - \xi t^2 - \frac{t^3}{3})\}$ leads to

$$1138 \quad (\text{B.4}) \quad \|T_t\|_{\mathcal{L}(L^2(\mathbb{R}))} \leq \exp\left(-\frac{t^3}{12}\right).$$

1139 It is then easy to get an upper bound for the resolvent. For $\lambda > 0$, we have

$$1140 \quad (\text{B.5}) \quad \|(\mathcal{A}^+ - \lambda)^{-1}\|_{\mathcal{L}(L^2(\mathbb{R}))} = \|(\widehat{\mathcal{A}}^+ - \lambda)^{-1}\|_{\mathcal{L}(L^2(\mathbb{R}))} \\ 1141 \quad \leq \int_0^{+\infty} \exp(t\lambda) \|T_t\|_{\mathcal{L}(L^2(\mathbb{R}))} dt \\ 1142 \quad \leq \int_0^{+\infty} \exp\left(t\lambda - \frac{t^3}{12}\right) dt.$$

1143 The right hand side can be estimated by using the Laplace integral method. Setting
1144 $t = \lambda^{\frac{1}{2}}s$, we have

$$1145 \quad (B.6) \quad \int_0^{+\infty} \exp\left(t\lambda - \frac{t^3}{12}\right) dt = \lambda^{\frac{1}{2}} \int_0^{+\infty} \exp\left(\lambda^{\frac{3}{2}}\left(s - \frac{s^3}{12}\right)\right) ds.$$

1146 We observe that $\hat{\phi}(s) = s - \frac{s^3}{12}$ admits a global non-degenerate maximum on $[0, +\infty)$
1147 at $s = 2$ with $\hat{\phi}(2) = \frac{4}{3}$ and $\hat{\phi}''(2) = -1$. The Laplace integral method gives the
1148 following equivalence as $\lambda \rightarrow +\infty$:

$$1149 \quad (B.7) \quad \int_0^{+\infty} \exp\left(\lambda^{\frac{3}{2}}\left(s - \frac{s^3}{12}\right)\right) ds \sim \sqrt{2\pi} \lambda^{-\frac{3}{4}} \exp\left(\frac{4}{3}\lambda^{\frac{3}{2}}\right).$$

1150 This completes the proof of the proposition. We note that this upper bound is not
1151 optimal in comparison with Bordeaux-Montrieux's formula (3.5).

1152 **B.2. Control in Hilbert-Schmidt norm.** In this part, we give a proof of
1153 Proposition 7. As in the previous subsection, we use the Fourier representation and
1154 analyze $\widehat{\mathcal{A}}^+$. Note that

$$1155 \quad (B.8) \quad \|(\widehat{\mathcal{A}}^+ - \lambda)^{-1}\|_{HS}^2 = \|(\mathcal{A}^+ - \lambda)^{-1}\|_{HS}^2.$$

1156 We have then an explicit description of the resolvent by

$$1157 \quad (\widehat{\mathcal{A}}^+ - \lambda)^{-1}u(\xi) = \int_{-\infty}^{\xi} u(\eta) \exp\left(\frac{1}{3}(\eta^3 - \xi^3) + \lambda(\xi - \eta)\right) d\eta.$$

1158 Hence, we have to compute

$$1159 \quad \|(\widehat{\mathcal{A}}^+ - \lambda)^{-1}\|_{HS}^2 = \int \int_{\eta < \xi} \exp\left(\frac{2}{3}(\eta^3 - \xi^3) + 2\lambda(\xi - \eta)\right) d\eta d\xi.$$

1160 After the change of variable $(\xi_1, \eta_1) = (\lambda^{-\frac{1}{2}}\xi, \lambda^{-\frac{1}{2}}\eta)$, we get

$$1161 \quad \|(\widehat{\mathcal{A}}^+ - \lambda)^{-1}\|_{HS}^2 = \lambda \int \int_{\eta_1 < \xi_1} \exp\left(\lambda^{\frac{3}{2}}\left[\frac{2}{3}(\eta_1^3 - \xi_1^3) + 2(\xi_1 - \eta_1)\right]\right) d\xi_1 d\eta_1.$$

1162 With

$$1163 \quad (B.9) \quad h = \lambda^{-\frac{3}{2}},$$

1164 we can write

$$1165 \quad (B.10) \quad \|(\widehat{\mathcal{A}}^+ - \lambda)^{-1}\|_{HS}^2 = h^{-\frac{2}{3}}\Phi(h),$$

1166 where

$$1167 \quad (B.11) \quad \Phi(h) = \int \int_{y < x} \exp\left(\frac{2}{h}[\phi(x) - \phi(y)]\right) dx dy,$$

1168 with

$$1169 \quad (B.12) \quad \phi(x) = x - \frac{x^3}{3}.$$

1170 $\Phi(h)$ can now be split in three terms

1171 (B.13)
$$\Phi(h) = I_1(h) + I_2(h) + I_3(h),$$

1172 with

1173
$$I_1(h) = \int \int_{\substack{y < x \\ y > 0}} \exp\left(\frac{2}{h}[\phi(x) - \phi(y)]\right) dx dy,$$

1174
$$I_2(h) = \int \int_{\substack{y < x \\ x < 0}} \exp\left(\frac{2}{h}[\phi(x) - \phi(y)]\right) dx dy,$$

1175
$$I_3(h) = \int \int_{\substack{x \in \mathbb{R}^+ \\ y \in \mathbb{R}^-}} \exp\left(\frac{2}{h}[\phi(x) - \phi(y)]\right) dx dy.$$

1176 We observe now that by the change of variable $(x, y) \mapsto (-y, -x)$, we get

1177
$$I_1(h) = I_2(h),$$

1178 and that

1179
$$I_3(h) = I_4(h)^2,$$

1180 with

1181
$$I_4(h) = \int_{\mathbb{R}^+} \exp\left(\frac{2}{h}\phi(x)\right) dx.$$

1182 Hence, it remains to estimate, as $h \rightarrow 0$, the integrals $I_1(h)$ and $I_4(h)$.

1183

1184 **B.2.1. Control of $I_1(h)$.**

1185 The function $\phi(x)$ is positive on $(0, \sqrt{3})$ and negative decreasing on $(\sqrt{3}, +\infty)$, with
 1186 $\phi(0) = \phi(\sqrt{3}) = 0$. It admits a unique non-degenerate maximum at $x = 1$ with
 1187 $\phi(1) = \frac{2}{3}$.

1188 We first observe the trivial estimates

1189
$$\exp\left(-\frac{2}{h}\phi(y)\right) \leq 1, \quad \forall y \in [0, \sqrt{3}],$$

1190 and

1191
$$\exp\left(\frac{2}{h}\phi(x)\right) \leq 1, \quad \forall x \in [\sqrt{3}, +\infty[.$$

1192 We will also have to estimate, for $x \geq \sqrt{3}$,

1193
$$J(h, x) := \int_{\sqrt{3}}^x \exp\left(-\frac{2}{h}\phi(y)\right) dy.$$

1194 For this purpose, we integrate by parts, observing that

1195
$$\exp\left(-\frac{2}{h}\phi(y)\right) = -\frac{h}{2} \frac{1}{\phi'(y)} \frac{d}{dy} \exp\left(-\frac{2}{h}\phi(y)\right).$$

1196 We get

$$1197 \quad J(h, x) = -\frac{h}{2} \frac{1}{\phi'(x)} \exp\left(-\frac{2}{h}\phi(x)\right) - \frac{h}{2} \frac{1}{\phi'(\sqrt{3})} + \frac{h}{2} \int_{\sqrt{3}}^x \left(\frac{1}{\phi'}\right)'(y) \exp\left(-\frac{2}{h}\phi(y)\right) dy.$$

1198 This implies

$$1199 \quad J(h, x) \leq \frac{h}{2} \frac{1}{x^2 - 1} \exp\left(-\frac{2}{h}\phi(x)\right) - \frac{h}{2} \frac{1}{\phi'(\sqrt{3})} + ChJ(h, x),$$

1200 and finally, for h small enough and another constant $C > 0$

$$1201 \quad (\text{B.14}) \quad J(h, x) \leq \frac{h}{2} \frac{1 + Ch}{x^2 - 1} \exp\left(-\frac{2}{h}\phi(x)\right) + Ch, \quad \forall x \in [\sqrt{3}, +\infty).$$

1202 Similarly, one can show that

$$1203 \quad (\text{B.15}) \quad \int_{\sqrt{3}}^{+\infty} \exp\left(\frac{2}{h}\phi(x)\right) dx \leq \frac{h}{4}.$$

1204 With these estimates, we can bound $I_1(h)$ from above in the following way

$$\begin{aligned} 1205 \quad I_1(h) &= \int_0^{\sqrt{3}} \exp\left(\frac{2}{h}\phi(x)\right) \left(\int_0^x \exp\left(-\frac{2}{h}\phi(y)\right) dy\right) dx \\ 1206 \quad &+ \int_{\sqrt{3}}^{+\infty} \exp\left(\frac{2}{h}\phi(x)\right) \left(\int_0^{\sqrt{3}} \exp\left(-\frac{2}{h}\phi(y)\right) dy\right) dx \\ 1207 \quad &+ \int_{\sqrt{3}}^{+\infty} \exp\left(\frac{2}{h}\phi(x)\right) J(h, x) \\ 1208 \quad &\leq \int_0^{\sqrt{3}} \exp\left(\frac{2}{h}\phi(x)\right) \left(\int_0^{\sqrt{3}} \exp\left(-\frac{2}{h}\phi(y)\right) dy\right) dx \\ 1209 \quad &+ (\sqrt{3} + Ch) \int_{\sqrt{3}}^{+\infty} \exp\left(\frac{2}{h}\phi(x)\right) dx \\ 1210 \quad &+ (1 + Ch) \frac{h}{2} \int_{\sqrt{3}}^{+\infty} \frac{1}{x^2 - 1} dx \\ 1211 \quad &\leq 3 \sup_{[0, \sqrt{3}]} \left\{ \exp\left(\frac{2}{h}\phi(x)\right) \right\} \\ 1212 \quad &+ \frac{(\sqrt{3} + Ch)h}{4} \\ 1213 \quad &+ (1 + Ch) \frac{h}{2} \int_{\sqrt{3}}^{+\infty} \frac{1}{x^2 - 1} dx \\ 1214 \quad &\leq 3 \exp\left(\frac{4}{3h}\right) + \frac{\sqrt{3}h(1 + Ch)}{4} + \frac{h}{2}(1 + Ch) \int_{\sqrt{3}}^{+\infty} \frac{1}{x^2 - 1} dx. \end{aligned}$$

1215 Hence we have shown the existence of $\hat{C} > 0$ and $h_0 > 0$ such that, for $h \in (0, h_0)$,

$$1216 \quad (\text{B.16}) \quad I_1(h) \leq \hat{C} \exp\left(\frac{4}{3h}\right).$$

1217 Consequently, $I_1(h)$ and $I_2(h)$ appear as remainder terms.

1218

1219 **B.2.2. Asymptotic of $I_4(h)$.** Here, using the properties of ϕ , we get by the
 1220 standard Laplace integral method

$$1221 \quad (B.17) \quad I_4(h) \sim \sqrt{\pi/2} \sqrt{h} \exp\left(\frac{4}{3h}\right).$$

1222 Hence, putting altogether the estimates, we get, as $h \rightarrow 0$,

$$1223 \quad (B.18) \quad \Phi(h) \sim \frac{\pi h}{2} \exp\left(\frac{8}{3h}\right)$$

1224 Coming back to (B.8), (B.9) and (B.10), this achieves the proof of Proposition 7.

1225 **Appendix C. Analysis of the resolvent for the Dirichlet realization in**
 1226 **the half-line.**

1227 **C.1. Main statement.** The aim of this appendix is to give the proof of Propo-
 1228 sition 11. Although it is not used in our main text, it is interesting to get the main
 1229 asymptotic for the Hilbert-Schmidt norm of the resolvent in Proposition 11.

1230 PROPOSITION 50. *As $\lambda \rightarrow +\infty$, we have*

$$1231 \quad (C.1) \quad \|\mathcal{G}^{-,D}(\lambda)\|_{HS} \sim \frac{\sqrt{3}}{2\sqrt{2}} \lambda^{-\frac{1}{4}} (\log \lambda)^{\frac{1}{2}}.$$

1232 **C.2. The Hilbert-Schmidt norm of the resolvent for real λ .** The Hilbert-
 1233 Schmidt norm of the resolvent can be written as

$$1234 \quad (C.2) \quad \|\mathcal{G}^{-,D}\|_{HS}^2 = \int_{\mathbb{R}_+^2} |\mathcal{G}^{-,D}(x, y; \lambda)|^2 dx dy = 8\pi^2 \int_0^\infty Q(x; \lambda) dx,$$

1235 where

$$1236 \quad (C.3) \quad Q(x; \lambda) = \frac{|\text{Ai}(e^{-i\alpha}(ix + \lambda))|^2}{|\text{Ai}(e^{-i\alpha}\lambda)|^2} \times \\ \times \int_0^x |\text{Ai}(e^{i\alpha}(iy + \lambda))\text{Ai}(e^{-i\alpha}\lambda) - \text{Ai}(e^{-i\alpha}(iy + \lambda))\text{Ai}(e^{i\alpha}\lambda)|^2 dy.$$

1237 Using the identity (A.4), we observe that

$$1238 \quad (C.4) \quad \text{Ai}(e^{i\alpha}(iy + \lambda))\text{Ai}(e^{-i\alpha}\lambda) - \text{Ai}(e^{-i\alpha}(iy + \lambda))\text{Ai}(e^{i\alpha}\lambda) \\ = e^{-i\alpha} (\text{Ai}(e^{-i\alpha}(iy + \lambda))\text{Ai}(\lambda) - \text{Ai}(iy + \lambda)\text{Ai}(e^{-i\alpha}\lambda)).$$

1239 Hence we get
 (C.5)

$$1240 \quad Q(x; \lambda) = |\text{Ai}(e^{-i\alpha}(ix + \lambda))|^2 \int_0^x \left| \text{Ai}(e^{-i\alpha}(iy + \lambda)) \frac{\text{Ai}(\lambda)}{\text{Ai}(e^{-i\alpha}\lambda)} - \text{Ai}(iy + \lambda) \right|^2 dy.$$

1241 **C.3. More facts on Airy expansions.** As a consequence of (A.5), we can
 1242 write for $\lambda > 0$ and $x > 0$

$$1243 \quad (C.6) \quad |\text{Ai}(e^{-i\alpha}(ix + \lambda))| = \frac{\exp(-\frac{2}{3}\lambda^{\frac{3}{2}}u(x/\lambda))}{2\sqrt{\pi}(\lambda^2 + x^2)^{\frac{1}{8}}}(1 + \mathcal{O}(\lambda^{-\frac{3}{2}})),$$

1244 where

$$1245 \quad (C.7) \quad u(s) = -(1 + s^2)^{\frac{3}{4}} \cos\left(\frac{3}{2} \tan^{-1}(s)\right) = \frac{\sqrt{\sqrt{1 + s^2} + 1}(\sqrt{1 + s^2} - 2)}{\sqrt{2}}.$$

1246 We note indeed that $|e^{-i\alpha}(ix + \lambda)| = \sqrt{x^2 + \lambda^2} \geq \lambda \geq \lambda_0$ and that we have a control
 1247 of the argument $\arg(e^{-i\alpha}(ix + \lambda)) \in [-\frac{2\pi}{3}, -\frac{\pi}{6}]$ which permits to apply (A.5).
 1248 Similarly, we obtain

$$1249 \quad (C.8) \quad |\text{Ai}(ix + \lambda)| = \frac{\exp(\frac{2}{3}\lambda^{\frac{3}{2}}u(x/\lambda))}{2\sqrt{\pi}(\lambda^2 + x^2)^{\frac{1}{8}}}(1 + \mathcal{O}(\lambda^{-\frac{3}{2}})).$$

1250 We note indeed that $|ix + \lambda| = \sqrt{x^2 + \lambda^2}$ and $\arg((ix + \lambda)) \in [0, +\frac{\pi}{2}]$ so that one
 1251 can then again apply (A.5). In particular the function $|\text{Ai}(ix + \lambda)|$ grows super-
 1252 exponentially as $x \rightarrow +\infty$.

1253 Figure 3 illustrates that, for large λ , both equations (C.6) and (C.8) are very accurate
 1254 approximations for $|\text{Ai}(e^{-i\alpha}(ix + \lambda))|$ and $|\text{Ai}(ix + \lambda)|$, respectively.

1255 The control of the next order term (as given in (A.5)) implies that there exist $C > 0$
 1256 and $\epsilon_0 > 0$, such that, for any $\epsilon \in (0, \epsilon_0]$, any $\lambda > \epsilon^{-\frac{2}{3}}$ and any $x \geq 0$, one has

$$1257 \quad (C.9) \quad (1 - C\epsilon) \frac{\exp(-\frac{2}{3}\lambda^{\frac{3}{2}}u(x/\lambda))}{2\sqrt{\pi}(\lambda^2 + x^2)^{\frac{1}{8}}} \leq |\text{Ai}(e^{-i\alpha}(ix + \lambda))| \leq (1 + C\epsilon) \frac{\exp(-\frac{2}{3}\lambda^{\frac{3}{2}}u(x/\lambda))}{2\sqrt{\pi}(\lambda^2 + x^2)^{\frac{1}{8}}}$$

1258 and

$$1259 \quad (C.10) \quad (1 - C\epsilon) \frac{\exp(\frac{2}{3}\lambda^{\frac{3}{2}}u(x/\lambda))}{2\sqrt{\pi}(\lambda^2 + x^2)^{\frac{1}{8}}} \leq |\text{Ai}(ix + \lambda)| \leq (1 + C\epsilon) \frac{\exp(\frac{2}{3}\lambda^{\frac{3}{2}}u(x/\lambda))}{2\sqrt{\pi}(\lambda^2 + x^2)^{\frac{1}{8}}},$$

1260 where the function u is explicitly defined in Eq. (C.7).

1261

1262 Basic properties of u .

1263 Note that

$$1264 \quad (C.11) \quad u'(s) = \frac{3}{2\sqrt{2}} \frac{s}{\sqrt{1 + \sqrt{1 + s^2}}} \geq 0 \quad (s \geq 0),$$

1265 and u has the following expansion at the origin

$$1266 \quad (C.12) \quad u(s) = -1 + \frac{3}{8}s^2 + \mathcal{O}(s^4).$$

1267 For large s , one has

$$1268 \quad (C.13) \quad u(s) \sim \frac{s^{\frac{3}{2}}}{\sqrt{2}}, \quad u'(s) \sim \frac{3s^{\frac{1}{2}}}{2\sqrt{2}}.$$

1269 One concludes that the function u is monotonously increasing on $[0, +\infty)$ with
 1270 $u(0) = -1$ and $\lim_{s \rightarrow +\infty} u(s) = +\infty$.

1271 **C.4. Upper bound.** We start from the simple upper bound (for any $\epsilon > 0$)

1272 (C.14)
$$Q(x, \lambda) \leq \left(1 + \frac{1}{\epsilon}\right) Q_1(x, \lambda) + (1 + \epsilon) Q_2(x, \lambda),$$

1273 with

1274
$$Q_1(x, \lambda) := |\text{Ai}(e^{-i\alpha}(ix + \lambda))|^2 \frac{|\text{Ai}(\lambda)|^2}{|\text{Ai}(e^{-i\alpha}\lambda)|^2} \int_0^x |\text{Ai}(e^{-i\alpha}(iy + \lambda))|^2 dy$$

1275 and

1276
$$Q_2(x, \lambda) := |\text{Ai}(e^{-i\alpha}(ix + \lambda))|^2 \int_0^x |\text{Ai}(iy + \lambda)|^2 dy.$$

1277 We then write

1278
$$Q_1(x, \lambda) \leq |\text{Ai}(e^{-i\alpha}(ix + \lambda))|^2 \frac{|\text{Ai}(\lambda)|^2}{|\text{Ai}(e^{-i\alpha}\lambda)|^2} \int_0^{+\infty} |\text{Ai}(e^{-i\alpha}(iy + \lambda))|^2 dy$$

1279 and integrating over x

1280
$$\int_0^{+\infty} Q_1(x, \lambda) dx \leq I_0(\lambda)^2 \frac{|\text{Ai}(\lambda)|^2}{|\text{Ai}(e^{-i\alpha}\lambda)|^2},$$

1281 where $I_0(\lambda)$ is given by (8.8).

1282

1283 Using (8.10) and (A.5), we obtain

1284 (C.15)
$$\int_0^{+\infty} Q_1(x, \lambda) dx \leq C\lambda^{-\frac{1}{2}}.$$

1285 Hence at this stage, we have proven the existence of $C > 0$, $\epsilon_0 > 0$ and λ_0 such
1286 that for any $\epsilon \in (0, \epsilon_0]$ and any $\lambda \geq \lambda_0$

1287 (C.16)
$$\|\mathcal{G}^{-,D}\|_{HS}^2 \leq (1 + \epsilon) \left(8\pi^2 \int_0^{\infty} Q_2(x; \lambda) dx\right) + C\lambda^{-\frac{1}{2}}\epsilon^{-1}.$$

1288 It remains to estimate

1289 (C.17)
$$\int_0^{+\infty} Q_2(x, \lambda) dx = \int_0^{+\infty} dx \int_0^x |\text{Ai}(e^{-i\alpha}(ix + \lambda))\text{Ai}(iy + \lambda)|^2 dy.$$

1290 Using the estimates (C.6) and (C.8), we obtain

1291 **LEMMA 51.** *There exist C and ϵ_0 , such that, for any $\epsilon \in (0, \epsilon_0)$, for $\lambda > \epsilon^{-\frac{2}{3}}$, the*
1292 *integral of $Q_2(x; \lambda)$ can be bounded as*

1293 (C.18)
$$\frac{1}{2}(1 - C\epsilon) I(\lambda) \leq 8\pi^2 \int_0^{+\infty} Q_2(x, \lambda) dx \leq \frac{1}{2}(1 + C\epsilon) I(\lambda),$$

1294 *where*

$$1295 \quad (C.19) \quad I(\lambda) = \int_0^\infty dx \frac{\exp(-\frac{4}{3}\lambda^{\frac{3}{2}}u(x/\lambda))}{(\lambda^2 + x^2)^{\frac{1}{4}}} \int_0^x dy \frac{\exp(\frac{4}{3}\lambda^{\frac{3}{2}}u(y/\lambda))}{(\lambda^2 + y^2)^{\frac{1}{4}}}.$$

1296 **Control of $I(\lambda)$.**

1297 It remains to control $I(\lambda)$ as $\lambda \rightarrow +\infty$. Using a change of variables, we get

$$1298 \quad (C.20) \quad I(\lambda) = \lambda \int_0^\infty dx \frac{\exp(-\frac{4}{3}\lambda^{\frac{3}{2}}u(x))}{(1 + x^2)^{\frac{1}{4}}} \int_0^x dy \frac{\exp(\frac{4}{3}\lambda^{\frac{3}{2}}u(y))}{(1 + y^2)^{\frac{1}{4}}}.$$

1299 Hence, introducing

$$1300 \quad (C.21) \quad t = \frac{4}{3}\lambda^{\frac{3}{2}},$$

1301 we reduce the analysis to $\hat{I}(t)$ defined for $t \geq t_0$ by

$$1302 \quad (C.22) \quad \hat{I}(t) := \int_0^\infty dx \frac{1}{(1 + x^2)^{\frac{1}{4}}} \int_0^x dy \frac{\exp(t(u(y) - u(x)))}{(1 + y^2)^{\frac{1}{4}}},$$

1303 with

$$1304 \quad (C.23) \quad I(\lambda) = \lambda \hat{I}\left(\frac{4}{3}\lambda^{\frac{3}{2}}\right).$$

1305 The following analysis is close to that of the asymptotic behavior of a Laplace integral.

1306

1307 **Asymptotic upper bound of $\hat{I}(t)$.**

1308 Although $u(y) \leq u(x)$ in the domain of integration in Eq. (C.22), a direct use of this
1309 upper bound will lead to an upper bound by $+\infty$.

1310 Let us start by a heuristic discussion. The maximum of $u(y) - u(x)$ should be
1311 on $x = y$. For $x - y$ small, we have $u(y) - u(x) \sim (y - x)u'(x)$. This suggests a
1312 concentration near $x = y = 0$, whereas a contribution for large x is of smaller order.

1313 More rigorously, we write

$$1314 \quad (C.24) \quad \hat{I}(t) = \hat{I}_1(t, \epsilon) + \hat{I}_2(t, \epsilon, \xi) + \hat{I}_3(t, \epsilon),$$

1315 with, for $0 < \epsilon < \xi$,

$$\begin{aligned} \hat{I}_1(t, \epsilon) &= \int_0^\epsilon dx \frac{1}{(1 + x^2)^{\frac{1}{4}}} \int_0^x dy \frac{\exp(t(u(y) - u(x)))}{(1 + y^2)^{\frac{1}{4}}}, \\ 1316 \quad (C.25) \quad \hat{I}_2(t, \epsilon, \xi) &= \int_\epsilon^\xi dx \frac{1}{(1 + x^2)^{\frac{1}{4}}} \int_0^x dy \frac{\exp(t(u(y) - u(x)))}{(1 + y^2)^{\frac{1}{4}}}, \\ \hat{I}_3(t, \xi) &= \int_\xi^{+\infty} dx \frac{1}{(1 + x^2)^{\frac{1}{4}}} \int_0^x dy \frac{\exp(t(u(y) - u(x)))}{(1 + y^2)^{\frac{1}{4}}}. \end{aligned}$$

1317 We now observe that $u(s)$ has the form $u(s) = v(s^2)$ where $v' > 0$, so that
 1318 (C.26)

$$\forall x, y \text{ s.t. } 0 \leq y \leq x \leq \tau_0, \\ \left(\sup_{\tau \in [0, \tau_0]} v'(\tau) \right) (x^2 - y^2) \geq u(x) - u(y) \geq \left(\inf_{\tau \in [0, \tau_0]} v'(\tau) \right) (x^2 - y^2).$$

1319 **Analysis of $\widehat{I}_1(t, \epsilon)$.**

1320 Using the right hand side of inequality (C.26) with $\tau_0 = \epsilon$, we show the existence of
 1321 constants C and $\epsilon_0 > 0$, such that, $\forall \epsilon \in (0, \epsilon_0)$

$$1322 \quad (C.27) \quad (1 - C\epsilon)J_\epsilon \left((1 + C\epsilon)\frac{3}{8}t \right) \leq \widehat{I}_1(t, \epsilon) \leq (1 + C\epsilon)J_\epsilon \left((1 - C\epsilon)\frac{3}{8}t \right),$$

1323 with

$$1324 \quad J_\epsilon(\sigma) := \int_0^\epsilon dx \int_0^x \exp(\sigma(y^2 - x^2)) dy,$$

1325 which has now to be estimated for large σ .

1326 For $\frac{1}{\sqrt{\epsilon\sigma}} \leq \epsilon$, we write

$$1327 \quad J_\epsilon(\sigma) = J_\epsilon^1(\sigma) + J_\epsilon^2(\sigma),$$

1328 with

$$1329 \quad J_\epsilon^1(\sigma) := \int_0^{\frac{1}{\sqrt{\epsilon\sigma}}} dx \int_0^x \exp(\sigma(y^2 - x^2)) dy, \\ J_\epsilon^2(\sigma) := \int_{\frac{1}{\sqrt{\epsilon\sigma}}}^\epsilon dx \int_0^x \exp(\sigma(y^2 - x^2)) dy.$$

1330 Using the trivial estimate

$$1331 \quad \int_0^x \exp(\sigma(y^2 - x^2)) dy \leq x,$$

1332 we get

$$1333 \quad (C.28) \quad J_\epsilon^1(\sigma) \leq \frac{1}{2\epsilon\sigma}.$$

1334 We have now to analyze $J_\epsilon^2(\sigma)$.

1335 The formula giving $J_\epsilon^2(\sigma)$ can be expressed by using the Dawson function (cf [1], p.
 1336 295 and 319)

$$1337 \quad s \mapsto D(s) := \int_0^s \exp(y^2 - s^2) dy$$

1338 and its asymptotics as $s \rightarrow +\infty$,

1339

$$1340 \quad (C.29) \quad D(s) = \frac{1}{2s}(1 + \delta(s)),$$

1341 where the function $\delta(s)$ satisfies $\delta(s) = \mathcal{O}(s^{-1})$.

1342 We get indeed

$$1343 \quad J_\epsilon^2(\sigma) = \frac{1}{\sigma} \int_{\epsilon^{-\frac{1}{2}}}^{\epsilon\sigma^{\frac{1}{2}}} D(s) ds.$$

1344 By taking ϵ small enough to use the asymptotics of $D(\cdot)$, we get

$$\begin{aligned}
 1345 \quad (C.30) \quad J_\epsilon^2(\sigma) &= \frac{1}{2\sigma} \left(\int_{\epsilon^{-\frac{1}{2}}}^{\sigma^{\frac{1}{2}}\epsilon} \frac{1}{s} ds + \int_{\epsilon^{-\frac{1}{2}}}^{\sigma^{\frac{1}{2}}\epsilon} \frac{\delta(s)}{s} ds \right) \\
 &= \frac{1}{4} \frac{\log \sigma}{\sigma} + \frac{C}{\sigma} (\log \epsilon + \mathcal{O}(1)).
 \end{aligned}$$

1346 Hence we have shown the existence of constants $C > 0$ and ϵ_0 such that if $t \geq C\epsilon^{-3}$
 1347 and $\epsilon \in (0, \epsilon_0)$, then

$$1348 \quad (C.31) \quad \widehat{I}_1(t, \epsilon) \leq \frac{2}{3} \frac{\log t}{t} + C \left(\epsilon \frac{\log t}{t} + \frac{1}{\epsilon t} \right).$$

1349 **Analysis of $\widehat{I}_3(t, \xi)$**

1350 We start from

$$1351 \quad \widehat{I}_3(t, \xi) = \int_\xi^{+\infty} dx \frac{1}{(1+x^2)^{\frac{1}{4}}} \int_0^x dy \frac{\exp(t(u(y) - u(x)))}{(1+y^2)^{\frac{1}{4}}}.$$

1352 Having in mind the properties of u , we can choose ξ large enough in order to have for
 1353 some $c_\xi > 0$ the property that for $x \geq \xi$ and $\frac{x}{2} \leq y \leq x$,

$$\begin{aligned}
 1354 \quad (C.32) \quad u(x) &\geq c_\xi x^{\frac{3}{2}}, \\
 u(x) - u(x/2) &\geq c_\xi x^{\frac{3}{2}}, \\
 u(x) - u(y) &\geq c_\xi x^{\frac{1}{2}} (x - y).
 \end{aligned}$$

1355 This determines our choice of ξ . Using these inequalities, we rewrite $\widehat{I}_3(t, \xi)$ as the
 1356 sum

$$1357 \quad \widehat{I}_3(t, \xi) = \widehat{I}_{31}(t) + \widehat{I}_{32}(t),$$

1358 with

$$\begin{aligned}
 1359 \quad \widehat{I}_{31}(t) &= \int_\xi^{+\infty} dx \frac{1}{(1+x^2)^{\frac{1}{4}}} \int_0^{\frac{x}{2}} dy \frac{\exp(t(u(y) - u(x)))}{(1+y^2)^{\frac{1}{4}}}, \\
 \widehat{I}_{32}(t) &= \int_\xi^{+\infty} dx \frac{1}{(1+x^2)^{\frac{1}{4}}} \int_{\frac{x}{2}}^x dy \frac{\exp(t(u(y) - u(x)))}{(1+y^2)^{\frac{1}{4}}}.
 \end{aligned}$$

1360 Using the monotonicity of u , we obtain the upper bound

$$\begin{aligned}
 \widehat{I}_{31}(t) &\leq \int_{\xi}^{+\infty} dx \frac{1}{(1+x^2)^{\frac{1}{4}}} \int_0^{\frac{x}{2}} dy \exp(t(u(y) - u(x))) \\
 &\leq \frac{1}{2} \int_{\xi}^{+\infty} x^{\frac{1}{2}} \exp(t(u(x/2) - u(x))) dx \\
 1361 \quad &\leq \frac{1}{2} \int_{\xi}^{+\infty} x^{\frac{1}{2}} \exp(-c_{\xi} t x^{\frac{3}{2}}) dx \\
 &\leq \frac{1}{3} \int_{\xi^{\frac{3}{2}}}^{+\infty} \exp(-c_{\xi} t s) ds \\
 &\leq \frac{1}{3c_{\xi} t} \exp(-c_{\xi} \xi^{\frac{3}{2}} t).
 \end{aligned}$$

1362 Hence, there exists $\epsilon_{\xi} > 0$ such that as $t \rightarrow +\infty$,

$$1363 \quad (\text{C.33}) \quad \widehat{I}_{31}(t) = \mathcal{O}(\exp(-\epsilon_{\xi} t)).$$

1364 The last term to control is $\widehat{I}_{32}(t)$. Using (C.32) and

$$1365 \quad (1+y^2)^{-\frac{1}{4}} \leq (1+(x/2)^2)^{-\frac{1}{4}} \leq \sqrt{2}(4+x^2)^{-\frac{1}{4}} \leq \sqrt{2}(1+x^2)^{-\frac{1}{4}}$$

1366 for $x/2 \leq y \leq x$, we get

$$\begin{aligned}
 \widehat{I}_{32}(t) &\leq \sqrt{2} \int_{\xi}^{+\infty} dx \frac{1}{(1+x^2)^{\frac{1}{2}}} \int_{\frac{x}{2}}^x dy \exp(t(u(y) - u(x))) \\
 1367 \quad (\text{C.34}) \quad &\leq \sqrt{2} \int_{\xi}^{+\infty} dx \frac{1}{(1+x^2)^{\frac{1}{2}}} \int_{\frac{x}{2}}^x dy \exp(-c_{\xi} t x^{\frac{1}{2}}(x-y)) \\
 &\leq \frac{\sqrt{2}}{c_{\xi} t} \int_{\xi}^{+\infty} x^{-\frac{3}{2}} dx = \frac{2\sqrt{2}}{\sqrt{\xi} c_{\xi} t}.
 \end{aligned}$$

1368 Hence putting together (C.33) and (C.34) we have, for this choice of ξ , the existence
 1369 of $\widehat{C}_{\xi} > 0$ and $t_{\xi} > 0$ such that

$$1370 \quad (\text{C.35}) \quad \forall t \geq t_{\xi}, \quad \widehat{I}_3(t) \leq \widehat{C}_{\xi}/t.$$

1371 **Analysis of $\widehat{I}_2(t, \epsilon, \xi)$.**

1372 We recall that

$$1373 \quad \widehat{I}_2(t, \epsilon, \xi) = \int_{\epsilon}^{\xi} dx \frac{1}{(1+x^2)^{\frac{1}{4}}} \int_0^x dy \frac{\exp(t(u(y) - u(x)))}{(1+y^2)^{\frac{1}{4}}}.$$

1374 We first observe that

$$1375 \quad \widehat{I}_2(t, \epsilon, \xi) \leq \int_{\epsilon}^{\xi} dx \int_0^x dy \exp(t(u(y) - u(x))) \leq \int_{\epsilon}^{\xi} dx \int_0^x dy \exp(c_{\xi} t(y^2 - x^2)),$$

1376 with

$$1377 \quad c_\xi = \inf_{[0, \xi]} v' > 0.$$

1378 Using now

$$1379 \quad \int_0^x \exp(c_\xi t(y^2 - x^2)) dy \leq \int_0^x \exp(c_\xi t x(y - x)) dy = \frac{1}{c_\xi t x} (1 - \exp(-c_\xi t x^2)) \leq \frac{1}{c_\xi t x},$$

1380 we get

$$1381 \quad (\text{C.36}) \quad \widehat{I}_2(t, \epsilon, \xi) \leq \frac{1}{c_\xi t} (\log \xi - \log \epsilon).$$

1382 Putting together (C.24), (C.31), (C.35) and (C.36), we have shown the existence of
1383 $C > 0$ and ϵ_0 such that if $t \geq C\epsilon^{-3}$ and $\epsilon \in (0, \epsilon_0)$, then

$$1384 \quad (\text{C.37}) \quad \widehat{I}(t) \leq \frac{2}{3} \frac{\log t}{t} + C \left(\epsilon \frac{\log t}{t} + \frac{1}{\epsilon} \frac{1}{t} \right).$$

1385 Coming back to (C.23) and using (C.16), we show the existence of $C > 0$ and ϵ_0 such
1386 that if $\lambda \geq C\epsilon^{-2}$, then

$$1387 \quad \|\mathcal{G}^{-,D}(\lambda)\|_{HS}^2 \leq \frac{3}{8} \lambda^{-\frac{1}{2}} \log \lambda + C \left(\epsilon \lambda^{-\frac{1}{2}} \log \lambda + \frac{1}{\epsilon} \lambda^{-\frac{1}{2}} \right).$$

1388 Taking $\epsilon = (\log \lambda)^{-\frac{1}{2}}$, we obtain

1389 LEMMA 52. *There exist $C > 0$ and λ_0 such that for $\lambda \geq \lambda_0$*

$$1390 \quad \|\mathcal{G}^{-,D}(\lambda)\|_{HS}^2 \leq \frac{3}{8} \lambda^{-\frac{1}{2}} (1 + C (\log \lambda)^{-\frac{1}{2}}) \log \lambda.$$

1391 **C.5. Lower bound.** Once the upper bounds are established, the proof of the
1392 lower bound is easy. We start from the simple lower bound (for any $\epsilon > 0$)

$$1393 \quad (\text{C.38}) \quad Q(x, \lambda) \geq -\frac{1}{\epsilon} Q_1(x, \lambda) + (1 - \epsilon) Q_2(x, \lambda),$$

1394 and consequently

$$1395 \quad (\text{C.39}) \quad \int_0^{+\infty} Q(x, \lambda) dx \geq (1 - \epsilon) \int_0^{+\infty} Q_2(x, \lambda) dx - \frac{1}{\epsilon} \int_0^{+\infty} Q_1(x, \lambda) dx.$$

1396 Taking $\epsilon = (\log \lambda)^{-\frac{1}{2}}$ and using the upper bound (C.15), it remains to find a lower
1397 bound for $\int_0^{+\infty} Q_2(x, \lambda) dx$, which can be worked out in the same way as for the upper
1398 bound. We can use (C.18), (C.27), (C.30) and

$$1399 \quad (\text{C.40}) \quad \widehat{I}(t) \geq \widehat{I}_1(t, \epsilon) \geq \frac{2}{3} \frac{\log t}{t} - C \left(\epsilon \frac{\log t}{t} + \frac{1}{\epsilon} \frac{1}{t} \right).$$

1400 This gives the proof of

1401 LEMMA 53. *There exist $C > 0$ and λ_0 such that for $\lambda \geq \lambda_0$*

$$1402 \quad \|\mathcal{G}^{-,D}(\lambda)\|_{HS}^2 \geq \frac{3}{8} \lambda^{-\frac{1}{2}} (1 - C (\log \lambda)^{-\frac{1}{2}}) \log \lambda.$$

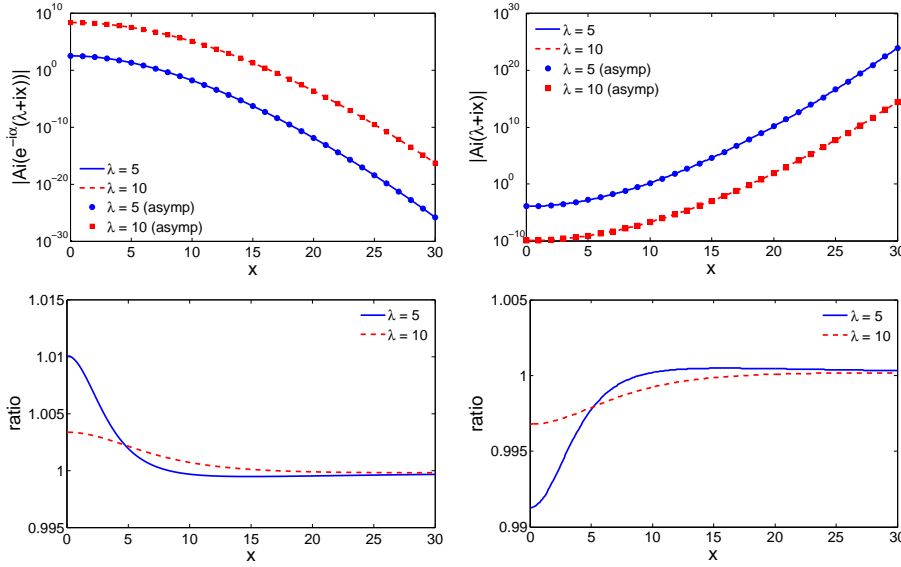


FIG. 3. (Top) Asymptotic behavior of $|\text{Ai}(e^{-i\alpha}(ix + \lambda))|$ (left) and $|\text{Ai}(ix + \lambda)|$ (right) for large λ . (Bottom) The ratio between these functions and their asymptotics given by (C.6) and (C.8).

1403 Appendix D. Phragmen-Lindelöf theorem.

1404 The Phragmen-Lindelöf Theorem (see Theorem 16.1 in [2]) reads

1405 THEOREM 54. (Phragmen-Lindelöf) Let us assume that there exist two rays

$$1406 \mathcal{R}_1 = \{re^{i\theta_1} : r \geq 0\} \text{ and } \mathcal{R}_2 = \{re^{i\theta_2} : r \geq 0\}$$

1407 with (θ_1, θ_2) such that $|\theta_1 - \theta_2| = \frac{\pi}{\alpha}$, and a continuous function F in the closed sector
1408 delimited by the two rays, holomorphic in the open sector, satisfying the properties

1409 (i)

$$1410 \exists C > 0, \quad \exists N \in \mathbb{R}, \text{ s. t. } \forall \lambda \in \mathcal{R}_1 \cup \mathcal{R}_2, \quad |F(\lambda)| \leq C(|\lambda|^2 + 1)^{N/2}.$$

1411 (ii) There exist an increasing sequence (r_k) tending to $+\infty$, and C such that

$$1412 \text{(D.1)} \quad \forall k, \quad \max_{|\lambda|=r_k} |F(\lambda)| \leq C \exp(r_k^\beta),$$

1413 with $\beta < \alpha$.

1414 Then we have

$$1415 |F(\lambda)| \leq C(|\lambda|^2 + 1)^{N/2}$$

1416 for all λ between the two rays \mathcal{R}_1 and \mathcal{R}_2 .

1417 Appendix E. Numerical computation of eigenvalues.

1418 In order to compute numerically the eigenvalues of the realization $\mathcal{A}_{1,L}^{+,D}$ of the
1419 complex Airy operator $\mathcal{A}_0^+ := D_x^2 + ix = -\frac{d^2}{dx^2} + ix$ on the real line with a transmission
1420 condition, we impose auxiliary Dirichlet boundary conditions at $x = \pm L$, i.e., we

1421 search for eigenpairs $\{\lambda_L, u_L(\cdot)\}$ of the following problem:

$$1422 \quad (\text{E.1}) \quad \begin{aligned} & \left(-\frac{d^2}{dx^2} + ix\right) u_L(x) = \lambda_L u_L(x), \quad (-L < x < L), \\ & u_L(\pm L) = 0, \quad u'_L(0_+) = u'_L(0_-) = \kappa(u_L(0_+) - u_L(0_-)), \end{aligned}$$

1423 with a positive parameter κ .

1424 Since the interval $[-L, L]$ is bounded, the spectrum of the above differential op-
 1425 erator is discrete. To compute its eigenvalues, one can either discretize the second
 1426 derivative, or represent this operator in an appropriate basis in the form of an infinite-
 1427 dimensional matrix. Following [21], we choose the second option and use the basis
 1428 formed by the eigenfunctions of the Laplace operator $-\frac{d^2}{dx^2}$ with the above boundary
 1429 conditions. Once the matrix representation is found, it can be truncated to compute
 1430 the eigenvalues numerically. Finally, one considers the limit $L \rightarrow +\infty$ to remove the
 1431 auxiliary boundary conditions at $x = \pm L$.

1432 There are two sets of Laplacian eigenfunctions in this domain:

1433 (i) symmetric eigenfunctions

$$1434 \quad (\text{E.2}) \quad v_{n,1}(x) = \sqrt{1/L} \cos(\pi(n+1/2)x/L), \quad \mu_{n,1} = \pi^2(n+1/2)^2/L^2,$$

1435 enumerated by the index $n \in \mathbb{N}$.

1436 (ii) antisymmetric eigenfunctions

$$1437 \quad (\text{E.3}) \quad v_{n,2}(x) = \begin{cases} +(\beta_n/\sqrt{L}) \sin(\alpha_n(1-x/L)) & (x > 0), \\ -(\beta_n/\sqrt{L}) \sin(\alpha_n(1+x/L)) & (x < 0), \end{cases}$$

1438 with $\mu_{n,2} = \alpha_n^2/L^2$, where α_n ($n = 0, 1, 2, \dots$) satisfy the equation

$$1439 \quad (\text{E.4}) \quad \alpha_n \operatorname{ctan}(\alpha_n) = -2\kappa L,$$

1440 while the normalization constant β_n is

$$1441 \quad (\text{E.5}) \quad \beta_n = \left(1 + \frac{2\kappa L}{\alpha_n^2 + 4\kappa^2 L^2}\right)^{-\frac{1}{2}}.$$

1442 The solutions α_n of Eq. (E.4) lie in the intervals $(\pi n + \pi/2, \pi n + \pi)$, with $n \in \mathbb{N}$.

1443 In what follows, we use the double index (n, j) to distinguish symmetric and anti-
 1444 symmetric eigenfunctions and to enumerate eigenvalues, eigenfunctions, as well as the
 1445 elements of governing matrices and vectors. We introduce two (infinite-dimensional)
 1446 matrices Λ and \mathcal{B} to represent the Laplace operator and the position operator in the
 1447 Laplacian eigenbasis:

$$1448 \quad (\text{E.6}) \quad \Lambda_{n,j;n',j'} = \delta_{n,n'} \delta_{j,j'} \mu_{n,j},$$

1449 and

$$1450 \quad (\text{E.7}) \quad \mathcal{B}_{n,j;n',j'} = \int_{-L}^L dx v_{n,j}(x) x v_{n',j'}(x).$$

1451 The symmetry of eigenfunctions $v_{n,j}$ implies $\mathcal{B}_{n,1;n',1} = \mathcal{B}_{n,2;n',2} = 0$, while

$$1452 \quad (\text{E.8}) \quad \begin{aligned} & \mathcal{B}_{n,1;n',2} = \mathcal{B}_{n',2;n,1} \\ & = -2L\beta_{n'} \frac{\sin(\alpha_{n'}) (\alpha_{n'}^2 + \pi^2(n+1/2)^2) - (-1)^n (2n+1)\pi\alpha_{n'}}{(\alpha_{n'}^2 - \pi^2(n+1/2)^2)^2}. \end{aligned}$$

	L	$\lambda_{1,L}$	$\lambda_{3,L}$	$\lambda_{5,L}$
$\kappa = 0$	4	0.5161 - 0.8918i	1.2938 - 2.1938i	3.7675 - 1.9790i
	6	0.5094 - 0.8823i	1.1755 - 3.9759i	1.6066 - 2.7134i
	8	0.5094 - 0.8823i	1.1691 - 5.9752i	1.6233 - 2.8122i
	10	0.5094 - 0.8823i	1.1691 - 7.9751i	1.6241 - 2.8130i
	∞	0.5094 - 0.8823i		1.6241 - 2.8130i
$\kappa = 1$	4	1.0516 - 1.0591i	1.3441 - 2.0460i	4.1035 - 1.7639i
	6	1.0032 - 1.0364i	1.1725 - 3.9739i	1.7783 - 2.7043i
	8	1.0029 - 1.0363i	1.1691 - 5.9751i	1.8364 - 2.8672i
	10	1.0029 - 1.0363i	1.1691 - 7.9751i	1.8390 - 2.8685i
	∞	1.0029 - 1.0363i		1.8390 - 2.8685i

TABLE 1

The convergence of the eigenvalues $\lambda_{n,L}$ computed by diagonalization of the matrix $\Lambda + i\mathcal{B}$ truncated to the size 100×100 . Due to the reflection symmetry of the interval, all eigenvalues appear in complex conjugate pairs: $\lambda_{2n,L} = \bar{\lambda}_{2n-1,L}$. The last line presents the poles of the resolvent of the complex Airy operator \mathcal{A}_1^+ obtained by solving numerically the equation (6.17). The intermediate column shows the eigenvalue $\lambda_{3,L}$ coming from the auxiliary boundary conditions at $x = \pm L$ (as a consequence, it does not depend on the transmission coefficient κ). Since the imaginary part of these eigenvalues diverges as $L \rightarrow +\infty$, they can be easily identified and discarded.

1453 The infinite-dimensional matrix $\Lambda + i\mathcal{B}$ represents the complex Airy operator $\mathcal{A}_{1,L}^{+,D}$
 1454 on the interval $[-L, L]$ in the Laplacian eigenbasis. As a consequence, the eigenvalues
 1455 and eigenfunctions can be numerically obtained by truncating and diagonalizing this
 1456 matrix. The obtained eigenvalues are ordered according to their increasing real part:

1457
$$\operatorname{Re} \lambda_{1,L} \leq \operatorname{Re} \lambda_{2,L} \leq \dots$$

1458 Table 1 illustrates the rapid convergence of these eigenvalues to the eigenvalues of the
 1459 complex Airy operator \mathcal{A}_1^+ on the whole line with transmission, as L increases. The
 1460 same matrix representation was used for plotting the pseudospectrum of \mathcal{A}_1^+ (Fig. 2).

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