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Comparison of a spectral method with volume penalization and a finite volume method with body fitted grids for turbulent flows

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Abstract

We consider a turbulent flow past periodic hills at Reynolds number 1400 and compare two numerical methods: A Fourier pseudo-spectral scheme with volume penalization to model the no-slip boundary conditions and a finite volume method with body fitted grids. A detailed comparison of the results is presented for mean velocity profiles and Reynolds stress and confronted with those obtained by Breuer et al. [1]. In addition higher order statistics are performed and their scale-dependence is analyzed using orthogonal wavelets. Moreover, for the Fourier pseudo-spectral scheme, the influence of the Reynolds number is investigated.

Keywords: computational fluid dynamics, periodic hill, volume penalization, Fourier spectral method, finite volumes, wavelets

1. Introduction

Computational fluid dynamics in complex geometries is still a challenge especially for high Reynolds numbers. The thin boundary layers which destabilize, subsequently detach and thus generate coherent vortices are a key feature to understand turbulence generated by walls. Two main approaches

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6 have been developed so far, either body fitted grids using, *e.g.*, finite vol-
7 ume or finite element discretizations, or immersed boundary methods (IBM)
8 which are becoming more and more attractive due to their high flexibility. In-
9 deed, for the latter the complex geometry is typically embedded into a larger
10 computational domain of simple shape, *e.g.*, a rectangle. Hence classical dis-
11 cretizations, using Cartesian grids, for which efficient solvers are available,
12 can be employed. The influence of the geometry and the no-slip boundary
13 conditions is modeled by modifying the underlying Navier–Stokes equations.
14 The origin of IBM can be traced back to Courant (1943) [2] in the context of
15 constraint optimization. A large range of immersed boundary methods can
16 be found since then, for example Lagrangian multipliers [4], level-set meth-
17 ods [5], fictitious domain approaches and surface [6] and volume penalization
18 approaches [7]. For reviews we refer the reader to [8] and [9].

19
20 Validation and benchmarking are an essential step in code development.
21 Comparing the results of different numerical methods enables us to know the
22 advantages and drawbacks. Their domain of validity can thus be checked
23 and their precision can be assessed.

24
25 In this study, we have chosen the volume penalization method [7] cou-
26 pled with a Fourier pseudo-spectral method [10]. Investigations on the Gibbs
27 oscillations which appear in Fourier based schemes can be found in [11],
28 as well as filtering techniques for removing this Gibbs phenomenon. The
29 code has been benchmarked in two space dimensions [12] and applied to
30 moving obstacles [13]. To evaluate the performance of this new method in
31 three-dimensional, turbulent fluid flow, it is compared here to that of a well-
32 established second-order finite-volume method in terms of accuracy and effi-
33 ciency. The latter, called “Fastest” [14] is based on a finite volume discretiza-
34 tion and uses body fitted grids. The flow configuration is a well documented
35 benchmark “flow over periodic hills” [1]. This benchmark is of interest, since
36 it is a geometrically simple test case and there is no difficulty to specify
37 inflow/outflow boundary conditions. Moreover, the physical mechanisms of
38 separation on curved surfaces and the streamwise decorrelation enhanced, by
39 choosing a sufficiently large distance between the two hills, are an attractive
40 features of this test case. An other class of IBM was investigated recently
41 using this benchmark in [15]. The three-dimensional Fourier pseudo-spectral
42 code including volume penalization (denoted by “Pen4Flow”) to impose non-
43 periodic boundary conditions is thus applied for the first time to compute

44 fully developed turbulent flows. The aim of the paper is to show a detailed
45 benchmark of the volume penalization method for three-dimensional turbu-
46 lent flows and to compare the efficiency with respect to a classical finite
47 volume code Fastest.

48

49 For the two solvers several grid levels are employed. As the considered
50 flow is turbulent, statistical analysis are performed and compared with each
51 other. In addition we also use orthogonal wavelets to decompose the flow
52 into different scales of motion. Statistical quantities can thus be defined as
53 a function of scale and direction.

54

55 The manuscript is organized as follows. In section 2 the “Flow over pe-
56 riodic hills” configuration is explained and the two numerical methods are
57 shortly presented. Then in section 3, the results for the different codes are
58 described and the flow statistics are compared. Scale dependent analysis of
59 the results are also performed. Finally, the influence of the Reynolds num-
60 ber is investigated for the pseudo-spectral code with the volume penalization
61 method. Some conclusions and perspectives are exposed at the end of the
62 manuscript.

63

64 **2. Flow configuration and Numerical methods**

65 *2.1. Flow configuration*

66 The classical benchmark “Flow over periodic hills” [1], is considered in
67 detail at a Reynolds number of $Re = 1400$ which is based on the bulk ve-
68 locity u_b at $x = 0$ and the bump height H . To check the influence of the
69 Reynolds number, we also considered $Re = 700$ and $Re = 2800$. The bound-
70 ary conditions are periodic along the streamwise x -direction and the spanwise
71 z -direction. The flow is confined by a hill at the bottom and by a wall plane
72 at the top (y -direction). The flow configuration is illustrated in Fig. 1.

73 *2.2. Pen4Flow: Fourier pseudo-spectral code with “volume penalization”* 74 *method*

75 In this subsection, we present the volume penalization method and the
76 methodology to include the periodic hill configuration. More details on the
77 numerical and mathematical validation of this method can be found [13, 16].

78

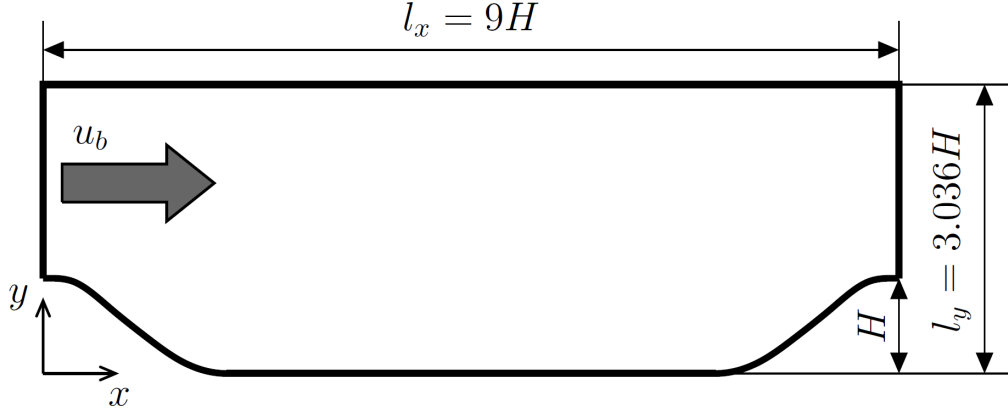


Figure 1: Sketch of the 2d hill geometry. The dimension in spanwise z -direction is $l_z = 4.5H$.

79 The Fourier pseudo-spectral code solves the incompressible Navier-Stokes
 80 equations in velocity-vorticity formulation. The volume penalization method
 81 modifies the Navier-Stokes equations by adding a penalization term to the
 82 right hand side of the momentum equation:

83

$$\frac{\partial \mathbf{u}}{\partial t} + \boldsymbol{\omega} \times \nabla \mathbf{u} + \nabla \Pi - \nu \nabla^2 \mathbf{u} - \mathbf{F} = -\frac{1}{\eta} \chi(\mathbf{u}), \quad \nabla \cdot \mathbf{u} = 0, \quad \mathbf{x} \in \Omega \quad (1)$$

84 with

$$-\nabla^2 \Pi = \nabla \cdot \left(\boldsymbol{\omega} \times \mathbf{u} + \mathbf{F} + \frac{1}{\eta} \chi(\mathbf{u}) \right) \quad (2)$$

85 where $\mathbf{u} = (u_x, u_y, u_z)$ is the velocity, ν the kinematic viscosity, Π is the
 86 modified pressure, $\boldsymbol{\omega}$ the vorticity, \mathbf{F} an external force, η the permeability
 87 (or penalization parameter) and χ the mask function which equals to 1 in Ω_s
 88 (solid domain) and 0 in Ω_f (fluid domain). Figure 2 (a) shows the computa-
 89 tional domain $\Omega = \Omega_f \cup \Omega_s$ where Ω_f is the fluid domain (in white) and Ω_s
 90 is the solid domain (in black). The solution of the penalized Navier-Stokes
 91 equations does converge towards the solution of the Navier-Stokes equations
 92 with no-slip boundary conditions [7]. The difference between the exact solu-
 93 tion of the penalized equation and the exact solution of the Navier-Stokes
 94 equations, called modeling error is proportional to $\sqrt{\eta}$ and for sufficiently
 95 small values of η the solution is precise enough.

96

97 In space all variables are represented as truncated Fourier series, prod-
 98 uct and nonlinear terms are evaluated in physical space while derivatives
 99 are evaluated in spectral space. For time integration a second order Adams-
 100 Bashforth scheme is used for the nonlinear and the penalization term. The
 101 viscous term is integrated exactly using the semi-group formulation [13],
 102 which means that the semi-group of the heat kernel is used for time integra-
 103 tion. This corresponds to the exact integration of the linear viscous term.
 104 The numerical study was carried out on three different grids that differed
 105 in the number of nodes employed, see Table 1. As the volume penalization
 106 method takes care of generating the boundary layer, the numerical grid has
 107 to be chosen sufficiently fine to be able to resolve these thin layers. For each
 108 grid, the size l_y in the y -direction of the computational domain is chosen such
 109 that $y = 3.036$ coincides with a collocation point and that the solid domain
 110 on the top contains 10 points along the y -direction. In all simulations, the
 111 penalization parameter is $\eta = 10^{-3}$ and the CFL constant is equal to 0.1.
 112 The choice of the penalization parameter is a compromise which works well
 113 for the resolutions considered here, more details on the influence of the penal-
 114 ization parameter can be found in [11]. For Pen4Flow, we observe that the
 115 time step limitation comes from the CFL condition and not for the stability
 116 limit ($\Delta t < \eta$) imposed by the penalization term.

117
 118 The flow is driven by an external force $\mathbf{F} = f\mathbf{e}_x \in \Omega$. In order to keep the
 119 desired mass flow rate constant throughout the whole simulation, a control
 120 function similar to [17], updates the forcing term on every time step:

$$f_{n+1} = f_n + (u_b - 2\tilde{u}_n + \tilde{u}_{n-1})/(\alpha_f \Delta t) \quad (3)$$

121 where \tilde{u}_n is the measured bulk velocity at time level n , α_f a damping factor
 122 ($\alpha_f = 10$ for Pen4Flow and $\alpha_f = 1$ for Fastest). In all computations the
 123 time-step for Pen4Flow is $\Delta t = 10^{-3}$ and for Fastest $\Delta t = 1.5 \cdot 10^{-3}$.

124 The flow is initialized with $\mathbf{u}(t = 0) = 0$. To trigger the instability, an
 125 impulsion is imposed during the first time interval $[0, 2]$, *i.e.*, the forcing term
 126 is extended to $\mathbf{F} = f\mathbf{e}_x + f_{\text{imp}}\mathbf{e}_z$ with $f_{\text{imp}} = A \sin(2x2\pi/l_x) * \exp(1 - 2(y -$
 127 $l_y/2)^2) * \sin(2z2\pi/l_z)$ and $A = 1$ being a suitable amplitude.

128 2.3. Fastest: Finite volume method

129 To solve the transient, incompressible Navier-Stokes equations with the
 130 finite volume method on body-fitted grids, we employ the solver Fastest

131 [14]. This code is based on a cell-centered finite volume method on block-
132 structured grids. Discretization of the convective term is carried out using
133 a multi-dimensional Taylor series expansion which preserves second-order
134 accuracy on strongly distorted grids [18], while the time is advanced by the
135 Crank-Nicolson scheme. This numerical procedure is of second order in space
136 and in time. The SIMPLE procedure is employed for the solution of the
137 pressure. Velocity and pressure fields are coupled via the Rhie-Chow inter-
138 polation. The discretized system of equations is implicitly solved by Stone's
139 method.

140 Two grids were generated for the finite volume computations, the first
141 of which, subsequently referred to as fine grid, was constructed to fulfill
142 the requirements of a DNS, while a coarse grid was derived from that by
143 omitting every second node in each direction. A precursor RANS simulation
144 employing the ζ - f model of Hanjalić [19] gave an estimate of the Kolmogorov
145 length scale η . Following Pope's recommendations [20], a maximum ratio of
146 $\Delta x/\eta = 2.1$ was ensured over the entire domain for the construction of the
147 fine grid. For the boundary layer resolution, the normalized wall distance
148 of the first cell center y_1^+ was kept below 0.1 everywhere. A coarse grid
149 was derived from the fine one by omitting every second node in every index
150 direction.

151 The time step size was chosen such that $CFL < 1$ for all time steps.
152 During the computation, the mass flux was kept constant utilizing (3) as
153 external forcing. However, $u(t = 0)/u_b = 1.0$ was chosen as initial condition,
154 and during the initial simulation period no additional forcing was necessary
155 to excite the flow instabilities.

156 3. Results

157 First we consider the periodic hill configuration at Reynolds number
158 $Re = 1400$ and compare Fastest and Pen4Flow with the results obtained
159 by Breuer et al. [1]. Second, we focus on Pen4Flow and study the influence
160 of the Reynolds number.

162 3.1. Case $Re = 1400$

163 In the following we present the results at $Re = 1400$ and study the influ-
164 ence of the resolution.

165

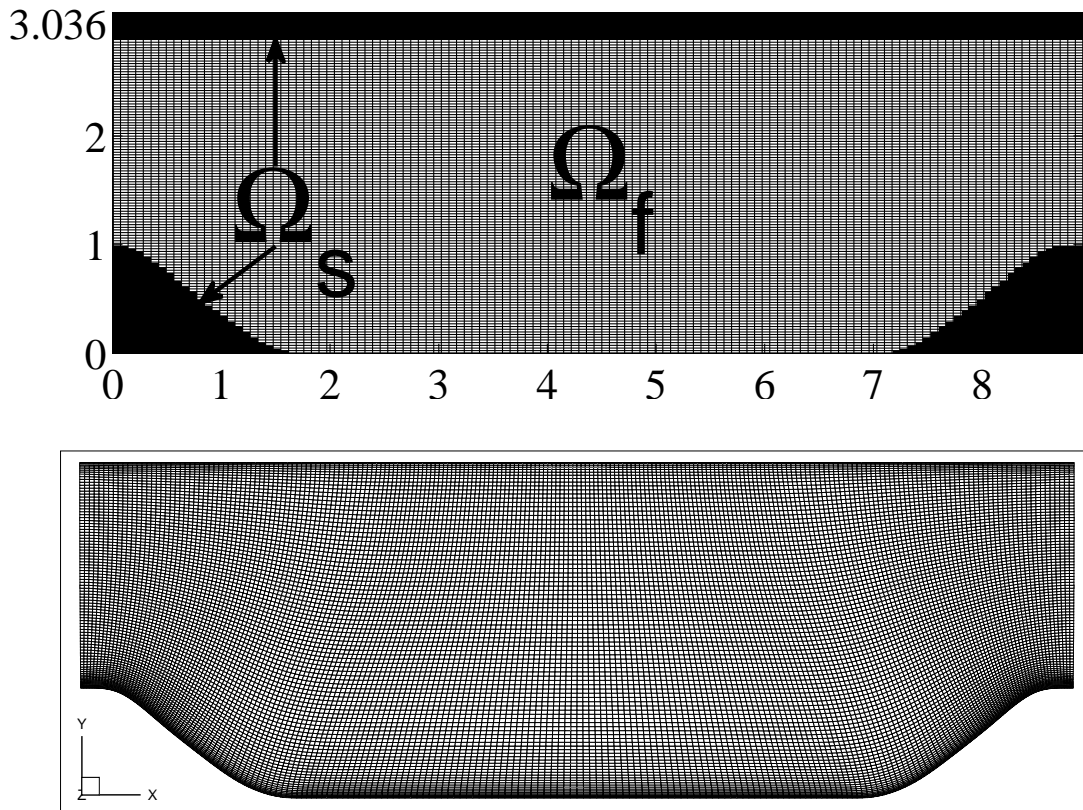


Figure 2: Numerical grids: (top) Pen4Flow (128^3 for visualization), (bottom) Fastest for the coarsest grid ($442 \times 184 \times 200$).

166 Table 1 summarizes the numerical and physical parameters for both codes.
 167 Scaling of the CPU time with respect to the number of processors is given
 168 in Fig. 3 for both codes using coarse resolution, obtained on the HHLR su-
 169 percomputer ("Hessischer Hochleistungsrechner"). Note that for Pen4Flow,
 170 the fluid domain height is constant and equal to $y_f = 3.036$ and the solid

171 domain height is chosen such that the solid contains 10 grid points and the
 172 boundary of the fluid coincides with a grid point.

173

	N_x	N_y	N_z	y_l/H	FTT
Pen4Flow 256^3	256	256	256	3.15941	48.88
Pen4Flow $512^2 \times 256$	512	512	256	3.09647	14.69
Pen4Flow 512^3	512	512	512	3.09647	29.19
Fastest coarse	221	92	100	3.05	215
Fastest fine	442	184	200	3.05	213

Table 1: Numerical parameters, flow domain height y_l/H , flow-through times (FTT = L/u_b) used for statistical evaluation for $Re = 1400$.

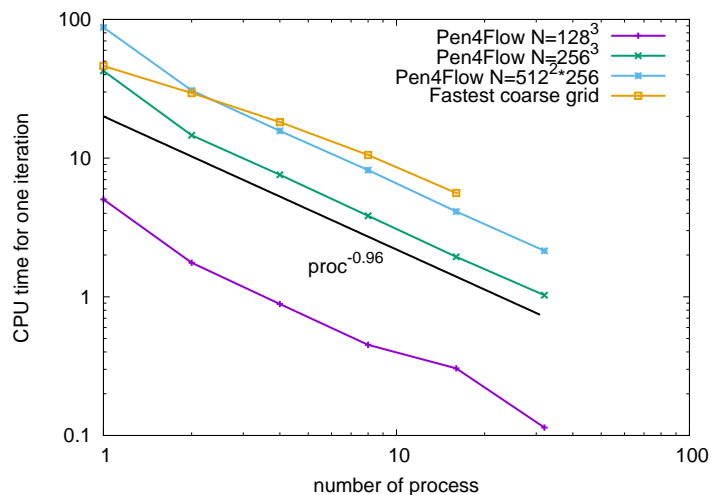


Figure 3: CPU time(s) for one iteration as function of the number of processors and for different resolutions using Pen4Flow and Fastest codes.

174 Figure 4 shows a three-dimensional representation of isosurfaces of the
 175 vorticity norm colored with the streamwise velocity. The computation has
 176 been obtained with the volume penalization method at resolution 512^3 . We
 177 observe that the flow is turbulent, characterized by the presence of vortices
 178 of different size. We can also identify a recirculation zone corresponding to
 179 an upstream (negative) velocity. In the following we will consider statisti-
 180 cal quantities only. Table 2 shows that both codes yield similar values for

181 the separation and reattachment point. A slight difference can however be
 182 observed for the reattachment point which might be due to the penalization
 183 method.

184

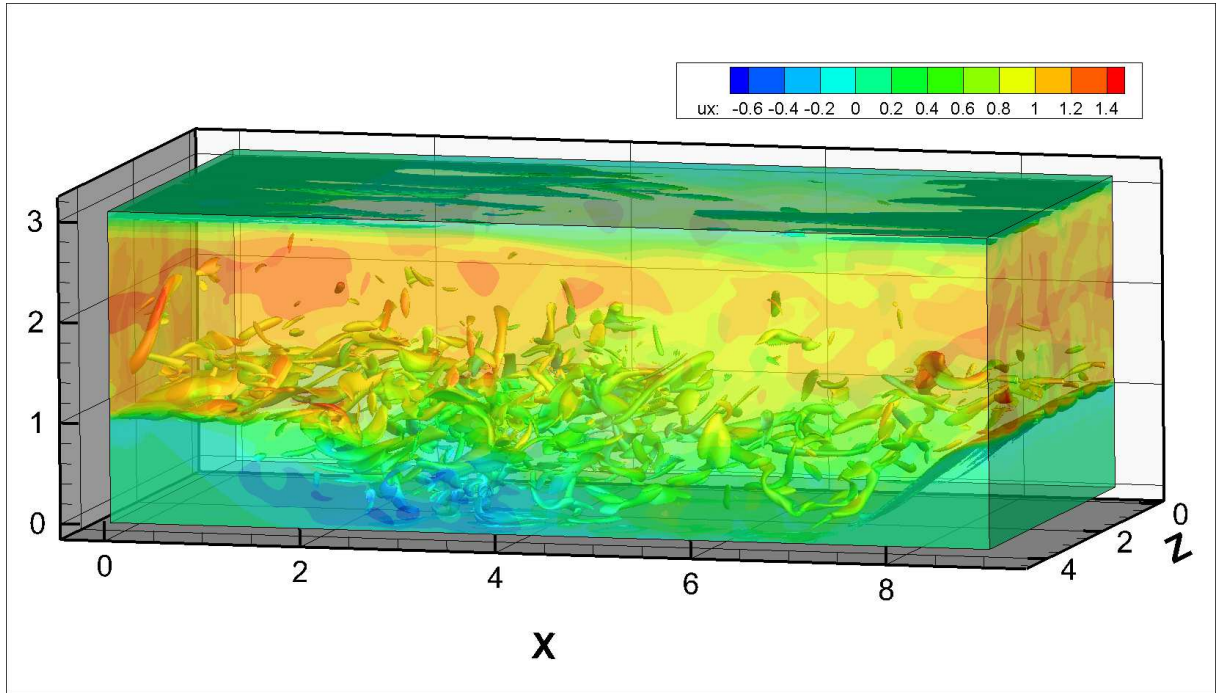


Figure 4: Isosurface of the vorticity norm $\|\boldsymbol{\omega}\| = \sqrt{\omega_x^2 + \omega_y^2 + \omega_z^2}$ colored with the horizontal velocity u_x computed with the Pen4Flow code at resolution 512^3 for $Re = 1400$.

	N_x	N_y	N_z	x_{sep}/H	x_{reatt}/H
Pen4Flow 512^3	512	512	512	0.26	5.26
Fastest coarse	221	92	100	0.26	5.20
Fastest fine	442	184	200	0.26	5.21
BPRM09 LESOCC	N_{tot}	=	$13.6 \cdot 10^6$	0.26	5.19

Table 2: Numerical parameters, separation point x_{sep}/H and reattachment point x_{reatt}/H .

185 *3.1.1. Moment statistics*

186 Table 3 summarizes the first and second order moments together with
 187 the skewness and flatness values of the velocity fields for both codes at a
 188 given time instant in the statistically steady state. We can note that aver-
 189 aging the results over time does not change the results and slight differences
 190 ($\leq 2\%$) appear only for high order moment statistics. The flatness reflects
 191 the intermittency in a turbulent flow while the skewness factor of velocity is
 192 related to the energy transfer, according to the Kolmogorov 4/5 law. The
 193 results show that all values are in good agreement, even for the higher order
 194 statistics. In all cases the difference is less than 5%.

	M_1	M_2	Flatness: M_4/M_2^2	Skewness: $M_3/M_2^{3/2}$
Pen4Flow 512^3	0.218	0.228	4.57	2.025
Fastest fine grid 512^3	0.221	0.233	4.36	1.966

Table 3: Moments of the velocity field (averaged over the three components) for the Pen4Flow code (512^3) and for the Fastest code, at a given time instant in the statistically steady state. The data of the finite volume computed on the fine grid are interpolated onto an uniform grid with 512^3 grid points.

196 Figure 5 (top) shows the probability density functions (PDF), estimated
 197 with a histogram using 50 bins, of the velocity at a given time instant in
 198 the statistically steady state. We find that the PDFs do almost collapse ex-
 199 cept for weak negative values where we can observe a small deviation. Note
 200 that the PDFs are plotted in log-lin representation, hence the difference is
 201 negligible. Spectra of turbulent kinetic energy along the spanwise direction
 202 are shown in figure 5 (bottom). A small inertial range with a $k^{-5/3}$ behavior
 203 can be identified for $k = 3$ to 10, followed by a faster decay. The spectra
 204 of both simulations collapse for low wavenumbers up to $k = 60$. For the
 205 Fastest code we then find a saturation, while for the spectral code energy
 206 further decays with increasing wavenumber. The saturation might be due to
 207 the linear interpolation of the finite volume computation.

208
 209 The statistical confidence of the second order moments can be quantified
 210 by the fourth order moments which are of the same order as the second order
 211 moments. Statistical convergence is also reflected by the velocity PDF shown

212 in Fig. 5 (top). The tails do not show any fluctuations at the extreme values
 213 and seem thus well converged.

214

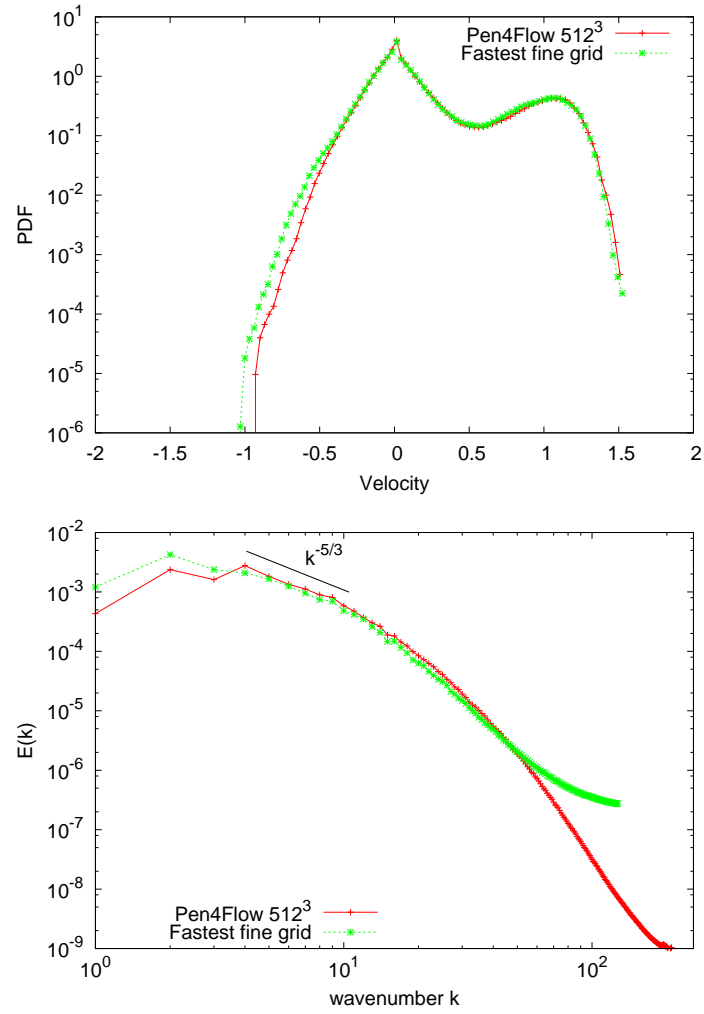


Figure 5: Top: PDF of the velocity (averaged over the three components) at a given time instant in the statistically steady state. Bottom: Turbulent kinetic energy spectrum versus wavenumber k in the spanwise direction. The codes used are Pen4Flow and Fastest.

215 *3.1.2. Mean velocity and Reynolds stress profiles*

216 First we consider the volume penalization code and we study the influence
 217 of the resolution on the statistics. Figure 6 shows mean velocity and Reynolds
 218 stress profiles at four different streamwise locations, $x = 0.5, 2, 4$ and 6 for
 219 three different resolutions, 256^3 , $512^2 \times 256$ and 512^3 . Note that the profiles
 220 are all averaged over time (between 30 and 40 flow through times) and over
 221 the spanwise direction. We observe that the influence of the resolution is
 222 weak for all mean velocity profiles (Fig. 6 only shows the x -component $\langle u_x \rangle$)
 223 while it is much more sensitive for the second order moments, illustrated in
 224 Fig. 6 (right) for the cross Reynolds stress $\langle u'_x u'_y \rangle$. Hence in the following we
 225 will consider only the two higher resolutions for Pen4Flow.

226
 227 In Fig. 7 and 8, the coarse grid computations using Fastest show a good
 228 agreement for mean velocities with the fine grid computations, similar to
 229 what has been observed for Pen4Flow. Concerning the Reynolds stresses the
 230 differences are less pronounced than for Pen4Flow, but are still slightly more
 231 visible, in particular for the $\langle u'_x u'_y \rangle$ (see Fig. 7, bottom, right).

232

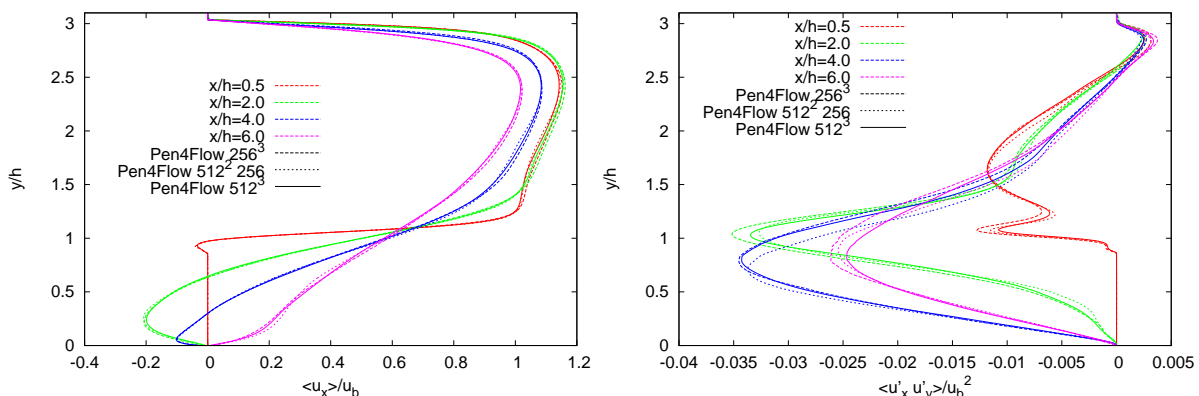


Figure 6: Mean velocity (left) and Reynolds stress (right) profiles at four different locations $x = 0.5, 2, 4$ and 6 for Pen4Flow using different resolutions.

233 Now, we compare the statistics for the two codes for different resolutions
 234 and different streamwise positions. We also plot the results from [1] and we
 235 use the notation: BPRM09 for LES (adapted grid) and DNS (MGLET: non
 236 uniform spatial grid).

237

238 The comparison of the statistics for the two codes is shown for the mean
239 velocity and Reynolds stress profiles at location $x = 0.5$ and $x = 4$ in figure 7
240 and figure 8, respectively. The profiles are averaged over time and over
241 the spanwise direction. We observe that, the mean velocities $\langle u_x \rangle$ are the
242 same, the mean velocities $\langle u_y \rangle$, the Reynolds stress $\langle u'_x u'_x \rangle$ and $\langle u'_y u'_y \rangle$ almost
243 coincide, except at $x = 0.5$ where we observe a small deviation for $\langle u'_y u'_y \rangle$.
244 The profiles of the Reynolds stress $\langle u'_z u'_z \rangle$ and $\langle u'_x u'_y \rangle$ exhibit slight differences
245 for $x = 2$ and $x = 4$. Significant differences in $\langle u'_z u'_z \rangle$ are observed if we
246 compare with the results from [1], however Pen4Flow and Fastest do still
247 match. The difference is probably due to the fact that, in [1], Breuer et al.
248 plotted the turbulent kinetic energy k instead of the Reynolds stress $\langle u'_z u'_z \rangle$.
249 We checked the turbulent kinetic energy computed from our data and found
250 a good agreement with the $\langle u'_z u'_z \rangle$ obtained in [1].

251 We can note also that for the mean vertical velocity $\langle u_y \rangle$, at $x = 6$ some
252 fluctuations and an overestimation around $y = 1$ are observed for Pen4Flow.
253 However, the fluctuations tend to disappear by increasing the resolution. The
254 higher moments are more sensitive when we change the methods, but we can
255 conclude that the two codes yield similar results which are in reasonable
256 agreement.

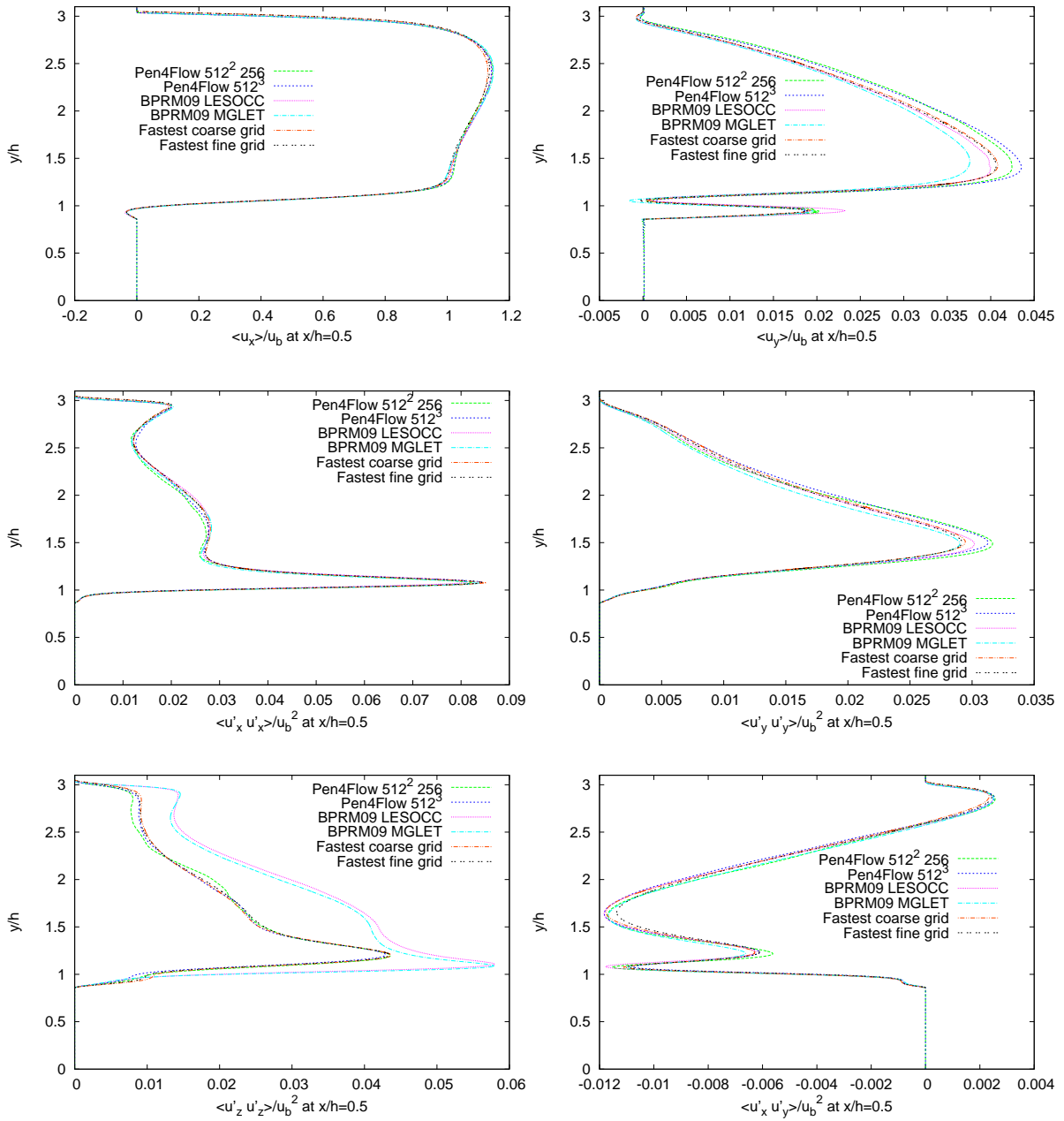


Figure 7: Mean velocity (top) and Reynolds stress profiles (middle and bottom) at $x = 0.5$. The profiles are averaged over time and over the spanwise direction. The codes used are Pen4Flow and Fastest with fine and coarse grids. The results from [1] are also plotted and denoted by BPRM09 for LES (adapted grid) and DNS (MGLET: non uniform spatial grid).

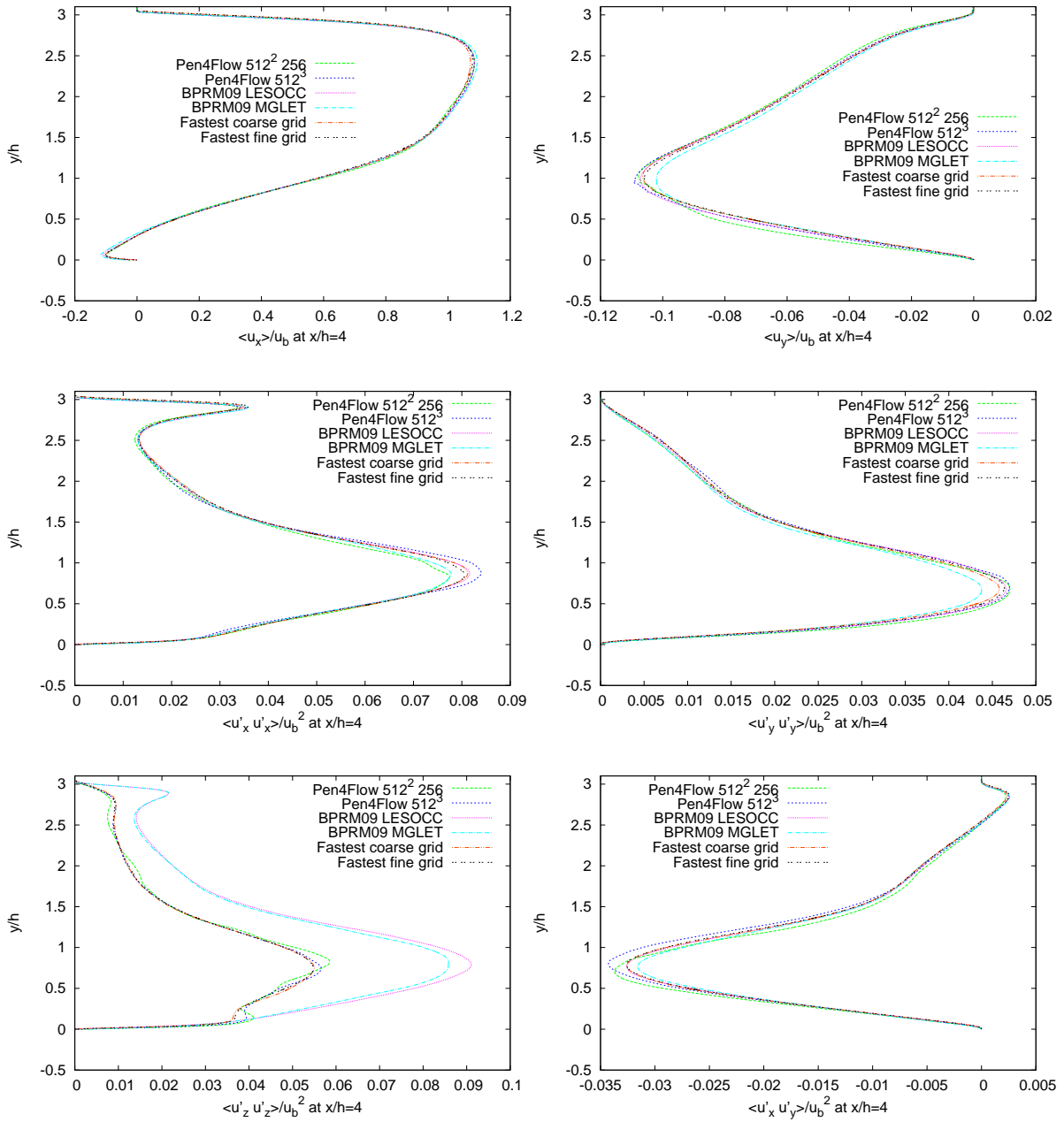


Figure 8: Mean velocity (top) and Reynolds stress profiles (middle and bottom) at $x = 4$. The profiles are averaged over time and over the spanwise direction. The codes used are Pen4Flow and Fastest with fine and coarse grids. The results from [1] are also plotted and denoted by BPRM09 for LES (adapted grid) and DNS (MGLET: non uniform spatial grid).

257 *3.1.3. Wavelet analysis for the velocity fields*

258 Wavelet decomposition of flow fields yields information on scale, position
 259 and direction. Scale-dependent statistical analysis of turbulent flows can thus
 260 be performed by considering the wavelet coefficients for example of velocity
 261 or vorticity, see, *e.g.*, [21]. Wavelet-based directional energies can be defined
 262 which capture the properties of velocity gradients. The intermittency of the
 263 flow in different directions can be quantified with scale-dependent directional
 264 flatness. The scale-dependent velocity flatness was shown to quantify the spa-
 265 tial variability of the energy spectrum [22]. Here we will present applications
 266 of scale dependent statistics of the velocity field for the periodic hill flow
 267 computed with both methods presented above.

268
 269 The velocity field, $\mathbf{u}(\mathbf{x}) = (u_x, u_y, u_z)$, is decomposed into an orthogonal
 270 wavelet series: with $\mathbf{x} = (x, y, z) \in [0, x_l] \times [0, y_l] \times [0, z_l]$ rescaled onto
 271 $[0, 2\pi]$ and given at resolution $N = 2^{3J}$, where J is the number of octaves in
 272 each spatial direction. The decomposition of \mathbf{u} into an orthogonal wavelet
 273 series yields:

$$\mathbf{u}(\mathbf{x}) = \sum_{\boldsymbol{\lambda} \in \Lambda} \tilde{\mathbf{u}}_{\boldsymbol{\lambda}} \psi_{\boldsymbol{\lambda}}(\mathbf{x}), \quad (4)$$

274 where the multi-index $\boldsymbol{\lambda} = (j, i_x, i_y, i_z, d)$ denotes the scale j , the position
 275 $\mathbf{i} = (i_x, i_y, i_z)$, and the directions $d = 1, 2, \dots, 7$ of the wavelets. Indeed, the
 276 wavelet decomposition in three dimensions yields seven directions due to the
 277 tensor product construction [23]. The corresponding index set Λ is

$$\Lambda = \{ \boldsymbol{\lambda} = (j, i_x, i_y, i_z, d), \quad j = 0, \dots, J - 1; \quad i_x, i_y, i_z = 0, \dots, 2^j - 1 \quad (5) \\ \text{and } d = 1, \dots, 7 \}.$$

278 Due to the orthogonality the wavelet coefficients are given by $\tilde{\mathbf{u}}_{\boldsymbol{\lambda}} =$
 279 $\langle \mathbf{u}, \psi_{\boldsymbol{\lambda}} \rangle$ where $\langle \cdot, \cdot \rangle$ denotes the L^2 -inner product, defined by
 280 $\langle f, g \rangle = \int_{[0, 2\pi]^3} f(\mathbf{x})g(\mathbf{x})d\mathbf{x}$, for each velocity component. The coefficients
 281 measure fluctuations of \mathbf{u} around scale 2^{-j} and around position $2\pi\mathbf{i}/2^j$ in one
 282 of the seven possible directions. The N wavelet coefficients $\tilde{\mathbf{u}}_{\boldsymbol{\lambda}}$ are efficiently
 283 computed from the N grid point values of $\boldsymbol{\omega}$ using the fast wavelet transform,
 284 which has linear complexity [24]. In this study, we have chosen the Coiflet
 285 30 wavelet, which has 10 vanishing moments ($\int \mathbf{x}^p \psi(\mathbf{x})d\mathbf{x} = 0, p = 0, \dots, 9$)
 286 and which is well adapted to represent the current flow simulations.

287 The a -order moments dependent on the scale index j and the direction d
 288 are defined as:

$$M_{(j,d)}^a(u_i) = \langle \tilde{u}_i^a \rangle_{(j,d)}, \quad \text{where } \mathbf{i} = i_x, i_y, i_z \text{ and } a \in \mathbb{N}. \quad (6)$$

289 $\langle \rangle_{(j,d)}$ is the average over all positions \mathbf{i} of the wavelets for a given direction
 290 d and a given scale j .

291

292 At a given time instant in the statistically steady state, we applied the
 293 wavelet based multi-scale analysis. Scale-dependent moments are shown in
 294 figure 9 for Pen4Flow with 512^3 collocation grid points and for the finite
 295 volume code with a fine grid interpolated on an uniform Cartesian grid 512^3 .
 296 Large scale corresponds to $J = 0$ and small scale to $J = 8$. The second order
 297 moments show the distribution of turbulent kinetic energy at different scales
 298 and yield similar results for both codes. The scale-dependent flatness, which
 299 is a measure for intermittency of the flow, obeys to the same behavior, but
 300 the difference between the two curves is small. The skewness (not shown
 301 here) gives for both cases similar values. Both codes show thus the same be-
 302 havior for the scale dependent statistics from the largest to the smallest scale.

303

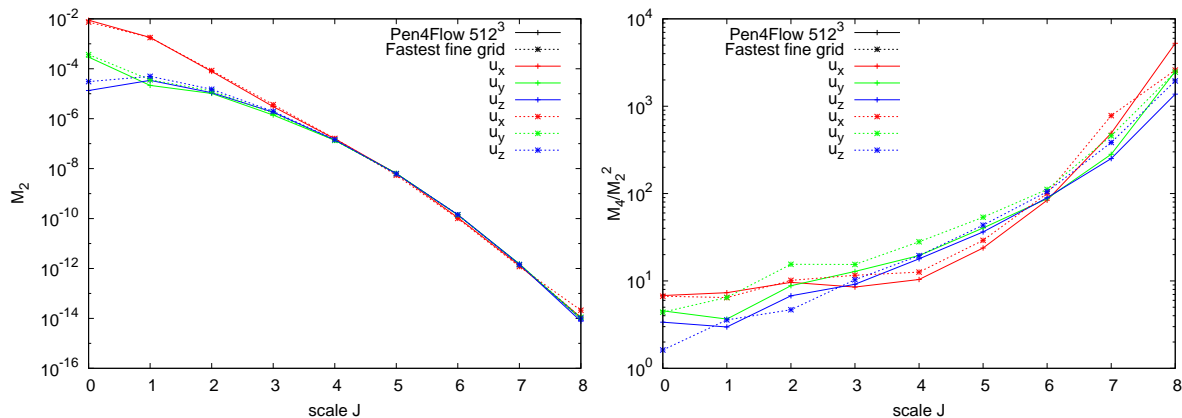


Figure 9: Scale-dependent second order moments (left) and scale dependent flatness (right) for the three different velocity components for Pen4Flow with 512^3 collocation grid points and for Fastest with a fine grid interpolated on an uniform Cartesian grid 512^3 .

304 To get further insight into the scale dependence of the Pen4Flow and the
 305 Fastest computations, we consider in Fig. 10 scale dependent moments of the

306 velocity fields in different spatial directions, defined in eq. (6) using an or-
307 thogonal wavelet decomposition. In Fig. 10 (top) the second order moment
308 of the three velocity components in the longitudinal and transverse direc-
309 tions, *i.e.*, for u_x the longitudinal direction corresponds to x while y and z
310 are the transverse directions, is shown. Large scale corresponds to $J = 0$
311 and small scale to $J = 8$. For clarity, we decided to omit the mixed direc-
312 tions xy, xz, yz, xyz . The scale dependent first (not shown here) and second
313 order moments, are almost the same for both codes. The behaviors of the
314 scale-dependent flatness are also similar; indeed the difference between the
315 two curves is small for all scales, except for the smallest scale $J = 8$ where
316 the deviation is more pronounced.

317

318 3.2. Influence of the Reynolds number

319 In the following, we investigate the influence of the Reynolds number us-
320 ing the Pen4Flow code and compare the results again with those obtained
321 by Breuer et al. [1]. In Table 4, the separation and reattachment points
322 are compared for three Reynolds numbers, $Re = 700, 1400$ and 2800 . Slight
323 differences are observed for the separation point, while the discrepancies are
324 larger for the reattachment by comparison with those obtained with BPRM09
325 LESOCC code.

326

	x_{sep}/H	x_{reatt}/H
$Re = 700$		
Pen4Flow $N = 256^3$	0.30	5.30
BPRM09 LESOCC	0.29	5.24
$Re = 1400$		
Pen4Flow $N = 512^3$	0.26	5.26
BPRM09 LESOCC	0.26	5.19
$Re = 2800$		
Pen4Flow $N = 512^3$	0.24	5.44
BPRM09 LESOCC	0.21	5.41

Table 4: Reynolds numbers, separation point x_{sep}/H and reattachment point x_{reatt}/H . The results from [1] are also plotted and denoted by BPRM09 for LES (adapted grid).

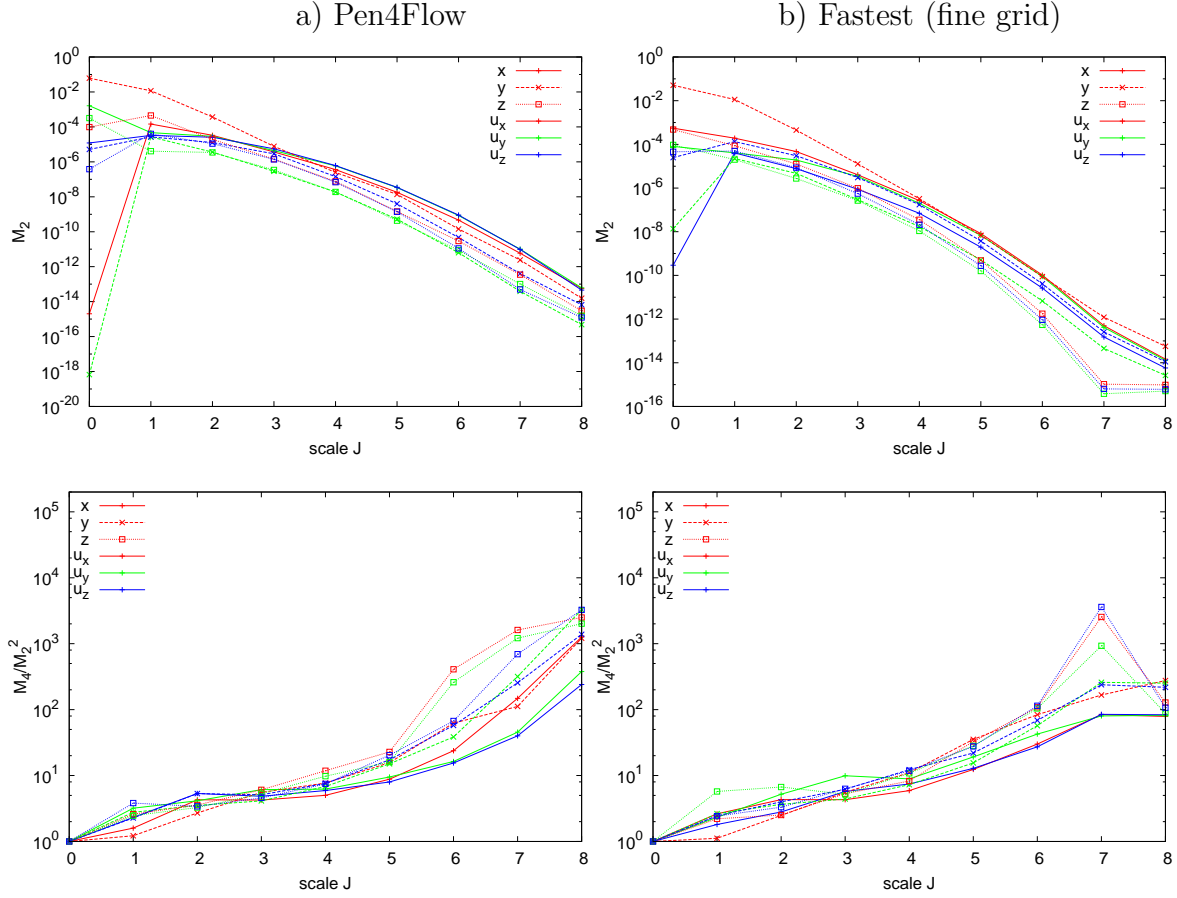


Figure 10: Scale-dependent second order moments (top) and scale dependent flatness (bottom) for the three velocity components in the three main directions. The codes used are Pen4Flow with 512^3 collocation grid points and Fastest with a fine grid interpolated on a Cartesian uniform grid 512^3 .

327 As example in Fig. 11, we selected the position $x/h = 4$ as the compari-
 328 son of the results for the other positions shows the similar tendencies. Again
 329 we start with the mean velocity profile for $\langle u_x \rangle$ and $\langle u_y \rangle$, we observe that for
 330 the three Reynolds numbers all curves agree reasonably well. The Reynolds
 331 stress profiles, Fig. 11 middle and bottom, reflect the same tendency, how-
 332 ever for $\langle u'_x u'_x \rangle$ we observe larger differences compared to the mean velocity.
 333 Nevertheless, for the worst case, $Re = 2800$, the differences are below 10%.

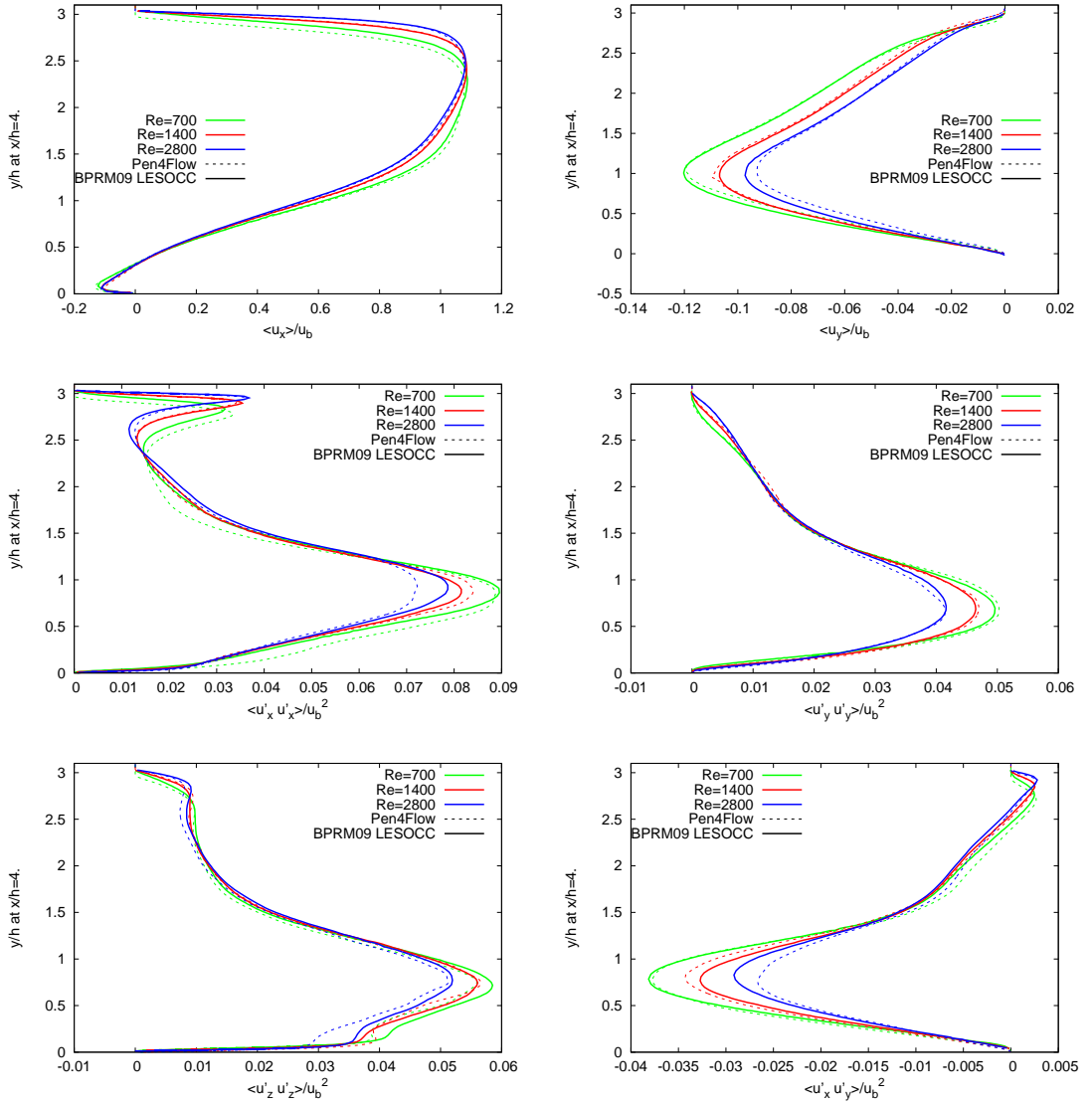


Figure 11: Mean velocity (top) and Reynolds stress profiles (middle and bottom) at $x = 4$ for Reynolds number $Re = 700, 1400, 2800$. The profiles are averaged over time and over the spanwise direction. The code used is Pen4Flow with fine and coarse grids. The results from [1] are also plotted and denoted by BPRM09 for LES (adapted grid).

334 4. Conclusion

335 The flow over a periodic hill configuration has been revisited using a
 336 Fourier pseudo-spectral code with volume penalization and a finite volume

337 code with body fitted grids. This classical benchmark involves complex,
338 non-Cartesian geometries and turbulent flow conditions. The presence of a
339 recirculation zone caused by the presence of the hill constitutes an additional
340 difficulty. This detailed study showed that the volume penalization code and
341 the finite volume code reproduce well the physics of this flow and produce
342 similar results. Wavelet analysis enabled us to show that the multi-scale
343 physics is almost the same for both codes. The volume penalization method
344 is a powerful technique to impose Dirichlet boundary conditions in complex
345 geometries using still a Fourier pseudo-spectral method, which has been con-
346 fronted here for three different Reynolds numbers.

347

348 The pros and the cons for Pen4Flow and Fastest can be summarized as
349 follows. Due to the volume penalization method, Pen4Flow is very flexible
350 for changing the geometry and the convergence of the method is mathemat-
351 ically justified and proved in [7]. The penalization method becomes more
352 efficient for higher Re numbers [25] because the penalization boundary layer
353 is proportional to $\sqrt{\eta\nu}$. Moreover for this code, no linear system has to be
354 solved (differential operators are diagonal) and most computational cost is
355 due to the FFT, for which highly efficient parallel implementations are avail-
356 able. We can note also that the Gibbs oscillations are not amplified [11] and
357 the method does not show numerical diffusion and dispersion. However, the
358 penalization term models boundary conditions with first order accuracy only
359 which is also the case for the geometrical errors due to the mask function
360 (staircase effect). Equidistant grids are required in Pen4Flow, which is less
361 efficient as fine grid size is required close to the boundary. Furthermore, the
362 volume penalization technique requires more grid points as the governing
363 equations have to be solved in the whole domain, including both solid and
364 fluid domains. Finally, the time step is limited due to stability restrictions
365 imposed by the penalization term in addition to the classical CFL limit. In
366 conclusion, considering these latter points, the volume penalization method
367 is an attractive alternative to simulate turbulent flows in complex geometries.
368 Fastest takes advantage of body fitted grids which are well adapted to the
369 geometry and involve no interpolation error. As a consequence less grid
370 points are required compared to Pen4Flow. The curvilinear grid with block-
371 structured grids enables to have boundary conditions fulfilled exactly. Nev-
372 ertheless, Fastest adopts the SIMPLE projection scheme for imposing the
373 incompressibility, which is not exactly fulfilled. The finite volume scheme,
374 used in this code, is second order accurate in space and time, and an alge-

375 braic system has to be solved in each time step. The parallelization is also
376 less efficient, as shown in Fig. 3. Finally, Fastest is less flexible compared to
377 Pen4Flow since the grid has to be generated for a given geometry.

378

379 Interesting perspectives of Pen4Flow are the application to compute tur-
380 bulent flows in complex geometries which are simply defined by the mask
381 function and which may even vary in time, as it is the case in fluid-structure
382 interaction problems. The implementation of additional transport equations,
383 e.g. for passive scalars, or the Boussinesq approximation is another promising
384 direction. The necessary Neumann boundary conditions can be implemented
385 using the technique proposed in [16].

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392

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