Coupling of radial and translational motion in small viscous bubbles

Stéphane Popinet, Stéphane Zaleski

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Stéphane Popinet
Laboratoire de Modélisation en Mécanique
8, rue du Capitaine Scott
75015 Paris
Email: popinet@lmm.jussieu.fr

Stéphane Zaleski
Laboratoire de Modélisation en Mécanique
8, rue du Capitaine Scott
75015 Paris
Email: zaleski@lmm.jussieu.fr

ABSTRACT

In this article we are interested in the coupling between the translational and radial velocity of a bubble trapped in an acoustic field. We write a simple model for the momentum equation of the center of gravity of the bubble. An analytic solution is derived in the case of small radial and translational oscillations. This solution is compared to numerical solutions of the model equation and to full axisymmetric Navier-Stokes numerical simulations. Satisfying agreement is found and the non-linear model equations are applied to the case of sonoluminescing bubbles. In this case, the amplitude of the bubble motion is found to be negligible. This suggests that the influence of the varying pressure gradient on the translational motion is not as significant as originally anticipated in the case of sonoluminescence.

INTRODUCTION

The acoustic levitation of gas bubbles has been used widely to study bubble dynamics. The acoustic pressure gradient averaged over one oscillation cycle, or Bjerknes force, allows to pin the bubble near an antinode of the acoustic field. This technique is used in sonoluminescence experiments (Barber and Putterman, 1991; Gaitan *et al.*, 1992; Crum, 1994). While detailed studies of the equilibrium position of the levitating bubble are available (Crum and Prosperetti, 1983; Matula *et al.*, 1997a), little attention has been given to the amplitude of the vertical motion of the bubble. Indeed, while the Bjerknes force compensates buoyancy on average over one cycle of oscillation, the instantaneous balance between buoyancy and acoustic pressure gradient is not verified. As a result the bubble oscillates vertically at the frequency of the acoustic forcing. This effect could be potentially significant for all matters regarding the shape stability of the acoustically-levitated bubble. Building on this observation, Prosperetti has proposed a very seducing explanation of the sonoluminescence phenomenon (Prosperetti, 1997). It is well known that a moving and contracting bubble cannot remain spherical and that a jet forms in the direction of motion. Prosperetti proposed that in the case of sonoluminescence the vertical motion of the bubble coupled with the violent collapse could lead to the formation of a very high-speed jet. Its impact with the bubble surface would be responsible for the light emission.

The two most actively discussed theories of the light emission mechanism are the shock wave theory (Greenspan and Nadim, 1993; Wu and Roberts, 1993; Moss *et al.*, 1997) and the adiabatic heating theory (Hilgenfeldt *et al.*, 1999). The shock wave theory relies on the bubble remaining spherical, while in the adiabatic heating theory the bubble deformation seems to play no essential role. The only alternative in which the loss of spherical symmetry plays a central role for the light emission mechanism is the jet formation theory.

Another motivation for the study of bubble shape instabilities and thus of its possible forcing by vertical translational motion arises when considering published data on the physical parameters for sonoluminescence. Most reports locate sonoluminescence at pressures and equilibrium radii for which the bubble is actually *unstable* (Prosperetti and Hao, 1999). Thus even a small forcing by translational motion of bubble deformation modes could lead to large amplitude effects.

A final motivation for the detailed study of Bjerknes forces in sonoluminescence conditions is the discrepancy observed by Matula et al. (1997a) between their measurements of the bubble equilibrium position and predictions of an inviscid theory.

In this paper we are interested in a prediction of the amplitude of the vertical motion of an acoustically-levitated bubble. We will use both physical modelling and direct numerical simulations to assess the accuracy of the models. Implications for the stability of sonoluminescing bubbles will be discussed.

PHYSICAL MODEL

We want to derive the equation of motion of the center of gravity of the bubble. The forces acting on the bubble are essentially viscous drag and buoyancy. Following Magnaudet and Legendre (1998) we can express the viscous drag force on a spherical bubble with a time-dependent radius as

$$F = 12\pi\rho\nu R U + \frac{2}{3}\pi\rho(3R^3\dot{U} + 3R^2 U\dot{R}) + \frac{4}{3}\pi\rho R^3\dot{V}_0, \quad (1)$$

where the dot denotes the time derivative, ρ is the liquid density, ν the kinematic viscosity, R the radius of the bubble and $U = U_\infty - V_0$. V_0 is the velocity of the center of gravity of the bubble and U_∞ is the velocity of the fluid far from the bubble. This expression is valid only in the case of Reynolds numbers much larger than unity. The Reynolds number is based either on the translational relative velocity U or on the radial velocity $\dot{R}$. When these two Reynolds numbers are both much smaller than unity, the history force due to the diffusion of vorticity is no longer negligible and the expression of the force becomes

$$\begin{aligned} F = {} & 4\pi\rho\nu R U + \frac{2}{3}\pi\rho(3R^3\dot{U} + 3R^2 U\dot{R}) + \frac{4}{3}\pi\rho R^3\dot{V}_0 + \quad (2)\\ & 8\pi\rho\nu \int_0^t \exp\left[9\nu\int_\tau^t R(t')^{-2}dt'\right] \times \\ & \mathrm{erfc}\left[\sqrt{9\nu\int_\tau^t R(t')^{-2}dt'}\right] d[R(\tau)U(\tau)]. \end{aligned}$$

In the case of acoustic bubble levitation, we consider an intermediate scale much larger than the bubble size but much smaller than the acoustic wavelength. At this scale the acoustic pressure gradient ∇p can be considered spatially constant and the velocity of the fluid is then given by

$$\dot{U}_\infty = \frac{-\nabla p}{\rho} = -\nabla\phi. \quad (3)$$

The buoyancy force acting on the bubble is given by $-4/3\pi R^3(\rho - \rho_{gas})g$ where g is the acceleration of gravity. If we neglect the mass of the gas contained in the bubble, (1) and (3) give the equation of motion in the case of high Re

$$\dot{U} + \left(\frac{18\nu}{R^2} + 3\frac{\dot{R}}{R}\right)U - 2\nabla\phi - 2g = 0. \quad (4)$$

At this point we can write a system of two coupled ordinary differential equations (4) plus the Rayleigh-Plesset equation (Plesset and Prosperetti, 1977), which describes both the vertical and the radial oscillation of the levitating bubble. This system can be solved numerically but it is interesting to try to find an analytical solution in a particular case.

SMALL AMPLITUDE OSCILLATIONS: THEORY

We will try to find an analytical solution in the case of small vertical and radial oscillations. When the amplitude of the radial oscillation is small, the Rayleigh-Plesset equation is shown to yield a first order solution for the radius of the form

$$R = R_0[1 + \varepsilon\cos(\omega t + \theta)], \quad (5)$$

where R_0 is the equilibrium radius, ε the relative amplitude of the radial oscillation, ω the frequency of the acoustic forcing and θ the phase-shift between the radial oscillation and the acoustic forcing. Using (5) in (4) and retaining the first order terms in ε yields

$$\begin{aligned} \dot{U} + \left[18\frac{\nu}{R_0^2} - 36\frac{\nu}{R_0^2}\varepsilon\cos(\omega t + \theta) - 3\varepsilon\omega\sin(\omega t + \theta)\right]U \quad (6)\\ -2|\nabla\phi|\cos(\omega t) - 2g = 0. \end{aligned}$$

We look for a solution of the form $U = a\cos(\omega t) + b\sin(\omega t)$ which yields the system of equations

$$\omega a + 18\omega\nu^\star b - 2|\nabla\phi| = 0, \quad (7)$$

$$18\nu^\star a - b = 0, \quad (8)$$

$$18\nu^\star\sin\theta a - \tfrac{3}{2}\cos\theta a - 18\nu^\star\cos\theta b - \tfrac{3}{2}\sin\theta b - \tfrac{2g}{\omega\varepsilon} = 0, \quad (9)$$

$$\varepsilon(12\nu^\star a + b) = 0, \quad (10)$$

$$\varepsilon(a - 12\nu^\star b) = 0, \quad (11)$$

where we have introduced the non-dimensional viscosity $\nu^\star = \nu/\omega R_0^2$. Equations (7) and (8) correspond to terms of frequency ω and give the leading-order solution for the bubble motion

$$U = 2\frac{|\nabla\phi|}{\omega}\cos(\psi)\sin(\omega t + \psi) \quad \text{with} \quad \tan\psi = 18\nu^\star. \quad (12)$$

Equation (9) corresponds to the condition when the secular term in (6) vanishes, which gives the equilibrium position defined by

$$\frac{g}{|\nabla\phi|} = -\frac{\varepsilon}{1+(18\nu^\star)^2}\left(\frac{3}{2}\cos\theta + 9\nu^\star\sin\theta + (18\nu^\star)^2\cos\theta\right). \tag{13}$$

In the case of a vanishing viscosity, equation (13) yields the result of Crum and Prosperetti $g/|\nabla\phi| = -3/2\varepsilon\cos\theta$ (Crum and Prosperetti, 1983). Equations (10) and (11) correspond to terms of frequency 2ω which vanish with the amplitude of the radial oscillation.

Equation (12) gives a maximum relative velocity U of twice the velocity of the fluid far from the bubble $U_\infty = |\nabla\phi|/\omega$, obtained when viscosity $\nu^\star$ vanishes. As the viscosity increases the relative velocity decreases and ultimately vanishes.

SMALL AMPLITUDE OSCILLATIONS: NUMERICS

It is important to note that equations (12) and (13) are valid only for small amplitudes of the radial and vertical oscillations. Moreover, in the case of an oscillating bubble, the translational velocity changes sign and consequently the translational Reynolds number can be arbitrarily small. In these conditions neither expression (1) nor expression (3) is valid throughout the oscillation cycle. As it is difficult to derive an analytical solution taking into account the history force we have used a numerical technique to solve the system of equations (1) or (3) coupled with the Rayleigh-Plesset equation. Figure 1 and 2 illustrate the results. An excellent agreement is found between the numerical solution of (1) (white squares) and the linear theory; the relative amplitude of the radial oscillation in the numerical solution is $\varepsilon = 0.01$. The agreement remains very satisfactory for values of ε up to 0.1. The numerical solution of equation (3) is shown using black disks. As expected the three solutions ((1), (3) and linear theory) are close when $\nu^\star$ is small. The limits are also the same when $\nu^\star$ becomes large. The history force is seen to act as an anti-dissipative term, which is obviously not the case when the radius of the bubble is constant in time.

As noted above neither of the two model equations (with or without history force) is valid throughout the oscillation cycle. We have used our 2D axisymmetric free-surface code (Popinet and Zaleski, 1998a; Popinet and Zaleski, 1998b) to solve the Navier-Stokes equations for a bubble in an acoustic field in order to verify the relevance of equations (1) and (3) for the description of the bubble motion. The simulations were done for different values of $\nu^\star$ and the results are summarized by the white triangles in figure 1 and 2. The error bars represent the standard deviation for a series of several periods of oscillation. The liquid density was $\rho = 1000$ kg/m^3, the surface tension coefficient $\sigma = 0$, the speed of sound in the liquid $C_l = 1481$ m/s, acceleration of gravity $g = 0.01$ m/s^2, equilibrium radius $R_0 = 1$ mm, equilibrium pressure $p_0 = 10^5$ Pascals, amplitude of the pressure

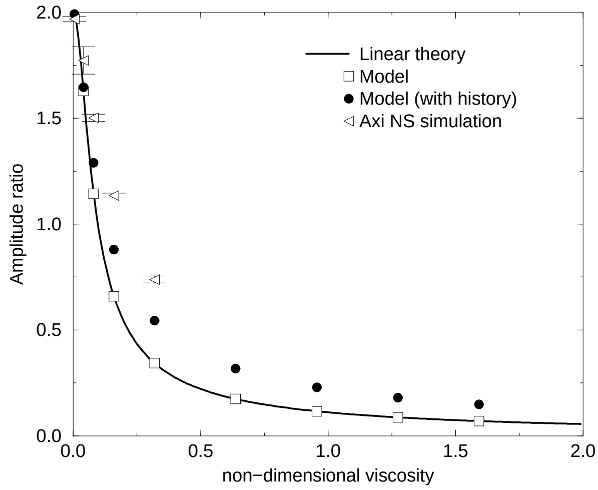

Figure 1. RATIO OF THE AMPLITUDE OF THE RELATIVE BUBBLE VELOCITY U TO THE FLUID VELOCITY U_∞.

fluctuation $p_a = 10^4$ Pascals, frequency of the acoustic forcing $f = 500$ Hz. For small values of $\nu^\star$ the agreement with the linear theory is very satisfactory. For larger values of the viscosity, the numerical resolution becomes difficult, our code being mainly designed to solve high-Reynolds number flows.

LARGE AMPLITUDE OSCILLATIONS AND APPLICATION TO SONOLUMINESCENCE

We can now be confident that our model gives a consistent prediction for the amplitude of bubble motion. The essential result of this analysis is that the translational velocity of the bubble (relative to the fluid) can never be larger than $2|\nabla\phi|/\omega$ in the linear regime. In the case of sonoluminescence $\nu^\star$ is approximately 0.1 ($R_0 = 10$ μm, $\nu = 10^{-6}$ m^2/s, $f = 25$ kHz) which gives a smaller amplitude of $|\nabla\phi|/\omega$.

The value of $|\nabla\phi|$ is defined through the equilibrium position of the bubble. Matula *et al.* (1997a) measured the equilibrium position of a bubble in sonoluminescence conditions. They report a characteristic distance from the antinode smaller than 1 mm which gives for the parameters used in their experiment ($R_0 = 3$ μm, $f = 19.5$ kHz, $p_a = 1.4$ bar) a value of $|\nabla\phi|$ of approximately 10 m/s^2 (consistently of the order of the acceleration of gravity). In the linear approximation this gives a maximum translational relative velocity of $2|\nabla\phi|/\omega = 0.16$ mm/s, the associated vertical displacement being $2|\nabla\phi|/\omega^2 = 1.3$ nm. One might argue however that we have used a linear theory well beyond its domain of validity (the radial oscillations of the sonoluminescing bubble being strongly non-linear).

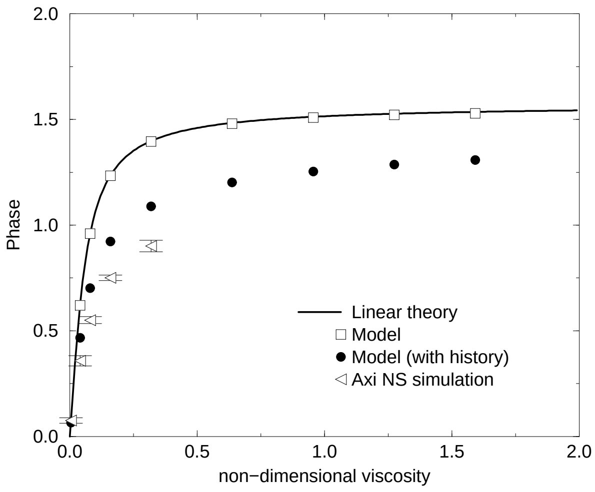

Figure 2. PHASE SHIFT BETWEEN THE RELATIVE BUBBLE VELOCITY AND THE FLUID VELOCITY AS A FUNCTION OF THE NON-DIMENSIONAL VISCOSITY $\nu^{\star}$.

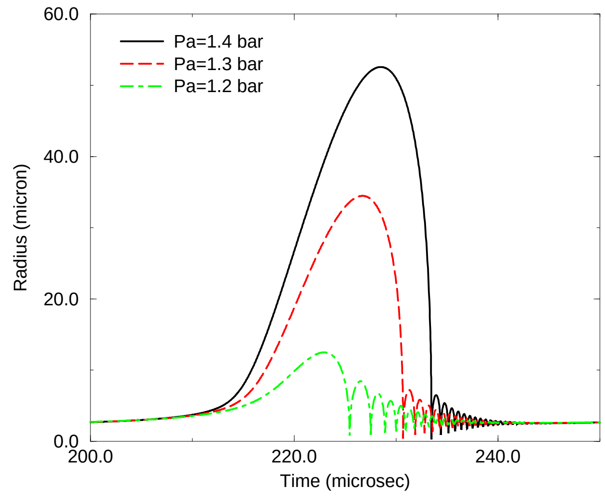

Figure 3. NUMERICAL SOLUTION OF THE RAYLEIGH-PLESSET EQUATION FOR PARAMETERS USED IN (Matula *et al.*, 1997a).

To answer this concern, we used a numerical solution of equation (4) coupled with a modified Rayleigh-Plesset equation in which liquid compressibility effects are included. The parameters used are those of Matula *et al*: $R_0 = 3$ μm, $f = 19.5$ kHz, $\sigma = 0.072$ kg/s^2, $C_l = 1490$ m/s, $g = -9.81$ m/s^2, $\mu = 0.001$ kg/m.s^2, $\rho = 1000$ kg/m^3 and different values of the amplitude of the acoustic forcing p_a. The results are illustrated on figures 3 and 4. The equilibrium positions relative to the antinode of the bubble are then obtained from our model. We find 79 μm, 59 μm, 49 μm for pressure amplitudes of 1.4 bar, 1.3 bar and 1.2 bar respectively, comparable to the positions found by the inviscid theory but still far from the experimental measurements (Matula *et al.*, 1997a). Notice that the amplitude of the relative vertical displacement of the bubble is very small (of the order of 1 nanometer) as predicted by the linear theory (figure 4). Velocities however may be large for a short time (up to 1 m/s). Full Navier-Stokes simulations (which however include no liquid compressibility effects) confirm this result, showing no sign of any significant translational motion.

This suggests that the Bjerknes force and its effect on the bubble translational motion is not as significant as originally anticipated. The exact effect of this translational motion on bubble shape depends on the coupling with shape modes of the bubble. While this is currently under investigation, one has to consider the possibility that this translational motion, which leads to displacements a thousand times smaller than the bubble radius, has no significant effect on the bubble shape.

In particular, in view of this hypothesis it would be unlikely that the Bjerknes force could explain the variation of the intensity of the emitted light found when doing sonoluminescence experiments in microgravity (Matula *et al.*, 1996; Matula *et al.*, 1997b). As far as the spherical stability of the bubble is concerned (including the eventual jet formation), other effects such as inhomogeneities of the sound field, deviation from spherical geometry in the experiments etc... would be more significant.

CONCLUSION

We have developed a simple model for the translational momentum of a bubble in an acoustic field. An analytical solution both for the amplitude of the bubble translational oscillation and for the equilibrium position has been found in the case of small radial and translational motion. The result regarding the amplitude of the bubble motion has been validated using both numerical solutions of the model equations and axisymmetric Navier-Stokes simulations. The model equation result for the equilibrium position of the bubble gives viscous corrections to the previous inviscid models (Crum and Prosperetti, 1983). However these corrections are too small to account for the discrepancy found for the bubble equilibrium position between theory and measurement (Matula *et al.*, 1997a). This discrepancy remains a mystery that could be investigated by further full Navier-Stokes simulations.

It has been shown that in the parameter range where sonoluminescence has been observed, the translational motion of the bubble due to the pressure gradient is small and may have only a minor role (if any) to play as far as spherical stability is con-

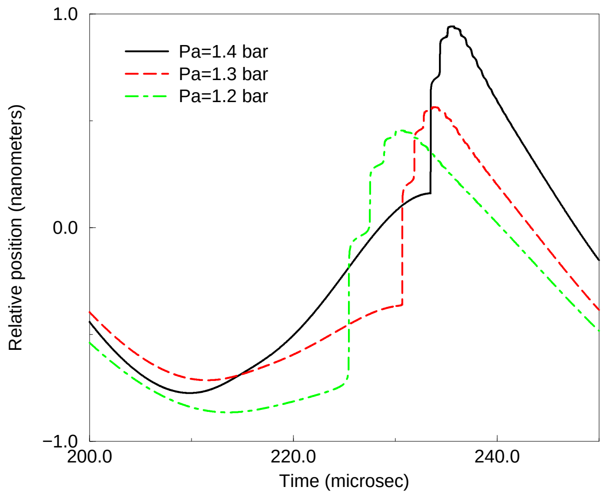

Figure 4. NUMERICAL SOLUTION OF THE COUPLED SYSTEM OF EQUATIONS (4) PLUS THE RAYLEIGH-PLESSET EQUATION FOR PARAMETERS USED IN (Matula *et al.*, 1997a). RELATIVE POSITION OF THE CENTER OF GRAVITY OF THE BUBBLE VERSUS TIME.

cerned. However, when considering larger bubbles (and consequently lower driving frequencies) this motion will become significant and should be taken into account if considering sonoluminescence upscaling (Hilgenfeldt and Lohse, 1999).

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