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CYCLICITY OF NON VANISHING FUNCTIONS IN THE POLYDISC AND IN THE BALL

ERIC AMAR AND PASCAL J. THOMAS

Abstract. We use a special version of the Corona Theorem in several variables, valid when all but one of the data functions are smooth, to generalize to the polydisc and to the ball results obtained by El Fallah, Kellay and Seip about cyclicity of non vanishing bounded holomorphic functions in large enough Banach spaces of analytic functions determined either by weighted sums of powers of Taylor coefficients or by radially weighted integrals of powers of the modulus of the function.

1. Introduction

The Hardy space can be seen as a space of square integrable functions on the circle with vanishing Fourier coefficients for the negative integers, a space of holomorphic functions on the unit disk, or the space of complex valued series with square summable moduli, and the interaction between those viewpoints has generated a long and rich history of works in harmonic analysis, complex function theory and operator theory.

The present work aims at generalizing one particular aspect of this to several complex variables: the study of cyclicity of some bounded holomorphic functions under the shift operator in large enough Banach spaces containing the Hardy space.

1.1. Definitions.

Definition 1. *Let $\omega : \mathbb{N}^d \longrightarrow (0,\infty)$, where $d \in \mathbb{N}^*$, and $p \geq 1$. We define the Banach space of power series in several variables*

$$X_{\omega,p} := \left\{ f(z) := \sum_{I\in\mathbb{N}^d} a_I z^I : \|f\|^p_{X_{\omega,p}} := \sum_{I\in\mathbb{N}^d} \left(\frac{|a_I|}{\omega(I)}\right)^p < \infty \right\},$$

with the usual multiindex notation, $z = (z_1,\dots,z_d) \in \mathbb{C}^d$, $I = (i_1,\dots,i_d) \in \mathbb{N}^d$, $z^I := z_1^{i_1}\cdots z_d^{i_d}$.

We also write $|I| := i_1 + \cdots + i_d$, $I! := i_1!\cdots i_d!$. We say that ω is *nondecreasing* if for any I, J, $\omega(I+J) \geq \omega(J)$.

Recall that domains of convergence of power series are logarithmically convex complete Reinhardt domains (for a definition and those terms and proofs, see e.g. [9], [8]). In what follows, we shall restrict our attention to the cases of the polydisc $\mathbb{D}^d := \{z \in \mathbb{C}^d : \max_{1\le j\le d} |z_j| < 1\}$ and the unit ball $\mathbb{B}^d := \{z \in \mathbb{C}^d : \sum_{1\le j\le d} |z_j|^2 < 1\}$. The letter Ω will stand for either one of those two domains, except in the more general Theorem 11.

If $\omega(I) = 1$ for any I, then we obtain the Hardy space $H^2(\mathbb{D}^d)$, which can also be described as the set of functions in the Nevanlinna class of the polydisc with boundary values (radial limits a.e.) on the torus $(\partial\mathbb{D})^d$ which are in $L^2((\partial\mathbb{D})^d)$, and

$$\|f\|^2_{H^2(\mathbb{D}^d)} = \sum_{I\in\mathbb{N}^d} |a_I|^2 = \frac{1}{(2\pi)^d}\int_{(\partial\mathbb{D})^d} |f|^2 d\theta_1 \dots d\theta_d.$$

The standard references for Hardy spaces on polydiscs is [11].

There is a Hardy space for $\mathbb{B}^d$, which is most easily described as as the set of functions in the Nevanlinna class of the ball with boundary values (radial limits a.e.) on the sphere $\partial\mathbb{B}^d$ which are in $L^2(\partial\mathbb{B}^d)$, and

$$\|f\|^2_{H^2(\mathbb{B}^d)} = \int_{\partial\mathbb{B}^d} |f|^2 d\sigma,$$

where σ is the $(2d-1)$-real dimensional Lebesgue measure normalized so that $\sigma(\partial\mathbb{B}^d) = 1$. The standard reference for function theory on the unit ball is [12]. Lemma 12 gives a description of $H^2(\mathbb{B}^d)$ in terms of the coefficients in the Taylor expansion.

Definition 2. *We set* $\omega_2^{\mathbb{D}^d}(J) := 1$, *and*

$$\omega_2^{\mathbb{B}^d}(J) := \frac{1}{\|z^J\|_{H^2(\mathbb{B}^d)}} = \left(\frac{(|J|+d-1)!}{(d-1)!J!}\right)^{1/2}.$$

We sometimes use the notation $\omega_2(J)$ (without superscript) when either of those quantities is meant.

We still have to understand in what sense a power series f can be understood as a function of $z \in \Omega$. We will want to consider weights which satisfy the following relative monotonicity condition : there exists a constant $C_m \ge 1$ such that, for any $I, J \in \mathbb{N}^d$,

$$C_m\omega(I+J) \ge \omega(J)\omega_2^{\Omega}(I). \tag{1}$$

One can check that $\omega = \omega_2^{\mathbb{B}^d}$ itself verifies condition (1).

When $\Omega = \mathbb{D}^d$, then $\omega_2^{\Omega}(I) = 1$ and if $C_m = 1$, we recover the usual monotonicity. We observe that for the polydisc, we can reduce ourselves to the case $C_m = 1$.

Lemma 3. *Let $\Omega = \mathbb{D}^d$. If ω satisfies Condition* (1)*, then $X_{\omega,p}$ admits an equivalent norm given by the* nondecreasing *weight*

$$\hat{\omega}(I) := \inf_{J\in\mathbb{N}^d} \omega(I+J).$$

Proof. Since $0 \in \mathbb{N}^d$ and we have (1), $1 \geq \frac{\hat{\omega}(I)}{\omega(I)} \geq C_m^{-1}$, and

$$\hat{\omega}(I+K) = \inf_{J\in\mathbb{N}^d} \omega(I+K+J) = \inf_{J\in K+\mathbb{N}^d} \omega(I+J) \geq \inf_{J\in\mathbb{N}^d} \omega(I+J) = \hat{\omega}(I).$$

□

Since the new norm is equivalent to the original one, the problem is unchanged and there is no loss of generality in assuming that ω has been modified and made nondecreasing, and we shall do so henceforth.

Lemma 4. *Let $\Omega = \mathbb{D}^d$ or $= \mathbb{B}^d$.*

If ω verifies the relative monotonicity condition (1) *and*

$$\log \omega_2^{\Omega}(I) \leq \log \omega(I) \leq \log \omega_2^{\Omega}(I) + o(|I|), \tag{2}$$

for any $f \in X_{\omega,p}$, the series defining f converges on Ω, and the map $X_{\omega,p} \ni f \mapsto f(z)$ is continuous with respect to the norm $\|\cdot\|_{X_{\omega,p}}$. In particular, $X_{\omega,p}$ can be seen as a subset of the space $\mathcal{H}(\Omega)$ of holomorphic functions on Ω.

Furthermore, there is no larger domain on which every $f \in X_{\omega,p}$ has to be holomorphic.

This lemma will be proved in Section 2.

In the ball case, consider λ a probability measure on $[0,1)$.

Definition 5. *The* radially weighted Bergman space *associated to λ is*

$$\mathcal{B} = \mathcal{B}^p(\lambda) = \mathcal{B}^p(\lambda)(\mathbb{B}^d)$$
$$:= \left\{ f \in \mathcal{H}(\mathbb{B}^d) : \|f\|_p^p := \int_0^1 \int_{\partial\mathbb{B}^d} |f(r\zeta)|^p \, d\sigma(\zeta) d\lambda(r) < \infty \right\}.$$

Typical examples are provided by $d\lambda(r) = c_\alpha (1-r^2)^\alpha r^{2d-1} dr$, where $\alpha > -1$ and c_α is an appropriate normalizing constant; they correspond to a weight $c_\alpha(1 - \sum_{1\leq j\leq d} |z_j|^2)^\alpha$, $z \in \mathbb{B}^d$.

Let λ be a probability measure on $[0,1)^d$, the elements of which are denoted $r := (r_1, \ldots, r_d)$. The torus $(\partial\mathbb{D})^d$ is endowed with its normalized Haar measure denoted by $d\theta$.

Definition 6. *The* weighted Bergman space *associated to λ is*

$$\mathcal{B} = \mathcal{B}^p(\lambda) = \mathcal{B}^p(\lambda)(\mathbb{D}^d)$$
$$:= \left\{ f \in \mathcal{H}(\mathbb{D}^d) : \|f\|_p^p := \int_{[0,1)^d} \int_{\mathbb{T}^d} \left| f(r_1 e^{i\theta_1}, \ldots, r_d e^{i\theta_d}) \right|^p d\theta d\lambda(r) < \infty \right\}.$$

Let $H^\infty(\Omega)$ stand for the set of bounded holomorphic functions on Ω. In each case, the conditions on λ ensure that $H^\infty(\Omega) \subset \mathcal{B}^p(\lambda)$.

Note that the norms of the monomials are given by moments of the measure λ. In the case where $\Omega = \mathbb{D}^d$,

$$\|z^I\|_p^p = \int_{[0,1)^d} r^{pI} d\lambda(r),$$

so that $\log(\|z^I\|_p^{-1})$ is a concave function of I.

When $p = 2$, $\mathcal{B}^2(\lambda)$ is a Hilbert space and the monomials z^I form an orthogonal system. Notice that in $X_{\omega,2}$, $\|z^I\|_{\omega,2} = \omega(I)^{-1}$, so that

$$\mathcal{B}^2(\lambda)(\mathbb{D}^d) = X_{\omega,2} \text{ with } \omega(I) = \left(\int_{[0,1)^d} r^{2I} d\lambda(r)\right)^{-1/2}.$$

In the case where $\Omega = \mathbb{B}^d$,

$$\|z^I\|_p^p = \left(\int_0^1 r^{p|I|} d\lambda(r)\right)\left(\int_{\partial\mathbb{B}^d} |\zeta^I|^p d\sigma(\zeta)\right).$$

When $p = 2$, since the surface measure $d\sigma$ on $\partial\mathbb{B}^d$ desintegrates as an integral of Haar measures on tori, the monomials (z^J) again form an orthogonal system, and in this case

$$\|z^I\|_2^2 = \left(\int_0^1 r^{2|I|} d\lambda(r)\right) \omega_2^{\mathbb{B}^d}(J)^{-2},$$

so that

$$\mathcal{B}^2(\lambda)(\mathbb{B}^d) = X_{\omega,2} \text{ with } \omega(I) = \left(\int_0^1 r^{2|I|} d\lambda(r)\right)^{-1/2} \omega_2^{\mathbb{B}^d}(J).$$

In general, whenever we consider a space X, we define the corresponding weight by $\omega(J) := 1/\left\|z^J\right\|_X$.

1.2. **Main results.** Let X be a Banach space as above, defined by power series or as a weighted Bergman space.

Definition 7. *We say that a function $f \in X$ is* cyclic *if for any $g \in X$, there exists a sequence of holomorphic polynomials (P_n) such that $\lim_{n\to\infty} \|g - P_n f\|_X = 0$.*

Note that using the word "cyclic" is a slight abuse of language, since for $d \geq 2$ we are not iterating a single operator, but taking compositions of the multiplication operators by each of the coordinate functions $z_1, \ldots, z_d$. It is, however, a straightforward generalization of the usual notion of cyclicity under the shift operator $f(z) \mapsto zf(z)$.

By Lemma 4 in the case of power series spaces, or by the mean value inequality in the case of Bergman spaces, the point evaluations are

continuous, therefore any cyclic f must verify that $f(z) \neq 0$ for any $z \in \Omega$.

Definition 8. *(1) When $\Omega = \mathbb{D}^d$, for any $k \in \mathbb{N}$, let*

$$\frac{1}{\tilde{\omega}(k)} := \sum_{j=1}^{d} \|z_j^k\|_X = \sum_{j=1}^{d} \|z^{ke_j}\|_X,$$

where (e_j) stands for the elementary multiindices of $\mathbb{N}^d$: $e_1 = (1, 0, \ldots, 0)$, $e_2 = (0, 1, 0, \ldots, 0)$, etc, so $ke_j = (0, \ldots, 0, k, 0, \ldots, 0)$, with k in the j-th place.

(2) When $\Omega = \mathbb{B}^d$, for any $k \in \mathbb{N}$, let

$$\frac{1}{\tilde{\omega}(k)} := \sum_{J,\ |J|=k} \frac{p_J}{\omega(J)},$$

where p_J is the multinomial coefficient, $p_J := \frac{|J|!}{J!}$.

When $X = X_{\omega,p}$ and $\Omega = \mathbb{D}^d$, notice that

$$d^{-1} \min_{1 \leq j \leq d} \omega(ke_j) \leq \tilde{\omega}(k) \leq \min_{1 \leq j \leq d} \omega(ke_j).$$

Note that if ω satisfies (2) then $\log \tilde{\omega}(k) = o(k)$, but the converse does not hold when $d > 1$.

Here are two interesting special cases of our results.

Theorem 9. *Let $\Omega := \mathbb{D}^d$ or $\mathbb{B}^d$.*

Suppose that $\lim_{k\to\infty} \tilde{\omega}(k) = \infty$, and ω satisfies (1) and (2), and that

$$\sum_{k \geq 1} \left(\frac{\log \tilde{\omega}(k)}{k} \right)^2 = \infty. \tag{3}$$

Let $U \in H^\infty(\Omega)$, verifying $U(z) \neq 0$ for any $z \in \Omega$.

- *(i) If $d \geq 1$, $p \geq 1$ and $X = \mathcal{B}^p(\lambda)$, then U is cyclic in X.*
- *(ii) If $\Omega = \mathbb{D}^d$ and $X = X_{\omega,2}$, then U is cyclic in X.*

When we demand a growth condition of a slightly stronger nature on $\tilde{\omega}$, we can expand the range of spaces where the result applies.

Theorem 10. *Let $X = X_{\omega,p}$, with $p \geq 2$, or $X = \mathcal{B}^p(\lambda)$. If $\lim_{k\to\infty} \tilde{\omega}(k) = \infty$, and ω satisfies (1), (2) and*

$$\limsup_{k} \frac{\log \tilde{\omega}(k)}{\sqrt{k}} = \infty, \tag{4}$$

then any zero-free $U \in H^\infty(\Omega)$ is cyclic in X.

1.3. **Previous results.** Many results have been proved for the case $d = 1$, and even more for $p = 2$. In one dimension, $\Omega = \mathbb{D}$ and $\omega = \tilde{\omega}$ of course. When furthermore $p = 2$, $X_{\omega,2}$ has a norm equivalent to that of a Bergman space if and only if $\log \omega(n)$ is a concave function of n [3, Theorem A.2 and Proposition 4.1].

In his seminal monograph [10], N. K. Nikolski proved that if ω is non-decreasing, $\lim_{k\to\infty} \omega(k) = \infty$, $\log \omega(k) = o(k)$, $\log \omega(n)$ is a concave function of n and

$$\sum_{k\geq 1} \frac{\log \tilde{\omega}(k)}{k^{3/2}} = \infty, \tag{5}$$

then any zero-free $f \in H^\infty(\mathbb{D})$ is cyclic in $X_{\omega,2}$.

Our main inspiration comes from [5], where O. El Fallah, K. Kellay and K. Seip show, still for $d = 1$ and $p = 2$, that (3), with no condition of concavity, is enough to imply cyclicity of any nonvanishing bounded function. Even though (3) is a stronger condition than (5), the concavity condition means that there exist weights to which the new result applies while Nikolski's cannot [5, Remark 2].

The novelty in the present work is of course that we have several variables, and exponents $p \neq 2$. We also notice that it is not necessary to make use of the inner-outer factorization: the much easier Harnack inequality suffices.

1.4. **A Corona-like Theorem.** As in [5], our main tool is a version of the Corona Theorem. In full generality, this is still a vexingly open question in several variables, be it in the ball or the polydisc. However, following an earlier result of Cegrell [4], a simpler proof [1] gives a Corona-type result in the special case where most of the given generating functions are smooth. That result is enough to yield the required estimates in this instance. For Ω a bounded domain in $\mathbb{C}^d$, let $A^1(\Omega) := \mathcal{H}(\Omega) \cap \mathcal{C}^1(\overline{\Omega})$.

Theorem 11. *Let Ω be a bounded pseudoconvex domain in $\mathbb{C}^d$, such that the equation $\bar{\partial} u = \omega$, $1 \leq q \leq n$, admits a solution $u = S_q\omega \in L^\infty_{(0,q-1)}(\Omega)$ when $\bar{\partial}\omega = 0$, $\omega \in L^\infty_{(0,q)}(\Omega)$, with the bounds :*

$$\|u\|_\infty \leq E_q \|\omega\|_\infty.$$

There exists a constant $C = C(d, \Omega)$ such that if $N \in \mathbb{N}$, $N \geq 2$, and if $f_j \in A^1(\Omega)$, $1 \leq j \leq N-1$, $f_N \in H^\infty(\Omega)$, verify

$$\sup_{z\in\Omega} \max_{1\leq j\leq N} |f_j(z)| \leq 1, \quad \inf_{z\in\Omega} \sum_{j=1}^{N} |f_j(z)| \geq \delta > 0,$$

then there exist $g_1, \dots, g_N \in H^\infty(\Omega)$ *such that* $\sum_{j=1}^N f_j(z)g_j(z) = 1$ *and for* $1 \le j \le d$,

$$\max_{1\le j\le N} \|g_j\|_\infty \le C(d,\Omega)N^{4d+2}\frac{\max_{1\le j\le N-1}\|\nabla f_j\|_\infty^d}{\delta^{2d+1}}.$$

Note that the polydisc and the ball verify the hypotheses of the theorem.

1.5. **Structure of the paper.** First we clarify the easy relationship between weights and domains of convergence in Section 2. Then we gather some preliminary results and a first reduction of the problem in Section 3. Theorem 11 is proved in Section 6, and used in the proofs of the two main theorems. The relatively easy proof of Theorem 10 is given in Section 4. Theorem 9 will follow from a more general and more technical result, Theorem 19, which is stated and proved in Section 5.

2. Domains of convergence

Lemma 12. *Let* f *be holomorphic on the unit ball* $\mathbb{B}^d$*, represented by the Taylor expansion* $f(z) = \sum_J a_J z^J$*. Then* $f \in H^2(\mathbb{B}^d)$ *if and only if*

$$\sum_{J\in\mathbb{N}^d}\left(\frac{|a_J|}{\omega_2^{\mathbb{B}^d}(J)}\right)^2 < \infty, \text{ where } (\omega_2^{\mathbb{B}^d}(J))^{-2} = \frac{(d-1)!J!}{(|J|+d-1)!}.$$

Proof. The surface measure $d\sigma$ on $\partial\mathbb{B}^d$ desintegrates as an integral of Haar measures on tori, so the monomials (z^J) form an orthogonal system in $H^2(\mathbb{B}^d)$, which is a basis since the polynomials are dense in the space. Then $\|f\|^2_{H^2(\mathbb{B}^d)} = \sum_J |a_J|^2\|z^J\|^2_{H^2(\mathbb{B}^d)}$. The explicit value of $\|z^J\|_{H^2(\mathbb{B}^d)} = (\omega_2^{\mathbb{B}^d}(J))^{-1}$ (Definition 2) can be found in [12, p. 12]. □

As an immediate consequence of this Lemma and of the remarks before Definition 2, if $X_{\omega,p}$ is as in Definition 1 and $p \ge 2$, and if for all J, $\omega_2^\Omega(J) \le \omega(J)$, then $H^2(\Omega) \subset X_{\omega,p}$.

Definition 13. *For* $z \in \mathbb{C}^d$*, let* $|z|_{\mathbb{D}^d} := \max_{1\le j\le d}|z_j|$, $|z|^2_{\mathbb{B}^d} := \sum_{1\le j\le d}|z_j|^2$.

In each case, $\Omega = \{z : |z|_\Omega < 1\}$.

Proof of Lemma 4. The last statement follows from the fact that (1) implies that $H^2(\Omega) \subset X_{\omega,p}$, as in the remark before the Definition.

To prove the convergence of the series, write $f(z) = \sum_J a_J z^J$ and take a point z such that $|z|_\Omega = \rho < 1$. Since $X_{\omega,p} \subset X_{\omega,\infty}$, $|a_J| \preceq \omega(J)$ and it will be enough to prove the convergence of $\sum_J \omega(J)|z^J|$.

In the case where $\Omega = \mathbb{D}^d$, then

$$\log(\omega(J)|z^J|) \le |J|\log\rho + o(|J|) \le -\eta|J|$$

for some $\eta > 0$ when $|J|$ is large enough, so the general term is dominated by the general term of a convergent geometric (multi)series.

In the case where $\Omega = \mathbb{B}^d$, for any $k \in \mathbb{N}$,

$$|z|_\Omega^{2k} = \sum_{J:|J|=k} \frac{|J|!}{J!} |z^{2J}|.$$

First consider only sums of terms with all powers even:

$$\sum_{J:|J|=k} \omega(2J)|z^{2J}| \le \sup_{J:|J|=k} \left(\frac{\omega(2J)J!}{|J|!} \right) \sum_{J:|J|=k} \frac{|J|!}{J!} |z^{2J}|$$
$$\le \sup_{J:|J|=k} \left(\frac{\omega(2J)J!}{|J|!} \right) \rho^{2k},$$

so using (2), we need to estimate $\frac{\omega_2(2J)^2(J!)^2}{(|J|!)^2}$.

Stirling's formula implies that for any $n \in \mathbb{N}$,

$$\log(n!) = n(\log n - 1) + o(n),$$

so we have, for $|J| = k$,

$$\log\left(\frac{\omega_2(2J)^2(J!)^2}{(|J|!)^2} \right) = \log\left(\frac{(2k+d-1)!(J!)^2}{(d-1)!(2J)!(k!)^2} \right) =$$
$$= (2k+d-1)\left(\log(2k+d-1) - 1\right) - 2k(\log k - 1)$$
$$+ 2\sum_{i=1}^d j_i(\log j_i - 1) - \sum_{i=1}^d 2j_i(\log(2j_i) - 1) + o(k)$$
$$= 2k\log 2 - 2\sum_{i=1}^d j_i \log 2 + 2k\left(\log(k + \frac{d-1}{2}) - \log k \right) + o(k)$$
$$= 0 + O(1) + o(k) = o(k).$$

This proves that $\sum_{J:|J|=k} \omega(2J)|z^{2J}|$ is dominated by the general term of a convergent geometric series for k large enough.

Now consider a general J such that $|J| = k$: then $J = 2J' + K$, with $j_i' = 2\,[j_i/2]$, and $K \in \{0,1\}^d$. Let $2J'' := J + K$. Condition (1) shows that $\omega(J) \asymp \omega(2J') \asymp \omega(2J'')$, $|z^J| \le |z^{2J'}|$, and each J' corresponds to at most 2^d different multi-indices J. So $\sum_{J:|J|=k} \omega(J)|z^J|$ is dominated by the general term of a convergent geometric series for k large enough.

Continuity of the evaluation map follows, for instance, from the Dominated Convergence Theorem applied to the series. □

Lemma 14. *Suppose that $H^\infty(\Omega)$ is a multiplier space for X; or that $X = X_{\omega,p}$, with $p \ge 2$ and ω verifying* (1)*. Let $g \in H^\infty(\Omega)$, $K \in \mathbb{N}^d$.*

Then

$$\|z^K g\|_{X_{\omega,p}} \leq \frac{C_m}{\omega(K)} \|g\|_{H^\infty(\Omega)}.$$

Observe that in the special case $K = 0$, $X = X_{\omega,p}$, we get back the fact that $H^2(\Omega) \subset X_{\omega,p}$.

Proof. Under the first assumption, we immediately have

$$\|z^K g\|_X^p \leq \|z^K\|_{X_{\omega,p}}^p \|g\|_{H^\infty(\Omega)} = \frac{1}{\omega(K)} \|g\|_{H^\infty(\Omega)}.$$

Under the second assumption, by scaling we may assume $1 = \|g\|_{H^2(\Omega)} \leq \|g\|_{H^\infty(\Omega)}$. Let $g(z) = \sum_J a_J z^J$. Then $\sup_J \frac{|a_J|^2}{\omega_2(J)^2} \leq 1$. Then

$$\begin{aligned}\|z^K g\|_{X_{\omega,p}}^p &= \sum_J \frac{|a_J|^p}{\omega(J+K)^p} \leq \left(\sup_J \frac{\omega_2(J)}{\omega(J+K)}\right)^p \sum_J \frac{|a_J|^p}{\omega_2(J)^p} \\ &\leq \frac{C_m^p}{\omega(K)^p} \sum_J \frac{|a_J|^2}{\omega_2(J)^2} = \frac{C_m^p}{\omega(K)^p} \leq \frac{C_m^p}{\omega(K)^p} \|g\|_{H^\infty(\Omega)}^p.\end{aligned}$$

□

3. Auxiliary results

3.1. **Multiplier property.**

Definition 15. *We shall say that $H^\infty(\Omega)$ is a* multiplier algebra *for X if there exists $C_m > 0$ such that*

$$\forall f \in X, \forall g \in H^\infty(\Omega), gf \in X \text{ and } \|gf\|_X \leq C_m \|g\|_\infty \|f\|_X.$$

Notice that, since constants are in X, this implies that $H^\infty(\Omega) \subset X$.

It is immediate that $H^\infty(\Omega)$ is a multiplier algebra for each $\mathcal{B}^p(\lambda)(\Omega)$, with $C_m = 1$. In the case of $X_{\omega,p}$, writing $\omega_\infty^\Omega(I) := \|z^I\|_{L^\infty(\Omega)}^{-1}$, an obvious necessary condition is that

$$C_m \omega(I+J) \geq \omega_\infty^\Omega(I)\omega(J), \tag{6}$$

but sufficient conditions are not so easy to state in general.

Observe that (6) is very similar to (1). In fact, $\omega_\infty^{\mathbb{D}^d}(I) = \omega_2^{\mathbb{D}^d}(I) = 1$ for all I, and one can show that

$$\omega_2^{\mathbb{B}^d}(I) \geq \omega_\infty^{\mathbb{B}^d}(I) \geq \omega_2^{\mathbb{B}^d}(I) - O(\log|I|),$$

by an appropriate minoration of $|z^I|$ on a strip of $\partial\mathbb{B}^d$ of width comparable to $|I|$ around its maximum modulus set (we omit the details; this can provide an alternate proof of Lemma 4 without recourse to Stirling's formula).

3.2. **Some tools.** Our first technical tool is a bound from below for the modulus of a zero-free bounded holomorphic function.

For $z \in \Omega$, $z^* := z/|z|_\Omega \in \partial\Omega$, where $|z|_\Omega$ is as in Definition 13.

Lemma 16. *Let U be a zero-free holomorphic function on Ω such that $\|U\|_\infty \leq 1$, and $z \in \Omega$. Let $c^2 := \log \frac{1}{|U(0)|}$. Suppose $k \geq 4c^2$. Then, for $\Omega = \mathbb{D}^d$,*

$$|U(z)| + |z_1|^k + \cdots + |z_d|^k \geq e^{-2c\sqrt{k}},$$

and for $\Omega = \mathbb{B}^d$,

$$|U(z)| + \sum_{|J|=k} |f_J(z)| \geq e^{-2c\sqrt{2k}},$$

where $f_J(z) := p_J z^{2J} = \frac{|J|!}{J!} z^{2J}$.

Proof. The conclusion is obvious if $z = 0$. If not, define a holomorphic function on $\mathbb{D}$ by $f_{z^*}(\zeta) := U(\zeta z^*)$. Then

- $\|f_{z^*}\|_\infty \leq 1$;
- $f_{z^*}(0) = U(0)$;
- $\forall \zeta \in \mathbb{D},\ f_{z^*}(\zeta) \neq 0$;
- $f_{z^*}(|z|_\Omega) = U(z)$.

The Harnack inequality applied to the positive harmonic function $\log\ |f_{z^*}|^{-1}$ shows that

$$|f_{z^*}(\zeta)| \geq \exp\left(-\frac{1+|\zeta|}{1-|\zeta|} \log\ \frac{1}{|U(0)|}\right) \geq \exp\left(-\frac{2}{1-|\zeta|} \log\ \frac{1}{|U(0)|}\right).$$

The computation implicit at the beginning of the proof of [5, Lemma 3] shows that $\inf_{\mathbb{D}} |f_{z^*}(\zeta)| + |\zeta|^k \geq e^{-2c\sqrt{k}}$ as soon as $k \geq 4c^2$; applying this to $\zeta = |z|_\Omega$, we find

$$|U(z)| + |z|_\Omega^k \geq f_{z^*}(|z|_\Omega) + |z|_\Omega^k \geq e^{-2c\sqrt{k}}.$$

In the case where $\Omega = \mathbb{D}^d$, this yields

$$|U(z)| + |z_1|^k + \cdots + |z_d|^k \geq |U(z)| + (\max_{1\leq j\leq d} |z_j|)^k \geq e^{-2c\sqrt{k}}.$$

In the case where $\Omega = \mathbb{B}^d$, substituting $2k$ for k, we obtain

$$|U(z)| + (|z_1|^2 + \cdots + |z_d|^2)^k = |U(z)| + \sum_{|J|=k} |f_J(z)| \geq e^{-2c\sqrt{2k}}.$$

□

Lemma 17. *If $X = X_{\omega,p}$ or $\mathcal{B}^p(\lambda)$ from Definitions 1 or 6 or 5 respectively, then the space of polynomials $\mathbb{C}[Z] := \mathbb{C}[Z_1, \ldots, Z_d]$ is dense in X.*

Proof. By construction, the polynomials are dense in $X_{\omega,p}$.

For $f \in \mathcal{B}^p(\lambda)(\mathbb{D}^d)$, $r = (r_1, \dots, r_d) \in [0,\infty)^d$, let

$$m_{r,p}(f) := \int_{\mathbb{T}^d} \left|f(r_1 e^{i\theta_1}, \dots, r_d e^{i\theta_d})\right|^p d\theta$$

denote the mean of $|f|^p$ on the torus $\mathbb{T}(r)$ of multiradius r. Since $|f|^p$ is plurisubharmonic, this is an increasing function with respect to each component of r. In particular, if we set for any $\gamma \in (0,1)$, $f_\gamma(z) := f(\gamma z)$, $m_{r,p}(f_\gamma) \le m_{r,p}(f)$ for each r.

We claim that $\lim_{\gamma\to 1} \|f - f_\gamma\|_{\mathcal{B}^p(\lambda)} = 0$. Indeed, $\|f - f_\gamma\|_{\mathcal{B}^p(\lambda)} = \int_{[0,1)^d} F_\gamma(r) d\lambda(r)$, where

$$F_\gamma(r) := \int_{\mathbb{T}^d} \left|f(r_1 e^{i\theta_1}, \dots, r_d e^{i\theta_d}) - f_\gamma(r_1 e^{i\theta_1}, \dots, r_d e^{i\theta_d})\right|^p d\theta.$$

Since $|f - f_\gamma|^p \le C_p(|f|^p + |f_\gamma|^p)$,

$$F_\gamma(r) \le C_p(m_{r,p}(f) + m_{r,p}(f_\gamma)) \le 2C_p m_{r,p}(f) \in L^1(d\lambda).$$

Since $f_\gamma \to f$ uniformly on the torus $\mathbb{T}(r)$ for each r as $\gamma \to 1$, $F_\gamma(r) \to 0$ for each r, and we can apply Lebesgue's Dominated Convergence theorem.

For each $\gamma \in (0,1)$, f_γ is holomorphic on a larger polydisc, so can be uniformly approximated by truncating its Taylor series.

When $\Omega = \mathbb{B}^d$, we can perform an analogous (and simpler) argument. □

3.3. **First reduction.** We begin by showing that it is enough to obtain a relaxed version of the conclusion.

Lemma 18. *Let $U \in H^\infty(\Omega)$ be a non-vanishing function.*

If either:

- *(i) $H^\infty(\Omega)$ is a multiplier algebra for X,*
- *or (ii) $X = X_{\omega,p}$, $p \ge 2$ and (1) is satisfied,*

and if there exists a sequence $(f_n) \subset H^\infty(\Omega)$ such that

$$\lim_{n\to\infty} \|1 - f_n U\|_X = 0,$$

then U is cyclic in X.

Proof. By Lemma 17, it is enough to show that we can approximate any polynomial P.

Let us show that it is enough to prove that for any $\varepsilon > 0$, there exists $Q \in \mathbb{C}[Z]$ such that $\|1 - QU\|_X \le \varepsilon$.

Let $P(z) := \sum_{|J|\le N} a_J z^J$, then

$$\|P - PQU\|_X = \|P(1 - QU)\|_X \le \|P\|_\infty \|1 - QU\|_X$$

in the case of assumption (i), and

$$\|P(1-QU)\|_X \leq \sum_{|J|\leq N} |a_J| \|z^J(1-QU)\|_X \leq C_m \left(\sum_{|J|\leq N} \frac{|a_J|}{\omega_2^\Omega(J)} \right) \|1-QU\|_X,$$

in the case of assumption (ii), and each upper bound can be made arbitrarily small by choosing Q.

In the case of assumption (i), let us then show that the constant function 1 can be approximated. Let $\varepsilon > 0$. Take $f \in H^\infty(\mathbb{D}^d)$ such that $\|1 - fU\|_X < \varepsilon/2$. By Lemma 17, we can choose $Q \in \mathbb{C}[Z]$ such that $\|f - Q\|_X \leq \frac{1}{C_m \|U\|_\infty} \frac{\varepsilon}{2}$, where C_m is as in Definition 15. Then

$$\|1-QU\|_X \leq \|1-fU\|_X + \|U(f-Q)\|_X < \frac{\varepsilon}{2} + C_m \|U\|_\infty \|f-Q\|_X \leq \varepsilon.$$

In the case of assumption (ii), again take f so that $\|1 - fU\|_{\omega,p}$ is small, then because $H^2(\Omega) \subset X_{\omega,p}$,

$$\|fU - QU\|_{\omega,p} \leq C\|fU - QU\|_{H^2} \leq C\|U\|_\infty \|f - Q\|_{H^2},$$

and this last quantity can be made arbitrarily small by taking Q a Taylor expansion of f for instance. □

4. Proof of Theorem 10

Proof of Theorem 10.

Observe that if $X = \mathcal{B}^p(\lambda)$, then $H^\infty(\Omega)$ is a multiplier algebra, so Lemma 14 always applies here.

Case 1: $\Omega = \mathbb{D}^d$.

Let $c^2 := -\log\ |U(0)|$ and $B > 2c(2d+1)$. By the hypothesis of the theorem, there exists a strictly increasing sequence $(n_k)_{k\geq 1}$ such that for all k, $\log \tilde{\omega}(n_k) \geq B\sqrt{n_k}$.

By Lemma 16 and Theorem 11, we get $g_j \in H^\infty(\mathbb{D}^d)$, for $j = 1, \ldots, d+1$, such that

$$g_{d+1}U + g_1 z_1^{n_k} + \cdots + g_d z_d^{n_k} = 1, \tag{7}$$

and

$$\forall j = 1, \ldots,\ d+1,\ \|g_j\|_\infty \leq C(d) n_k^d e^{2c(2d+1)\sqrt{n_k}}. \tag{8}$$

Set $f_k := g_{d+1}$, we get, using Lemma 14,

(9)

$$\|1 - f_k U\|_X \leq \sum_{j=1}^d \left\| g_j z_j^{n_k} \right\|_X \leq C_m \sum_{j=1}^d \frac{\|g_j\|_\infty}{\omega(n_k e_j)} \leq C_m \frac{C(d) n_k^d e^{2c(2d+1)\sqrt{n_k}}}{\tilde{\omega}(n_k)}.$$

By the choice of B, this tends to 0 as $k \to \infty$. It only remains to apply lemma 18 to conclude.

Case 2: $\Omega = \mathbb{B}^d$.

Let c, (n_k) be as above and $B > 2c\sqrt{2}(2d+1)$.

By Theorem 11, we will get $g_0, g_J \in H^\infty(\mathbb{D}^d)$, for $|J| = n_k$, such that

$$g_0 U + \sum_{|J|=n_k} g_J f_J \equiv 1, \tag{10}$$

where f_J is as in Lemma 16.

We need to estimate the size of the g_J, g_0.

First $\sum_{|J|=n_k} |f_J(z)| = |z|_\Omega^{2n_k} < 1$.

The number of terms in the Bezout equation is

$$N = N(d,k) = \#\left\{J \in \mathbb{N}^d : |J| = n_k\right\} = \frac{(n_k+d-1)!}{n_k!(d-1)!} \leq n_k^{d-1}.$$

We also need $\|\nabla f_J\|_\infty$. We have

$$\frac{\partial}{\partial z_i} f_J = p_J 2j_i \frac{z^{2J}}{z_i} \Rightarrow \nabla f_J = p_J z^{2J}(\frac{2j_1}{z_1}, ..., \frac{2j_d}{z_d}) = 2f_J(z)(\frac{j_1}{z_1}, ..., \frac{j_d}{z_d}).$$

If we set $\tilde{J} := (\max(0, 2j_1 - 1), ..., \max(0, 2j_d - 1))$ and $\tilde{z}_i := z_1 \cdots z_{i-1}\hat{z}_i z_{i+1} \cdots z_d$, where $\hat{z}_i$ is omitted, then

$$\nabla f_J(z) = 2p_J z^{\tilde{J}}(j_1\tilde{z}_1, ..., j_d\tilde{z}_d).$$

So we get, because $|\tilde{z}_i| \leq 1$ in the ball,

$$|\nabla f_J(z)| \leq 2p_J \left|z^{\tilde{J}}\right| \sum_{i=1}^d j_i |\tilde{z}_i| \leq 2p_J \left|z^{\tilde{J}}\right| |J|.$$

But if we write $J' := ((j_1-1)_+, \ldots, (j_d-1)_+)$, then $|z^{\tilde{J}}| \leq |z^{2J'}|$ and $\sum_{|J'|=n_k-d} p_{J'}|z^{2J'}| < 1$. Furthermore,

$$p_J \leq \frac{n_k(n_k-1)\cdots(n_k-d+1)}{j_1\cdots j_d} p_{J'} \leq n_k^d p_{J'}.$$

All together then, $\|\nabla f_J(z)\|_\infty \leq C(d) n_k^{d+1}$.

By Lemma 16 (in the case of the ball),

$$\delta = \inf_{z\in\mathbb{B}^d} \left(|U(z)| + \sum_{|J|=n_k} |f_J(z)| \right) \geq e^{-2c\sqrt{2n_k}}.$$

Gathering the estimates, we get

$$\|g_J\|_\infty \leq C(d)N(d,k)^{4d+2} e^{2c(2d+1)\sqrt{2n_k}} \leq C(d) n_k^{5d^2} e^{2c(2d+1)\sqrt{2n_k}}. \tag{11}$$

Then let $f_k = g_0$ (at the n_k step)

$$\|1 - f_k U\|_X \leq \sum_{|J|=n_k} \|g_J f_J\|_X \leq C_m \sum_{|J|=n_k} p_J \|z^{2J}\|_X \|g_J\|_\infty \leq C_m \frac{C(d) n_k^{5d^2} e^{2c(2d+1)\sqrt{2n_k}}}{\tilde{\omega}(n_k)}, \tag{12}$$

and we finish as before. □

5. Proof of Theorem 9

5.1. Main intermediate result.

Theorem 19. *Let X be a Banach space as in Definitions 1, 5 or 6. Suppose that $H^\infty(\mathbb{D}^d)$ is a multiplier algebra for X. Suppose also that $\lim_{k\to\infty} \tilde{\omega}(k) = \infty$, that $\log \tilde{\omega}(k) = o(k)$, and that conditions (1), (2) and (3) hold.*

Then any $U \in H^\infty(\mathbb{D}^d)$, verifying $U(z) \neq 0$ for any $z \in \mathbb{D}^d$ is cyclic in X.

Proof. Now we need to distinguish two cases according to the growth of $\omega(k)$.

Case 1: $\sup_k \frac{\log \tilde{\omega}(k)}{\sqrt{k}} = \infty$.

Then Theorem 10 applies.

Case 2: $\sup_k \frac{\log \tilde{\omega}(k)}{\sqrt{k}} = B < \infty$. To deal with this more delicate case, we shall need the full power of the proof scheme in [5]. Since our Corona-like estimates are slightly different from those in dimension 1, we first need a refined version of [5, Lemma 1].

Lemma 20. *Let $\tilde{\omega}$ be as in Theorem 19. Let $C_0 > 0$. Then there exists a strictly increasing sequence $(n_k)_{k\geq 1}$ such that*

$$\sum_{k\geq 1} \frac{(\log \tilde{\omega}(n_k))^2}{n_k} = \infty,$$

and for all k, $\log \tilde{\omega}(n_{k+1}) \geq 2 \log \tilde{\omega}(n_k)$ and $\log \tilde{\omega}(n_k) \geq C_0 \log n_k$.

The last condition is the only novelty with respect to [5, Lemma 1].

Proof. First notice that there exists an infinite set $E \subset \mathbb{N}^*$ such that for all $n \in E$, $\log \tilde{\omega}(n) \geq C_0 \log n$. Indeed, if not, for n large enough, we would have

$$\log \tilde{\omega}(n) \leq C_0 \log n \leq n^{1/4},$$

and (3) would be violated.

Now let $n_0 = 1$ and if n_j is defined, let

$$n'_{j+1} := \min\left\{n > n_j : \log\tilde{\omega}(n) \geq 2\log\tilde{\omega}(n_j)\right\},$$
$$n_{j+1} := \min\left\{n > n_j, n \in E : \log\tilde{\omega}(n) \geq 2\log\tilde{\omega}(n_j)\right\}.$$

Obviously, $n_j < n'_{j+1} \leq n_{j+1}$. We claim that

$$S := \sum_{j\geq 0}\sum_{k=n'_j}^{n_j-1}\left(\frac{\log\tilde{\omega}(k)}{k}\right)^2 < \infty.$$

Accepting the claim, the proof finishes as in [5]:

$$\sum_{k\geq 1}\left(\frac{\log\tilde{\omega}(k)}{k}\right)^2 \leq S + \sum_{j\geq 0}\sum_{k=n_j}^{n'_{j+1}-1}\left(\frac{\log\tilde{\omega}(k)}{k}\right)^2$$
$$\leq S + \sum_{j\geq 0} 4\left(\log\tilde{\omega}(n_j)\right)^2\sum_{k=n_j}^{n'_{j+1}-1}\frac{1}{k^2} \leq S + 4\sum_{j\geq 0}\frac{\left(\log\tilde{\omega}(n_j)\right)^2}{n_j - 1},$$

so the last sum must diverge.

We now prove the claim. If $n'_j \leq k < n_j$, then $n \notin E$, so for j large enough and $n'_j \leq k < n_j$, $\log\tilde{\omega}(k) \leq k^{1/4}$, thus

$$\sum_{k=n'_j}^{n_j-1}\left(\frac{\log\tilde{\omega}(k)}{k}\right)^2 \leq \sum_{k\geq n'_j}\frac{1}{k^{3/2}} \leq \frac{2}{\sqrt{n'_j} - 1}. \tag{13}$$

The definition of n_j implies that $\tilde{\omega}(n_j) \geq C_0 2^j$, and $n'_{j+1} \notin E$ (if it is distinct from n_{j+1}) so

$$C_0\log n'_{j+1} > \log\tilde{\omega}(n'_{j+1}) \geq 2\log\tilde{\omega}(n_j) \geq C_0 2^j,$$

and the series with general term the last expression in (13) must converge. □

We follow the proof of [5, Theorem 1], with a couple of wrinkles.

Choose $A := \max(2, \log C(d))$ where $C(d)$ is the constant in (8) when $\Omega = \mathbb{D}^d$ (resp. (11) when $\Omega = \mathbb{B}^d$). Then for $c^2 = -\log|U(0)|$ as above, let $C_1^2 := (8\sqrt{2}(2d+1)Ac)^2 + B^2$. We choose $C_0 \geq C_1/c$ when $\Omega = \mathbb{D}^d$ (resp. $C_0 \geq \frac{5d^2C_1}{2\sqrt{2}(2d+1)c}$ when $\Omega = \mathbb{B}^d$), and define the sequence (n_j) as in Lemma 20 above. For any given $j_0 \in \mathbb{N}$, let $\alpha_j^2 := \frac{\left(\log\tilde{\omega}(n_{j_0+j})\right)^2}{n_{j_0+j}}$,

$$N := \min\left\{M : \sum_{j=1}^{M}\alpha_j^2 \geq (8\sqrt{2}(2d+1)Ac)^2\right\},$$

$\lambda_j := \alpha_j \left(\sum_{i=1}^N \alpha_i^2\right)^{-1/2}$.

Notice that for any j, $\alpha_j \le B$ by the hypothesis of Case 2, and that

$$\sum_{i=1}^N \alpha_i^2 \le \sum_{i=1}^{N-1} \alpha_i^2 + \alpha_N^2 \le (8\sqrt{2}(2d+1)Ac)^2 + B^2 = C_1^2,$$

so that $\lambda_j \ge \alpha_j/C_1$. Clearly, $\lambda_j \le \alpha_j/(8\sqrt{2}(2d+1)Ac)$.

We write $U_j := U^{\lambda_j^2}$, so that $U = \prod_{j=1}^N U_j$. As above, choose $f_j := g_{d+1}$ satisfying (7) and (8), but with U_j instead of U and n_{j_0+j} instead of n_k. The quantity c must then be replaced by $c\lambda_j$.

When $\Omega = \mathbb{D}^d$, the bound (8) can be rewritten

$$\|f_j\|_\infty \le \exp\left(2c(2d+1)\lambda_j\sqrt{n_{j_0+j}} + d\log n_{j_0+j} + \log C(d)\right). \tag{14}$$

Notice that

$$c\lambda_j\sqrt{n_{j_0+j}} \ge \frac{c}{C_1}\log\tilde{\omega}(n_{j_0+j}) \ge \frac{cC_0}{C_1}\log n_{j_0+j} \ge \log n_{j_0+j},$$

by our choice of C_0, so that

$$\|f_j\|_\infty \le \exp A\left(2c(2d+1)\lambda_j\sqrt{n_{j_0+j}} + 1\right). \tag{15}$$

We finish as in [5]. Let $f := \prod_{j=1}^N f_j$. Since

$$1 - fU = 1 - \prod_{j=1}^N f_jU_j = \sum_{k=1}^N (1 - U_kf_k)\prod_{j=1}^{k-1} f_jU_j,$$

$$\|1 - fU\|_X \le C_m \sum_{k=1}^N \|1 - U_kf_k\|_X \prod_{j=1}^{k-1} \|f_jU_j\|_\infty,$$

which by (9) becomes

$$\begin{aligned} &\le C_m \sum_{k=1}^N \frac{C(d)n_{j_0+k}^d e^{2c(2d+1)\sqrt{n_{j_0+k}}}}{\tilde{\omega}(n_{j_0+k})} \prod_{j=1}^{k-1} \|f_j\|_\infty \\ &\le C_m \sum_{k=1}^N \frac{1}{\tilde{\omega}(n_{j_0+k})} \exp\left(A\sum_{j=1}^k (2c(2d+1)\lambda_j\sqrt{n_{j_0+j}} + 1)\right), \end{aligned} \tag{16}$$

and using the growth of $\log \tilde{\omega}(n_j)$ obtained in Lemma 20,

$$\leq C_m \sum_{k=1}^{N} \frac{1}{\tilde{\omega}(n_{j_0+k})} \exp\left(Ak + \sum_{j=1}^{k} \frac{1}{4} \log \tilde{\omega}(n_{j_0+j}) \right)$$
$$\leq C_m \sum_{k=1}^{N} \exp\left(Ak - \frac{1}{2} \log \tilde{\omega}(n_{j_0+k}) \right).$$

Now choose j_0 such that $\log \tilde{\omega}(n_{j_0}) \geq A$, the sum above has terms with better than geometric decrease, so is bounded by $\tilde{\omega}(n_{j_0+1})^{-1/2}$, which can be made arbitrarily small by choosing j_0 large enough.

When $\Omega = \mathbb{B}^d$, we need to make the changes indicated at the beginning of the argument, and replace the bound (14) by the following:

$$\|f_j\|_\infty \leq \exp\left(2c\sqrt{2}(2d+1)\lambda_j \sqrt{n_{j_0+j}} + 5d^2 \log n_{j_0+j} + \log C(d) \right).$$

Then the choice (for $\Omega = \mathbb{B}^d$) of C_0 implies that $2c\sqrt{2}(2d+1)\lambda_j \sqrt{n_{j_0+j}} \geq 5d^2 \log n_{j_0+j}$, and this leads again to (15). In the succession of majorations that follow, (16) becomes

$$\leq C_m \sum_{k=1}^{N} \frac{C(d) n_{j_0+k}^{5d^2} e^{2c\sqrt{2}(2d+1)\sqrt{n_{j_0+k}}}}{\tilde{\omega}(n_{j_0+k})} \prod_{j=1}^{k-1} \|f_j\|_\infty$$
$$\leq C_m \sum_{k=1}^{N} \frac{1}{\tilde{\omega}(n_{j_0+k})} \exp\left(A \sum_{j=1}^{k} (2c(2d+1)\lambda_j \sqrt{n_{j_0+j}} + 1) \right),$$

and the proof concludes in the same way. □

5.2. **Proof of Theorem 9.** We now obtain cyclicity results as soon as we can prove that $H^\infty(\mathbb{D}^d)$ is a multiplier algebra on the space X. As remarked after Definition 15, this is always the case when $X = \mathcal{B}^p(\lambda)$. So we obtain Theorem 9 (i).

When $X = X_{\omega,2}$ and ω is relatively nondecreasing, then Lemma 3 reduces us to the nondecreasing case, where the multiplication operators by each z_j are commuting contractions on a Hilbert space. Von Neumann's inequality was generalized by Ando in the case of two contractions, and to an arbitrary number of weighted shifts by Michael Hartz [6]: this is precisely our situation. It implies that for any polynomial f, and thus for any $f \in H^\infty(\mathbb{D}^d)$, $\|fg\|_X \leq \|f\|_\infty \|g\|_X$. So we obtain Theorem 9 (ii).

6. Proof of the Corona theorem with smooth data

We begin by constructing a partition of unity which exploits the smoothness of the data.

Because of the corona hypothesis, and f_j is continuous up to the boundary of Ω, for $1 \leq j \leq N-1$, we have that $g(z) := \sum_{j=1}^{N-1} |f_j(z)|$ is continuous in Ω, and even Lipschitz with a constant controlled by $\max_{1\leq j\leq N-1} \|\nabla f_j\|_\infty$.

Set

$$U'_N := \{z \in \bar{\Omega} : g(z) < \frac{N-1}{4N}\delta\} \text{ and } U_N := \{z \in \bar{\Omega} : g(z) < \frac{N-1}{2N}\delta\},$$

and

$$U_j := \{z \in \bar{\Omega} : |f_j| > \frac{\delta}{5N}\} \text{ and } U'_j := \{z \in \bar{\Omega} : |f_j| > \frac{\delta}{4N}\}.$$

Then $U'_j \Subset U_j$, $1 \leq j \leq N$.

Lemma 21. *There exist $C_1 > 0$ and $\chi_j \in \mathcal{C}_c^\infty(U_j)$, $j = 1, ..., N$, such that for $z \in \Omega$, $0 \leq \chi_j \leq 1$, $\sum_{j=1}^N \chi_j(z) = 1$, and*

$$\left|\frac{\chi_j}{f_j}\right| \leq \frac{C_1 N}{\delta}, \ \|\nabla \chi_j\|_\infty \leq \frac{C_1 N^2}{\delta} \max_{1\leq i\leq N-1} \|\nabla f_i\|_\infty, j = 1, \ldots, N,$$

$$\max_{1\leq j\leq N} \sup_{z\in\Omega} \frac{|\nabla \chi_j(z)|}{|f_j(z)|} \leq C_1 \frac{N^3}{\delta^2} \max_{1\leq i\leq N-1} \|\nabla f_i\|_\infty,$$

where C_1 is an absolute constant.

Proof. We can construct a function $\psi_N \in \mathcal{C}_c^\infty(U_N)$ such that $0 \leq \psi_N \leq 1$ and $\psi_N \equiv 1$ on U'_N, with $\|\nabla \psi_N\|_\infty \leq \frac{C}{\delta}$, for instance by composing $|g|$ with an appropriate smooth one-variable function.

We have

$$\mathcal{O} := \bar{U}'_N \cup \bigcup_{j=1}^{N-1} U'_j \supset \bar{\Omega},$$

because for $z \notin \bigcup_{j=1}^{N-1} U'_j$, then

$$\forall j = 1, ..., N-1, \ |f_j(z)| \leq \frac{\delta}{4N} \Rightarrow \sum_{j=1}^{N-1} |f_j(z)| \leq \frac{N-1}{4N}\delta \Rightarrow z \in \bar{U}'_N.$$

Now we construct a partition of unity $\{\chi_j\}_{j=1,\ldots,N}$ subordinated to $\{U_j\}$ in the usual way: we take a nonnegative function $\psi_j \in \mathcal{C}_c^\infty(U_j)$ such that $\psi_j \leq 1$ everywhere and $\psi_j \equiv 1$ on U'_j, with $\|\nabla \psi_j\|_\infty \leq C\frac{N}{\delta}\|\nabla f_j\|_\infty$. We set

$$\chi_j := \frac{\psi_j}{\sum_{k=1}^N \psi_k}.$$

Since $\sum_{k=1}^{N}\psi_k \geq 1$, we have $0 \leq \chi_j \leq 1$, $\chi_j \in \mathcal{C}_c^\infty(U_j)$ and $\chi_1 + \cdots + \chi_N = 1$ on $\bar{\Omega}$ and

$$\|\nabla \chi_j\|_\infty \leq C\frac{N^2}{\delta}\max_{1\leq i\leq N-1}\|\nabla f_i\|_\infty.$$

This yields a partition of unity such that $\frac{\chi_j}{f_j} \in \mathcal{C}^\infty(\bar{\Omega})$ for $1 \leq j \leq N$ and for $j \leq N-1$, $\left|\frac{\chi_j}{f_j}\right| \leq \frac{5N}{\delta}$, because supp $\chi_j \subset U_j$, where $|f_j| > \frac{\delta}{5N}$ and $\chi_j \leq 1$.

For $j = N$ on the other hand, we have supp $\chi_N \subset U_N$ and, by the corona hypothesis,

$$z \in U_N \Rightarrow |f_N(z)| \geq \delta - g(z) = \delta - \frac{N-1}{2N}\delta = \frac{N+1}{2N}\delta$$

hence

$$\left|\frac{\chi_N}{f_N}\right| \leq \frac{2N}{(N+1)\delta} \leq \frac{5N}{\delta}.$$

An analogous reasoning yields the bound on $\frac{|\nabla\chi_j|}{|f_j|}$, $1 \leq j \leq N$. □

Proof of Theorem 11. We shall now go through the Koszul complex method, introduced in this context by Hörmander [7], to obtain the explicit bounds we need. We follow the notations of [2].

Let $\wedge^k(\mathbb{C}^N)$ be the exterior algebra on $\mathbb{C}^N$, let e_j, $j = 1, ..., N$, be the canonical basis of $\wedge^1(\mathbb{C}^N)$, and $e_\alpha := e_{\alpha_1} \wedge \cdots \wedge e_{\alpha_k}$, $\alpha_j \in \{1, \ldots, N\}$, the associated basis of $\wedge^k(\mathbb{C}^N)$.

Let L_r^k be the space of bounded and infinitely differentiable differential forms in Ω of type $(0, r)$ with values in $\wedge^k(\mathbb{C}^N)$. The norm on these spaces is defined to be the maximum of the uniform norms of the coefficients.

We define two linear operators on L_r^k.

$$\forall \omega \in L_r^k, \quad R_f(\omega) := \omega \wedge \sum_{j=1}^{N}\frac{\chi_j}{f_j}e_j \in L_r^{k+1}.$$

We see that $\|R_f\omega\| \leq C_f\|\omega\|$, with

$$C_f := N\sup_{1\leq j\leq N, z\in\Omega}\left|\frac{\chi_j(z)}{f_j(z)}\right|. \tag{17}$$

The operator $d_f : L_r^{k+1} \longrightarrow L_r^k$ is defined by induction and linearity. For $\omega \in L_r^0$, $d_f\omega = 0$. To define the operator on L_r^1, set $d_f(e_j) := f_j$ and extend by linearity.

To define d_f on L_r^{k+1}, for $e_\alpha \in \wedge^k(\mathbb{C}^N)$, $1 \leq j \leq d$, set

$$d_f(e_\alpha \wedge e_j) := f_j e_\alpha - d_f(e_\alpha) \wedge e_j \in L_r^k.$$

It follows that $\|d_f\|_{\mathcal{L}(L_r^{k+1},L_r^k)} \leq C(k)\max_{1\leq j\leq N}\|f_j\|_\infty$.

It is easily seen by induction that $d_f^2=0$, $\bar\partial d_f\omega = d_f\bar\partial\omega$ and

$$d_f\omega = 0 \Rightarrow d_f(R_f\omega)=\omega,$$

i.e. $\lambda = R_f\omega$ is a solution to the equation $d_f\lambda=\omega$ when the necessary condition $d_f\omega=0$ is verified.

Together with the operator $\bar\partial : L_r^k \longrightarrow L_{r+1}^k$, we have a double complex, whose elementary squares are commutative diagrams.

We now construct by induction, for $0\leq k\leq N$, forms $\omega_{k,l}\in L_l^k$ and $\alpha_{k,l}\in L_l^{k+1}$, where $l\leq k\leq l+1$.

We start with $\omega_{0,0}=1$,

$$\omega_{1,0} := R_f(\omega_{0,0}) = \sum_{j=1}^N \frac{\chi_j}{f_j}e_j \in L_0^1.$$

Then, if $\omega_{k,k-1}$ is given, we set $\omega_{k,k} := \bar\partial\omega_{k,k-1}$; if $\omega_{k,k}$ is given, we set $\omega_{k+1,k} := R_f\omega_{k,k}$. This construction stops for $k=d$ since there are no $(0,d+1)$ forms on $\mathbb{C}^d$.

Claim. For any $k\geq 0$, $d_f\omega_{k+1,k}=\omega_{k,k}$.

We prove the claim by induction. It is enough to see that $d_f\omega_{k,k}=0$. For $k=0$, this is true by construction. For $k\geq 1$, assume the property holds at rank $k-1$. Then

$$d_f\omega_{k,k} = d_f\bar\partial\omega_{k,k-1} = \bar\partial d_f\omega_{k,k-1} = \bar\partial\omega_{k-1,k-1} = \bar\partial^2\omega_{k-1,k-2}=0.$$

From the construction, we have $\|\omega_{k+1,k}\|\leq C_f\|\omega_{k,k}\|$, with C_f defined in (17). Since

$$\omega_{k,k} = \bar\partial(R_f\omega_{k-1,k-1}) = \bar\partial\left(\omega_{k-1,k-1}\wedge\sum_{j=1}^N\frac{\chi_j}{f_j}e_j\right) = \omega_{k-1,k-1}\wedge\bar\partial\left(\sum_{j=1}^N\frac{\chi_j}{f_j}e_j\right)$$

because $\omega_{k-1,k-1}$ is $\bar\partial$-exact, we find $\|\omega_{k,k}\|\leq D_f'\|\omega_{k-1,k-1}\|$, with

$$D_f' := N\sup_{1\leq j\leq N, z\in\Omega}\frac{\|\nabla\chi_j(z)\|}{|f_j(z)|}.$$

By an immediate induction, $\|\omega_{k,k}\|\leq (D_f')^k$, $\|\omega_{k+1,k}\|\leq C_f(D_f')^k$.

We proceed with the construction of the forms $\alpha_{k,l}$, by descending induction. Set $\alpha_{d+2,d}=\alpha_{d+1,d}=0$. Since $\bar\partial\omega_{d+1,d}=0$ by degree reasons, there exists $u\in L_{d-1}^{d+1}$ such that $\bar\partial u=\omega_{d+1,d}$, and $\|u\|\leq E_d\|\omega_{d+1,d}\|$. We set $\alpha_{d+1,d-1}=u$.

Suppose given $\alpha_{k+1,k}=d_f\alpha_{k+2,k}$, with $\bar\partial\omega_{k+1,k}-\bar\partial\alpha_{k+1,k}=0$ (this is trivially verified when $k=d$). Then the hypothesis on Ω implies that

there exists $u \in L^{k+1}_{k-1}$ such that

$$\|u\| \leq E_k \|\omega_{k+1,k} - \alpha_{k+1,k}\| \text{ and } \bar\partial u = \omega_{k+1,k} - \alpha_{k+1,k}.$$

Then we set $\alpha_{k+1,k-1} := u$.

Finally, we put $\alpha_{k,k-1} := d_f \alpha_{k+1,k-1}$. We need to check the condition on $\bar\partial$:

$$\bar\partial \alpha_{k,k-1} = d_f \bar\partial \alpha_{k+1,k-1} = d_f \left(\omega_{k+1,k} - d_f \alpha_{k+2,k}\right) = d_f \omega_{k+1,k} = \omega_{k,k} = \bar\partial \omega_{k,k-1}.$$

The following diagram, where $\mathcal{S}$ stands for the operator solving the $\bar\partial$ equation, describes the whole complex for $n = 2$, $N = 3$.

$$\begin{tikzcd}
 & \alpha_{3,1} \arrow[d, "d_f"'] & & \\
 & & \omega_{3,2} \arrow[ul, "S"'] \arrow[r, "\bar\partial"] & 0 \\
\alpha_{2,0} \arrow[d, "d_f"'] & \alpha_{2,1} \arrow[ul, "S"'] \arrow[dr, "\bar\partial"] & & \\
 & \omega_{2,1} \arrow[r, "\bar\partial"] & \omega_{2,2} \arrow[uu, "R_f"] \arrow[r, "\bar\partial"] \arrow[d, "d_f"] & 0 \\
\alpha_{1,0} \arrow[dr, "\bar\partial"] & & & \\
 & \omega_{1,1} \arrow[r, "\bar\partial"] \arrow[d, "d_f"] & 0 & \\
\omega_{1,0} - \alpha_{1,0} \arrow[dr, "d_f"'] & \omega_{1,0} \arrow[r, "\bar\partial"] & & \\
 & \omega_{0,0} \arrow[r, "\bar\partial"] \arrow[uu, "R_f"] & 0 &
\end{tikzcd}$$

The bounds on the solution of the Cauchy-Riemann equation $\bar\partial$ and those on $\omega_{k,l}$ imply that

$$\|\alpha_{k+1,k-1}\| \leq E_k \left(C_f (D'_f)^k + \|d_f\| \|\alpha_{k+2,k}\|\right),$$

from which we deduce by induction

$$\|\alpha_{k+1,k-1}\| \leq C_f \sum_{j=k}^{d-1} (D'_f)^j \|d_f\|^{j-k} \prod_{i=k}^{j} E_j + \|d_f\|^d \left(\prod_{j=k}^{d-1} E_j\right) \|\alpha_{d+1,d-1}\|,$$

so taking into account the bound $\|\alpha_{d+1,d-1}\| \leq E_d \|\omega_{d+1,d}\|$, we have for any k

$$\|\alpha_{k+1,k-1}\| \leq C(d) \|f\|^d_\infty \left(\prod_{j=1}^{d} E_j\right) C_f (D'_f)^d.$$

Finally, we claim that a solution to the Bezout equation is given by the components of $\gamma_{1,0} := \omega_{1,0} - \alpha_{1,0} =: \sum_{j=1}^N g_j e_j$.

Indeed, $\bar\partial(\alpha_{1,0} - \omega_{1,0}) = 0$, so the coefficients of $\gamma_{1,0}$ are holomorphic functions, and

$$\sum_{j=1}^N g_j f_j = d_f(\gamma_{1,0}) = d_f\left(\omega_{1,0} - d_f\alpha_{2,0}\right) = d_f(\omega_{1,0}) = \omega_{0,0} = 1.$$

The bound on the g_j follows from the bounds on $\|\alpha_{1,0}\|$ and $\|\omega_{1,0}\|$ and Lemma 21, which gives $C_f \le C\frac{N^2}{\delta}$, $D'_f \le C\frac{N^4}{\delta^2}$. □

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