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Universal time-dependent dispersion properties for diffusion in a one-dimensional critically tilted potential

T. Guérin and D. S. Dean

Laboratoire Ondes et Matière d'Aquitaine (LOMA), CNRS, UMR 5798/Université de Bordeaux, F-33400 Talence, France

We consider the time-dependent dispersion properties of overdamped tracer particles diffusing in a one-dimensional periodic potential under the influence of an additional constant tilting force F . The system is studied in the region where the force is close to the critical value F_c at which the barriers separating neighboring potential wells disappear. We show that, when F crosses the critical value, the shape of the mean-square displacement (MSD) curves is strongly modified. We identify a diffusive regime at intermediate-time scales with an effective diffusion coefficient which is much larger than the late-time diffusion coefficient for $F > F_c$, whereas for $F < F_c$ the late-time and intermediate-time diffusive regimes are indistinguishable. Explicit asymptotic regimes for the MSD curves are identified at all time scales.

I. INTRODUCTION

In a variety of physical systems, the motion of tracer particles can be described by Fokker-Planck equations or their associated stochastic differential or Langevin equations [1–3]. In such systems, the motion results from the combined action of deterministic, the so called drift, and stochastic forces whose amplitudes are given by a diffusivity or local diffusion constant. In a periodic heterogeneous medium in which the local transport coefficients are constant in time but spatially periodic, the late-time large scale dynamics can be characterized by a mean velocity and an effective diffusion tensor which characterize the mean drift and the spatial extension of a cloud of initially close tracer particles. These effective transport coefficients can be widely different from typical microscopic ones and are important in a number of phenomena, such as mixing, pollutant spreading, or chemical reactions [4–8].

The late-time effective transport coefficients have theoretically been characterized in a number of systems. For example, they can be calculated in the case of tracer particles diffusing in incompressible hydrodynamic flows [9–14] or in porous media [15–20] using homogenization theory. In another context, dispersion properties were derived using methods of statistical physics for particles diffusing in periodic potentials with uniform molecular diffusivity [21–26]. The problem of dispersion in one-dimensional (1D) systems also was investigated at length [27–39]. A remarkable prediction in the case of diffusion in a one-dimensional tilted potential is a huge increase in the effective diffusivity when the tilting force approaches a critical value [27], a phenomenon observed in various experiments [40–43] which was recently used to estimate the energy barrier opposing the steps of a rotary molecular motor [44]. More recently, analytical results demonstrated how an external force can influence dispersion in periodic systems in higher dimensions [45].

These results, however, characterize only the late-time effective diffusivity, whereas dispersion can also be characterized by the time-dependent mean-square displacement (MSD) which is routinely measured for example in single particle tracking experiments. The general theory developed in Ref. [45] shows that, starting from steady state initial

conditions, the average drift is independent of the time whereas the MSD actually evolves in time, eventually attaining the late-time diffusive limit. The approach to the diffusive limit has been calculated in equilibrium systems [46,47]. Apart from approximate forms of the distribution of particles [48,49] in tilted potentials, little is known about the temporal evolution of dispersion coefficients in general nonequilibrium periodic media.

Recently, a very general formalism was proposed to calculate the MSD in a wide class of nonequilibrium systems [45,50]. This formalism examines the full temporal behavior of dispersion, and here we use it to analytically calculate the time-dependent MSD of overdamped particles diffusing in a 1D tilted periodic potential. We derive exact and explicit asymptotic expressions for the dispersion at various time scales in the case where the external force is close to its critical value (at which the barriers between successive potential wells disappear). In earlier studies [27,29], it has been shown that, when the tilting force is close to its critical value, the tracer particles spend most of their time in a very narrow window of positions and thus the effective diffusivity takes a universal form depending only on the properties of the potential near these positions. Here we will see that this property carries over to the time-dependent MSD, which admits simple and universal forms at various time scales.

The outline of the paper is as follows. The model is briefly introduced in Sec. II. In Sec. III, we derive a formula for the MSD in terms of first-passage time (FPT) densities. This formula is analyzed asymptotically in Sec. IV where we show that the shape of the MSD curves show a remarkable change when F crosses its critical value. Our predictions are in excellent agreement with results obtained by simulating the stochastic trajectories.

II. MODEL AND QUANTITIES OF INTEREST

We consider the motion of an overdamped tracer particle of position $X(t)$ at time t in a 1D space, moving in a periodic potential $V(x)$ (of period L with x as the spatial coordinate) and subject to an additional external *tilting* force F at finite temperature T [see Fig. 1(a)]. The overdamped dynamics

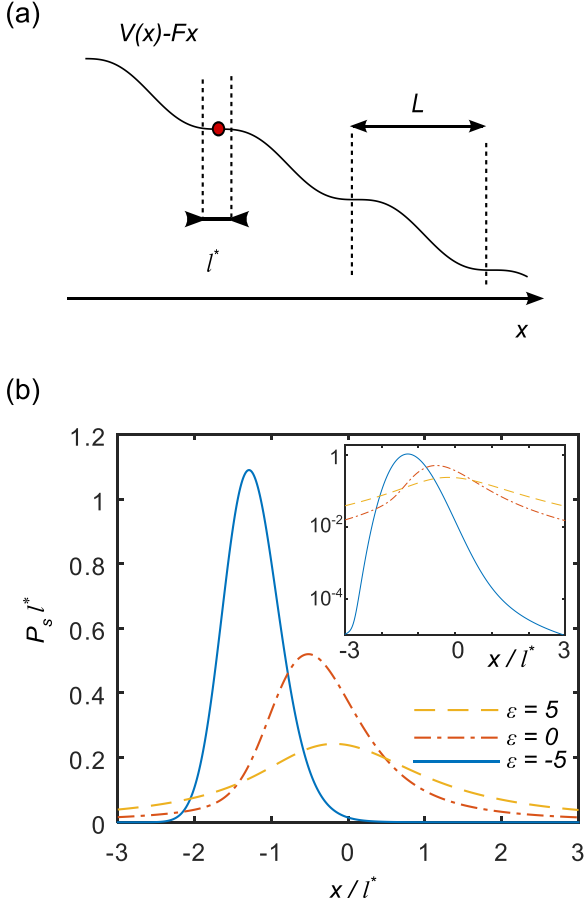


FIG. 1. (a) Sketch of the problem investigated in this paper: We calculate the dispersion properties of particles diffusing in periodic tilted potentials. When the noise is weak and the force is close to its critical value, the tracer particles spend most of their time in narrow regions, termed the slow regions, of typical size l^* much smaller than period L . (b) Value of the stationary PDF $P_s(x)$ inside the slow region obtained by evaluating Eqs. (19) and (20) for $\varepsilon = -5$ (F below the critical force F_c), $\varepsilon = 0$ ($F = F_c$), and $\varepsilon = 5$ ($F > F_c$). The inset: same curves on a semilogarithmic scale.

satisfies the force balance equation, which in Langevin form reads

$$\zeta \partial_t X(t) = -V'(X(t)) + F + \xi(t), \quad (1)$$

where ζ is the frictional drag coefficient and the thermal fluctuating forces $\xi(t)$ have zero mean white noise Gaussian statistics characterized by the correlation function $\langle \xi(t)\xi(t') \rangle = 2k_B T \zeta \delta(t - t')$. We denote $D_0 = k_B T / \zeta$ as the local molecular diffusivity of the tracer particle, and we define the drift field,

$$u(x) = \zeta^{-1} [-\partial_x V(x) + F]. \quad (2)$$

Equivalently, the process can be described by the Fokker-Planck equation,

$$\partial_t P = -\partial_x [u(x)P] + D_0 \partial_x^2 P, \quad (3)$$

where $P(x, t)$ is the probability density function of particles at positions x at time t .

In this paper, we aim to calculate the time-dependent dispersion quantified by the MSD function $\psi(t)$ defined as

$$\psi(t) = \langle [X(t) - X(0)]^2 \rangle - \langle X(t) - X(0) \rangle^2, \quad (4)$$

where $\langle \cdot \cdot \cdot \rangle$ denotes ensemble averaging over realizations of the white noise. Note that we assume here that the system has reached a steady state at time $t = 0$ (in the sense that the probability distribution over a unit cell of one period is the stationary one).

III. GENERAL EXPRESSION OF THE MEAN-SQUARE DISPLACEMENT

Our starting point is the following Kubo formula, derived in Refs. [45,50]:

$$\hat{\psi}(s) = \frac{2D_0}{s^2} - \frac{2}{s^2} \int_0^L dx \int_0^L dx_0 u(x) u^*(x_0) \times \left[\hat{P}(x, s | x_0) - \frac{P_s(x)}{s} \right] P_s(x_0). \quad (5)$$

Here, $\hat{\psi}(s)$ is the Laplace transform of the MSD, i.e., $\hat{\psi}(s) = \int_0^\infty \psi(t) e^{-st} dt$, whereas $P(x, t | x_0)$ is the propagator of the process mod L (with periodic boundary conditions), i.e., the probability density function for the tracer particle at $x \pmod L$ at t given an initial position x_0 (also mod L), and \hat{P} is its temporal Laplace transform. Moreover, $P_s(x) = \lim_{t \rightarrow \infty} P(x, t | x_0)$ is the probability density function of the position (mod L) in the steady state. Note that P_s is not the equilibrium-Boltzmann distribution which is only applicable for finite systems with reflecting or confining rather than periodic boundary conditions. Here it describes a nonequilibrium steady state and is characterized by a nonzero flux J_s , which has both a convective and a diffusive (Fickian) component,

$$J_s = u(x)P_s(x) - D_0 \partial_x P_s(x). \quad (6)$$

Note that J_s must be constant in a 1D problem. Finally, in Eq. (5), $u^*(x)$ represents the drift of the time-reversed stochastic process [50],

$$u^*(x) = u(x) - 2J_s / P_s(x). \quad (7)$$

The first step of the present analysis consists of expressing the MSD in terms of FPT densities rather than propagators; this will greatly simplify the asymptotic analysis in the next sections. Consider $f(x, t | x_0)$ as the probability density of reaching position $x \pmod L$ for the first time at t , starting from the initial position x_0 . The propagators and the FPT densities can be linked by the following, well known, *renewal* equation [1]:

$$P(x, t | x_0) = \int_0^t dt' f(x, t' | x_0) P(x, t - t' | x). \quad (8)$$

Physically, the above equation states that, if a particle reaches x at t , it also means that x was reached for the first time at some earlier instant t' and that the particle subsequently reached x again in a time $t - t'$. The Laplace transform of Eq. (8) is as follows:

$$\hat{P}(x, s | x_0) = \hat{f}(x, s | x_0) \hat{P}(x, s | x). \quad (9)$$

We also consider the FPT density averaged over initial conditions,

$$\bar{f}_s(x, t) \equiv \int_0^L dx_0 f(x, t | x_0) P_s(x_0), \quad (10)$$

and we remark that averaging (9) over the stationary distribution for x_0 gives

$$\frac{P_s(x)}{s} = \hat{f}_s(x, s) \hat{P}(x, s | x). \quad (11)$$

Using the above expressions, Eq. (5) for the MSD becomes

$$\hat{\psi}(s) = \frac{2D_0}{s^2} - \frac{2}{s^3} \int_0^L dx \int_0^L dx_0 u(x) u^*(x_0) P_s(x_0) P_s(x) \times \left[\frac{\hat{f}(x, s | x_0)}{\hat{f}_s(x, s)} - 1 \right]. \quad (12)$$

This expression for the MSD is well adapted for the asymptotic analysis of dispersion presented in the next section.

IV. DISPERSION IN A CRITICAL TILTED POTENTIAL AT DIFFERENT TIME SCALES

A. Regions of fast and slow motion

We now focus on the case where the external force is very close to the critical tilt force $F_c = \max[V'(x)]$ [Fig. 1(a)]. When $F \simeq F_c$, the late-time effective diffusivity varies as $D_0^{1/3}$ [27,29] and can therefore be much larger than the molecular diffusivity D_0 when the latter is small. Our goal here is to predict the approach to this diffusive limit, which is universal in the sense that it does not depend on the detailed shape of the potential. Following the notations of Refs. [27,29], we introduce the parameter μ defined as

$$\mu = -V'''(0)/6, \quad (13)$$

where we chose the origin of coordinates such that $V'(x)$ is maximum at $x = 0$. Note that we assume here that the potential admits a third derivative; more general cases could be treated with our approach but are not considered here for simplicity. We also introduce a parameter ε to measure the distance to the critical force,

$$\varepsilon = \frac{F - F_c}{\mu^{1/3} (k_B T)^{2/3}}. \quad (14)$$

The limit considered here is that of weak noise $k_B T \rightarrow 0$ while keeping the parameter ε constant. In this limit, since $D_0 \propto k_B T$, the convective terms dominate over the diffusive terms in Eq. (3) almost everywhere except in a narrow region of characteristic size l^* located around $x = 0$. In this region, the convective flux is $J_c = u P_s \simeq \zeta^{-1} \mu x^2 P_s$ whereas the diffusive flux is on the order of $J_d = D_0 \nabla P_s \simeq D_0 P_s / l^*$. The fluxes J_c and J_d are on the same order of magnitude when $x \sim l^*$ if we set the characteristic size l^* to

$$l^* = (k_B T / \mu)^{1/3}. \quad (15)$$

The characteristic time in the inner region is

$$\tau^* = \frac{(l^*)^2}{D_0} = \frac{\zeta}{\mu^{2/3} (k_B T)^{1/3}}. \quad (16)$$

The time τ^* diverges in the small temperature limit, and we therefore refer to the region around $x = 0$ as a *slow* region as opposed to the outer region called the *fast* region.

Let us briefly derive the stationary probability density and steady state flux; although these quantities are known in the literature, we show below that they play an important role in the dispersion properties. Since each particle spends most of its time in one of the slow regions, the steady state stationary density $P_s(x)$ of positions (mod L) is localized in these regions where it satisfies

$$-\zeta^{-1} \partial_x [(3\mu x^2 + F - F_c) P_s] + D_0 \partial_x^2 P_s = 0, \quad (17)$$

where we have used a Taylor expansion of the potential $V(x)$ at next to leading order around $x = 0$. We introduce the dimensionless position $y = x/l^*$, probability density $\tilde{P}_s(y) = P_s l^*$, and flux $\tilde{J}_s = \tau^* J_s$. Using these notations, integrating Eq. (17) and comparing with Eq. (6) leads to

$$-(3y^2 + \varepsilon) \tilde{P}_s + \partial_y \tilde{P}_s = -\tilde{J}_s. \quad (18)$$

Assuming that $\tilde{P}_s(y)$ vanishes for $y \rightarrow \infty$ (that is, for $x \gg l^*$), the solution of this equation is

$$\tilde{P}_s(y) = \tilde{J}_s \int_y^\infty du e^{-u^3 - \varepsilon u + y^3 + \varepsilon y}. \quad (19)$$

The normalization imposes that $1 = \int_{-L/2}^{L/2} dx P_s(x) \simeq \int_{-\infty}^\infty dy \tilde{P}_s(y)$ and thus leads to the identification of the dimensionless current,

$$\tilde{J}_s^{-1} = \int_{-\infty}^\infty dy \int_0^\infty du e^{-(u+y)^3 - \varepsilon u + y^3}. \quad (20)$$

The rescaled stationary PDF in the slow region is represented in Fig. 1(b) where one observes the transition from narrow distributions shifted in the region $x < 0$ when $\varepsilon < 0$ (that is, for forces $F < F_c$) to broader and more centered distributions when $F > F_c$. The dimensionless flux is represented in Fig. 2(a), it almost vanishes for $F < F_c$, and then significantly increases when $F > F_c$.

In the outer region $|x| \gg l^*$, the convective term takes over the diffusive term in Eq. (6), and we simply obtain

$$P_s(x) \simeq J_s / u(x), \quad (21)$$

physically this means that, in the regions where drift dominates, the probability of presence is inversely proportional to the speed. We note that the expressions in the outer and the inner regions [Eqs. (19) and (21)] can be matched in the region $l^* \ll |x| \ll L$ where both approximations give $P_s(x) \simeq J_s \zeta / (3\mu x^2 + F - F_c)$.

B. MSD at intermediate-time scales: ballistic and diffusive regimes

We now investigate the properties of the MSD $\psi(t)$ at different time scales. First, at very short times, $t \rightarrow 0$ (or $s \rightarrow \infty$), we have $\psi(t) = 2D_0 t$ where the effective diffusivity is exactly equal to the molecular diffusivity. We do not discuss this regime any further and consider now longer times with the additional condition that $t \ll \tau^*$ (or equivalently $s \tau^* \gg 1$). At this time scale, one can consider that the events of crossing the slow region or escaping from it take an infinite time.

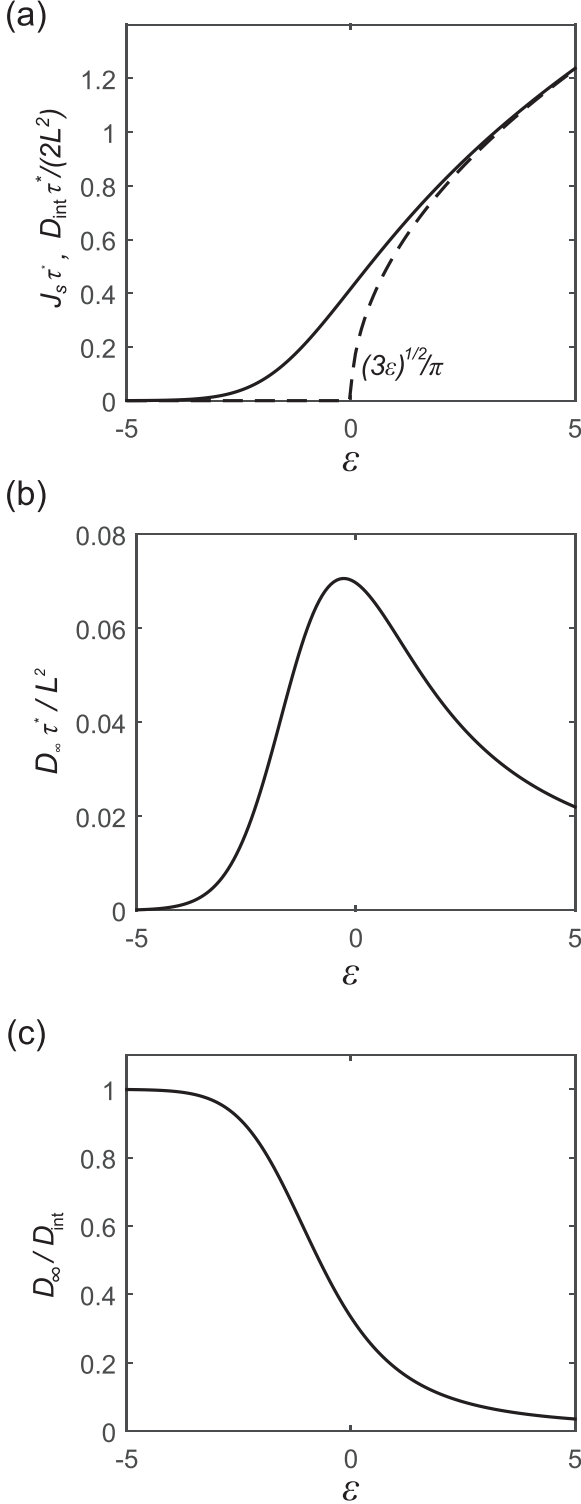


FIG. 2. (a) Value of the dimensionless flux $\tilde{J}_s = J_s \tau^*$ for near critical forces, which is also proportional to the effective diffusion coefficient in the intermediate-time regime [see Eq. (33)]. (b) Rescaled late-time effective diffusivity D_∞ , deduced from Eq. (A9). (c) Ratio of the late-time effective diffusivity over the intermediate-time effective diffusivity.

In this regime, when x, x_0 are in the *fast* region with $0 < x_0 < x < L$, we can approximate the FPT to x from x_0 by

the duration of the deterministic trajectory that links x_0 and x ,

$$f(x, t|x_0) \simeq \delta\left(t - \int_{x_0}^x \frac{dy}{u(y)}\right). \quad (22)$$

In the opposite direction where $x_0 > x$, reaching x from x_0 requires crossing the slow region, which takes a time infinite compared to the time scale considered here. Hence $f(x, t|x_0)$ is simply approximated by 0 in this case. Taking all this into account, the FPT density in Laplace space reads

$$\hat{f}(x, s|x_0) \simeq \begin{cases} \exp\left[-s \int_{x_0}^x \frac{dy}{u(y)}\right], & \text{if } 0 < x_0 < x < L, \\ 0, & \text{if } 0 < x < x_0 < L. \end{cases} \quad (23)$$

We now calculate $\hat{f}_s(s, x)$, the average FPT density to x , with stationary initial conditions,

$$\hat{f}_s(s, x) \simeq \int_{x_-}^x dx_0 P_s(x_0) \hat{f}(s, x|x_0). \quad (24)$$

Here the upper integration limit has been set to x because if $x_0 > x$ the FPT from x_0 to x is infinite at this time scale. In turn, the lower integration limit has been set to x_- , which is a position such that $l^* \ll x_- \ll L$ (its precise value will not change the result, see below). The reason for this is that, if $x_0 \sim l^*$ is in the slow region, its escape time is $\sim \tau^*$ and therefore almost no trajectory can bring it to x at the considered time scale. For $x_0 \gg l^*$, we can approximate $P_s(x_0) \simeq J_s/u(x_0)$ [see Eq. (21)] and therefore,

$$\hat{f}_s(s, x) \simeq \int_{x_-}^x dx_0 \frac{J_s}{u(x_0)} \exp\left[-s \int_{x_0}^x \frac{dy}{u(y)}\right]. \quad (25)$$

The integration over x_0 is now straightforward,

$$\hat{f}_s(s, x) \simeq \frac{J_s}{s}, \quad (26)$$

and the result does not depend on the lower bound of the integral x_- as soon as $x_- \ll L$.

Furthermore, when x, x_0 are in the fast regions, the approximation $u(x)P_s(x) = J_s = -u^*(x_0)P_s(x_0)$ holds [cf. Eq. (21)]. Hence, inserting (23) and (26) into Eq. (12) and keeping only the dominant contribution to the MSD (coming from the x, x_0 in the fast region), the expression of ψ near criticality is considerably simplified,

$$\hat{\psi}(s) \simeq \frac{2J_s^2}{s^3} \int_0^L dx \int_0^x dx_0 \left[\frac{s \exp\left[-s \int_{x_0}^x \frac{dy}{u(y)}\right]}{J_s} - 1 \right]. \quad (27)$$

Here we investigate the regime $t \ll \tau^*$ or equivalently $s\tau^* \gg 1$. Since $J_s \sim 1/\tau^*$, we obtain $s/J_s \gg 1$ and

$$\hat{\psi}(s) \simeq \frac{2J_s}{s^2} \int_0^L dx \int_0^x dx_0 \exp\left[-s \int_{x_0}^x \frac{dy}{u(y)}\right]. \quad (28)$$

The above equation describes the MSD at time scales smaller than τ^* . Let us consider its asymptotics. First, for $s \rightarrow \infty$, the terms $x \simeq x_0$ essentially contribute in the integral, and therefore

$$\hat{\psi}(s) \simeq \frac{2J_s}{s^2} \int_0^L dx \int_0^x dx_0 \exp\left[\frac{-s(x-x_0)}{u(x)}\right]. \quad (29)$$

Performing the integral, we obtain

$$\hat{\psi}(s) \simeq \frac{2J_s}{s^3} \int_0^L dx u(x) = \frac{2J_s F_c L}{\zeta s^3}. \quad (30)$$

If we invert the Laplace transform, we obtain

$$\psi(t) = \frac{J_s F_c L}{\zeta} t^2, \quad (31)$$

and thus the regime identified here is a ballistic regime with a MSD $\psi(t) \simeq \alpha t^2$. The nontrivial coefficient α identified in Eq. (31) is the product of two velocities; the first velocity is the average velocity $J_s L$, whereas the second velocity is the characteristic velocity in the fast region F_c/ζ . Note the nontrivial temperature dependence [as $(k_B T)^{1/3}$] of the coefficient of the MSD in this ballistic regime.

Now, let us take the small s limit of Eq. (28), for which

$$\hat{\psi}(s) = \frac{2J_s}{s^2} \int_0^L dx x \simeq \frac{L^2 J_s}{s^2}. \quad (32)$$

Inverting the Laplace transform leads to $\psi(t) = 2D_{\text{int}}t$ with D_{int} as an effective diffusion coefficient at intermediate times,

$$D_{\text{int}} = L^2 J_s / 2. \quad (33)$$

Thus, the analysis reveals the existence of a diffusive regime at intermediate-time scales. The effective diffusivity D_{int} is proportional to the flux of particles and therefore increases significantly when the force becomes larger than the critical force [that is, for increasing ε , see Fig. 2(a)].

When $t \gg \tau^*$, the motion becomes effectively diffusive with an effective diffusion coefficient $D_\infty = (L^2/\tau^*)G(\varepsilon)$, where G is the same dimensionless function identified in Refs. [27,29]. It is represented in Fig. 2(b) and shows a maximum for $\varepsilon \simeq 0$, leading to the giant enhancement of diffusivity at the critical force. For completeness we provide in Appendix A a derivation of D_∞ within our formalism, which is an alternative to the approach of Refs. [27,29].

The ratio of the late-time over the intermediate-time diffusivity D_∞/D_{int} is shown in Fig. 2(c). It is almost equal to 1 for forces below the critical force ($\varepsilon < 0$) but then vanishes for larger values of ε . This means that, when the critical force is reached, the shape of the MSD curves changes drastically. For $F < F_c$, one observes a direct transition between a ballistic and the long-time ballistic regime. When $F > F_c$, one observes the intermediate diffusive regime with a diffusivity larger than the effective diffusivity, which translates by an *overshoot* of the MSD and an apparent regime of subdiffusion.

If we summarize all the results, we obtain

$$\psi(t) \simeq \begin{cases} 2D_0 t, & \text{if } t \ll t_2, \\ J_s v L t^2 & \text{if } t_2 \ll t \ll L/v, \\ 2D_{\text{int}} t, & \text{if } L/v \ll t \ll \tau^*, \\ 2D_\infty t, & \text{if } \tau^* \ll t, \end{cases} \quad (34)$$

with $v = F_c/\zeta$ as the velocity in the fast region and $t_2 = k_B T / (F_c L J_s)$. In order to check these predictions, we performed stochastic simulations of the Langevin equation (1) in the case of a sine potential (see Appendix B for details on the simulation algorithm). The results presented in Fig. 3 confirm the validity of the asymptotic regimes described by Eq. (34)

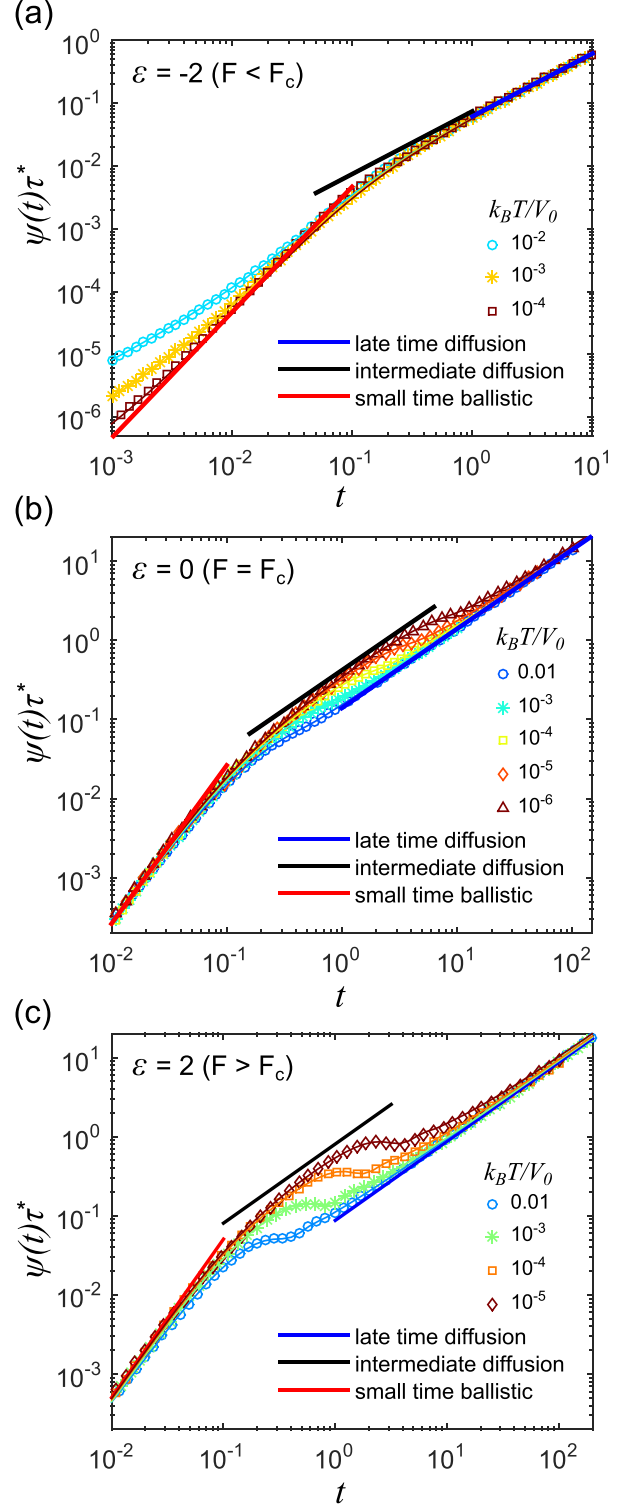


FIG. 3. MSD curves for the diffusion in a sine potential $V(x) = V_0 \sin(2\pi x/L)$ for (a) a tilting force below the critical force ($\varepsilon = -2$), (b) at the critical force ($\varepsilon = 0$), and (c) above the critical force ($\varepsilon = 2$). Different values of the noise amplitude are indicated in the legend. Symbols: results of the simulations of the Langevin equation (1) (see Appendix B for details). Lines: results of the numerical evaluation of the Kubo formula (5). Bold lines: asymptotic results of Eq. (34) (with no free parameters). Note that the regime $\psi = 2D_0 t$ is not represented but is present at short times. Units of length and times are L and $\zeta L^2/V_0$, respectively.

for tilting forces that are below, above, or equal to the critical force.

V. CONCLUSION

In this paper we have studied the time-dependent dispersion properties of particles diffusing in a near critically tilted one-dimensional periodic potential. We have derived explicit asymptotic expressions for the MSD at different time scales [Eq. (34)]. The approach to the late-time diffusive limit depends only on a small number of parameters which characterize the potential in particular regions where the dynamics is slow. The approach to the diffusive limit is therefore universal in the sense that it does not depend on the details of the potential shape.

It has been proposed that the giant increase in the late-time diffusivity that occurs at the critical force [27] can be used to estimate barriers of potential energy [40,44]. Here we have shown that the time-dependent dispersion properties are strongly modified when the force becomes larger than F_c , in particular, we have found the presence of a second diffusive regime at intermediate-time scales. The strong difference between these two diffusion coefficients could be used as another signature of the effect of the crossing of the critical force.

In this paper, we have quantified the transition between the short-time regime of molecular diffusion and the late-time effective diffusion. The mean-square displacement between these regimes can be regarded as ‘‘anomalous’’ in the sense that it is a nonlinear function of time. Anomalous diffusion can have a variety of origins in different physical systems and may occur in the late-time regime for example for particles undergoing jumps whose sizes or durations follow large distributions [51] when the distribution of energy barriers leads to diverging mean occupation times in the local energy minima or when the convective velocity field has long range correlations [52]. Anomalous diffusion also arises in fractal media [53] when considering time-dependent microscopic diffusivities [54] or when the tracer trajectory results from a collective dynamics, such as in polymer systems [55,56] or complex fluids [57,58]. This paper where we study out-of-equilibrium tracer particles in periodic media is an example where one can entirely characterize the anomalous time-dependent dispersion properties at all intermediate-time scales between the molecular diffusion regime and the final late-time regime, which is however one of normal diffusion.

APPENDIX A: LATE-TIME EFFECTIVE DIFFUSIVITY

In this appendix we briefly derive an expression for the late-time effective diffusivity D_∞ . If we expand the temporal Laplace transform of the FPT densities for $s \rightarrow 0$ we obtain

$$\hat{f}(x, s|x_0) \simeq 1 - \tau(x|x_0)s + \dots, \quad (\text{A1})$$

$$\hat{f}_s(x, s) \simeq 1 - \tau_s(x)s + \dots, \quad (\text{A2})$$

where $\tau(x|x_0) = \int_0^\infty dt f(x, t|x_0)t$ is the mean first-passage time (MFPT) to reach the position $x \bmod L$ starting from x_0 , whereas $\tau_s(x)$ is the MFPT to x with x_0 averaged over stationary initial conditions. Using Eq. (12), we then find that

$\hat{\psi} \simeq 2D_\infty/s^2$ where the effective diffusivity D_∞ reads

$$D_\infty = D_0 + \int_0^L dx \int_0^L dx_0 u(x) P_s(x) u^*(x_0) P_s(x_0) \times [\tau(x|x_0) - \tau_s(x)]. \quad (\text{A3})$$

For forces close to the critical force, this expression can be simplified considerably. The dominant contribution to this integral comes from the values of x, x_0 that are outside the slow region in which the diffusive component of the flux is negligible, and thus $u(x)P_s(x) = J_s = -u^*(x_0)P_s(x_0)$ [cf. Eq. (21)]. Furthermore, taking x and x_0 in $[0, L]$, we realize that $\tau(x|x_0)$ is negligible when $x_0 < x$ (because the convection brings the particle to x almost immediately), whereas for $x < x_0$ this time is equal to the time to cross the slow region, which is exactly the inverse of the flux J_s .

We define τ_e as the time to escape the slow region starting from stationary initial conditions, and we note that $\tau_s(x) \simeq \tau_e$ is independent of x . Following these considerations, and neglecting the term D_0 in Eq. (A3), we obtain

$$D_\infty \simeq J_s^2 L^2 \left[\frac{1}{2J_s} - \tau_e \right]. \quad (\text{A4})$$

Now let us consider $\tau_\infty(x_0)$, the mean time to escape the slow region, starting from x_0 from which $\tau_e = \langle \tau_\infty(x_0) \rangle$ can be deduced. We pose $y_0 = x_0/l^*$ and $\tau_\infty(x_0) = \tau^* \tilde{\tau}_\infty$. Then, the dimensionless MFPT satisfies the backward equation [1,3],

$$(3y_0^2 + \varepsilon) \partial_{y_0} \tilde{\tau}_\infty + \partial_{y_0}^2 \tilde{\tau}_\infty = -1. \quad (\text{A5})$$

Noting that

$$\partial_{y_0} [e^{y_0^3 + \varepsilon y_0} \partial_{y_0} \tilde{\tau}_\infty] = -e^{y_0^3 + \varepsilon y_0}, \quad (\text{A6})$$

the differential equation (A5) can be integrated twice, leading to

$$\tilde{\tau}_\infty(y_0) = \int_{y_0}^\infty dz \int_0^\infty du e^{-z^3 + (z-u)^3 - \varepsilon u}, \quad (\text{A7})$$

where we took into account the condition that $\tilde{\tau}_\infty(y_0)$ vanishes at infinity. Now, the average escape time from the slow region can be calculated by averaging over P_s given in Eq. (19), leading to

$$\tilde{\tau}_e = \tilde{J}_s \int_{-\infty}^\infty dy \int_0^\infty dv \int_0^\infty dw \int_0^\infty du e^{-(v+y)^3 - \varepsilon v + y^3} \times e^{-(y+w)^3 + (y+w-u)^3 - \varepsilon u}. \quad (\text{A8})$$

Taking $y_0 \rightarrow -\infty$ in (A7) we can check that the average time to cross the slow region is also $1/J_s$. Therefore, we obtain at the end,

$$D_\infty \simeq \frac{L^2}{\tau^*} \tilde{J}_s^2 \left[\frac{1}{2\tilde{J}_s} - \tilde{\tau}_e \right] = \frac{L^2}{\tau^*} G(\varepsilon), \quad (\text{A9})$$

where the dimensionless function $G(\varepsilon)$ is deduced from Eqs. (20) and (A8), is represented in Fig. 2, and is in excellent agreement with the results of Refs. [27,29].

APPENDIX B: DETAILS ON SIMULATIONS

We performed numerical simulations of the Langevin equation (1) by using the algorithm $x_{i+1} = x_i + u(x_i)\Delta + \sqrt{2\Delta D_0} w_i$ with w_i as a Gaussian random variable of zero

mean and variance 1, Δ as the time step, and x_i as the position at time $t = i\Delta$. Each trajectory was simulated during a time t_m , the position of the particle at the end of a trajectory being used as the initial position for the next trajectory. The MSD was then computed with Eq. (4) by averaging over distinct trajectories. For each parameter, Δ was chosen small enough so that we could observe the regime $\psi(t) \simeq 2D_0t$ for small times (not shown in Fig. 3) and we carefully controlled that

MSD curves, obtained with different Δ 's, overlapped. In order to ensure that different runs are independent, the time t_m was chosen large enough to be located in the late-time diffusive regime. The number of runs was always larger than 8000 for each parameter set. In the late-time regime, where $X(t) - X(0)$ is Gaussian distributed, the 95% confidence intervals can be evaluated to $(1 \pm \sqrt{8/N_{\text{run}}})\psi$ so that the precision on ψ is approximately $\pm 3\%$.

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