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Georgi Vodev

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# PARABOLIC TRANSMISSION EIGENVALUE-FREE REGIONS IN THE DEGENERATE ISOTROPIC CASE

GEORGI VODEV

ABSTRACT. We study the location of the transmission eigenvalues in the isotropic case when the restrictions of the refraction indices on the boundary coincide. Under some natural conditions we show that there exist parabolic transmission eigenvalue-free regions.

## 1. INTRODUCTION AND STATEMENT OF RESULTS

Let  $\Omega \subset \mathbf{R}^d$ ,  $d \geq 2$ , be a bounded, connected domain with a  $C^\infty$  smooth boundary  $\Gamma = \partial\Omega$ . A complex number  $\lambda \neq 0$ ,  $\operatorname{Re} \lambda \geq 0$ , will be said to be a transmission eigenvalue if the following problem has a non-trivial solution:

$$\begin{cases} (\Delta + \lambda^2 n_1(x)) u_1 = 0 & \text{in } \Omega, \\ (\Delta + \lambda^2 n_2(x)) u_2 = 0 & \text{in } \Omega, \\ u_1 = u_2, \quad \partial_\nu u_1 = \partial_\nu u_2 & \text{on } \Gamma, \end{cases} \quad (1.1)$$

where  $\nu$  denotes the Euclidean unit inner normal to  $\Gamma$ ,  $n_j \in C^\infty(\overline{\Omega})$ ,  $j = 1, 2$  are strictly positive real-valued functions called refraction indices. In the non-degenerate isotropic case when

$$n_1(x) \neq n_2(x) \quad \text{on } \Gamma \quad (1.2)$$

it has been recently proved in [16] that there are no transmission eigenvalues in the region

$$\{\lambda \in \mathbf{C} : \operatorname{Re} \lambda \geq 0, |\operatorname{Im} \lambda| \geq C\} \quad (1.3)$$

for some constant  $C > 0$ . Moreover, it follows from the analysis in [6] (see Section 4) that the eigenvalue-free region (1.3) is optimal and cannot be improved in general. In the present paper we will consider the degenerate isotropic case when

$$n_1(x) \equiv n_2(x) \quad \text{on } \Gamma. \quad (1.4)$$

We suppose that there is an integer  $j \geq 1$  such that

$$\partial_\nu^s n_1(x) \equiv \partial_\nu^s n_2(x) \quad \text{on } \Gamma, \quad 0 \leq s \leq j-1, \quad (1.5)$$

$$\partial_\nu^j n_1(x) \neq \partial_\nu^j n_2(x) \quad \text{on } \Gamma. \quad (1.6)$$

It was proved in [1] (see Theorem 4.2) that in this case the eigenvalue-free region (1.3) is no longer valid. On the other hand, it follows from [5] that under the conditions (1.5) and (1.6) there are no transmission eigenvalues in  $|\arg \lambda| \geq \varepsilon$ ,  $|\lambda| \geq C_\varepsilon \gg 1$ ,  $\forall 0 < \varepsilon \ll 1$ . Our goal in the present paper is to improve this result showing that in this case we have a much larger parabolic eigenvalue-free region. Our main result is the following

**Theorem 1.1.** *Under the conditions (1.5) and (1.6) there exists a constant  $C > 0$  such that there are no transmission eigenvalues in the region*

$$\left\{ \lambda \in \mathbf{C} : \operatorname{Re} \lambda \geq 0, |\operatorname{Im} \lambda| \geq C (\operatorname{Re} \lambda + 1)^{1-\kappa_j} \right\}, \quad (1.7)$$

where  $\kappa_j = 2(3j + 2)^{-1}$ .

To prove this theorem we make use of the semi-classical parametrix for the interior Dirichlet-to-Neumann (DN) map built in [14]. It is proved in [14] that for  $|\operatorname{Im} \lambda| \geq (\operatorname{Re} \lambda + 1)^{1/2+\epsilon}$ ,  $0 < \epsilon \ll 1$ , the DN map is an  $h - \Psi$ DO of class  $\operatorname{OPS}_{1/2-\epsilon}^1(\Gamma)$ , where  $0 < h \ll 1$  is a semi-classical parameter such that  $h \sim |\lambda|^{-1}$ . A direct consequence of this fact is the existence of a transmission eigenvalue-free region of the form

$$|\operatorname{Im} \lambda| \geq C_\epsilon (\operatorname{Re} \lambda + 1)^{1/2+\epsilon}, \quad \forall 0 < \epsilon \ll 1, \quad (1.8)$$

under the condition (1.2). The most difficult part of the parametrix construction in [14] is near the glancing region (see Section 3 for the definition). Indeed, outside an arbitrary neighbourhood of the glancing region the parametrix construction in [14] works for  $|\operatorname{Im} \lambda| \geq (\operatorname{Re} \lambda + 1)^\epsilon$  and the corresponding parametrix belongs to the class  $\operatorname{OPS}_0^1(\Gamma)$ . In other words, to improve the eigenvalue-free region (1.8) one has to improve the parametrix in the glancing region. Such an improved parametrix has been built in [15] for strictly concave domains and as a consequence (1.8) was improved to

$$|\operatorname{Im} \lambda| \geq C_\epsilon (\operatorname{Re} \lambda + 1)^\epsilon, \quad \forall 0 < \epsilon \ll 1, \quad (1.9)$$

in this case. In fact, it turns out that to get larger eigenvalue-free regions under the condition (1.2) no parametrix construction in the glancing region is needed. It suffices to show that the norm of the DN map microlocalized in a small neighbourhood of the glancing region gets small if  $|\operatorname{Im} \lambda|$  and  $\operatorname{Re} \lambda$  are large. Indeed, this strategy has been implemented in [16] to get the optimal transmission eigenvalue-free region (1.3) for an arbitrary domain. In fact, the main point in the approach in [16] is the construction of a parametrix in the hyperbolic region valid for  $1 \ll C_\epsilon \leq |\operatorname{Im} \lambda| \leq (\operatorname{Re} \lambda)^{1-\epsilon}$ ,  $\operatorname{Re} \lambda \geq C'_\epsilon \gg 1$ ,  $0 < \epsilon \ll 1$ . The strategy of [16], however, does not work any more when we have the condition (1.4). In this case the parametrix in the glancing region turns out to be essential to get eigenvalue-free regions like (1.7). In Section 3 we revisit the parametrix construction of [14] and we study carefully the way in which it depends on the restriction on the boundary of the normal derivatives of the refraction index (see Theorem 3.1). In Section 4 we improve Theorem 3.1. In Section 5 we show how Theorem 4.1 implies Theorem 1.1. We also show that to improve (1.7) it suffices to improve the parametrix in the glancing region, only (see Proposition 5.2).

As in [9] one can study in this case the counting function  $N(r, C) = \#\{\lambda - \text{trans. eig.} : C \leq |\lambda| \leq r\}$ , where  $r \gg C > 0$ . We have the following

**Corollary 1.2.** *Under the conditions of Theorem 1.1, there exists a constant  $C > 0$  such that the counting function of the transmission eigenvalues satisfies the asymptotics*

$$N(r, C) = \tau r^d + \mathcal{O}_\varepsilon(r^{d-\kappa_j+\varepsilon}), \quad \forall 0 < \varepsilon \ll 1, \quad (1.10)$$

where

$$\tau = \frac{\omega_d}{(2\pi)^d} \int_\Omega \left( n_1(x)^{d/2} + n_2(x)^{d/2} \right) dx,$$

$\omega_d$  being the volume of the unit ball in  $\mathbf{R}^d$ .

Note that the eigenvalue-free region (1.3) implies (1.10) with  $\kappa_j$  replaced by 1. Note also that asymptotics for the counting function  $N(r, C)$  with remainder  $o(r^d)$  have been previously obtained in [3], [7], [12] still under the condition (1.2).

## 2. BASIC PROPERTIES OF THE $h - \Psi$ DOS

In this section we will recall some basic properties of the  $h - \Psi$ DOs on a compact manifold without boundary. Let  $\Gamma$ ,  $\dim \Gamma = d - 1$ , be as in the previous section and recall that given a

symbol  $a \in C^\infty(T^*\Gamma)$ , the  $h$ - $\Psi$ DO,  $\text{Op}_h(a)$ , is defined as follows

$$(\text{Op}_h(a)f)(x') = (2\pi h)^{-d+1} \int_{T^*\Gamma} e^{-\frac{i}{h}\langle x'-y', \xi' \rangle} a(x', \xi') f(y') dy' d\xi'.$$

We have the following criteria of  $L^2$ -boundedness.

**Proposition 2.1.** *Let the function  $a$  satisfy the bounds*

$$|\partial_{x'}^\alpha a(x', \xi')| \leq C_\alpha, \quad \forall (x', \xi') \in T^*\Gamma, \quad (2.1)$$

for all multi-indices  $\alpha$ . Then the operator  $\text{Op}_h(a)$  is bounded on  $L^2(\Gamma)$  and satisfies

$$\|\text{Op}_h(a)\|_{L^2(\Gamma) \rightarrow L^2(\Gamma)} \leq C \sum_{|\alpha| \leq d} C_\alpha \quad (2.2)$$

with a constant  $C > 0$  independent of  $h$  and  $C_\alpha$ .

Let the function  $a$  satisfy the bounds

$$\left| \partial_{x'}^\alpha \partial_{\xi'}^\beta a(x', \xi') \right| \leq C_{\alpha, \beta} h^{-(|\alpha|+|\beta|)/2}, \quad \forall (x', \xi') \in T^*\Gamma, \quad (2.3)$$

for all multi-indices  $\alpha$  and  $\beta$ . Then the operator  $\text{Op}_h(a)$  is bounded on  $L^2(\Gamma)$  and satisfies

$$\|\text{Op}_h(a)\|_{L^2(\Gamma) \rightarrow L^2(\Gamma)} \leq C \sum_{|\alpha|+|\beta| \leq s_d} C_{\alpha, \beta} \quad (2.4)$$

with a constant  $C > 0$  independent of  $h$  and  $C_{\alpha, \beta}$ , where  $s_d > 0$  is an integer depending only on the dimension.

Given  $\ell \in \mathbf{R}$ ,  $\delta_1, \delta_2 \geq 0$  and a function  $m > 0$  on  $T^*\Gamma$ , we denote by  $S_{\delta_1, \delta_2}^\ell(m)$  the set of all functions  $a \in C^\infty(T^*\Gamma)$  satisfying

$$\left| \partial_{x'}^\alpha \partial_{\xi'}^\beta a(x', \xi') \right| \leq C_{\alpha, \beta} m^{\ell - \delta_1|\alpha| - \delta_2|\beta|}$$

for all multi-indices  $\alpha$  and  $\beta$  with constants  $C_{\alpha, \beta} > 0$  independent of  $m$ . Given  $k \in \mathbf{R}$ ,  $0 \leq \delta < 1/2$ , we also denote by  $\mathcal{S}_\delta^k$  the space of all symbols  $a \in C^\infty(T^*\Gamma)$  satisfying

$$\left| \partial_{x'}^\alpha \partial_{\xi'}^\beta a(x', \xi') \right| \leq C_{\alpha, \beta} h^{-\delta(|\alpha|+|\beta|)} \langle \xi' \rangle^{k-|\beta|}$$

for all multi-indices  $\alpha$  and  $\beta$  with constants  $C_{\alpha, \beta} > 0$  independent of  $h$ . It is well-known that the  $h$ - $\Psi$ DOs of class  $\text{OPS}_\delta^k$  have nice calculus (e.g. see Section 7 of [2]). The next proposition is very useful for inverting such operators depending on additional parameters (see also Proposition 2.2 of [14]).

**Proposition 2.2.** *Let  $h^{\ell_\pm} a^\pm \in \mathcal{S}_\delta^{\pm k}$ ,  $0 \leq \delta < 1/2$ , where  $\ell_\pm \geq 0$  are some numbers. Assume in addition that the functions  $a^\pm$  satisfy*

$$\left| \partial_{x'}^{\alpha_1} \partial_{\xi'}^{\beta_1} a^+(x', \xi') \partial_{x'}^{\alpha_2} \partial_{\xi'}^{\beta_2} a^-(x', \xi') \right| \leq \mu C_{\alpha_1, \beta_1, \alpha_2, \beta_2} h^{-(|\alpha_1|+|\beta_1|+|\alpha_2|+|\beta_2|)/2}, \quad (2.5)$$

$\forall (x', \xi') \in T^*\Gamma$ , for all multi-indices  $\alpha_1, \beta_1, \alpha_2, \beta_2$  such that  $|\alpha_j| + |\beta_j| \geq 1$ ,  $j = 1, 2$ , with constants  $C_{\alpha_1, \beta_1, \alpha_2, \beta_2} > 0$  independent of  $h$  and  $\mu$ . Then we have

$$\left\| \text{Op}_h(a^+) \text{Op}_h(a^-) - \text{Op}_h(a^+ a^-) \right\|_{L^2(\Gamma) \rightarrow L^2(\Gamma)} \leq C(\mu + h) \quad (2.6)$$

with a constant  $C > 0$  independent of  $h$  and  $\mu$ .

Given any real  $s$ , we define the semi-classical Sobolev norm by

$$\|f\|_{H_h^s(\Gamma)} := \|\text{Op}_h(\langle \xi' \rangle^s) f\|_{L^2(\Gamma)}.$$

Using the calculus of the  $h$ - $\Psi$ DOs one can derive from (2.4) the following

**Proposition 2.3.** *Let  $a \in \mathcal{S}_\delta^{-k}$ ,  $0 \leq \delta < 1/2$ . Then, for every  $s$ , we have*

$$\text{Op}_h(a) = \mathcal{O}_s(1) : H_h^s(\Gamma) \rightarrow H_h^{s+k}(\Gamma).$$

Proposition 2.2 implies the following

**Proposition 2.4.** *Let  $a^\pm \in \mathcal{S}_0^{\pm k}$ . Then, for every  $s$ , we have*

$$\text{Op}_h(a^+) \text{Op}_h(a^-) - \text{Op}_h(a^+ a^-) = \mathcal{O}_s(h) : H_h^s(\Gamma) \rightarrow H_h^s(\Gamma).$$

### 3. THE PARAMETRIX CONSTRUCTION REVISITED

In this section we will build a parametrix for the semi-classical Dirichlet-to-Neumann map following [14]. Note that in [14] there is a gap due to a missing term in the transport equations (4.11), which however does not affect the proof of the main results. Here we will correct this gap making some slight modifications.

Given  $f \in H^{m+1}(\Gamma)$ , let  $u$  solve the equation

$$\begin{cases} (h^2 \Delta + zn(x)) u = 0 & \text{in } \Omega, \\ u = f & \text{on } \Gamma, \end{cases} \quad (3.1)$$

where  $n \in C^\infty(\overline{\Omega})$  is a strictly positive function,  $0 < h \ll 1$  is a semi-classical parameter and  $z \in Z_1 \cup Z_2 \cup Z_3$ , where  $Z_1 = \{z \in \mathbf{C} : \text{Re } z = 1, 0 < |\text{Im } z| \leq 1\}$ ,  $Z_2 = \{z \in \mathbf{C} : \text{Re } z = -1, |\text{Im } z| \leq 1\}$ ,  $Z_3 = \{z \in \mathbf{C} : |\text{Re } z| \leq 1, |\text{Im } z| = 1\}$ . Given  $\varepsilon > 0$  we also set  $Z_1(\varepsilon) = \{z \in Z_1 : h^\varepsilon \leq |\text{Im } z| \leq 1\}$ . We define the semi-classical Dirichlet-to-Neumann map

$$\mathcal{N}(h, z) : H^{m+1}(\Gamma) \rightarrow H^m(\Gamma)$$

by

$$\mathcal{N}(h, z) f := -ih \partial_\nu u|_\Gamma$$

where  $\nu$  denotes the Euclidean unit inner normal to  $\Gamma$ . Given an integer  $m \geq 0$ , denote by  $H_h^m(\Omega)$  the Sobolev space equipped with the semi-classical norm

$$\|v\|_{H_h^m(\Omega)} = \sum_{|\alpha| \leq m} h^{|\alpha|} \|\partial_x^\alpha v\|_{L^2(\Omega)}.$$

We define similarly the Sobolev space  $H_h^m(\Gamma)$ . Note that this norm is equivalent to that one defined in Section 2. Throughout this section we will use the normal coordinates  $(x_1, x')$  with respect to the Euclidean metric near the boundary  $\Gamma$ , where  $0 < x_1 \ll 1$  denotes the Euclidean distance to the boundary and  $x'$  are coordinates on  $\Gamma$ . We denote by  $\Delta_\Gamma$  the negative Laplace-Beltrami operator on  $\Gamma$  equipped with the Riemannian metric induced by the Euclidean one in  $\Omega$ . Let  $r_0(x', \xi') \geq 0$  be the principal symbol of  $-\Delta_\Gamma$  written in the coordinates  $(x', \xi') \in T^*\Gamma$ . Since the function  $n$  is smooth up to the boundary we can expand it as

$$n(x) = \sum_{k=0}^{N-1} x_1^k n_k(x') + x_1^N \mathcal{M}_N(x)$$

for every integer  $N \geq 1$ , where  $n_k = (k!)^{-1} \partial_\nu^k n|_\Gamma$ ,  $n_0 > 0$ , and  $\mathcal{M}_N(x)$  is a real-valued smooth function. Set

$$\rho(x', \xi', z) = \sqrt{-r_0(x', \xi') + zn_0(x')}, \quad \text{Im } \rho > 0.$$

The glancing region for the problem (3.1) is defined by

$$\Sigma := \{(x', \xi') \in T^*\Gamma : r_{\sharp}(x', \xi') = 1\}, \quad r_{\sharp} = n_0^{-1}r_0.$$

Let  $\phi \in C_0^\infty(\mathbf{R})$ ,  $0 \leq \phi \leq 1$ ,  $\phi(\sigma) = 1$  for  $|\sigma| \leq 1$ ,  $\phi(\sigma) = 0$  for  $|\sigma| \geq 2$ , and set  $\eta(x', \xi') = \phi(r_0(x', \xi')/\delta_0)$ . Clearly, taking  $\delta_0 > 0$  small enough we can arrange that  $|\rho| \geq C\langle r_0 \rangle^{1/2} \geq C\langle \xi' \rangle$  on  $\text{supp}(1 - \eta)$ . We also define the function  $\chi(x', \xi') = \phi((1 - r_{\sharp}(x', \xi'))/\delta_1)$ , where  $0 < \delta_1 \ll 1$  is independent of  $h$  and  $z$ . Clearly,  $\chi = 1$  in a neighbourhood of  $\Sigma$ ,  $\chi = 0$  outside another neighbourhood of  $\Sigma$ .

We will say that a function  $a \in C^\infty(T^*\Gamma)$  belongs to  $S_{\delta_1, \delta_2}^{\ell_1}(m_1) + S_{\delta_3, \delta_4}^{\ell_2}(m_2)$  if  $\eta a \in S_{\delta_1, \delta_2}^{\ell_1}(m_1)$  and  $(1 - \eta)a \in S_{\delta_3, \delta_4}^{\ell_2}(m_2)$ . Given any integer  $k$ , it follows from Lemma 3.2 of [14] that

$$\rho^k, |\rho|^k \in S_{2,2}^k(|\rho|) + S_{0,1}^k(|\rho|). \quad (3.2)$$

In particular, (3.2) implies that

$$(1 - \eta)\rho^k, (1 - \eta)|\rho|^k \in \mathcal{S}_0^k. \quad (3.3)$$

Since  $\rho = i\sqrt{r_0}(1 + \mathcal{O}(r_0^{-1}))$  as  $r_0 \rightarrow \infty$ , it is easy to check that

$$(1 - \eta)\rho^k - (1 - \eta)(i\sqrt{r_0})^k \in \mathcal{S}_0^{k-2} \quad (3.4)$$

for every integer  $k$ . Since  $|\rho| \geq C\sqrt{|\text{Im } z|}$  for  $z \in Z_1$ ,  $(x', \xi') \in \text{supp } \chi$  and  $|\rho| \geq C > 0$  for  $z \in Z_2 \cup Z_3$  or  $z \in Z_1$ ,  $(x', \xi') \in \text{supp}(1 - \chi)$  (see Lemma 3.1 of [14]), it also follows from (3.2) that

$$(1 - \chi)\rho^k, (1 - \chi)|\rho|^k \in \mathcal{S}_0^k, \quad (3.5)$$

$$h^{\frac{k_-}{4}}\chi\rho^k, h^{\frac{k_-}{4}}\chi|\rho|^k \in \mathcal{S}_{1/2-\epsilon}^{-N}, \quad z \in Z_1(1/2 - \epsilon), \quad (3.6)$$

$$\chi\rho^k, \chi|\rho|^k \in \mathcal{S}_0^{-N}, \quad z \in Z_2 \cup Z_3, \quad (3.7)$$

for every integer  $N \geq 0$  and  $0 < \epsilon \ll 1$ , where  $k_- = 0$  if  $k \geq 0$ ,  $k_- = |k|$  if  $k < 0$ . Our goal in this section is to prove the following

**Theorem 3.1.** *Let  $z \in Z_1(1/2 - \epsilon)$ ,  $0 < \epsilon \ll 1$ . Then, for every integer  $s \geq 0$  there is a function  $b_s \in \mathcal{S}_{1/2-\epsilon}^0$  independent of all  $n_k$  with  $k \geq s$  such that*

$$\left\| \mathcal{N}(h, z) - \text{Op}_h(\rho + b_s + c_s h^s \rho^{-s-1} z n_s) \right\|_{L^2(\Gamma) \rightarrow H_h^{s+1}(\Gamma)} \leq C_s h^{s+1} |\text{Im } z|^{-2s-3/2} \quad (3.8)$$

where  $c_s = 0$  if  $s = 0$ , and  $c_s = -is!(-2i)^{-s-1}$  for  $s \geq 1$ . If  $z \in Z_2 \cup Z_3$ , then (3.8) holds with  $|\text{Im } z|$  replaced by 1. Moreover, for  $z \in Z_1(1 - \epsilon)$  we have

$$\left\| \mathcal{N}(h, z) \text{Op}_h(1 - \chi) - \text{Op}_h(\rho(1 - \chi) + \tilde{b}_s + c_s h^s (1 - \chi)\rho^{-s-1} z n_s) \right\|_{L^2(\Gamma) \rightarrow H_h^{s+1}(\Gamma)} \leq C_s h^{s+1} \quad (3.9)$$

where the function  $\tilde{b}_s \in \mathcal{S}_0^0$  is independent of all  $n_k$  with  $k \geq s$ .

*Proof.* We will recall the parametrix construction in [14]. We will proceed locally and then we will use partition of the unity to get the global parametrix. Fix a point  $x^0 \in \Gamma$  and let  $\mathcal{U}_0 \subset \Gamma$  be a small open neighbourhood of  $x^0$ . Let  $(x_1, x')$ ,  $x_1 > 0$ ,  $x' \in \mathcal{U}_0$ , be the normal coordinates. In these coordinates the Laplacian can be written as follows

$$\Delta = \partial_{x_1}^2 + r(x, \partial_{x'}) + q(x, \partial_x)$$

where  $r(x, \xi') = \langle R(x)\xi', \xi' \rangle \geq 0$ ,  $R = (R_{ij})$  being a symmetric  $(d-1) \times (d-1)$  matrix-valued function with smooth real-valued entries,  $q(x, \xi) = \langle q(x), \xi \rangle = q^\sharp(x)\xi_1 + \langle q^\flat(x), \xi' \rangle$ ,  $q^\sharp$  and  $q^\flat$  being smooth functions. We can expand them as follows

$$\begin{aligned} R(x) &= \sum_{k=0}^{N-1} x_1^k R_k(x') + x_1^N \mathcal{R}_N(x), \\ q^\sharp(x) &= \sum_{k=0}^{N-1} x_1^k q_k^\sharp(x') + x_1^N \mathcal{Q}_N^\sharp(x), \\ q^\flat(x) &= \sum_{k=0}^{N-1} x_1^k q_k^\flat(x') + x_1^N \mathcal{Q}_N^\flat(x), \end{aligned}$$

for every integer  $N \geq 1$ . Clearly,  $r_0(x', \xi') = r(0, x', \xi') = \langle R_0(x')\xi', \xi' \rangle$ .

Take a function  $\psi^0 \in C_0^\infty(\mathcal{U}_0)$ . In what follows  $\psi$  will denote either the function  $\psi^0$  or the function  $\psi^0(1 - \chi)$ . Following [14], we will construct a parametrix  $\tilde{u}_\psi$  of the solution of (3.1) with  $\tilde{u}_\psi|_{x_1=0} = \text{Op}_h(\psi)f$  in the form

$$\tilde{u}_\psi = (2\pi h)^{-d+1} \int \int e^{\frac{i}{h}(\langle y', \xi' \rangle + \varphi(x, \xi', z))} \Phi_\delta(x, \xi', z) a(x, \xi', h, z) f(y') d\xi' dy'$$

where  $\Phi_\delta = \phi(x_1/\delta)\phi(x_1/\rho_1\delta)$ , with  $\rho_1 = |\rho|^3$  if  $z \in Z_1(1/2 - \epsilon)$ ,  $\psi = \psi^0$ , and  $\rho_1 = 1$  if  $z \in Z_2 \cup Z_3$  or  $z \in Z_1(1 - \epsilon)$ ,  $\psi = \psi^0(1 - \chi)$ . Here  $0 < \delta \ll 1$  is a parameter independent of  $h$  and  $z$  to be fixed later on. The phase  $\varphi$  is complex-valued such that  $\varphi|_{x_1=0} = -\langle x', \xi' \rangle$  and satisfies the eikonal equation mod  $\mathcal{O}(x_1^N)$ :

$$(\partial_{x_1}\varphi)^2 + \langle R(x)\nabla_{x'}\varphi, \nabla_{x'}\varphi \rangle - zn(x) = x_1^N \Psi_N \quad (3.10)$$

where  $N \gg 1$  is an arbitrary integer and the function  $\Psi_N$  is smooth up to the boundary  $x_1 = 0$ . It is shown in [14], Section 4, that the equation (3.10) has a smooth solution of the form

$$\varphi = \sum_{k=0}^N x_1^k \varphi_k(x', \xi', z), \quad \varphi_0 = -\langle x', \xi' \rangle,$$

satisfying

$$\partial_{x_1}\varphi|_{x_1=0} = \varphi_1 = \rho. \quad (3.11)$$

More generally, the functions  $\varphi_k$  satisfy the relations

$$\sum_{k+j=K} (k+1)(j+1)\varphi_{k+1}\varphi_{j+1} + \sum_{k+j+\ell=K} \langle R_\ell \nabla_{x'}\varphi_k, \nabla_{x'}\varphi_j \rangle - zn_K = 0 \quad (3.12)$$

for every integer  $0 \leq K \leq N-1$ . Then equation (3.10) is satisfied with

$$\begin{aligned} \Psi_N &= \langle \mathcal{R}_N(x)\nabla_{x'}\varphi, \nabla_{x'}\varphi \rangle - z\mathcal{M}_N(x) \\ &+ \sum_{k+j \geq N} x_1^{k+j-N} (k+1)(j+1)\varphi_{k+1}\varphi_{j+1} + \sum_{k+j+\ell \geq N} x_1^{k+j+\ell-N} \langle R_\ell \nabla_{x'}\varphi_k, \nabla_{x'}\varphi_j \rangle \end{aligned}$$

where  $\varphi_\nu = 0$  for  $\nu \geq N+1$  so that the above sums are finite. Using (3.12) one can prove by induction the following lemma (see Lemma 4.1 of [14]).

**Lemma 3.2.** *We have*

$$\varphi_k \in S_{2,2}^{4-3k}(|\rho|) + S_{0,1}^1(|\rho|), \quad 1 \leq k \leq N, \quad (3.13)$$

$$\partial_{x_1}^k \Psi_N \in S_{2,2}^{2-3N-3k}(|\rho|) + S_{0,1}^2(|\rho|), \quad k \geq 0, \quad (3.14)$$

uniformly in  $z$  and  $0 \leq x_1 \leq 2\delta \min\{1, |\rho|^3\}$ . Moreover, if  $\delta > 0$  is small enough, independent of  $\rho$ , we have

$$\operatorname{Im} \varphi \geq x_1 \operatorname{Im} \rho / 2 \quad \text{for } 0 \leq x_1 \leq 2\delta \min\{1, |\rho|^3\}. \quad (3.15)$$

One can also easily prove by induction the following

**Lemma 3.3.** *For every integer  $k \geq 1$  the functions  $\varphi_k$  and  $\varphi_{k+1} - \frac{zn_k}{2(k+1)\rho}$  are independent of all  $n_\ell$  with  $\ell \geq k$ .*

It follows from (3.13) that  $(1-\eta)\varphi_k \in \mathcal{S}_0^1$  for all  $k$ . Define now the functions  $\tilde{\varphi}_k$  independent of all  $n_\ell$ ,  $\ell \geq 0$ , satisfying the relations

$$\sum_{k+j=K} (k+1)(j+1)\tilde{\varphi}_{k+1}\tilde{\varphi}_{j+1} + \sum_{k+j+l=K} \langle R_\ell \nabla_{x'} \tilde{\varphi}_k, \nabla_{x'} \tilde{\varphi}_j \rangle = 0, \quad (3.16)$$

$1 \leq k \leq N-1$ , and  $\tilde{\varphi}_0 = -\langle x', \xi' \rangle$ ,  $\tilde{\varphi}_1 = i\sqrt{r_0}$ . Using (3.4) together with (3.12) and (3.16), one can easily prove by induction the following

**Lemma 3.4.** *For every integer  $k \geq 1$ , we have  $(1-\eta)(\varphi_k - \tilde{\varphi}_k) \in \mathcal{S}_0^{-1}$ .*

The amplitude  $a$  is of the form

$$a = \sum_{j=0}^{N-1} h^j a_j(x, \xi', z)$$

where the functions  $a_j$  satisfy the transport equations mod  $\mathcal{O}(x_1^N)$ :

$$2i\partial_{x_1}\varphi\partial_{x_1}a_j + 2i\langle R(x)\nabla_{x'}\varphi, \nabla_{x'}a_j \rangle + i(\Delta\varphi)a_j + \Delta a_{j-1} = x_1^N A_N^{(j)}, \quad 0 \leq j \leq N-1, \quad (3.17)$$

$a_0|_{x_1=0} = \psi$ ,  $a_j|_{x_1=0} = 0$  for  $j \geq 1$ , where  $a_{-1} = 0$  and the functions  $A_N^{(j)}$  are smooth up to the boundary  $x_1 = 0$ . We will be looking for the solutions to (3.17) in the form

$$a_j = \sum_{k=0}^N x_1^k a_{k,j}(x', \xi', z).$$

We can write

$$\Delta\varphi = \sum_{k=0}^{N-1} x_1^k \varphi_k^\Delta + x_1^N E_N(x)$$

with

$$\varphi_k^\Delta = (k+1)(k+2)\varphi_{k+2} + \sum_{\ell+\nu=k} \left( \langle R_\ell \nabla_{x'} \cdot, \nabla_{x'} \varphi_\nu \rangle + q_\ell^\sharp(\nu+1)\varphi_{\nu+1} + \langle q_\ell^b, \nabla_{x'} \varphi_\nu \rangle \right),$$

$$\begin{aligned} E_N &= \langle \mathcal{R}_N \nabla_{x'} \cdot, \nabla_{x'} \varphi \rangle + \mathcal{Q}_N^\sharp \partial_{x_1} \varphi + \langle \mathcal{Q}_N^b, \nabla_{x'} \varphi \rangle \\ &+ \sum_{\ell+\nu \geq N} x_1^{\ell+\nu-N} \left( \langle R_\ell \nabla_{x'} \cdot, \nabla_{x'} \varphi_\nu \rangle + q_\ell^\sharp(\nu+1)\varphi_{\nu+1} + \langle q_\ell^b, \nabla_{x'} \varphi_\nu \rangle \right), \end{aligned}$$

where  $\varphi_\nu = 0$  for  $\nu \geq N+1$ . Similarly

$$\Delta a_{j-1} = \sum_{k=0}^{N-1} x_1^k a_{k,j-1}^\Delta + x_1^N F_N^{(j-1)}(x)$$



with

$$\begin{aligned} a_{k,j-1}^{\Delta} &= (k+1)(k+2)a_{k+2,j-1} \\ &+ \sum_{\ell+\nu=k} \left( \langle R_{\ell} \nabla_{x'} \cdot, \nabla_{x'} a_{\nu,j-1} \rangle + q_{\ell}^{\sharp}(\nu+1)a_{\nu+1,j-1} + \langle q_{\ell}^{\flat}, \nabla_{x'} a_{\nu,j-1} \rangle \right), \\ F_N^{(j-1)} &= \langle \mathcal{R}_N \nabla_{x'} \cdot, \nabla_{x'} a_{j-1} \rangle + \mathcal{Q}_N^{\sharp} \partial_{x_1} a_{j-1} + \langle \mathcal{Q}_N^{\flat}, \nabla_{x'} a_{j-1} \rangle \\ &+ \sum_{\ell+\nu \geq N} x_1^{\ell+\nu-N} \left( \langle R_{\ell} \nabla_{x'} \cdot, \nabla_{x'} a_{\nu,j-1} \rangle + q_{\ell}^{\sharp}(\nu+1)a_{\nu+1,j-1} + \langle q_{\ell}^{\flat}, \nabla_{x'} a_{\nu,j-1} \rangle \right), \end{aligned}$$

where  $a_{\nu,j-1} = 0$  for  $\nu \geq N+1$ . We also have

$$(\Delta \varphi) a_j = \sum_{k=0}^{N-1} x_1^k \sum_{k_1+k_2=k} \varphi_{k_1}^{\Delta} a_{k_2,j} + x_1^N \mathcal{E}_N^{(j)}$$

with

$$\mathcal{E}_N^{(j)} = E_N a_j + \sum_{k_1+k_2 \geq N} x_1^{k_1+k_2-N} \varphi_{k_1}^{\Delta} a_{k_2,j},$$

$$\partial_{x_1} \varphi \partial_{x_1} a_j = \sum_{k=0}^{N-1} x_1^k \sum_{k_1+k_2=k} (k_1+1)(k_2+1) \varphi_{k_1+1} a_{k_2+1,j} + x_1^N \mathcal{F}_N^{(j)}$$

with

$$\begin{aligned} \mathcal{F}_N^{(j)} &= \sum_{k_1+k_2 \geq N} x_1^{k_1+k_2-N} (k_1+1)(k_2+1) \varphi_{k_1+1} a_{k_2+1,j}, \\ \langle R(x) \nabla_{x'} \varphi, \nabla_{x'} a_j \rangle &= \sum_{k=0}^{N-1} x_1^k \sum_{k_1+k_2+k_3=k} \langle R_{k_1} \nabla_{x'} \varphi_{k_2}, \nabla_{x'} a_{k_3,j} \rangle + x_1^N \mathcal{G}_N^{(j)} \end{aligned}$$

with

$$\mathcal{G}_N^{(j)} = \langle \mathcal{R}_N(x) \nabla_{x'} \varphi, \nabla_{x'} a_j \rangle + \sum_{k_1+k_2+k_3 \geq N} x_1^{k_1+k_2+k_3-N} \langle R_{k_1} \nabla_{x'} \varphi_{k_2}, \nabla_{x'} a_{k_3,j} \rangle,$$

where  $\varphi_{\nu} = 0$ ,  $a_{\nu,j} = 0$  for  $\nu \geq N+1$  so that the above sums are finite. Inserting the above identities into equation (3.17) and comparing the coefficients of all powers  $x_1^k$ ,  $0 \leq k \leq N-1$ , we get that the functions  $a_{k,j}$  must satisfy the relations

$$\begin{aligned} &\sum_{k_1+k_2=k} 2i(k_1+1)(k_2+1) \varphi_{k_1+1} a_{k_2+1,j} + \sum_{k_1+k_2+k_3=k} 2i \langle R_{k_1} \nabla_{x'} \varphi_{k_2}, \nabla_{x'} a_{k_3,j} \rangle \\ &+ \sum_{k_1+k_2=k} i \varphi_{k_1}^{\Delta} a_{k_2,j} = -a_{k,j-1}^{\Delta}, \quad \text{for } 0 \leq k \leq N-1, \quad 0 \leq j \leq N-1, \end{aligned} \quad (3.18)$$

and  $a_{0,0} = \psi$ ,  $a_{0,j} = 0$ ,  $j \geq 1$ ,  $a_{k,-1} = 0$ ,  $k \geq 0$ . Then equation (3.17) is satisfied with

$$A_N^{(j)} = 2i \mathcal{F}_N^{(j)} + 2i \mathcal{G}_N^{(j)} + i \mathcal{E}_N^{(j)} + F_N^{(j-1)}.$$

Let us calculate  $a_{1,0}$ . By (3.18) with  $j=0$ ,  $k=0$ , we get

$$a_{1,0} = -\varphi_1^{-1} \langle B_0 \xi', \nabla_{x'} \psi \rangle - (\varphi_1^{-1} \varphi_2 + 2^{-1} q_0^{\sharp} - (2\varphi_1)^{-1} \langle q_0^{\flat}(x'), \xi' \rangle) \psi. \quad (3.19)$$

On the other hand, by (3.12) with  $K=1$  we get

$$\varphi_2 = -(2\rho)^{-1} \langle B_0 \xi', \nabla_{x'} \rho \rangle - (4\rho)^{-1} \langle B_1 \xi', \xi' \rangle + z(4\rho)^{-1} n_1.$$

Using the identity

$$2\rho \nabla_{x'} \rho = -\nabla_{x'} r_0 + z \nabla_{x'} n_0$$

we can write  $\varphi_2$  in the form

$$\begin{aligned} \varphi_2 &= (2\rho)^{-2} \langle B_0 \xi', \nabla_{x'} r_0 \rangle - (4\rho)^{-1} \langle B_1 \xi', \xi' \rangle \\ &\quad - z(2\rho)^{-2} \langle B_0 \xi', \nabla_{x'} n_0 \rangle + z(4\rho)^{-1} n_1. \end{aligned} \quad (3.20)$$

By (3.19) and (3.20),

$$\begin{aligned} a_{1,0} &= -\rho^{-1} \langle B_0 \xi', \nabla_{x'} \psi \rangle - 2^{-1} \psi q_0^\# + (2\rho)^{-1} \psi \langle q_0^b(x'), \xi' \rangle \\ &\quad - 4^{-1} \rho^{-3} \psi \langle B_0 \xi', \nabla_{x'} r_0 \rangle + 4^{-1} \rho^{-2} \psi \langle B_1 \xi', \xi' \rangle \\ &\quad + z 4^{-1} \rho^{-3} \psi \langle B_0 \xi', \nabla_{x'} n_0 \rangle - z 4^{-1} \rho^{-2} \psi n_1. \end{aligned} \quad (3.21)$$

By (3.2) and (3.21) we conclude

$$a_{1,0} \in S_{2,2}^{-3}(|\rho|) + S_{0,1}^0(|\rho|). \quad (3.22)$$

The next lemma follows from Lemma 3.2 and (3.22) together with equations (3.18) and can be proved in the same way as Lemma 4.2 of [14]. We will sketch the proof.

**Lemma 3.5.** *We have*

$$a_{k,j} \in S_{2,2}^{-3k-4j}(|\rho|) + S_{0,1}^{-j}(|\rho|), \quad \text{for } k \geq 1, j \geq 0, \quad (3.23)$$

$$\partial_{x_1}^k A_N^{(j)} \in S_{2,2}^{-3N-3k-4j-2}(|\rho|) + S_{0,1}^{1-j}(|\rho|), \quad \text{for } k \geq 0, j \geq 0, \quad (3.24)$$

uniformly in  $z$  and  $0 \leq x_1 \leq 2\delta \min\{1, |\rho|^3\}$ .

*Proof.* Recall that  $\nabla_{x'} \varphi_0 = -\xi'$ . By (3.13) we have

$$\begin{aligned} \nabla_{x'} \varphi_k &\in S_{2,2}^{2-3k}(|\rho|) + S_{0,1}^1(|\rho|), \quad k \geq 1, \\ \varphi_k^\Delta &\in S_{2,2}^{-2-3k}(|\rho|) + S_{0,1}^1(|\rho|), \quad k \geq 0. \end{aligned}$$

We will prove (3.23) by induction. In view of (3.22) we have (3.23) with  $k = 1, j = 0$ . Suppose now that (3.23) is true for all  $j \leq J - 1$  and all  $k \geq 1$ , and for  $j = J$  and  $k \leq K$ . We have to show that it is true for  $j = J$  and  $k = K + 1$ . To this end, we will use equation (3.18) with  $j = J$  and  $k = K$ . Indeed, the LHS is equal to  $2i(K+1)\rho a_{K+1,J}$  modulo  $S_{2,2}^{-3K-4J-2}(|\rho|) + S_{0,1}^{-J+1}(|\rho|)$ , while the RHS belongs to  $S_{2,2}^{-3K-4J-2}(|\rho|) + S_{0,1}^{-J+1}(|\rho|)$ . In other words,  $\rho a_{K+1,J}$  belongs to  $S_{2,2}^{-3K-4J-2}(|\rho|) + S_{0,1}^{-J+1}(|\rho|)$ . This implies that  $a_{K+1,J}$  belongs to  $S_{2,2}^{-3K-4J-3}(|\rho|) + S_{0,1}^{-J}(|\rho|)$ , as desired. Furthermore, (3.24) follows from (3.13) and (3.23) since the functions  $A_N^{(j)}$  are expressed in terms of  $\varphi_k$  and  $a_{k,j}$ . One needs the simple observation that

$$a \in S_{2,2}^{\ell_1}(|\rho|) + S_{0,1}^{\ell_2}(|\rho|)$$

implies

$$x_1^k a \in S_{2,2}^{\ell_1+3k}(|\rho|) + S_{0,1}^{\ell_2}(|\rho|).$$

□

Using Lemma 3.3 we will prove the following

**Lemma 3.6.** *For all  $k \geq 1, j \geq 0$ , the function*

$$a_{k,j} - \frac{(k+j)!}{k!} \frac{z\psi n_{k+j}}{(-2i\rho)^{j+2}}$$

is independent of all  $n_\ell$  with  $\ell \geq k+j$ .

*Proof.* It follows from Lemma 3.3 that the function

$$\varphi_k^\Delta - (2\rho)^{-1}(k+1)zn_{k+1}$$

is independent of all  $n_\ell$  with  $\ell \geq k+1$ . We will first prove the assertion for  $j=0$  and all  $k \geq 1$  by induction in  $k$ . In view of (3.21) it is true for  $k=1$ . Suppose it is true for all integers  $k \leq K$  with some integer  $K \geq 1$ . We will prove it for  $k=K+1$ . To this end, we will use equation (3.18) with  $j=0$  and  $k=K$ . Since the RHS is zero, we get that the function

$$2i(K+1)\rho a_{K+1,0} + i\varphi_K^\Delta \psi$$

is independent of all  $n_\ell$  with  $\ell \geq K+1$ . Hence, so is the function

$$a_{K+1,0} + (2\rho)^{-2}(K+1)z\psi n_{K+1}$$

as desired. We will now prove the assertion for all  $k \geq 1, j \geq 0$  by induction in  $j$ . Suppose it is true for  $j \leq J$  and all  $k \geq 1$  with some integer  $J \geq 1$ . We will prove it for  $j=J+1$  and all  $k \geq 1$ . To this end, we will use equation (3.18) with  $j=J+1$  and  $k$  replaced by  $k-1, k \geq 1$ . We have that, modulo functions independent of all  $n_\ell$  with  $\ell \geq k+J+1$ , the LHS is equal to  $2ik\rho a_{k,J+1}$ , while the RHS is equal to  $-k(k+1)a_{k+1,J}$ . Hence the function

$$a_{k,J+1} + (2i\rho)^{-1}(k+1)a_{k+1,J}$$

is independent of all  $n_\ell$  with  $\ell \geq k+J+1$ , which clearly implies the desired assertion.  $\square$

It follows from (3.23) that  $(1-\eta)a_{k,j} \in \mathcal{S}_0^{-j}$  for all  $k \geq 1, j \geq 0$ . Define now the functions  $\tilde{a}_{k,j}$  independent of all  $n_\ell, \ell \geq 0$ , satisfying the relations

$$\begin{aligned} \sum_{k_1+k_2=k} 2i(k_1+1)(k_2+1)\tilde{\varphi}_{k_1+1}\tilde{a}_{k_2+1,j} + \sum_{k_1+k_2+k_3=k} 2i\langle R_{k_1}\nabla_{x'}\tilde{\varphi}_{k_2}, \nabla_{x'}\tilde{a}_{k_3,j} \rangle \\ + \sum_{k_1+k_2=k} i\tilde{\varphi}_{k_1}^\Delta \tilde{a}_{k_2,j} = -\tilde{a}_{k,j-1}^\Delta, \end{aligned} \quad (3.25)$$

and  $\tilde{a}_{0,0} = \psi, \tilde{a}_{0,j} = 0, j \geq 1, \tilde{a}_{k,-1} = 0, k \geq 0$ , where  $\tilde{\varphi}_k^\Delta$  is defined by replacing in the definition of  $\varphi_k^\Delta$  all functions  $\varphi_j$  by  $\tilde{\varphi}_j$ . Using Lemma 3.4 we will prove the following

**Lemma 3.7.** *For all  $k \geq 1, j \geq 0$ , we have  $(1-\eta)(a_{k,j} - \tilde{a}_{k,j}) \in \mathcal{S}_0^{-j-1}$ .*

*Proof.* By Lemma 3.4 together with (3.18) and (3.25) we obtain that the relations

$$\begin{aligned} \sum_{k_1+k_2=k} 2i(k_1+1)(k_2+1)(1-\eta)\tilde{\varphi}_{k_1+1}(a_{k_2+1,j} - \tilde{a}_{k_2+1,j}) \\ + \sum_{k_1+k_2+k_3=k} 2i(1-\eta)\langle R_{k_1}\nabla_{x'}\tilde{\varphi}_{k_2}, \nabla_{x'}(a_{k_3,j} - \tilde{a}_{k_3,j}) \rangle \\ + \sum_{k_1+k_2=k} i(1-\eta)\tilde{\varphi}_{k_1}^\Delta (a_{k_2,j} - \tilde{a}_{k_2,j}) = -(1-\eta)(a_{k,j-1}^\Delta - \tilde{a}_{k,j-1}^\Delta) \end{aligned} \quad (3.26)$$

are satisfied modulo  $\mathcal{S}_0^{-j-1}$ . We will proceed by induction. Suppose now that the assertion is true for all  $j \leq J-1$  and all  $k \geq 1$ , and for  $j=J$  and  $k \leq K$ . This implies that the LHS of (3.26) with  $k=K$  and  $j=J$  is equal to  $2i(K+1)(1-\eta)\tilde{\varphi}_1(a_{K+1,J} - \tilde{a}_{K+1,J})$  modulo  $\mathcal{S}_0^{-J}$ , while the RHS belongs to  $\mathcal{S}_0^{-J}$ . Hence,  $(1-\eta)(a_{K+1,J} - \tilde{a}_{K+1,J})$  belongs to  $\mathcal{S}_0^{-J-1}$ , as desired.  $\square$

In view of (3.11) we have

$$-ih\partial_{x_1}\tilde{u}_\psi|_{x_1=0} = \mathcal{T}_\psi(h, z)f = \text{Op}_h(\tau_\psi)f$$

where

$$\tau_\psi = a \frac{\partial \varphi}{\partial x_1} \Big|_{x_1=0} - ih \frac{\partial a}{\partial x_1} \Big|_{x_1=0} = \rho \psi - i \sum_{j=0}^{N-1} h^{j+1} a_{1,j}.$$

**Lemma 3.8.** *For every integer  $m \geq 0$  there are  $N_m > 1$  and  $\ell_m > 0$  such that for all  $N \geq N_m$  we have the estimate*

$$\|\mathcal{N}(h, z) \text{Op}_h(\psi) - \mathcal{T}_\psi(h, z)\|_{L^2(\Gamma) \rightarrow H_h^m(\Gamma)} \leq C_{N,m} h^{\epsilon N - \ell_m} \quad (3.27)$$

if  $\psi = \psi^0$ ,  $z \in Z_1(1/2 - \epsilon)$ , or  $\psi = \psi^0(1 - \chi)$ ,  $z \in Z_1(1 - \epsilon)$ . If  $\psi = \psi^0$ ,  $z \in Z_2 \cup Z_3$ , then (3.27) holds with  $\epsilon$  replaced by 1.

*Proof.* Denote by  $G_D$  the Dirichlet self-adjoint realization of the operator  $-n^{-1}\Delta$  on the Hilbert space  $L^2(\Omega; n(x)dx)$ . It is easy to see that

$$(h^2 G_D - z)^{-1} = \mathcal{O}(\theta(z)^{-1}) : L^2(\Omega) \rightarrow L^2(\Omega)$$

where  $\theta(z) = |\text{Im } z|$  if  $z \in Z_1$ ,  $\theta(z) = 1$  if  $z \in Z_2 \cup Z_3$ . Clearly, under the conditions of Lemma 3.8, we have  $h < \theta(z) \leq 1$ . The above bound together with the coercivity of  $G_D$  imply

$$(h^2 G_D - z)^{-1} = \mathcal{O}_s(\theta(z)^{-1}) : H^s(\Omega) \rightarrow H^s(\Omega) \quad (3.28)$$

for every integer  $s \geq 0$ . We also have the identity

$$\mathcal{N}(h, z) \text{Op}_h(\psi) f - \mathcal{T}_\psi(h, z) f = -ih\gamma \partial_\nu((h^2 G_D - z)^{-1} V) \quad (3.29)$$

where  $\gamma$  denotes the restriction on  $\Gamma$ , and

$$\begin{aligned} V &= (h^2 \Delta + zn) \tilde{u}_\psi \\ &= \mathcal{K}(h, z) f = (2\pi h)^{-d+1} \int \int e^{\frac{i}{h}(\langle y', \xi' \rangle + \varphi(x, \xi', z))} K(x, \xi', h, z) f(y') d\xi' dy', \end{aligned}$$

where  $K = K_1 + K_2$  with

$$\begin{aligned} K_1 &= [h^2 \Delta, \Phi_\delta] a, \quad K_2 = (x_1^N A_N + h^N B_N) \Phi_\delta, \\ A_N &= \Psi_N a + \sum_{j=0}^{N-1} h^{j+1} A_N^{(j)}, \quad B_N = \Delta a_{N-1} = \sum_{k=0}^{N-1} x_1^k a_{k, N-1}^\Delta + x_1^N F_N^{(N-1)}. \end{aligned}$$

By the trace theorem we get from (3.28) and (3.29),

$$\begin{aligned} \|\mathcal{N}(h, z) \text{Op}_h(\psi) f - \mathcal{T}_\psi(h, z) f\|_{H_h^m(\Gamma)} &\leq \mathcal{O}(h^{-1}) \|(h^2 G_D - z)^{-1} V\|_{H_h^{m+1}(\Omega)} \\ &\leq \mathcal{O}((h\theta(z))^{-1}) \|V\|_{H_h^{m+1}(\Omega)} \leq \mathcal{O}(h^{-2}) \|V\|_{H_h^{m+1}(\Omega)}. \end{aligned} \quad (3.30)$$

To bound the norm of  $V$  we need to bound the kernel of the operator

$$\mathcal{K}_\alpha := \partial_x^\alpha \mathcal{K}(h, z) : L^2(\Gamma) \rightarrow L^2(\Omega).$$

By Lemma 3.1 of [14] we have

$$\text{Im } \rho \geq \frac{|\text{Im } z|}{2|\rho|} \quad \text{on } \text{supp } \eta, z \in Z_1,$$

$$\text{Im } \rho \geq C \langle \xi' \rangle \quad \text{for } z \in Z_2 \cup Z_3 \quad \text{and on } \text{supp}(1 - \eta), z \in Z_1,$$

where  $C > 0$  is some constant. Hence, by (3.15), for  $0 \leq x_1 \leq 2\delta \min\{1, |\rho|^3\}$  we have

$$x_1^N \left| e^{i\varphi/h} \right| \leq x_1^N e^{-\text{Im } \varphi/h} \leq x_1^N e^{-x_1 \text{Im } \rho/2h} \leq C_N \left( \frac{h}{\text{Im } \rho} \right)^N$$

$$\leq \begin{cases} C_N \left( \frac{h|\rho|}{|\operatorname{Im} z|} \right)^N, & z \in Z_1, (x', \xi') \in \operatorname{supp} \eta, \\ C_N \left( \frac{h}{\langle \xi' \rangle} \right)^N, & \text{otherwise.} \end{cases} \quad (3.31)$$

On the other hand, by Lemmas 3.2 and 3.5, for  $|\rho|^4 \geq h$  and  $0 \leq x_1 \leq 2\delta \min\{1, |\rho|^3\}$  we have

$$|\partial_x^\alpha A_N| \leq \begin{cases} C_{\alpha, N} h^{-\ell_\alpha} |\rho|^{-3N}, & \text{on } \operatorname{supp} \eta, \\ C_{\alpha, N} \langle \xi' \rangle^2, & \text{on } \operatorname{supp} (1 - \eta), \end{cases} \quad (3.32)$$

$$|\partial_x^\alpha B_N| \leq \begin{cases} C_{\alpha, N} h^{-\ell_\alpha} |\rho|^{-4N}, & \text{on } \operatorname{supp} \eta, \\ C_{\alpha, N} \langle \xi' \rangle^{-N+1}, & \text{on } \operatorname{supp} (1 - \eta), \end{cases} \quad (3.33)$$

for every multi-index  $\alpha$  with some  $\ell_\alpha > 0$  independent of  $N$ . By (3.31), (3.32) and (3.33), using that  $|\rho|^2 \geq C|\operatorname{Im} z|$ ,  $C > 0$ , on  $\operatorname{supp} \eta$ , we conclude

$$\left| \partial_x^\alpha \left( e^{i\varphi/h} K_2 \right) \right| \leq C_{\alpha, N} h^{-\ell_\alpha} \left( \frac{h}{|\rho|^2 |\operatorname{Im} z|} + \frac{h}{|\rho|^4} \right)^N \leq C_{\alpha, N} h^{-\ell_\alpha} \left( \frac{h}{|\operatorname{Im} z|^2} \right)^N \quad (3.34)$$

for  $z \in Z_1$ ,  $(x', \xi') \in \operatorname{supp} \eta$ , and

$$\left| \partial_x^\alpha \left( e^{i\varphi/h} K_2 \right) \right| \leq C_{\alpha, N} \left( \frac{h}{\langle \xi' \rangle} \right)^{N-\ell_\alpha} \quad (3.35)$$

otherwise, with possibly a new  $\ell_\alpha > 0$  independent of  $N$ . Similar estimates hold for the function  $K_1$ , too. Indeed, observe that on  $\operatorname{supp} [\Delta, \Phi_\delta]$  we have  $\delta \min\{1, |\rho|^3\} \leq x_1 \leq 2\delta \min\{1, |\rho|^3\}$ , and hence

$$\begin{aligned} \left| e^{i\varphi/h} \right| &\leq e^{-\operatorname{Im} \varphi/h} \leq e^{-x_1 \operatorname{Im} \rho/2h} \\ &\leq \begin{cases} e^{-C|\rho|^2 |\operatorname{Im} z|/h}, & z \in Z_1, (x', \xi') \in \operatorname{supp} \eta, \\ e^{-C\langle \xi' \rangle/h}, & \text{otherwise,} \end{cases} \end{aligned} \quad (3.36)$$

with some constant  $C > 0$ . Using (3.36) one can easily get that the estimates (3.34) and (3.35) are satisfied with  $K_2$  replaced by  $K_1$ . Therefore, the function  $K$  satisfies the bounds

$$\left| \partial_x^\alpha \left( e^{i\varphi/h} K \right) \right| \leq \begin{cases} C_{\alpha, N} h^{2\epsilon N - \ell_\alpha}, & z \in Z_1(1/2 - \epsilon), (x', \xi') \in \operatorname{supp} \eta, \\ C_{\alpha, N} \left( \frac{h}{\langle \xi' \rangle} \right)^{N-\ell_\alpha}, & \text{otherwise.} \end{cases} \quad (3.37)$$

Moreover, since  $|\rho| \geq \operatorname{Const} > 0$  on  $\operatorname{supp}(1 - \chi)$ , in the case when  $\psi = \psi^0(1 - \chi)$  we obtain that (3.37) holds with  $Z_1(1/2 - \epsilon)$  replaced by  $Z_1(1 - \epsilon)$  and  $2\epsilon$  replaced by  $\epsilon$ . Note now that the kernel,  $L_\alpha$ , of the operator  $\mathcal{K}_\alpha$  is given by

$$L_\alpha(x, y') = (2\pi h)^{-d+1} \int e^{\frac{i}{h}\langle y', \xi' \rangle} \partial_x^\alpha \left( e^{\frac{i}{h}\varphi(x, \xi', z)} K(x, \xi', h, z) \right) d\xi'.$$

If  $N$  is taken large enough, (3.37) implies the bounds

$$|L_\alpha(x, y')| \leq \begin{cases} C_{\alpha, N} h^{2\epsilon N - \ell_\alpha}, & z \in Z_1(1/2 - \epsilon), \\ C_{\alpha, N} h^{N - \ell_\alpha}, & z \in Z_2 \cup Z_3, \end{cases} \quad (3.38)$$

with a new  $\ell_\alpha > 0$  independent of  $N$ . When  $\psi = \psi^0(1 - \chi)$ , (3.38) holds with  $Z_1(1/2 - \epsilon)$  replaced by  $Z_1(1 - \epsilon)$  and  $2\epsilon$  replaced by  $\epsilon$ . Clearly, (3.27) follows from (3.30) and (3.38).  $\square$

In the case when  $\psi = \psi^0$ , by (3.23) we have

$$\left| \partial_{x'}^\alpha \partial_{\xi'}^\beta a_{k, j} \right| \leq C_{k, j, \alpha, \beta} |\rho|^{-3k - 4j - 2(|\alpha| + |\beta|)}$$

on  $\text{supp } \eta$ , and

$$\left| \partial_{x'}^\alpha \partial_{\xi'}^\beta a_{k,j} \right| \leq C_{k,j,\alpha,\beta} \langle \xi' \rangle^{-j-|\beta|}$$

on  $\text{supp}(1-\eta)$ . Since  $|\rho| \geq C\sqrt{|\text{Im } z|}$  for  $z \in Z_1$ ,  $(x', \xi') \in \text{supp } \chi$  and  $|\rho| \geq C > 0$  for  $z \in Z_2 \cup Z_3$  or  $z \in Z_1$ ,  $(x', \xi') \in \text{supp}(1-\chi)$ , we get

$$\left| \partial_{x'}^\alpha \partial_{\xi'}^\beta a_{k,j} \right| \leq C_{k,j,\alpha,\beta} |\text{Im } z|^{-3k/2-2j-|\alpha|-|\beta|}$$

for  $z \in Z_1$ ,  $(x', \xi') \in \text{supp } \chi$ , and

$$\left| \partial_{x'}^\alpha \partial_{\xi'}^\beta a_{k,j} \right| \leq C_{k,j,\alpha,\beta} \langle \xi' \rangle^{-j-|\beta|}$$

otherwise. Hence  $(1-\chi)a_{k,j} \in \mathcal{S}_0^{-j}$ ,  $h^{k+j}a_{k,j} \in \mathcal{S}_{1/2-\epsilon}^{-j}$  uniformly in  $z \in Z_1(1/2-\epsilon)$ ,  $a_{k,j} \in \mathcal{S}_0^{-j}$  for  $z \in Z_2 \cup Z_3$ . Therefore, we have

$$\text{Op}_h(\eta a_{k,j}) - \text{Op}_h(\eta_1) \text{Op}_h(\eta a_{k,j}) = \mathcal{O}(h^\infty) : L^2(\Gamma) \rightarrow H_h^m(\Gamma)$$

for every integer  $m \geq 0$ , where  $\eta_1 \in C_0^\infty(T^*\Gamma)$  is such that  $\eta_1 = 1$  on  $\text{supp } \eta$ . In view of (2.4) this implies

$$\begin{aligned} \|\text{Op}_h(\eta a_{k,j})\|_{L^2(\Gamma) \rightarrow H_h^m(\Gamma)} &\leq \|\text{Op}_h(\eta a_{k,j})\|_{L^2(\Gamma) \rightarrow L^2(\Gamma)} \|\text{Op}_h(\eta_1)\|_{L^2(\Gamma) \rightarrow H_h^m(\Gamma)} + \mathcal{O}(h^\infty) \\ &\leq \begin{cases} C_{k,j,m} |\text{Im } z|^{-3k/2-2j}, & z \in Z_1(1/2-\epsilon), \\ C_{k,j,m}, & z \in Z_2 \cup Z_3, \end{cases} \end{aligned} \quad (3.39)$$

for every integer  $m \geq 0$ . In view of Proposition 2.3 we also have

$$\|\text{Op}_h((1-\eta)a_{k,j})\|_{L^2(\Gamma) \rightarrow H_h^j(\Gamma)} \leq C_{k,j}. \quad (3.40)$$

By (3.39) and (3.40) we conclude

$$\|\text{Op}_h(a_{k,j})\|_{L^2(\Gamma) \rightarrow H_h^j(\Gamma)} \leq \begin{cases} C_{k,j} |\text{Im } z|^{-3k/2-2j}, & z \in Z_1(1/2-\epsilon), \\ C_{k,j}, & z \in Z_2 \cup Z_3. \end{cases} \quad (3.41)$$

By Lemma 3.7 we also have

$$\|\text{Op}_h((1-\eta)(a_{k,j} - \tilde{a}_{k,j}))\|_{L^2(\Gamma) \rightarrow H_h^{j+1}(\Gamma)} \leq C_{k,j}, \quad z \in Z_1 \cup Z_2 \cup Z_3. \quad (3.42)$$

In the case when  $\psi = \psi^0(1-\chi)$ , the functions  $a_{k,j}$  vanish on  $\text{supp } \chi$ , and hence  $a_{k,j} \in \mathcal{S}_0^{-j}$  for  $z \in Z_1$ . Therefore, in this case the estimate (3.41) holds with  $|\text{Im } z|$  replaced by 1 and  $Z_1(1/2-\epsilon)$  replaced by  $Z_1$ .

We are ready now to prove Theorem 3.1. If  $s = 0$  we put  $b_0^\psi = -ih(1-\eta)\tilde{a}_{1,0}$ , and if  $s \geq 1$  we put

$$b_s^\psi = -i \sum_{j=0}^{s-1} h^{j+1} a_{1,j} - c_s h^s \rho^{-s-1} z n_s \psi - ih^{s+1} (1-\eta) \tilde{a}_{1,s}.$$

In view of Lemma 3.6, the function  $b_s^\psi$  is independent of all  $n_\ell$  with  $\ell \geq s$ . If we take  $N$  big enough, we can decompose the function  $\tau_\psi$  as

$$\tau_\psi = \rho\psi + b_s^\psi + c_s h^s \rho^{-s-1} z n_s \psi + \tilde{b}_s^\psi$$

where

$$\tilde{b}_s^\psi = -ih^{s+1} \eta a_{1,s} - ih^{s+1} (1-\eta) (a_{1,s} - \tilde{a}_{1,s}) - i \sum_{j=s+1}^{N-1} h^{j+1} a_{1,j}.$$

By (3.39), (3.41) and (3.42) we have

$$\left\| \text{Op}_h(\tilde{b}_s^\psi) \right\|_{L^2(\Gamma) \rightarrow H_h^{s+1}(\Gamma)} \leq \begin{cases} C_s h^{s+1} |\text{Im } z|^{-3/2-2s}, & z \in Z_1(1/2 - \epsilon), \\ C_s h^{s+1}, & z \in Z_2 \cup Z_3. \end{cases} \quad (3.43)$$

Moreover, if  $\psi = \psi^0(1 - \chi)$ , the estimate (3.43) holds with  $|\text{Im } z|$  replaced by 1 and  $Z_1(1/2 - \epsilon)$  replaced by  $Z_1$ . We would like to apply Lemma 3.8 with  $m = s + 1$ . To this end we take  $N$  big enough to arrange that

$$\epsilon N - \ell_{s+1} > s + 1.$$

By (3.27) and (3.43) we get

$$\begin{aligned} & \left\| \mathcal{N}(h, z) \text{Op}_h(\psi) - \text{Op}_h(\rho\psi + b_s^\psi + c_s h^s \rho^{-s-1} z n_s \psi) \right\|_{L^2(\Gamma) \rightarrow H_h^{s+1}(\Gamma)} \\ & \leq \begin{cases} C_s h^{s+1} |\text{Im } z|^{-3/2-2s}, & z \in Z_1(1/2 - \epsilon), \\ C_s h^{s+1}, & z \in Z_2 \cup Z_3, \end{cases} \end{aligned} \quad (3.44)$$

if  $\psi = \psi^0$ . Moreover, if  $\psi = \psi^0(1 - \chi)$ , the estimate (3.44) holds with  $|\text{Im } z|$  replaced by 1 and  $Z_1(1/2 - \epsilon)$  replaced by  $Z_1(1 - \epsilon)$ .

We will now use a partition of the unity on  $\Gamma$ . We can find functions  $\{\psi_j^0\}_{j=1}^J$  such that  $\sum_{j=1}^J \psi_j^0 = 1$  and (3.44) is valid with  $\psi$  replaced by each  $\psi_j$ , where  $\psi_j$  is defined by replacing in the definition of  $\psi$  the function  $\psi^0$  by  $\psi_j^0$ . Summing up all the estimates we get (3.8) and (3.9), respectively.  $\square$

#### 4. IMPROVED ESTIMATES

To prove Theorem 1.1 we actually need the following improved version of Theorem 3.1.

**Theorem 4.1.** *Let  $z \in Z_1(1/2 - \epsilon)$ ,  $0 < \epsilon \ll 1$ . Then, for every integer  $s \geq 1$  there are an operator  $\mathcal{B}_s$  independent of all  $n_k$  with  $k \geq s$  and an operator*

$$\mathcal{A}_s = \mathcal{O}_s(h^{-s}) : H_h^{s+1}(\Gamma) \rightarrow L^2(\Gamma) \quad (4.1)$$

independent of all  $n_k$  with  $k \geq 1$  such that

$$\|\mathcal{A}_s \mathcal{N}(h, z) - \mathcal{B}_s - n_s I\|_{L^2(\Gamma) \rightarrow L^2(\Gamma)} \leq C_s h |\text{Im } z|^{-3s/2-1} \quad (4.2)$$

where  $I$  denotes the identity. If  $z \in Z_2 \cup Z_3$ , then (4.2) holds with  $|\text{Im } z|$  replaced by 1.

*Proof.* Recall that by (3.5), (3.6), (3.7), we have that for every integer  $k$ ,  $h^{\frac{k-}{4}} \rho^k \in \mathcal{S}_{1/2-\epsilon}^k$  uniformly in  $z \in Z_1(1/2 - \epsilon)$  and  $\rho^k \in \mathcal{S}_0^k$  if  $z \in Z_2 \cup Z_3$ . We would like to apply Proposition 2.2 with

$$a_s^+ = (c_s h^s \rho^{-s-1} z)^{-1}, \quad a_s^- = c_s h^s \rho^{-s-1} z.$$

Using (3.2) one can easily check that (2.5) is satisfied with  $\mu = h |\text{Im } z|^{-2} = \mathcal{O}(h^{2\epsilon})$ . By (2.6) we get

$$\left\| \text{Op}_h(a_s^+) \text{Op}_h(a_s^-) - I \right\|_{L^2(\Gamma) \rightarrow L^2(\Gamma)} \leq C h^{2\epsilon} \leq 1/2 \quad (4.3)$$

if  $h$  is taken small enough. It follows from (4.3) that the operator  $\text{Op}_h(a_s^-)$  is invertible with an inverse

$$\mathcal{A}_s := (\text{Op}_h(a_s^-))^{-1} = (\text{Op}_h(a_s^+) \text{Op}_h(a_s^-))^{-1} \text{Op}_h(a_s^+).$$

Since  $h^s a_s^+ \in \mathcal{S}_{1/2-\epsilon}^{s+1}$  uniformly in  $z$ , by Proposition 2.3 we have

$$h^s \text{Op}_h(a_s^+) = \mathcal{O}_s(1) : H_h^{s+1}(\Gamma) \rightarrow L^2(\Gamma)$$

which implies (4.1). By (3.27) and (4.1),

$$\|\mathcal{A}_s \mathcal{N}(h, z) - \mathcal{A}_s \mathcal{T}(h, z)\|_{L^2(\Gamma) \rightarrow L^2(\Gamma)} \leq C_{N,s} h^{\epsilon N - s - \ell_s + 1} \leq h \quad (4.4)$$

if  $N$  is taken large enough, where  $\mathcal{T} = \sum_{j=1}^J \mathcal{T}_{\psi_j}$ . On the other hand, we can write

$$\mathcal{A}_s \mathcal{T} = \mathcal{B}_s + n_s I + \tilde{\mathcal{B}}_s$$

where

$$\mathcal{B}_s = \mathcal{A}_s \text{Op}_h(\rho + b_s), \quad b_s = \sum_{j=1}^J b_s^{\psi_j},$$

$$\tilde{\mathcal{B}}_s = \mathcal{A}_s \text{Op}_h(\tilde{b}_s), \quad \tilde{b}_s = \sum_{j=1}^J \tilde{b}_s^{\psi_j}.$$

Clearly, the operator  $\mathcal{B}_s$  is independent of all  $n_k$  with  $k \geq s$  because so is the function  $b_s$ . Therefore, it follows from (4.4) that to prove (4.2) it suffices to prove the bound

$$\left\| \text{Op}_h(a_s^+) \text{Op}_h(\tilde{b}_s) \right\|_{L^2(\Gamma) \rightarrow L^2(\Gamma)} \leq \begin{cases} C_s h |\text{Im } z|^{-3s/2-1}, & z \in Z_1(1/2 - \epsilon), \\ C_s h, & z \in Z_2 \cup Z_3. \end{cases} \quad (4.5)$$

In view of Lemmas 3.5 and 3.7, we have  $\tilde{b}_s = h^{s+1} g_s$  with  $g_s \in S_{2,2}^{-3-4s}(|\rho|) + S_{0,1}^{-s-1}(|\rho|)$  uniformly in  $h$  as long as  $|\rho|^4 \geq h$ . Thus, (4.5) is equivalent to

$$\left\| \text{Op}_h(\rho^{s+1}) \text{Op}_h(g_s) \right\|_{L^2(\Gamma) \rightarrow L^2(\Gamma)} \leq \begin{cases} C_s |\text{Im } z|^{-3s/2-1}, & z \in Z_1(1/2 - \epsilon), \\ C_s, & z \in Z_2 \cup Z_3. \end{cases} \quad (4.6)$$

To prove (4.6) observe that  $\rho^{s+1} g_s \in S_{2,2}^{-2-3s}(|\rho|) + S_{0,1}^0(|\rho|)$  uniformly in  $h$ , which yields the bounds

$$\left| \partial_{x'}^\alpha \partial_{\xi'}^\beta (\rho^{s+1} g_s) \right| \leq \begin{cases} C_{s,\alpha,\beta} |\text{Im } z|^{-3s/2-1-|\alpha|-|\beta|}, & z \in Z_1(1/2 - \epsilon), \\ C_{s,\alpha,\beta}, & z \in Z_2 \cup Z_3. \end{cases} \quad (4.7)$$

By (2.4) and (4.7) we get

$$\left\| \text{Op}_h(\rho^{s+1} g_s) \right\|_{L^2(\Gamma) \rightarrow L^2(\Gamma)} \leq \begin{cases} C_s |\text{Im } z|^{-3s/2-1}, & z \in Z_1(1/2 - \epsilon), \\ C_s, & z \in Z_2 \cup Z_3. \end{cases} \quad (4.8)$$

On the other hand, applying Proposition 2.2 with  $a^+ = \rho^{s+1}$  and  $a^- = g_s$  yields the bound

$$\begin{aligned} & \left\| \text{Op}_h(\rho^{s+1}) \text{Op}_h(g_s) - \text{Op}_h(\rho^{s+1} g_s) \right\|_{L^2(\Gamma) \rightarrow L^2(\Gamma)} \\ & \leq \begin{cases} C_s h |\text{Im } z|^{-3s/2-3}, & z \in Z_1(1/2 - \epsilon), \\ C_s h, & z \in Z_2 \cup Z_3. \end{cases} \end{aligned} \quad (4.9)$$

Clearly, (4.6) follows from (4.8) and (4.9).  $\square$



## 5. PROOF OF THEOREM 1.1

Define the DN maps  $\mathcal{N}_j(\lambda)$ ,  $j = 1, 2$ , by

$$\mathcal{N}_j(\lambda)f = \partial_\nu u_j|_\Gamma$$

where  $\nu$  is the Euclidean unit inner normal to  $\Gamma$  and  $u_j$  is the solution to the equation

$$\begin{cases} (\Delta + \lambda^2 n_j(x)) u_j = 0 & \text{in } \Omega, \\ u_j = f & \text{on } \Gamma, \end{cases} \quad (5.1)$$

and consider the operator

$$T(\lambda) = \mathcal{N}_1(\lambda) - \mathcal{N}_2(\lambda).$$

Clearly,  $\lambda$  is a transmission eigenvalue if there exists a non-trivial function  $f$  such that  $T(\lambda)f = 0$ . Thus Theorem 1.1 is a consequence of the following

**Theorem 5.1.** *Under the conditions of Theorem 1.1, the operator  $T(\lambda)$  sends  $L^2(\Gamma)$  into  $H^{j+1}(\Gamma)$ . Moreover, there exists a constant  $C > 0$  such that  $T(\lambda)$  is invertible for  $|\operatorname{Im} \lambda| \geq C(\operatorname{Re} \lambda + 1)^{1-\kappa_j}$  with an inverse satisfying in this region the bound*

$$\|T(\lambda)^{-1}\|_{H^{j+1}(\Gamma) \rightarrow L^2(\Gamma)} \lesssim |\lambda|^{j-1} \quad (5.2)$$

where the Sobolev space is equipped with the classical norm.

*Proof.* We make our problem semi-classical by putting  $h = |\operatorname{Re} \lambda^2|^{-1/2}$ ,  $z = h^2 \lambda^2 = \pm 1 + i \operatorname{Im} z$ , if  $|\operatorname{Re} \lambda^2| \geq |\operatorname{Im} \lambda^2|$ ,  $\pm \operatorname{Re} \lambda^2 > 0$ , and  $h = |\operatorname{Im} \lambda^2|^{-1/2}$ ,  $z = h^2 \lambda^2 = \operatorname{Re} z + i$ , if  $|\operatorname{Re} \lambda^2| \leq |\operatorname{Im} \lambda^2|$ . Clearly,  $h \sim |\lambda|^{-1}$ . We set  $\mathcal{N}_j(h, z) = -ih\mathcal{N}_j(\lambda)$  and

$$T(h, z) = \mathcal{N}_1(h, z) - \mathcal{N}_2(h, z).$$

We now apply Theorem 4.1 with  $s = j \geq 1$ . In view of the conditions (1.5) and (1.6), we get

$$\left\| \mathcal{A}_j T(h, z) - (n_j^{(1)} - n_j^{(2)}) I \right\|_{L^2(\Gamma) \rightarrow L^2(\Gamma)} \leq C_j h |\operatorname{Im} z|^{-3j/2-1} \quad (5.3)$$

for  $z \in Z_1(1/2 - \epsilon)$ , where

$$n_j^{(\ell)} = (j!)^{-1} \partial_\nu^j n_\ell|_\Gamma, \quad \ell = 1, 2.$$

When  $z \in Z_2 \cup Z_3$ , the estimate (5.3) holds with  $|\operatorname{Im} z|$  replaced by 1. It follows from (5.3) that the operator  $(n_j^{(1)} - n_j^{(2)})^{-1} \mathcal{A}_j T(h, z)$  is invertible for  $z \in Z_1(1/2 - \epsilon)$ ,  $|\operatorname{Im} z| \geq (C'_j h)^{1/(3j/2+1)}$ , and for  $z \in Z_2 \cup Z_3$ ,  $h$  small enough. Hence so is  $T(h, z)$  and we have the bound

$$\|T(h, z)^{-1}\|_{H_h^{j+1}(\Gamma) \rightarrow L^2(\Gamma)} \leq \mathcal{O}(1) \|\mathcal{A}_j\|_{H_h^{j+1}(\Gamma) \rightarrow L^2(\Gamma)} \leq \mathcal{O}(h^{-j}). \quad (5.4)$$

Now (5.2) follows from (5.4) after passing from  $(h, z)$  to  $\lambda$  and using the fact that the semi-classical norm in  $H_h^{j+1}(\Gamma)$  is bounded from above by the classical norm in  $H^{j+1}(\Gamma)$ .  $\square$

It is worth noticing that it follows from the estimate (3.9) that the operator  $T(h, z)$  can be inverted outside the glancing region for much smaller  $|\operatorname{Im} z|$ . In other words, to improve the eigenvalue-free region (1.7) one has to improve the parametrix in the glancing region, only. More precisely, we have the following

**Proposition 5.2.** *Let  $z \in Z_1(1 - \epsilon)$ . Then, under the conditions of Theorem 1.1, there exists an operator*

$$\tilde{\mathcal{A}}_j = \mathcal{O}(h^{-j}) : H_h^{j+1}(\Gamma) \rightarrow L^2(\Gamma)$$

such that

$$\left\| T(h, z) \tilde{\mathcal{A}}_j - \operatorname{Op}_h(1 - \chi) \right\|_{H_h^{j+1}(\Gamma) \rightarrow H_h^{j+1}(\Gamma)} \leq Ch. \quad (5.5)$$

When  $z \in Z_2 \cup Z_3$ , the estimate (5.5) holds with  $\chi$  replaced by 0.

*Proof.* By (3.9) with  $s = j$  we have

$$\left\| T(h, z) \text{Op}_h(1 - \chi) - \text{Op}_h \left( (1 - \chi) c_j h^j \rho^{-j-1} z (n_j^{(1)} - n_j^{(2)}) \right) \right\|_{L^2(\Gamma) \rightarrow H_h^{j+1}(\Gamma)} \leq C_j h^{j+1} \quad (5.6)$$

for  $z \in Z_1(1 - \epsilon)$ . Let  $\chi_1 \in C_0^\infty(T^*\Gamma)$  be such that  $\chi = 1$  on  $\text{supp } \chi_1$ ,  $\chi_1 = 1$  in a neighbourhood of  $\Sigma$ , and set

$$\tilde{a}_j^+ = (1 - \chi) \rho^{-j-1} c_j z (n_j^{(1)} - n_j^{(2)}), \quad \tilde{a}_j^- = (1 - \chi_1) \rho^{j+1} \left( c_j z (n_j^{(1)} - n_j^{(2)}) \right)^{-1}.$$

We have  $\tilde{a}_j^+ \in \mathcal{S}_0^{-j-1}$ ,  $\tilde{a}_j^- \in \mathcal{S}_0^{j+1}$  and  $\tilde{a}_j^+ \tilde{a}_j^- = 1 - \chi$ . We now apply Proposition 2.4 with  $\tilde{a}_j^+$  and  $\tilde{a}_j^-$  in place of  $a^+$  and  $a^-$ . We have

$$\left\| \text{Op}_h(\tilde{a}_j^+) \text{Op}_h(\tilde{a}_j^-) - \text{Op}_h(1 - \chi) \right\|_{H_h^{j+1}(\Gamma) \rightarrow H_h^{j+1}(\Gamma)} \leq Ch. \quad (5.7)$$

Clearly, (5.5) follows from (5.6) and (5.7) with  $\tilde{\mathcal{A}}_j = h^{-j} \text{Op}_h(1 - \chi) \text{Op}_h(\tilde{a}_j^-)$ .  $\square$

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UNIVERSITÉ DE NANTES, LABORATOIRE DE MATHÉMATIQUES JEAN LERAY, 2 RUE DE LA HOUSSINIÈRE, BP 92208, 44322 NANTES CEDEX 03, FRANCE

*E-mail address:* Georgi.Vodev@univ-nantes.fr