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► **To cite this version:**

Paola Cantu. Peano and Gödel. Gabriella Crocco and Eva-Maria Engelen. Kurt Gödel: Philosopher-Scientist, Presses Universitaires de Provence, pp.107-126, 2016, 9782853999762. hal-01451603

HAL Id: hal-01451603

<https://hal.science/hal-01451603>

Submitted on 28 Feb 2023

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Peano and Gödel

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This paper will take into account several passages from Gödel's unpublished philosophical manuscripts, the *Max Phil*, and compare them with relevant passages from the *Formulary* and from Russell's *Principia Mathematica* on definite descriptions, definitions and functions. Gödel's remarks together with a reconstruction of Peano's general notion of function will be used to prove how intensively Gödel had worked on Peano's writings and to oppose, or at least to restrict, the conceptual continuity between Peano and Russell outlined in recent literature. In particular, I will claim that a definite description that fails to satisfy the conditions of existence and uniqueness is meaningful in Peano, although it can not play the role of a definition. This is possible because 1) the function that characterizes definite descriptions is introduced as the inverse of the iota function, and 2) functions can be intended in a general sense, without a determined domain, or in a strict sense, and 3) only in the second case the conditions of existence and uniqueness need to be proved in order to use the definite description as a definition. I will conclude by suggesting some reasons that might explain Gödel's interest in Peano's approach.

1. Introduction

The paper will take into account several passages from Gödel's unpublished philosophical manuscripts, called *Max Phil*, which are written in German shorthand (Gabelsberger), and have been partially transcribed by a research group directed by Gabriella Crocco at the University of Aix-Marseille.¹ Special attention will be devoted to some excerpts where Gödel discusses Peano's understanding of definite descriptions, functions and definitions. By comparison with relevant passages from the fifth edition of Peano's *Formulary* and from the *Principia Mathematica*, I will discuss some philosophical issues concerning the role of definitions, definite descriptions and functions. In particular, I will use Gödel's reading of the *Formulary* together with my reconstruction of Peano's understanding of the notion of function to oppose, or at least to restrict, the historical and conceptual continuity between Peano and Russell that has often been mentioned in recent literature: see for example the studies by Ivor Grattan-Guinness (2003, p. 65) or Francisco Rodriguez-Consuegra (2000, p. 23).

¹ This paper was presented at the 14th *International Congress of Logic, Methodology and Philosophy of Sciences*, July 19–26, 2011, Nancy (France), but had already been discussed with colleagues at the Meeting of the participants to the ANR project “Kurt Gödel philosopher: from logic to cosmology” directed by Gabriella Crocco. I thank all participants and Gabriella Crocco in particular for the precious comments and fruitful discussions on the topic of definite descriptions.

It is well known that Peano was mentioned several times in Gödel's paper *Russell's mathematical logic* (1990), yet Gödel limits himself to general praise of Peano, whom he considers either as a follower of Leibniz's² or as a precursor of Russell and Whitehead's *Principia Mathematica*.³ There is no evidence in the published papers that Gödel had worked intensively on the *Formulary*, nor that he had noticed any relevant differences between Peano's and Russell's conceptions. Given the passage mentioned in footnote 2, there is general agreement on the fact that Gödel saw a certain continuity between the symbolism used by Peano and the one used by Russell. There seems to be no reason to doubt that Gödel might have seen a similar continuity between Russell's analysis of definite descriptions and Peano's remarks on the topic.

This continuity between Peano and Russell has often been asserted in recent literature: even authors who maintain that Russell introduced his theory independently from the work of Peano, remark that there are similarities between the two theories. Discussing Russell's treatment of definite descriptions, Grattan-Guinness observes, for example, that the conditions imposed by Russell are the same introduced in Peano (1897) as conditions for a function to be single-valued.⁴ Rodriguez-Consuegra remarked that of the three main features of Russell's theory of definite descriptions—1) existence and uniqueness are the conditions expressing the logical power of the definite article, 2) the definite article is eliminable and 3) sentences containing the definite article but not satisfying the two mentioned conditions are false—the first two are explicitly stated by Peano, and the third is stated in a slightly different form (sentences that fail to satisfy the given conditions are meaningless rather than false) [Rodriguez-Consuegra, 2000, p. 5–6].⁵

The aim of this paper however will on the contrary be that of showing that Gödel's unpublished manuscripts reveal not only how intensively he had worked on Peano's *Formulary*,⁶ but also that he clearly distinguished some ideas suggested by Peano from the solutions proposed by Russell. In particular, basing my analysis on the 1897 paper mentioned both by Grattan-Guinness and by

² "It seems reasonable to suspect that it is this incomplete understanding of the foundations which is responsible for the fact that mathematical logic has up to now remained so far behind the high expectations of Peano and others who (in accordance with Leibniz's claims) had hoped that it would facilitate theoretical mathematics to the same extent as the decimal system of numbers has facilitated numerical computations". For a comparison of Leibniz's project of a characteristic with the writings by Peano and Gödel see (Cantù 2014).

³"It was in this line of thought of Frege and Peano that Russell's work set in. Frege, in consequence of his painstaking analysis of the proofs, had not gotten beyond the most elementary properties of the series of integers, while Peano had accomplished a big collection of mathematical theorems expressed in the new symbolism, but without proofs. It was only in *Principia mathematica* that full use was made of the new method for actually deriving large parts of mathematics from a very few logical concepts and axioms." (Gödel, 1990, p. 119–120).

⁴"The search for a satisfactory theory of denoting functions continued, and in 1905 Russell laid down in *Mind* conditions for the non-existence of definite descriptions in propositions such as 'the present King of France is bald': there should be one and only one entity involved, and it should indeed have the property required; otherwise the proposition was false, as in this example (OD). These criteria were exactly those assumed by Cauchy and his successors for a mathematical function to be single-valued; indeed, Peano had stated them explicitly in that context in a survey (1897) of mathematical logic in the second edition of the *Formulaire*. This was one of the materials sent to Russell by Peano in 1900; presumably Russell had forgotten it when proposing his own criterion (in a more general setting, of course)" (Grattan-Guinness, 2003, p. 65).

⁵Rodriguez-Consuegra grounds his interpretations on three passages by Peano that I will discuss further on in the paper: (Peano, 1896–97, p. 582), (Peano, 1897, p. 50) and (Peano, 1898, p. 523).

⁶Apart from the passages extracted from the *MaxPhil* that I will mention later on, this can be proved also by Gödel's detailed annotations on Peano's *Arithmetices Principia* and on Peano's *Formulary* that are contained in another unpublished notebook called *Logik und Grundlagen* and written before 1952 (see 10a39, p. 105–119). I thank Mark van Atten for drawing my attention to this.

Rodriguez-Consuegra, I will claim that Gödel was right in remarking that the example used by Peano—the maximum of a function—illustrates a relevant difference between Peano’s and Russell’s conception of definite descriptions:⁷ the conditions of existence and uniqueness were certainly stated by Peano, but it is not at all evident that they played the same role as in Russell. Besides, Gödel’s remarks will outline other relevant differences between Peano and Russell, concerning the understanding of falsity as a null object—an object that has contradictory properties—rather than as the empty class, and the definability of the inverse iota—i.e. of Peano’s symbol for the definite article (§ 2). Finally, I will rely on Gödel’s analysis of the notions of definition and function in order to support my reconstruction of Peano’s distinction between defined and undefined functions, which is in turn related to his peculiar understanding of the notion of equality (§ 3).

I will restrict myself here to some passages from *Max Phil X* (March 1943–January 1944)—given that this notebook has been transcribed entirely and will be published soon—but there are several other passages from *Max Phil IV* (May 1941–April 1942) and *Max Phil IX* (November 1942–February 1943) that go in the same direction.⁸

2. Definite descriptions and definitions

2.1 Gödel’s remarks on the definability of the inverse iota

In the following remark on logic and the foundations of mathematics, Gödel compares several kind of definitions: typographical abbreviations (defined here as sameness of sense and reference), synonymical functions (defined here as sameness of denotation), and descriptive functions (described here as sameness of sense).

There are three kind of definitions to consider: 1) statements of the form: a has the same sense and the same reference as b (where infinitely many cases might be comprised by the variables a and b), i.e. as mere typographical abbreviations; 2) statements of the form: a has the same reference as b and is a name of what is described by b ; 3) statements of the form: $\varphi(a)$, i.e. descriptions (yet existence and uniqueness should then be proved). To this effect the axioms of geometry could be definitions of the primitive concepts. (Gödel, forthcoming, *Max Phil X*, p. 25–26, my transl.).⁹

As Gabriella Crocco remarks in the commentary to this passage, the first conception is near to that of Frege, and the second to that of Russell. What about the third? My claim is that the analysis of a further unpublished remark suggests a connection with Peano and applies to the symbol used in definite descriptions: the inverse iota.

⁷ I will not discuss in detail Gödel’s own conception of definite descriptions, which has been deeply analyzed by Gabriella Crocco (2012) in an interesting paper first presented at the 2010 ANR meeting and to which this article is deeply indebted.

⁸See footnote 22 on page 18. An interesting historical question that might receive an answer only when all the *Max Phil* manuscripts will have been transcribed is why Gödel did not explicitly oppose Peano and Russell in the 1944 paper, given that he clearly opposed them in the unpublished manuscripts.

⁹“Es gibt drei Arten Def<inition> aufzufassen: 1. Als Aussagen der Form: a ist sinn- und bedeutungsgleich mit b (wobei durch Variablen in a und b unendlich viele Fälle zusammengefaßt werden können), d.h. als bloß typogr<aphische> Abkürzungen. 2. Als Aussagen der Form: a ist bedeutungsgleich mit b und ist ein Name des mit b Beschriebenen. 3. Als Aussagen der Form $\varphi(a)$, d.h. Beschreibungen (dann muß aber Existenz und Eindeutigkeit bewiesen werden). In diesem Sinn könnten die Ax<iome> der Geometrie Def<initionen> der Grundbegriffen sein.”

In the *Principia Mathematica* by Russell and Whitehead the expression of a definite description $(\iota x)\varphi(x)$ is not definable, because it is an incomplete symbol rather than a proper name, so one might define its use, but not its meaning (Whitehead and Russell, 1910-13, p. 65–66). Yet, in the following passage, Gödel remarks that Peano has a way to define the inverse iota:

When $(\iota x)\varphi(x)$ designates the respective x , neither it nor the statements in which it occurs can be defined from the other logical concepts (i.e. one cannot compose symbols that have the same reference)—[this holds] for propositions having the same reference (in Russell’s sense), in case propositions denote truth values, but not for propositions having the same sense. Yet, apparently, one might actually define $(\iota x)\varphi(x)$, as Peano does, in the following way: $(\exists! x)\varphi(x). \varphi(a) \supset (\iota x)\varphi(x) = a$. Here the $=$ cannot denote the identity of reference, because a , as a variable, does not have any determinate reference. Then it should be the name of a variable, i.e. the label [...] (Gödel, forthcoming, Max Phil X, p. 26–27, my transl.).¹⁰

Gödel remarks that one might define the expression of a definite description by means of a proposition containing a bound (or apparent) variable and a name for variables. So, the second part of the quoted formula cannot be reduced to one of the former cases, because there is not identity of denotation, given that a is a variable name and not a variable—in other words, ‘the only x such that $\varphi(x)$ ’ denotes an individual, whereas ‘ a ’ denotes a variable. Therefore, this kind of definition of the inverse iota cannot be reduced either to the first or to the second kind of definition mentioned in the passage above, because there is no identity of reference.

The link between definitions and definite descriptions is already traced here: different understandings of what is a definition will allow different understandings of what is the expression of a definite description. Now, in order to better understand the meaning and the interest of Gödel’s remarks, I will comment upon some relevant passages from the *Principia* and from the *Formulary*, in order to show something that is not so evident from the text, but that is relevant for the determination of the condition of truth or falsity of a definite description when the grammatical subject is empty, and for the analysis of null objects. I will claim that Peano, unlike Russell, introduced the symbol used in definite descriptions as the inverse of the singleton operator, and thus as a function that can be defined, while Russell takes it to be part of an incomplete expression that cannot be defined.

2.2 Whitehead and Russell’s *Principia Mathematica*

Whitehead and Russell’s *Principia Mathematica* are based on the distinction between propositional function and function. A propositional function is an incomplete expression that can acquire meaning only when it is completed in some way, for example when one determines the domain in which the values of the function might vary. What is nowadays understood by mathematical function, requiring definability as a necessary criterion, is not a propositional function but rather a function tout court. We will see in §3.2 that this conception is quite different from the conception of function introduced by Peano in the *Formulary*.

¹⁰“Wenn $(\iota x)\varphi(x)$ das betreffende x bezeichnet, so ist es (und auch die Aussagen, in denen es vorkommt) aus den übrigen logischen Begriffen undefinierbar (d.h. man kann keine bedeutungsgleichen Symbole zusammensetzen)—für Sätze <findet man> wohl bedeutungsgleich<e> (in der Russellschen Weise), falls Sätze Wahrheitswerte bedeuten, aber jedenfalls nicht sinnliche Sätze. Aber scheinbar kann man sogar $(\iota x)\varphi(x)$ definieren in Peanoscher Weise durch: $(\exists! x)\varphi(x). \varphi(a) \supset (\iota x)\varphi(x) = a$. Hier kann $=$ nicht die Bedeutungsgleichheit bedeuten, weil a als Variable gar keine bestimmte Bedeutung hat. Da müsste es ein Variablenname sein, d.h. die Wertnahme. [...]

Given the primitive notions of propositional function, truth and falsity, the empty class—i.e. the class that has no members—can be defined as the result of any propositional function that becomes a falsity for any assignation. For example, the class determined by the function “being non-identical with itself”. The universe class, i.e. the class that contains all elements, can similarly be defined as the result of any propositional function that becomes a truth for any assignation. For example it is the class determined by the function “being identical with itself”. The concepts of empty class and universal class can thus be defined by means of the primitive concept of propositional function (Whitehead and Russell, 1910-13, p. 29ff.).

The concept of a definite description can be expressed by means of the notion of propositional function too. It is not a primitive concept, but it cannot be defined. What can be defined, is its use in determinate contexts (Whitehead and Russell, 1910- 13, p. 65ff.). The expression “*the* x such that” is correctly used only when there is a propositional function that is satisfied by just one x . The inverse iota symbol is soon abandoned and a more general notation based on relations is used in the following pages:

$$R'y$$

i.e. the R of y is the only x that is in the relation R with y . According to the *Principia*, ordinary mathematical functions are of this kind, because they satisfy the condition of functionality, so they are descriptions and are defined on a given domain (Whitehead and Russell, 1910-13, p. 30ff.).

2.3 The inverse iota symbol in the *Formulary*

Even if Russell and Whitehead initially used Peano’s symbol \imath for the inverse iota operator, the way the symbol was introduced is different. In the fifth edition of the *Formulary*—the latin version quoted in Gödel’s *Max Phil*—Peano introduced two operators that are each the inverse of the other.

The iota operator ι , whose symbol is an abbreviation of the greek term “isos” (equal), corresponds to the operator that associates to an element the class of all elements that are equal to it. Properly speaking, the iota operator associates to an element its class of equivalence with respect to a relation of equality. So, for example, if the relation is an identity, then “isos” is the singleton operator: $\iota : x \rightarrow \{x\}$.

$$7.1.0 \quad \iota x = y \ni (y = x) \quad Df\iota$$

The inverse iota operator \imath is defined as the inverse operation of the iota operator and is read as “illo” (the): given any singleton, the function extracts the element of the singleton; more generally, given a class of equivalence with respect to the equality relation the function extracts its representant, which is unique, because all elements are equal to it. So, given a singleton $a = z$, then $\imath : a \rightarrow z$ (Peano, 1906, §7.2.0, p. 13).

$$7.2.0 \quad a \in Cls. \exists a: x, y \in a. \supset_{x,y}. x = y: \supset:: z = \imath a. =. a = \iota z \quad Df\imath$$

The author of these definitions, first introduced in the second edition of the *Formulary* and then substantially maintained with only minor changes to the annotations in the third and the fifth editions, is Alessandro Padoa, who had worked intensively on the notion of equivalence class,

and wanted to distinguish the sets that are also elements. Peano used the notation *Elm* instead of the notation *Un* introduced by Burali Forti, because he wanted to highlight the fact that this notion does not depend on the concept of number (Padoa, 1896-1899, p. 117).¹¹

Peano uses the iota operator ι to suggest an alternative possible definition of the negation (Peano, 1906, §7.1.4, p. 13)

$$7.1.4 \quad a \in Cls. \supset. -a = x \ni (\iota x \neg a = \Lambda) \quad Dfp -$$

What about the inverse iota operator? Can it also be used to define any other notions? Firstly, one should remark that Peano's empty set is not defined as the set such that no set is a member of it, but rather as the class that contains the objects that are common to any class, because it aims at describing the role of the number 0 in arithmetics (Peano, 1906, §6.1.0, p. 12).

$$6.1.0 \quad \Lambda = x \ni (a \in Cls. \supset. x \in a) \quad Df\Lambda$$

Secondly, one should note that the definition of Λ by means of the inverse iota operator allows us to define it as an individual object rather than as a set: it is the element associated to a set that contains all the x such that $(a) x = a \wedge \neg a$, i.e. the elements that, for any property, satisfy that property and its contradictory (Peano, 1906, §7.2.5, p. 13).

$$7.2.5 \quad a \in Cls. \exists a: x, y \in a. \supset. x = y: \supset. \Lambda = \iota x \ni [a \in Cls. \supset. a - a = x] \quad Dfp\Lambda$$

The inverse iota operator \imath is thus used to suggest an alternative possible definition of Λ as a null object, so that it might be considered as an individual rather than as a class.

2.4 The interest of Gödel's remarks with respect to the comparison between Russell and Peano

Our detailed analysis of the two texts reveals several differences between Peano and Russell, contrary to what is claimed in the literature.

1. The inverse iota is a symbol used to denote the inverse of the *isos* function in Peano, while it is a symbol that is part of an incomplete expression in the *Principia Mathematica*.
2. The inverse iota function can be defined by means of the primitive notions of class, equality, *isos* in Peano.¹² The expression of a definite description cannot be properly defined, if not in context, according to Russell.

¹¹See also the tribute to Padoa in (Peano, 1901, § 7, p. 31).

¹²Actually, Peano is more cautious on this point: he says that what we define is not the expression $\imath a$ but rather the whole formula $x = \imath a$. See for example the following passage, where the symbol \imath had not yet been introduced, and the inverse of the ι function was expressed by $\bar{\iota}$: "Actually this definition gives the meaning of the whole formula $x = \bar{\iota}a$ ", and not only of the group $\bar{\iota}a$. But any proposition containing $\bar{\iota}a$ can be reduced to the form $\bar{\iota}a \in b$, where b is a class, and this can be reduced to $a \supset b$, where the symbol $\bar{\iota}$ has vanished, even if we are not able to form an equality whose first member is $\bar{\iota}a$ and the second member a group of known symbols." In other words, Peano is saying that there are ways to express what is meant by the $\bar{\iota}$ symbol without using it, but it cannot be defined by means of a typographical abbreviation. For this very reason, I disagree with Francisco Rodriguez-Consuegra when he interprets this passage as suggesting that Peano believed the symbol could be "eliminated" (Rodriguez-Consuegra, 2000, p. 14). But why should Peano have maintained the symbol among the primitives, had he really believed in the

3. In Peano's *Formulary* the inverse iota function is a symbol that visually explains its meaning: the symbol is obtained by a 180° rotation of the iota symbol and denotes the inverse of the function expressed by the iota symbol, which in turn is an abbreviation for the Greek word *isos*, meaning equal or identical (Peano, 1906, I, § 7, p. 13). The symbolism is intended thus to be an image of the conceptual. In Russell's *Principia* there is no iota function whatsoever and it is not at all clear why Russell uses the inverse iota in the expression of definite description. The only reason is that he takes the symbolism from the preceding literature, but, as we have seen, abstracting it from its original context.
4. According to Peano, not only can iota and inverse iota be defined by means of other logical primitives, such as that of class, equality of individuals and implication, but they can also be used as primitive concepts in order to define other logical concepts such as the complement of a class and the empty class (Peano, 1906, 7.1.4 and 7.2.5, p. 13).
5. The empty class is defined as a class both in Peano and in Russell. But Peano considers also an alternative definition, based on the notion of inverse iota operator \imath , such that the empty class might be considered as an element (*illo x*).
6. According to Russell, the mathematical notion of function needs to be defined on a given domain, because the notion of function includes the condition of functionality. On the contrary, the notion of function in Peano is as general as the notion of correspondance between two domains. Functionality is not assumed as part of the definition of function but is derived from the properties of equality.

Most of the mentioned differences are not explicitly outlined by Gödel in the above quotation, but seem to constitute the background of his analysis, proving that he had a very deep knowledge of the *Formulary*. First of all, Gödel remarks upon relevant differences between Russell and Peano that often go unmentioned in recent literature. A deeper analysis of several other passages from the *Max Phil* will show that he had not only read the *Formulary* in details but also shared several traits of Peano's conception.¹³

One of this passages (Gödel, forthcoming, X, p. 51–53), concerns the definition of Λ as a null object and is inspired by Peano's introduction of the two operators. In particular, if one defines Λ by means of the inverse iota operator, it can be conceived as the result of an operation, i.e. as the intersection of a class and its complement. This allows a better understanding of Gödel's hypothesis: he considered an empty definite description as a null object (*Unding*), i.e. as something that is neither a class nor a definite function, but rather an individual concept obtained

possibility of its elimination? Rodriguez-Consuegra argues that reasons of convenience may suffice to explain this, especially because Peano did not understand the philosophical importance of such an elimination (Rodriguez-Consuegra, 2000, p. 14). I agree that the iota symbol and its inverse are not mentioned by Peano among the primitive logical symbols, but I disagree on the point that Peano's remarks on the elimination of the symbol might be based on a conception that is similar to that of Russell. Firstly, as I mentioned before, the symbol could not be eliminated as such, but only as part of a formula, because it could not be defined as such—it could not be introduced by a typographical abbreviation. Secondly, as we have seen, the symbol could be used to give an alternative definition of Λ . Thirdly, rather than insisting on the eliminability of the symbol in order to explain why it was wrongly taken to be always part of a meaningful expression, Peano aimed to introduce it because it was relevant in several arithmetical notions.

¹³Or at least of the conception I attribute to Peano in my reconstruction of his notion of function and on his attitude towards the Leibnizian project of a characteristic (Cantù 2010, Cantù 2014).

by inverting an operation that is undefined on some parts of the universe. I mention this passage in order to show the depth of Gödel's knowledge of the *Formulary*, and the way he applied an idea taken from arithmetical notions to the more general case of definite descriptions.

Referring to a passage of the *Formulary* where Peano defines the arithmetical notion of maximum and proves that, given a numerical class u and a natural number m , if no successor of m belongs to u , then the maximum of u belongs to u (Peano, 1906, p. 47),¹⁴ Gödel remarks that Peano apparently assumes 1) that functions have a meaning independent of the values they are defined for, because the proof that the maximum of a class of numbers u is a number is developed without any assumption about the domain to which u is applied, and 2) that there is some thing U (the null object) that does not belong to any of the following categories: individual, function, number: e.g. ιa , which is a null object when the class a has no elements and many elements.

Peano apparently assumes that

1. all functions for arbitrary arguments have a meaning (the proof of $\max u$ occurs without any assumption about u).
2. There is some thing U (the null object) such that it does not belong to any of the following categories: class, individuum, function, number, etc., and that ιa when a has no elements and several elements, is this null object (Proof, p. 46, § 9, Form. V), and that the presuppositions $\iota m \in Q$ always mean: the limit exists.
3. On the other hand for this U also holds that $U \neq U$, but never that $x \in U$ (nor $U \in x$), evidently also $f(U) = U$ for any function U (this derives from the same passage).
4. But this U [is] obviously included in the universal quantifier (because of the rules of substitution). Or at least certain rules of transformation [are] general, e.g. $x \in \iota a. \equiv x = a$. [The] rule of substitution of the Def[iniens] by means of the Def[iniendum] is also limited by that, except when propositions that contain a U are devoid of meaning?
5. But instead of 2, 3, 4 [it is] also possible (in order to explain his assertions) to define $\max(u)$ by means of $x \in N_0 \rightarrow \max u = x \equiv \varphi(u, x)$. Only when one wants to assign a reference to $x \in \iota a. \equiv x = a$ in all circumstances, does it become necessary to assume things with peculiar properties such as U . But it is also possible to assume that any class is defined only within a limited domain (N_0, Q etc.) and that it might contain beyond it other things, like a [?] function can be extended arbitrarily but not absolutely. To each stage of the extension there is a univocally determined 'natural' or 'rational' extension. Similarly to: final realization of a legal state after no matter how many iniquities. Is this the return of multiplicity to unity? Is any multiplicity after all a fall of mankind or an iniquity? (Gödel, forthcoming, X, p. 51–53, my transl.)¹⁵

¹⁴See below footnote 15.

¹⁵“Peano scheint anzunehmen, dass

1. alle Funktionen für beliebige Argumente einen Sinn haben (*Bew* $\max u \in N_0$ tritt ohne Voraussetzung über u auf).
2. Es irgendein Ding U (das Unding) gibt, welches zu keiner der Kategorien: Klasse, Individuum, Funktion, Zahl etc. gehört, dass das ιa im Fall, <dass> a keine und mehrere Elemente hat, dieses Unding ist (*Bew*, p. 46, § 9, Form. V), und dass die Voraussetzungen $\iota m \in Q$ immer bedeuten: der Limes existiert.
3. Andererseits gilt auch für dieses U $U \neq U$, aber niemals $x \in U$ (auch <nicht> $U \in x$), offenbar auch $f(U) = U$ für jede Funktion U (folgt aus derselben Stelle).
4. Aber dieses U offenbar im Allquantor eingeschlossen (wegen Substitutionsregel). ? oder zumindest gewisse Transformationsregel allgemein, z.B. $x \in \iota a. \equiv x = a$. <Ist die> Regel der Ersetzung des Definiens durch das Definiendum dadurch auch beschränkt, ausser wenn Sätze, die ein U enthalten sinnlos sind?

Let's recall now the distinction of three kinds of definitions made by Gödel in the quotation mentioned on page and compare it with Gödel's remarks on the null object. It is well known that the empty class and the idea of a non-actual object have been used to guarantee a denotation to syntagms that are meaningful but lack reference. This strategy was followed respectively by Frege and by Meinong, as they claimed that denotative syntagms always denote something: if there is not something real that is denoted, then they denote the empty class (Frege) or an object that is non-actual (Meinong).

This was not the way followed by Russell, who managed to do without this use of the empty class or of a particular non-existent individual, because he assumed that the logical form of definite descriptions differs from that of proper names and can be analyzed once the syntagms are correctly transformed, making the conditions of existence and uniqueness explicit.

According to Gödel's understanding, Peano's strategy differs both from Frege's and from Russell's conception: some denotative syntagms derive their meaning from contextual definitions in which they occur, i.e. from the axioms, and there is no need to make the conditions of existence explicit in order to understand the meaning of an expression containing the inverse iota operator. The inverse iota operator applied to a generic class ιa is conceived most generally as the inverse of the function ιa and does not have existential import as such: conditions on the existence of the class a will determine the possibility of assigning or not assigning a denotation to the syntagm, but its meaning as the inverse of a certain function can be grasped, even when the domain of the function is not defined. Even if Gödel's analysis is based on excerpts from the fifth edition of the *Formulary*, there is a passage from the second edition of the *Formulary* that can be fruitfully related to Gödel's remarks, because it discusses the same example in more detail. As already mentioned in footnote 10, in the second edition of the *Formulary* the inverse iota is expressed by the symbol $\bar{\iota}$.

The denotation of the symbol $\bar{\iota} a$ is given by proposition 430, only under the condition that the mentioned hypotheses are satisfied. We cannot assign a reference [signification]¹⁶ to that symbol when

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5. Aber statt 2, 3, 4 <ist es> auch möglich (zur Erklärung seiner Behauptungen), $\max(u)$ zu definieren durch $x \in N_0 \rightarrow \max u = x \equiv \varphi(u, x)$. Erst wenn man $x \in \iota a. \equiv. x = a$ unter allen Umständen eine Bedeutung geben will, tritt die Notwendigkeit ein, Dinge mit so sonderbaren Eigenschaften wie U anzunehmen. Aber <es ist> auch möglich, anzunehmen, dass jede Klasse nur innerhalb eines beschränkten Bereiches (N_0, Q etc.) definiert ist, und ausserhalb noch andere Dinge enthalten kann, ähnlich wie eine Einer-<?> Funktion zwar beliebig weit „übeliebig“ fortgesetzt werden kann, aber nicht für das Absolute. Für jedes Stadium der Fortsetzung gibt es eine eindeutig bestimmte „natürliche“ oder „vernünftige“ Fortsetzung. Ähnlich wie: schliesslich Realisierung des Rechtszustands auch nach beliebig vielen Ungerechtigkeiten. Ist das <die> Rückkehr der Vielheit zur Einheit? Ist jede Vielheit letzten Endes ein Sündenfall oder eine Ungerechtigkeit?”

¹⁶The difference between my interpretation and the one suggested by Rodriguez-Consuegra is based on a different understanding of Peano's conception that becomes evident in the translation of the word “signification” that occurs here: I translate with “reference” and he translates with “meaning” (Rodriguez-Consuegra, 2000, p. 15). Rodriguez-Consuegra remarks that when the conditions of uniqueness and existence are not satisfied, the symbol is meaningless; I claim that the proposition it occurs in cannot be used as a proper definition. Rodriguez-Consuegra also remarks that a similar idea is again expressed in Peano's paper on vectors, where, after having defined the division of a vector by a natural number using the $\bar{\iota}$ function, Peano observes: “In order for this definition to be applicable, it is necessary that the conditions be satisfied under which one can use the symbol $\bar{\iota}$, i.e. that there exist vectors v that satisfy the condition $av = u$ and that there exists only one such vector” (Peano, 1898, p. 523, my transl.). This passage does not say that the proposition is meaningless, if the conditions are not satisfied, but only that it can not be applied as a definition.

the class a is empty or when it contains more than one individual. But it is not necessary to repeat those hypotheses each time the symbol \bar{i} occurs. For example

$$u \in K'q. \supset. \max u = \bar{i}\{u \bar{x} \bar{\epsilon} (u \bar{x} (x + Q) = \wedge)\}$$

Given a class of real numbers u , we call maximum of u the number of the class u such that there are no numbers of the class u that are bigger than x . This is a definition of $\max u$. As we give that definition, we do not assert that the class

$$u \cap \bar{x} \bar{\epsilon} (u \bar{x} (x + Q) = \wedge)$$

actually exists, i.e. we do not assert the existence of the maximum (Peano, 1897, p. 50, my transl.).¹⁷

According to my reconstruction the use of the inverse iota symbol in the definition of \max is an example of what Gödel called “descriptions” or “statements of the form: $\varphi(a)$ ”, where “existence and uniqueness should then be proved” (see quotation on page). The conditions of existence and uniqueness are not themselves part of the logical form of the expression, but of course the reference of the symbol, its denotation of something that exists, depends on that proof.

Gödel’s remarks then are particularly useful to help us understand the philosophical difference between Peano’s and Russell’s aims. Peano wants to axiomatize mathematics and knows that mathematics uses expressions concerning “the unique x such that”, whereas he believes that a term for the inverse iota is missing in the ordinary language. Since the inverse iota operator is usually introduced as the inverse function of the singleton, its denotation will of course depend on the domain on which the direct function is defined, but it will be unique in the sense that whenever it picks some element, then this will be unique. Russell has a completely different aim, because he wants to give a logical analysis of ordinary language, and shows that some apparently denotative expressions of the ordinary language, are logically non denotative. Russell assumes the conditions of existence and uniqueness as part of the logical form of the term, while Peano demonstrates them afterwards (and he does it by means of the propositions on equality), as is clearly remarked upon by Gödel.

Beside the difference concerning ordinary and mathematical language, there is a deeper logical difference between Peano’s and Russell’s perspectives that concerns the understanding of the notions of function and equality. What is expressed by the inverse iota symbol? A function, according to Peano; part of a propositional function, according to Russell. Where are the roots of such difference? I will try to answer this question based on my interpretation of Peano’s works, which is grounded on the individuation of deep similarities between his conception of function and that of Hermann Grassmann, and on a distinction between the universal notion of equality defended by Frege, Russell and even by some of the members of the Peano school (such as Alessandro Padoa) and Peano’s local notion of equality. Russell admits a unique and fixed notion of identity from the beginning onwards, while Peano does not; as a consequence, Russell conceives a function as a correspondence between fixed domains (as in modern mathematics), whereas Peano distinguishes between a specific notion of function, where domain and co-domain are preliminarily determined, and a general notion of function, that can be introduced without reference to a given domain. I will conclude by quoting several excerpts from Gödel’s *Max Phil* that suggest, according to my reading, a similar difference between Russell and Peano.

¹⁷The mentioned proposition 430 is the definition of the inverse iota and the example of the maximum is analogous to the passage from the fifth edition to which Gödel explicitly refers (Peano, 1906, p. 47) and that I will quote in section 3.1.

3. The inverse iota as an indefinite function

3.1 Gödel on Peano's notion of function

Let's recall the passage by Gödel quoted above in section 2.4 and concerning Peano's notion of function. Gödel attributes to Peano the belief 1) that any function might be meaningful for any value. He then considers two possible alternatives in order to explain this apparently paradoxical thesis, for mathematical functions are usually identified by the property of functionality, and are thus defined only on a domain of values which satisfy that condition.

The first alternative is that of accepting 2) that there is something (the null object expressed by the inverse iota operator) that is neither a class nor an individual nor a function nor a number, 3) that the null object is contradictory, 4) and that the universal quantifier ranges on the null object too.

The second alternative is that of assuming 5) that every function is associated to a specific domain, and that nonetheless a function might have a meaning even when it is applied beyond that domain, as is the case when a function is extended "naturally" in order to achieve a greater generality and a reunification of several different functions into a single function.

To understand how Gödel interprets Peano's notion of a function that is incompletely [unvollständig] defined, the following passage from Gödel's *Max Phil* might be valuable, because Gödel remarks that Peano admits the possibility that a function might be defined for some values, undefined for other values, and only partially defined for yet other values. Examples are the function *max*, the function *num* and the *differentia*. All three definitions contain either the iota or the inverse iota operator.

In the case of an incomplete definition of 'function' (Peano), there is also the possibility that it might be defined for certain arguments, undefined for certain arguments, partially defined for certain arguments: e.g. from the definition of *num* follows by induction for infinite sets M that $num M \notin N_0$; analogously in the case of the definition of *max* by means of $n = max M \equiv \dots$. Similarly in the case of the definition of *DF* (*differentia*) the value is a function whose value is determined only for certain arguments on which F is definable (even if it might have a value also for other arguments). But a function is only determined when it is defined for *all* arguments whether there is a value, and if so, what value it is. Even in the case of creative definitions there is at least something (the equality, etc.) that is defined for the newly created elements (Gödel, forthcoming, p. 55–56, my transl.).¹⁸

¹⁸“Im Fall von unvollständiger Definition von “Funktion” (Peano) gibt es auch die Möglichkeit, dass für gewisse Argumente definiert, für gewisse nicht definiert, für gewisse teilweise definiert <ist>, z.B. bei der Definition von *num* durch Induktion folgt für unendliche Menge M , $num M \notin N_0$ ebenso bei der Definition von *max* durch $n = max M \equiv \dots$. Ebenso bei der Definition von *DF* (*differentia*) ist der Wert eine Funktion, deren Wert nur bestimmt ist für Argumente, an denen F definierbar (obwohl sie vielleicht auch für andere einen Wert haben). Eine Funktion ist aber erst bestimmt, wenn für *alle* Argumente definiert ist, ob es einen Wert gibt und was (gegebenenfalls) der Wert ist. Auch bei den schöpferischen Definitionen wird bloss Manches (die Gleichheit etc.) für neue geschaffene Elemente definiert.”

Before further commenting on Gödel’s remarks, I would like to quote some passages from Peano’s *Formulary*, in particular the introduction of the notion of *maximum* of a set mentioned by Gödel,¹⁹ and present my interpretation of the notion of function in Peano.

In the fifth edition of the *Formulary*, Peano defines the notion of maximum (greatest element) of a subset u of the natural numbers as “the” element $x \in u$ that is greater than any other element of the subset (Peano, 1906, § 9, p. 46):

$$9.1.0 \quad u \in Cls'N_0. a, b \in N_0. \supset: max u = u + +1u \frown x \ni (y \in u - ix. \supset_y. y < x) \quad Df \ max$$

Only afterwards does he demonstrate the existence of the maximum by proving inductively that any subset of N has a maximum (Peano, 1906, § 9, p. 47):

$$9.1.8 \quad u \in Cls'N_0. a, b \in N_0. \supset: \exists u. m \in N_0. \neg \exists u \frown (m + N_1). \supset. max u \in u$$

As already mentioned in section 2.4, Peano’s definition of the function *max* is based on a use of the inverse iota operator that allows to consider the expression “the element of a subset that is greater than any other element of the subset” as meaningful even when it refers to no elements or to several elements. When Peano says “the x ”, he apparently means “that x that is unique” if it exists, or if it exists in a different set.

3.2 Peano’s distinction between indefinite and definite functions

Peano’s approach can be understood only on the basis of his distinction between a general notion of function, for which no equality is defined, and the notion of a definite function, which is relative to a domain and for which an equality relation can be introduced. Similarly to Grassmann, who admitted a very general notion of operation that need not be defined on a specific domain, Peano introduced a notion of function that might be partially undetermined, so that the equality between two functions cannot always be established.

The domain in which the function is defined, which is called variation domain, is not connected to the sign of function, and thus we can always restrict or dilate it. Therefore, we cannot talk about the equality of two functions; two functions can indeed produce identical results on one domain and different results on another domain. Two arbitrary functions u and v always have a domain of coincidence which is expressed by: $x \ni (ux = vx)$. We cannot talk of the number of functions that satisfy some condition. No function is invertible; and so on. When the mathematicians talk about equality, number, invertibility of a function, the word “function” corresponds to the system $(u; a)$, where w is the function considered in §1 and a is the domain of variation. We call it “definite function” and we indicate it by means of the symbol F , or *Funct* (Peano, 1906, III, § 4, p. 79ff, my transl.).

Equality cannot be properly defined. Peano introduces an equality between individuals (based on

¹⁹I will not discuss the examples of *num* and *differentia*, but I include the passages by Peano to which Gödel refers, so that the reader might easily note the presence of the iota and the inverse iota symbols and compare these examples with the example concerning the maximum.

Peano’s definition of *num* corresponds to the modern notion of the cardinality of a set (Peano, 1906, II, § 8, p. 46):

$$0. \quad u \in Cls. \supset: num u = 0. =. u = \Lambda \quad Df$$

$$1. \quad u \in Cls. m \in N_0. \supset: num u = m + 1. =. \exists u: x \in u. \supset_u. num(u - ix) = m \quad Df$$

Peano’s definition of *Differentia* corresponds to the operation of subtraction between numbers (Peano, 1906, II, § 6, p. 44):

$$1.0 \quad a \in N_0. b \in a + N_0. \supset. b - a = \iota[N_0 \frown x \ni (x + a = b)] \quad Df \text{ --}''$$

reflexivity, symmetry and transitivity) and an equality between classes (defined as mutual inclusion), but unlike his disciple Padoa²⁰ he does not explicitly introduce the definition of identity by means of the reflexivity and substitutivity property that had been used by Hilbert (1905, p. 178, Engl. transl. p. 132). Peano, unlike Padoa and Frege, does not believe that the symbol of equality should have a fixed interpretation in all domains (Cantù, 2010, p. 552–53). The definition of equality as well as the definition of function are introduced for each domain. This is the reason why equality expresses existence: it defines what there is in the domain, and the reason why it allows the definition of definite functions.

On the contrary, equality between two general *functions* cannot be defined. The general notion of a function is only partly determined. If the domain and the codomain of the function are precisely defined, then the notion of equality can be introduced. Similarly in Grassmann there are general notions of fundamental operations (one is characterized by associativity, the second by distributivity with respect to the former, and so on) that are not defined on a specific domain. For this reason they are more general and abstract: logic includes such general concepts that are not defined on a fixed domain, because there is no universal domain.

One might object that this notion of function is useless in mathematics, and that all mathematical functions share the property of functionality, so if one limits oneself to mathematics, there is no reason for such a general notion of function. I will consider the example of Grassmann's regressive product, which was defined independently of a domain, and whose result depends on the domain it is applied to (Cantù, 2010, p. 94ff). Given that Grassmann deeply influenced Peano, who developed and translated into Italian Grassmann's extension calculus, the example is not inappropriate.

The idea of considering as more 'general' the product relative to a variable domain—a domain that is not closed under the operation but rather a result of our carrying out the operation itself—is one of Grassmann's most interesting philosophical ideas that lacks an adequate representation in set-theoretical approaches to mathematics. It has been claimed that this general notion of product (relative to a variable domain) introduced by Grassmann in the first edition of his *Ausdehnungslehre* (*Extension Theory*) was substituted in the second edition by a general notion of product (relative to a principal domain) because of technical mathematical difficulties. I have elsewhere argued why this is not the case (Cantù, 2010, p. 100). Besides, recent studies in mathematics and in cognitive sciences have proved that Grassmann's insights have been interestingly developed, especially in algebra.²¹ The reason why the restriction to a specific domain is not introduced from the beginning is related to the distinction between formal sciences—where no constraint on the domain is taken for granted, and the forms are one and the same with their construction—and real sciences, where some constraints are accepted from the outset, and forms are thus 'embodied' in a fixed domain. Grassmann's epistemological approach, which I have defined as a form of constructivism, is based on the idea that there are some fixed fundamental operations rather than a fixed domain; besides, a general notion is particularized when further conditions are fixed, as in the case of the regressive product which is less general, if considered as relative to a unique domain.

²⁰See the correspondence between Giovanni Vailati and Alessandro Padoa discussed in Cantù (2007).

²¹See in particular the section 'Present and future of Hermann Grassmann's ideas in mathematics' in Petsche et al. (2011).

My claim is that the notion of function in Peano is similar to this kind of general and abstract notion of operation that can be found in Grassmann. The condition of functionality is not part of the definition of function in the *Formulary*. After all the definition of function does not appear neither in the section on Mathematical Logic nor in the section on Arithmetic, but rather at the beginning of the section on Algebra.

‘Function, operation, correspondence’ are identical or similar words. In the ordinary language each relative word, such as ‘father, son, ...’ indicates a function [...]. We consider two classes a and b ; and we write $u \in ab$, and we read “ u is a transformation of a in b ” [...]

$$0. \quad a, b \in \text{Cls.} \supset \cdot u \in ab. =: x \in a. \supset_x xu \in b \quad \text{Df } u$$

And we write $u \in afb$, and we read “ u is a sign of pre-function that to each a lets correspond some b ” [...]

$$01. \quad a, b \in \text{Cls.} \supset \cdot u \in bfa. =: x \in a. \supset_x xu \in b \quad \text{Df } f \text{ (Peano, 1906, III, § 1, p. 73, my transl.)}$$

If one accepts this general notion of function as a correspondence, or as an operation between any two entities, then the usual understanding of logic as based on a domain with a fixed identity might be revised. Equality is necessary if one wants to compare such functions, because they can be defined as equal or not equal only when they are restricted to a specific domain. But the meaning of the functions is determined and can be grasped independently from the domain restriction. This point is particularly interesting for two reasons: uncompletely defined functions are neither classes nor functions nor objects (that can be said to be equal or not equal), and yet they are meaningful. In a reconstruction of the evolution of the *Formulary* that I developed elsewhere, I discussed the different contributions by Burali-Forti, Peano, and Padoa on this issue (Cantù, 2010). My claim is that Peano was strongly influenced by Grassmann’s idea that there are some fundamental operations (theory of general operations) that get embodied in different kinds of operations once their respective domains are determined (classes, numbers, vectors, and so on). This is the deep reason for Peano’s piecemeal approach to definitions: whereas Russell and Frege consider a unique domain onto which all functions have to be defined completely (one should determine exactly for all values of the domain if the function is defined or not defined on it), Peano believes that certain notions should be introduced in a more general way, and only afterwards should they be specified case by case.²² See for example the following passage by Peano:

In mathematics there is no single definition for example of “multiplication”, nor does there exist in the *Formulary* an equality of the form $\times =$ (expression composed by other signs). But there exists a definition of multiplication between two natural numbers, then between two relative numbers, then between two rational numbers, and so on. In the *Formulary* one can easily find more than 30 definitions of $x \times y$, with different hypotheses. Thus to the sign of function it is not connected a domain onto which the function is determined, which is also called variability domain of the function. As a matter of fact, it is not possible to talk about equality of two functions, because two functions might produce identical results in one domain and different results in another domain. But two arbitrary functions u and v always have a common domain that can be expressed by $x \ni (ux = vx)$. We cannot talk about the number of functions that satisfy a given condition: no function is invertible, and so on. When mathematicians talk about equality, number, inverse of a function, the term “function” denotes the system $(u; a)$, where u is the function considered in § 1 and a is the variability domain. We call it “definite function” (Peano, 1906, III, § 3, p. 79, my transl.).

²²This is, I believe, a further argument against the reconstruction by Consuegra-Rodriguez, who tends to underline the similarities between Peano and Russell by attributing a form of logicism to Peano.

This passage is particularly interesting, because it suggests a deep similarity with Grassmann's approach. Grassmann considers multiplication as applied to a given domain when he wants to consider specific applications of a general mathematical concept. Generalization is not conceived as an enlargement of a given domain by means of the addition of new elements; rather it is obtained by modifying the defining conditions of the fundamental operations. In particular, he distinguishes between the 'general' notion of product defined for any two magnitudes with respect to the domain generated by the magnitudes themselves (regressive product in its general formulation), and the 'particular', regressive, applied product defined with respect to a system of n dimensions, or, as in the case of geometry, to a system of 3 dimensions. Geometry is an application of extension theory, essentially because multiplication in space is relative to the fixed number of dimensions of the space itself. This is the primary sense in which geometrical magnitudes are 'embodied' extensive forms: the operations on geometrical figures are relative to a fixed system, the 3-dimensional space. Here, similarly, Peano distinguishes the general notion of multiplication, which cannot be properly defined, from the various specifications of it in the *Formulary*, which can be defined and differ from one another, because the domains onto which they are defined differ from one another. This approach is perfectly coherent with Peano's general view on axiomatics. There cannot be a unique theory in which we talk about all mathematical objects. Each theory has its own mathematical axioms, and the notion of function does not belong to the common logical axioms.

3.3 A general notion of operation

We have seen that Peano's piecemeal definitions and distinction between function and definite functions recalls Grassmann's distinction between a general theory of forms, where he identified fundamental kinds of operation, such as additive and multiplicative operations (i.e. operations of first and second order respectively, algebraically characterized by their reciprocal properties, such as distributivity of the latter with respect to the former) and the specific branches of mathematics.

Let's come back now to Gödel in order to understand the mentioned passages where he rightly attributes to Peano the idea that a function is completely defined only when a domain is determined, so that the function might be defined for all values that range in that domain (see quotation on page 12). Gödel remarked upon the general notion of function introduced by Peano, and associated it to what Peano says about the operations *max*, *num* and *differentia*. Does Gödel share Peano and Grassmann's idea of a general operation that might be partially undefined and get determined only when it is applied to a specific domain? The following passage is particularly interesting in this respect because Gödel considers a general operation of combination of two things, an operation that might assume different meanings depending on the things it is applied to:

I. What is the most general operation that can be applied to two arbitrary things a , b and that contains as special cases "ε" and "application" —and that corresponds to the operation of moving side by side [Nebeneinanderschiebens]? That there is such an operation follows from the fact that each concept has an extension in almost all directions and that the application is the extension of ε. This relation [is] maybe definable by: 1) the simplest kind of combination, 2) that which owes its existence immediately to the existence of this pair, 3) that which is perceived (that towards which one turns its gaze) when one directs its vision to a and b , 4) the relation that subsists between a and b (or the connection that subsists between them), e.g. the vector $a - b$. In particular it is to be expected that for two things emerges a pair, for two propositions a product, for something that needs completion what is obtained by the completion (as for example by an operation), for classes and numbers maybe the sum, for

concepts the product. Notably, depending on the type, ab would denote $a \in b$, $a \sim b$, $b \in a$, $a \mid b$ (for relations and functions).

II. If one explains the concept of function in Peano's way on the basis of this operation of combination, then each thing is both a function and a transformation, because each thing can be combined with other things (the concept application, \in , etc., in this operation of combination is "limited" to certain classes of things). Yet, on the other hand there are functions in a strict sense, i.e. objects whose essence consists in the possibility of being combined, whereas for the others the fact of being combined is so to say something 'external' (Gödel, forthcoming, X, p. 53–54, my transl.).²³

It is interesting to compare this last remark by Gödel with Grassmann's extensive product, which Grassmann restricts to a given domain when he wants to compose it with other operations. Similarly, Gödel discusses here the fact that the complete definition of a function is required in order to calculate the composition of functions. Actually, Grassmann discussed the possibility of finding a common domain to two functions in order to compose them and to determine the inverse of a function. Peano, as we have seen, mentioned the fact that "two arbitrary functions u and v always have a common domain".

4. Gödel's interest in Peano

Gödel's interest in the Grassmann-Peano perspective on operations and functions might explain his lack of sympathy for Russell's typed universe, which rigidly determines from the outset the domain of any function that can be introduced in the theory. It is presumably because he was interested in possible revisions of Russell's typed universe that Gödel read Peano's *Formulary* so carefully before sending the last revision of the Russell paper to the editors. The mentioned passages in Gödel's *Max Phil X*, together with several other passages from *Max Phil IX* (March 1943--January 1944) and *IV* (May 1941--April 1942),²⁴ explain that when Gödel refers in his published works to Peano as one of the most important authors in mathematical logic, he is not only generally hailing him as one of the fathers of logic, but he also has a specific interest in some of Peano's solutions, that he considered as profoundly different both from Russell's and

²³"I. Was ist die allgemeinste Operation, welche auf zwei beliebige Dinge a , b angewendet werden kann, und welche "ε" und "Anwendung" als spezielle Fälle enthält – und welche der Operation des Nebeneinanderschiebens entspricht? Dass es eine solche gibt, folgt daraus, daß jeder Begriff fast überallhin eine Fortsetzung hat und daß die Anwendung die Fortsetzung von \in ist. Diese Relation <ist> vielleicht definierbar durch: 1) die einfachste Art der Kombination, 2) das, was seine Existenz unmittelbar der Existenz dieses Paares verdankt, 3) das, was dann wahrgenommen wird (worauf der Blick gerichtet ist), wenn man die Blickrichtung durch a und b legt, 4) das zwischen a und b bestehende Verhältnis (oder die zwischen ihnen bestehende Verbindung), z.B. der Vektor $a - b$. Insbesondere <ist> zu erwarten, daß für zwei einfache Dinge das Paar rauskommt, für zwei Sätze das Produkt, für etwas, das Ergänzung bedürfte, ist (wie z.B. eine Operation) das durch die Ergänzung Erhaltene, für Klassen und Zahlen vielleicht die Summe, für Begriffe das Produkt. Insbesondere könnte je nach dem Typus ab bedeutend $a \in b$, $a \sim b$, $b \in a$, $a \mid b$ (für Relationen und Funktionen).

II. Wenn man aufgrund dieser Zusammensetzungsoperation den Funktionsbegriff in Peanoscher Weise erklärt, dann ist jedes Ding sowohl eine Funktion, als eine Transformation, da jedes Ding mit anderen kombiniert werden kann (<ist> der Begriff Anwendung, \in , . etc. in dieser Operation der Kombination "beschränkt" auf gewisse Klassen von Dingen). Andererseits gibt es aber doch Funktionen im engeren Sinn, das heißt, Gegenstände, deren Wesen im Kombiniert-werden-Können bestehen, während bei den anderen das Kombiniertwerden sozusagen etwas "äußeres" ist."

²⁴See in particular (Gödel, forthcoming, IX, p. 14, 42 and 69 and X, p. 78). Further relevant passages where Gödel discusses the definition of the inverse iota operation can be found in the not yet fully transcribed *Max Phil IV*, especially on pages 191–193 and 211–212.

Frege's points of view, and yet worth a deeper investigation. Gödel's detailed knowledge and serious consideration of some of the solutions proposed by Peano suggests that he read the *Formulary* in the hope of finding definitions and solutions to classical problems that differed from those that had become mainstream, and also a non universalistic approach to logic that diverged radically from that of the Frege-Russell tradition. In other words, Peano attracted Gödel for the very same reason for which he was generally despised at the time by the logical community: Peano's logic and mathematics cannot easily be translated into a set-theoretic approach.

As a conclusion, I would like to mention the fact that a complete transcription of the *Max Phil* notebooks will allow a better understanding of the role played by Peano, and by Gödel's reading of the fifth edition of the *Formulary*, on the Leibnizian turn in Gödel's philosophy. My provisional claim, which of course will have to be tested once further transcriptions are available, is that Peano's influence, although played down in the published works, was relevant, especially to the search for a different solution to the question of definite descriptions, and to the development of logic as a general theory of concepts in the Leibnizian tradition.

I have not mentioned Peano's or Grassmann's deep relation with the logical and mathematical works by Leibniz, but it is well known that Peano repeatedly insisted on the necessity of developing logic in order to satisfy the Leibnizian project of a universal characteristic and Gödel himself recognizes it in the opening part of the Russell paper. Grassmann also described his calculus as a realization of Leibniz's geometrical characteristic. I will not discuss this point in detail here, but I would like to mention the fact that the examples analyzed in this paper (the iota and its inverse), are very important to understand the deep relation between Peano, Gödel and Leibniz on the search for primitive concepts that might mirror reality.

Leibniz's idea of a characteristic containing 'real' characters is not completely abandoned in Peano's perspective, because Peano's choice of symbolism reveals an effort to mirror the concepts by means of the symbols used to denote them. This is already clear in the *Formulary* but even more evident in Peano's investigations on a universal language, because the *latino sine flexione* is based on symbols (roots of Latin words) that should preserve the essential relation to the denoted concept, independently from grammatical variations (Cantù, 2014). Gödel's strong interest in the *Formulary*, and in particular in the fifth edition (written in *latino sine flexione*) might thus be related to Gödel's more general interest in Leibniz and in his project of a characteristic.

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