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Riesz potentials of Radon measures associated to reflection groups

Léonard GALLARDO,^{*} Chaabane REJEB[†] and Mohamed SIFI[‡]

Abstract

For a root system R on \mathbb{R}^d and a nonnegative multiplicity function k on R , we consider the heat kernel $p_k(t, x, y)$ associated to the Dunkl-Laplacian operator Δ_k . For $\beta \in]0, d + 2\gamma[$, where $\gamma = \frac{1}{2} \sum_{\alpha \in R} k(\alpha)$, we study the Δ_k -Riesz kernel of index β defined by $R_{k,\beta}(x, y) = \frac{1}{\Gamma(\beta/2)} \int_0^{+\infty} t^{\beta/2-1} p_k(t, x, y) dt$ and the corresponding Δ_k -Riesz potential $I_{k,\beta}[\mu]$ of a Radon measure μ on \mathbb{R}^d . According to the values of β , we study the Δ_k -superharmonicity of these functions and we give some applications like the Δ_k -Riesz measure of $I_{k,\beta}[\mu]$, the uniqueness principle and a pointwise Hedberg's inequality.

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Key words: Reflection groups, Dunkl-Laplace operator, Dunkl heat kernel, Generalized volume mean operator, Dunkl subharmonic functions, Riesz kernel and potentials, Hedberg's inequality.

1 Introduction

Let R be a normalized root system in \mathbb{R}^d . That is, for every $\alpha \in R$, $\|\alpha\|^2 = 2$, $R \cap \mathbb{R}\alpha = \{\pm\alpha\}$ and $\sigma_\alpha(R) = R$, where σ_α is the reflection with respect to the hyperplane H_α orthogonal to α (see [15] and [17]). We fix $k \geq 0$ a multiplicity function (i.e. $k : R \rightarrow [0, +\infty[$ invariant under the action of the Coxeter-Weyl group W associated to R) and we consider the associated Dunkl-Laplacian operator Δ_k given by

$$\Delta_k f(x) = \Delta f(x) + 2 \sum_{\alpha \in R_+} k(\alpha) \left(\frac{\langle \nabla f(x), \alpha \rangle}{\langle x, \alpha \rangle} - \frac{f(x) - f(\sigma_\alpha(x))}{\langle x, \alpha \rangle^2} \right), \quad f \in \mathcal{C}^2(\mathbb{R}^d), \quad (1.1)$$

^{*}Laboratoire de Mathématiques et Physique Théorique CNRS-UMR 7350, Université de Tours, Campus de Grandmont, 37200 Tours, FRANCE; Email: Leonard.Gallardo@lmpt.univ-tours.fr

[†]Université de Tunis El Manar, Faculté des Sciences de Tunis, Laboratoire d'Analyse Mathématiques et Applications LR11ES11, 2092 El Manar I, Tunis, Tunisie and Laboratoire de Mathématiques et Physique Théorique CNRS-UMR 7350, Université de Tours, Campus de Grandmont, 37200 Tours, FRANCE; Email: chaabane.rejeb@gmail.com

[‡]Université de Tunis El Manar, Faculté des Sciences de Tunis, Laboratoire d'Analyse Mathématiques et Applications LR11ES11, 2092 El Manar I, Tunis, Tunisie; Email: mohamed.sifi@fst.rnu.tn

with R_+ a positive subsystem (see [7]).

Acting on $\mathcal{C}^\infty(\mathbb{R}^d)$, it is related to the classical Laplacian operator Δ by means of the so-called Dunkl intertwining operator V_k (see [6], [7], [30]) as follows:

$$\Delta_k V_k = V_k \Delta. \quad (1.2)$$

In [23], M. Rösler has proved that for any $x \in \mathbb{R}^d$, there exists a compactly supported probability measure μ_x on \mathbb{R}^d (which we call Rösler's measure at point x) such that

$$\forall f \in \mathcal{C}^\infty(\mathbb{R}^d), \quad V_k(f)(x) = \int_{\mathbb{R}^d} f(y) d\mu_x(y), \quad (1.3)$$

with

$$\text{supp } \mu_x \subset C(x) = \text{co}\{gx, g \in W\} \quad (1.4)$$

(the convex hull of the orbit of x under the group W). We note that, according to [9], the support of μ_x contains the point x and it is W -invariant under the hypothesis that the multiplicity function is positive.

Let $p_k(t, x, y)$ ($t > 0, x, y \in \mathbb{R}^d$) be the heat kernel of the Dunkl Laplacian Δ_k which is given by (see [21] and [25])

$$p_k(t, x, y) := \frac{1}{(2t)^{d/2+\gamma} c_k} e^{-(\|x\|^2 + \|y\|^2)/4t} E_k\left(\frac{x}{\sqrt{2t}}, \frac{y}{\sqrt{2t}}\right), \quad (1.5)$$

where

$$E_k(x, y) = V_k(e^{\langle \cdot, y \rangle})(x) \quad (x, y \in \mathbb{R}^d) \quad (1.6)$$

is the Dunkl kernel (see [5] and [7]), c_k is the Macdonald-Mehta constant (see [19]) given by

$$c_k := \int_{\mathbb{R}^d} e^{-\frac{\|x\|^2}{2}} \omega_k(x) dx, \quad (1.7)$$

and ω_k is the Dunkl weight function

$$\omega_k(x) = \prod_{\alpha \in R_+} |\langle \alpha, x \rangle|^{2k(\alpha)} \quad (1.8)$$

which is homogeneous of degree 2γ .

It is also known (see [21]) that for all fixed $x \in \mathbb{R}^d$, the function $p_k(t, x, \cdot)$ solves the Dunkl heat equation

$$(\Delta_k - \partial_t)p_k(t, x, \cdot) = 0. \quad (1.9)$$

Let $\gamma = \sum_{\alpha \in R_+} k(\alpha)$ and suppose that $d + 2\gamma > 2$. For $\beta \in]0, d + 2\gamma[$, we define the Δ_k -Riesz kernel of index β as follows

$$R_{k,\beta}(x, y) := \frac{1}{\Gamma(\beta/2)} \int_0^{+\infty} t^{\frac{\beta}{2}-1} p_k(t, x, y) dt.$$

We note that when $\beta = 2$, we obtain the Dunkl-Newton kernel which has been introduced and studied in [10].

Let $x \in \mathbb{R}^d$, $x \neq 0$, be fixed and $W.x$ be its W -orbit. If $y \notin W.x$, $R_{k,\beta}(x, y)$ is finite. But when $y \in W.x$, it seems hard, except in the case $y = x$, to decide in general if $R_{k,\beta}(x, y)$ is finite or infinite. These difficulties are illustrated by the particular case of the root system of type $A_1 \times A_1 \times \cdots \times A_1$ (m times, $1 \leq m \leq d$), where we manage to give a complete description of the singularities of the function $R_{k,\beta}(x, \cdot)$.

The aim of this paper is the study, when $d + 2\gamma > 2$, of the Δ_k -Riesz kernel $R_{k,\beta}$ and the corresponding potential

$$I_{k,\beta}[\mu](x) = \int_{\mathbb{R}^d} R_{k,\beta}(x, y) d\mu(y)$$

of a signed Radon measure μ on \mathbb{R}^d .

In particular, we will study the sub-or-superharmonicity of these functions in the sense of the Dunkl-Laplace operator and we will describe explicitly their Δ_k -Riesz measures. This notion of subharmonicity, which generalizes the classical one¹ has been introduced and studied in some details in [10]. More precisely, let Ω be a W -invariant open subset of \mathbb{R}^d . A function $u : \Omega \rightarrow [-\infty, +\infty[$ is called Δ_k -subharmonic (D-subharmonic) on Ω if

- u is upper semi-continuous (u.s.c.) on Ω ,
- u is not identically $-\infty$ on each connected component of Ω ,
- it satisfies the sub-mean volume property: for every closed ball $B(x, r) \subset \Omega$, we have

$$u(x) \leq M_B^r(u)(x) := \frac{1}{m_k[B(0, r)]} \int_{\mathbb{R}^d} u(y) h_k(r, x, y) \omega_k(y) dy, \quad (1.10)$$

where m_k is the measure $\omega_k(x) dx$ and $h_k(r, x, y)$ is a kernel of the form

$$h_k(r, x, y) := \int_{\mathbb{R}^d} \mathbf{1}_{[0, r]}(\sqrt{\|x\|^2 + \|y\|^2 - 2\langle x, z \rangle}) d\mu_y(z) \quad (1.11)$$

with μ_y Rösler's measure at point y . The function $y \mapsto h_k(r, x, y)$ is a generalized translate of the indicator function $\mathbf{1}_{B(0, r)}$ of the ball $B(0, r)$ called harmonic kernel, introduced and studied in [8] and which properties will be recalled in the next section. Moreover the harmonic kernel is a crucial tool to get quite explicit expressions of the Δ_k -Riesz kernel (see section 3).

Naturally, a function u is D-superharmonic on Ω if $-u$ is D-subharmonic on Ω .

Finally, we study some applications. The main one is the following version of the uniqueness principle: if μ and ν are finite and nonnegative Radon measures on \mathbb{R}^d and if $I_{k,\beta}[\mu] = I_{k,\beta}[\nu]$ a.e. on \mathbb{R}^d , then $\mu = \nu$. We also prove a pointwise Hedberg's inequality in the sense of the operator Δ_k and we deduce L^p -boundedness properties of the Δ_k -Riesz potentials.

2 Generalities in Dunkl Theory

In order to help the reader, we have collected in this section some basics from Dunkl theory which will be used in the sequel.

¹see for example [2],[12],[14] and [18].

Notations: Let us introduce the following functional spaces which are present throughout the paper:

- Ω a W -invariant open subset of \mathbb{R}^d .
- $L_k^p(\Omega)$ (resp. $L_{k,loc}^p(\Omega)$), $1 \leq p < +\infty$ the space of measurable functions $f : \Omega \rightarrow \mathbb{C}$ such that $\|f\|_{L_k^p(\Omega)}^p := \int_{\Omega} |f(x)|^p \omega_k(x) dx < +\infty$ (resp. $\int_K |f(x)|^p \omega_k(x) dx < +\infty$ for any compact set $K \subset \Omega$).
- $L_k^\infty(\Omega)$ the space of measurable and essentially bounded functions on Ω .
- When $\Omega = \mathbb{R}^d$, the norm of the space $L_k^p(\mathbb{R}^d)$, $1 \leq p \leq +\infty$, will be denoted $\|\cdot\|_{k,p}$ instead of $\|\cdot\|_{L_k^p(\mathbb{R}^d)}$.
- $\mathcal{D}(\Omega)$ the space of C^∞ -functions on Ω with compact support.
- $\mathcal{D}'(\Omega)$ the space of distributions on Ω (i.e. the topological dual of $\mathcal{D}(\Omega)$ carrying the Fréchet topology).
- $\mathcal{S}(\mathbb{R}^d)$ the Schwartz space of C^∞ -functions on \mathbb{R}^d which are rapidly decreasing together with their derivatives.
- $\mathcal{S}'(\mathbb{R}^d)$ the space of tempered distributions.

2.1 The Dunkl transform

The Dunkl transform of a function $f \in L_k^1(\mathbb{R}^d)$ is defined by (see [16] and [25])

$$\mathcal{F}_k(f)(\lambda) := \int_{\mathbb{R}^d} f(x) E_k(-i\lambda, x) \omega_k(x) dx, \quad \lambda \in \mathbb{R}^d, \quad (2.1)$$

where $E_k(x, y)$ is the Dunkl kernel (1.6) which is analytically extendable to $\mathbb{C}^d \times \mathbb{C}^d$ and satisfies the following properties (see [5], [7], [16])

1. for all $x, y \in \mathbb{R}^d$, we have

$$|E_k(-ix, y)| \leq 1. \quad (2.2)$$

2. for all $a \in \mathbb{C}$, $x, y \in \mathbb{C}^d$ and all $g \in W$, we have

$$E_k(ax, y) = E_k(x, ay), \quad E_k(x, y) = E_k(y, x) \quad \text{and} \quad E_k(gx, gy) = E_k(x, y).$$

It is well known (see [16]) that the Dunkl transform \mathcal{F}_k is an isomorphism of $\mathcal{S}(\mathbb{R}^d)$ onto itself and its inverse is given by

$$\mathcal{F}_k^{-1}(f)(x) = c_k^{-2} \int_{\mathbb{R}^d} f(\lambda) E_k(ix, \lambda) \omega_k(\lambda) d\lambda, \quad x \in \mathbb{R}^d, \quad (2.3)$$

where c_k is the constant given by (1.7).

We note that for $f, g \in \mathcal{S}(\mathbb{R}^d)$ the following relation holds

$$\int_{\mathbb{R}^d} \mathcal{F}_k(f)(x) g(x) \omega_k(x) dx = \int_{\mathbb{R}^d} f(x) \mathcal{F}_k(g)(x) \omega_k(x) dx. \quad (2.4)$$

Moreover, the transformation $c_k^{-1} \mathcal{F}_k$ extends uniquely to an isometric isomorphism of $L_k^2(\mathbb{R}^d)$ (Plancherel theorem, see [16]).

We will also need the Dunkl transform $\mathcal{F}_k(S)$ of a tempered distribution $S \in \mathcal{S}'(\mathbb{R}^d)$ which is the distribution defined by

$$\langle \mathcal{F}_k(S), \phi \rangle := \langle S, \mathcal{F}_k(\phi) \rangle, \quad \phi \in \mathcal{S}(\mathbb{R}^d).$$

It is known that \mathcal{F}_k is a topological isomorphism of $\mathcal{S}'(\mathbb{R}^d)$ onto itself (see [31]). Note that if μ is a bounded Radon measure on \mathbb{R}^d , $\mu \in \mathcal{S}'(\mathbb{R}^d)$ and its distributional Dunkl transform can be identified to the continuous function $\xi \mapsto \int_{\mathbb{R}^d} E_k(-ix, \xi) d\mu(x) \omega_k(\xi)$. In the literature, the function

$$\mathcal{F}_k(\mu) : \xi \mapsto \int_{\mathbb{R}^d} E_k(-ix, \xi) d\mu(x) \tag{2.5}$$

is called the Dunkl transform of the measure μ . This transformation is injective on the space of bounded Radon measures on \mathbb{R}^d (see [22]).

We recall also that the Dunkl-Laplace operator Δ_k leaves the spaces $\mathcal{D}'(\mathbb{R}^d)$ and $\mathcal{S}'(\mathbb{R}^d)$ invariant where the Δ_k -action on S in $\mathcal{D}'(\mathbb{R}^d)$ (resp. in $\mathcal{S}'(\mathbb{R}^d)$) is defined as in the classical case by

$$\langle \Delta_k S, \phi \rangle = \langle S, \Delta_k \phi \rangle, \quad \phi \in \mathcal{D}(\mathbb{R}^d) \text{ (resp. } \phi \in \mathcal{S}(\mathbb{R}^d)\text{)}. \tag{2.6}$$

2.2 Dunkl's translation operators and heat kernel properties

- The Dunkl translation operators $\tau_x, x \in \mathbb{R}^d$, are defined on $\mathcal{C}^\infty(\mathbb{R}^d)$ by (see [31])

$$\forall y \in \mathbb{R}^d, \quad \tau_x f(y) = \int_{\mathbb{R}^d} V_k \circ T_z \circ V_k^{-1}(f)(y) d\mu_x(z), \tag{2.7}$$

where T_x is the classical translation operator given by $T_x f(y) = f(x + y)$. The operators $\tau_x, x \in \mathbb{R}^d$, satisfy the following properties:

- 1) For all $x \in \mathbb{R}^d$, the operator τ_x is continuous from $\mathcal{C}^\infty(\mathbb{R}^d)$ into itself.
- 2) For all $f \in \mathcal{C}^\infty(\mathbb{R}^d)$ and all $x, y \in \mathbb{R}^d$, we have

$$\tau_x f(0) = f(x), \quad \tau_x f(y) = \tau_y f(x).$$

- 3) The Dunkl-Laplace operator Δ_k commutes with the Dunkl translations, i.e.

$$\tau_x(\Delta_k f) = \Delta_k(\tau_x f), \quad x \in \mathbb{R}^d, f \in \mathcal{C}^\infty(\mathbb{R}^d).$$

- 4) If $f \in \mathcal{C}^\infty(\mathbb{R}^d)$ is radial, M. Rösler ([26]) has proved the useful formula

$$\forall x \in \mathbb{R}^d, \quad \tau_x f(y) = \int_{\mathbb{R}^d} \tilde{f}(\sqrt{\|x\|^2 + \|y\|^2 + 2\langle x, z \rangle}) d\mu_y(z), \tag{2.8}$$

where \tilde{f} is the profile of f and μ_y is the measure defined by (1.3).

In the particular case when $f \in \mathcal{S}(\mathbb{R}^d)$, $\tau_x f \in \mathcal{S}(\mathbb{R}^d)$ and using the Dunkl transform we have (see [31]):

$$\tau_x f(y) = \mathcal{F}_k^{-1}[E_k(ix, \cdot) \mathcal{F}_k(f)](y) = c_k^{-2} \int_{\mathbb{R}^d} \mathcal{F}_k(f)(\lambda) E_k(ix, \lambda) E_k(iy, \lambda) \omega_k(\lambda) d\lambda, \quad y \in \mathbb{R}^d.$$

• Using (2.8), the Dunkl heat kernel can also be written

$$p_k(t, x, y) = \frac{1}{(2t)^{d/2+\gamma} c_k} \tau_{-x} \left(e^{-\frac{\|\cdot\|^2}{4t}} \right) (y) \quad (2.9)$$

$$= \frac{1}{(2t)^{d/2+\gamma} c_k} \int_{\mathbb{R}^d} e^{-\frac{1}{4t} (\|x\|^2 + \|y\|^2 - 2 \langle x, z \rangle)} d\mu_y(z). \quad (2.10)$$

For later use, we record also the following properties of the heat kernel (see [21] and [25])

1. The Dunkl heat kernel is symmetric in x and y i.e. $p_k(t, x, y) = p_k(t, y, x)$, $t > 0$.
2. For every $t > 0$ and $x \in \mathbb{R}^d$, we have

$$\|p_k(t, x, \cdot)\|_{k,1} = \int_{\mathbb{R}^d} p_k(t, x, y) \omega_k(y) dy = 1. \quad (2.11)$$

3. For every $t > 0$ and $x, y \in \mathbb{R}^d$,

$$p_k(t, x, y) = \mathcal{F}_k^{-1} \left(E_k(-ix, \cdot) e^{-t\|\cdot\|^2} \right) (y) \quad (2.12)$$

$$= c_k^{-2} \int_{\mathbb{R}^d} e^{-t\|\xi\|^2} E_k(-ix, \xi) E_k(iy, \xi) \omega_k(\xi) d\xi. \quad (2.13)$$

4. For every $t > 0$, the following inequality holds

$$\forall x, y \in \mathbb{R}^d, \quad p_k(t, x, y) \leq \frac{1}{(2t)^{d/2+\gamma} c_k} e^{-\frac{1}{4t} \min_{g \in W} \|x - gy\|^2}. \quad (2.14)$$

5. For all $t, s > 0$, the Dunkl heat kernel satisfies the semi-group property

$$\forall x, y \in \mathbb{R}^d, \quad p_k(t + s, x, y) = \int_{\mathbb{R}^d} p_k(t, x, z) p_k(s, y, z) \omega_k(z) dz. \quad (2.15)$$

2.3 The harmonic kernel and Δ_k -subharmonic functions

For $r > 0$ and $x, y \in \mathbb{R}^d$, let $h_k(r, x, y)$ be the harmonic kernel defined by (1.11). In the classical case (i.e. $k = 0$), we have $\mu_y = \delta_y$ (the Dirac measure at y) and then $h_0(r, x, y) = \mathbf{1}_{[0,r]}(\|x - y\|) = \mathbf{1}_{B(x,r)}(y)$. This implies, in particular, that the Dunkl-volume operator defined by (1.10) generalizes the usual one.

The harmonic kernel has the following properties (see [8]):

- 1) For all $r > 0$ and $x, y \in \mathbb{R}^d$, $0 \leq h_k(r, x, y) \leq 1$.

2) For all fixed $x, y \in \mathbb{R}^d$, the function $r \mapsto h_k(r, x, y)$ is right-continuous and nondecreasing.

3) Let $r > 0$ and $x \in \mathbb{R}^d$. If $k(\alpha) > 0$ for every $\alpha \in R$, then

$$\text{supp } h_k(r, x, \cdot) = B^W(x, r) := \cup_{g \in W} B(gx, r)$$

and if the function k vanishes somewhere then

$$B(x, r) \subset \text{supp } h_k(r, x, \cdot) \subset B^W(x, r)$$

(see [8] and [9]).

4) For all $r > 0$ and $x, y \in \mathbb{R}^d$, we have

$$h_k(r, x, y) = h_k(r, y, x). \quad (2.16)$$

5) Let $r > 0$ and $x, y \in \mathbb{R}^d$. Then, for all $g \in W$, we have

$$h_k(r, gx, gy) = h_k(r, x, y) \quad \text{and} \quad h_k(r, gx, y) = h_k(r, x, g^{-1}y). \quad (2.17)$$

6) For all $r > 0$ and $x \in \mathbb{R}^d$, we have

$$\|h_k(r, x, \cdot)\|_{k,1} := \int_{\mathbb{R}^d} h_k(r, x, y) \omega_k(y) dy = m_k(B(0, r)) = \frac{d_k r^{d+2\gamma}}{d+2\gamma}, \quad (2.18)$$

where we recall that $dm_k(y) = \omega_k(y) dy$ and d_k is the constant

$$d_k = \int_{S^{d-1}} \omega_k(\xi) d\sigma(\xi) = \frac{c_k}{2^{d/2+\gamma-1} \Gamma(d/2 + \gamma)}. \quad (2.19)$$

Here $d\sigma(\xi)$ is the surface measure of the unit sphere S^{d-1} of \mathbb{R}^d .

Finally, we recall that

- a function u of class C^2 on Ω is D-subharmonic in the sense of (1.10) if and only if $\Delta_k u \geq 0$ on Ω (see [10]).
- if u is D-subharmonic on Ω , then $u\omega_k \in L_{loc}^1(\Omega)$ (that is $u \in L_{k,loc}^1(\Omega)$) and its distributional Dunkl-Laplacian $\Delta_k(u\omega_k)$ is a nonnegative distribution on Ω in the sense that for any nonnegative function $\phi \in \mathcal{D}(\Omega)$ we have

$$\langle \Delta_k(u\omega_k), \phi \rangle := \langle u\omega_k, \Delta_k \phi \rangle = \int_{\mathbb{R}^d} u(x) \Delta_k \phi(x) \omega_k(x) dx \geq 0. \quad (2.20)$$

The nonnegative distribution $\Delta_k(u\omega_k)$ is then a nonnegative Radon measure on Ω called the Δ_k -Riesz measure of the D-subharmonic function u (see [10]). In particular, if $u \in C^2(\Omega)$ its Δ_k -Riesz measure is equal to $\Delta_k u(x) \omega_k(x) dx$.

3 The Δ_k -Riesz kernel

In this section, we will study some properties of the Δ_k -Riesz kernel. Recalling that for $x, y \in \mathbb{R}^d$ and $0 < \beta < d + 2\gamma$, the Δ_k -Riesz kernel is defined by

$$R_{k,\beta}(x, y) := \frac{1}{\Gamma(\beta/2)} \int_0^{+\infty} t^{\frac{\beta}{2}-1} p_k(t, x, y) dt. \quad (3.1)$$

Remark 3.1 1) *Since the Dunkl heat kernel is positive, we have $0 < R_{k,\beta}(x, y) \leq +\infty$ for all $x, y \in \mathbb{R}^d$.*

2) *Let $x \in \mathbb{R}^d$ be fixed. From (2.14), we can see that if $y \notin \mathbb{R}^d \setminus W.x$, then for any $\beta \in]-\infty, d + 2\gamma[$ the function $t \mapsto t^{\frac{\beta}{2}-1} p_k(t, x, y)$ is integrable on $]0, +\infty[$. Thus, using the properties of the Gamma function, the function $y \mapsto \frac{1}{\Gamma(\beta/2)} \int_0^{+\infty} t^{\frac{\beta}{2}-1} p_k(t, x, y) dt$ is well defined on $\mathbb{R}^d \setminus W.x$ whenever $\beta \in]-\infty, d + 2\gamma[\setminus -2\mathbb{N}$. In this case, we will continue denoting it $y \mapsto R_{k,\beta}(x, y)$.*

In the following result, we will show that the Δ_k -Riesz kernel can be expressed in terms of the harmonic kernel. This new formula will be a crucial tool in the sequel of the paper.

Proposition 3.1 *For every $x, y \in \mathbb{R}^d$, we have*

$$R_{k,\beta}(x, y) = \kappa \int_{\mathbb{R}^d} \left(\|x\|^2 + \|y\|^2 - 2 \langle x, z \rangle \right)^{\frac{\beta-(d+2\gamma)}{2}} d\mu_y(z) \quad (3.2)$$

$$= \frac{\kappa}{d + 2\gamma - \beta} \int_0^{+\infty} t^{\beta-d-2\gamma} h_k(t, x, y) \frac{dt}{t}, \quad (3.3)$$

where

$$\kappa = \kappa(d, \gamma, \beta) = \frac{2^{1-\beta} \Gamma(\frac{d+2\gamma-\beta}{2})}{d_k \Gamma(\beta/2) \Gamma(d/2 + \gamma)} = \frac{2^{\frac{d}{2}+\gamma-\beta} \Gamma(\frac{d+2\gamma-\beta}{2})}{c_k \Gamma(\beta/2)}, \quad (3.4)$$

c_k and d_k being the constants given by (1.7) and (2.19) respectively.

Proof: Using the change of variables $1/4t \leftrightarrow t$, the relation (2.10) can be rewritten

$$R_{k,\beta}(x, y) = \frac{2^{\frac{d}{2}+\gamma-\beta}}{\Gamma(\beta/2) c_k} \int_0^{+\infty} t^{\frac{d+2\gamma-\beta}{2}-1} \int_{\mathbb{R}^d} e^{-t(\|x\|^2 + \|y\|^2 - 2 \langle x, z \rangle)} d\mu_x(z) dt.$$

Now, by Fubini's theorem and the identity

$$\forall a \geq 0, \quad \forall \theta > 0, \quad a^{-\theta/2} = \frac{1}{\Gamma(\theta/2)} \int_0^{+\infty} s^{\frac{\theta}{2}-1} e^{-sa} ds$$

(notice that if we take $a = 0$, the both terms are equal $+\infty$), we deduce that (3.2) holds.

• Let us now prove (3.3). Starting from (3.2) and applying again Fubini's theorem, we get

$$\begin{aligned}
R_{k,\beta}(x, y) &= \kappa \int_{\mathbb{R}^d} \left(\|x\|^2 + \|y\|^2 - 2 \langle x, z \rangle \right)^{\frac{\beta-(d+2\gamma)}{2}} d\mu_y(z) \\
&= \frac{\kappa}{d+2\gamma-\beta} \int_{\mathbb{R}^d} \int_{\sqrt{\|x\|^2+\|y\|^2-2\langle x,z \rangle}}^{+\infty} t^{\beta-d-2\gamma} \frac{dt}{t} d\mu_y(z) \\
&= \frac{\kappa}{d+2\gamma-\beta} \int_0^{+\infty} t^{\beta-d-2\gamma} \left(\int_{\mathbb{R}^d} \mathbf{1}_{[0,t]}(\sqrt{\|x\|^2 + \|y\|^2 - 2 \langle x, z \rangle}) d\mu_y(z) \right) \frac{dt}{t} \\
&= \frac{\kappa}{d+2\gamma-\beta} \int_0^{+\infty} t^{\beta-d-2\gamma} h_k(t, x, y) \frac{dt}{t}.
\end{aligned}$$

This gives the desired relation. \square

Example 3.1 1) When $k = 0$, as $\mu_y = \delta_y$ we have $R_{0,\beta}(x, y) = \kappa(d, 0, \beta) \|x - y\|^{\beta-d}$ the classical Riesz kernel (see [18]).

2) Since $\mu_0 = \delta_0$, for any choice of the Coxeter-Weyl group and of a nonnegative multiplicity function, we have $R_{k,\beta}(x, 0) = \kappa(d, \gamma, \beta) \|x\|^{\beta-d-2\gamma}$.

3) We consider \mathbb{R}^d ($d \geq 1$) with the root system $R_m := \{\pm e_1, \dots, \pm e_m\}$, where m is a fixed integer in $\{1, \dots, d\}$ and $(e_j)_{1 \leq j \leq d}$ is the canonical basis of \mathbb{R}^d . For $\xi \in \mathbb{R}^d$, we will denote $\xi = (\xi^{(m)}, \xi') \in \mathbb{R}^m \times \mathbb{R}^{d-m}$.

We note that the Coxeter-Weyl group is $W = \mathbb{Z}_2^m$ and the \mathbb{Z}_2^m -orbit of a point $\xi \in \mathbb{R}^d$ is as follows

$$\mathbb{Z}_2^m \cdot \xi := \{ \varepsilon \cdot \xi := (\varepsilon_1 \xi_1, \dots, \varepsilon_m \xi_m, \xi'), \quad \varepsilon = (\varepsilon_i)_{1 \leq i \leq m} \in \{\pm 1\}^m \}.$$

The multiplicity function can be represented by the m -multidimensional parameter $k = (k_1, \dots, k_m)$ with $k_j = k(e_j) > 0$. Moreover, the Rösler measure is of the form $\mu_y = \mu_{(y^{(m)}, y')} = \mu_{y_1} \otimes \dots \otimes \mu_{y_m} \otimes \delta_{y'}$ with μ_{y_i} the \mathbb{Z}_2 -Rösler measure at point y_i . If $y_i = 0$, we know that $\mu_0 = \delta_0$ and if $y_i \neq 0$, we have

$$\langle \mu_{y_i}, f \rangle := \int_{-1}^1 f(ty_i) \phi_{k_i}(t) dt, \quad f \in \mathcal{C}(\mathbb{R}),$$

where ϕ_{k_i} is the \mathbb{Z}_2 -Dunkl density function of parameter k_i given by (see [5] or [25] p.104)

$$\phi_{k_i}(t) := \frac{\Gamma(k_i + 1/2)}{\sqrt{\pi} \Gamma(k_i)} (1-t)^{k_i-1} (1+t)^{k_i} \mathbf{1}_{[-1,1]}(t). \quad (3.5)$$

In this case, the Δ_k -Riesz kernel is of the form

$$\begin{aligned}
R_{k,\beta}(x, y) &= \kappa \int_{[-1,1]^m} \left(\|x^{(m)}\|^2 + \|y^{(m)}\|^2 - 2 \sum_{j=1}^m t_j x_j y_j + \|x' - y'\|^2 \right)^{\frac{\beta-d-2\gamma}{2}} \\
&\quad \times \prod_{i=1}^m \phi_{k_i}(t_i) dt_1 \dots dt_m. \quad (3.6)
\end{aligned}$$

Proposition 3.2 *Suppose that $\gamma > 0$. Let $0 < \beta < d + 2\gamma$ and $x, y \in \mathbb{R}^d$.*

- 1) *If $y \notin W.x$, then $R_{k,\beta}(x, y) < +\infty$.*
- 2) *Assume that $x \in \mathbb{R}^d \setminus \bigcup_{\alpha \in R} H_\alpha$. Then $R_{k,\beta}(x, x) = +\infty$ if and only if $d \geq \beta$.*
- 3) *If $x \in \bigcup_{\alpha \in R} H_\alpha$ and $\beta \leq d$, then $R_{k,\beta}(x, x) = +\infty$.*

Proof: At first we note that

$$\forall x, y \in \mathbb{R}^d, \quad \forall t > 0, \quad t^{\frac{\beta}{2}-1} p_k(t, x, y) \leq Ct^{\frac{\beta-d-2\gamma}{2}-1}$$

Hence, as $\beta < d + 2\gamma$, the function $t \mapsto t^{\frac{\beta}{2}-1} p_k(t, x, y)$ is integrable on $[1, +\infty[$ for every $x, y \in \mathbb{R}^d$.

1) We obtain the result by using (2.14).

2) Fix $x \in \mathbb{R}^d$ such that x is not in any hyperplane H_α , $\alpha \in R$ (i.e. x lives in a Weyl chamber). We will use the following short-time asymptotic result of the Dunkl type heat kernel which has been established in ([24], Corollary 2): Let C be a fixed Weyl chamber. If $x, y \in C$, then

$$p_k(t, x, y) \sim_{t \rightarrow 0} (\omega_k(x)\omega_k(y))^{-1/2} (4\pi t)^{-d/2} e^{-\frac{\|x-y\|^2}{4t}}. \quad (3.7)$$

Taking $y = x$, we deduce that the function $t \mapsto t^{\frac{\beta}{2}-1} p_k(t, x, x)$ is not integrable near 0 if and only if $d \geq \beta$.

3) Let $x \in H_\alpha$ for some $\alpha \in R$. One can see that the function $\psi : \xi \mapsto R_{k,\beta}(\xi, \xi)$ is the increasing limit of the sequence of continuous functions $\xi \mapsto \int_{1/n}^n t^{\frac{\beta}{2}-1} p_k(t, \xi, \xi) dt$. This implies that ψ is lower semi-continuous on \mathbb{R}^d . Consequently, when $\beta \leq d$ we have $R_{k,\beta}(x, x) = \liminf_{\xi \rightarrow x} R_{k,\beta}(\xi, \xi) = +\infty$. \square

As already mentioned, for $g \neq id$, it is much more difficult to see if $R_{k,\beta}(x, gx)$ is finite or infinite. This new phenomena will be illustrated by the following complete characterization of the singularities of the Δ_k -Riesz kernel in the case of the \mathbb{Z}_2^m -Coxeter-Weyl group acting on \mathbb{R}^d . More precisely, we have:

Proposition 3.3 *Let $x \in \mathbb{R}^d \setminus \{0\}$. Using the same notations of Example 3.1, 3), denoting H_i the hyperplane orthogonal to e_i and recalling $\varepsilon.x = (\varepsilon_1 x_1, \dots, \varepsilon_m x_m, x') \in \mathbb{Z}_2^m .x$, we have*

1. *If $x \in \bigcap_{i=1}^m H_i$, then $x = \varepsilon.x$ and $R_{k,\beta}(x, x) = +\infty$.*
2. *Assume that $x \notin \bigcap_{i=1}^m H_i$. Set $A := \{i \in \{1, \dots, m\}, x_i \neq 0\}$ and $\varepsilon^{(n)}.x = (\varepsilon_1 x_1, \dots, \varepsilon_m x_m, x')$ the point of \mathbb{Z}_2^m -orbit of x such that $|\{j \in A, \varepsilon_j = 1\}| = n$ i.e the point $\varepsilon^{(n)}.x$ has exactly n among the nonzero coordinates $(x_j)_{j \in A}$ that have not been changed under the action of \mathbb{Z}_2^m . Then,*

$$R_{k,\beta}(x, \varepsilon^{(n)}.x) = +\infty \iff d \geq 2(|A| - n + \sum_{j \in A} k_j - \gamma) + \beta. \quad (3.8)$$

3. Assume that $x \notin \cup_{i=1}^m H_i$. Then,

$$R_{k,\beta}(x, \varepsilon^{(n)}.x) = +\infty \iff d \geq 2(m-n) + \beta. \quad (3.9)$$

In this case, we have $\sum_{n=\max(0, \lfloor m-\frac{d}{2}+\frac{\beta}{2} \rfloor)}^m \binom{m}{n}$ singularities living in $\mathbb{R}^d \setminus \cup_{i=1}^m H_i$.

Proof: For abbreviation, we will use the following constants

$$C_1 := 2^{\frac{\beta-d-2\gamma}{2}} \kappa, \quad C(k) := \frac{\Gamma(k+1/2)}{\sqrt{\pi}\Gamma(k)}. \quad (3.10)$$

From (3.6), it is easy to see that

$$R_{k,\beta}(x, \varepsilon.x) = C_1 \int_{[-1,1]^m} \left(\sum_{j=1}^m (1 - \varepsilon_j t_j) x_j^2 \right)^{\frac{\beta-d-2\gamma}{2}} \prod_{j=1}^m \phi_{k_j}(t_j) \otimes_{j=1}^m dt_j. \quad (3.11)$$

1) Clearly, from (3.11), the condition $x \in \cap_{i=1}^m H_i$ i.e. $x^{(m)} = 0$ implies that $x = \varepsilon.x = (0, x')$ and $R_{k,\beta}(x, \varepsilon.x) = +\infty$.

2) Suppose that $x \notin \cap_{i=1}^m H_i$. Using the notations of the Proposition, Fubini's theorem and the fact that ϕ_{k_j} are probability densities, (3.11) can be written in the following form

$$R_{k,\beta}(x, \varepsilon.x) = C_1 \int_{[-1,1]^{|A|}} \left(\sum_{j \in A} (1 - \varepsilon_j t_j) x_j^2 \right)^{\frac{\beta-d-2\gamma}{2}} \prod_{j \in A} \phi_{k_j}(t_j) \otimes_{j \in A} dt_j. \quad (3.12)$$

We will distinguish two cases:

First case $|A| = 1$. Let $i \in \{1, \dots, m\}$ such that $x_i \neq 0$. In this case, using (3.5) and (3.10), we deduce that (3.12) takes the form

$$\begin{aligned} R_{k,\beta}(x, \varepsilon.x) &= C_1 \int_{-1}^1 \left((1 - \varepsilon_i s) x_i^2 \right)^{\frac{\beta-d-2\gamma}{2}} \phi_{k_i}(s) ds \\ &= C(k_i) C_1 |x_i|^{\beta-d-2\gamma} \int_{-1}^1 (1 - \varepsilon_i s)^{\frac{\beta-d-2\gamma}{2}} (1-s)^{k_i-1} (1+s)^{k_i} ds. \end{aligned}$$

• If $\varepsilon_i = 1$, then according to our notations, we have $n = |A| = 1$, $\varepsilon.x = \varepsilon^{(1)}.x = x$ and

$$R_{k,\beta}(x, \varepsilon^{(1)}.x) = C(k_i) C_1 |x_i|^{\beta-d-2\gamma} \int_{-1}^1 (1-s)^{k_i+\frac{\beta-d-2\gamma}{2}-1} (1+s)^{k_i} ds.$$

Consequently, $R_{k,\beta}(x, \varepsilon^{(1)}.x) = +\infty$ if and only if $d \geq \beta + 2k_i - 2\gamma$. Then, the result is proved in this case.

• When $\varepsilon_i = -1$, we have $n = 0$, $\varepsilon.x = \varepsilon^{(0)}.x$ and

$$R_{k,\beta}(x, \varepsilon^{(0)}.x) = C(k_i) |x_i|^{\beta-d-2\gamma} \int_{-1}^1 (1+s)^{k_i+\frac{\beta-d-2\gamma}{2}} (1-s)^{k_i-1} ds.$$

Thus, as $k_i > 0$ we have $R_{k,\beta}(x, \varepsilon^{(0)}.x) = +\infty$ if and only if $d \geq 2(1 + k_i - \gamma) + \beta$.

Second case $|A| = r \geq 2$. Using (3.12) and the change of variables $t_j \leftrightarrow 1 - \varepsilon_j t_j$, we obtain

$$\begin{aligned} R_{k,\beta}(x, \varepsilon.x) &= C_1 \int_{]0,2[|^A|} \left(\sum_{j \in A} t_j x_j^2 \right)^{\frac{\beta-d-2\gamma}{2}} \prod_{j \in A} \phi_{k_j}(\varepsilon_j - \varepsilon_j t_j) \otimes_{j \in A} dt_j \\ &= C_1 \int_{]0,2[|^A| \cap B_r} + C_1 \int_{]0,2[|^A| \setminus B_r} \\ &= C_1 I(x, \varepsilon.x) + C_1 J(x, \varepsilon.x), \end{aligned}$$

where B_r is the open unit ball in $\mathbb{R}^{|A|} = \mathbb{R}^r$.

The singularities of these integrals being at point 0 and thus it is clear that $J(x, \varepsilon.x) < +\infty$. Thus, we need to know when the integral $I(x, \varepsilon.x)$ diverges. To do this, we will identify $(t_j)_{j \in A}$ with $v = (v_1, \dots, v_r) \in \mathbb{R}^r$ and use the spherical coordinates in \mathbb{R}^r :

$$\rho = \|v\|, \quad v_1 = \rho a_1, \quad \dots, v_{r-1} = \rho a_{r-1} \quad \text{and} \quad v_r = \rho a_r,$$

where

$$a_1 = \cos \theta_1, \dots, a_{r-1} = \prod_{i=1}^{r-2} \sin \theta_i \cos \theta_{r-1}, \quad a_r = \prod_{i=1}^{r-1} \sin \theta_i.$$

Notice that all a_j are positive.

$$I(x, \varepsilon.x) = \int_{S_+^{r-1}} \psi(a^{(r)}, x^{(r)}) \left(\int_0^1 \prod_{j \in A} \phi_{k_j}(\varepsilon_j - \varepsilon_j a_j \rho) \rho^{r + \frac{\beta-d-2\gamma}{2} - 1} d\rho \right) d\sigma_r(a^{(r)}), \quad (3.13)$$

where $S_+^{r-1} :=]0, 2[\cap S^{r-1}$, $d\sigma_r$ is the surface measure of the unit sphere S^{r-1} of \mathbb{R}^r , $a^{(r)} = (a_j)_{j \in A}$, $x^{(r)} = (x_j)_{j \in A}$ and

$$\psi(a^{(r)}, x^{(r)}) := \left(\sum_{j \in A} a_j x_j^2 \right)^{\frac{\beta-d-2\gamma}{2}}.$$

We have

$$\phi_{k_j}(\varepsilon_j - \varepsilon_j a_j \rho) = C(k_j) (1 - \varepsilon_j + \varepsilon_j a_j \rho)^{k_j - 1} (1 + \varepsilon_j - \varepsilon_j a_j \rho)^{k_j}.$$

Hence,

$$\phi_{k_j}(\varepsilon_j - \varepsilon_j a_j \rho) = \begin{cases} C(k_j) a_j^{k_j - 1} \rho^{k_j - 1} (2 - a_j \rho)^{k_j}, & \text{if } \varepsilon_j = 1 \\ C(k_j) a_j^{k_j} \rho^{k_j} (2 - a_j \rho)^{k_j - 1}, & \text{if } \varepsilon_j = -1. \end{cases} \quad (3.14)$$

Define

$$A_1 := \{j \in A, \quad \varepsilon_j = 1\}, \quad A_2 = A \setminus A_1.$$

According to our notations, we have $|A_1| = |\{j, \quad \varepsilon_j = 1\}| = n$.

Then, from (3.13), (3.14) and recalling the definition of the vector $\varepsilon^{(n)}.x$, we deduce that

$$I(x, \varepsilon^{(n)}.x) = \int_{S_+^{r-1}} \psi(a^{(r)}, x^{(r)}) \left(\int_0^1 f(a^{(r)}, \rho) \rho^{\lambda+r+\frac{\beta-d-2\gamma}{2}-1} d\rho \right) d\sigma_r(a^{(r)}), \quad (3.15)$$

with

$$f(a^{(r)}, \rho) := \prod_{j \in A_1} C(k_j) a_j^{k_j-1} (2 - a_j \rho)^{k_j} \prod_{j \in A_2} C(k_j) a_j^{k_j} (2 - a_j \rho)^{k_j-1}.$$

and

$$\lambda := \sum_{j \in A_1} (k_j - 1) + \sum_{j \in A_2} k_j = \sum_{j \in A} k_j - n.$$

The function $\rho \mapsto f(a^{(r)}, \rho)$ is continuous and does not vanish on the compact set $[0, 1]$. So that the singularity in the $d\rho$ -integral is only in the term of

$$\rho^{\lambda+r+\frac{\beta-d-2\gamma}{2}-1} = \rho^{(\sum_{j \in A} k_j) - n + r + \frac{\beta-d-2\gamma}{2} - 1}.$$

Finally, we conclude that

$$R_{k,\beta}(x, \varepsilon^{(n)}.x) = +\infty \quad \Leftrightarrow \quad I(x, \varepsilon^{(n)}.x) = +\infty \quad \Leftrightarrow \quad d \geq 2(|A| - n + \sum_{j \in A} k_j - \gamma) + \beta.$$

This completes the proof of the assertion 2).

3) When $x \notin \cup_{i=1}^m H_i$, we have $A = \{1, \dots, m\}$ and then the result is a particular case of the statement 2). \square

Proposition 3.4 *The Riesz kernel $R_{k,\beta}(\cdot, \cdot)$ satisfies the following properties*

1) *For every $x, y \in \mathbb{R}^d$ and $g \in W$, we have*

$$R_{k,\beta}(x, y) = R_{k,\beta}(y, x), \quad R_{k,\beta}(gx, y) = R_{k,\beta}(x, g^{-1}y). \quad (3.16)$$

2) *Let $\beta, \theta > 0$ such that $\beta + \theta < d + 2\gamma$. Then we have the following generalized Riesz composition formula*

$$\int_{\mathbb{R}^d} R_{k,\beta}(x, z) R_{k,\theta}(y, z) \omega_k(z) dz = R_{k,\beta+\theta}(x, y). \quad (3.17)$$

3) *Let $x \in \mathbb{R}^d$. Then, for every $y \in \mathbb{R}^d \setminus W.x$, we have*

$$\kappa \min_{g \in W} (\|x - gy\|^{\beta-d-2\gamma}) \leq R_{k,\beta}(x, y) \leq \kappa \max_{g \in W} (\|x - gy\|^{\beta-d-2\gamma}) \quad (3.18)$$

4) *Let $y \in \mathbb{R}^d$. Then, the function $x \mapsto R_{k,\beta}(x, y)$ is*
-lower semi-continuous (l.s.c.) on \mathbb{R}^d .
-of class C^∞ on $\mathbb{R}^d \setminus W.x$ and we have

$$\partial_j R_{k,\beta}(x, y) = (\beta - d - 2\gamma) \kappa \int_{\mathbb{R}^d} (x_j - z_j) (\|x\|^2 + \|y\|^2 - 2 \langle x, z \rangle)^{\frac{\beta-2-d-2\gamma}{2}} d\mu_y(z). \quad (3.19)$$

Proof: 1) The result follows from (3.3), (2.16) and (2.17).

2) The result follows from the Fubini's theorem and the semi-group property of the Dunkl-heat kernel (2.15).

3) Let $y \in \mathbb{R}^d$. From (1.4) for any $z \in \text{supp } \mu_y$, we can write $z = \sum_{g \in W} \lambda_g(z)gy$, where $\lambda_g(z) \in [0, 1]$ are such that $\sum_{g \in W} \lambda_g(z) = 1$. Then, we have

$$\|x\|^2 + \|y\|^2 - 2 \langle x, z \rangle = \sum_{g \in W} \lambda_g(z) \|x - gy\|^2. \quad (3.20)$$

As $\psi : t \mapsto t^{\frac{\beta-d-2\gamma}{2}}$ is a convex function on $]0, +\infty[$, by (3.20) we have

$$\begin{aligned} \left(\|x\|^2 + \|y\|^2 - 2 \langle x, z \rangle \right)^{\frac{\beta-d-2\gamma}{2}} &= \left(\sum_{g \in W} \lambda_g(z) \|x - gy\|^2 \right)^{\frac{\beta-d-2\gamma}{2}} \\ &\leq \max_{g \in W} (\|x - gy\|^{\beta-d-2\gamma}). \end{aligned}$$

This implies the right inequality. Again by convexity, Jensen's inequality and (3.20), we get

$$\begin{aligned} R_{k,\beta}(x, y) &\geq \kappa \left(\int_{\mathbb{R}^d} (\|x\|^2 + \|y\|^2 - 2 \langle x, z \rangle) d\mu_y(z) \right)^{\frac{\beta-d-2\gamma}{2}} \\ &\geq \kappa \left(\sum_{g \in W} \left(\int_{\mathbb{R}^d} \lambda_g(z) d\mu_y(z) \right) \|x - gy\|^2 \right)^{\frac{\beta-d-2\gamma}{2}} \\ &\geq \kappa \left(\max_{g \in W} \|x - gy\|^2 \right)^{\frac{\beta-d-2\gamma}{2}} = \kappa \min_{g \in W} (\|x - gy\|^{\beta-(d+2\gamma)}), \end{aligned}$$

where in the last line we have used the fact that ψ is a decreasing function.

4) The function $x \mapsto R_{k,\beta}(x, y)$ is l.s.c. on \mathbb{R}^d as being the increasing limit of the sequence (f_n) of continuous functions defined by $f_n : x \mapsto \int_{1/n}^n t^{\frac{\beta}{2}-1} p_k(t, x, y) dt$.

Fix $y \in \mathbb{R}^d$. Using the fact that μ_y is with compact support and the fact that the function

$$(x, z) \mapsto (\|x\|^2 + \|y\|^2 - 2 \langle x, z \rangle)^{\frac{\beta-d-2\gamma}{2}}$$

is of class C^∞ on $\mathbb{R}^d \setminus W.y \times \mathbb{R}^d$, we can differentiate under the integral in the relation (3.2) and we obtain the result. \square

In the following result, we study the $L_{k,loc}^p(\mathbb{R}^d)$ -integrability of the function $R_{k,\beta}(x, \cdot)$, for fixed $x \in \mathbb{R}^d$.

Proposition 3.5 *Let $0 < \beta < d + 2\gamma$ and $p \in [1, \frac{d+2\gamma}{d+2\gamma-\beta}]$. Then, for every $R > 0$, there exists a positive constant $C = C(R, p, d, \gamma, \beta)$ such that*

$$\forall x \in \mathbb{R}^d, \quad \|R_{k,\beta}(x, \cdot)\|_{L_k^p(B(0,R))} \leq C. \quad (3.21)$$

In particular, for every $x \in \mathbb{R}^d$, $R_{k,\beta}(x, \cdot)$ is in $L_{k,loc}^p(\mathbb{R}^d)$.

Proof: By Jensen's inequality and (3.2), we have

$$(R_{k,\beta}(x, y))^p \leq \kappa^p \int_{\mathbb{R}^d} (\|x\|^2 + \|y\|^2 - 2\langle x, z \rangle)^{\frac{p(\beta-d-2\gamma)}{2}} d\mu_y(z).$$

Using the same idea as in the proof of (3.3), we can write the previous inequality as follows

$$\begin{aligned} (R_{k,\beta}(x, y))^p &\leq \frac{\kappa^p}{p(d+2\gamma-\beta)} \int_0^{+\infty} t^{p(\beta-d-2\gamma)} h_k(t, x, y) \frac{dt}{t} \\ &= C_1 \int_0^1 t^{p(\beta-d-2\gamma)} h_k(t, x, y) \frac{dt}{t} + C_1 \int_1^{+\infty} t^{p(\beta-d-2\gamma)} h_k(t, x, y) \frac{dt}{t} \\ &\leq C_1 \int_0^1 t^{p(\beta-d-2\gamma)} h_k(t, x, y) \frac{dt}{t} + \frac{C_1}{p(d+2\gamma-\beta)}, \end{aligned}$$

where $C_1 = \frac{\kappa^p}{p(d+2\gamma-\beta)}$ and we have used the fact that $h_k(t, x, y) \leq 1$ in the last inequality. Let then $R > 0$. From (2.18), Fubini's theorem and our hypothesis, we deduce that

$$\int_{B(0,R)} \int_0^1 t^{p(\beta-d-2\gamma)} h_k(t, x, y) \frac{dt}{t} \omega_k(y) dy \leq \frac{d_k}{d+2\gamma} \int_0^1 t^{p(\beta-d-2\gamma)} t^{d+2\gamma} \frac{dt}{t} := C_2 < +\infty.$$

This proves the desired inequality where we can take

$$C = \left(C_1 C_2 + \frac{C_1 m_k[B(0, R)]}{p(d+2\gamma-\beta)} \right)^{1/p}.$$

□

Proposition 3.6 *Let $0 < \beta < d + 2\gamma$ and $x_0 \in \mathbb{R}^d$. Then, the function $R_{k,\beta}(x_0, \cdot)$ is*

- i) *D-superharmonic on \mathbb{R}^d when $\beta \geq 2$,*
- ii) *D-harmonic on $\mathbb{R}^d \setminus W.x_0$ when $\beta = 2$,*
- iii) *D-subharmonic on $\mathbb{R}^d \setminus W.x_0$ when $\beta \leq 2$*

Proof: The case $\beta = 2$ (i.e. the case of the Dunkl-Newton kernel) has been done in [10]. So, we will deal with the case $\beta \neq 2$.

i) Suppose that $\beta > 2$. We consider the function $S_{x_0,\beta,r}$

$$S_{x_0,\beta,r}(x) := \frac{1}{\Gamma(\beta/2)} \int_r^{+\infty} t^{\frac{\beta}{2}-1} p_k(t, x_0, x) dt.$$

By the monotone convergence theorem, we see that the function $R_{k,\beta}(x_0, \cdot)$ is the pointwise increasing limit of the sequence $\left(S_{x_0,\beta,\frac{1}{n}} \right)_n$. Hence, by Proposition 3.3 in [10], it suffices to prove that for every $r > 0$, $S_{x_0,\beta,r}$ is D-superharmonic on \mathbb{R}^d . To do this, we have only to show that $S_{x_0,\beta,r}$ is of class C^2 on \mathbb{R}^d and $\Delta_k S_{x_0,\beta,r} \leq 0$ on \mathbb{R}^d (see [10], Proposition 4.1).

The function $p_k(t, x_0, \cdot)$ is of class C^∞ on \mathbb{R}^d and we can differentiate under the integral sign in the relation (2.10) to obtain

$$\partial_j p_k(t, x_0, \cdot)(x) = -\frac{1}{2t} \frac{1}{(2t)^{\frac{d}{2} + \gamma} c_k} \int_{\mathbb{R}^d} (x_j - z_j) e^{-\frac{1}{4t}(\|x\|^2 + \|x_0\|^2 - 2\langle x, z \rangle)} d\mu_{x_0}(z) \quad (3.22)$$

and

$$\begin{aligned} \partial_i \partial_j p_k(t, x_0, \cdot)(x) &= -\delta_{ij} \frac{1}{2t} p_k(t, x_0, x) \\ &+ \frac{1}{4t^2} \frac{1}{(2t)^{\frac{d}{2} + \gamma} c_k} \int_{\mathbb{R}^d} (x_j - z_j)(x_i - z_i) e^{-\frac{1}{4t}(\|x\|^2 + \|x_0\|^2 - 2\langle x, z \rangle)} d\mu_{x_0}(z), \end{aligned} \quad (3.23)$$

where δ_{ij} is the Kronecker symbol.

Using the fact that $\text{supp } \mu_{x_0} \subset B(0, \|x_0\|)$, we deduce from (3.22) and (3.23) that

$$\begin{aligned} |\partial_j p_k(t, x_0, \cdot)(x)| &\leq \frac{\|x\| + \|x_0\|}{(2t)^{1 + \frac{d}{2} + \gamma} c_k}, \\ |\partial_i \partial_j p_k(t, x_0, \cdot)(x)| &\leq \frac{1}{(2t)^{1 + \frac{d}{2} + \gamma} c_k} + \frac{(\|x\| + \|x_0\|)^2}{(2t)^{2 + \frac{d}{2} + \gamma} c_k}. \end{aligned}$$

Let $R > 0$. The previous inequalities and the differentiation theorem under the integral sign imply that $S_{x_0, \beta, r}$ is of class C^2 on the open ball $\mathring{B}(0, R)$ and as $x \mapsto p_k(t, x_0, x)$ is a solution of the Dunkl-heat equation (1.9), we deduce that

$$\begin{aligned} \forall x \in \mathring{B}(0, R), \quad \Delta_k S_{x_0, \beta, r}(x) &= \frac{1}{\Gamma(\beta/2)} \int_r^{+\infty} t^{\frac{\beta}{2} - 1} \Delta_k (p_k(t, x_0, \cdot))(x) dt \\ &= \frac{1}{\Gamma(\beta/2)} \int_r^{+\infty} t^{\frac{\beta}{2} - 1} \partial_t p_k(t, x_0, x) dt \\ &= -\frac{r^{\frac{\beta}{2} - 1}}{\Gamma(\beta/2)} p_k(r, x_0, x) - \frac{\beta - 2}{2\Gamma(\beta/2)} \int_r^{+\infty} t^{\frac{\beta}{2} - 2} p_k(t, x_0, x) dt < 0. \end{aligned}$$

Therefore, $S_{x_0, \beta, r}$ is D-superharmonic on $\mathring{B}(0, R)$. As $R > 0$ is arbitrary, we conclude that $S_{x_0, \beta, r}$ is D-superharmonic on \mathbb{R}^d as desired.

iii) Let $\beta \in]0, 2[$. Using (3.22), (3.23) and (3.20), we can see that

$$\begin{aligned} |\partial_j p_k(t, x_0, \cdot)(x)| &\leq \frac{\|x\| + \|x_0\|}{(2t)^{1 + \frac{d}{2} + \gamma} c_k} e^{-\frac{\min_{g \in W} (\|x - gx_0\|^2)}{4t}}, \\ |\partial_i \partial_j p_k(t, x_0, \cdot)(x)| &\leq \left(\frac{1}{(2t)^{1 + \frac{d}{2} + \gamma} c_k} + \frac{(\|x\| + \|x_0\|)^2}{(2t)^{2 + \frac{d}{2} + \gamma} c_k} \right) e^{-\frac{\min_{g \in W} (\|x - gx_0\|^2)}{4t}}. \end{aligned}$$

Fix an arbitrary open Dunkl ball $O^W(a, R) := \cup_{g \in W} \mathring{B}(ga, R)$ such that its closure is contained in $\mathbb{R}^d \setminus W.x_0$. The previous inequalities imply that we can differentiate with

respect to $x \in O^W(a, R)$ under the integral sign in the relation (3.1). Furthermore, using the heat equation (1.9) and integrating by parts, we obtain

$$\begin{aligned} \forall x \in O^W(a, R), \quad \Delta_k(R_{k,\beta}(x_0, \cdot))(x) &= \frac{1}{\Gamma(\beta/2)} \int_0^{+\infty} t^{\frac{\beta}{2}-1} \partial_t p_k(t, x_0, x) dt \\ &= -\frac{\beta-2}{2\Gamma(\beta/2)} \int_0^{+\infty} t^{\frac{\beta}{2}-2} p_k(t, x_0, x) dt \geq 0. \end{aligned}$$

According to Remark 3.1-2), the above relation can be written as

$$\forall x \in O^W(a, R), \quad \Delta_k(R_{k,\beta}(x_0, \cdot))(x) = -R_{k,\beta-2}(x_0, x) \geq 0. \quad (3.24)$$

Therefore, the function $R_{k,\beta}(x_0, \cdot)$ is D-subharmonic on $O^W(a, R)$ and so on $\mathbb{R}^d \setminus W.x_0$. \square

Proposition 3.7 *Let $\beta \in]0, d+2\gamma[$ and $x_0 \in \mathbb{R}^d$. Then, the function $x \mapsto R_{k,\beta}(x_0, x)\omega_k(x)$ defines a tempered distribution and we have*

$$\mathcal{F}_k(R_{k,\beta}(x_0, \cdot)\omega_k) = E_k(-ix_0, \cdot) \|\cdot\|^{-\beta} \omega_k \quad \text{in } \mathcal{S}'(\mathbb{R}^d). \quad (3.25)$$

Proof: Let $m \in \mathbb{N}$ such that $m > d + 2\gamma$. We claim that there exists a constant $C_m = C(d, \gamma, \beta, m) > 0$ such that

$$\forall x_0 \in \mathbb{R}^d, \quad \int_{\mathbb{R}^d} (1 + \|x\|^2)^{-m} R_{k,\beta}(x_0, x)\omega_k(x) dx \leq C_m. \quad (3.26)$$

From (3.3), we can write

$$\begin{aligned} R_{k,\beta}(x_0, x) &= \frac{\kappa}{d + 2\gamma - \beta} \left(\int_0^1 t^{\beta-d-2\gamma-1} h_k(t, x_0, x) dt + \int_1^{+\infty} t^{\beta-d-2\gamma-1} h_k(t, x_0, x) dt \right) \\ &:= A(x_0, x) + B(x_0, x). \end{aligned}$$

• Using Fubini's theorem and the relation (2.18), for any $x_0 \in \mathbb{R}^d$ we obtain

$$\begin{aligned} \int_{\mathbb{R}^d} (1 + \|x\|^2)^{-m} A(x_0, x)\omega_k(x) dx &\leq \int_{\mathbb{R}^d} A(x_0, x)\omega_k(x) dx \\ &= \frac{\kappa}{d + 2\gamma - \beta} \int_0^1 t^{\beta-d-2\gamma-1} \|h_k(t, x_0, \cdot)\|_{k,1} dt \\ &= \frac{d_k \kappa}{\beta(d + 2\gamma)(d + 2\gamma - \beta)} := C_{1,m}. \end{aligned}$$

• Now, using the inequality $h_k(t, x_0, x) \leq 1$, we deduce that

$$\forall x_0 \in \mathbb{R}^d, \quad B(x_0, x) \leq \frac{\kappa}{(d + 2\gamma - \beta)^2}.$$

This relation and the choice of m imply that

$$\begin{aligned} \forall x_0 \in \mathbb{R}^d, \quad \int_{\mathbb{R}^d} (1 + \|x\|^2)^{-m} B(x_0, x)\omega_k(x) dx &\leq \frac{\kappa}{(d + 2\gamma - \beta)^2} \int_{\mathbb{R}^d} (1 + \|x\|^2)^{-m} \omega_k(x) dx \\ &:= C_{2,m} < +\infty. \end{aligned}$$

This proves (3.26) and this implies that the function $R_{k,\beta}(x_0, \cdot)\omega_k$ defines a tempered distribution (see [27], Theorem VII, p. 242).

Let us now prove (3.25). For $\phi \in \mathcal{S}(\mathbb{R}^d)$, we have

$$\langle \mathcal{F}_k(R_{k,\beta}(x_0, \cdot)\omega_k), \phi \rangle = \frac{1}{\Gamma(\beta/2)} \int_{\mathbb{R}^d} \left(\int_0^{+\infty} t^{\frac{\beta}{2}-1} p_k(t, x_0, x) dt \right) \mathcal{F}_k(\phi)(x) \omega_k(x) dx.$$

Multiplying and dividing by $(1 + \|x\|^2)^m$ (the integer m is chosen as above) and using the fact that $\mathcal{F}_k(\phi) \in \mathcal{S}(\mathbb{R}^d)$, we see that we can use Fubini's theorem in the above relation. Moreover, from (2.4) and (2.12), we obtain

$$\begin{aligned} \langle \mathcal{F}_k(R_{k,\beta}(x_0, \cdot)\omega_k), \phi \rangle &= \frac{1}{\Gamma(\beta/2)} \int_0^{+\infty} t^{\frac{\beta}{2}-1} \left(\int_{\mathbb{R}^d} \mathcal{F}_k(p_k(t, x_0, \cdot))(x) \phi(x) \omega_k(x) dx \right) dt \\ &= \frac{1}{\Gamma(\beta/2)} \int_0^{+\infty} t^{\frac{\beta}{2}-1} \left(\int_{\mathbb{R}^d} E_k(-ix_0, x) e^{-t\|x\|^2} \phi(x) \omega_k(x) dx \right) dt. \end{aligned}$$

Applying again Fubini's theorem, we deduce that

$$\langle \mathcal{F}_k(R_{k,\beta}(x_0, \cdot)\omega_k), \phi \rangle = \int_{\mathbb{R}^d} E_k(-ix_0, x) \|x\|^{-\beta} \phi(x) \omega_k(x) dx.$$

This completes the proof. \square

Corollary 3.1 *For every $x_0 \in \mathbb{R}^d$, we have*

$$\lim_{\beta \rightarrow 0} R_{k,\beta}(x_0, \cdot)\omega_k = \delta_{x_0} \quad \text{in } \mathcal{S}'(\mathbb{R}^d). \quad (3.27)$$

Proof: We can see that for every $\xi \in \mathbb{R}^d$,

$$\|\xi\|^{-\beta} \leq \mathbf{1}_{\mathbb{R}^d \setminus B(0,1)}(\xi) + \|\xi\|^{-d-2\gamma} \mathbf{1}_{B(0,1)}(\xi).$$

Consequently, we can use the dominated convergence theorem to obtain from (3.25)

$$\lim_{\beta \rightarrow 0} \mathcal{F}_k(R_{k,\beta}(x_0, \cdot)\omega_k) = E_k(-ix_0, \cdot)\omega_k = \mathcal{F}_k(\delta_{x_0}) \quad \text{in } \mathcal{S}'(\mathbb{R}^d).$$

Thus, we deduce the result by using the properties of the Dunkl transform on $\mathcal{S}'(\mathbb{R}^d)$. \square

From the formula (3.24), we see that the Δ_k -Riesz measure related to the D-subharmonic function $R_{k,\beta}(x_0, \cdot)$, $\beta < 2$, is given by $-R_{k,\beta-2}(x_0, x)\omega_k(x)dx$. In the following result, we will compute the Δ_k -Riesz measure of the D-superharmonic function $R_{k,\beta}(x_0, \cdot)$ with $\beta \in [2, d + 2\gamma]$.

Corollary 3.2 *Let $2 \leq \beta < d + 2\gamma$ and $x_0 \in \mathbb{R}^d$. If $m \in [1, \beta/2]$ be an integer, then the function $x \mapsto R_{k,\beta}(x_0, x)$ satisfies*

$$(-\Delta_k)^m (R_{k,\beta}(x_0, \cdot)\omega_k) = \begin{cases} R_{k,\beta-2m}(x_0, \cdot)\omega_k & \text{in } \mathcal{S}'(\mathbb{R}^d) \quad \text{if } \beta > 2m, \\ \delta_{x_0} & \text{in } \mathcal{S}'(\mathbb{R}^d) \quad \text{if } \beta = 2m, \end{cases} \quad (3.28)$$

where δ_{x_0} is the Dirac measure at x_0 .

Proof: At first, we remark that if $U \in \mathcal{S}'(\mathbb{R}^d)$, then

$$\mathcal{F}_k(\Delta_k U) = -\|\cdot\|^2 \mathcal{F}_k(U), \quad (3.29)$$

as easily follows from the relation $\Delta_k \mathcal{F}_k(f) = -\mathcal{F}_k(\|\cdot\|^2 f)$ for all $f \in \mathcal{S}(\mathbb{R}^d)$.

From (3.29) and (3.25), we obtain

$$\begin{aligned} \mathcal{F}_k((-\Delta_k)^m (R_{k,\beta}(x_0, \cdot)\omega_k)) &= E_k(-ix_0, \cdot)\|\cdot\|^{\beta-2m}\omega_k \\ &= \begin{cases} \mathcal{F}_k(R_{k,\beta-2m}(x_0, \cdot)\omega_k) & \text{in } \mathcal{S}'(\mathbb{R}^d) \text{ if } \beta > 2m, \\ \mathcal{F}_k(\delta_{x_0}) & \text{in } \mathcal{S}'(\mathbb{R}^d) \text{ if } \beta = 2m. \end{cases} \end{aligned}$$

Hence, we deduce the result by the fact that \mathcal{F}_k is a topological isomorphism of $\mathcal{S}'(\mathbb{R}^d)$ onto itself. \square

Remark 3.2 Let $1 \leq m < \gamma + d/2$ an integer. Taking $x_0 = 0$ in (3.28), we deduce that the function $S : y \mapsto R_{k,2m}(0, y)\omega_k(y) = \kappa\|y\|^{2m-d-2\gamma}\omega_k(y)$ is the fundamental solution of the Dunkl-polylaplacian of order m $(-\Delta_k)^m$ i.e. $(-\Delta_k)^m S = \delta_0$ in $\mathcal{S}'(\mathbb{R}^d)$.

4 Riesz potentials of Radon measures

The sets $\mathcal{M}(\mathbb{R}^d)$ and $\mathcal{M}^+(\mathbb{R}^d)$ denote respectively the space of signed Radon measures on \mathbb{R}^d and the convex cone of nonnegative Radon measures on \mathbb{R}^d .

Definition 4.1 Let $\mu \in \mathcal{M}^+(\mathbb{R}^d)$ and $\beta \in]0, d + 2\gamma[$. The β - Δ_k -Riesz potential of μ is defined by

$$I_{k,\beta}[\mu](x) = \int_{\mathbb{R}^d} R_{k,\beta}(x, y)d\mu(y), \quad x \in \mathbb{R}^d. \quad (4.1)$$

Proposition 4.1 Let $\mu \in \mathcal{M}^+(\mathbb{R}^d)$ and $\beta \in]0, d + 2\gamma[$.

1. If μ is bounded, then $I_{k,\beta}[\mu] \in L_{k,loc}^p(\mathbb{R}^d)$ whenever $p \in [1, \frac{d+2\gamma}{d+2\gamma-\beta}]$. In particular, $I_{k,\beta}[\mu]$ is finite a.e. in \mathbb{R}^d .

2. The following statements are equivalent

- i) $I_{k,\beta}[\mu]$ is finite a.e. in \mathbb{R}^d ,
- ii) the measure μ satisfies

$$\int_{\mathbb{R}^d} (1 + \|y\|)^{\beta-d-2\gamma} d\mu(y) < +\infty, \quad (4.2)$$

- iii) $I_{k,\beta}[\mu](x_0) < +\infty$ for some $x_0 \in \mathbb{R}^d$.

If ii) holds, then $I_{k,\beta}[\mu] \in L_{k,loc}^1(\mathbb{R}^d)$.

Proof: **1)** Assume that μ is a probability measure on \mathbb{R}^d . Let p as in the proposition and $R > 0$. Using respectively (4.1), Jensen's inequality, Fubini's theorem, the fact that the Riesz kernel is symmetric and (3.21), we get

$$\begin{aligned} \int_{B(0,R)} (I_{k,\beta}[\mu](x))^p(x) \omega_k(x) dx &\leq \int_{B(0,R)} \left(\int_{\mathbb{R}^d} (R_{k,\beta}(x,y))^p d\mu(y) \right) \omega_k(x) dx \\ &= \int_{\mathbb{R}^d} \left(\int_{B(0,R)} (R_{k,\beta}(x,y))^p \omega_k(x) dx \right) d\mu(y) \\ &\leq C < +\infty, \end{aligned}$$

where C is the constant in (3.21).

2) ii) \Rightarrow i) Assume that the condition (4.2) holds. We will prove that $x \mapsto I_{k,\beta}[\mu](x)$ is in $L^1_{k,loc}(\mathbb{R}^d)$. Let $R > 1$. By Fubini's theorem, we have

$$\begin{aligned} A_R &:= \int_{B(0,R)} I_{k,\beta}[\mu](x) \omega_k(x) dx = \int_{\mathbb{R}^d} \int_{B(0,R)} R_{k,\beta}(x,y) \omega_k(x) dx d\mu(y) \\ &= \int_{\|y\| \leq 2R} \int_{B(0,R)} R_{k,\beta}(x,y) \omega_k(x) dx d\mu(y) + \int_{\|y\| > 2R} \int_{B(0,R)} R_{k,\beta}(x,y) \omega_k(x) dx d\mu(y) \\ &:= A_{1,R} + A_{2,R}. \end{aligned}$$

Applying the assertion 1) with the finite measure $\mu|_{B(0,R)}$, we get $A_{1,R} < +\infty$.

Now, from (3.18) we deduce that

$$A_{2,R} \leq \kappa \int_{\|y\| > 2R} \int_{B(0,R)} \max_{g \in W} (\|x - gy\|^{\beta-d-2\gamma}) \omega_k(x) dx d\mu(y).$$

But, for every $x \in B(0,R)$ and every $y \in \mathbb{R}^d \setminus B(0,2R)$, we have $\|x - gy\| \geq \|y\| - \|x\| \geq \frac{\|y\|}{2}$. Moreover, as $R > 1$, we see that $\|y\| \geq \frac{1}{2}(1 + \|y\|)$ whenever $\|y\| \geq 2R$. In other words, the inequality

$$\max_{g \in W} (\|x - gy\|^{\beta-d-2\gamma}) \leq 4^{\beta-d-2\gamma} (1 + \|y\|)^{\beta-d-2\gamma}$$

holds for every $x \in B(0,R)$ and every $y \in \mathbb{R}^d \setminus B(0,2R)$. Hence, by our hypothesis we conclude that

$$A_{2,R} \leq 4^{\beta-d-2\gamma} \kappa m_k[B(0,R)] \int_{\|y\| \geq 2R} (1 + \|y\|)^{\beta-d-2\gamma} d\mu(y) < +\infty$$

and thus the function $x \mapsto I_{k,\beta}[\mu](x) \omega_k(x)$ is locally integrable on \mathbb{R}^d . In particular, $I_{k,\beta}[\mu](x) < +\infty$ a.e. on \mathbb{R}^d .

i) \Rightarrow iii) It is obvious.

iii) \Rightarrow ii) Let $x_0 \in \mathbb{R}^d$ such that $I_{k,\beta}[\mu](x_0) < +\infty$. From (3.18), we can see that

$$\begin{aligned} I_{k,\beta}[\mu](x_0) &\geq \kappa \int_{\mathbb{R}^d} \min_{g \in W} (\|x_0 - gy\|^{\beta-d-2\gamma}) d\mu(y) \\ &\geq \kappa \int_{\mathbb{R}^d} (\|x_0\| + \|y\|)^{\beta-d-2\gamma} d\mu(y). \end{aligned}$$

If $\|x_0\| \leq 1$, we deduce immediately from the previous inequality that (4.2) holds. If $\|x_0\| > 1$, using the fact that

$$\|x_0\| + \|y\| \leq \|x_0\|(1 + \|y\|)$$

and using again the above inequality, we obtain that (4.2) holds.

This finishes the proof. \square

Remark 4.1 Let $\beta \in]0, d + 2\gamma[$.

• Let $\mu \in \mathcal{M}(\mathbb{R}^d)$ and $\mu = \mu^+ - \mu^-$ its Hahn-Jordan decomposition. If μ^+ and μ^- satisfy (4.2), then the Δ_k -Riesz potential of μ is well defined almost everywhere by setting $I_{k,\beta}[\mu](x) = I_{k,\beta}[\mu^+](x) - I_{k,\beta}[\mu^-](x)$. Moreover, the function $I_{k,\beta}[\mu] \in L^1_{k,loc}(\mathbb{R}^d)$.

• Let us introduce the following notations

$$\mathcal{M}^+_{k,\beta}(\mathbb{R}^d) := \left\{ \mu \in \mathcal{M}^+(\mathbb{R}^d), \quad \mu \text{ satisfies (4.2)} \right\} \quad (4.3)$$

and

$$\mathcal{M}_{k,\beta}(\mathbb{R}^d) := \left\{ \mu = \mu^+ - \mu^- \in \mathcal{M}(\mathbb{R}^d), \quad \mu^+, \mu^- \in \mathcal{M}^+_{k,\beta}(\mathbb{R}^d) \right\}. \quad (4.4)$$

We note that if $0 < \beta_1 \leq \beta_2 < d + 2\gamma$, then $\mathcal{M}^+_{k,\beta_2}(\mathbb{R}^d) \subset \mathcal{M}^+_{k,\beta_1}(\mathbb{R}^d)$ and $\mathcal{M}_{k,\beta_2}(\mathbb{R}^d) \subset \mathcal{M}_{k,\beta_1}(\mathbb{R}^d)$.

• Let $\beta, \theta > 0$ be such that $\beta + \theta < d + 2\gamma$. Then using the generalized Riesz composition formula (3.17) and Fubini's theorem we can see that

$$\forall \mu \in \mathcal{M}^+_{k,\beta+\theta}(\mathbb{R}^d), \quad I_{k,\beta+\theta}[\mu] = I_{k,\beta}[I_{k,\theta}[\mu](y)\omega_k(y)dy]. \quad (4.5)$$

In the following result we will establish that any measure $\mu \in \mathcal{M}^+_{k,\beta}(\mathbb{R}^d)$ doesn't charge the singularities of the function $R_{k,\beta}(x, \cdot)$ whenever its β - Δ_k -Riesz potential valued at x is finite. More precisely, we have

Proposition 4.2 Let $\mu \in \mathcal{M}^+_{k,\beta}(\mathbb{R}^d)$ and $x \in \mathbb{R}^d$ such that $I_{k,\beta}[\mu](x) < +\infty$. Then $\mu(\{gx\}) = 0$ whenever the point gx , $g \in W$, is a singularity of $R_{k,\beta}(x, \cdot)$.

In particular, if $\beta \leq d$, the condition $I_{k,\beta}[\mu](x) < +\infty$ implies that $\mu(\{x\}) = 0$.

Proof: Let $g \in W$ such that $R_{k,\beta}(x, gx) = +\infty$ and let $n \in \mathbb{N}$. Since $R_{k,\beta}(x, \cdot)$ is l.s.c. at gx , there exists $r > 0$ such that $R_{k,\beta}(x, y) \geq n$ for all $y \in B(gx, r)$. This implies that

$$I_{k,\beta}[\mu](x) \geq \int_{B(gx,r)} R_{k,\beta}(x, y) d\mu(y) \geq n\mu(B(gx, r)) \geq n\mu(\{gx\}).$$

This proves the first part.

Now, since $\beta \leq d$, we know that x is always a singularity of the function $R_{k,\beta}(x, \cdot)$. Thus the second part follows from the first one. \square

Now, we establish a boundedness principle for the potential of a compactly supported measure which generalizes the known result in the classical case (i.e. $k = 0$) (see [18], Theorem 1.5).

Proposition 4.3 *Let $0 < \beta < d + 2\gamma$ and μ be a compactly supported nonnegative Radon measure on \mathbb{R}^d . If $I_{k,\beta}[\mu] \leq M$ holds on $W.\text{supp } \mu$, then*

$$I_{k,\beta}[\mu] \leq 2^{d+2\gamma-\beta} M \quad \text{on } \mathbb{R}^d. \quad (4.6)$$

Proof: Let $x \notin W.\text{supp } \mu$ and $x_0 \in W.\text{supp } \mu$ such that $\|x - x_0\| = \text{dist}(x, W.\text{supp } \mu)$. We have

$$\forall y \in \text{supp } \mu, \quad \forall g \in W, \quad \|x_0 - gy\| \leq \|x_0 - x\| + \|x - gy\| \leq 2\|x - gy\|.$$

Hence, by (3.20) we deduce that

$$\forall y \in \text{supp } \mu, \quad \forall z \in \text{supp } \mu_y, \quad \|x_0\|^2 + \|y\|^2 - 2\langle x_0, z \rangle \leq 4(\|x\|^2 + \|y\|^2 - 2\langle x, z \rangle).$$

Now, using (2.10), we obtain

$$\forall y \in \text{supp } \mu, \quad 4^{-\frac{d}{2}-\gamma} p_{t/4}(x, y) \leq p_t(x_0, y).$$

From (3.1), the above inequality implies that

$$\forall y \in \text{supp } \mu, \quad 2^{-d-2\gamma+\beta} R_{k,\beta}(x, y) \leq R_{k,\beta}(x_0, y).$$

Finally, if we integrate with respect to the measure $d\mu(y)$ and use our hypothesis, the inequality (4.6) follows. \square

In the following result, we will study some continuity properties of the β - Δ_k -Riesz potentials:

Proposition 4.4 *Let $\beta \in]0, d + 2\gamma[$ and $\mu \in \mathcal{M}_{k,\beta}^+(\mathbb{R}^d)$ with compact support.*

- 1) *The function $I_{k,\beta}[\mu]$ is lower semi-continuous on \mathbb{R}^d and continuous on $\mathbb{R}^d \setminus W.\text{supp } \mu$.*
- 2) *If the restriction of the function $I_{k,\beta}[\mu]$ on $W.\text{supp } \mu$ is continuous on $W.\text{supp } \mu$, then $I_{k,\beta}[\mu]$ is continuous on \mathbb{R}^d .*

Proof: 1) • Consider the function F_n given by

$$F_n(x) = \frac{1}{\Gamma(\beta/2)} \int_{\text{supp } \mu} \left(\int_{1/n}^n t^{\frac{\beta}{2}-1} p_k(t, x, y) dt \right) d\mu(y).$$

As $t^{\frac{\beta}{2}-1} p_k(t, x, y) \leq 2^{-\frac{d}{2}-\gamma} c_k^{-1} t^{-\frac{\beta-d-2\gamma}{2}-1}$, by the continuity theorem under the integral sign, we see that F_n is continuous on \mathbb{R}^d . Moreover, from the monotone convergence theorem, we deduce that the function $I_{k,\beta}[\mu]$ is l.s.c. on \mathbb{R}^d as being the pointwise increasing limit of the sequence (F_n) .

• Let us prove the second part of 1). Fix a closed ball $B(x_0, R)$ in $\mathbb{R}^d \setminus W.\text{supp } \mu$ and set

$$\eta := \text{dist}(B(x_0, R), W.\text{supp } \mu) > 0.$$

From (2.14), we deduce that

$$\forall (x, y) \in B(x_0, R) \times \text{supp } \mu, \quad p_k(t, x, y) \leq \frac{1}{(2t)^{\frac{d}{2} + \gamma} c_k} e^{-\frac{\eta^2}{4t}}.$$

Then, writing

$$I_{k,\beta}[\mu](x) = \frac{1}{\Gamma(\beta/2)} \int_{\text{supp } \mu} \left(\int_0^{+\infty} t^{\frac{\beta}{2}-1} p_k(t, x, y) dt \right) d\mu(y)$$

and using the continuity theorem under the integral sign, it follows that $I_{k,\beta}[\mu]$ is continuous on $B(x_0, R)$. As the ball $B(x_0, R)$ is arbitrary, the result follows.

2) Fix $x_0 \in W.\text{supp } \mu$ and $\varepsilon > 0$. Let (x_n) be a sequence which converges to x_0 . For $R > 0$ (small), set $\mu_R := \mu|_{B(x_0, R)}$ and $\nu_R := \mu - \mu_R$. In particular, we note that

$$I_{k,\beta}[\mu] = I_{k,\beta}[\mu_R] + I_{k,\beta}[\nu_R].$$

We have

$$|I_{k,\beta}[\mu](x_n) - I_{k,\beta}[\mu](x_0)| \leq I_{k,\beta}[\mu_R](x_n) + I_{k,\beta}[\mu_R](x_0) + |I_{k,\beta}[\nu_R](x_n) - I_{k,\beta}[\nu_R](x_0)|. \quad (4.7)$$

• As $x_0 \notin W.\text{supp } \nu_R$, by the assertion 1, the function $I_{k,\beta}[\nu_R]$ is continuous at x_0 . Hence, there exists $N_1 \in \mathbb{N}$ such that

$$\forall n \geq N_1, \quad |I_{k,\beta}[\nu_R](x_n) - I_{k,\beta}[\nu_R](x_0)| \leq \varepsilon. \quad (4.8)$$

• For every n , let $x'_n \in K := W.\text{supp } \mu_R = W.(\text{supp } \mu \cap B(x_0, R))$ such that $\|x_n - x'_n\| = \text{dist}(x_n, K) = \inf\{\|x_n - \xi\|, \xi \in K\}$. As $x_0 \in K$, we can see that $\|x_n - x_0\| \geq \|x_n - x'_n\|$. This implies that $\|x'_n - x_0\| \leq \|x_n - x'_n\| + \|x_n - x_0\| \leq 2\|x_n - x_0\|$ and thus $x'_n \rightarrow x_0$ as $n \rightarrow +\infty$. Using the inequality (4.6), we deduce that

$$I_{k,\beta}[\mu_R](x_n) \leq 2^{d+2\gamma-\beta} I_{k,\beta}[\mu_R](x'_n) = 2^{d+2\gamma-\beta} (I_{k,\beta}[\mu](x'_n) - I_{k,\beta}[\nu_R](x'_n)). \quad (4.9)$$

But, $x'_n \in W.\text{supp } \mu$ and the restriction of $I_{k,\beta}[\mu]$ on $W.\text{supp } \mu$ is continuous. Thus, $\lim_{n \rightarrow +\infty} I_{k,\beta}[\mu](x'_n) = I_{k,\beta}[\mu](x_0)$. Again by continuity of $I_{k,\beta}[\nu_R]$ at x_0 , we also get $\lim_{n \rightarrow +\infty} I_{k,\beta}[\nu_R](x'_n) = I_{k,\beta}[\nu_R](x_0)$. Therefore, $I_{k,\beta}[\mu_R](x'_n) \rightarrow I_{k,\beta}[\mu_R](x_0)$ as $n \rightarrow +\infty$. Let then $N_2 \in \mathbb{N}$ such that

$$\forall n \geq N_2, \quad I_{k,\beta}[\mu_R](x'_n) \leq I_{k,\beta}[\mu_R](x_0) + \varepsilon. \quad (4.10)$$

Finally from (4.7), (4.8), (4.9) and (4.10), the inequality

$$|I_{k,\beta}[\mu](x_n) - I_{k,\beta}[\mu](x_0)| \leq (2^{d+2\gamma-\beta} + 1)(\varepsilon + I_{k,\beta}[\mu_R](x_0)) \quad (4.11)$$

holds for every $n \geq \max(N_1, N_2)$ and every $R > 0$.

But, since $x_0 \in W.\text{supp } \mu$ and $I_{k,\beta}[\mu]$ is continuous on $W.\text{supp } \mu$, we must have $I_{k,\beta}[\mu](x_0) < +\infty$. Furthermore, because x_0 is a singularity of $R_{k,\beta}(x_0, \cdot)$, Proposition 4.2 imply that

$$\lim_{R \rightarrow 0} I_{k,\beta}[\mu_R](x_0) = \lim_{R \rightarrow 0} \int_{B(x_0, R)} R_{k,\beta}(x_0, y) d\mu(y) = 0. \quad (4.12)$$

Finally, by the relations (4.11) and (4.12) we deduce that $I_{k,\beta}[\mu]$ is continuous at x_0 . \square

Theorem 4.1 Let $\beta \in]0, d + 2\gamma[$ and $\mu \in \mathcal{M}_{k,\beta}^+(\mathbb{R}^d)$ with compact support. Then, the function $I_{k,\beta}[\mu]$ is

- i) D -superharmonic on \mathbb{R}^d if $\beta \geq 2$,
- ii) D -harmonic on $\mathbb{R}^d \setminus W.\text{supp } \mu$ if $\beta = 2$,
- iii) D -subharmonic on $\mathbb{R}^d \setminus W.\text{supp } \mu$ if $\beta \leq 2$.

We need the following lemma:

Proof of Theorem 4.1: i) Let $\beta > 2$. Using Fubini's theorem and the D -superharmonicity of the Δ_k -Riesz kernel (see Proposition 3.6), we can easily see that $I_{k,\beta}[\mu]$ satisfies the super-mean property i.e. for all $x \in \mathbb{R}^d$ and all $r > 0$, $M_B^r(I_{k,\beta}[\mu])(x) \leq I_{k,\beta}[\mu](x)$. Since $I_{k,\beta}[\mu]$ is l.s.c and finite a.e., we deduce that the function $I_{k,\beta}[\mu]$ is D -superharmonic on \mathbb{R}^d .

ii) If $\beta = 2$, we are in the case of the Dunkl-Newton potential and the result has been proved in [10].

iii) Let $\beta < 2$. From Lemma 4.4, we know that $I_{k,\beta}[\mu]$ is a continuous function on $\mathbb{R}^d \setminus W.\text{supp } \mu$. Furthermore, by Proposition 3.6 and Fubini's theorem, the sub-mean property is satisfied by the function $I_{k,\beta}[\mu]$ on $\mathbb{R}^d \setminus W.\text{supp } \mu$. Thus, $I_{k,\beta}[\mu]$ is D -subharmonic on $\mathbb{R}^d \setminus W.\text{supp } \mu$. \square

Corollary 4.1 Let $\beta \in [2, d + 2\gamma[$. If $\mu \in \mathcal{M}_{k,\beta}^+(\mathbb{R}^d)$, then the function $I_{k,\beta}[\mu]$ is D -superharmonic on \mathbb{R}^d .

Proof: Let Φ_n the function defined by $\Phi_n(x) = \int_{B(0,n)} R_{k,\beta}(x,y) d\mu(y)$. From Theorem 4.1, the function Φ_n is D -superharmonic on \mathbb{R}^d . Thus, as $I_{k,\beta}[\mu]$ is not identically $+\infty$ by hypothesis, the function $I_{k,\beta}[\mu]$ is D -superharmonic on \mathbb{R}^d as being an increasing pointwise limit of the sequence $(\Phi_n)_n$ of D -superharmonic functions (see [10], Proposition 3.3). \square

Proposition 4.5 Let $\mu \in \mathcal{M}_{k,\beta}^+(\mathbb{R}^d)$ with $\beta \in [2, d + 2\gamma[$ and $m \in \mathbb{N}$ be such that $1 \leq m \leq \beta/2$. Then, the function $x \mapsto I_{k,\beta}[\mu](x)\omega_k(x)$ satisfies

$$(-\Delta_k)^m (I_{k,\beta}[\mu]\omega_k) = \begin{cases} I_{k,\beta-2m}[\mu]\omega_k & \text{in } \mathcal{D}'(\mathbb{R}^d) \text{ if } \beta > 2m, \\ \mu & \text{in } \mathcal{D}'(\mathbb{R}^d) \text{ if } \beta = 2m, \end{cases} \quad (4.13)$$

Proof: Let $\phi \in \mathcal{D}(\mathbb{R}^d)$. We will only prove the result in the case $\beta > 2m$ and by the same arguments it can be obtained when $\beta = 2m$. We have

$$\begin{aligned} \langle (-\Delta_k)^m (I_{k,\beta}[\mu]\omega_k), \phi \rangle &= \int_{\mathbb{R}^d} \left(\int_{\mathbb{R}^d} R_{k,\beta}(x,y) (-\Delta_k)^m \phi(x) \omega_k(x) dx \right) d\mu(y) \\ &= \int_{\mathbb{R}^d} \left(\int_{\mathbb{R}^d} R_{k,\beta-2m}(x,y) \phi(x) \omega_k(x) dx \right) d\mu(y) \\ &= \int_{\mathbb{R}^d} I_{k,\beta-2m}[\mu](x) \phi(x) \omega_k(x) dx, \end{aligned}$$

where we have used

- Fubini's theorem in the first and the last lines (it is possible because $I_{k,\beta}[\mu] \in L^1_{k,loc}(\mathbb{R}^d)$ and by Remark 4.1, $I_{k,\beta-2m}[\mu]$ is also in $L^1_{k,loc}(\mathbb{R}^d)$);
- the fact that the Δ_k -Riesz kernel is symmetric and the relation (3.28) in the second line. \square

From the previous proposition, we obtain immediately the uniqueness principle for Δ_k -Riesz potential of index $2m$:

Corollary 4.2 *Let $m \in]0, \frac{d}{2} + \gamma[$ be an integer and $\mu, \nu \in \mathcal{M}^+_{k,2m}(\mathbb{R}^d)$. If $I_{k,2m}[\mu] = I_{k,2m}[\nu]$ a.e, then $\mu = \nu$.*

For an arbitrary index $\beta \in]0, d + 2\gamma[$, we have the following version of the uniqueness principle for finite measures:

Theorem 4.2 *Let $\beta \in]0, d + 2\gamma[$ and let μ, ν be two finite and nonnegative Radon measures on \mathbb{R}^d . If $I_{k,\beta}[\mu] = I_{k,\beta}[\nu]$ a.e. on \mathbb{R}^d , then $\mu = \nu$.*

We start by the following result

Lemma 4.1 *Let μ be a finite and nonnegative Radon measure on \mathbb{R}^d . Then, $I_{k,\beta}[\mu]\omega_k$ is a tempered distribution and its distributional Dunkl transform is given by*

$$\mathcal{F}_k(I_{k,\beta}[\mu]\omega_k) = \|\cdot\|^{-\beta} \mathcal{F}_k(\mu)\omega_k \quad \text{in } \mathcal{S}'(\mathbb{R}^d). \quad (4.14)$$

Here, $\mathcal{F}_k(\mu)$ is the function defined by (2.5).

Proof: Let $m > d + 2\gamma$ an integer and C_m as in (3.26). By Fubini's theorem, the symmetric property of the Δ_k -Riesz kernel and the relation (3.26), we get

$$\begin{aligned} \int_{\mathbb{R}^d} (1 + \|x\|^2)^{-m} I_{k,\beta}[\mu](x)\omega_k(x)dx &= \int_{\mathbb{R}^d} \left(\int_{\mathbb{R}^d} (1 + \|x\|^2)^{-m} R_{k,\beta}(x, y)\omega_k(x)dx \right) d\mu(y) \\ &\leq C_m \mu(\mathbb{R}^d) < +\infty. \end{aligned}$$

This shows that $I_{k,\beta}[\mu]\omega_k \in \mathcal{S}'(\mathbb{R}^d)$.

Let $\phi \in \mathcal{S}(\mathbb{R}^d)$. We have

$$\begin{aligned} \langle \mathcal{F}_k(I_{k,\beta}[\mu]\omega_k), \phi \rangle &= \int_{\mathbb{R}^d} \left(\int_{\mathbb{R}^d} R_{k,\beta}(x, y)d\mu(y) \right) \mathcal{F}_k(\phi)(x)\omega_k(x)dx \\ &= \int_{\mathbb{R}^d} \left(\int_{\mathbb{R}^d} R_{k,\beta}(x, y)\mathcal{F}_k(\phi)(x)\omega_k(x)dx \right) \mu(y) \\ &= \int_{\mathbb{R}^d} \left(\int_{\mathbb{R}^d} E_k(-iy, x)\|x\|^{-\beta}\omega_k(x)\phi(x)dx \right) d\mu(y) \\ &= \int_{\mathbb{R}^d} \|x\|^{-\beta} \mathcal{F}_k(\mu)(x)\omega_k(x)\phi(x)dx, \end{aligned}$$

where we have used

- Fubini's theorem in the first second line: it is possible because $\mathcal{F}_k(\phi) \in \mathcal{S}(\mathbb{R}^d)$ and then

the function $x \mapsto (1 + \|x\|^2)^m \mathcal{F}_k(\phi)(x)$ is bounded with m the integer chosen as above;
 -the relations (3.25) and $R_{k,\beta}(x, y) = R_{k,\beta}(y, x)$ in the third line;
 -the boundedness of the function $(x, y) \mapsto E_k(iy, x)$ (see (2.2)), Fubini's theorem and (2.5) in the last line. \square

Proof of Theorem 4.2: By our hypothesis and Lemma 4.1, we have $I_{k,\beta}[\mu] = I_{k,\beta}[\nu]$ in $\mathcal{S}'(\mathbb{R}^d)$. Applying Dunkl transform to the both terms and using the relation (4.14), we deduce that

$$\|\cdot\|^{-\beta} \mathcal{F}_k(\mu)\omega_k = \|\cdot\|^{-\beta} \mathcal{F}_k(\nu)\omega_k \quad \text{in } \mathcal{S}'(\mathbb{R}^d).$$

As the functions $\|\cdot\|^{-\beta} \mathcal{F}_k(\mu)\omega_k$ and $\|\cdot\|^{-\beta} \mathcal{F}_k(\nu)\omega_k$ are locally integrable on \mathbb{R}^d , we get

$$\|\cdot\|^{-\beta} \mathcal{F}_k(\mu)\omega_k = \|\cdot\|^{-\beta} \mathcal{F}_k(\nu)\omega_k \quad \text{a.e. on } \mathbb{R}^d.$$

Now, by continuity it follows that the functions $\mathcal{F}_k(\mu)$ and $\mathcal{F}_k(\nu)$ coincide everywhere on \mathbb{R}^d . Finally, by the injectivity of the Dunkl transform on the space of finite Radon measures on \mathbb{R}^d , we conclude that $\mu = \nu$. \square

In order to extend the pointwise Hedberg inequality in Dunkl setting, in the following result we give the link between the Δ_k -Riesz potential and the volume mean of a nonnegative Radon measure.

Proposition 4.6 *Let μ be a nonnegative Radon measure on \mathbb{R}^d . Then, for all $\beta \in]0, d + 2\gamma[$, we have*

$$I_{k,\beta}[\mu](x) = \frac{d_k \kappa}{(d + 2\gamma)(d + 2\gamma - \beta)} \int_0^{+\infty} t^\beta M_B^t(\mu)(x) \frac{dt}{t}, \quad (4.15)$$

where

$$M_B^t(\mu)(x) := \frac{1}{m_k[B(0, t)]} \int_{\mathbb{R}^d} h_k(t, x, y) d\mu(y). \quad (4.16)$$

Proof: The result follows from (3.3), Fubini's theorem, (2.18) and (4.16). \square

In the following result, we will extend the pointwise Hedberg inequality (see [13]). We recall that the Dunkl-Hardy-Littlewood maximal operator is defined for $f \in L_{k,loc}^1(\mathbb{R}^d)$ by (see [28])

$$M_k(f)(x) = \sup_{r>0} \frac{1}{m_k[B(0, r)]} \int_{\mathbb{R}^d} |f(y)| \tau_{-x}(\mathbf{1}_{B(0,r)})(y) \omega_k(y) dy, \quad (4.17)$$

where $\tau_{-x}(\mathbf{1}_{B(0,r)})$ denotes the $L_k^2(\mathbb{R}^d)$ -function with Dunkl transform

$$\xi \mapsto E_k(-ix, \xi) \mathcal{F}_k(\mathbf{1}_{B(0,r)})(\xi).$$

According to [20], we have $h_k(r, x, \cdot) = \tau_{-x}(\mathbf{1}_{B(0,r)})$ a.e. on \mathbb{R}^d . Thus, we will take this remark into account in the formula (4.17) and in the sequel of the paper.

Moreover, when $d\mu(y) = |f(y)|\omega_k(y)dy$, $f \in L_{k,loc}^1(\mathbb{R}^d)$, we will use the notation $I_{k,\beta}[|f|]$ instead of $I_{k,\beta}[|f(y)|\omega_k(y)dy]$.

Theorem 4.3 For $0 < \beta < d+2\gamma$, $1 \leq p < \frac{d+2\gamma}{\beta}$, there exists constants $C = C(d, \gamma, \beta, p) > 0$ such that for any measurable function f and any $x \in \mathbb{R}^d$, we have

$$I_{k,\beta}[|f|](x) \leq C \|f\|_{k,p}^{\frac{\beta p}{d+2\gamma}} (M_k(f)(x))^{1-\frac{\beta p}{d+2\gamma}}, \quad (4.18)$$

Proof: For every $A > 0$, by (4.15) where we take $d\mu(y) = |f(y)|\omega_k(y)dy$, we can write

$$\begin{aligned} I_{k,\beta}[|f|](x) &= I_{k,\beta}[|f|\omega_k](x) = C \int_0^A t^{\beta-1} M_B^t(|f|)(x) dt + C \int_A^{+\infty} t^{\beta-1} M_B^t(|f|)(x) dt \\ &:= I_1(x) + I_2(x). \end{aligned}$$

• Clearly, we see that

$$I_1(x) \leq CA^\beta M_k(f)(x). \quad (4.19)$$

• We have

$$\begin{aligned} I_2(x) &= C \sum_{n=0}^{+\infty} \int_{2^n A}^{2^{n+1} A} t^{\beta-d-2\gamma-1} \int_{\mathbb{R}^d} |f(y)| h_k(t, x, y) \omega_k(y) dy dt \\ &\leq C \|f\|_{k,p} \sum_{n=0}^{+\infty} \int_{2^n A}^{2^{n+1} A} t^{\beta-d-2\gamma-1} t^{d+2\gamma(1-1/p)} dt \\ &\leq C \|f\|_{k,p} \sum_{n=0}^{+\infty} (2^n A)^{\beta-\frac{d+2\gamma}{p}}, \end{aligned}$$

where we have used Hölder's inequality and the relation (2.18) in the second line. Therefore, we have

$$I_2(x) \leq CA^{\beta-\frac{d+2\gamma}{p}} \|f\|_{k,p}. \quad (4.20)$$

Now, using (4.19), (4.20) and choosing

$$A = A(x) = \left(\frac{\|f\|_{k,p}}{M_k(f)(x) + \varepsilon} \right)^{\frac{p}{d+2\gamma}},$$

we obtain

$$I_{k,\beta}[|f|](x) \leq C \|f\|_{k,p}^{\frac{\beta p}{d+2\gamma}} (M_k(f)(x) + \varepsilon)^{1-\frac{\beta p}{d+2\gamma}}.$$

Letting $\varepsilon \rightarrow 0$, we get (4.18). \square

Using the Hedberg inequality (4.18), the L_k^p -boundedness properties of the Dunkl-Hardy-Littlewood maximal function (see [4] or [28]) and following the same proof as in the classical case (see Theorem 3.1.4 in [1]), we obtain the Sobolev inequality:

Corollary 4.3 Let $0 < \beta < d + 2\gamma$, $1 \leq p < \frac{d+2\gamma}{\beta}$ and $p^* = \frac{p(d+2\gamma)}{d+2\gamma-\beta p}$.

1) If $p = 1$, then $I_{k,\beta}$ is of weak type $(1, p^*)$ i.e. there exists a constant $C = C(\beta, d, \gamma)$ such that

$$\forall \lambda > 0, \quad \forall f \in L_k^1(\mathbb{R}^d), \quad \int_{\{x: I_{k,\beta}[|f|] > \lambda\}} \omega_k(x) dx \leq C \left(\frac{\|f\|_{k,1}}{\lambda} \right)^{p^*}. \quad (4.21)$$

1) If $p > 1$, then $I_{k,\beta}$ is of strong type (p, p^*) i.e. $I_{k,\beta} : L_k^p(\mathbb{R}^d) \rightarrow L_k^{p^*}(\mathbb{R}^d)$ is bounded.

Remark 4.2 *The previous result has been obtain in [11] by another proof using interpolation methods and in the particular case when the Coxeter-Weyl group is \mathbb{Z}_2^d in [29].*

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