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Léonard GALLARDO,* **Chaabane REJEB**† and **Mohamed SIFI**‡

Abstract

For a root system R on $\mathbb{R}^d$ and a nonnegative multiplicity function k on R, we consider the heat kernel $p_k(t,x,y)$ associated to the Dunkl-Laplacian operator Δ_k. For $\beta \in]0, d+2\gamma[$, where $\gamma = \frac{1}{2}\sum_{\alpha\in R} k(\alpha)$, we study the Δ_k-Riesz kernel of index β defined by $R_{k,\beta}(x,y) = \frac{1}{\Gamma(\beta/2)}\int_0^{+\infty} t^{\frac{\beta}{2}-1} p_k(t,x,y)dt$ and the corresponding Δ_k-Riesz potential $I_{k,\beta}[\mu]$ of a Radon measure μ on $\mathbb{R}^d$. According to the values of β, we study the Δ_k-superharmonicity of these functions and we give some applications like the Δ_k-Riesz measure of $I_{k,\beta}[\mu]$, the uniqueness principle and a pointwise Hedberg's inequality.

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1 Introduction

Let R be a normalized root system in $\mathbb{R}^d$. That is, for every $\alpha \in R$, $\|\alpha\|^2 = 2$, $R \cap \mathbb{R}\alpha = \{\pm\alpha\}$ and $\sigma_\alpha(R) = R$, where σ_α is the reflection with respect to the hyperplane H_α orthogonal to α (see [15] and [17]). We fix $k \geq 0$ a multiplicity function (i.e. $k : R \to [0, +\infty[$ invariant under the action of the Coxeter-Weyl group W associated to R) and we consider the associated Dunkl-Laplacian operator Δ_k given by

$$\Delta_k f(x) = \Delta f(x) + 2\sum_{\alpha\in R_+} k(\alpha)\left(\frac{\langle \nabla f(x), \alpha\rangle}{\langle x,\alpha\rangle} - \frac{f(x) - f(\sigma_\alpha(x)}{\langle x,\alpha\rangle^2}\right), \quad f \in \mathcal{C}^2(\mathbb{R}^d), \quad (1.1)$$

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with R_+ a positive subsystem (see [7]).
Acting on $\mathcal{C}^\infty(\mathbb{R}^d)$, it is related to the classical Laplacian operator Δ by means of the so-called Dunkl intertwining operator V_k (see [6], [7], [30]) as follows:

$$\Delta_k V_k = V_k \Delta. \tag{1.2}$$

In [23], M. Rösler has proved that for any $x \in \mathbb{R}^d$, there exists a compactly supported probability measure μ_x on $\mathbb{R}^d$ (which we call Rösler's measure at point x) such that

$$\forall\ f \in \mathcal{C}^\infty(\mathbb{R}^d), \quad V_k(f)(x) = \int_{\mathbb{R}^d} f(y) d\mu_x(y), \tag{1.3}$$

with

$$\text{supp}\ \mu_x \subset C(x) = co\{gx,\ g \in W\} \tag{1.4}$$

(the convex hull of the orbit of x under the group W). We note that, according to [9], the support of μ_x contains the point x and it is W-invariant under the hypothesis that the multiplicity function is positive.

Let $p_k(t, x, y)$ ($t > 0$, $x, y \in \mathbb{R}^d$) be the heat kernel of the Dunkl Laplacian Δ_k which is given by (see [21] and [25])

$$p_k(t, x, y) := \frac{1}{(2t)^{d/2+\gamma} c_k} e^{-(\|x\|^2+\|y\|^2)/4t} E_k(\frac{x}{\sqrt{2t}}, \frac{y}{\sqrt{2t}}), \tag{1.5}$$

where

$$E_k(x, y) = V_k(e^{\langle ., y\rangle})(x) \quad (x, y \in \mathbb{R}^d) \tag{1.6}$$

is the Dunkl kernel (see [5] and [7]), c_k is the Macdonald-Mehta constant (see [19]) given by

$$c_k := \int_{\mathbb{R}^d} e^{-\frac{\|x\|^2}{2}} \omega_k(x) dx, \tag{1.7}$$

and ω_k is the Dunkl weight function

$$\omega_k(x) = \prod_{\alpha \in R_+} |\ \langle \alpha, x\rangle\ |^{2k(\alpha)} \tag{1.8}$$

which is homogeneous of degree 2γ.
It is also known (see [21]) that for all fixed $x \in \mathbb{R}^d$, the function $p_k(t, x, .)$ solves the Dunkl heat equation

$$(\Delta_k - \partial_t) p_k(t, x, .) = 0. \tag{1.9}$$

Let $\gamma = \sum_{\alpha \in R_+} k(\alpha)$ and suppose that $d + 2\gamma > 2$. For $\beta \in]0, d + 2\gamma[$, we define the Δ_k-Riesz kernel of index β as follows

$$R_{k,\beta}(x, y) := \frac{1}{\Gamma(\beta/2)} \int_0^{+\infty} t^{\frac{\beta}{2}-1} p_k(t, x, y) dt.$$

We note that when $\beta = 2$, we obtain the Dunkl-Newton kernel which has been introduced and studied in [10].

Let $x \in \mathbb{R}^d$, $x \neq 0$, be fixed and $W.x$ be its W-orbit. If $y \notin W.x$, $R_{k,\beta}(x,y)$ is finite. But when $y \in W.x$, it seams hard, except in the case $y = x$, to decide in general if $R_{k,\beta}(x,y)$ is finite or infinite. These difficulties are illustrated by the particular case of the root system of type $A_1 \times A_1 \times \cdots \times A_1$ (m times, $1 \leq m \leq d$), where we manage to give a complete description of the singularities of the function $R_{k,\beta}(x,.)$.
The aim of this paper is the study, when $d + 2\gamma > 2$, of the Δ_k-Riesz kernel $R_{k,\beta}$ and the corresponding potential

$$I_{k,\beta}[\mu](x) = \int_{\mathbb{R}^d} R_{k,\beta}(x,y) d\mu(y)$$

of a signed Radon measure μ on $\mathbb{R}^d$.
In particular, we will study the sub-or-superharmonicity of these functions in the sense of the Dunkl-Laplace operator and we will describe explicitly their Δ_k-Riesz measures. This notion of subharmonicity, which generalizes the classical one[1] has been introduced and studied in some details in [10]. More precisely, let Ω be a W-invariant open subset of $\mathbb{R}^d$. A function $u : \Omega \longrightarrow [-\infty, +\infty[$ is called Δ_k-subharmonic (D-subharmonic) on Ω if
• u is upper semi-continuous (u.s.c.) on Ω,
• u is not identically $-\infty$ on each connected component of Ω,
• it satisfies the sub-mean volume property: for every closed ball $B(x,r) \subset \Omega$, we have

$$u(x) \leq M_B^r(u)(x) := \frac{1}{m_k[B(0,r)]} \int_{\mathbb{R}^d} u(y) h_k(r,x,y) \omega_k(y) dy, \tag{1.10}$$

where m_k is the measure $\omega_k(x)dx$ and $h_k(r,x,y)$ is a kernel of the form

$$h_k(r,x,y) := \int_{\mathbb{R}^d} \mathbf{1}_{[0,r]}(\sqrt{\|x\|^2 + \|y\|^2 - 2\langle x,z\rangle}) d\mu_y(z) \tag{1.11}$$

with μ_y Rösler's measure at point y. The function $y \mapsto h_k(r,x,y)$ is a generalized translate of the indicator function $\mathbf{1}_{B(0,r)}$ of the ball $B(0,r)$ called harmonic kernel, introduced and studied in [8] and which properties will be recalled in the next section. Moreover the harmonic kernel is a crucial tool to get quite explicit expressions of the Δ_k-Riesz kernel (see section 3).
Naturally, a function u is D-superharmonic on Ω if $-u$ is D-subharmonic on Ω.

Finally, we study some applications. The main one is the following version of the uniqueness principle: if μ and ν are finite and nonnegative Radon measures on $\mathbb{R}^d$ and if $I_{k,\beta}[\mu] = I_{k,\beta}[\nu]$ a.e. on $\mathbb{R}^d$, then $\mu = \nu$. We also prove a pointwise Hedberg's inequality in the sense of the operator Δ_k and we deduce L^p-boundedness properties of the Δ_k-Riesz potentials.

2 Generalities in Dunkl Theory

In order to help the reader, we have collected in this section some basics from Dunkl theory which will be used in the sequel.

[1] see for example [2],[12],[14] and [18].

Notations: Let us introduce the following functional spaces which are present throughout the paper:
• Ω a W-invariant open subset of $\mathbb{R}^d$.
• $L^p_k(\Omega)$ (resp. $L^p_{k,loc}(\Omega)$), $1 \leq p < +\infty$ the space of measurable functions $f : \Omega \longrightarrow \mathbb{C}$ such that $\|f\|^p_{L^p_k(\Omega)} := \int_\Omega |f(x)|^p \omega_k(x)dx < +\infty$ (resp. $\int_K |f(x)|^p \omega_k(x)dx < +\infty$ for any compact set $K \subset \Omega$).
• $L^\infty_k(\Omega)$ the space of measurable and essentially bounded functions on Ω.
• When $\Omega = \mathbb{R}^d$, the norm of the space $L^p_k(\mathbb{R}^d)$, $1 \leq p \leq +\infty$, will be denoted $\|.\|_{k,p}$ instead of $\|.\|_{L^p_k(\mathbb{R}^d)}$.
• $\mathcal{D}(\Omega)$ the space of C^∞-functions on Ω with compact support.
• $\mathcal{D}'(\Omega)$ the space of distributions on Ω (i.e. the topological dual of $\mathcal{D}(\Omega)$ carrying the Fréchet topology).
• $\mathcal{S}(\mathbb{R}^d)$ the Schwartz space of C^∞-functions on $\mathbb{R}^d$ which are rapidly decreasing together with their derivatives.
• $\mathcal{S}'(\mathbb{R}^d)$ the space of tempered distributions.

2.1 The Dunkl transform

The Dunkl transform of a function $f \in L^1_k(\mathbb{R}^d)$ is defined by (see [16] and [25])

$$\mathcal{F}_k(f)(\lambda) := \int_{\mathbb{R}^d} f(x)E_k(-i\lambda, x)\omega_k(x)dx, \quad \lambda \in \mathbb{R}^d, \tag{2.1}$$

where $E_k(x, y)$ is the Dunkl kernel (1.6) which is analytically extendable to $\mathbb{C}^d \times \mathbb{C}^d$ and satisfies the following properties (see [5], [7], [16])

1. for all $x, y \in \mathbb{R}^d$, we have

$$|E_k(-ix, y)| \leq 1. \tag{2.2}$$

2. for all $a \in \mathbb{C}$, $x, y \in \mathbb{C}^d$ and all $g \in W$, we have

$$E_k(ax, y) = E_k(x, ay), \quad E_k(x, y) = E_k(y, x) \quad \text{and} \quad E_k(gx, gy) = E_k(x, y).$$

It is well known (see [16]) that the Dunkl transform $\mathcal{F}_k$ is an isomorphism of $\mathcal{S}(\mathbb{R}^d)$ onto itself and its inverse is given by

$$\mathcal{F}^{-1}_k(f)(x) = c_k^{-2} \int_{\mathbb{R}^d} f(\lambda)E_k(ix, \lambda)\omega_k(\lambda)d\lambda, \quad x \in \mathbb{R}^d, \tag{2.3}$$

where c_k is the constant given by (1.7).
We note that for $f, g \in \mathcal{S}(\mathbb{R}^d)$ the following relation holds

$$\int_{\mathbb{R}^d} \mathcal{F}_k(f)(x)g(x)\omega_k(x)dx = \int_{\mathbb{R}^d} f(x)\mathcal{F}_k(g)(x)\omega_k(x)dx. \tag{2.4}$$

Moreover, the transformation $c_k^{-1}\mathcal{F}_k$ extends uniquely to an isometric isomorphism of $L^2_k(\mathbb{R}^d)$ (Plancherel theorem, see [16]).

We will also need the Dunkl transform $\mathcal{F}_k(S)$ of a tempered distribution $S \in \mathcal{S}'(\mathbb{R}^d)$ which is the distribution defined by

$$\langle \mathcal{F}_k(S), \phi \rangle := \langle S, \mathcal{F}_k(\phi) \rangle, \quad \phi \in \mathcal{S}(\mathbb{R}^d).$$

It is known that $\mathcal{F}_k$ is a topological isomorphism of $\mathcal{S}'(\mathbb{R}^d)$ onto itself (see [31]).
Note that if μ is a bounded Radon measure on $\mathbb{R}^d$, $\mu \in \mathcal{S}'(\mathbb{R}^d)$ and its distributional Dunkl transform can be identified to the continuous function $\xi \mapsto \int_{\mathbb{R}^d} E_k(-ix, \xi) d\mu(x) \omega_k(\xi)$. In the literature, the function

$$\mathcal{F}_k(\mu) : \xi \mapsto \int_{\mathbb{R}^d} E_k(-ix, \xi) d\mu(x) \tag{2.5}$$

is called the Dunkl transform of the measure μ. This transformation is injective on the space of bounded Radon measures on $\mathbb{R}^d$ (see [22]).
We recall also that the Dunkl-Laplace operator Δ_k leaves the spaces $\mathcal{D}'(\mathbb{R}^d)$ and $\mathcal{S}'(\mathbb{R}^d)$ invariant where the Δ_k-action on S in $\mathcal{D}'(\mathbb{R}^d)$) (resp. in $\mathcal{S}'(\mathbb{R}^d)$) is defined as in the classical case by

$$\langle \Delta_k S, \phi \rangle = \langle S, \Delta_k \phi \rangle, \quad \phi \in \mathcal{D}(\mathbb{R}^d) \text{ (resp. } \phi \in \mathcal{S}(\mathbb{R}^d)). \tag{2.6}$$

2.2 Dunkl's translation operators and heat kernel properties

• The Dunkl translation operators $\tau_x, x \in \mathbb{R}^d$, are defined on $\mathcal{C}^\infty(\mathbb{R}^d)$ by (see [31])

$$\forall\, y \in \mathbb{R}^d, \quad \tau_x f(y) = \int_{\mathbb{R}^d} V_k \circ T_z \circ V_k^{-1}(f)(y) d\mu_x(z), \tag{2.7}$$

where T_x is the classical translation operator given by $T_x f(y) = f(x + y)$. The operators τ_x, $x \in \mathbb{R}^d$, satisfy the following properties:

1) For all $x \in \mathbb{R}^d$, the operator τ_x is continuous from $\mathcal{C}^\infty(\mathbb{R}^d)$ into itself.

2) For all $f \in \mathcal{C}^\infty(\mathbb{R}^d)$ and all $x, y \in \mathbb{R}^d$, we have

$$\tau_x f(0) = f(x), \quad \tau_x f(y) = \tau_y f(x).$$

3) The Dunkl-Laplace operator Δ_k commutes with the Dunkl translations, i.e.

$$\tau_x(\Delta_k f) = \Delta_k(\tau_x f), \quad x \in \mathbb{R}^d, f \in \mathcal{C}^\infty(\mathbb{R}^d).$$

4) If $f \in \mathcal{C}^\infty(\mathbb{R}^d)$ is radial, M. Rösler ([26]) has proved the useful formula

$$\forall\, x \in \mathbb{R}^d, \quad \tau_x f(y) = \int_{\mathbb{R}^d} \widetilde{f}(\sqrt{\|x\|^2 + \|y\|^2 + 2\langle x, z \rangle}) d\mu_y(z), \tag{2.8}$$

where $\widetilde{f}$ is the profile of f and μ_y is the measure defined by (1.3).

In the particular case when $f \in \mathcal{S}(\mathbb{R}^d)$, $\tau_x f \in \mathcal{S}(\mathbb{R}^d)$ and using the Dunkl transform we have (see [31]):

$$\tau_x f(y) = \mathcal{F}_k^{-1}[E_k(ix,.)\mathcal{F}_k(f)](y) = c_k^{-2}\int_{\mathbb{R}^d} \mathcal{F}_k(f)(\lambda)E_k(ix,\lambda)E_k(iy,\lambda)\omega_k(\lambda)d\lambda, \quad y \in \mathbb{R}^d.$$

• Using (2.8), the Dunkl heat kernel can also be written

$$p_k(t,x,y) = \frac{1}{(2t)^{d/2+\gamma}c_k}\ \tau_{-x}\big(e^{-\frac{\|.\|^2}{4t}}\big)(y) \tag{2.9}$$

$$= \frac{1}{(2t)^{d/2+\gamma}c_k}\int_{\mathbb{R}^d} e^{-\frac{1}{4t}\left(\|x\|^2+\|y\|^2-2\,\langle x,z\rangle\right)}d\mu_y(z). \tag{2.10}$$

For later use, we record also the following properties of the heat kernel (see [21] and [25])

1. The Dunkl heat kernel is symmetric in x and y i.e. $p_k(t,x,y) = p_k(t,y,x)$, $t > 0$.

2. For every $t > 0$ and $x \in \mathbb{R}^d$, we have

$$\|p_k(t,x,.)\|_{k,1} = \int_{\mathbb{R}^d} p_k(t,x,y)\omega_k(y)dy = 1. \tag{2.11}$$

3. For every $t > 0$ and $x, y \in \mathbb{R}^d$,

$$p_k(t,x,y) = \mathcal{F}_k^{-1}\big(E_k(-ix,.)e^{-t\|.\|^2}\big)(y) \tag{2.12}$$

$$= c_k^{-2}\int_{\mathbb{R}^d} e^{-t\|\xi\|^2}E_k(-ix,\xi)E_k(iy,\xi)\omega_k(\xi)d\xi. \tag{2.13}$$

4. For every $t > 0$, the following inequality holds

$$\forall\ x,y \in \mathbb{R}^d, \quad p_k(t,x,y) \le \frac{1}{(2t)^{d/2+\gamma}c_k}e^{-\frac{1}{4t}\min_{g\in W}\|x-gy\|^2}. \tag{2.14}$$

5. For all $t, s > 0$, the Dunkl heat kernel satisfies the semi-group property

$$\forall\ x,y \in \mathbb{R}^d, \quad p_k(t+s,x,y) = \int_{\mathbb{R}^d} p_k(t,x,z)p_k(s,y,z)\omega_k(z)dz. \tag{2.15}$$

2.3 The harmonic kernel and Δ_k-subharmonic functions

For $r > 0$ and $x, y \in \mathbb{R}^d$, let $h_k(r,x,y)$ be the harmonic kernel defined by (1.11). In the classical case (i.e. $k = 0$), we have $\mu_y = \delta_y$ (the Dirac measure at y) and then $h_0(r,x,y) = \mathbf{1}_{[0,r]}(\|x-y\|) = \mathbf{1}_{B(x,r)}(y)$. This implies, in particular, that the Dunkl-volume operator defined by (1.10) generalizes the usual one.
The harmonic kernel has the following properties (see [8]):

1) For all $r > 0$ and $x, y \in \mathbb{R}^d$, $0 \le h_k(r,x,y) \le 1$.

2) For all fixed $x, y \in \mathbb{R}^d$, the function $r \longmapsto h_k(r, x, y)$ is right-continuous and nondecreasing.

3) Let $r > 0$ and $x \in \mathbb{R}^d$. If $k(\alpha) > 0$ for every $\alpha \in R$, then

$$\text{supp } h_k(r, x, .\ ) = B^W(x, r) := \cup_{g \in W} B(gx, r)$$

and if the function k vanishes somewhere then

$$B(x, r) \subset \text{supp } h_k(r, x, .\ ) \subset B^W(x, r)$$

(see [8] and [9]).

4) For all $r > 0$ and $x, y \in \mathbb{R}^d$, we have

$$h_k(r, x, y) = h_k(r, y, x). \tag{2.16}$$

5) Let $r > 0$ and $x, y \in \mathbb{R}^d$. Then, for all $g \in W$, we have

$$h_k(r, gx, gy) = h_k(r, x, y) \quad \textit{and} \quad h_k(r, gx, y) = h_k(r, x, g^{-1}y). \tag{2.17}$$

6) For all $r > 0$ and $x \in \mathbb{R}^d$, we have

$$\|h_k(r, x, .)\|_{k,1} := \int_{\mathbb{R}^d} h_k(r, x, y)\omega_k(y)dy = m_k(B(0, r)) = \frac{d_k r^{d+2\gamma}}{d + 2\gamma}, \tag{2.18}$$

where we recall that $dm_k(y) = \omega_k(y)dy$ and d_k is the constant

$$d_k = \int_{S^{d-1}} \omega_k(\xi)d\sigma(\xi) = \frac{c_k}{2^{d/2+\gamma-1}\Gamma(d/2 + \gamma)}. \tag{2.19}$$

Here $d\sigma(\xi)$ is the surface measure of the unit sphere S^{d-1} of $\mathbb{R}^d$.

Finally, we recall that
• a function u of class C^2 on Ω is D-subharmonic in the sense of (1.10) if and only if $\Delta_k u \geq 0$ on Ω (see [10]).
• if u is D-subharmonic on Ω, then $u\omega_k \in L^1_{loc}(\Omega)$ (that is $u \in L^1_{k,loc}(\Omega)$) and its distributional Dunkl-Laplacian $\Delta_k(u\omega_k)$ is a nonnegative distribution on Ω in the sense that for any nonnegative function $\phi \in \mathcal{D}(\Omega)$ we have

$$\langle \Delta_k(u\omega_k), \phi\rangle := \langle u\omega_k, \Delta_k\phi\rangle = \int_{\mathbb{R}^d} u(x)\Delta_k\phi(x)\omega_k(x)dx \geq 0. \tag{2.20}$$

The nonnegative distribution $\Delta_k(u\omega_k)$ is then a nonnegative Radon measure on Ω called the Δ_k-Riesz measure of the D-subharmonic function u (see [10]). In particular, if $u \in \mathcal{C}^2(\Omega)$ its Δ_k-Riesz measure is equal to $\Delta_k u(x)\omega_k(x)dx$.

3 The Δ_k-Riesz kernel

In this section, we will study some properties of the Δ_k-Riesz kernel. Recalling that for $x, y \in \mathbb{R}^d$ and $0 < \beta < d + 2\gamma$, the Δ_k-Riesz kernel is defined by

$$R_{k,\beta}(x,y) := \frac{1}{\Gamma(\beta/2)} \int_0^{+\infty} t^{\frac{\beta}{2}-1} p_k(t,x,y) dt. \tag{3.1}$$

Remark 3.1 *1) Since the Dunkl heat kernel is positive, we have $0 < R_{k,\beta}(x,y) \leq +\infty$ for all $x, y \in \mathbb{R}^d$.*

2) Let $x \in \mathbb{R}^d$ be fixed. From (2.14), we can see that if $y \notin \mathbb{R}^d \backslash W.x$, then for any $\beta \in]-\infty, d+2\gamma[$ the function $t \mapsto t^{\frac{\beta}{2}-1} p_k(t,x,y)$ is integrable on $]0,+\infty[$. Thus, using the properties of the Gamma function, the function $y \mapsto \frac{1}{\Gamma(\beta/2)} \int_0^{+\infty} t^{\frac{\beta}{2}-1} p_k(t,x,y) dt$ is well defined on $\mathbb{R}^d \setminus W.x$ whenever $\beta \in]-\infty, d+2\gamma[\setminus -2\mathbb{N}$. In this case, we will continue denoting it $y \mapsto R_{k,\beta}(x,y)$.

In the following result, we will show that the Δ_k-Riesz kernel can be expressed in terms of the harmonic kernel. This new formula will be a crucial tool in the sequel of the paper.

Proposition 3.1 *For every $x, y \in \mathbb{R}^d$, we have*

$$R_{k,\beta}(x,y) = \kappa \int_{\mathbb{R}^d} \Big(\|x\|^2 + \|y\|^2 - 2\langle x, z \rangle \Big)^{\frac{\beta-(d+2\gamma)}{2}} d\mu_y(z) \tag{3.2}$$

$$= \frac{\kappa}{d+2\gamma-\beta} \int_0^{+\infty} t^{\beta-d-2\gamma} h_k(t,x,y) \frac{dt}{t}, \tag{3.3}$$

where

$$\kappa = \kappa(d,\gamma,\beta) = \frac{2^{1-\beta}\Gamma(\frac{d+2\gamma-\beta}{2})}{d_k \Gamma(\beta/2)\Gamma(d/2+\gamma)} = \frac{2^{\frac{d}{2}+\gamma-\beta}\Gamma(\frac{d+2\gamma-\beta}{2})}{c_k \Gamma(\beta/2)}, \tag{3.4}$$

c_k and d_k being the constants given by (1.7) and (2.19) respectively.

Proof: Using the change of variables $1/4t \leftrightarrow t$, the relation (2.10) can be rewritten

$$R_{k,\beta}(x,y) = \frac{2^{\frac{d}{2}+\gamma-\beta}}{\Gamma(\beta/2)c_k} \int_0^{+\infty} t^{\frac{d+2\gamma-\beta}{2}-1} \int_{\mathbb{R}^d} e^{-t\left(\|x\|^2+\|y\|^2-2\langle x,z\rangle\right)} d\mu_x(z) dt.$$

Now, by Fubuni's theorem and the identity

$$\forall\ a \geq 0, \quad \forall\ \theta > 0, \quad a^{-\theta/2} = \frac{1}{\Gamma(\theta/2)} \int_0^{+\infty} s^{\frac{\theta}{2}-1} e^{-sa} ds$$

(notice that if we take $a=0$, the both terms are equal $+\infty$), we deduce that (3.2) holds.
• Let us now prove (3.3). Starting from (3.2) and applying again Fubini's theorem, we get

$$\begin{aligned}
R_{k,\beta}(x,y) &= \kappa \int_{\mathbb{R}^d} \Big(\|x\|^2 + \|y\|^2 - 2\langle x,z\rangle \Big)^{\frac{\beta-(d+2\gamma)}{2}} d\mu_y(z) \\
&= \frac{\kappa}{d+2\gamma-\beta} \int_{\mathbb{R}^d} \int_{\sqrt{\|x\|^2+\|y\|^2-2\langle x,z\rangle}}^{+\infty} t^{\beta-d-2\gamma} \frac{dt}{t} d\mu_y(z) \\
&= \frac{\kappa}{d+2\gamma-\beta} \int_0^{+\infty} t^{\beta-d-2\gamma} \left(\int_{\mathbb{R}^d} \mathbf{1}_{[0,t]}(\sqrt{\|x\|^2+\|y\|^2-2\langle x,z\rangle}) d\mu_y(z) \right) \frac{dt}{t} \\
&= \frac{\kappa}{d+2\gamma-\beta} \int_0^{+\infty} t^{\beta-d-2\gamma} h_k(t,x,y) \frac{dt}{t}.
\end{aligned}$$

This gives the desired relation. □

Example 3.1 ***1)*** *When $k=0$, as $\mu_y=\delta_y$ we have $R_{0,\beta}(x,y)=\kappa(d,0,\beta)\|x-y\|^{\beta-d}$ the classical Riesz kernel (see [18]).*

2) *Since $\mu_0=\delta_0$, for any choice of the Coxeter-Weyl group and of a nonnegative multiplicity function, we have $R_{k,\beta}(x,0)=\kappa(d,\gamma,\beta)\|x\|^{\beta-d-2\gamma}$.*

3) *We consider $\mathbb{R}^d$ ($d\geq 1$) with the root system $R_m:=\{\pm e_1,\ldots,\pm e_m\}$, where m is a fixed integer in $\{1,\ldots,d\}$ and $(e_j)_{1\leq j\leq d}$ is the canonical basis of $\mathbb{R}^d$. For $\xi\in\mathbb{R}^d$, we will denote $\xi=(\xi^{(m)},\xi')\in\mathbb{R}^m\times\mathbb{R}^{d-m}$.*
We note that the Coxeter-Weyl group is $W=\mathbb{Z}_2^m$ and the $\mathbb{Z}_2^m$-orbit of a point $\xi\in\mathbb{R}^d$ is as follows

$$\mathbb{Z}_2^m.\xi := \big\{ \varepsilon.\xi := (\varepsilon_1\xi_1,\ldots,\varepsilon_m\xi_m,\xi'), \quad \varepsilon=(\varepsilon_i)_{1\leq i\leq m}\in\{\pm 1\}^m \big\}.$$

The multiplicity function can be represented by the m-multidimensional parameter $k=(k_1,\ldots,k_m)$ with $k_j=k(e_j)>0$. Moreover, the Rösler measure is of the form $\mu_y=\mu_{(y^{(m)},y')}=\mu_{y_1}\otimes\cdots\otimes\mu_{y_m}\otimes\delta_{y'}$ with μ_{y_i} the $\mathbb{Z}_2$-Rösler measure at point y_i. If $y_i=0$, we know that $\mu_0=\delta_0$ and if $y_i\neq 0$, we have

$$\langle \mu_{y_i}, f\rangle := \int_{-1}^{1} f(ty_i)\phi_{k_i}(t)dt, \quad f\in\mathcal{C}(\mathbb{R}),$$

where ϕ_{k_i} is the $\mathbb{Z}_2$-Dunkl density function of parameter k_i given by (see [5] or [25] p.104)

$$\phi_{k_i}(t) := \frac{\Gamma(k_i+1/2)}{\sqrt{\pi}\Gamma(k_i)}(1-t)^{k_i-1}(1+t)^{k_i}\mathbf{1}_{[-1,1]}(t). \tag{3.5}$$

In this case, the Δ_k-Riesz kernel is of the form

$$\begin{aligned}
R_{k,\beta}(x,y) = \kappa \int_{[-1,1]^m} \Big(\|x^{(m)}\|^2 + \|y^{(m)}\|^2 - 2\sum_{j=1}^m t_j x_j y_j + \|x'-y'\|^2 \Big)^{\frac{\beta-d-2\gamma}{2}} \\
\times \prod_{i=1}^m \phi_{k_i}(t_i) dt_1\ldots dt_m. \quad (3.6)
\end{aligned}$$

Proposition 3.2 *Suppose that $\gamma > 0$. Let $0 < \beta < d + 2\gamma$ and $x, y \in \mathbb{R}^d$.*

1) *If $y \notin W.x$, then $R_{k,\beta}(x, y) < +\infty$.*

2) *Assume that $x \in \mathbb{R}^d \backslash \bigcup_{\alpha \in R} H_\alpha$. Then $R_{k,\beta}(x, x) = +\infty$ if and only if $d \geq \beta$.*

3) *If $x \in \bigcup_{\alpha \in R} H_\alpha$ and $\beta \leq d$, then $R_{k,\beta}(x, x) = +\infty$.*

Proof: At first we note that

$$\forall\ x, y \in \mathbb{R}^d, \quad \forall\ t > 0, \quad t^{\frac{\beta}{2}-1} p_k(t, x, y) \leq C t^{\frac{\beta-d-2\gamma}{2}-1}$$

Hence, as $\beta < d + 2\gamma$, the function $t \mapsto t^{\frac{\beta}{2}-1} p_k(t, x, y)$ is integrable on $[1, +\infty[$ for every $x, y \in \mathbb{R}^d$.

1) We obtain the result by using (2.14).

2) Fix $x \in \mathbb{R}^d$ such that x is not in any hyperplane H_α, $\alpha \in R$ (i.e. x lives in a Weyl chamber). We will use the following short-time asymptotic result of the Dunkl type heat kernel which has been established in ([24], Corollary 2): Let C be a fixed Weyl chamber. If $x, y \in C$, then

$$p_k(t, x, y) \sim_{t \to 0} \big(\omega_k(x)\omega_k(y)\big)^{-1/2} (4\pi t)^{-d/2} e^{-\frac{\|x-y\|^2}{4t}}. \tag{3.7}$$

Taking $y = x$, we deduce that the function $t \mapsto t^{\frac{\beta}{2}-1} p_k(t, x, x)$ is not integrable near 0 if and only if $d \geq \beta$.

3) Let $x \in H_\alpha$ for some $\alpha \in R$. One can see that the function $\psi : \xi \longmapsto R_{k,\beta}(\xi, \xi)$ is the increasing limit of the sequence of continuous functions $\xi \longmapsto \int_{1/n}^{n} t^{\frac{\beta}{2}-1} p_k(t, \xi, \xi) dt$. This implies that ψ is lower semi-continuous on $\mathbb{R}^d$. Consequently, when $\beta \leq d$ we have $R_{k,\beta}(x, x) = \liminf_{\xi \to x} R_{k,\beta}(\xi, \xi) = +\infty$. □

As already mentioned, for $g \neq id$, it is much more difficult to see if $R_{k,\beta}(x, gx)$ is finite or infinite. This new phenomena will be illustrated by the following complete characterization of the singularities of the Δ_k-Riesz kernel in the case of the $\mathbb{Z}_2^m$-Coxeter-Weyl group acting on $\mathbb{R}^d$. More precisely, we have:

Proposition 3.3 *Let $x \in \mathbb{R}^d \backslash \{0\}$. Using the same notations of Example 3.1, 3), denoting H_i the hyperplane orthogonal to e_i and recalling $\varepsilon.x = (\varepsilon_1 x_1, \ldots, \varepsilon_m x_m, x') \in \mathbb{Z}_2^m.x$, we have*

1. *If $x \in \cap_{i=1}^m H_i$, then $x = \varepsilon.x$ and $R_{k,\beta}(x, x) = +\infty$.*
2. *Assume that $x \notin \cap_{i=1}^m H_i$. Set $A := \{i \in \{1, \ldots, m\}, \quad x_i \neq 0\}$ and $\varepsilon^{(n)}.x = (\varepsilon_1 x_1, \ldots, \varepsilon_m x_m, x')$ the point of $\mathbb{Z}_2^m$-orbit of x such that $\big|\{j \in A, \quad \varepsilon_j = 1\}\big| = n$ i.e the point $\varepsilon^{(n)}.x$ has exactly n among the nonzero coordinates $(x_j)_{j \in A}$ that have not been changed under the action of $\mathbb{Z}_2^m$. Then,*

$$R_{k,\beta}(x, \varepsilon^{(n)}.x) = +\infty \quad \Longleftrightarrow \quad d \geq 2\big(|A| - n + \textstyle\sum_{j \in A} k_j - \gamma\big) + \beta. \tag{3.8}$$

3. *Assume that $x \notin \cup_{i=1}^m H_i$. Then,*

$$R_{k,\beta}(x, \varepsilon^{(n)}.x) = +\infty \quad \Longleftrightarrow \quad d \geq 2(m-n) + \beta. \tag{3.9}$$

In this case, we have $\sum_{n=\max(0,\lfloor m-\frac{d}{2}+\frac{\beta}{2}\rfloor)}^{m} \binom{m}{n}$ singularities living in $\mathbb{R}^d \setminus \bigcup_{i=1}^m H_i$.

Proof: For abbreviation, we will use the following constants

$$C_1 := 2^{\frac{\beta-d-2\gamma}{2}}\kappa, \quad C(k) := \frac{\Gamma(k+1/2)}{\sqrt{\pi}\Gamma(k)}. \tag{3.10}$$

From (3.6), it is easy to see that

$$R_{k,\beta}(x, \varepsilon.x) = C_1 \int_{[-1,1]^m} \Big(\sum_{j=1}^m (1-\varepsilon_j t_j)x_j^2\Big)^{\frac{\beta-d-2\gamma}{2}} \prod_{j=1}^m \phi_{k_j}(t_j) \otimes_{j=1}^m dt_j. \tag{3.11}$$

1) Clearly, from (3.11), the condition $x \in \cap_{i=1}^m H_i$ i.e. $x^{(m)} = 0$ implies that $x = \varepsilon.x = (0, x')$ and $R_{k,\beta}(x, \varepsilon.x) = +\infty$.
2) Suppose that $x \notin \cap_{i=1}^m H_i$. Using the notations of the Proposition, Fubini's theorem and the fact that ϕ_{k_j} are probability densities, (3.11) can be written in the following form

$$R_{k,\beta}(x, \varepsilon.x) = C_1 \int_{[-1,1]^{|A|}} \Big(\sum_{j\in A} (1-\varepsilon_j t_j)x_j^2\Big)^{\frac{\beta-d-2\gamma}{2}} \prod_{j\in A} \phi_{k_j}(t_j) \otimes_{j\in A} dt_j. \tag{3.12}$$

We will distinguish two cases:
First case $|A| = 1$. Let $i \in \{1, \ldots, m\}$ such that $x_i \neq 0$. In this case, using (3.5) and (3.10), we deduce that (3.12) takes the form

$$\begin{aligned} R_{k,\beta}(x, \varepsilon.x) &= C_1 \int_{-1}^1 \Big((1-\varepsilon_i s)x_i^2\Big)^{\frac{\beta-d-2\gamma}{2}} \phi_{k_i}(s)ds \\ &= C(k_i)C_1|x_i|^{\beta-d-2\gamma} \int_{-1}^1 (1-\varepsilon_i s)^{\frac{\beta-d-2\gamma}{2}}(1-s)^{k_i-1}(1+s)^{k_i}ds. \end{aligned}$$

• If $\varepsilon_i = 1$, then according to our notations, we have $n = |A| = 1$, $\varepsilon.x = \varepsilon^{(1)}.x = x$ and

$$R_{k,\beta}(x, \varepsilon^{(1)}.x) = C(k_i)C_1|x_i|^{\beta-d-2\gamma} \int_{-1}^1 (1-s)^{k_i+\frac{\beta-d-2\gamma}{2}-1}(1+s)^{k_i}ds.$$

Consequently, $R_{k,\beta}(x, \varepsilon^{(1)}.x) = +\infty$ if and only if $d \geq \beta + 2k_i - 2\gamma$. Then, the result is proved in this case.
• When $\varepsilon_i = -1$, we have $n = 0$, $\varepsilon.x = \varepsilon^{(0)}.x$ and

$$R_{k,\beta}(x, \varepsilon^{(0)}.x) = C(k_i)|x_i|^{\beta-d-2\gamma} \int_{-1}^1 (1+s)^{k_i+\frac{\beta-d-2\gamma}{2}}(1-s)^{k_i-1}ds.$$

Thus, as $k_i > 0$ we have $R_{k,\beta}(x, \varepsilon^{(0)}.x) = +\infty$ if and only if $d \geq 2(1 + k_i - \gamma) + \beta$.
Second case $|A| = r \geq 2$. Using (3.12) and the change of variables $t_j \leftrightarrow 1 - \varepsilon_j t_j$, we obtain

$$\begin{aligned} R_{k,\beta}(x, \varepsilon.x) &= C_1 \int_{]0,2[^{|A|}} \Big(\sum_{j\in A} t_j x_j^2 \Big)^{\frac{\beta-d-2\gamma}{2}} \prod_{j\in A} \phi_{k_j}(\varepsilon_j - \varepsilon_j t_j) \otimes_{j\in A} dt_j \\ &= C_1 \int_{]0,2[^{|A|}\cap B_r} \quad + \quad C_1 \int_{]0,2[^{|A|}\setminus B_r} \\ &= C_1 I(x, \varepsilon.x) \quad + \quad C_1 J(x, \varepsilon.x), \end{aligned}$$

where B_r is the open unit ball in $\mathbb{R}^{|A|} = \mathbb{R}^r$.
The singularities of these integrals being at point 0 and thus it is clear that $J(x, \varepsilon.x) < +\infty$. Thus, we need to know when the integral $I(x, \varepsilon.x)$ diverges. To do this, we will identify $(t_j)_{j\in A}$ with $v = (v_1, \ldots, v_r) \in \mathbb{R}^r$ and use the spherical coordinates in $\mathbb{R}^r$:

$$\rho = \|v\|, \quad v_1 = \rho a_1, \quad \ldots, v_{r-1} = \rho a_{r-1} \quad \text{and} \quad v_r = \rho a_r,$$

where

$$a_1 = \cos\theta_1, \ldots, a_{r-1} = \prod_{i=1}^{r-2} \sin\theta_i \cos\theta_{r-1}, \quad a_r = \prod_{i=1}^{r-1} \sin\theta_i.$$

Notice that all a_j are positive.

$$I(x, \varepsilon.x) = \int_{S_+^{r-1}} \psi(a^{(r)}, x^{(r)}) \Big(\int_0^1 \prod_{j\in A} \phi_{k_j}(\varepsilon_j - \varepsilon_j a_j \rho) \rho^{r + \frac{\beta-d-2\gamma}{2} - 1} d\rho \Big) d\sigma_r(a^{(r)}), \quad (3.13)$$

where $S_+^{r-1} :=]0,2[^r \cap S^{r-1}$, $d\sigma_r$ is the surface measure of the unit sphere S^{r-1} of $\mathbb{R}^r$, $a^{(r)} = (a_j)_{j\in A}$, $x^{(r)} = (x_j)_{j\in A}$ and

$$\psi(a^{(r)}, x^{(r)}) := \Big(\sum_{j\in A} a_j x_j^2 \Big)^{\frac{\beta-d-2\gamma}{2}}.$$

We have

$$\phi_{k_j}(\varepsilon_j - \varepsilon_j a_j \rho) = C(k_j)(1 - \varepsilon_j + \varepsilon_j a_j \rho)^{k_j - 1} (1 + \varepsilon_j - \varepsilon_j a_j \rho)^{k_j}.$$

Hence,

$$\phi_{k_j}(\varepsilon_j - \varepsilon_j a_j \rho) = \begin{cases} C(k_j) a_j^{k_j - 1} \rho^{k_j - 1} (2 - a_j \rho)^{k_j}, & \text{if} \quad \varepsilon_j = 1 \\ C(k_j) a_j^{k_j} \rho^{k_j} (2 - a_j \rho)^{k_j - 1}, & \text{if} \quad \varepsilon_j = -1. \end{cases} \quad (3.14)$$

Define

$$A_1 := \Big\{ j \in A, \quad \varepsilon_j = 1 \Big\}, \quad A_2 = A \setminus A_1.$$

According to our notations, we have $|A_1| = |\{j, \quad \varepsilon_j = 1\}| = n$.
Then, from (3.13), (3.14) and recalling the definition of the vector $\varepsilon^{(n)}.x$, we deduce that

$$I(x, \varepsilon^{(n)}.x) = \int_{S_+^{r-1}} \psi(a^{(r)}, x^{(r)}) \Big(\int_0^1 f(a^{(r)}, \rho) \rho^{\lambda + r + \frac{\beta-d-2\gamma}{2} - 1} d\rho \Big) d\sigma_r(a^{(r)}), \quad (3.15)$$

with

$$f(a^{(r)},\rho) := \prod_{j\in A_1} C(k_j)a_j^{k_j-1}(2-a_j\rho)^{k_j} \prod_{j\in A_2} C(k_j)a_j^{k_j}(2-a_j\rho)^{k_j-1}.$$

and

$$\lambda := \sum_{j\in A_1}(k_j-1) + \sum_{j\in A_2} k_j = \sum_{j\in A} k_j - n.$$

The function $\rho \longmapsto f(a^{(r)},\rho)$ is continuous and does not vanish on the compact set $[0,1]$. So that the singularity in the $d\rho$-integral is only in the term of

$$\rho^{\lambda+r+\frac{\beta-d-2\gamma}{2}-1} = \rho^{(\sum_{j\in A} k_j)-n+r+\frac{\beta-d-2\gamma}{2}-1}.$$

Finally, we conclude that

$$R_{k,\beta}(x,\varepsilon^{(n)}.x) = +\infty \quad \Leftrightarrow \quad I(x,\varepsilon^{(n)}.x) = +\infty \quad \Leftrightarrow \quad d \geq 2(|A| - n + \sum_{j\in A} k_j - \gamma) + \beta.$$

This completes the proof of the assertion 2).
3) When $x \notin \cup_{i=1}^m H_i$, we have $A = \{1,\ldots,m\}$ and then the result is a particular case of the statement 2). □

Proposition 3.4 *The Riesz kernel $R_{k,\beta}(.,.)$ satisfies the following properties*

1) *For every $x, y \in \mathbb{R}^d$ and $g \in W$, we have*

$$R_{k,\beta}(x,y) = R_{k,\beta}(y,x), \quad R_{k,\beta}(gx,y) = R_{k,\beta}(x,g^{-1}y). \tag{3.16}$$

2) *Let $\beta,\theta > 0$ such that $\beta+\theta < d+2\gamma$. Then we have the following generalized Riesz composition formula*

$$\int_{\mathbb{R}^d} R_{k,\beta}(x,z)R_{k,\theta}(y,z)\omega_k(z)dz = R_{k,\beta+\theta}(x,y). \tag{3.17}$$

3) *Let $x \in \mathbb{R}^d$. Then, for every $y \in \mathbb{R}^d\backslash W.x$, we have*

$$\kappa \min_{g\in W}\left(\|x-gy\|^{\beta-d-2\gamma}\right) \leq R_{k,\beta}(x,y) \leq \kappa \max_{g\in W}\left(\|x-gy\|^{\beta-d-2\gamma}\right) \tag{3.18}$$

4) *Let $y \in \mathbb{R}^d$. Then, the function $x \mapsto R_{k,\beta}(x,y)$ is*
-lower semi-continuous (l.s.c.) on $\mathbb{R}^d$.
-of class C^∞ on $\mathbb{R}^d\backslash W.x$ and we have

$$\partial_j R_{k,\beta}(x,y) = (\beta-d-2\gamma)\kappa \int_{\mathbb{R}^d} (x_j-z_j)\left(\|x\|^2+\|y\|^2-2\langle x,z\rangle\right)^{\frac{\beta-2-d-2\gamma}{2}} d\mu_y(z). \tag{3.19}$$

Proof: **1)** The result follows from (3.3), (2.16) and (2.17).
2) The result follows from the Fubini's theorem and the semi-group property of the Dunkl-heat kernel (2.15).
3) Let $y \in \mathbb{R}^d$. From (1.4) for any $z \in \text{supp } \mu_y$, we can write $z = \sum_{g\in W} \lambda_g(z)gy$, where $\lambda_g(z) \in [0,1]$ are such that $\sum_{g\in W} \lambda_g(z) = 1$. Then, we have

$$\|x\|^2 + \|y\|^2 - 2\langle x, z\rangle = \sum_{g\in W} \lambda_g(z)\|x - gy\|^2. \tag{3.20}$$

As $\psi : t \longmapsto t^{\frac{\beta-d-2\gamma}{2}}$ is a convex function on $]0, +\infty[$, by (3.20) we have

$$\begin{aligned}\Big(\|x\|^2 + \|y\|^2 - 2\langle x, z\rangle\Big)^{\frac{\beta-d-2\gamma}{2}} &= \Big(\sum_{g\in W} \lambda_g(z)\|x - gy\|^2\Big)^{\frac{\beta-d-2\gamma}{2}} \\ &\leq \max_{g\in W}\big(\|x - gy\|^{\beta-d-2\gamma}\big).\end{aligned}$$

This implies the right inequality. Again by convexity, Jensen's inequality and (3.20), we get

$$\begin{aligned}R_{k,\beta}(x,y) &\geq \kappa\left(\int_{\mathbb{R}^d}(\|x\|^2 + \|y\|^2 - 2\langle x, z\rangle)d\mu_y(z)\right)^{\frac{\beta-d-2\gamma}{2}} \\ &\geq \kappa\left(\sum_{g\in W}\left(\int_{\mathbb{R}^d}\lambda_g(z)d\mu_y(z)\right)\|x - gy\|^2\right)^{\frac{\beta-d-2\gamma}{2}} \\ &\geq \kappa\left(\max_{g\in W}\|x - gy\|^2\right)^{\frac{\beta-d-2\gamma}{2}} = \kappa\min_{g\in W}\left(\|x - gy\|^{\beta-(d+2\gamma)}\right),\end{aligned}$$

where in the last line we have used the fact that ψ is a decreasing function.
4) The function $x \mapsto R_{k,\beta}(x,y)$ is l.s.c. on $\mathbb{R}^d$ as being the increasing limit of the sequence (f_n) of continuous functions defined by $f_n : x \mapsto \int_{1/n}^n t^{\frac{\beta}{2}-1}p_k(t,x,y)dt$.
Fix $y \in \mathbb{R}^d$. Using the fact that μ_y is with compact support and the fact that the function

$$(x,z) \longmapsto \big(\|x\|^2 + \|y\|^2 - 2\langle x, z\rangle\big)^{\frac{\beta-d-2\gamma}{2}}$$

is of class C^∞ on $\mathbb{R}^d\backslash W.y \times \mathbb{R}^d$, we can differentiate under the integral in the relation (3.2) and we obtain the result. □

In the following result, we study the $L^p_{k,loc}(\mathbb{R}^d)$-integrability of the function $R_{k,\beta}(x,.)$, for fixed $x \in \mathbb{R}^d$.

Proposition 3.5 *Let $0 < \beta < d + 2\gamma$ and $p \in [1, \frac{d+2\gamma}{d+2\gamma-\beta}[$. Then, for every $R > 0$, there exists a positive constant $C = C(R, p, d, \gamma, \beta)$ such that*

$$\forall\ x \in \mathbb{R}^d, \quad \|R_{k,\beta}(x,.)\|_{L^p_k(B(0,R))} \leq C. \tag{3.21}$$

In particular, for every $x \in \mathbb{R}^d$, $R_{k,\beta}(x,.)$ is in $L^p_{k,loc}(\mathbb{R}^d)$.

Proof: By Jensen's inequality and (3.2), we have

$$(R_{k,\beta}(x,y))^p \leq \kappa^p \int_{\mathbb{R}^d} \left(\|x\|^2 + \|y\|^2 - 2\langle x,z\rangle\right)^{\frac{p(\beta-d-2\gamma)}{2}} d\mu_y(z).$$

Using the same idea as in the proof of (3.3), we can write the previous inequality as follows

$$\begin{aligned}(R_{k,\beta}(x,y))^p &\leq \frac{\kappa^p}{p(d+2\gamma-\beta)} \int_0^{+\infty} t^{p(\beta-d-2\gamma)} h_k(t,x,y)\frac{dt}{t} \\ &= C_1 \int_0^1 t^{p(\beta-d-2\gamma)} h_k(t,x,y)\frac{dt}{t} + C_1 \int_1^{+\infty} t^{p(\beta-d-2\gamma)} h_k(t,x,y)\frac{dt}{t} \\ &\leq C_1 \int_0^1 t^{p(\beta-d-2\gamma)} h_k(t,x,y)\frac{dt}{t} + \frac{C_1}{p(d+2\gamma-\beta)},\end{aligned}$$

where $C_1 = \frac{\kappa^p}{p(d+2\gamma-\beta)}$ and we have used the fact that $h_k(t,x,y) \leq 1$ in the last inequality. Let then $R > 0$. From (2.18), Fubini's theorem and our hypothesis, we deduce that

$$\int_{B(0,R)} \int_0^1 t^{p(\beta-d-2\gamma)} h_k(t,x,y)\frac{dt}{t}\omega_k(y)dy \leq \frac{d_k}{d+2\gamma} \int_0^1 t^{p(\beta-d-2\gamma)} t^{d+2\gamma}\frac{dt}{t} := C_2 < +\infty.$$

This proves the desired inequality where we can take

$$C = \left(C_1 C_2 + \frac{C_1 m_k[B(0,R)]}{p(d+2\gamma-\beta)}\right)^{1/p}.$$

□

Proposition 3.6 *Let $0 < \beta < d+2\gamma$ and $x_0 \in \mathbb{R}^d$. Then, the function $R_{k,\beta}(x_0,.)$ is*

i) *D-superharmonic on $\mathbb{R}^d$ when $\beta \geq 2$,*

ii) *D-harmonic on $\mathbb{R}^d \backslash W.x_0$ when $\beta = 2$,*

iii) *D-subharmonic on $\mathbb{R}^d \backslash W.x_0$ when $\beta \leq 2$*

Proof: The case $\beta = 2$ (i.e. the case of the Dunkl-Newton kernel) has been done in [10]. So, we will deal with the case $\beta \neq 2$.
i) Suppose that $\beta > 2$. We consider the function $S_{x_0,\beta,r}$

$$S_{x_0,\beta,r}(x) := \frac{1}{\Gamma(\beta/2)} \int_r^{+\infty} t^{\frac{\beta}{2}-1} p_k(t,x_0,x)dt.$$

By the monotone convergence theorem, we see that the function $R_{k,\beta}(x_0,.)$ is the pointwise increasing limit of the sequence $\left(S_{x_0,\beta,\frac{1}{n}}\right)_n$. Hence, by Proposition 3.3 in [10], it suffices to prove that for every $r > 0$, $S_{x_0,\beta,r}$ is D-superharmonic on $\mathbb{R}^d$. To do this, we have only to show that $S_{x_0,\beta,r}$ is of class C^2 on $\mathbb{R}^d$ and $\Delta_k S_{x_0,\beta,r} \leq 0$ on $\mathbb{R}^d$ (see [10], Proposition 4.1).

The function $p_k(t,x_0,.)$ is of class C^∞ on $\mathbb{R}^d$ and we can differentiate under the integral sign in the relation (2.10) to obtain

$$\partial_j p_k(t,x_0,.)(x) = -\frac{1}{2t}\frac{1}{(2t)^{\frac{d}{2}+\gamma}c_k}\int_{\mathbb{R}^d}(x_j-z_j)e^{-\frac{1}{4t}(\|x\|^2+\|x_0\|^2-2\langle x,z\rangle)}d\mu_{x_0}(z) \tag{3.22}$$

and

$$\begin{aligned}\partial_i\partial_j p_k(t,x_0,.)(x) &= -\delta_{ij}\frac{1}{2t}p_k(t,x_0,x)\\ &\quad+\frac{1}{4t^2}\frac{1}{(2t)^{\frac{d}{2}+\gamma}c_k}\int_{\mathbb{R}^d}(x_j-z_j)(x_i-z_i)e^{-\frac{1}{4t}(\|x\|^2+\|x_0\|^2-2\langle x,z\rangle)}d\mu_{x_0}(z),\end{aligned} \tag{3.23}$$

where δ_{ij} is the Kronecker symbol.
Using the fact that supp $\mu_{x_0}\subset B(0,\|x_0\|)$, we deduce from (3.22) and (3.23) that

$$|\partial_j p_k(t,x_0,.)(x)| \le \frac{\|x\|+\|x_0\|}{(2t)^{1+\frac{d}{2}+\gamma}c_k},$$

$$|\partial_i\partial_j p_k(t,x_0,.)(x)| \le \frac{1}{(2t)^{1+\frac{d}{2}+\gamma}c_k}+\frac{(\|x\|+\|x_0\|)^2}{(2t)^{2+\frac{d}{2}+\gamma}c_k}.$$

Let $R>0$. The previous inequalities and the differentiation theorem under the integral sign imply that $S_{x_0,\beta,r}$ is of class C^2 on the open ball $\overset{\circ}{B}(0,R)$ and as $x\mapsto p_k(t,x_0,x)$ is a solution of the Dunkl-heat equation (1.9), we deduce that

$$\begin{aligned}\forall\, x\in\overset{\circ}{B}(0,R),\quad \Delta_k S_{x_0,\beta,r}(x) &= \frac{1}{\Gamma(\beta/2)}\int_r^{+\infty}t^{\frac{\beta}{2}-1}\Delta_k\left(p_k(t,x_0,.)\right)(x)dt\\ &= \frac{1}{\Gamma(\beta/2)}\int_r^{+\infty}t^{\frac{\beta}{2}-1}\partial_t p_k(t,x_0,x)dt\\ &= -\frac{r^{\frac{\beta}{2}-1}}{\Gamma(\beta/2)}p_k(r,x_0,x)-\frac{\beta-2}{2\Gamma(\beta/2)}\int_r^{+\infty}t^{\frac{\beta}{2}-2}p_k(t,x_0,x)dt<0.\end{aligned}$$

Therefore, $S_{x_0,\beta,r}$ is D-superharmonic on $\overset{\circ}{B}(0,R)$. As $R>0$ is arbitrary, we conclude that $S_{x_0,\beta,r}$ is D-superharmonic on $\mathbb{R}^d$ as desired.

iii) Let $\beta\in]0,2[$. Using (3.22), (3.23) and (3.20), we can see that

$$|\partial_j p_k(t,x_0,.)(x)| \le \frac{\|x\|+\|x_0\|}{(2t)^{1+\frac{d}{2}+\gamma}c_k}e^{-\frac{\min_{g\in W}(\|x-gx_0\|^2)}{4t}},$$

$$|\partial_i\partial_j p_k(t,x_0,.)(x)| \le \Big(\frac{1}{(2t)^{1+\frac{d}{2}+\gamma}c_k}+\frac{(\|x\|+\|x_0\|)^2}{(2t)^{2+\frac{d}{2}+\gamma}c_k}\Big)e^{-\frac{\min_{g\in W}(\|x-gx_0\|^2)}{4t}}.$$

Fix an arbitrary open Dunkl ball $O^W(a,R):=\cup_{g\in W}\overset{\circ}{B}(ga,R)$ such that its closure is contained in $\mathbb{R}^d\backslash W.x_0$. The previous inequalities imply that we can differentiate with

respect to $x \in O^W(a,R)$ under the integral sign in the relation (3.1). Furthermore, using the heat equation (1.9) and integrating by parts, we obtain

$$\forall\ x \in O^W(a,R), \quad \Delta_k\left(R_{k,\beta}(x_0,.)\right)(x) = \frac{1}{\Gamma(\beta/2)}\int_0^{+\infty} t^{\frac{\beta}{2}-1}\partial_t p_k(t,x_0,x)dt$$
$$= -\frac{\beta-2}{2\Gamma(\beta/2)}\int_0^{+\infty} t^{\frac{\beta}{2}-2}p_k(t,x_0,x)dt \geq 0.$$

According to Remark 3.1-***2)***, the above relation can be written as

$$\forall\ x \in O^W(a,R), \quad \Delta_k\left(R_{k,\beta}(x_0,.)\right)(x) = -R_{k,\beta-2}(x_0,x) \geq 0. \tag{3.24}$$

Therefore, the function $R_{k,\beta}(x_0,.)$ is D-subharmonic on $O^W(a,R)$ and so on $\mathbb{R}^d\backslash W.x_0$. □

Proposition 3.7 *Let $\beta \in]0, d+2\gamma[$ and $x_0 \in \mathbb{R}^d$. Then, the function $x \mapsto R_{k,\beta}(x_0,x)\omega_k(x)$ defines a tempered distribution and we have*

$$\mathcal{F}_k\left(R_{k,\beta}(x_0,.)\omega_k\right) = E_k(-ix_0,.)\|.\|^{-\beta}\omega_k \quad in \quad \mathcal{S}'(\mathbb{R}^d). \tag{3.25}$$

Proof: Let $m \in \mathbb{N}$ such that $m > d+2\gamma$. We claim that there exists a constant $C_m = C(d,\gamma,\beta,m) > 0$ such that

$$\forall\ x_0 \in \mathbb{R}^d, \quad \int_{\mathbb{R}^d}(1+\|x\|^2)^{-m}R_{k,\beta}(x_0,x)\omega_k(x)dx \leq C_m. \tag{3.26}$$

From (3.3), we can write

$$R_{k,\beta}(x_0,x) = \frac{\kappa}{d+2\gamma-\beta}\Big(\int_0^1 t^{\beta-d-2\gamma-1}h_k(t,x_0,x)dt + \int_1^{+\infty} t^{\beta-d-2\gamma-1}h_k(t,x_0,x)dt\Big)$$
$$:= A(x_0,x) \quad + \quad B(x_0,x).$$

• Using Fubini's theorem and the relation (2.18), for any $x_0 \in \mathbb{R}^d$ we obtain

$$\int_{\mathbb{R}^d}(1+\|x\|^2)^{-m}A(x_0,x)\omega_k(x)dx \leq \int_{\mathbb{R}^d}A(x_0,x)\omega_k(x)dx$$
$$= \frac{\kappa}{d+2\gamma-\beta}\int_0^1 t^{\beta-d-2\gamma-1}\|h_k(t,x_0,.)\|_{k,1}dt$$
$$= \frac{d_k\kappa}{\beta(d+2\gamma)(d+2\gamma-\beta)} := C_{1,m}.$$

• Now, using the inequality $h_k(t,x_0,x) \leq 1$, we deduce that

$$\forall\ x_0 \in \mathbb{R}^d, \quad B(x_0,x) \leq \frac{\kappa}{(d+2\gamma-\beta)^2}.$$

This relation and the choice of m imply that

$$\forall\ x_0 \in \mathbb{R}^d, \quad \int_{\mathbb{R}^d}(1+\|x\|^2)^{-m}B(x_0,x)\omega_k(x)dx \leq \frac{\kappa}{(d+2\gamma-\beta)^2}\int_{\mathbb{R}^d}(1+\|x\|^2)^{-m}\omega_k(x)dx$$
$$:= C_{2,m} < +\infty.$$

This proves (3.26) and this implies that the function $R_{k,\beta}(x_0,.)\omega_k$ defines a tempered distribution (see [27], Theorem VII, p. 242).

Let us now prove (3.25). For $\phi \in \mathcal{S}(\mathbb{R}^d)$, we have

$$\langle \mathcal{F}_k\left(R_{k,\beta}(x_0,.)\omega_k\right),\phi\rangle = \frac{1}{\Gamma(\beta/2)}\int_{\mathbb{R}^d}\left(\int_0^{+\infty} t^{\frac{\beta}{2}-1}p_k(t,x_0,x)dt\right)\mathcal{F}_k(\phi)(x)\omega_k(x)dx.$$

Multiplying and dividing by $(1+\|x\|^2)^m$ (the integer m is chosen as above) and using the fact that $\mathcal{F}_k(\phi)\in\mathcal{S}(\mathbb{R}^d)$, we see that we can use Fubini's theorem in the above relation. Moreover, from (2.4) and (2.12), we obtain

$$\begin{aligned}\langle \mathcal{F}_k\left(R_{k,\beta}(x_0,.)\omega_k\right),\phi\rangle &= \frac{1}{\Gamma(\beta/2)}\int_0^{+\infty} t^{\frac{\beta}{2}-1}\left(\int_{\mathbb{R}^d}\mathcal{F}_k(p_k(t,x_0,.))(x)\phi(x)\omega_k(x)dx\right)dt\\ &= \frac{1}{\Gamma(\beta/2)}\int_0^{+\infty} t^{\frac{\beta}{2}-1}\left(\int_{\mathbb{R}^d}E_k(-ix_0,x)e^{-t\|x\|^2}\phi(x)\omega_k(x)dx\right)dt.\end{aligned}$$

Applying again Fubini's theorem, we deduce that

$$\langle \mathcal{F}_k\left(R_{k,\beta}(x_0,.)\omega_k\right),\phi\rangle = \int_{\mathbb{R}^d}E_k(-ix_0,x)\|x\|^{-\beta}\phi(x)\omega_k(x)dx.$$

This completes the proof. □

Corollary 3.1 *For every $x_0\in\mathbb{R}^d$, we have*

$$\lim_{\beta\to 0} R_{k,\beta}(x_0,.)\omega_k = \delta_{x_0} \quad in \quad \mathcal{S}'(\mathbb{R}^d). \tag{3.27}$$

Proof: We can see that for every $\xi\in\mathbb{R}^d$,

$$\|\xi\|^{-\beta} \le \mathbf{1}_{\mathbb{R}^d\setminus B(0,1)}(\xi) + \|\xi\|^{-d-2\gamma}\mathbf{1}_{B(0,1)}(\xi).$$

Consequently, we can use the dominated convergence theorem to obtain from (3.25)

$$\lim_{\beta\to 0}\mathcal{F}_k\left(R_{k,\beta}(x_0,.)\omega_k\right) = E_k(-ix_0,.)\omega_k = \mathcal{F}_k(\delta_{x_0}) \quad \text{in} \quad \mathcal{S}'(\mathbb{R}^d).$$

Thus, we deduce the result by using the properties of the Dunkl transform on $\mathcal{S}'(\mathbb{R}^d)$. □

From the formula (3.24), we see that the Δ_k-Riesz measure related to the D-subharmonic function $R_{k,\beta}(x_0,.)$, $\beta<2$, is given by $-R_{k,\beta-2}(x_0,x)\omega_k(x)dx$. In the following result, we will compute the Δ_k-Riesz measure of the D-superharmonic function $R_{k,\beta}(x_0,.)$ with $\beta\in[2,d+2\gamma[$.

Corollary 3.2 *Let $2\le\beta<d+2\gamma$ and $x_0\in\mathbb{R}^d$. If $m\in[1,\beta/2]$ be an integer, then the function $x\mapsto R_{k,\beta}(x_0,x)$ satisfies*

$$(-\Delta_k)^m\left(R_{k,\beta}(x_0,.)\omega_k\right) = \begin{cases} R_{k,\beta-2m}(x_0,.)\omega_k & in \quad \mathcal{S}'(\mathbb{R}^d) \quad if \quad \beta>2m,\\ \\ \delta_{x_0} & in \quad \mathcal{S}'(\mathbb{R}^d) \quad if \quad \beta=2m,\end{cases} \tag{3.28}$$

where δ_{x_0} is the Dirac measure at x_0.

Proof: At first, we remark that if $U \in \mathcal{S}'(\mathbb{R}^d)$, then

$$\mathcal{F}_k(\Delta_k U) = -\|.\|^2 \mathcal{F}_k(U), \tag{3.29}$$

as easily follows from the relation $\Delta_k \mathcal{F}_k(f) = -\mathcal{F}_k(\|.\|^2 f)$ for all $f \in \mathcal{S}(\mathbb{R}^d)$.
From (3.29) and (3.25), we obtain

$$\begin{aligned}\mathcal{F}_k\big((-\Delta_k)^m \left(R_{k,\beta}(x_0,.)\omega_k\right)\big) &= E_k(-ix_0,.)\|.\|^{\beta-2m}\omega_k \\ &= \begin{cases} \mathcal{F}_k\big(R_{k,\beta-2m}(x_0,.)\omega_k\big) & \text{in} \quad \mathcal{S}'(\mathbb{R}^d) \quad \text{if} \quad \beta > 2m, \\ \mathcal{F}_k(\delta_{x_0}) & \text{in} \quad \mathcal{S}'(\mathbb{R}^d) \quad \text{if} \quad \beta = 2m. \end{cases}\end{aligned}$$

Hence, we deduce the result by the fact that $\mathcal{F}_k$ is a topological isomorphism of $\mathcal{S}'(\mathbb{R}^d)$ onto itself. □

Remark 3.2 *Let $1 \leq m < \gamma + d/2$ an integer. Taking $x_0 = 0$ in (3.28), we deduce that the function $S : y \mapsto R_{k,2m}(0,y)\omega_k(y) = \kappa\|y\|^{2m-d-2\gamma}\omega_k(y)$ is the fundamental solution of the Dunkl-polylaplacian of order m $(-\Delta_k)^m$ i.e. $(-\Delta_k)^m S = \delta_0$ in $\mathcal{S}'(\mathbb{R}^d)$.*

4 Riesz potentials of Radon measures

The sets $\mathcal{M}(\mathbb{R}^d)$ and $\mathcal{M}^+(\mathbb{R}^d)$ denote respectively the space of signed Radon measures on $\mathbb{R}^d$ and the convex cone of nonnegative Radon measures on $\mathbb{R}^d$.

Definition 4.1 *Let $\mu \in \mathcal{M}^+(\mathbb{R}^d)$ and $\beta \in]0, d+2\gamma[$. The β-Δ_k-Riesz potential of μ is defined by*

$$I_{k,\beta}[\mu](x) = \int_{\mathbb{R}^d} R_{k,\beta}(x,y)d\mu(y), \quad x \in \mathbb{R}^d. \tag{4.1}$$

Proposition 4.1 *Let $\mu \in \mathcal{M}^+(\mathbb{R}^d)$ and $\beta \in]0, d+2\gamma[$.*

1. *If μ is bounded, then $I_{k,\beta}[\mu] \in L^p_{k,loc}(\mathbb{R}^d)$ whenever $p \in [1, \frac{d+2\gamma}{d+2\gamma-\beta}[$. In particular, $I_{k,\beta}[\mu]$ is finite a.e. in $\mathbb{R}^d$.*

2. *The following statements are equivalent*

i) *$I_{k,\beta}[\mu]$ is finite a.e. in $\mathbb{R}^d$,*

ii) *the measure μ satisfies*

$$\int_{\mathbb{R}^d} (1+\|y\|)^{\beta-d-2\gamma} d\mu(y) < +\infty, \tag{4.2}$$

iii) *$I_{k,\beta}[\mu](x_0) < +\infty$ for some $x_0 \in \mathbb{R}^d$.*

If ii) holds, then $I_{k,\beta}[\mu] \in L^1_{k,loc}(\mathbb{R}^d)$.

Proof: **1)** Assume that μ is a probability measure on $\mathbb{R}^d$. Let p as in the proposition and $R > 0$. Using respectively (4.1), Jensen's inequality, Fubini's theorem, the fact that the Riesz kernel is symmetric and (3.21), we get

$$\begin{aligned}\int_{B(0,R)} (I_{k,\beta}[\mu](x))^p\,(x)\omega_k(x)dx &\leq \int_{B(0,R)} \Big(\int_{\mathbb{R}^d} (R_{k,\beta}(x,y))^p\,d\mu(y)\Big)\omega_k(x)dx\\ &= \int_{\mathbb{R}^d} \Big(\int_{B(0,R)} (R_{k,\beta}(x,y))^p\,\omega_k(x)dx\Big)d\mu(y)\\ &\leq C < +\infty,\end{aligned}$$

where C is the constant in (3.21).

2) ii) $\Rightarrow$ i) Assume that the condition (4.2) holds. We will prove that $x \mapsto I_{k,\beta}[\mu](x)$ is in $L^1_{k,loc}(\mathbb{R}^d)$. Let $R > 1$. By Fubini's theorem, we have

$$\begin{aligned}A_R &:= \int_{B(0,R)} I_{k,\beta}[\mu](x)\omega_k(x)dx = \int_{\mathbb{R}^d}\int_{B(0,R)} R_{k,\beta}(x,y)\omega_k(x)dxd\mu(y)\\ &= \int_{\|y\|\leq 2R}\int_{B(0,R)} R_{k,\beta}(x,y)\omega_k(x)dxd\mu(y) + \int_{\|y\|>2R}\int_{B(0,R)} R_{k,\beta}(x,y)\omega_k(x)dxd\mu(y)\\ &:= A_{1,R} \quad + \quad A_{2,R}.\end{aligned}$$

Applying the assertion 1) with the finite measure $\mu_{|B(0,R)}$, we get $A_{1,R} < +\infty$.
Now, from (3.18) we deduce that

$$A_{2,R} \leq \kappa \int_{\|y\|>2R}\int_{B(0,R)} \max_{g\in W}\Big(\|x-gy\|^{\beta-d-2\gamma}\Big)\,\omega_k(x)dxd\mu(y).$$

But, for every $x \in B(0,R)$ and every $y \in \mathbb{R}^d\backslash B(0,2R)$, we have $\|x-gy\| \geq \|y\|-\|x\| \geq \frac{\|y\|}{2}$. Moreover, as $R > 1$, we see that $\|y\| \geq \frac{1}{2}(1+\|y\|)$ whenever $\|y\| \geq 2R$. In other words, the inequality

$$\max_{g\in W}\Big(\|x-gy\|^{\beta-d-2\gamma}\Big) \leq 4^{\beta-d-2\gamma}(1+\|y\|)^{\beta-d-2\gamma}$$

holds for every $x \in B(0,R)$ and every $y \in \mathbb{R}^d \setminus B(0,2R)$. Hence, by our hypothesis we conclude that

$$A_{2,R} \leq 4^{\beta-d-2\gamma}\kappa\; m_k[B(0,R)] \int_{\|y\|\geq 2R} (1+\|y\|)^{\beta-d-2\gamma}d\mu(y) < +\infty$$

and thus the function $x \mapsto I_{k,\beta}[\mu](x)\omega_k(x)$ is locally integrable on $\mathbb{R}^d$. In particular, $I_{k,\beta}[\mu](x) < +\infty$ a.e. on $\mathbb{R}^d$.

i) $\Rightarrow$ iii) It is obvious.

iii) $\Rightarrow$ ii) Let $x_0 \in \mathbb{R}^d$ such that $I_{k,\beta}[\mu](x_0) < +\infty$. From (3.18), we can see that

$$\begin{aligned}I_{k,\beta}[\mu](x_0) &\geq \kappa \int_{\mathbb{R}^d} \min_{g\in W}\Big(\|x_0-gy\|^{\beta-d-2\gamma}\Big)\,d\mu(y)\\ &\geq \kappa \int_{\mathbb{R}^d} (\|x_0\|+\|y\|)^{\beta-d-2\gamma}d\mu(y).\end{aligned}$$

If $\|x_0\| \leq 1$, we deduce immediately from the previous inequality that (4.2) holds.
If $\|x_0\| > 1$, using the fact that

$$\|x_0\| + \|y\| \leq \|x_0\|(1 + \|y\|)$$

and using again the above inequality, we obtain that (4.2) holds.
This finishes the proof. □

Remark 4.1 *Let $\beta \in]0, d + 2\gamma[$.*

• *Let $\mu \in \mathcal{M}(\mathbb{R}^d)$ and $\mu = \mu^+ - \mu^-$ its Hahn-Jordan decomposition. If μ^+ and μ^- satisfy (4.2), then the Δ_k-Riesz potential of μ is well defined almost everywhere by setting $I_{k,\beta}[\mu](x) = I_{k,\beta}[\mu^+](x) - I_{k,\beta}[\mu^-](x)$. Moreover, the function $I_{k,\beta}[\mu] \in L^1_{k,loc}(\mathbb{R}^d)$.*

• *Let us introduce the following notations*

$$\mathcal{M}^+_{k,\beta}(\mathbb{R}^d) := \Big\{\mu \in \mathcal{M}^+(\mathbb{R}^d), \quad \mu \text{ satisfies (4.2)}\Big\} \tag{4.3}$$

and

$$\mathcal{M}_{k,\beta}(\mathbb{R}^d) := \Big\{\mu = \mu^+ - \mu^- \in \mathcal{M}(\mathbb{R}^d), \quad \mu^+, \mu^- \in \mathcal{M}^+_{k,\beta}(\mathbb{R}^d)\Big\}. \tag{4.4}$$

We note that if $0 < \beta_1 \leq \beta_2 < d + 2\gamma$, then $\mathcal{M}^+_{k,\beta_2}(\mathbb{R}^d) \subset \mathcal{M}^+_{k,\beta_1}(\mathbb{R}^d)$ and $\mathcal{M}_{k,\beta_2}(\mathbb{R}^d) \subset \mathcal{M}_{k,\beta_1}(\mathbb{R}^d)$.

• *Let $\beta, \theta > 0$ be such that $\beta + \theta < d + 2\gamma$. Then using the generalized Riesz composition formula (3.17) and Fubini's theorem we can see that*

$$\forall\ \mu \in \mathcal{M}^+_{k,\beta+\theta}(\mathbb{R}^d), \quad I_{k,\beta+\theta}[\mu] = I_{k,\beta}\big[I_{k,\theta}[\mu](y)\omega_k(y)dy\big]. \tag{4.5}$$

In the following result we will establish that any measure $\mu \in \mathcal{M}^+_{k,\beta}(\mathbb{R}^d)$ doesn't charge the singularities of the function $R_{k,\beta}(x,.)$ whenever its β-Δ_k-Riesz potential valued at x is finite. More precisely, we have

Proposition 4.2 *Let $\mu \in \mathcal{M}^+_{k,\beta}(\mathbb{R}^d)$ and $x \in \mathbb{R}^d$ such that $I_{k,\beta}[\mu](x) < +\infty$. Then $\mu(\{gx\}) = 0$ whenever the point gx, $g \in W$, is a singularity of $R_{k,\beta}(x,.)$.*
In particular, if $\beta \leq d$, the condition $I_{k,\beta}[\mu](x) < +\infty$ implies that $\mu(\{x\}) = 0$.

Proof: Let $g \in W$ such that $R_{k,\beta}(x, gx) = +\infty$ and let $n \in \mathbb{N}$. Since $R_{k,\beta}(x,.)$ is l.s.c. at gx, there exists $r > 0$ such that $R_{k,\beta}(x, y) \geq n$ for all $y \in B(gx, r)$. This implies that

$$I_{k,\beta}[\mu](x) \geq \int_{B(gx,r)} R_{k,\beta}(x, y)d\mu(y) \geq n\mu(B(gx, r)) \geq n\mu(\{gx\}).$$

This proves the first part.
Now, since $\beta \leq d$, we know that x is always a singularity of the function $R_{k,\beta}(x,.)$. Thus the second part follows from the first one. □

Now, we establish a boundedness principle for the potential of a compactly supported measure which generalizes the known result in the classical case (i.e. $k = 0$) (see [18], Theorem 1.5).

Proposition 4.3 *Let $0 < \beta < d + 2\gamma$ and μ be a compactly supported nonnegative Radon measure on $\mathbb{R}^d$. If $I_{k,\beta}[\mu] \leq M$ holds on W.supp μ, then*

$$I_{k,\beta}[\mu] \leq 2^{d+2\gamma-\beta} M \quad on \quad \mathbb{R}^d. \tag{4.6}$$

Proof: Let $x \notin W.\text{supp } \mu$ and $x_0 \in W.\text{supp } \mu$ such that $\|x - x_0\| = dist(x, W.\text{supp } \mu)$. We have

$$\forall\ y \in \text{supp } \mu, \quad \forall\ g \in W, \quad \|x_0 - gy\| \leq \|x_0 - x\| + \|x - gy\| \leq 2\|x - gy\|.$$

Hence, by (3.20) we deduce that

$$\forall\ y \in \text{supp } \mu, \quad \forall\ z \in \text{supp } \mu_y, \quad \|x_0\|^2 + \|y\|^2 - 2\langle x_0, z\rangle \leq 4(\|x\|^2 + \|y\|^2 - 2\langle x, z\rangle).$$

Now, using (2.10), we obtain

$$\forall\ y \in \text{supp } \mu, \quad 4^{-\frac{d}{2}-\gamma} p_{t/4}(x, y) \leq p_t(x_0, y).$$

From (3.1), the above inequality implies that

$$\forall\ y \in \text{supp } \mu, \quad 2^{-d-2\gamma+\beta} R_{k,\beta}(x, y) \leq R_{k,\beta}(x_0, y).$$

Finally, if we integrate with respect to the measure $d\mu(y)$ and use our hypothesis, the inequality (4.6) follows. □

In the following result, we will study some continuity properties of the β-Δ_k-Riesz potentials:

Proposition 4.4 *Let $\beta \in]0, d + 2\gamma[$ and $\mu \in \mathcal{M}^+_{k,\beta}(\mathbb{R}^d)$ with compact support.*

1) *The function $I_{k,\beta}[\mu]$ is lower semi-continuous on $\mathbb{R}^d$ and continuous on $\mathbb{R}^d \setminus W.supp\ \mu$.*

2) *If the restriction of the function $I_{k,\beta}[\mu]$ on W.supp μ is continuous on W.supp μ, then $I_{k,\beta}[\mu]$ is continuous on $\mathbb{R}^d$.*

Proof: **1)** • Consider the function F_n given by

$$F_n(x) = \frac{1}{\Gamma(\beta/2)} \int_{\text{supp } \mu} \Big(\int_{1/n}^{n} t^{\frac{\beta}{2}-1} p_k(t, x, y) dt \Big) d\mu(y).$$

As $t^{\frac{\beta}{2}-1} p_k(t, x, y) \leq 2^{-\frac{d}{2}-\gamma} c_k^{-1} t^{-\frac{\beta-d-2\gamma}{2}-1}$, by the continuity theorem under the integral sign, we see that F_n is continuous on $\mathbb{R}^d$. Moreover, from the monotone convergence theorem, we deduce that the function $I_{k,\beta}[\mu]$ is l.s.c. on $\mathbb{R}^d$ as being the pointwise increasing limit of the sequence (F_n).

• Let us prove the second part of 1). Fix a closed ball $B(x_0, R)$ in $\mathbb{R}^d \setminus W.\text{supp } \mu$ and set

$$\eta := dist\,(B(x_0, R), W.\text{supp } \mu) > 0.$$

From (2.14), we deduce that

$$\forall (x,y) \in B(x_0,R) \times \text{supp } \mu, \quad p_k(t,x,y) \leq \frac{1}{(2t)^{\frac{d}{2}+\gamma} c_k} e^{-\frac{\eta^2}{4t}}.$$

Then, writing

$$I_{k,\beta}[\mu](x) = \frac{1}{\Gamma(\beta/2)} \int_{\text{supp } \mu} \Big(\int_0^{+\infty} t^{\frac{\beta}{2}-1} p_k(t,x,y) dt \Big) d\mu(y)$$

and using the continuity theorem under the integral sign, it follows that $I_{k,\beta}[\mu]$ is continuous on $B(x_0,R)$. As the ball $B(x_0,R)$ is arbitrary, the result follows.

2) Fix $x_0 \in W.\text{supp } \mu$ and $\varepsilon > 0$. Let (x_n) be a sequence which converges to x_0. For $R > 0$ (small), set $\mu_R := \mu_{|B(x_0,R)}$ and $\nu_R := \mu - \mu_R$. In particular, we note that

$$I_{k,\beta}[\mu] = I_{k,\beta}[\mu_R] + I_{k,\beta}[\nu_R].$$

We have

$$\big| I_{k,\beta}[\mu](x_n) - I_{k,\beta}[\mu](x_0) \big| \leq I_{k,\beta}[\mu_R](x_n) + I_{k,\beta}[\mu_R](x_0) + \big| I_{k,\beta}[\nu_R](x_n) - I_{k,\beta}[\nu_R](x_0) \big|. \quad (4.7)$$

• As $x_0 \notin W.\text{supp } \nu_R$, by the assertion 1, the function $I_{k,\beta}[\nu_R]$ is continuous at x_0. Hence, there exists $N_1 \in \mathbb{N}$ such that

$$\forall\ n \geq N_1, \quad \big| I_{k,\beta}[\nu_R](x_n) - I_{k,\beta}[\nu_R](x_0) \big| \leq \varepsilon. \quad (4.8)$$

• For every n, let $x_n' \in K := W.\text{supp } \mu_R = W.\big(\text{supp } \mu \cap B(x_0,R)\big)$ such that $\|x_n - x_n'\| = dist(x_n, K) = \inf\{\|x_n - \xi\|,\ \xi \in K\}$. As $x_0 \in K$, we can see that $\|x_n - x_0\| \geq \|x_n - x_n'\|$. This implies that $\|x_n' - x_0\| \leq \|x_n - x_n'\| + \|x_n - x_0\| \leq 2\|x_n - x_0\|$ and thus $x_n' \longrightarrow x_0$ as $n \to +\infty$. Using the inequality (4.6), we deduce that

$$I_{k,\beta}[\mu_R](x_n) \leq 2^{d+2\gamma-\beta} I_{k,\beta}[\mu_R](x_n') = 2^{d+2\gamma-\beta} \big(I_{k,\beta}[\mu](x_n') - I_{k,\beta}[\nu_R](x_n') \big). \quad (4.9)$$

But, $x_n' \in W.\text{supp } \mu$ and the restriction of $I_{k,\beta}[\mu]$ on $W.\text{supp } \mu$ is continuous. Thus, $\lim_{n\to+\infty} I_{k,\beta}[\mu](x_n') = I_{k,\beta}[\mu](x_0)$. Again by continuity of $I_{k,\beta}[\nu_R]$ at x_0, we also get $\lim_{n\to+\infty} I_{k,\beta}[\nu_R](x_n') = I_{k,\beta}[\nu_R](x_0)$. Therefore, $I_{k,\beta}[\mu_R](x_n') \longrightarrow I_{k,\beta}[\mu_R](x_0)$ as $n \to +\infty$. Let then $N_2 \in \mathbb{N}$ such that

$$\forall\ n \geq N_2, \quad I_{k,\beta}[\mu_R](x_n') \leq I_{k,\beta}[\mu_R](x_0) + \varepsilon. \quad (4.10)$$

Finally from (4.7), (4.8), (4.9) and (4.10), the inequality

$$\big| I_{k,\beta}[\mu](x_n) - I_{k,\beta}[\mu](x_0) \big| \leq \big(2^{d+2\gamma-\beta} + 1 \big) \big(\varepsilon + I_{k,\beta}[\mu_R](x_0) \big) \quad (4.11)$$

holds for every $n \geq \max(N_1, N_2)$ and every $R > 0$.
But, since $x_0 \in W.\text{supp } \mu$ and $I_{k,\beta}[\mu]$ is continuous on $W.\text{supp } \mu$, we must have $I_{k,\beta}[\mu](x_0) < +\infty$. Furthermore, because x_0 is a singularity of $R_{k,\beta}(x_0,.)$, Proposition 4.2 imply that

$$\lim_{R\to 0} I_{k,\beta}[\mu_R](x_0) = \lim_{R\to 0} \int_{B(x_0,R)} R_{k,\beta}(x_0,y) d\mu(y) = 0. \quad (4.12)$$

Finally, by the relations (4.11) and (4.12) we deduce that $I_{k,\beta}[\mu]$ is continuous at x_0. □

Theorem 4.1 *Let $\beta \in]0, d+2\gamma[$ and $\mu \in \mathcal{M}^+_{k,\beta}(\mathbb{R}^d)$ with compact support. Then, the function $I_{k,\beta}[\mu]$ is*

i) *D-superharmonic on $\mathbb{R}^d$ if $\beta \geq 2$,*

ii) *D-harmonic on $\mathbb{R}^d \setminus W.supp\ \mu$ if $\beta = 2$,*

iii) *D-subharmonic on $\mathbb{R}^d \setminus W.supp\ \mu$ if $\beta \leq 2$.*

We need the following lemma:

Proof of Theorem 4.1: **i)** Let $\beta > 2$. Using Fubini's theorem and the D-superharmonicty of the Δ_k-Riesz kernel (see Proposition 3.6), we can easily see that $I_{k,\beta}[\mu]$ satisfies the super-mean property i.e. for all $x \in \mathbb{R}^d$ and all $r > 0$, $M^r_B\left(I_{k,\beta}[\mu]\right)(x) \leq I_{k,\beta}[\mu](x)$.
Since $I_{k,\beta}[\mu]$ is l.s.c and finite a.e., we deduce that the function $I_{k,\beta}[\mu]$ is D-superharmonic on $\mathbb{R}^d$.

ii) If $\beta = 2$, we are in the case of the Dunkl-Newton potential and the result has been proved in [10].

iii) Let $\beta < 2$. From Lemma 4.4, we know that $I_{k,\beta}[\mu]$ is a continuous function on $\mathbb{R}^d \setminus W.\text{supp}\ \mu$. Furthermore, by Proposition 3.6 and Fubini's theorem, the sub-mean property is satisfied by the function $I_{k,\beta}[\mu]$ on $\mathbb{R}^d \setminus W.\text{supp}\ \mu$. Thus, $I_{k,\beta}[\mu]$ is D-subharmonic on $\mathbb{R}^d \setminus W.\text{supp}\ \mu$. □

Corollary 4.1 *Let $\beta \in [2, d+2\gamma[$. If $\mu \in \mathcal{M}^+_{k,\beta}(\mathbb{R}^d)$, then the function $I_{k,\beta}[\mu]$ is D-superharmonic on $\mathbb{R}^d$.*

Proof: Let Φ_n the function defined by $\Phi_n(x) = \int_{B(0,n)} R_{k,\beta}(x,y)d\mu(y)$. From Theorem 4.1, the function Φ_n is D-superharmonic on $\mathbb{R}^d$. Thus, as $I_{k,\beta}[\mu]$ is not identically $+\infty$ by hypothesis, the function $I_{k,\beta}[\mu]$ is D-superharmonic on $\mathbb{R}^d$ as being an increasing pointwise limit of the sequence $(\Phi_n)_n$ of D-superharmonic functions (see [10], Proposition 3.3). □

Proposition 4.5 *Let $\mu \in \mathcal{M}^+_{k,\beta}(\mathbb{R}^d)$ with $\beta \in [2, d+2\gamma[$ and $m \in \mathbb{N}$ be such that $1 \leq m \leq \beta/2$. Then, the function $x \mapsto I_{k,\beta}[\mu](x)\omega_k(x)$ satisfies*

$$(-\Delta_k)^m \left(I_{k,\beta}[\mu]\omega_k\right) = \begin{cases} I_{k,\beta-2m}[\mu]\omega_k & in \quad \mathcal{D}'(\mathbb{R}^d) \quad if \quad \beta > 2m, \\ \\ \mu & in \quad \mathcal{D}'(\mathbb{R}^d) \quad if \quad \beta = 2m, \end{cases} . \tag{4.13}$$

Proof: Let $\phi \in \mathcal{D}(\mathbb{R}^d)$. We will only prove the result in the case $\beta > 2m$ and by the same arguments it can be obtained when $\beta = 2m$. We have

$$\begin{aligned} \left\langle (-\Delta_k)^m \left(I_{k,\beta}[\mu]\omega_k\right), \phi \right\rangle &= \int_{\mathbb{R}^d} \Big(\int_{\mathbb{R}^d} R_{k,\beta}(x,y)(-\Delta_k)^m \phi(x)\omega_k(x)dx \Big) d\mu(y) \\ &= \int_{\mathbb{R}^d} \Big(\int_{\mathbb{R}^d} R_{k,\beta-2m}(x,y)\phi(x)\omega_k(x)dx \Big) d\mu(y) \\ &= \int_{\mathbb{R}^d} I_{k,\beta-2m}[\mu](x)\phi(x)\omega_k(x)dx, \end{aligned}$$

where we have used
-Fubini's theorem in the first and the last lines (it is possible because $I_{k,\beta}[\mu] \in L^1_{k,loc}(\mathbb{R}^d)$ and by Remark 4.1, $I_{k,\beta-2m}[\mu]$ is also in $L^1_{k,loc}(\mathbb{R}^d)$);
- the fact that the Δ_k-Riesz kernel is symmetric and the relation (3.28) in the second line.
□

From the previous proposition, we obtain immediately the uniqueness principle for Δ_k-Riesz potential of index $2m$:

Corollary 4.2 *Let $m \in]0, \frac{d}{2} + \gamma[$ be an integer and $\mu, \nu \in \mathcal{M}^+_{k,2m}(\mathbb{R}^d)$. If $I_{k,2m}[\mu] = I_{k,2m}[\nu]$ a.e, then $\mu = \nu$.*

For an arbitrary index $\beta \in]0, d + 2\gamma[$, we have the following version of the uniqueness principle for finite measures:

Theorem 4.2 *Let $\beta \in]0, d+2\gamma[$ and let μ, ν be two finite and nonnegative Radon measures on $\mathbb{R}^d$. If $I_{k,\beta}[\mu] = I_{k,\beta}[\nu]$ a.e. on $\mathbb{R}^d$, then $\mu = \nu$.*

We start by the following result

Lemma 4.1 *Let μ be a finite and nonnegative Radon measure on $\mathbb{R}^d$. Then, $I_{k,\beta}[\mu]\omega_k$ is a tempered distribution and its distributional Dunkl transform is given by*

$$\mathcal{F}_k\left(I_{k,\beta}[\mu]\omega_k\right) = \|.\|^{-\beta}\mathcal{F}_k(\mu)\omega_k \quad in \quad \mathcal{S}'(\mathbb{R}^d). \tag{4.14}$$

Here, $\mathcal{F}_k(\mu)$ is the function defined by (2.5).

Proof: Let $m > d+2\gamma$ an integer and C_m as in (3.26). By Fubini's theorem, the symmetric property of the Δ_k-Riesz kernel and the relation (3.26), we get

$$\begin{aligned}\int_{\mathbb{R}^d}(1+\|x\|^2)^{-m} I_{k,\beta}[\mu](x)\omega_k(x)dx &= \int_{\mathbb{R}^d}\Big(\int_{\mathbb{R}^d}(1+\|x\|^2)^{-m} R_{k,\beta}(x,y)\omega_k(x)dx\Big)d\mu(y)\\ &\leq C_m\mu(\mathbb{R}^d) < +\infty.\end{aligned}$$

This shows that $I_{k,\beta}[\mu]\omega_k \in \mathcal{S}'(\mathbb{R}^d)$.
Let $\phi \in \mathcal{S}(\mathbb{R}^d)$. We have

$$\begin{aligned}\left\langle \mathcal{F}_k\left(I_{k,\beta}[\mu]\omega_k\right), \phi\right\rangle &= \int_{\mathbb{R}^d}\Big(\int_{\mathbb{R}^d} R_{k,\beta}(x,y)d\mu(y)\Big)\mathcal{F}_k(\phi)(x)\omega_k(x)dx\\ &= \int_{\mathbb{R}^d}\Big(\int_{\mathbb{R}^d} R_{k,\beta}(x,y)\mathcal{F}_k(\phi)(x)\omega_k(x)dx\Big)\mu(y)\\ &= \int_{\mathbb{R}^d}\Big(\int_{\mathbb{R}^d} E_k(-iy,x)\|x\|^{-\beta}\omega_k(x)\phi(x)dx\Big)d\mu(y)\\ &= \int_{\mathbb{R}^d}\|x\|^{-\beta}\mathcal{F}_k(\mu)(x)\omega_k(x)\phi(x)dx,\end{aligned}$$

where we have used
-Fubini's theorem in the first second line: it is possible because $\mathcal{F}_k(\phi) \in \mathcal{S}(\mathbb{R}^d)$ and then

the function $x \mapsto (1+\|x\|^2)^m \mathcal{F}_k(\phi)(x)$ is bounded with m the integer chosen as above;
-the relations (3.25) and $R_{k,\beta}(x,y) = R_{k,\beta}(y,x)$ in the third line;
-the boundedness of the function $(x,y) \mapsto E_k(iy,x)$ (see (2.2)), Fubini's theorem and (2.5) in the last line. □

Proof of Theorem 4.2: By our hypothesis and Lemma 4.1, we have $I_{k,\beta}[\mu] = I_{k,\beta}[\nu]$ in $\mathcal{S}'(\mathbb{R}^d)$. Applying Dunkl transform to the both terms and using the relation (4.14), we deduce that

$$\|.\|^{-\beta}\mathcal{F}_k(\mu)\omega_k = \|.\|^{-\beta}\mathcal{F}_k(\nu)\omega_k \quad \text{in} \quad \mathcal{S}'(\mathbb{R}^d).$$

As the functions $\|.\|^{-\beta}\mathcal{F}_k(\mu)\omega_k$ and $\|.\|^{-\beta}\mathcal{F}_k(\nu)\omega_k$ are locally integrable on $\mathbb{R}^d$, we get

$$\|.\|^{-\beta}\mathcal{F}_k(\mu)\omega_k = \|.\|^{-\beta}\mathcal{F}_k(\nu)\omega_k \quad \text{a.e. on} \quad \mathbb{R}^d.$$

Now, by continuity it follows that the functions $\mathcal{F}_k(\mu)$ and $\mathcal{F}_k(\nu)$ coincide everywhere on $\mathbb{R}^d$. Finally, by the injectivity of the Dunkl transform on the space of finite Radon measures on $\mathbb{R}^d$, we conclude that $\mu = \nu$. □

In order to extend the pointwise Hedbreg inequality in Dunkl setting, in the following result we give the link between the Δ_k-Riesz potential and the volume mean of a nonnegative Radon measure.

Proposition 4.6 *Let μ be a nonnegative Radon measure on $\mathbb{R}^d$. Then, for all $\beta \in]0, d+2\gamma[$, we have*

$$I_{k,\beta}[\mu](x) = \frac{d_k \kappa}{(d+2\gamma)(d+2\gamma-\beta)} \int_0^{+\infty} t^\beta M_B^t(\mu)(x) \frac{dt}{t}, \tag{4.15}$$

where

$$M_B^t(\mu)(x) := \frac{1}{m_k[B(0,t)]} \int_{\mathbb{R}^d} h_k(t,x,y) d\mu(y). \tag{4.16}$$

Proof: The result follows from (3.3), Fubini's theorem, (2.18) and (4.16). □

In the following result, we will extend the pointwise Hedberg inequality (see [13]). We recall that the Dunkl-Hardy-Littlewood maximal operator is defined for $f \in L^1_{k,loc}(\mathbb{R}^d)$ by (see [28])

$$M_k(f)(x) = \sup_{r>0} \frac{1}{m_k[B(0,r)]} \int_{\mathbb{R}^d} |f(y)| \tau_{-x}(\mathbf{1}_{B(0,r)})(y) \omega_k(y) dy, \tag{4.17}$$

where $\tau_{-x}(\mathbf{1}_{B(0,r)})$ denotes the $L^2_k(\mathbb{R}^d)$-function with Dunkl transform

$$\xi \mapsto E_k(-ix,\xi)\mathcal{F}_k\left(\mathbf{1}_{B(0,r)}\right)(\xi).$$

According to [20], we have $h_k(r,x,.) = \tau_{-x}(\mathbf{1}_{B(0,r)})$ a.e. on $\mathbb{R}^d$. Thus, we will take this remark into account in the formula (4.17) and in the sequel of the paper.

Moreover, when $d\mu(y) = |f(y)|\omega_k(y)dy$, $f \in L^1_{k,loc}(\mathbb{R}^d)$, we will use the notation $I_{k,\beta}[|f|]$ instead of $I_{k,\beta}[|f(y)|\omega_k(y)dy]$.

Theorem 4.3 *For $0 < \beta < d+2\gamma$, $1 \leq p < \frac{d+2\gamma}{\beta}$, there exists constants $C = C(d,\gamma,\beta,p) > 0$ such that for any measurable function f and any $x \in \mathbb{R}^d$, we have*

$$I_{k,\beta}[|f|](x) \leq C\|f\|_{k,p}^{\frac{\beta p}{d+2\gamma}} \left(M_k(f)(x)\right)^{1-\frac{\beta p}{d+2\gamma}}, \tag{4.18}$$

Proof: For every $A > 0$, by (4.15) where we take $d\mu(y) = |f(y)|\omega_k(y)dy$, we can write

$$\begin{aligned} I_{k,\beta}[|f|](x) = I_{k,\beta}[|f|\omega_k](x) &= C\int_0^A t^{\beta-1} M_B^t(|f|)(x)dt + C\int_A^{+\infty} t^{\beta-1} M_B^t(|f|)(x)dt \\ &:= I_1(x) + I_2(x). \end{aligned}$$

• Clearly, we see that

$$I_1(x) \leq CA^{\beta} M_k(f)(x). \tag{4.19}$$

• We have

$$\begin{aligned} I_2(x) &= C\sum_{n=0}^{+\infty} \int_{2^n A}^{2^{n+1}A} t^{\beta-d-2\gamma-1} \int_{\mathbb{R}^d} |f(y)| h_k(t,x,y)\omega_k(y)dydt \\ &\leq C\|f\|_{k,p} \sum_{n=0}^{+\infty} \int_{2^n A}^{2^{n+1}A} t^{\beta-d-2\gamma-1} t^{d+2\gamma(1-1/p)} dt \\ &\leq C\|f\|_{k,p} \sum_{n=0}^{+\infty} (2^n A)^{\beta-\frac{d+2\gamma}{p}}, \end{aligned}$$

where we have used Hölder's inequality and the relation (2.18) in the second line. Therefore, we have

$$I_2(x) \leq CA^{\beta-\frac{d+2\gamma}{p}} \|f\|_{k,p}. \tag{4.20}$$

Now, using (4.19), (4.20) and choosing

$$A = A(x) = \left(\frac{\|f\|_{k,p}}{M_k(f)(x)+\varepsilon}\right)^{\frac{p}{d+2\gamma}},$$

we obtain

$$I_{k,\beta}[|f|](x) \leq C\|f\|_{k,p}^{\frac{\beta p}{d+2\gamma}} \left(M_k(f)(x)+\varepsilon\right)^{1-\frac{\beta p}{d+2\gamma}}.$$

Letting $\varepsilon \longrightarrow 0$, we get (4.18). □

Using the Hedberg inequality (4.18), the L_k^p-boundedness properties of the Dunkl-Hardy-Littlewood maximal function (see [4] or [28]) and following the same proof as in the classical case (see Theorem 3.1.4 in [1]), we obtain the Sobolev inequality:

Corollary 4.3 *Let $0 < \beta < d+2\gamma$, $1 \leq p < \frac{d+2\gamma}{\beta}$ and $p^* = \frac{p(d+2\gamma)}{d+2\gamma-\beta p}$.*

1) *If $p = 1$, then $I_{k,\beta}$ is of weak type $(1,p^*)$ i.e. there exists a constant $C = C(\beta,d,\gamma)$ such that*

$$\forall\ \lambda > 0, \quad \forall\ f \in L_k^1(\mathbb{R}^d), \quad \int_{\{x:\ I_{k,\beta}[|f|]>\lambda\}} \omega_k(x)dx \leq C\left(\frac{\|f\|_{k,1}}{\lambda}\right)^{p^*}. \tag{4.21}$$

1) *If $p > 1$, then $I_{k,\beta}$ is of strong type (p,p^*) i.e. $I_{k,\beta} : L_k^p(\mathbb{R}^d) \longrightarrow L_k^{p^*}(\mathbb{R}^d)$ is bounded.*

Remark 4.2 *The previous result has been obtain in [11] by another proof using interpolation methods and in the particular case when the Coxeter-Weyl group is* $\mathbb{Z}_2^d$ *in [29].*

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