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Israël Chedjou Tankam, Plaire Tchinda Mouofo, Jean Jules Tewa

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Israël Tankam[a, d, *]- M. P. Tchinda[b, d]- JJ. Tewa[c, d]

a,* Department of Mathematics, University of Yaoundé I, PO Box 812 Yaoundé, Cameroon, israeltankam@gmail.com, Corresponding author, Tel.+(237) 698 74 58 64
b Department of Mathematics, University of Yaoundé I, PO Box 812 Yaoundé, Cameroon, tchindaplaire@yahoo.fr
c National Advanced School of Engineering University of Yaoundé I, Department of Mathematics and Physics P.O. Box 8390 Yaoundé, Cameroon, tewajules@gmail.com
d UMI 209 IRD/UPMC UMMISCO, University of Yaoundé I, Faculty of Science, CETIC Project team GRIMCAPE, University of Yaoundé I, Faculty of Science P.O. Box 812, Yaoundé, Cameroon

ABSTRACT. In this paper we study the properties of the Hopf bifurcations obtained for a delayed predator-prey model with continuous prey harvesting. Considering delay as a parameter, we investigate the effect of delay on the stability of the coexisting equilibrium. It is observed that there are stability switches, and Hopf bifurcation occurs when the delay crosses some critical value τ_0. By applying the normal form theory and the center manifold theorem, the formulae that determine the stability and direction of the bifurcating periodic solutions are established. Numerical simulations are carried out to illustrate different analytical findings. For the considered parameters values, results indicate that the Hopf bifurcation is supercritical and the bifurcating periodic solution is stable.

RÉSUMÉ. Dans cet article nous faisons une étude sur les propriétés de la bifurcation de Hopf dans un modèle proie-prédateur retardé prenant en compte une récolte continue. Considérant le retard comme un paramètre de bifurcation, nous analysons l'effet de ce dernier sur la stabilité de l'équilibre de coexistence. Compte tenu du changement de stabilité et de l'apparition d'une bifurcation de Hopf lorsque le retard traverse une valeur critique τ_0, nous appliquons la théorie des formes normales et le théorème de la variété centrale pour étudier la stabilité et la direction des trajectoires périodiques. Des simulations numériques sont faites afin de confirmer les résultats analytiques obtenus. Pour des valeurs des paramètres considérés, les résultats indiquent que la bifurcation de Hopf est supercritique et la solution périodique qui en résulte est stable.

KEYWORDS : Delay; predator-prey; Hopf bifurcation; local bifurcations.

MOTS-CLÉS : Retard; prédateur-proie; bifurcation de Hopf; bifurcations locales.

1. Introduction

The combined effects of harvesting and time delay on the dynamics of predator-prey models have a strong impact on the dynamic evolution of populations subjected to them. Therefore, many works have been investigated by many authors taking in account time delay. We refer to [3, 6, 10, 12] for general delay biological systems and to [1, 4, 5, 13, 14, 16] for studies on delay predator-prey system. In general, delay differential equations exhibit much more complicated dynamics than ordinary differential equations since a time delay could cause a stable equilibrium to become unstable and cause the populations to fluctuate.

On the other hand, there are many work on predator-prey models which consider harvesting [8, 9, 11, 17]. The problem is that these models consider either constant or linear harvesting functions which easily leads to the extinction of the populations.

In this paper, we consider a system of delayed differential equations modelling the predator-prey dynamic with a continuous double threshold harvesting and a Holling response function of type III. Recently, Tankam & al. [15] considered the following model:

$$\begin{cases} \dot{x}(t) &= \varphi(x(t)) - my(t)p(x(t)) - H(x(t)), \\ \dot{y}(t) &= [-d + cmp(x(t-\tau))]y(t). \end{cases} \tag{1}$$

where x and y represent the population of preys and predators respectively. d is the natural mortality rate of the predators. c and m are positive constants. Here, there is a time delay τ in the predator response term $p(x(t))$ in the predator equation. This delay is introduced in order to take into account the gestation period or reaction time of the predators. The function

$$\varphi(x) = rx\left(1 - \frac{x}{K}\right) \ , \tag{2}$$

models the dynamics of prey in absence of predators, r is the growth rate of prey for small values of x, while K is the capacity of the environment to support the prey. The function $p(x)$ is the Holling response function of type III given by:

$$p(x) = \frac{x^2}{a\,x^2 + b\,x + 1}, \tag{3}$$

(where $a > 0$ is constant and b is nonnegative constant). This function reflects very small predation when the amount of prey is small ($p'(0) = 0$), and a group of advantages for preys when the amount of prey is high ($p(x)$ tends to $\frac{1}{a}$ when x tends to infinity).

$H(x)$ is the threshold harvesting function given by:

$$H(x) = \begin{cases} 0 & \text{if } x < T_1, \\ \dfrac{h(x - T_1)}{T_2 - T_1} & \text{if } T_1 \leq x \leq T_2, \\ h & \text{if } x \geq T_2, \end{cases} \tag{4}$$

For this harvesting function, T_1 and T_2 are the threshold values. In this way, once the prey population reaches the size $x = T_1$, then harvesting starts and increases linearly to a limit value h that will be the constant value of the harvest from the threshold T_2. This piecewise linear operator policy harvesting has been introduced in [2] in a predator-prey model without delay, where a Holling response function of type II was considered.

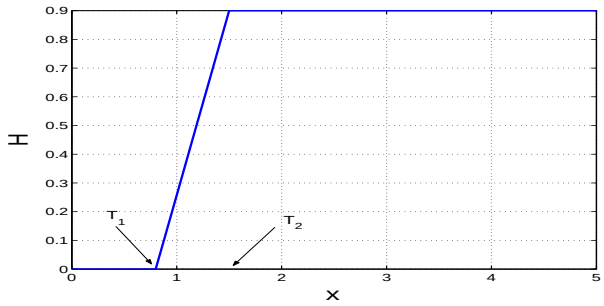

Figure 1. *Harvesting function ($T_1 = 0.8$, $T_2 = 1.5$ and $h = 0.9$) given by Eq.(4).*

Tankam et al. [15] have proven that a Hopf bifurcation occurs in the equation (1) considering the harvesting function (4) but the properties of this bifurcation have not been established. Yet, analysing properties of this bifurcation ensures us whether the bifurcating branch of periodic solution exists locally when the delay crosses a critical value.

The principal aim of this paper is to study the properties of the Hopf bifurcation obtained by Tankam et al. [15] and stability of bifurcated periodic solutions occurring through the Hopf bifurcation. The paper is organized as follows. Section 2 summarizes the results on equilibria and their stability. In Section 3 we determine the stability and direction of the bifurcating periodic solutions. Numerical results to illustrate the analytical findings are presented in Section 4 and, finally, a summary is presented in Section 5.

2. Results on equilibria and stability [15]

Let $x_0 = \dfrac{K[r(T_2 - T_1) - h]}{2r(T_2 - T_1)}$ and $d_1 = \dfrac{d}{cm}$. The following table summarizes the equilibria of the System 1.

Table 1 : Existence conditions of equilibrium points.

Number of equilibria	Equilibria	Existence conditions	Domain
1	$\tilde{F}_1(\tilde{x}_1, 0)$	$\varphi(T_2) > h$	$\tilde{x}_1 \in]T_2, K]$, $\tilde{x}_1 > \frac{K}{2}$
1	$\tilde{F}(T_2, 0)$	$\varphi(T_2) = h, T_2 \geq \frac{K}{2}$	$T_2 \in [\frac{K}{2}, K]$
1	$\bar{E}_1(\bar{x}_1, 0)$	$\varphi(T_2) < h, T_2 \geq \frac{K}{2}$	$\bar{x}_1 \in]T_1, T_2[$
1	$\bar{E}_2(\bar{x}_2, 0)$	$\varphi(T_2) < h, \varphi(\frac{K}{2}) < h$ and $x_0 \leq T_2 < \frac{K}{2}$	$\bar{x}_2 \in]T_1, T_2[$
2	$\tilde{\tilde{F}}(T_2, 0)$, $\tilde{F}_3(\tilde{x}_3, 0)$	$\varphi(T_2) = h$, $x_0 \leq T_2 < \frac{K}{2}$	$\tilde{x}_3 \in]T_2, K]$
2	$\bar{E}_3(\bar{x}_3, 0)$ and $\tilde{F}_4(\frac{K}{2}, 0)$	$\varphi(T_2) < h, \varphi(\frac{K}{2}) = h$ and $x_0 \leq T_2 < \frac{K}{2}$	$\bar{x}_3 \in]T_1, T_2[$
3	$\bar{E}_4(\bar{x}_4, 0)$, $\tilde{F}_5(\tilde{x}_5, 0)$, $\tilde{F}_6(\tilde{x}_6, 0)$	$\varphi(T_2) < h, \varphi(\frac{K}{2}) > h$ and $x_0 \leq T_2 < \frac{K}{2}$	$\tilde{x}_5 \in]T_2, \frac{K}{2}[$, $\bar{x}_4 \in]T_1, T_2[$ $\tilde{x}_6 \in]\frac{K}{2}, K[$

Concerning the stability of equilibria, when $\tau = 0$ we have the following results.

Theorem 2.1 *Let us consider the model system (1) and (4).*

- **(a)** *Equilibria $\tilde{F}_5(\tilde{x}_5, 0)$ and $\tilde{\tilde{F}}(T_2, 0)$ are always unstable.*
- **(b)** *If $1-ad_1 \leq 0$, then equilibria $\tilde{F}_1(\tilde{x}_1, 0)$, $\tilde{F}(T_2, 0)$, $\tilde{F}_4(\frac{K}{2}, 0)$, and $\tilde{F}_6(\tilde{x}_6, 0)$ when they exist are locally asymptotically stable. When $1-ad_1 > 0$, these equilibria are unstable if the first coordinate belongs to $]x^\star, K[$ and stable if the first coordinate is less than $x^\star$.*
- **(c)** *The equilibrium $\tilde{E}_3(\bar{x}_3, 0)$ when it exists is stable if and only if $1 - ad_1 \leq 0$ and $\bar{x}_3 > \frac{K}{2}$ and unstable if not.*

Theorem 2.2 *If $1 - ad_1 \leq 0$ and $\bar{x}_i > x_0$, $i = 1, ..., 4$, the equilibria $\bar{E}_i(\bar{x}_i, 0)$, $i = 1, ..., 4$, are locally asymptotically stable. These equilibria are unstable if the condition is reversed.*

Theorem 2.3 *Let*

$$\Delta_E = [\varphi'(x^\star) - mp'(x^\star)y^\star - H'(x^\star)]^2 - 4mdp'(x^\star)y^\star.$$

The stability of equilibrium $E(x^\star, y^\star)$ for $\tau = 0$ is given as follows:

- **(a)** *If $\Delta_E \geq 0$ then E is a node. The node is stable if $\varphi'(x^\star) - mp'(x^\star)y^\star - H'(x^\star) < 0$ and unstable if the inequality is reversed.*
- **(b)** *If $\Delta_E < 0$ then E is a focus. The focus is stable if $\varphi'(x^\star) - mp'(x^\star)y^\star - H'(x^\star) < 0$ and unstable if the inequality is reversed.*
- **(c)** *If $\varphi'(x^\star) - mp'(x^\star)y^\star - H'(x^\star) = 0$ then E is center-type.*

Concerning the stablility of equilibria, when $\tau > 0$ we have the following theorem.

Theorem 2.4 *Suppose that a positive equilibrium E exists and is locally asymptotically stable for (1) with $\tau = 0$. Also let $\eta_0 = w_0^2$ be a positive root of $\eta^2 + \Big[\varphi'(x^\star) - H'(x^\star) - mp'(x^\star)y^\star\Big]^2 \eta - dmp'(x^\star)y^{\star 2} = 0$. Then there exists a $\tau = \tau^0$ such that E is locally asymptotically stable for $\tau \in (0, \tau^0]$ and unstable for $\tau > \tau^0$. Furthermore, the system undergoes a Hopf bifurcation at E when $\tau = \tau^0$, where $\tau^0 = \frac{1}{w_0} \arccos\left(\frac{w_0^2}{dmp'(x^\star)y^\star}\right)$.*

The next section deals with the properties of the Hopf Bifurcation.

3. Properties of Hopf Bifurcation

In this section, we analyse the properties of the Hopf bifurcation using normal forms theory as in Hassard et al.[7]. The main result is given in Theorem 3.1 after having been proved by pre-calculations.
Considering the equations (1) and $x_1(t) = x(t) - x^\star$ and $x_2(t) = y(t) - y^\star$; then system (1) is equivalent to the following two-dimensional system:

$$\begin{cases} \dot{x}_1(t) &= \Big[\varphi'(x^\star) - my^\star p'(x^\star) - H'(x^\star)\Big]\, x_1(t) - mp(x^\star)\, x_2(t) + f_1\big(x_1(t), x_2(t)\big), \\ \dot{x}_2(t) &= cmy^\star p'(x^\star)\, x_1(t-\tau) + f_2\big(x_1(t), x_2(t), x_1(t-\tau)\big). \end{cases} \tag{5}$$

where

$$\begin{aligned} f_1\big(x_1(t), x_2(t)\big) &= \varphi\big(x_1(t)+x^\star\big) - m(x_2(t)+y^\star)p\big(x_1(t)+x^\star\big) - H\big(x_1+x^\star\big) \\ &- \Big[\varphi'(x^\star) - my^\star(t)p'(x^\star) - H'(x^\star)\Big]\, x_1(t) + mp(x^\star)\, x_2(t) \end{aligned}$$

and

$$\begin{aligned} f_2\big(x_1(t), x_2(t), x_1(t-\tau)\big) &= \Big[-d + cmp\big(x_1(t-\tau)+x^\star\big)\Big]\big(x_2(t)+y^\star\big) \\ &- y^\star cmp\big(x^\star\big)x_1(t-\tau) \end{aligned}$$

let $\tau = \tau_j^0 + \mu$; then $\mu = 0$ is the Hopf bifurcation value of system (1) at the positive equilibrium $G(x^\star, y^\star)$. Since system (1) is equivalent to system (5), in the following discussion we shall consider mainly system (5).

In system (5), let $\bar{x}_k(t) = x_k(\tau t)$ and drop the bars for simplicity of notation. Then system (5) can be rewritten as a system of RFDEs in $\mathcal{C}\big([-1,0], \mathbb{R}^2\big)$ of the form:

$$\begin{cases} \dot{x}_1(t) &= (\tau_j^0+\mu)\Big[\varphi'(x^\star) - my^\star p'(x^\star) - H'(x^\star)\Big]\, x_1(t) - (\tau_j^0+\mu)mp(x^\star)\, x_2(t) \\ &+ (\tau_j^0+\mu) f_1\big(x_1(t), x_2(t)\big), \\ \dot{x}_2(t) &= (\tau_j^0+\mu)cmy^\star p'(x^\star)\, x_1(t-\tau) + (\tau_j^0+\mu) f_2\big(x_1(t), x_2(t), x_1(t-\tau)\big). \end{cases} \tag{6}$$

Let us consider the following lemma proved in Annex 1.

Lemma 1 *The system [6] is equivalent to*

$$\dot{x}(t) = A(\mu)x_t + R(\mu)x_t, \tag{7}$$

where $A(\mu)$ is linear. Besides, there exists an inner product $< \cdot, \cdot >$ and eigenvectors $q(\theta)$ and $q^\star(s)$ respectively of $A(0)$ and $A^\star$ such as $< q^\star(s), q(\theta) >= 1$, where $A^\star$ is the associate operator of A.

Using the same notations as in [7], we first compute the coordinates to describe the center manifold $\mathcal{C}_0$ at $\mu = 0$. Let x_t be the solution of Equation (5) when $\mu = 0$. Define

$$\begin{aligned} z(t) &= \langle q^* \ , \ x_t \rangle \\ W(t,\theta) &= x_t(\theta) - 2\mathcal{R}_e\Big(z(t)q(\theta)\Big) \\ &= x_t(\theta) - \Big(z(t)q(\theta) + \bar{z}(t)\bar{q}(\theta)\Big) \end{aligned} \tag{8}$$

On the center manifold $\mathcal{C}_0$ we have

$$W(t,\theta) = W(z, \bar{z}, \theta) \tag{9}$$

where

$$W(z,\bar{z},\theta) = W_{20}(\theta)\frac{z^2}{2} + W_{11}(\theta)z\bar{z} + W_{02}\frac{\bar{z}^2}{2} + W_{30}(\theta)\frac{z^3}{6} + \cdots \tag{10}$$

z and $\bar{z}$ are local coordinates for center manifold C_0 in the direction of q^* and $\bar{q}^*$. Note that W is real if x_t is real. We only consider real solutions. For solution $x_t \in \mathcal{C}_0$ of (5), since $\mu = 0$, we have

$$\dot{z}(t) = iw_0\tau_j^0 z + \bar{q}^*(0) f\Big(0, W(z,\bar{z},0) + 2\mathcal{R}_e\big(z(t)q(\theta)\big)\Big) \equiv iw_0\tau_j^0 z + \bar{q}^*(0) f_0(z,\bar{z})$$

We rewrite this equation as

$$\dot{z}(t) = iw_0\tau_j^0 z + g\big(z,\bar{z}\big) \tag{11}$$

where

$$g\big(z,\bar{z}\big) = g_{20}(\theta)\frac{z^2}{2} + g_{11}(\theta)z\bar{z} + g_{02}\frac{\bar{z}^2}{2} + g_{21}(\theta)\frac{z^2\bar{z}}{2} + \cdots \tag{12}$$

The following lemma gives the values of the coefficients of $g(z,\bar{z})$ and is proven in Annex 2.

Lemma 2

$$\begin{aligned}
g_{20} &= 2\tau_j^0\bar{D}\left[-\left(\tfrac{r}{K} + mp'(x^\star)\nu_1 + \tfrac{my^\star p''(x^\star)}{2}\right)\right.\\
&+ \left.\bar{\nu}_1\left(\tfrac{y^\star cmp''(x^\star)e^{-2iw_0\tau_j^0}}{2} + cmp'(x^\star)\nu_1 e^{-iw_0\tau_j^0}\right)\right]\\
g_{02} &= 2\tau_j^0\bar{D}\left[-\left(\tfrac{r}{K} + mp'(x^\star)\bar{\nu}_1 + \tfrac{my^\star p''(x^\star)}{2}\right)\right.\\
&+ \left.\bar{\nu}_1\left(\tfrac{y^\star cmp''(x^\star)e^{2iw_0\tau_j^0}}{2} + cmp'(x^\star)\bar{\nu}_1 e^{iw_0\tau_j^0}\right)\right]\\
g_{11} &= 2\tau_j^0\bar{D}\left[-\left(\tfrac{r}{K} + mp'(x^\star)\mathcal{R}_e\{\nu_1\} + \tfrac{my^\star p''(x^\star)}{2}\right)\right.\\
&+ \left.\bar{\nu}_1\left(\tfrac{y^\star cmp''(x^\star)}{2} + cmp'(x^\star)\mathcal{R}_e\left\{\nu_1 e^{iw_0\tau_j^0}\right\}\right)\right]\\
g_{21} &= \tau_j^0\bar{D}\left[-\tfrac{r}{K}\left(4W_{11}^{(1)}(0) + 2W_{20}^{(1)}(0)\right)\right.\\
&- mp'(x^\star)\left(2W_{11}^{(2)}(0) + W_{20}^{(2)}(0) + \bar{\nu}_1 W_{20}^{(1)}(0) + 2\nu_1 W_{11}^{(1)}(0)\right)\\
&- \tfrac{mp''(x^\star)}{2}(2\bar{\nu}_1 + 4\nu_1) - \tfrac{my^\star p''(x^\star)}{2}\left(4W_{11}^{(1)}(0) + 2W_{20}^{(1)}(0)\right)\\
&+ \bar{\nu}_1 my^\star p''(x^\star)\Big(2W_{11}^{(1)}(-1) + W_{20}^{(1)}(-1)e^{iw_0\tau_j^0}\Big)\\
&+ \bar{\nu}_1 cmp'(x^\star)\left(\bar{\nu}_1 W_{20}^{(1)}(-1) + W_{20}^{(2)}(0)e^{iw_0\tau_j^0} + 2W_{11}^{(2)}(0)e^{-iw_0\tau_j^0} + 2\nu_1 W_{11}^{(1)}(-1)\right)\\
&+ \left.\tfrac{cmp''(x^\star)}{2}\Big(4\nu_1 + 2\bar{\nu}_1 e^{-2iw_0\tau_j^0}\Big)\right]
\end{aligned} \tag{13}$$

Since there are $W_{20}(\theta)$ and $W_{11}(\theta)$ in g_{21}, we still need to compute them. From (36) (*cf Annex 1*) and (8), we have:

$$\begin{aligned}\dot{W} &= \dot{x}_t - \dot{z}q - \dot{\bar{z}}\bar{q} \\ &= \begin{cases} AW - 2\mathcal{R}_e\Big\{\bar{q}^*(0)f_0 q(\theta)\Big\}, & \theta \in [-1;0); \\ AW - 2\mathcal{R}_e\Big\{\bar{q}^*(0)f_0 q(\theta)\Big\} + f_0, & \theta = 0. \end{cases} \\ &\equiv^{\text{def}} AW + \mathcal{H}\big(z,\bar{z},\theta\big)\end{aligned} \tag{14}$$

where

$$\mathcal{H}\big(z,\bar{z},\theta\big) = \mathcal{H}_{20}(\theta)\frac{z^2}{2} + \mathcal{H}_{11}(\theta)z\bar{z} + \mathcal{H}_{02}(\theta)\frac{\bar{z}^2}{2} + \cdots \tag{15}$$

Substituting the corresponding series into (14) and comparing the coefficients, we obtain

$$\begin{aligned}(A - 2iw_0\tau_j^0)W_{20}(\theta) &= -\mathcal{H}_{20}(\theta) \\ AW_{11}(\theta) &= -\mathcal{H}_{11}(\theta)\end{aligned} \tag{16}$$

From (14), we know that for $\theta \in [-1, 0)$,

$$\mathcal{H}\big(z,\bar{z},\theta\big) = -\bar{q}^*(0)f_0 q(\theta) - q^*(0)\bar{f}_0\bar{q}(\theta) = -g(z,\bar{z})q(\theta) - \bar{g}(z,\bar{z})\bar{q}(\theta) \tag{17}$$

Comparing the coefficient with (15), we get:

$$-g_{20}q(\theta) - \bar{g}_{02}\bar{q}(\theta) = H_{20}(\theta) \tag{18}$$

$$-g_{11}q(\theta) - \bar{g}_{11}\bar{q}(\theta) = H_{11}(\theta) \tag{19}$$

From (16) and (18) and the definition of A, it follows that

$$\dot{W}(\theta) = 2iw_0\tau_j^0 W_{20} + g_{20}q(\theta) + \bar{g}_{02}\bar{q}(\theta) \tag{20}$$

Notice that $q(\theta) = \Big(1, \nu_1\Big)^T e^{iw_0\tau_j^0\theta}$. Hence,

$$W_{20}(\theta) = \frac{ig_{20}}{w_0\tau_j^0}q(0)e^{iw_0\tau_j^0\theta} + \frac{i\bar{g}_{02}}{3w_0\tau_j^0}\bar{q}(0)e^{-iw_0\tau_j^0\theta} + E_1 e^{2iw_0\tau_j^0\theta} \tag{21}$$

where $E_1 = \Big(E_1^{(1)}, E_1^{(2)}\Big) \in \mathbb{R}^2$ is a constant vector. Similarly, from (16) and (19), we obtain

$$W_{11}(\theta) = -\frac{ig_{11}}{w_0\tau_j^0}q(0)e^{iw_0\tau_j^0\theta} + \frac{i\bar{g}_{11}}{w_0\tau_j^0}\bar{q}(0)e^{-iw_0\tau_j^0\theta} + E_2 \tag{22}$$

where $E_2 = \Big(E_2^{(1)}, E_2^{(2)}\Big) \in \mathbb{R}^2$ is also a constant vector.

In what follows, we will seek appropriate E_1 and E_2. From the definition of A and (16), we obtain

$$\int_{-1}^{0} d\eta(\theta)W_{20}(\theta) = 2iw_0\tau_j W_{20}(0) - H_{20}(0) \tag{23}$$

$$\int_{-1}^{0} d\eta(\theta) W_{11}(\theta) = -H_{11}(0) \tag{24}$$

where $\eta(\theta) = \eta(0, \theta)$. By (14), we have

$$H_{20}(0) = -g_{20}q(0) - \bar{g}_{02}\bar{q}(0) + 2\tau_j^0 \begin{pmatrix} -\frac{r}{K} - mp'(x^\star)\nu_1 - \frac{my^\star p''(x^\star)}{2} \\ \frac{y^\star cmp''(x^\star)}{2} e^{-2iw_0\tau_j^0} + cmp'(x^\star)\nu_1 e^{-iw_0\tau_j^0} \end{pmatrix} \tag{25}$$

$$H_{11}(0) = -g_{11}q(0) - \bar{g}_{11}\bar{q}(0) + 2\tau_j^0 \begin{pmatrix} -\frac{r}{K} - mp'(x^\star)\mathcal{R}_e\{\nu_1\} - \frac{my^\star p''(x^\star)}{2} \\ \frac{y^\star cmp''(x^\star)}{2} + cmp'(x^\star)\mathcal{R}_e\left\{\nu_1 e^{iw_0\tau_j^0}\right\} \end{pmatrix} \tag{26}$$

Substituting (21) and (25) into (23) and noticing that

$$\begin{array}{rcl} \left(iw_0\tau_j^0 I - \int_{-1}^{0} e^{iw_0\tau_j^0\theta} d\eta(\theta)\right) q(0) & = & 0 \\ \left(-iw_0\tau_j^0 I - \int_{-1}^{0} e^{-iw_0\tau_j^0\theta} d\eta(\theta)\right) \bar{q}(0) & = & 0 \end{array} \tag{27}$$

we obtain

$$\left(2iw_0\tau_j^0 I - \int_{-1}^{0} e^{2iw_0\tau_j^0\theta} d\eta(\theta)\right) E_1 =$$

$$2\tau_j^0 \begin{pmatrix} -\frac{r}{K} - mp'(x^\star)\nu_1 - \frac{my^\star p''(x^\star)}{2} \\ \frac{y^\star cmp''(x^\star)}{2} e^{-2iw_0\tau_j^0} + cmp'(x^\star)\nu_1 e^{-iw_0\tau_j^0} \end{pmatrix}$$

This leads to

$$\begin{pmatrix} 2iw_0 - \varphi'(x^\star) + my^\star p'(x^\star) + H'(x^\star) & mp(x^\star) \\ y^\star cmp'(x^\star) e^{-2iw_0\tau_j^0} & 2iw_0 \end{pmatrix} E_1$$

$$= 2 \begin{pmatrix} -\frac{r}{K} - mp'(x^\star)\nu_1 - \frac{my^\star p''(x^\star)}{2} \\ \frac{y^\star cmp''(x^\star)}{2} e^{-2iw_0\tau_j^0} + cmp'(x^\star)\nu_1 e^{-iw_0\tau_j^0} \end{pmatrix}$$

Solving this system for E_1, we obtain

$$E_1^{(1)} = \frac{2}{\sigma} \begin{vmatrix} -\frac{r}{K} - mp'(x^\star)\nu_1 - \frac{my^\star p''(x^\star)}{2} & mp(x^\star) \\ \frac{y^\star cmp''(x^\star)}{2} e^{-2iw_0\tau_j^0} + cmp'(x^\star)\nu_1 e^{-iw_0\tau_j^0} & 2iw_0 \end{vmatrix}$$

$$E_1^{(2)} =$$

$$\frac{2}{\sigma} \begin{vmatrix} 2iw_0 - \varphi'(x^\star) + my^\star p'(x^\star) + H'(x^\star) & -\frac{r}{K} - mp'(x^\star)\nu_1 - \frac{my^\star p''(x^\star)}{2} \\ y^\star cmp'(x^\star) e^{-2iw_0\tau_j^0} & \frac{y^\star cmp''(x^\star)}{2} e^{-2iw_0\tau_j^0} + cmp'(x^\star)\nu_1 e^{-iw_0\tau_j^0} \end{vmatrix}$$

where

$$\sigma = \begin{vmatrix} 2iw_0 - \varphi'(x^\star) + my^\star p'(x^\star) + H'(x^\star) & mp(x^\star) \\ y^\star cmp'(x^\star)e^{-2iw_0\tau_j^0} & 2iw_0 \end{vmatrix}$$

Similarly, substituting (22) and (26) into (24), we get

$$\begin{pmatrix} \varphi'(x^\star) - my^\star p'(x^\star) - H'(x^\star) & -mp(x^\star) \\ -y^\star cmp'(x^\star) & 0 \end{pmatrix} E_2$$

$$= 2 \begin{pmatrix} -\frac{r}{K} - mp'(x^\star)\mathcal{R}_e\{\nu_1\} - \frac{my^\star p''(x^\star)}{2} \\ \frac{y^\star cmp''(x^\star)}{2} + cmp'(x^\star)\mathcal{R}_e\left\{\nu_1 e^{iw_0\tau_j^0}\right\} \end{pmatrix}$$

and hence

$$E_2^{(1)} = \frac{2}{\varrho} \begin{vmatrix} -\frac{r}{K} - mp'(x^\star)\mathcal{R}_e\{\nu_1\} - \frac{my^\star p''(x^\star)}{2} & -mp(x^\star) \\ \frac{y^\star cmp''(x^\star)}{2} + cmp'(x^\star)\mathcal{R}_e\left\{\nu_1 e^{iw_0\tau_j^0}\right\} & 0 \end{vmatrix}$$

$$E_2^{(2)} = \frac{2}{\varrho} \begin{vmatrix} \varphi'(x^\star) - my^\star p'(x^\star) - H'(x^\star) & -\frac{r}{K} - mp'(x^\star)\mathcal{R}_e\{\nu_1\} - \frac{my^\star p''(x^\star)}{2} \\ -y^\star cmp'(x^\star) & \frac{y^\star cmp''(x^\star)}{2} + cmp'(x^\star)\mathcal{R}_e\left\{\nu_1 e^{iw_0\tau_j^0}\right\} \end{vmatrix}$$

where

$$\varrho = \begin{vmatrix} \varphi'(x^\star) - my^\star p'(x^\star) - H'(x^\star) & -mp(x^\star) \\ -y^\star cmp'(x^\star) & 0 \end{vmatrix}$$

Thus, we can determine W_{20} and W_{11} from (21) and (22). Furthermore, g_{21} in (13) can be expressed by the parameters and delay. Thus, we can compute the following values:

$$\begin{aligned} C_1(0) &= \frac{i}{2w_0\tau_j^0}\left(g_{20}g_{11} - 2|g_{11}|^2 - \frac{|g_{02}|^2}{3}\right) + \frac{g_{21}}{2} \\ \nu_2 &= -\frac{\mathcal{R}_e\{C_1(0)\}}{\mathcal{R}_e\left\{\lambda'(\tau_j^0)\right\}} \\ \beta_2 &= 2\mathcal{R}_e\{C_1(0)\} \\ P_2 &= -\frac{\mathcal{I}_m\{C_1(0)\} + \nu_2\mathcal{I}_m\left\{\lambda'(\tau_j^0)\right\}}{w_0\tau_j^0} \end{aligned} \tag{28}$$

which determine the qualities of bifurcating periodic solution in the center manifold at the critical value τ_j^0.

Theorem 3.1 *: In Eq. (28), the sign of ν_2 determines the direction of the Hopf bifurcation. Thus, if $\nu_2 > 0$, then the Hopf bifurcation is supercritical and the bifurcating*

periodic solution exists for $\tau_1 > \tau_1^0$. If $\nu_2 < 0$, then the Hopf bifurcation is subcritical and the bifurcating periodic solution exists for $\tau_1 < \tau_1^0$. β_2 determines the stability of the bifurcating periodic solution: The bifurcating periodic solutions are stable if $\beta_2 < 0$ and unstable if $\beta_2 > 0$. P_2 determines the period of the bifurcating periodic solutions: the period increases if $P_2 > 0$ and decreases if $P_2 < 0$.

4. Numerical simulations

In this section, we give some numerical simulations for a special case of system (1) with harvesting function (4) to support our analytical results. As an example, we consider systems (1) and (4) with the coefficients $r = 1$, $K = 40$; $a = 0.1$, $b = 0.6$, $m = 0.1$, $d = 0.67$, $c = 1$, $h = 0.1 * K$, $T_1 = 10$, $T_2 = 25$. From the above discussion, we may determine the direction of Hopf bifurcation and the direction of bifurcating periodic solution. So we consider the following systems:

$$\begin{cases} \dot{x}(t) &= 1 * x(t)\left(1 - \dfrac{x(t)}{40}\right) - \dfrac{0.1\, y(t)\, x^2(t)}{0.1\, x^2(t) + 0.6\, x(t) + 1} - H(x(t)), \\ \dot{y}(t) &= -0.67\, y(t) + \dfrac{1 * 0.1 * \ y(t)\, x^2(t-\tau)}{0.1x^2(t-\tau) + 0.6\, x(t-\tau) + 1}, \end{cases} \tag{29}$$

which has a positive equilibrium $G(13.67, 7.59)$. When τ passes through the critical value $\tau = \tau_0 = 1.1807$ and $\frac{d(R_e\lambda(\tau))}{d\tau})\Big|_{\lambda=iw_0,\tau=\tau_0} = 0.0959$, the equilibrium G losses its stability and the system (1) with (4) experiences Hopf bifurcation. When $\tau \geq 0$, the system satisfies all conditions of the Hopf bifurcation given by [15]. Consequently, the coexistence equilibrium point G becomes asymptotically stable when $\tau = 0$.

Figure 2 shows the behaviour of the system (29) when $\tau = 1.4 > \tau_0$ and Figure3 shows the behaviour of the same system when $\tau = 1 < \tau_0 = 1.1807$. As in Theorem, the system (29) becomes conditionally stable around the coexistence equilibrium point G for $\tau \in [0, \tau_0)$ (see Fig. 3), and unstable for $\tau > \tau_0$ (see Figure 2). From the previous section, we can determine the nature of the stability and direction of the periodic solution bifurcating from the interior equilibrium at the critical point τ_0.

Using (28), we can compute $C_1(0) = -0.0014 - 0.0010i$, $\nu_2 = 0.0146 > 0$, $\beta_2 = -0.0028 > 0$ and $T_2 = 0.0106$. Since $\nu_2 > 0$ and $\beta_2 < 0$, the Hopf bifurcation is supercritical and the bifurcating periodic solutions exist when τ crosses τ_1^0 from left to right. Also, the bifurcating periodic solution is stable (as $\beta_2 < 0$) and its period increases with τ (as $P_2 > 0$).

5. Conclusion

Prey-predator harvesting model has undergone different developments in theoretical and practical applications in the field of biomathematics.So, in this paper, we have studied properties of the Hopf bifurcation of a delayed predator-prey system with Holling Type III response function and continuous harvesting function with two thresholds. In a previous work, sufficient conditions on the parameters for which the delay-induced system is asymptotically stable around the positive equilibrium for all values of the delay parameter had been obtained; and if the conditions were not satisfied, then there existed a critical

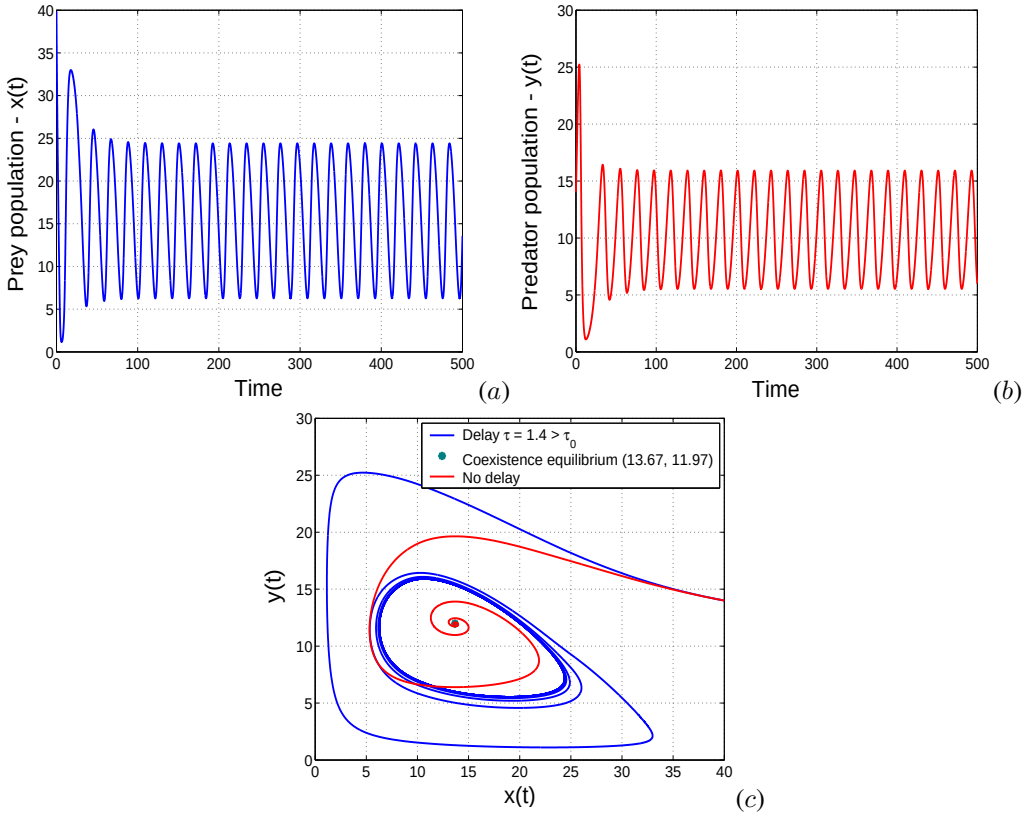

Figure 2. *When $\tau = 1.4 > \tau_0$, bifurcating periodic solutions from $G(13.67, 7.59)$ occur.*

value of the delay parameter below which the system is stable and above which the system is unstable. By applying the normal form theory and the center manifold theorem, we now determined the explicit formulae which determine the stability and direction of the bifurcating periodic solutions. Our analytical and simulation results show that when τ passes through the critical value τ_0, the coexisting equilibrium E losses its stability and a Hopf bifurcation occurs, that is, a family of periodic solutions bifurcate from E. Also, the amplitude of oscillations increases with increasing τ. For the considered parameter values, it is observed that the Hopf bifurcation is supercritical and the bifurcating periodic solution is stable.

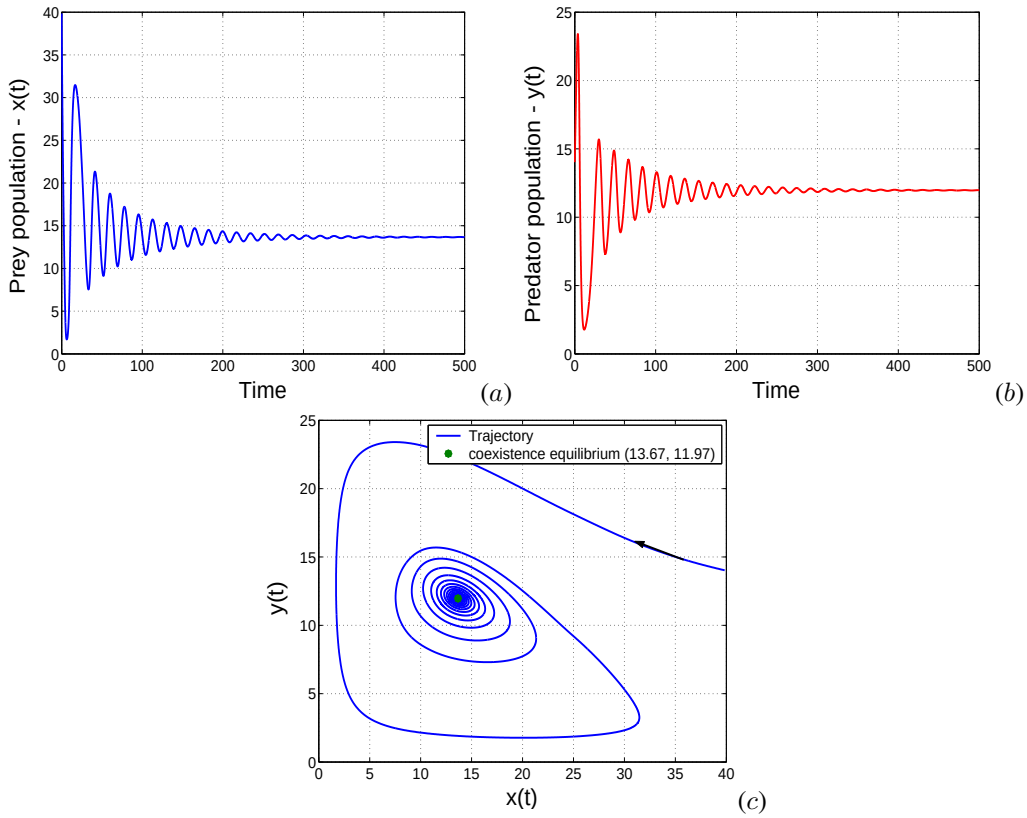

Figure 3. *The system (1) is locally asymptotically stable around the interior equilibrium G at $\tau = 1 < \tau_0 = 1.1807$. The other parameter values are given in the previous figure.*

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Annex 1: Proof of the Lemma 1

Define the linear operator $L(\mu) : \mathcal{C} \to \mathbb{R}^2$ and the nonlinear operator $f(\cdot, \mu) : \mathcal{C} \to \mathbb{R}^2$ by:

$$\begin{aligned} L_\mu(\phi) &= (\tau_j^0 + \mu) \begin{pmatrix} \varphi'(x^\star) - my^\star p'(x^\star) - H'(x^\star) & -mp(x^\star) \\ 0 & 0 \end{pmatrix} \begin{pmatrix} \phi_1(0) \\ \phi_2(0) \end{pmatrix} \\ &+ (\tau_j^0 + \mu) \begin{pmatrix} 0 & 0 \\ y^\star cmp'(x^\star) & 0 \end{pmatrix} \begin{pmatrix} \phi_1(-1) \\ \phi_2(-1) \end{pmatrix} \end{aligned} \tag{30}$$

and

$$f(\phi, \mu) = (\tau_j^0 + \mu) \begin{pmatrix} f_1\big(\phi_1(0), \phi_2(0)\big) \\ f_2\big(\phi_1(0), \phi_2(0), \phi_1(-1)\big) \end{pmatrix} \tag{31}$$

respectively, where $\phi = (\phi_1, \phi_2)^T \in \mathcal{C}$.

By the Riesz representation theorem, there exists a 2×2 matrix function $\eta(\theta, \mu)$, $-1 \leq \theta \leq 0$ whose elements are of bounded variation such that

$$L_\mu(\phi) = \int_{-1}^{0} d\eta(\theta, \mu)\phi(\theta) \quad \text{for} \quad \phi \in \mathcal{C}\big([-1, 0], \mathbb{R}^2\big). \tag{32}$$

In fact, we can choose

$$\begin{aligned} \eta(\theta, \mu) &= (\tau_j^0 + \mu) \begin{pmatrix} \varphi'(x^\star) - my^\star p'(x^\star) - H'(x^\star) & -mp(x^\star) \\ 0 & 0 \end{pmatrix} \delta\big(\theta\big) \\ &+ (\tau_j^0 + \mu) \begin{pmatrix} 0 & 0 \\ y^\star cmp'(x^\star) & 0 \end{pmatrix} \delta\big(\theta + 1\big) \end{aligned} \tag{33}$$

where δ is the Dirac delta function

For $\phi \in \mathcal{C}\big([-1, 0], \mathbb{R}^2\big)$, define

$$A(\mu)\phi = \begin{cases} \frac{d\phi(\theta)}{d\theta}, & \theta \in [-1, 0); \\ \int_1^0 d\eta(\mu, s)\phi(s), & \theta = 0. \end{cases} \tag{34}$$

and

$$R(\mu)\phi = \begin{cases} 0, & \theta \in [-1, 0); \\ f(\mu, \phi), & \theta = 0. \end{cases} \tag{35}$$

Then, the system (6) is equivalent to

$$\dot{x}(t) = A(\mu)x_t + R(\mu)x_t \tag{36}$$

where $x_t(\theta) = x(t + \theta)$, $\theta \in [-1, 0]$.

For $\psi \in \mathcal{C}^1\big([0, 1], \mathbb{R}^2\big)$, define

$$A^*\psi = \begin{cases} -\frac{d\psi(s)}{ds}, & s \in (0,1]; \\ \int_1^0 d\eta(t,0)\phi(-t), & s = 0. \end{cases} \tag{37}$$

and a bilinear inner product

$$\langle \psi(s), \phi(\theta) \rangle = \bar{\psi}(0)\phi(0) - \int_{-1}^{0}\int_{\xi=0}^{\theta} \bar{\psi}(\xi-\theta)d\eta(\theta)\phi(\xi)d\xi \tag{38}$$

where $\eta(\theta) = \eta(\theta, 0)$. In addition, by Theorem 2.4 we know that $\pm iw_0\tau_j^0$ are eigenvalues of $A(0)$. Thus, they are also eigenvalues of A^*. Let $q(\theta)$ be the eigenvector of $A(0)$ corresponding to $iw_0\tau_j^0$ and $q^*(s)$ be the eigenvector of A^* corresponding to $-iw_0\tau_j^0$.

Let $q(\theta) = \begin{pmatrix} 1, & \nu_1 \end{pmatrix}^T e^{iw_0\tau_j^0\theta}$ and $q^*(s) = D\begin{pmatrix} 1, & \nu_1^* \end{pmatrix}^T e^{iw_0\tau_j^0 s}$. From the above discussion, it is easy to know that $A(0)q(0) = iw_0\tau_j^0 q(0)$ and $A^*(0)q^*(0) = -iw_0\tau_j^0 q^*(0)$. That is

$$\tau_j^0 \begin{pmatrix} \varphi'(x^\star) - my^\star p'(x^\star) - H'(x^\star) & -mp(x^\star) \\ 0 & 0 \end{pmatrix} q(0)$$
$$+\tau_j^0 \begin{pmatrix} 0 & 0 \\ y^\star cmp'(x^\star) & 0 \end{pmatrix} q(-1) = iw_0\tau_j^0 q(0)$$

and

$$\tau_j^0 \begin{pmatrix} \varphi'(x^\star) - my^\star p'(x^\star) - H'(x^\star) & 0 \\ -mp(x^\star) & 0 \end{pmatrix} q^*(0)$$
$$+\tau_j^0 \begin{pmatrix} 0 & y^\star cmp'(x^\star) \\ 0 & 0 \end{pmatrix} q^*(-1) = -iw_0\tau_j^0 q^*(0)$$

Thus, we can easily obtain

$$q(\theta) = \begin{pmatrix} 1, & \dfrac{y^\star cmp'(x^\star)e^{-iw_0\tau_j^0}}{iw_0} \end{pmatrix}^T e^{iw_0\tau_j^0\theta} \tag{39}$$

$$q^*(s) = D\begin{pmatrix} 1, & \dfrac{mp(x^\star)}{iw_0} \end{pmatrix}^T e^{iw_0\tau_j^0 s} \tag{40}$$

In order to assure $\langle \bar{q}^*(s), q(\theta) \rangle = 1$, we need to determine the value of D. From (38), we have

$$\begin{aligned} \langle q^*(s), q(\theta) \rangle &= \bar{q}^*(0)q(0) - \textstyle\int_{-1}^{0}\int_{\xi=0}^{\theta} \bar{q}^*(\xi-\theta)d\eta(\theta)q(\xi)d\xi \\ &= \bar{q}^*(0)q(0) - \textstyle\int_{-1}^{0}\int_{\xi=0}^{\theta} \bar{D}\Big(1,\ \bar{\nu}_1^*\Big)e^{-iw_0\tau_j^0(\xi-\theta)}d\eta(\theta)\Big(1,\ \nu_1\Big)^T e^{iw_0\tau_j^0\xi}d\xi \\ &= \bar{q}^*(0)q(0) - \bar{q}^*(0)\textstyle\int_{-1}^{0}\theta e^{iw_0\tau_j^0\theta}d\eta(\theta)q(0) \\ &= \bar{q}^*(0)q(0) \\ &\quad -\bar{q}^*(0)\tau_j^0 \begin{pmatrix} \varphi'(x^\star) - my^\star p'(x^\star) - H'(x^\star) & -mp(x^\star) \\ 0 & 0 \end{pmatrix} \\ &\quad \Big(-e^{-iw_0\tau_j^0}\Big)q(0) \\ &= \bar{D}\Big[1 + \nu_1\bar{\nu}_1^* + \tau_j^0 e^{-iw_0\tau_j^0\bar{\nu}_1^*}y^\star cmp'(x^\star)\Big] \end{aligned}$$

So, we have

$$\begin{array}{rcl} \bar{D} & = & \frac{1}{1+\nu_1\bar{\nu}_1^*+\tau_j^0 e^{-iw_0\tau_j^0}\bar{\nu}_1^* y^\star cmp'(x^\star)} \\ D & = & \frac{1}{1+\bar{\nu}_1\nu_1^*+\tau_j^0 e^{iw_0\tau_j^0}\nu_1^* y^\star cmp'(x^\star)} \end{array} \tag{41}$$

That ends our proof.

Annex 2: Proof of the Lemma 2

We have $x_t(\theta) = \big(x_{1t}(\theta), x_{2t}(\theta)\big)$ and $q(\theta) = \big(1,\nu_1\big)^T e^{iw_0\tau_j^0\theta}$. So, from (8) and (10), it follows that

$$\begin{array}{rcl} x_t(\theta) & = & W(t,\theta) + 2\mathcal{R}_e\Big(z(t)q(\theta)\Big) \\ & = & W_{20}(\theta)\frac{z^2}{2} + W_{11}(\theta)z\bar{z} + W_{02}\frac{\bar{z}^2}{2} + \big(1,\nu_1\big)^T e^{iw_0\tau_j^0\theta}z(t) + \big(1,\bar{\nu}_1\big)^T e^{-iw_0\tau_j^0\theta}\bar{z}(t) + \cdots \end{array} \tag{42}$$

and then we have

$$\begin{array}{rcl} x_{1t}(0) & = & z + \bar{z} + W_{20}^{(1)}(0)\frac{z^2}{2} + W_{11}^{(1)}(0)z\bar{z} + W_{02}^{(1)}(0)\frac{\bar{z}^2}{2} + \cdots \\ x_{2t}(0) & = & \nu_1 z + \bar{\nu}_1\bar{z} + W_{20}^{(2)}(0)\frac{z^2}{2} + W_{11}^{(2)}(0)z\bar{z} + W_{02}^{(2)}(0)\frac{\bar{z}^2}{2} + \cdots \\ x_{1t}(-1) & = & ze^{-iw_0\tau_j^0} + \bar{z}e^{iw_0\tau_j^0} + W_{20}^{(1)}(-1)\frac{z^2}{2} + W_{11}^{(1)}(-1)z\bar{z} + W_{02}^{(1)}(-1)\frac{\bar{z}^2}{2} + \cdots \\ x_{2t}(-1) & = & \nu_1 ze^{-iw_0\tau_j^0} + \bar{\nu}_1\bar{z}e^{iw_0\tau_j^0} + W_{20}^{(2)}(-1)\frac{z^2}{2} + W_{11}^{(2)}(-1)z\bar{z} + W_{02}^{(2)}(-1)\frac{\bar{z}^2}{2} + \cdots \end{array} \tag{43}$$

It follows together with (31) that

$$\begin{array}{rcl} g(z,\bar{z}) & = & \frac{z^2}{2}\Big\{2\tau_j^0\bar{D}\Big[-\Big(\frac{r}{K} + mp'(x^\star)\nu_1 + \frac{my^\star p''(x^\star)}{2}\Big) \\ & + & \bar{\nu}_1\Big(\frac{y^\star cmp''(x^\star)e^{-2iw_0\tau_j^0}}{2} + cmp'(x^\star)\nu_1 e^{-iw_0\tau_j^0}\Big)\Big]\Big\} \\ & + & \frac{\bar{z}^2}{2}\Big\{2\tau_j^0\bar{D}\Big[-\Big(\frac{r}{K} + mp'(x^\star)\bar{\nu}_1 + \frac{my^\star p''(x^\star)}{2}\Big) \\ & + & \bar{\nu}_1\Big(\frac{y^\star cmp''(x^\star)e^{2iw_0\tau_j^0}}{2} + cmp'(x^\star)\bar{\nu}_1 e^{iw_0\tau_j^0}\Big)\Big]\Big\} \\ & + & z\bar{z}\Big\{2\tau_j^0\bar{D}\Big[-\Big(\frac{r}{K} + mp'(x^\star)\mathcal{R}_e\{\nu_1\} + \frac{my^\star p''(x^\star)}{2}\Big) \\ & + & \bar{\nu}_1\Big(\frac{y^\star cmp''(x^\star)}{2} + cmp'(x^\star)\mathcal{R}_e\Big\{\nu_1 e^{iw_0\tau_j^0}\Big\}\Big)\Big]\Big\} \\ & + & \frac{z^2\bar{z}}{2}\Big\{\tau_j^0\bar{D}\Big[-\frac{r}{K}\Big(4W_{11}^{(1)}(0) + 2W_{20}^{(1)}(0)\Big)\Big] \end{array}$$

$$
\begin{aligned}
- \quad & mp'(x^\star)\Big(2W_{11}^{(2)}(0) + W_{20}^{(2)}(0) + \bar{\nu}_1 W_{20}^{(1)}(0) + 2\nu_1 W_{11}^{(1)}(0)\Big) \\
- \quad & \tfrac{mp''(x^\star)}{2}(2\bar{\nu}_1 + 4\nu_1) - \tfrac{my^\star p''(x^\star)}{2}\Big(4W_{11}^{(1)}(0) + 2W_{20}^{(1)}(0)\Big) \\
+ \quad & \bar{\nu}_1 my^\star p''(x^\star)\Big(2W_{11}^{(1)}(-1) + W_{20}^{(1)}(-1)e^{iw_0\tau_j^0}\Big) \\
+ \quad & \bar{\nu}_1 cmp'(x^\star)\Big(\bar{\nu}_1 W_{20}^{(1)}(-1) + W_{20}^{(2)}(0)e^{iw_0\tau_j^0} \\
& + 2W_{11}^{(2)}(0)e^{-iw_0\tau_j^0} + 2\nu_1 W_{11}^{(1)}(-1)\Big) \\
+ \quad & \tfrac{cmp''(x^\star)}{2}\Big(4\nu_1 + 2\bar{\nu}_1 e^{-2iw_0\tau_j^0}\Big)\Big]\Big\}
\end{aligned}
$$

Where f and D are given in the proof of the lemma 1 respectively by (31) and (41).

Comparing the coefficients with (12), we obtain the coefficients of $g(z, \bar{z})$.
That ends our proof.