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► **To cite this version:**

Rémi Boutonnet, Ionut Chifan, Adrian Ioana. III1 Factors with non-isomorphic ultrapowers. Duke Mathematical Journal, 2017. hal-01450072

HAL Id: hal-01450072

<https://hal.science/hal-01450072>

Submitted on 31 Jan 2017

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II₁ FACTORS WITH NON-ISOMORPHIC ULTRAPOWERS

RÉMI BOUTONNET, IONUȚ CHIFAN, AND ADRIAN IOANA

ABSTRACT. We prove that there exist uncountably many separable II₁ factors whose ultrapowers (with respect to arbitrary ultrafilters) are non-isomorphic. In fact, we prove that the families of non-isomorphic II₁ factors originally introduced by McDuff [MD69a, MD69b] are such examples. This entails the existence of a continuum of non-elementarily equivalent II₁ factors, thus settling a well-known open problem in the continuous model theory of operator algebras.

1. INTRODUCTION

The ultrapower construction for II₁ factors, originally introduced in [Wr54, Sa62], first came to prominence following McDuff's work [MD69c]. It has since played a fundamental role in the study of von Neumann algebras. In particular, the analysis of ultrapowers of II₁ factors was a crucial ingredient in Connes' celebrated classification of injective factors [Co75]. In the same paper, ultrapowers were used by Connes to formulate his famous (still unsolved) *embedding problem*. More recently, ultrapower techniques have been instrumental in the advances made in the classification of II₁ factors by Popa's deformation/rigidity theory (see e.g. [Po04]). For more history on ultrapowers and ultraproducts of von Neumann algebras, see [AH13].

While ultrapowers of II₁ factors have been extremely useful in various applications, the following intrinsic problem remained open: how many ultrapowers (with respect to a fixed ultrafilter) of separable II₁ factors exist, up to isomorphism? Recently, a closely related problem has been considered in the emerging field of continuous model theory of operator algebras [FHS09, FHS10, FHS11]: how many elementary equivalence classes of II₁ factors exist? This problem has received a lot of attention (see e.g. [FHS11, GS14] and the survey [Fa14]). The connection between the two problems stems from the continuous version of the Keisler-Shelah theorem [Ke61, Sh71]. This asserts that two II₁ factors M and N are elementarily equivalent if and only if they have isomorphic ultrapowers, $M^{\mathcal{U}} \cong N^{\mathcal{V}}$, with respect to ultrafilters \mathcal{U} and \mathcal{V} on arbitrarily large sets.

At present, only three different elementary equivalence classes of II₁ factors appear in the literature. More precisely, it was noticed in [FGL06, FHS11] that, for separable II₁ factors, property Gamma and the property of being McDuff are elementary properties (i.e. they are remembered by ultrapowers). Thus, the hyperfinite II₁ factor R , the free group factor $L(\mathbb{F}_2)$, and any non-McDuff separable II₁ factor that has property Gamma (see [DL69]), are not elementarily equivalent.

In contrast, the existence of uncountably many non-isomorphic separable II₁ factors has been known for a long time [MD69b, Sa69]. This situation is partially explained by the fact that elementary equivalence of II₁ factors is a much coarser notion of equivalence than isomorphism. An illuminating explanation of this fact is provided by a result in [FHS11] which states that any II₁ factor is elementarily equivalent to uncountably many non-isomorphic II₁ factors.

R. B. was supported in part by NSF Career Grant DMS #1253402 and ANR Grant NEUMANN.

I. C. was supported in part by the NSF Grant DMS #1301370.

A.I. was supported in part by NSF Grant DMS #1161047, NSF Career Grant DMS #1253402 and a Sloan Foundation Fellowship.

In this paper we solve the above problems, by proving the existence of a continuum of separable II_1 factors whose ultrapowers, with respect to any ultrafilters, are non-isomorphic.

1.1. Construction and statement of the result. Our examples of II_1 factors with non-isomorphic ultrapowers come from McDuff's work [MD69a, MD69b]. The construction relies on two functors T_0 and T_1 , from the category of countable groups to itself, defined as follows.

Consider a countable group Γ . Let Γ_i , $i \geq 1$, be isomorphic copies of Γ , and Λ_i , $i \geq 1$, be isomorphic copies of \mathbb{Z} . We define $\tilde{\Gamma} = \bigoplus_{i \geq 1} \Gamma_i$ and denote by S_∞ the group of finite permutations of $\{1, 2, \dots\}$. We consider the semidirect product $\tilde{\Gamma} \rtimes S_\infty$ associated to the action of S_∞ on $\tilde{\Gamma}$ which permutes the copies of Γ . Following [MD69b],

- we define $T_0(\Gamma)$ as the group generated by $\tilde{\Gamma}$ and Λ_i , $i \geq 1$, with the only relations that Γ_i and Λ_j commute for every $i \geq j \geq 1$;
- we define $T_1(\Gamma)$ as the group generated by $\tilde{\Gamma} \rtimes S_\infty$ and Λ_i , $i \geq 1$, with the only relations that Γ_i and Λ_j commute for every $i \geq j \geq 1$.

The definition of T_0 is due to Dixmier and Lance in [DL69, §21], who were inspired by a construction in [MvN43].

The identification $\Gamma = \Gamma_1$ gives an embedding of Γ inside $T_\alpha(\Gamma)$, for $\alpha \in \{0, 1\}$. Moreover, every inclusion $\Sigma \subset \Sigma'$ of countable groups gives a canonical inclusion $T_\alpha(\Sigma) \subset T_\alpha(\Sigma')$. Hence, any sequence $\alpha = (\alpha_n)_{n \geq 1}$ of 0's and 1's, gives rise to a sequence of inclusions

$$\Gamma \subset T_{\alpha_1}(\Gamma) \subset T_{\alpha_1} \circ T_{\alpha_2}(\Gamma) \subset T_{\alpha_1} \circ T_{\alpha_2} \circ T_{\alpha_3}(\Gamma) \subset \dots$$

Definition 1.1. Given a sequence α of 0's and 1's we define

- $T_\alpha(\Gamma) := \Gamma$, if α is the empty sequence;
- $T_\alpha(\Gamma) := T_{\alpha_1} \circ T_{\alpha_2} \circ \dots \circ T_{\alpha_n}(\Gamma)$, if $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_n)$ is a finite sequence;
- $T_\alpha(\Gamma)$ is the inductive limit of the increasing sequence of groups $(K_{(\alpha_1, \dots, \alpha_n)}(\Gamma))_{n \geq 1}$, if $\alpha = (\alpha_n)_{n \geq 1}$ is an infinite sequence.

We denote by $M_\alpha(\Gamma) := L(T_\alpha(\Gamma))$ the associated von Neumann algebra.

The countable family of non-isomorphic II_1 factors constructed in [MD69a] is just $M_{\alpha_n}(\mathbb{F}_2)$, $n \geq 1$, where α_n denotes the finite 0-valued sequence of length n and \mathbb{F}_2 is the free group on two generators. The uncountable family $M_\alpha(\Gamma)$, indexed over infinite sequences α , is precisely the family of non-isomorphic II_1 factors constructed in [MD69b].

Theorem 1.2. *Consider a countable group Γ and two different sequences $\alpha \neq \beta$ of 0's and 1's. Assume that $\Gamma = \mathbb{F}_2$, or that the sequences α and β are infinite.*

Then $M_\alpha(\Gamma)^\mathcal{U}$ is not isomorphic to $M_\beta(\Gamma)^\mathcal{V}$, for any ultrafilters \mathcal{U} and \mathcal{V} .

We will actually deduce Theorem 1.2 from two results.

- Firstly, we prove that any ultrapower of $M_\alpha(\mathbb{F}_2)$ remembers the length of α . This is achieved by introducing a new invariant for von Neumann algebras, called *McDuff depth*, or *depth* for short, which quantifies property Gamma. We will show that if α is a sequence of 0's and 1's, then the depth of any ultrapower of $M_\alpha(\Gamma)$ is at least the length of α , with equality for $\Gamma = \mathbb{F}_2$. This first half can be found in Section 3; more precisely, see Theorem 3.10.

- Secondly, we show that two infinite different sequences give different ultrapowers. For this we generalize McDuff's *property V* [MD69b]. Using our notion of depth, we show that any ultrapowers of $M_{\alpha}(\Gamma)$ has property *V* at depth k if and only if $\alpha_k = 1$. Hence the sequence α is an invariant of $M_{\alpha}(\Gamma)$ and its ultrapowers. This part is done in Section 4; more precisely, see Theorem 4.2.

As a consequence of Theorem 1.2 we deduce the existence a continuous family of separable non-nuclear \mathcal{Z} -stable C^* -algebras with non-isomorphic ultrapowers. We thank Ilijas Farah for pointing this out to us. For any sequence α of 0's and 1's and any group Γ , define $A_{\alpha}(\Gamma) = C_r^*(T_{\alpha}(\Gamma)) \otimes \mathcal{Z}$, where \mathcal{Z} is the Jiang-Su algebra.

Corollary 1.3. *In the setting of Theorem 1.2, assume moreover that α and β are non-empty.*

Then the C^ -algebraic ultrapowers $A_{\alpha}(\Gamma)^{\mathcal{U}}$ and $A_{\beta}(\Gamma)^{\mathcal{V}}$ are not isomorphic, for any ultrafilters \mathcal{U} and \mathcal{V} .*

Note that the proof of Corollary 1.3 also implies that the reduced group C^* -algebras $C_r^*(T_{\alpha}(\Gamma))$ and $C_r^*(T_{\beta}(\Gamma))$ do not have isomorphic ultrapowers.

Throughout the article, we will use the above notations. In addition, we will often consider the following subgroups of $T_0(\Gamma)$ or $T_1(\Gamma)$:

$$\tilde{\Gamma}_n = \bigoplus_{i \geq n} \Gamma_i, \quad \text{and} \quad \tilde{\Gamma}_{n,n'} = \bigoplus_{n' > i \geq n} \Gamma_i, \quad \text{for every } n' > n \geq 1.$$

Acknowledgements. I.C. is very grateful to Isaac Goldbring for kindly showing to him that the results in [ZM69] can be used to produce a fourth elementary equivalence class of II₁ factors. A.I. would like to thank Sorin Popa and Thomas Sinclair for many stimulating discussions. We are also grateful to Ilijas Farah for various comments, and especially for pointing out Corollary 1.3 to us.

2. PRELIMINARIES

2.1. Terminology. Throughout this article we work with *tracial von Neumann algebras* (M, τ) , i.e. von Neumann algebras M endowed with a faithful normal trace $\tau : M \rightarrow \mathbb{C}$. We say that M is *separable* if it is separable with respect to the norm $\|x\|_2 = \tau(x^*x)^{1/2}$. We denote by $\mathcal{U}(M)$ the group of *unitaries* of M . If $n \geq 1$, then we denote by $M^{\otimes n}$ the tensor product von Neumann algebra $\bar{\otimes}_{i=1}^n M$. If $A, B \subset M$, then we denote $A' \cap B = \{x \in B \mid xy = yx, \text{ for all } y \in A\}$. A tracial von Neumann algebra M is a *II₁ factor* if it is infinite dimensional and has trivial center.

If Γ is a countable group, then we denote by $(u_g)_{g \in \Gamma} \subset \mathcal{U}(\ell^2(\Gamma))$ its left regular representation given by $u_g(\delta_h) = \delta_{gh}$, where $(\delta_h)_{h \in \Gamma}$ is the usual orthonormal basis of $\ell^2(\Gamma)$. The weak (operator) closure of the linear span of $(u_g)_{g \in \Gamma}$ is a tracial von Neumann algebra, which we denote by $L(\Gamma)$. The so-called *group von Neumann algebra* $L(\Gamma)$ is a II₁ factor precisely when Γ has infinite non-trivial conjugacy classes (icc).

2.2. Ultrafilters and ultraproducts. In this subsection we collect together several elementary facts regarding ultraproducts of von Neumann algebras.

An *ultrafilter* \mathcal{U} on a set S is collection of subsets of S which is closed under finite intersections, does not contain the empty set, and contains either S' or $S \setminus S'$, for every subset $S' \subset S$. An ultrafilter \mathcal{U} is called *free* if it contains the complements of all finite subsets of S . If $f \in \ell^\infty(S)$, then its limit along \mathcal{U} , denoted by $\lim_{\mathcal{U}} f(s)$, is the unique $\ell \in \mathbb{C}$ such that $\{s \in S \mid |f(s) - \ell| < \varepsilon\} \in \mathcal{U}$, for every $\varepsilon > 0$. The map $\ell^\infty(S) \ni f \mapsto \lim_{\mathcal{U}} f(s) \in \mathbb{C}$ is a $*$ -homomorphism, which allows

to identify \mathcal{U} with a point in the Stone-Ćech compactification βS of S . Via this identification, an ultrafilter \mathcal{U} is free if and only if it belongs to $\beta S \setminus S$.

Given an ultrafilter \mathcal{U} on a set S and a family of tracial von Neumann algebras (M_s, τ_s) , $s \in S$, we define the *ultraproduct* algebra $\prod_{\mathcal{U}} M_s$ as the quotient \mathcal{A}/\mathcal{I} , where \mathcal{A} is the C*-algebra $\mathcal{A} = \{(x_s)_s \in \prod M_s \mid \sup_s \|x_s\| < \infty\}$ and \mathcal{I} is the closed ideal of $(x_s)_s \in \mathcal{A}$ such that $\lim_{\mathcal{U}} \|x_s\|_2 = 0$. It turns out that $\prod_{\mathcal{U}} M_s$ is a tracial von Neumann algebra, with the canonical trace given by $\tau((x_s)_s) = \lim_{\mathcal{U}} \tau_s(x_s)$. When $(M_s)_s$ is the constant family $(M)_s$, we write $M^{\mathcal{U}}$ and call it the *ultrapower* von Neumann algebra. In this case, the map $\pi : M \rightarrow M^{\mathcal{U}}$ given by $\pi(x) = (x_s)_s$, where $x_s = x$, for all $s \in S$, is an injective *-homomorphism.

Next, we recall the known fact that depending on the ultrafilter \mathcal{U} , $M^{\mathcal{U}}$ is either non-separable or isomorphic to M (see Lemma 2.3).

Definition 2.1. An ultrafilter \mathcal{U} on a set S is called *countably cofinal* (or *countably incomplete*) if there is a sequence $\{A_n\}$ in \mathcal{U} such that $\bigcap_n A_n = \emptyset$. Otherwise, \mathcal{U} is called *countably complete*.

Any free ultrafilter on a countable set is countably cofinal, while any principal (i.e. non-free) ultrafilter is countably complete. The hypothesis that there exists a countably complete free ultrafilter is a very strong axiom which is not provable from ZFC (e.g. [Ke10, Section 5]). However, such set-theoretic issues will not be important here.

Lemma 2.2. *Let \mathcal{U} be a countably cofinal ultrafilter on a set S . For every $s \in S$, let M_s be a tracial von Neumann algebra and $\{M_{s,n}\}_{n \geq 1}$ be an increasing sequence of von Neumann subalgebras whose union is weakly dense in M_s . Let $Q \subset \prod_{\mathcal{U}} M_s$ be a separable subalgebra.*

Then for every $s \in S$ we can find an integer $n_s \geq 1$ such that $Q \subset \prod_{\mathcal{U}} M_{s,n_s}$.

Proof. Since \mathcal{U} is countably cofinal, we can find a sequence $\{A_n\}_{n \geq 2}$ in \mathcal{U} such that $\bigcap_n A_n = \emptyset$. Let $A_1 = S \setminus A_2$. For $s \in S$, let $f(s)$ be the largest integer $n \geq 1$ such that $s \in A_n$. It is clear that $f : S \rightarrow \mathbb{N}$ is well-defined and $\lim_{\mathcal{U}} f(s) = +\infty$.

Let $Q \subset \prod_{\mathcal{U}} M_s$ be a separable subalgebra. Let $\{x_k\}_{k \geq 1}$ be a $\|\cdot\|_2$ -dense sequence in Q . For $k \geq 1$, represent $x_k = (x_{k,s})_s$, where $x_{k,s} \in M_s$, for all $s \in S$. Let $s \in S$. Since $\bigcup_{n \geq 1} M_{s,n}$ is weakly dense in M and $M_{s,n} \subset M_{s,n+1}$, for all $n \geq 1$, we can find $n_s \geq 1$ such that

$$\|x_{k,s} - E_{M_{s,n_s}}(x_{k,s})\|_2 \leq \frac{1}{f(s)}, \quad \text{for all } 1 \leq k \leq f(s).$$

Since $\lim_{\mathcal{U}} f(s) = +\infty$, it follows that for every $k \geq 1$ we have that $\lim_{\mathcal{U}} \|x_{k,s} - E_{M_{s,n_s}}(x_{k,s})\|_2 = 0$. This implies that $x_k \in \prod_{\mathcal{U}} M_{s,n_s}$, for every $k \geq 1$, and hence that $Q \subset \prod_{\mathcal{U}} M_{s,n_s}$. \blacksquare

The first assertion of the next lemma is well-known [Fe56], while the second assertion follows from the proof of [GH01, Proposition 6.1(2)]. Nevertheless, we include a proof for completeness.

Lemma 2.3. *Let \mathcal{U} be an ultrafilter on a set S . Let (M, τ) be a tracial von Neumann algebra.*

- (1) *If \mathcal{U} is countably cofinal and M has a diffuse direct summand, then $M^{\mathcal{U}}$ is non-separable.*
- (2) *If \mathcal{U} is countably complete and M is separable, then $\pi : M \rightarrow M^{\mathcal{U}}$ given by $\pi(x) = (x_s)_s$, where $x_s = x$, for all $s \in S$, is a *-isomorphism.*

Proof. (1) If $p \in M$ is a projection, then $(pMp)^{\mathcal{U}}$ is a subalgebra of $M^{\mathcal{U}}$. If M is a diffuse von Neumann algebra, then M contains a copy of $A := L^\infty([0, 1])$, hence $M^{\mathcal{U}}$ contains a copy of $A^{\mathcal{U}}$.

We may therefore reduce to the case when $M = A$. Let $\{A_n\}_{n \geq 1}$ be an increasing sequence of finite dimensional subalgebras of A such that $\bigcup_n A_n$ is weakly dense in A . Assuming that $A^{\mathcal{U}}$ is

separable, Lemma 2.2 implies that $A^{\mathcal{U}} = \prod_{\mathcal{U}} A_{n_s}$, for some integers $n_s \geq 1, s \in S$. But since A is diffuse and A_{n_s} is finite dimensional, we can find $u_s \in \mathcal{U}(A)$ such that $E_{A_{n_s}}(u_s) = 0$, for every $s \in S$. Then $u = (u_s)_s \in \mathcal{U}(A^{\mathcal{U}})$ would be orthogonal onto $\prod_{\mathcal{U}} A_{n_s}$, which is a contradiction.

(2) Since \mathcal{U} is countably complete, $\bigcap_n A_n \neq \emptyset$, for any sequence $\{A_n\}$ in \mathcal{U} . Then the collection \mathcal{U}' of all sets of the form $\bigcap_n A_n$, where $\{A_n\}$ is a sequence in \mathcal{U} , is a filter on S . Since $\mathcal{U} \subset \mathcal{U}'$ and \mathcal{U} is an ultrafilter, we get that $\mathcal{U}' = \mathcal{U}$, hence $\bigcap_n A_n \in \mathcal{U}$, for any sequence $\{A_n\}$ in \mathcal{U} .

Let $f_m \in \ell^\infty(S)$, for $m \geq 1$, and put $\ell_m = \lim_{\mathcal{U}} f_m(s)$. Then the previous paragraph implies that

$$(2.1) \quad \{s \in S \mid f_m(s) = \ell_m \text{ for all } m \geq 1\} = \bigcap_{m,n \geq 1} \{s \in S \mid |f_m(s) - \ell_m| < \frac{1}{n}\} \in \mathcal{U}.$$

Assuming that M is separable, let us show that $\pi : M \rightarrow M^{\mathcal{U}}$ is onto. Indeed, let $x = (x_s)_s \in M^{\mathcal{U}}$. Let $\{z_m\}$ be a $\|\cdot\|_2$ -dense sequence in M . For every m , define $f_m \in \ell^\infty(S)$ by $f_m(s) = \tau(x_s z_m)$. By (2.1), there exists a set $A \in \mathcal{U}$ such that $f_m(s) = f_m(s')$, for all $s, s' \in A$ and every $m \geq 1$. Hence $x_s = x_{s'}$, for every $s, s' \in A$. Choosing $s_0 \in A$, it clearly follows that $\pi(x_{s_0}) = x$. This shows that π is onto. Since π is also injective, we conclude that π is a $*$ -isomorphism. \blacksquare

Let us also record a simple consequence of Lemma 2.2, more specific to our problem.

Corollary 2.4. *Let \mathcal{U} be a countably cofinal ultrafilter on a set S . Let Γ be a countable group, $\alpha \in \{0, 1\}$, and consider the notation from Section 1.1. Put $M = L(T_\alpha(\Gamma))$ and $P_n = L(\tilde{\Gamma}_n)$, for $n \geq 1$. For every $s \in S$, let $t_s \geq 1$ be an integer, and Q_s be a tracial von Neumann algebra.*

If $A \subset \prod_{\mathcal{U}} (M^{\otimes t_s} \bar{\otimes} Q_s)$ is a separable subalgebra, then there are integers $n_s \geq 1, s \in S$, satisfying

$$\prod_{\mathcal{U}} P_{n_s}^{\otimes t_s} \subset A' \cap \prod_{\mathcal{U}} M^{\otimes t_s}.$$

Proof. From the definition of T_0 and T_1 , we see that the increasing union $\bigcup_{n \geq 1} (P'_n \cap M)$ is weakly dense in M . Thus, the increasing union $\bigcup_{n \geq 1} [(P'_n)^{\otimes t_s} \cap M^{\otimes t_s}] \bar{\otimes} Q_s$ is weakly dense in $M^{\otimes t_s} \bar{\otimes} Q_s$, for any $s \in S$. Lemma 2.2 then gives integers $n_s \geq 1$, for any $s \in S$, such that $A \subset \prod_{\mathcal{U}} [(P_{n_s}^{\otimes t_s})' \cap M^{\otimes t_s}] \bar{\otimes} Q_s$. This clearly implies the conclusion. \blacksquare

2.3. Residual inclusions. A subalgebra P of a tracial von Neumann algebra M is called residual if it “absorbs” central sequences: any central sequence in M asymptotically lies in P . In this subsection, we define and use a quantitative notion of residual subalgebras.

Definition 2.5. Let (M, τ) be a tracial von Neumann algebra, $k \geq 1$ an integer, and $C > 0$. A von Neumann subalgebra $P \subset M$ is said to be (k, C) -residual if there exist unitary elements $u_1, u_2, \dots, u_k \in M$ such that for all $\xi \in M$ we have

$$\|\xi - E_P(\xi)\|_2^2 \leq C \sum_{i=1}^k \|[\xi, u_i]\|_2^2.$$

Lemma 2.6. *Let $k \geq 1$ and $C > 0$. Let \mathcal{U} be an ultrafilter on a set S . For any $s \in S$, let M_s and Q_s be tracial von Neumann algebras, and $P_s \subset M_s$ be a (k, C) -residual von Neumann subalgebra.*

Then there exists a separable von Neumann subalgebra $A \subset \prod_{\mathcal{U}} M_s$ such that

$$A' \cap \prod_{\mathcal{U}} (M_s \bar{\otimes} Q_s) \subset \prod_{\mathcal{U}} (P_s \bar{\otimes} Q_s).$$

Proof. Let $s \in S$. Let $u_1^s, \dots, u_k^s \in \mathcal{U}(M_s)$ such that $\|\xi - E_{P_s}(\xi)\|_2^2 \leq C \sum_{i=1}^k \|\xi, u_i^s\|_2^2$, for all $\xi \in M_s$. From this it follows easily that $\|\xi - E_{P_s \bar{\otimes} Q_s}(\xi)\|_2^2 \leq C \sum_{i=1}^k \|\xi, u_i^s\|_2^2$, for all $\xi \in M_s \bar{\otimes} Q_s$. Denote $u_i = (u_i^s)_s \in \mathcal{U}(\prod_{\mathcal{U}} M_s)$, for $1 \leq i \leq k$. Then the last inequality implies that

$$\|\xi - E_{\prod_{\mathcal{U}}(P_s \bar{\otimes} Q_s)}(\xi)\|_2^2 \leq C \sum_{i=1}^k \|\xi, u_i\|_2^2, \text{ for all } \xi \in \prod_{\mathcal{U}}(M_s \bar{\otimes} Q_s).$$

It is clear that the von Neumann algebra A generated by u_1, \dots, u_k satisfies the conclusion. \blacksquare

Definition 2.7. [MD69a] Let Γ be a countable group. A subgroup $\Lambda < \Gamma$ is called *strongly residual* if there exist elements $a, b \in \Gamma$ and a subset $F \subset \Gamma \setminus \Lambda$ satisfying the following properties:

- (i) $a\Lambda a^{-1} = \Lambda$,
- (ii) $aFa^{-1} \cup F = \Gamma \setminus \Lambda$, and
- (iii) $\{b^k F b^{-k}\}_{k \in \mathbb{Z}}$ is a family of disjoint subsets of $\Gamma \setminus \Lambda$.

Lemma 2.8. *Let $\Lambda < \Gamma$ be a strongly residual subgroup.*

Then $L(\Lambda) \subset L(\Gamma)$ is a $(2, 100)$ -residual subalgebra.

Proof. Let $a, b \in \Gamma$ and $F \subset \Gamma \setminus \Lambda$ be as in Definition 2.7. Let $\xi \in L(\Gamma)$. View $\xi \in \ell^2(\Gamma)$ and for any subset $A \subset \Gamma$, define $\nu(A) = \sum_{h \in A} |\xi(h)|^2$. Then ν is a finite measure on Γ and the Cauchy-Schwarz inequality implies that for every $g \in \Gamma$ and $A \subset \Gamma$ we have

$$\begin{aligned} |\nu(gAg^{-1}) - \nu(A)| &= \left| \sum_{h \in A} |\xi(ghg^{-1})|^2 - |\xi(h)|^2 \right| \\ &\leq \sum_{h \in A} (|\xi(ghg^{-1})| + |\xi(h)|) \left| |\xi(ghg^{-1})| - |\xi(h)| \right| \\ &\leq (\nu(gAg^{-1})^{1/2} + \nu(A)^{1/2}) \sum_{h \in A} |\xi(ghg^{-1}) - \xi(h)|^2. \end{aligned}$$

Hence

$$(2.2) \quad |\nu(gAg^{-1}) - \nu(A)| \leq (\nu(gAg^{-1})^{1/2} + \nu(A)^{1/2}) \|[u_g, \xi]\|_2.$$

On the other hand, by using conditions (ii) and (iii) from Definition 2.7 we get that

$$(2.3) \quad \nu(F) + \nu(aFa^{-1}) \geq \nu(\Gamma \setminus \Lambda) \geq \nu(F) + \nu(bFb^{-1}) + \nu(b^{-1}Fb).$$

By combining (2.2) and (2.3), and using that $F, aFa^{-1}, bFb^{-1}, b^{-1}Fb \subset \Gamma \setminus \Lambda$ we deduce that

$$\begin{aligned} \nu(\Gamma \setminus \Lambda) &\leq 3(\nu(F) + \nu(aFa^{-1})) - 2(\nu(F) + \nu(bFb^{-1}) + \nu(b^{-1}Fb)) \\ &\leq 3|\nu(aFa^{-1}) - \nu(F)| + 2|\nu(bFb^{-1}) - \nu(F)| + 2|\nu(b^{-1}Fb) - \nu(F)| \\ &\leq 6 \nu(\Gamma \setminus \Lambda)^{1/2} \|[u_a, \xi]\|_2 + 8 \nu(\Gamma \setminus \Lambda)^{1/2} \|[u_b, \xi]\|_2 \\ &\leq 10 \nu(\Gamma \setminus \Lambda)^{1/2} (\|[u_a, \xi]\|_2^2 + \|[u_b, \xi]\|_2^2)^{1/2}. \end{aligned}$$

Since $\nu(\Gamma \setminus \Lambda) = \|\xi - E_{L(\Lambda)}(\xi)\|_2^2$, the conclusion follows. \blacksquare

Lemma 2.9. *Let Γ be a countable group, and consider the notations from Section 1.1.*

Then $\tilde{\Gamma}_n$ is strongly residual in $T_\alpha(\Gamma)$, for every $n \geq 1$ and $\alpha \in \{0, 1\}$.

This statement is proven in [DL69, §21] in the case $\alpha = 0$ and $n = 1$, and is used (without proof) in full generality in [MD69a, MD69b]. For completeness, we provide a proof.

Proof. Let $n \geq 1$. In the case $\alpha = 0$, we define $\Sigma_n < T_0(\Gamma)$ to be the subgroup generated by $\tilde{\Gamma}, \Lambda_1, \Lambda_2, \dots, \Lambda_n$, and $\Delta_n < T_0(\Gamma)$ to be the subgroup generated by $\tilde{\Gamma}_n, \Lambda_{n+1}, \Lambda_{n+2}, \dots$. Similarly, in the case $\alpha = 1$, we define $\Sigma_n < T_1(\Gamma)$ to be the subgroup generated by $\tilde{\Gamma} \rtimes S_\infty, \Lambda_1, \Lambda_2, \dots, \Lambda_n$, and $\Delta_n < T_1(\Gamma)$ to be the subgroup generated by $\tilde{\Gamma}_n, \Lambda_{n+1}, \Lambda_{n+2}, \dots$. One then checks in both cases that Σ_n, Δ_n generate $T_\alpha(\Gamma)$, and $T_\alpha(\Gamma) = \Sigma_n *_{\tilde{\Gamma}_n} \Delta_n$. Moreover, if a and b denote the generators of Λ_1 and Λ_{n+1} , then a commutes with $\tilde{\Gamma}_n$, $a \in \Sigma_n \setminus \tilde{\Gamma}_n$ and $b^k \in \Delta_n \setminus \tilde{\Gamma}_n$, for all $k \geq 1$.

The conclusion is now a consequence of the following fact. Let $G = H_1 *_{H_0} H_2$ be an amalgamated free product group such that there exist $a \in H_1 \setminus H_0, b \in H_2 \setminus H_0$ satisfying $aH_0a^{-1} = H_0$ and $b^k \notin H_0$, for all $k \geq 1$. Then H_0 is a strongly residual subgroup of G . Indeed, let $F \subset G \setminus H_0$ be the set of reduced words of the form $g_1g_2\dots g_k$, for some $k \geq 1$ and $g_1 \in H_1 \setminus H_0, g_2 \in H_2 \setminus H_0, g_3 \in H_1 \setminus H_0, \dots$. It is easy to see that a, b, F verify conditions (i)-(iii) of Definition 2.7. ■

Lemma 2.10. [MD69a] *Let $\Lambda_i < \Gamma_i$ be strongly residual subgroups, for every $1 \leq i \leq n$.*

Then $\bigoplus_{i=1}^n \Lambda_i < \bigoplus_{i=1}^n \Gamma_i$ is a strongly residual subgroup.

For completeness, we recall the proof from [MD69a].

Proof. Let $a_i, b_i \in \Gamma_i$ and $F_i \subset \Gamma_i \setminus \Lambda_i$ be as in Definition 2.7. Denote $\Lambda = \bigoplus_{i=1}^n \Lambda_i$ and $\Gamma = \bigoplus_{i=1}^n \Gamma_i$. Put $a = (a_1, \dots, a_n)$ and $b = (b_1, \dots, b_n) \in \Gamma$. Set

$$F := \{(g_1, \dots, g_n) \in \Gamma \mid \exists i \text{ such that } g_1, \dots, g_{i-1} \in \Lambda \text{ and } g_i \in F_i\}.$$

Then a, b, F satisfy conditions (i)-(iii) from Definition 2.7 for $\Gamma < \Lambda$. ■

Combining the above results we have the following key corollary which will be used repeatedly.

Corollary 2.11. *Let \mathcal{U} be an ultrafilter on a set S . Let Γ be a countable group, $\alpha \in \{0, 1\}$, and consider the notation from Section 1.1. Put $M = L(T_\alpha(\Gamma))$ and $P_n = L(\tilde{\Gamma}_n)$, for every $n \geq 1$. For every $s \in S$, let $n_s \geq 1$ and $t_s \geq 1$ be integers, and Q_s be a tracial von Neumann algebra.*

Then there exists a separable subalgebra $A \subset \prod_{\mathcal{U}} M^{\otimes t_s}$ such that

$$A' \cap \prod_{\mathcal{U}} (M^{\otimes t_s} \bar{\otimes} Q_s) \subset \prod_{\mathcal{U}} (P_{n_s}^{\otimes t_s} \bar{\otimes} Q_s).$$

Proof. Let $s \in S$. By combining Lemmas 2.9 and 2.10, we get that $\bigoplus_{i=1}^{t_s} \tilde{\Gamma}_{n_s} < \bigoplus_{i=1}^{t_s} T_\alpha(\Gamma)$ is a strongly residual subgroup. Lemma 2.8 then implies that $P_{n_s}^{\otimes t_s} \subset M^{\otimes t_s}$ is a $(2, 100)$ -residual subalgebra. The conclusion now follows from Lemma 2.6. ■

3. MCDUFF DEPTH OF A VON NEUMANN ALGEBRA

3.1. Properties at depth k .

Definition 3.1. Let \mathcal{M} be a (typically non-separable) von Neumann algebra, and I be a directed set. On the set of subalgebras of \mathcal{M} , consider the partial order given by inclusion. A decreasing net $(A_i)_{i \in I}$ of subalgebras of \mathcal{M} is called a *residual net* if:

- For any separable subalgebra $Q \subset \mathcal{M}$, there exists $i \in I$ such that $A_i \subset Q' \cap \mathcal{M}$, and
- For any $i \in I$, there exists a separable subalgebra $Q \subset \mathcal{M}$ such that $Q' \cap \mathcal{M} \subset A_i$.

A residual net is called *trivial* if there exists $i \in I$ such that $A_i = \mathbb{C}$.

Example 3.2. Given a von Neumann algebra \mathcal{M} , consider the set I of separable subalgebras of \mathcal{M} , ordered by inclusion. Then the net $(Q' \cap \mathcal{M})_{Q \in I}$ is clearly a residual net, which we call the *standard residual net*.

The following result provides our main example of a residual net.

Lemma 3.3. *Let Γ be a countable group and $\alpha \in \{0, 1\}$. For $n' > n \geq 1$, put $M = L(T_\alpha(\Gamma))$, $P_n = L(\tilde{\Gamma}_n)$, $P_{n,n'} = L(\tilde{\Gamma}_{n,n'})$, where $\tilde{\Gamma}_{n,n'} < \tilde{\Gamma}_n < \tilde{\Gamma} < T_\alpha(\Gamma)$ are defined as in Section 1.1.*

Let \mathcal{U} be a countably cofinal ultrafilter on a set S . For every $s \in S$, let $t_s \geq 1$ be an integer. Define $\mathcal{M} = \prod_{\mathcal{U}} M^{\otimes t_s}$. Endow $I = \mathbb{N}^S$ with the following partial order: $(n_s)_s < (m_s)_s$ iff $n_s < m_s$, for all $s \in S$. For $i = (n_s)_{s \in S} \in I$, define $A_i = \prod_{\mathcal{U}} P_{n_s}^{\otimes t_s}$.

Then $(A_i)_{i \in I}$ is a residual net of \mathcal{M} . Moreover, if Γ is icc and $i = (n_s)_s < j = (m_s)_s$, then $A'_j \cap A_i = \prod_{\mathcal{U}} P_{n_s, m_s}^{\otimes t_s}$.

Proof. The fact that $(A_i)_{i \in I}$ is a residual net of \mathcal{M} follows easily from Lemmas 2.4 and 2.11. Now, if Γ is icc, then $L(\Gamma)$ is a II_1 factor. It follows that for $n' > n \geq 1$ we have $P'_{n'} \cap P_n = P_{n,n'}$, which clearly implies the moreover part. \blacksquare

Motivated by the moreover assertion of Lemma 3.3, we introduce the following definition.

Definition 3.4. Let \mathcal{P} be a property for von Neumann algebras. Let \mathcal{M} be a von Neumann algebra with a residual net $(A_i)_{i \in I}$. We say that \mathcal{M} has *property \mathcal{P} at depth 1* if for all $i_1 \in I$, there exists $i_2 > i_1$ such that for all $i_3 > i_2$ there exists $i_4 > i_3$, such that the inclusion $A'_{i_3} \cap A_{i_2} \subset A'_{i_4} \cap A_{i_1}$ contains an intermediate von Neumann subalgebra with property \mathcal{P} .

Remark 3.5. A von Neumann algebra \mathcal{M} is trivial at depth 1 if and only if it admits a separable subalgebra $Q \subset \mathcal{M}$ such that $Q' \cap \mathcal{M} = \mathbb{C}1$.

Lemma 3.6. *Having property \mathcal{P} at depth 1 is independent of the choice of a residual net.*

Proof. Consider a von Neumann algebra \mathcal{M} with two residual nets $(A_i)_{i \in I}$ and $(B_j)_{j \in J}$. From the definition of residual nets we see that for every $i \in I$ there exists $j \in J$ such that $B_j \subset A_i$. Symmetrically, for every $j \in J$ there exists $i \in I$ such that $A_i \subset B_j$.

Assume that \mathcal{M} has property \mathcal{P} at depth 1 with respect to $(A_i)_{i \in I}$.

Fix $j_1 \in J$. Then there exists $i_1 \in I$ such that $A_{i_1} \subset B_{j_1}$. Take $i_2 > i_1$ as in the definition of property \mathcal{P} at depth 1. Then there exists $j_2 > j_1$ such that $B_{j_2} \subset A_{i_2}$.

Take an arbitrary $j_3 > j_2$, and pick $i_3 > i_2$ such that $A_{i_3} \subset B_{j_3}$. Next, we find $i_4 > i_3$ as in the definition of property \mathcal{P} at depth 1. Then there exists $j_4 > j_3$ such that $B_{j_4} \subset A_{i_4}$.

Altogether, we have the inclusions

$$B_{j_4} \subset A_{i_4} \subset A_{i_3} \subset B_{j_3} \subset B_{j_2} \subset A_{i_2} \subset A_{i_1} \subset B_{j_1}.$$

From this we get that

$$B'_{j_3} \cap B_{j_2} \subset A'_{i_3} \cap A_{i_2} \subset A'_{i_4} \cap A_{i_1} \subset B'_{j_4} \cap B_{j_1}.$$

By our choice of $i_1, i_2, i_3, i_4 \in I$, there is an intermediate subalgebra with property \mathcal{P} inside the inclusion $A'_{i_3} \cap A_{i_2} \subset A'_{i_4} \cap A_{i_1}$. Thus, \mathcal{M} has property \mathcal{P} at depth 1 with respect to $(B_j)_{j \in J}$. \blacksquare

Definition 3.7. Let \mathcal{P} be a property of von Neumann algebras. We define inductively on $k \geq 0$ what it means for a von Neumann algebra \mathcal{M} to have *property \mathcal{P} at depth k* . We denote this property by $\mathcal{P}^{(k)}$.

- If $k = 0$, then we say that \mathcal{M} has property $\mathcal{P}^{(0)}$ if it has property \mathcal{P} .

- If $k \geq 0$, then we say that \mathcal{M} has $\mathcal{P}^{(k+1)}$ if it has property $\mathcal{P}^{(k)}$ at depth 1.

Definition 3.8. A von Neumann algebra \mathcal{M} is said to have *finite McDuff depth* if there exists k such that \mathcal{M} is trivial at depth k . The *McDuff depth* of \mathcal{M} is defined as the smallest $k \geq 0$ such that \mathcal{M} is trivial at depth $k + 1$. If \mathcal{M} does not have finite McDuff depth, then we define its McDuff depth to be infinite.

Examples 3.9. Let M be a separable II₁ factor and \mathcal{U} be a countably cofinal ultrafilter.

- By Remark 3.5, $M^{\mathcal{U}}$ has depth 0 if and only if M does not have property Gamma.
- If M has property Gamma but is non McDuff, then $M' \cap M^{\mathcal{U}}$ is abelian and non trivial. This easily implies that $M^{\mathcal{U}}$ has infinite depth.
- If M is the hyperfinite II₁ factor, then $M^{\mathcal{U}}$ has depth 1.

3.2. Computing the depth. The aim of this subsection is to prove the following result.

Theorem 3.10. *Let \mathcal{U} be a countably cofinal ultrafilter, and Γ be a non-trivial countable group. Let α be a (finite or infinite) sequence of 0's and 1's. Let $M_{\alpha}(\Gamma)$ be as defined in Section 1.1.*

Then the depth of $M_{\alpha}(\Gamma)^{\mathcal{U}}$ is at least the length of α . Moreover, if $\Gamma = \mathbb{F}_2$, then we have equality.

Let us fix a countably cofinal ultrafilter \mathcal{U} on a set S . Towards proving Theorem 3.10, we first provide an upper bound on depth when $\Gamma = \mathbb{F}_2$.

Lemma 3.11. *For all $k \geq 0$ and $\alpha \in \{0, 1\}^k$ and all integers $t_s \geq 1, s \in S$, the von Neumann algebra $\prod_{\mathcal{U}} M_{\alpha}(\mathbb{F}_2)^{\otimes t_s}$ is trivial at depth $k + 1$.*

Proof. We proceed by induction on the length k of the sequence α . First assume $k = 0$. Then α is the empty sequence and $M_{\alpha}(\mathbb{F}_2) = L(\mathbb{F}_2)$. Since the trivial subgroup is strongly residual in \mathbb{F}_2 (see e.g. the proof of Lemma 2.9), by combining Lemmas 2.10, 2.8, and 2.6, we deduce the existence of a separable subalgebra $Q \subset \prod_{\mathcal{U}} M_{\alpha}(\mathbb{F}_2)^{\otimes t_s}$ with trivial relative commutant. In other words, $\prod_{\mathcal{U}} M_{\alpha}(\mathbb{F}_2)^{\otimes t_s}$ is trivial at depth 1 (see Remark 3.5).

Assume that the conclusion holds for $k \geq 0$. Take integers $t_s \geq 1, s \in S$, and a sequence $\alpha \in \{0, 1\}^{k+1}$ of length $k + 1$. Put $\mathcal{M} = \prod_{\mathcal{U}} M_{\alpha}(\mathbb{F}_2)^{\otimes t_s}$. We want to show that, at depth 1, \mathcal{M} has the property of being trivial at depth $k + 1$.

By Lemma 3.6, in order to check this property at depth 1 we can use any residual net for \mathcal{M} . From Definition 1.1 we see that $T_{\alpha}(\mathbb{F}_2) = T_{\alpha_1}(T_{\beta}(\mathbb{F}_2))$, hence $M_{\alpha}(\mathbb{F}_2) = L(T_{\alpha_1}(T_{\beta}(\mathbb{F}_2)))$, where $\beta = (\alpha_2, \dots, \alpha_{k+1}) \in \{0, 1\}^k$. Applying Lemma 3.3 (to $\Gamma = T_{\beta}(\mathbb{F}_2)$ and $\alpha = \alpha_1$) we obtain that $A_i = \prod_{\mathcal{U}} P_{n_s}^{\otimes t_s}$, where $i = (n_s)_s \in I$, is a residual net for \mathcal{M} .

Now, for any indices $i_4 > i_3 > i_2 > i_1$, the inclusion $A'_{i_3} \cap A_{i_2} \subset A'_{i_4} \cap A_{i_1}$ has $A'_{i_4} \cap A_{i_1}$ as an intermediate algebra. Moreover, since $T_{\beta}(\mathbb{F}_2)$ is icc, by Lemma 3.3 we get that $A'_{i_4} \cap A_{i_1} = \prod_{\mathcal{U}} P_{n_{1s}, n_{4s}}^{\otimes t_s}$. Since $P_{m, m'} \cong M_{\beta}(\mathbb{F}_2)^{\otimes(m'-m)}$, for any $m' > m \geq 1$, we conclude that $A'_{i_4} \cap A_{i_1}$ is of the form $\prod_{\mathcal{U}} M_{\beta}(\mathbb{F}_2)^{\otimes v_s}$, for some integers $v_s \geq 1, s \in S$. By our induction assumption, $A'_{i_4} \cap A_{i_1}$ is trivial at depth $k + 1$. This shows that \mathcal{M} is trivial at depth $k + 2$, as desired. ■

Obtaining a lower bound on depth requires additional work. We start by defining some more general residual nets.

Definition 3.12. Let $\mathcal{N} \subset \mathcal{M}$ be an inclusion of von Neumann algebras, and I be a directed set. Let $(A_i)_{i \in I}$ and $(B_i)_{i \in I}$ be two decreasing nets of subalgebras of \mathcal{M} such that $A_i \subset \mathcal{N}$, $B_i \subset \mathcal{M}$ and $A_i \subset B_i$, for all $i \in I$. The net $(A_i \subset B_i)_{i \in I}$ is called a *residual pair* for the inclusion $\mathcal{N} \subset \mathcal{M}$ if the following two properties are satisfied:

- For any $i \in I$, there exists a separable subalgebra $Q \subset \mathcal{N}$ such that $Q' \cap \mathcal{M} \subset B_i$, and
- For any separable subalgebra $Q \subset \mathcal{M}$, there exists $i \in I$ such that $A_i \subset Q' \cap \mathcal{N}$.

If $\mathcal{N} = \mathcal{M}$, and $(A_i)_{i \in I}$ is a residual net of \mathcal{M} , then $(A_i \subset A_i)_{i \in I}$ is a residual pair for $\mathcal{N} \subset \mathcal{M}$. As in Lemma 3.3, we have the following key example.

Lemma 3.13. *Let Γ be a countable group and $\alpha \in \{0, 1\}$. For $n' > n \geq 1$, put $M = L(T_\alpha(\Gamma))$, $P_n = L(\tilde{\Gamma}_n)$, $P_{n,n'} = L(\tilde{\Gamma}_{n,n'})$, where $\tilde{\Gamma}_{n,n'} < \tilde{\Gamma}_n < \tilde{\Gamma} < T_\alpha(\Gamma)$ are defined as in Section 1.1.*

Recall that \mathcal{U} is a countably cofinal ultrafilter on the set S . Let $Q_s, s \in S$, be tracial von Neumann algebras. Define $\mathcal{N} = \prod_{\mathcal{U}} M$ and $\mathcal{M} = \prod_{\mathcal{U}} (M \bar{\otimes} Q_s)$. Consider $I = \mathbb{N}^S$ ordered as in Lemma 3.3. For $i = (n_s)_{s \in S} \in I$, define $A_i = \prod_{\mathcal{U}} P_{i_s} \subset \mathcal{N}$ and $B_i = \prod_{\mathcal{U}} (P_{i_s} \bar{\otimes} Q_s) \subset \mathcal{M}$.

Then the net $(A_i \subset B_i)_{i \in I}$ is a residual pair for the inclusion $\mathcal{N} \subset \mathcal{M}$.

Moreover, if Γ is icc and $i = (n_s)_s < j = (m_s)_s$, then we have $A'_j \cap B_i = \prod_{\mathcal{U}} (P_{n_s, m_s} \bar{\otimes} Q_s)$ and $B'_j \cap A_i = \prod_{\mathcal{U}} P_{n_s, m_s}$.

The following result is a simple variation of Lemma 3.6.

Lemma 3.14. *Let \mathcal{P} be a property of von Neumann algebras. Let $\mathcal{A} \subset \mathcal{M} \subset \mathcal{B}$ be von Neumann algebras. Assume that \mathcal{M} has property \mathcal{P} at depth 1.*

Then for any residual pair $(A_i \subset B_i)_{i \in I}$ for the inclusion $\mathcal{A} \subset \mathcal{B}$, we have:

For all $i_1 \in I$, there exists $i_2 > i_1$ such that for all $i_3 > i_2$ there exists $i_4 > i_3$ such that the inclusion $B'_{i_3} \cap A_{i_2} \subset A'_{i_4} \cap B_{i_1}$ contains an intermediate von Neumann algebra with property \mathcal{P} .

Proof. Fix $i_1 \in I$. Then there exists a separable subalgebra $Q_1 \subset \mathcal{A}$ such that $Q'_1 \cap \mathcal{B} \subset B_{i_1}$.

Since \mathcal{M} has property \mathcal{P} at depth 1, Lemma 3.6 implies that \mathcal{M} has property \mathcal{P} at depth 1 with respect to the standard residual net from Example 3.2. Thus, there exists a separable subalgebra $\mathcal{M} \supset Q_2 \supset Q_1$ such that for any separable subalgebra $\mathcal{M} \supset Q_3 \supset Q_2$ we can find a separable subalgebra $\mathcal{M} \supset Q_4 \supset Q_3$ such that the inclusion $(Q'_3 \cap \mathcal{M})' \cap (Q'_2 \cap \mathcal{M}) \subset (Q'_4 \cap \mathcal{M})' \cap (Q'_1 \cap \mathcal{M})$ contains an intermediate subalgebra with property \mathcal{P} .

Next, let $i_2 > i_1$ such that $A_{i_2} \subset Q'_2 \cap \mathcal{A}$. Pick any index $i_3 \in I$ with $i_3 > i_2$. Then there exists a separable subalgebra $Q \subset \mathcal{A}$ such that $Q' \cap \mathcal{B} \subset B_{i_3}$. Let Q_3 be the von Neumann algebra generated by Q and Q_2 . Then $\mathcal{M} \supset Q_3 \supset Q_2$ and $Q'_3 \cap \mathcal{B} \subset B_{i_3}$. Let $Q_4 \supset Q_3$ be as given by the previous paragraph. Since $Q_4 \subset \mathcal{M} \subset \mathcal{B}$, there exists $i_4 > i_3$ such that $A_{i_4} \subset Q'_4 \cap \mathcal{A}$.

Altogether, we have the inclusions

$$A_{i_4} \subset Q'_4 \cap \mathcal{M} \subset Q'_3 \cap \mathcal{M} \subset B_{i_3}$$

and

$$A_{i_2} \subset Q'_2 \cap \mathcal{M} \subset Q'_1 \cap \mathcal{M} \subset B_{i_1}.$$

These lead to

$$B'_{i_3} \cap A_{i_2} \subset (Q'_3 \cap \mathcal{M})' \cap (Q'_2 \cap \mathcal{M}) \subset (Q'_4 \cap \mathcal{M})' \cap (Q'_1 \cap \mathcal{M}) \subset A'_{i_4} \cap B_{i_1}.$$

It follows that the inclusion $B'_{i_3} \cap A_{i_2} \subset A'_{i_4} \cap B_{i_1}$ contains an intermediate von Neumann algebra with property \mathcal{P} , as claimed. \blacksquare

We are now ready to prove the second half of Theorem 3.10: the lower bound on the depth.

Lemma 3.15. *Fix a non-trivial countable group Γ . Then for any $k \geq 0$ and $\alpha \in \{0, 1\}^k$ and any family of tracial von Neumann algebras $Q_s, s \in S$, any intermediate von Neumann subalgebra $M_\alpha(\Gamma)^{\mathcal{U}} = \prod_{\mathcal{U}} M_\alpha(\Gamma) \subset \mathcal{M} \subset \prod_{\mathcal{U}} (M_\alpha(\Gamma) \bar{\otimes} Q_s)$ is not trivial at depth k .*

Proof. We proceed by induction on k . First assume $k = 0$. Then α is the empty sequence and $M_\alpha(\Gamma) = L(\Gamma) \neq \mathbb{C}1$, hence the ultrapower $M_\alpha(\Gamma)^\mathcal{U}$ is not trivial, so \mathcal{M} is not trivial either.

Assume that the result holds for some $k \geq 0$. Take a sequence α of length $k + 1$ and suppose by contradiction that there exist tracial von Neumann algebras $Q_s, s \in S$, and an intermediate von Neumann subalgebra \mathcal{M} as follows which is trivial at depth $k + 1$

$$M_\alpha(\Gamma)^\mathcal{U} \subset \mathcal{M} \subset \prod_{\mathcal{U}} (M_\alpha(\Gamma) \bar{\otimes} Q_s).$$

Therefore, at depth 1, \mathcal{M} has the property of being trivial at depth k . From Definition 1.1, we see that $M_\alpha(\Gamma) = L(T_{\alpha_1}(T_\beta(\Gamma)))$, where $\beta = (\alpha_2, \dots, \alpha_{k+1}) \in \{0, 1\}^k$. Let $(A_i \subset B_i)_{i \in I}$ be the residual pair for the inclusion $M_\alpha(\Gamma)^\mathcal{U} \subset \prod_{\mathcal{U}} (M_\alpha(\Gamma) \bar{\otimes} Q_s)$ obtained by applying Lemma 3.13 to $T_\beta(\Gamma)$ and α_1 instead of Γ and α .

Lemma 3.14 then implies that for all $i_1 \in I$, there exists $i_2 > i_1$ such that for all $i_3 > i_2$ there exists $i_4 > i_3$ such that the inclusion $B_{i_3}' \cap A_{i_2} \subset A_{i_4}' \cap B_{i_1}$ contains an intermediate subalgebra which is trivial at depth n . As $T_\beta(\Gamma)$ is icc, by Lemma 3.13 for all indices $i_4 > i_3 > i_2 > i_1$ we have

$$B_{i_3}' \cap A_{i_2} = \prod_{\mathcal{U}} P_{n_{2,s}, n_{3,s}} \quad \text{and} \quad A_{i_4}' \cap B_{i_1} = \prod_{\mathcal{U}} (P_{n_{1,s}, n_{4,s}} \bar{\otimes} Q_s).$$

Choose $i_3 = (n_{3,s})$ such that $n_{3,s} = n_{2,s} + 1$ for all $s \in S$. Since $P_{m,m'} \cong M_\beta(\Gamma)^{\bar{\otimes}(m'-m)}$, for any $m' > m \geq 1$, we see that the inclusion $B_{i_3}' \cap A_{i_2} \subset A_{i_4}' \cap B_{i_1}$ is of the form

$$\prod_{\mathcal{U}} P_{n_{2,s}, n_{2,s}+1} = \prod_{\mathcal{U}} M_\beta(\Gamma) \subset \prod_{\mathcal{U}} (P_{n_{1,s}, n_{4,s}} \bar{\otimes} Q_s) = \prod_{\mathcal{U}} (M_\beta(\Gamma) \bar{\otimes} \tilde{Q}_s),$$

for some tracial von Neumann algebras \tilde{Q}_s . Since by the induction assumption there is no intermediate subalgebra in this inclusion which is trivial at depth k , we get a contradiction. ■

4. DISTINGUISHING UNCOUNTABLY MANY ULTRAPOWERS

4.1. Property \tilde{V} and proof of the main results. In order to show that the II₁ factors $M_\alpha(\Gamma)$ are non-isomorphic, McDuff introduced [MD69b] a certain property for separable II₁ factors, called property V (cf. with the earlier notions of asymptotically abelian II₁ factors [Sa68, DL69, ZM69]). In this section, inspired by property V , we define the following new property for non-separable von Neumann algebras:

Definition 4.1. A non-separable von Neumann algebra \mathcal{M} has *property \tilde{V}* if there exists a separable subalgebra $A \subset \mathcal{M}$ such that for any separable subalgebra $B \subset A' \cap \mathcal{M}$ and any separable subalgebra $C \subset \mathcal{M}$, there exists a unitary $u \in \mathcal{M}$ such that $uBu^* \subset C' \cap \mathcal{M}$.

One can check that if a separable II₁ factor M has property V , then M^ω has property \tilde{V} , for any free ultrafilter ω on \mathbb{N} . Let Γ be a countable group. Then $L(T_1(\Gamma))$ has property V by [MD69b, Lemma 1], hence $L(T_1(\Gamma))^\omega$ has property \tilde{V} . On the other hand, if Γ is non-amenable, then we show that $L(T_0(\Gamma))^\omega$ does not have property \tilde{V} . More generally, we have the following theorem.

As in the previous section, we fix a countably cofinal ultrafilter \mathcal{U} on a set S .

Theorem 4.2. *Let Γ be a non-amenable countable group. Let $k \geq 1$ and $\alpha \in \{0, 1\}^k$. Let $M_\alpha(\Gamma)$ be as defined in Section 1.1.*

Then $M_\alpha(\Gamma)^\mathcal{U}$ has property \tilde{V} at depth $k - 1$ (see Definition 3.7) if and only if $\alpha_k = 1$.

Moreover, if instead α is a sequence of 0's and 1's with length greater than k , then the same conclusion holds for arbitrary Γ (possibly amenable).

Note that the moreover part of Theorem 4.2 follows from the first part of the statement. Indeed, if $k \geq 1$ and α has length greater than k , then Definition 1.1 implies that the factor $M_\alpha(\Gamma)$ is of the form $M_\beta(\Lambda)$, for some non-amenable group Λ and the truncated sequence $\beta = (\alpha_n)_{n=1}^k \in \{0, 1\}^k$.

Let us explain how this theorem implies our main results.

Proof of Theorem 1.2. Assume that $M_\alpha(\Gamma)^\mathcal{U} \simeq M_\beta(\Gamma)^\mathcal{V}$, for two ultrafilters \mathcal{U} and \mathcal{V} . If \mathcal{U} and \mathcal{V} are countably cofinal, then combining Theorem 3.10 and Theorem 4.2 leads to a contradiction. If one of the ultrafilters, say \mathcal{U} , is not countably cofinal, then Lemma 2.3 readily implies that $M_\alpha(\Gamma) \simeq M_\beta(\Gamma)$, and [MD69b] gives a contradiction. Alternatively, we may choose a free ultrafilter ω on \mathbb{N} , and derive a contradiction from $M_\alpha(\Gamma)^\omega \simeq M_\beta(\Gamma)^\omega$, as above. ■

Proof of Corollary 1.3. Denote by τ_1 the canonical trace on $C_r^*(T_\alpha(\Gamma))$, by τ_2 the unique trace on \mathcal{Z} , and put $\tau = \tau_1 \otimes \tau_2$. Note that the von Neumann algebra generated by $A_\alpha(\Gamma)$ acting via the GNS representation with respect to τ is $M_\alpha(\Gamma) \bar{\otimes} R \simeq M_\alpha(\Gamma)$. Indeed, $M_\alpha(\Gamma)$ is McDuff, as soon as α is non-empty.

It is an easy exercise to show that for any ultrafilter \mathcal{U} , the von Neumann algebra generated by $A_\alpha(\Gamma)^\mathcal{U}$ acting via the GNS representation associated with $\tau^\mathcal{U}$ is precisely $M_\alpha(\Gamma)^\mathcal{U}$.

Hence, in order to deduce Corollary 1.3 from Theorem 1.2, we only need to check that $\tau^\mathcal{U}$ is the only trace on $A_\alpha(\Gamma)^\mathcal{U}$. Note that for any group Γ and $\alpha \in \{0, 1\}$, $T_\alpha(\Gamma)$ is equal to the increasing union of the free product groups $G_n := \Sigma_n * \Lambda_{n+1}$, where Σ_n is the subgroup generated by $\Lambda_1 * \cdots * \Lambda_n$ and $\Gamma_1 \oplus \cdots \oplus \Gamma_n$ if $\alpha = 0$ (respectively, $(\Gamma_1 \oplus \cdots \oplus \Gamma_n) \rtimes S_n$ if $\alpha = 1$). In particular, $T_\alpha(\Gamma)$ is an increasing union of Powers groups (see e.g. [dlHS86]):

$$(4.1) \quad T_\alpha(\Gamma) = \bigcup_n G_n.$$

Since \mathcal{Z} has a unique trace, [dlHS86, Corollary 7] implies that $A_\alpha(\Gamma)$ has a unique trace. Moreover, if Γ is exact, so is $A_\alpha(\Gamma)$ and [Oz13, Theorem 8] shows that $A_\alpha(\Gamma)^\mathcal{U}$ has the unique trace property. Let us treat the general case, when Γ is not necessarily exact.

By (4.1), it is sufficient to show that for all family of integers $(k_s)_s$, the algebra $\prod_{\mathcal{U}} (C_r^*(G_{k_s}) \otimes \mathcal{Z})$ has a unique trace. Since $\mathcal{Z}^\mathcal{U}$ has a unique trace and all the G_{k_s} 's are Powers groups, this is an easy consequence of [dlHS86, Lemma 5]. ■

The rest of this section is devoted to the prove Theorem 4.2. As explained above we only need to prove the first statement. We will proceed by induction on k , and treat the base case and the inductive step in two separate subsections.

4.2. The base case. The case $k = 0$ of Theorem 4.2 is dealt with by the following two lemmas.

Lemma 4.3. *Let \mathcal{U} be a countably cofinal ultrafilter on a set S . Let Γ be a countable group and denote $M = L(T_1(\Gamma))$. For every $s \in S$, let $t_s \geq 1$ be an integer. Put $\mathcal{M} = \prod_{\mathcal{U}} M^{\otimes t_s}$.*

Then \mathcal{M} has property \tilde{V} .

Proof. Recall from Section 1.1 that $T_1(\Gamma)$ is generated by $\tilde{\Gamma} \rtimes S_\infty$ and $\Lambda_j, j \geq 1$, where $\Gamma_i, i \geq 1$, and $\Lambda_j, j \geq 1$, are isomorphic copies of Γ and \mathbb{Z} , respectively, and S_∞ acts on $\tilde{\Gamma} = \bigoplus_{i \geq 1} \Gamma_i$ by permutations, with the only relations that Γ_i and Λ_j commute whenever $i \geq j$. Put $P = L(\tilde{\Gamma})$. Then Corollary 2.11 provides a separable subalgebra $A \subset \mathcal{M}$ such that $A' \cap \mathcal{M} \subset \prod_{\mathcal{U}} P^{\otimes t_s}$.

For $n' > n \geq 1$, recall that $\tilde{\Gamma}_{n,n'} = \bigoplus_{n' > i \geq n} \Gamma_i$. For $n \geq 1$, let $H_n < T_1(\Gamma)$ be the subgroup generated by $\tilde{\Gamma}_{1,n+1} \rtimes S_n$ and $\Lambda_1, \dots, \Lambda_n$, where we view S_n as the group of permutations of $\{1, 2, \dots\}$ which leave any $k > n$ fixed. Put $R_n = L(\tilde{\Gamma}_{1,n+1})$ and $M_n = L(H_n)$.

Let $B \subset \prod_{\mathcal{U}} P^{\bar{\otimes} t_s}$ and $C \subset \mathcal{M}$ be separable subalgebras. Since $\cup_{m \geq 1} R_m$ is weakly dense in P and $\cup_{n \geq 1} M_n$ is weakly dense in M , Lemma 2.2 gives integers $m_s, n_s \geq 1$, for $s \in S$, such that

$$(4.2) \quad B \subset \prod_{\mathcal{U}} R_{m_s}^{\bar{\otimes} t_s} \quad \text{and} \quad C \subset \prod_{\mathcal{U}} M_{n_s}^{\bar{\otimes} t_s}.$$

Finally, for every $s \in S$, let $\sigma_s \in S_\infty$ be a permutation such that $\sigma_s(k) > n_s$, for any $1 \leq k \leq m_s$. Then $\sigma_s \tilde{\Gamma}_{1,m_s+1} \sigma_s^{-1} \subset \bigoplus_{i > n_s} \Gamma_i$, and hence $\sigma_s \tilde{\Gamma}_{1,m_s+1} \sigma_s^{-1}$ commutes with H_{n_s} . Thus, the unitary element $u_s = u_{\sigma_s} \in \mathcal{U}(M)$ satisfies $u_s R_{m_s} u_s^* \subset M_{n_s}' \cap M$. Therefore, if we let $u = (u_s^{\otimes t_s})_s \in \mathcal{U}(\mathcal{M})$, then equation (4.2) implies that $u B u^* \subset C' \cap \mathcal{M}$. This finishes the proof. \blacksquare

Lemma 4.4. *Let \mathcal{U} be an ultrafilter on a set S . Let Γ be a countable non-amenable group and denote $M = L(T_0(\Gamma))$. For every $s \in S$, let Q_s be a tracial von Neumann algebra.*

Then any intermediate subalgebra $M^{\mathcal{U}} \subset \mathcal{M} \subset \prod_{\mathcal{U}} (M \bar{\otimes} Q_s)$ does not have property \tilde{V} .

Lemma 4.4 strengthens [MD69b, Lemma 3]. In order to prove it, we will need two additional results.

Recall from Section 1.1 that $T_0(\Gamma)$ is generated by $\tilde{\Gamma} = \bigoplus_{i \geq 1} \Gamma_i$ and $\tilde{\Lambda} = *_{j \geq 1} \Lambda_j$, where Γ_i, Λ_j are isomorphic copies of Γ, \mathbb{Z} , respectively, with the only relations being that Γ_i and Λ_j commute whenever $i \geq j \geq 1$. For $n \geq 1$, we denote by $\pi_n : \Gamma \rightarrow \tilde{\Gamma}$ the canonical embedding with $\pi_n(\Gamma) = \Gamma_n$. We also let $\tilde{\Gamma}_n = \bigoplus_{i \geq n} \Gamma_i$.

Lemma 4.5. *Let $g \in T_0(\Gamma), g' \in \tilde{\Gamma}_{n+1}$, and $g'' \in \Gamma_n$, for some $n \geq 1$. Assume that $g' g g'' = g$.*

Then $g' = g'' = e$.

Proof. Fix n, g, g', g'' such that $g' g g'' = g$, as in the statement of the lemma.

Note that $T_0(\Gamma)$ splits as an amalgamated free product $T_0(\Gamma) = \Sigma *_{\tilde{\Gamma}_{n+1}} \Delta$, where

- $\Sigma < T_0(\Gamma)$ is the subgroup generated by $\tilde{\Gamma}, \Lambda_1, \dots, \Lambda_n$;
- $\Delta < T_0(\Gamma)$ is the subgroup generated by $\tilde{\Gamma}_{n+1}$ and $\Lambda_{n+1}, \Lambda_{n+2}, \dots$.

Then $g'' \in \Sigma$ is conjugate inside $T_0(\Gamma)$ to $g' \in \tilde{\Gamma}_{n+1}$. By the General fact below, this implies that g'' and is actually conjugate inside Σ to an element of $\tilde{\Gamma}_{n+1}$. Now note that $\tilde{\Gamma}_{n+1}$ is normal inside Σ (it is even in product position). This forces g'' to belong to $\Gamma_n \cap \tilde{\Gamma}_{n+1}$. Hence $g'' = e$, and further $g' = e$.

General fact. Consider an amalgamated free product of groups $A = A_1 *_{A_0} A_2$. Assume that two elements $a_1 \in A_1$ and $a_2 \in A_2$ are conjugate inside A . Then a_1 is conjugate inside A_1 to an element in A_0 .

Indeed, assume that $a_2 = h a_1 h^{-1}$ for some $h \in A$. If $h \in A_1$ then clearly $a_2 \in A_1 \cap A_2$ and we are done. Otherwise, write h as a product $h = h_0 b$ for some $b \in A_1$ and some reduced word $h_0 \in A$ with rightmost letter in $A_2 \setminus A_0$. Then $h_0 b a_1 b^{-1}$ lies inside A_0 , for if not $h_0 (b a_1 b^{-1}) h_0^{-1}$ would be a reduced word with a letter $h_0 b a_1 b^{-1} h_0^{-1} \in A_1 \setminus A_0$, so it could not lie inside A_2 . \blacksquare

Lemma 4.6. *There exist $g_1, \dots, g_m \in \Gamma$ and $C > 0$ such that the following holds:*

For any $n \geq 1$, any unitaries $v_1, \dots, v_m \in \mathcal{U}(L(\tilde{\Gamma}_{n+1}))$, and any $\xi \in M$ we have that

$$\|\xi\|_2 \leq C \sum_{k=1}^m \|u_{\pi_n(g_k)}\xi - \xi v_k\|_2.$$

Proof. Let $\lambda_n : \Gamma_n \rightarrow \mathcal{U}(\ell^2(T_0(\Gamma)/\tilde{\Gamma}_{n+1}))$ be the quasi-regular representation of Γ_n . Lemma 4.5 implies that Γ_n acts freely on $T_0(\Gamma)/\tilde{\Gamma}_{n+1}$. Thus, λ_n is a multiple of the left regular representation of Γ_n , hence $\lambda_n \circ \pi_n$ is a multiple of the left regular representation of Γ . Since Γ is non-amenable, there exist $g_1, \dots, g_m \in \Gamma$ and $C > 0$ such that for every $\xi \in \ell^2(T_0(\Gamma)/\tilde{\Gamma}_{n+1})$ and $n \geq 1$ we have

$$(4.3) \quad \|\xi\|_2 \leq C \sum_{k=1}^m \|\lambda_n(\pi_n(g_k))\xi - \xi\|_2.$$

We identify $L^2(M) \equiv \ell^2(T_0(\Gamma))$ as usual, via the unitary given by $u_g \mapsto \delta_g$, for any $g \in T_0(\Gamma)$. Fix $n \geq 1$. For $S \subset T_0(\Gamma)$, we denote by P_S the orthogonal projection from $\ell^2(T_0(\Gamma))$ onto the $\|\cdot\|_2$ -closed linear span of $\{\delta_g \mid g \in S\}$. We define $T : \ell^2(T_0(\Gamma)) \rightarrow \ell^2(T_0(\Gamma)/\tilde{\Gamma}_{n+1})$ by letting

$$T(\xi)(g\tilde{\Gamma}_{n+1}) = \|P_{g\tilde{\Gamma}_{n+1}}(\xi)\|_2, \text{ for every } \xi \in \ell^2(T_0(\Gamma)) \text{ and } g \in T_0(\Gamma).$$

Then for every $\xi, \eta \in \ell^2(T_0(\Gamma))$, $g \in \Gamma_n$, and $v \in \mathcal{U}(L(\tilde{\Gamma}_{n+1}))$ we have that

$$(4.4) \quad \|T(\xi) - T(\eta)\|_2 \leq \|\xi - \eta\|_2, \quad T(u_g\xi) = \lambda_n(g)(T(\xi)), \quad \text{and} \quad T(\xi v) = T(\xi).$$

For the last identity, just notice that since the $\|\cdot\|_2$ -closed linear span of $\{\delta_h \mid h \in g\tilde{\Gamma}_{n+1}\}$ is a right $L(\tilde{\Gamma}_{n+1})$ -module, we have that $P_{g\tilde{\Gamma}_{n+1}}(\xi v) = P_{g\tilde{\Gamma}_{n+1}}(\xi)v$, for every $g \in T_0(\Gamma)$ and $v \in \mathcal{U}(L(\tilde{\Gamma}_{n+1}))$.

Finally, let $\xi \in \ell^2(T_0(\Gamma))$ and $v_1, \dots, v_m \in \mathcal{U}(L(\tilde{\Gamma}_{n+1}))$. By applying (4.3) to $T(\xi)$ and using (4.4) we deduce

$$\|\xi\|_2 = \|T(\xi)\|_2 \leq C \sum_{k=1}^m \|T(u_{\pi_n(g_k)}\xi) - T(\xi v_k)\|_2 \leq C \sum_{k=1}^m \|u_{\pi_n(g_k)}\xi - \xi v_k\|_2.$$

This finishes the proof. ■

Proof of Lemma 4.4. Assume by contradiction that there exists an intermediate subalgebra $M^{\mathcal{U}} \subset \mathcal{M} \subset \prod_{\mathcal{U}}(M \bar{\otimes} Q_s)$ with property \tilde{V} . Let $A \subset \mathcal{M}$ be as in Definition 4.1. For $n \geq 1$, put $P_n = L(\tilde{\Gamma}_n)$. By applying Corollary 2.4, we can find integers $i_s \geq 1$, for every $s \in S$, such that

$$\prod_{\mathcal{U}} P_{i_s} \subset A' \cap \mathcal{M}.$$

Let $\rho : \Gamma \rightarrow \mathcal{U}(\prod_{\mathcal{U}} P_{i_s})$ be the homomorphism given by $\rho(g) = (u_{\pi_{i_s}(g)})_s$, for $g \in \Gamma$. Take for $B \subset A' \cap \mathcal{M}$ the (separable) subalgebra generated by $\rho(\Gamma)$. Also, use Corollary 2.11 to get a separable subalgebra $C \subset M^{\mathcal{U}}$ such that

$$C' \cap \mathcal{M} \subset \prod_{\mathcal{U}} (P_{i_s+1} \bar{\otimes} Q_s).$$

Since \mathcal{M} has property \tilde{V} , there exists a unitary $v \in \mathcal{M} \subset \prod_{\mathcal{U}}(M \bar{\otimes} Q_s)$ such that

$$v^* B v \subset C' \cap \mathcal{M} \subset \prod_{\mathcal{U}} (P_{i_s+1} \bar{\otimes} Q_s).$$

Represent $v = (v_s)_s$, where $v_s \in M \bar{\otimes} Q_s$ is a unitary, for any $s \in S$. Let $g_1, \dots, g_m \in \Gamma$ and $C > 0$ be given by Lemma 4.6. For $1 \leq k \leq m$, denote $u_k = v^* \rho(g_k) v$. Since $u_k \in \prod_{\mathcal{U}} (P_{i_s+1} \bar{\otimes} Q_s)$, we can represent $u_k = (u_{k,s})_s$, where $u_{k,s} \in P_{i_s+1} \bar{\otimes} Q_s$ is a unitary. Since $\rho(g_k) v = v u_k$, we get

$$\lim_{\mathcal{U}} \|u_{\pi_{i_s}(g_k)} v_s - v_s u_{k,s}\|_2 = 0, \text{ for every } 1 \leq k \leq m.$$

Since $\|v_s\|_2 = 1$, for every $s \in S$, this clearly contradicts the conclusion of Lemma 4.6. \blacksquare

4.3. The inductive step. Theorem 4.2 clearly follows from the next two lemmas.

Lemma 4.7. *Let Γ be a countable group. Use the notations of Section 1.1.*

For any $k \geq 1$ and $\alpha \in \{0, 1\}^k$ such that $\alpha_k = 1$, any integers $t_s \geq 1, s \in S$, we have that $\prod_{\mathcal{U}} M_{\alpha}(\Gamma)^{\bar{\otimes} t_s}$ has property \tilde{V} at depth $k - 1$.

Proof. We proceed by induction on k . If $k = 1$, then $M_{\alpha}(\Gamma) = L(T_1(\Gamma))$ and the conclusion follows from Lemma 4.3.

Assume that the conclusion holds for some $k \geq 1$. Let $\alpha \in \{0, 1\}^{k+1}$ be such that $\alpha_{k+1} = 1$. Then the sequence $\beta = (\alpha_{n+1})_{n=1}^k$ is of length k and $\beta_k = \alpha_{k+1} = 1$. Moreover, we have that $M_{\alpha}(\Gamma) = L(T_{\alpha_1}(T_{\beta}(\Gamma)))$. Let $t_s \geq 1, s \in S$, be integers and put $\mathcal{M} = \prod_{\mathcal{U}} M_{\alpha}(\Gamma)^{\bar{\otimes} t_s}$.

By applying Lemma 3.3 (to $T_{\beta}(\Gamma)$ and α_1 instead of Γ and α) we get that $A_i = \prod_{\mathcal{U}} P_{i_s}^{\bar{\otimes} t_s}$, where $i = (i_s)_s \in I$, is a residual net for \mathcal{M} . Since $\beta_k = 1$ and $P_{m,m'} \cong M_{\beta}(\Gamma)^{\bar{\otimes}(m'-m)}$, for any $m' > m \geq 1$, by using the inductive hypothesis and repeating the end of the proof of Lemma 3.11 it follows that, at depth 1, \mathcal{M} has property \tilde{V} at depth $k - 1$.

This shows that \mathcal{M} has property \tilde{V} at depth k , and finishes the proof. \blacksquare

Lemma 4.8. *Let Γ be a non-amenable countable group and keep the notations from Section 1.1.*

For any $k \geq 1$ and $\alpha \in \{0, 1\}^k$ such that $\alpha_k = 0$ and any family of tracial von Neumann algebras $Q_s \geq 1, s \in S$, no intermediate von Neumann subalgebra $M_{\alpha}(\Gamma)^{\mathcal{U}} = \prod_{\mathcal{U}} M_{\alpha}(\Gamma) \subseteq \mathcal{M} \subseteq \prod_{\mathcal{U}} (M_{\alpha}(\Gamma) \bar{\otimes} Q_s)$ has property \tilde{V} at depth $k - 1$.

Proof. We proceed by induction on k . If $k = 1$, then $M_{\alpha}(\Gamma) = L(T_0(\Gamma))$ and since Γ is non-amenable, so the conclusion follows from Lemma 4.4.

Now assume that the conclusion holds for some $k \geq 1$. Let $\alpha \in \{0, 1\}^{k+1}$ be such that $\alpha_{k+1} = 0$. Then the sequence $\beta = (\alpha_{n+1})_{n=1}^k$ is of length k and $\beta_k = \alpha_{k+1} = 0$. Moreover, $M_{\alpha}(\Gamma) = L(T_{\alpha_1}(T_{\beta}(\Gamma)))$.

Suppose by contradiction that there exist tracial von Neumann algebras $Q_s, s \in S$, and an algebra \mathcal{M} as follows which has property \tilde{V} at depth k .

$$M_{\alpha}(\Gamma)^{\mathcal{U}} \subset \mathcal{M} \subset \prod_{\mathcal{U}} (M_{\alpha}(\Gamma) \bar{\otimes} Q_s).$$

Then, at depth 1, \mathcal{M} has property \tilde{V} at depth $k - 1$. Let $(A_i \subset B_i)_{i \in I}$ be the residual pair for the inclusion $M_{\alpha}(\Gamma)^{\mathcal{U}} \subset \prod_{\mathcal{U}} (M_{\alpha}(\Gamma) \bar{\otimes} Q_s)$ described in Lemma 3.13, with $T_{\beta}(\Gamma)$ and α_1 instead of Γ and α . Since $\beta_{k+1} = 0$ and $P_{m,m'} \cong M_{\beta}(\Gamma)^{\bar{\otimes}(m'-m)}$, for any $m' > m \geq 1$, by using the inductive hypothesis and repeating the end of the proof of Lemma 3.15 we get a contradiction. \blacksquare

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