The periodic unfolding for a Fredholm alternative in perforated domains

D Cioranescu, A Damlamian, G Griso

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D. Cioranescu[1], A. Damlamian[2], G. Griso[3]

[1,3] Laboratoire J.-L. Lions–CNRS, Boîte courrier 187, Université Pierre et Marie Curie,
4 place Jussieu, 75005 Paris, France, Email: cioran@ann.jussieu.fr, griso@ann.jussieu.fr
[2] Université Paris-Est, Laboratoire d'Analyse et de Mathématiques Appliquées, CNRS UMR 8050
Centre Multidisciplinaire de Créteil, 94010 Créteil, Cedex, France, E-mail: damla@univ-paris12.fr

Abstract

We consider the homogenization of a pure Neumann boundary value problem in perforated domains. Contrary to the usual case, several obstructions occur in this case. We present an approximation of the domains for which the problem is well-posed and homogenization can be proved. The main tool is the unfolding method together with geometrical results used to establish uniform bounds.

Introduction

Consider a pure Neumann problem (i.e., without zero order term) which is traditionally associated with the Fredholm alternative. Its variational formulation in a domain $\mathcal{O}$ is to find u in $H^1(\mathcal{O})$ such that $\displaystyle\int_{\mathcal{O}} A\nabla u\,\nabla v\,dx = \int_{\mathcal{O}} fv\,dx, \quad \forall v \in H^1(\mathcal{O})$, where f is in $L^2(\mathcal{O})$ and $A(x) = (a_{ij}(x))_{1\le i,j\le n}$ is an elliptic matrix field. It is well-known that this problem has a unique solution (up to a constant), provided $\mathcal{O}$ is bounded, connected with Lipschitz boundary and if and only if $\displaystyle\int_{\mathcal{O}} f(x)\,dx = 0$ (Fredholm alternative).

We are interested in homogenizing this problem when $\mathcal{O}$ is the periodically perforated domain Ω^*_ε defined in (1.5).

In this geometry, the pure homogeneous Neumann problem with a zero order term was considered in [2] (see its appendix). The case with non homogeneous Neumann conditions on the boundary of the holes was treated in [4]. Because of the presence of a zero order term, there is no Fredholm alternative to be considered in these papers.

In the case without zero order term, there are at least two reasons why using the same geometry fails. First, it is not true in general that Ω^*_ε is connected. Second, the boundary of Ω^*_ε can be so irregular that ∇v in $L^2(\Omega^*_\varepsilon)$ does not imply $v \in H^1(\Omega^*_\varepsilon)$ but only $v \in H^1_{loc}(\Omega^*_\varepsilon)$. Consequently, the problem is not well-posed for a general f satisfying the Fredholm condition.

To avoid these difficulties, we construct a subset $\mathcal{C}^*_\varepsilon$ of Ω^*_ε which is connected, and a Hilbert space $W(\mathcal{C}^*_\varepsilon)$ on which the approximate problem is well-posed. The set obtained by "filling the holes" of $\mathcal{C}^*_\varepsilon$ converges to Ω in an appropriate sense (Proposition 1.12 and Remark 1.11). Furthermore, all the spaces $W(\mathcal{C}^*_\varepsilon)$ satisfy a Poincaré-Wirtinger inequality with a uniform constant, the key to obtaining uniform estimates (Theorem1.14).

This construction is rather involved and follows from the considerations of [4] for the implementation of the unfolding method in the case of perforated domains.

Plan

Section 1 presents a brief summary of this method, with some new results adapted to the Neumann case. The approximate problem is stated in Section 2. Its homogenization is performed in Section 3, including corrector results. In the Appendix, we give the proof of Lemma 1.13. This result is used to show the uniform boundedness of the Poincaré-Wirtinger constants in Theorem 1.14.

General notations

We follow the general notations of [4]. In particular, ε indicates the generic element of a bounded subset of $\mathbb{R}^*_+$ in the closure of which 0 lies. Convergence of ε to 0 is understood in this set. Also, c and C denote generic constants which do not depend upon ε.
As usual, 1_D denotes the characteristic function of the set D.
For a measurable set D in $\mathbb{R}^n$, $|D|$ denotes its Lebesgue measure.
For simplicity, the notation $L^p(\mathcal{O})$ will be used both for scalar and vector-valued functions defined on the set $\mathcal{O}$, since no ambiguity will arise.
For a set ω with positive finite measure and a function f integrable on ω, $\mathcal{M}_\omega(f)$ denote the average of f over ω.

We also refer to [4] for an expanded list of references.

1 A brief summary of the unfolding method in perforated domains

In this section, we begin by giving the setup and notations for periodically perforated domains. Then, we briefly recall the results of [4] concerning the periodic unfolding for such domains.

Let $\mathbf{b} = (b_1, \ldots, b_n)$ be a basis in $\mathbb{R}^n$. Set

$$\mathcal{G} = \Big\{ \xi \in \mathbb{R}^n \mid \xi = \sum_{i=1}^n k_i b_i,\ (k_1, \ldots, k_n) \in \mathbb{Z}^n \Big\}. \tag{1.1}$$

By Y, we denote the reference cell, which in the simplest case is the open parallelotop generated by the basis $\mathbf{b}$,

$$\Big\{y \in \mathbb{R}^n \mid y = \sum_{i=1}^{n} y_i b_i,\ (y_1, \dots, y_n) \in (0, 1)^n\Big\}. \tag{1.2}$$

For the general case of Y having the paving property, we refer to [4].

Let S be a closed strict subset of $\overline{Y}$ and denote by Y^* the part occupied by the material i.e. $Y^* = Y \setminus S$. The sets S and Y^* are the reference hole and perforated cell, respectively (see Figure 1).

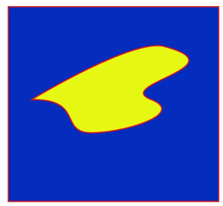


Figure 1. The $Y^* = Y \setminus S$

We introduce the following notations :

$$(\mathbb{R}^n)^* = \mathbb{R}^n \setminus \bigcup_{\xi \in \mathcal{G}} (\xi + \overline{S}), \quad (\mathbb{R}^n)^*_\varepsilon = \varepsilon(\mathbb{R}^n)^* = \mathbb{R}^n \setminus \bigcup_{\xi \in \mathcal{G}} \varepsilon(\xi + \overline{S}) \doteq \mathbb{R}^n \setminus \overline{S_\varepsilon}. \tag{1.3}$$

By this definition, $(\mathbb{R}^n)^*_\varepsilon$ is $\mathbb{R}^n$ $\varepsilon\mathcal{G}$-periodically perforated by εS.

Let now Ω be an open bounded subset of $\mathbb{R}^n$ with Lipschitz boundary, and consider the sets

$$\widehat{\Omega}_\varepsilon = \text{interior}\Big\{\bigcup_{\xi \in \Xi_\varepsilon} \varepsilon\big(\xi + \overline{Y}\big)\Big\}, \quad \Lambda_\varepsilon = \Omega \setminus \widehat{\Omega}_\varepsilon, \text{ where } \Xi_\varepsilon = \big\{\xi \in \mathcal{G}, \varepsilon(\xi + Y) \subset \Omega\big\}. \tag{1.4}$$

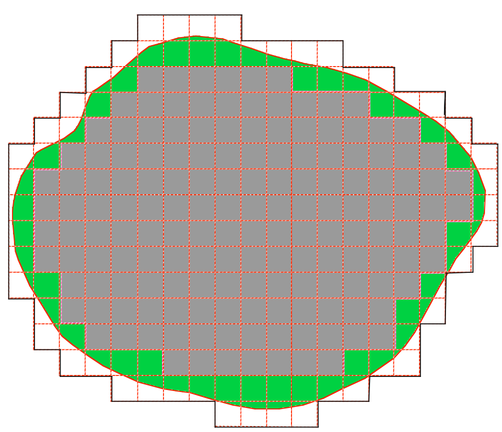


Figure 2. The sets $\widehat{\Omega}_\varepsilon$ (in grey) and Λ_ε (in green)

The set $\widehat{\Omega}_\varepsilon$ is the interior of the largest union of $\varepsilon(\xi + \overline{Y})$ cells ($\xi \in \mathbf{G}$), such that $\varepsilon(\xi + Y)$ are included in Ω, while Λ_ε is the subset of Ω containing the parts from $\varepsilon\big(\xi + \overline{Y}\big)$ cells intersecting the boundary $\partial\Omega$ (see Figure 2).

The perforated domain Ω_ε^* is obtained by removing from Ω the set of holes S_ε (see Figure 3 for the two-dimensional case),

$$\Omega_\varepsilon^* = (\mathbb{R}^n)_\varepsilon^* \cap \Omega. \tag{1.5}$$

The following notations will be used (see Figure 3):

$$\widehat{\Omega}_\varepsilon^* = \widehat{\Omega}_\varepsilon \cap \Omega_\varepsilon^*, \qquad \Lambda_\varepsilon^* = \Lambda_\varepsilon \cap \Omega_\varepsilon^*. \tag{1.6}$$

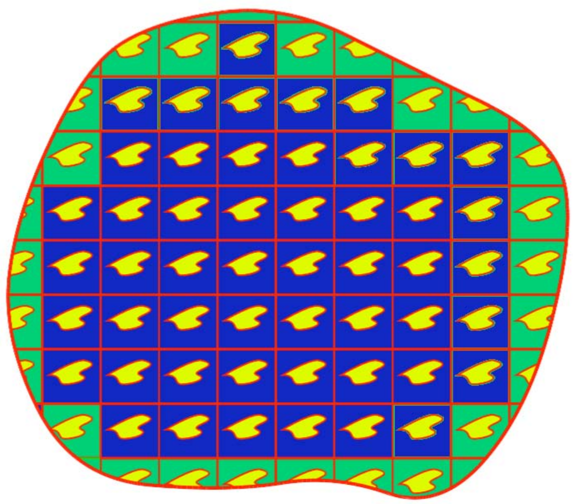


Figure 3. The sets Ω_ε^*, $\widehat{\Omega}_\varepsilon^*$ (in dark blue) and Λ_ε^* (in light green)

For $z \in \mathbb{R}^n$, $[z]_Y$ denotes the unique (up to a set of measure zero) integer combination $\sum_{j=1}^n k_j b_j$ of the periods such that $z - [z]_Y$ belongs to Y (see Figure 4).

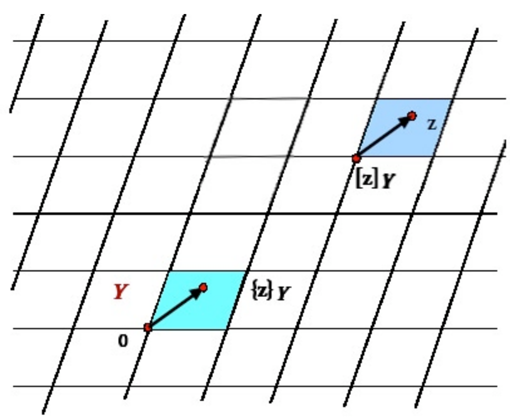


Figure 4. Definition of $[z]_Y$ and $\{z\}_Y$

Set now

$$\{z\}_Y = z - [z]_Y \in Y \quad \text{a.e. for} \quad z \in \mathbb{R}^n.$$

The unfolding operator $\mathcal{T}_\varepsilon^*$ for functions defined on the perforated domain Ω_ε^* was introduced in [4] as follows:

Definition 1.1. *For any function ϕ Lebesgue-measurable on $\widehat{\Omega}^*_\varepsilon$, the unfolding operator $\mathcal{T}^*_\varepsilon$ is defined by*

$$\mathcal{T}^*_\varepsilon(\phi)(x,y) = \begin{cases} \phi\Big(\varepsilon\Big[\dfrac{x}{\varepsilon}\Big]_Y + \varepsilon y\Big) & a.e.\ for \quad (x,y) \in \widehat{\Omega}_\varepsilon \times Y^*, \\ 0 & a.e.\ for \quad (x,y) \in \Lambda_\varepsilon \times Y^*. \end{cases} \tag{1.7}$$

For ϕ Lebesgue-measurable on Ω^*_ε, we denote $\mathcal{T}^*_\varepsilon(\phi_{|_{\widehat{\Omega}^*_\varepsilon}})$ simply by $\mathcal{T}^*_\varepsilon(\phi)$.

This operator maps functions defined on the oscillating domain $\widehat{\Omega}^*_\varepsilon$, to functions defined on the fixed domain $\Omega \times Y^*$. Its main properties are recalled in the two propositions below (see [4] for their proofs).

Proposition 1.2. *For $p \in [1, +\infty[$, the operator $\mathcal{T}^*_\varepsilon$ is linear and continuous from $L^p(\Omega^*_\varepsilon)$ to $L^p(\Omega \times Y^*)$. For every ϕ in $L^1(\Omega^*_\varepsilon)$ and w in $L^p(\Omega^*_\varepsilon)$,*

$$(i) \quad \frac{1}{|Y|}\int_{\Omega\times Y^*} \mathcal{T}^*_\varepsilon(\phi)(x,y)\,dx\,dy = \int_{\widehat{\Omega}^*_\varepsilon} \phi(x)\,dx,$$

$$(ii) \quad \|\mathcal{T}^*_\varepsilon(w)\|_{L^p(\Omega\times Y^*)} \le |\,Y\,|^{1/p}\,\|w\|_{L^p(\Omega^*_\varepsilon)}.$$

Proposition 1.3. *Let p belong to $[1, +\infty[$.*

(i) For $w \in L^p(\Omega)$,

$$\mathcal{T}^*_\varepsilon(w) \to w \quad \text{strongly in } L^p(\Omega \times Y^*).$$

(ii) Let w_ε be in $L^p(\Omega^*_\varepsilon)$ such that $||w_\varepsilon||_{L^p(\Omega^*_\varepsilon)} \le C$. If

$$\mathcal{T}^*_\varepsilon(w_\varepsilon) \rightharpoonup \widehat{w} \quad \text{weakly in } L^p(\Omega \times Y^*),$$

then

$$\widetilde{w_\varepsilon} \rightharpoonup \frac{|Y^*|}{|Y|}\mathcal{M}_{Y^*}(\widehat{w}) \quad \text{weakly in } L^p(\Omega).$$

(iii) For w_ε in $L^p(\Omega^*_\varepsilon)$ such that $\|w_\varepsilon\|_{L^p(\Omega^*_\varepsilon)}$ is bounded, the following are equivalent:

a) There is $w \in L^p(\Omega)$ such that

$$\widetilde{w_\varepsilon} \rightharpoonup \frac{|Y^*|}{|Y|}\, w \quad \text{weakly in } L^p(\Omega).$$

b) All the weak limit points W in $L^p(\Omega \times Y^*)$ of the sequence $\{\mathcal{T}^*_\varepsilon(w_\varepsilon)\}$ have the same average over Y^* (this average $\mathcal{M}_{Y^*}(W)$ being just w).

The next results, which will be used in the sequel, are consequences of these propositions.

Corollary 1.4. *Let p be in $[1,\infty]$ and p' be its conjugate.*

(i) Both sequences $\{1_{\Omega_\varepsilon^*}\}$ and $\{1_{\widehat{\Omega}_\varepsilon^*}\}$ converge weakly-$*$ in $L^\infty(\Omega)$ to the constant $\dfrac{|Y^*|}{|Y|}$ and consequently

$$\lim_{\varepsilon\to 0}|\Omega_\varepsilon^*| = \lim_{\varepsilon\to 0}|\widehat{\Omega}_\varepsilon^*| = \frac{|Y^*|}{|Y|}|\Omega|. \tag{1.8}$$

(ii) Let w_ε be in $L^p(\Omega_\varepsilon^*)$ such that

$$\widetilde{w_\varepsilon} \rightharpoonup \frac{|Y^*|}{|Y|}\, w \quad \textit{weakly in } L^p(\Omega).$$

Then,

$$\lim_{\varepsilon\to 0}\mathcal{M}_{\Omega_\varepsilon^*}(w_\varepsilon) = \lim_{\varepsilon\to 0}\mathcal{M}_{\widehat{\Omega}_\varepsilon^*}(w_\varepsilon\, 1_{\widehat{\Omega}_\varepsilon^*}) = \mathcal{M}_\Omega(w). \tag{1.9}$$

In particular, for Ψ in $L^p(\Omega)$,

$$\mathcal{M}_{\Omega_\varepsilon^*}(\Psi\, 1_{\Omega_\varepsilon^*}) \to \mathcal{M}_\Omega(\Psi). \tag{1.10}$$

Proof. (i) Since

$$\mathcal{T}_\varepsilon^*(1_{\Omega_\varepsilon^*}) = \mathcal{T}_\varepsilon^*(1_{\widehat{\Omega}_\varepsilon^*}) = 1_{\widehat{\Omega}_\varepsilon\times Y^*} \to 1 \text{ strongly in } L^1(\Omega\times Y^*),$$

statement (i) follows from Proposition 1.3 (ii).

(ii) By Proposition 1.3 (iii), every weak limit point W of the sequence $\{\mathcal{T}_\varepsilon^*(w_e)\}$ satisfies $w = \mathcal{M}_{Y^*}(W)$. Now, taking the corresponding subsequence,

$$\int_{\widehat{\Omega}_\varepsilon^*} w_\varepsilon dx = \frac{1}{|Y|}\int_{\Omega\times Y^*}\mathcal{T}_\varepsilon^*(w_\varepsilon)dx\,dy \to \frac{1}{|Y|}\int_{\Omega\times Y^*} W\,dx\,dy = \frac{|Y^*|}{|Y|}\mathcal{M}_\Omega(w).$$

This holds for every subsequence. Consequently, for the full sequence, and using (1.8) above,

$$\lim_{\varepsilon\to 0}\int_{\widehat{\Omega}_\varepsilon^*} w_\varepsilon = \frac{|Y^*|}{|Y|}\mathcal{M}_\Omega(w) \quad \text{and} \quad \mathcal{M}_{\widehat{\Omega}_\varepsilon^*}(w_\varepsilon) \to \mathcal{M}_\Omega(w).$$

On the other hand, $\displaystyle\int_{\Omega_\varepsilon^*} w_\varepsilon dx - \int_{\widehat{\Omega}_\varepsilon^*} w_\varepsilon dx = \int_\Omega \widetilde{w}_\varepsilon 1_{\Omega_\varepsilon^*\setminus\widehat{\Omega}_\varepsilon^*}dx$ which goes to zero by equicontinuity since $|\Omega_\varepsilon^*\setminus\widehat{\Omega}_\varepsilon^*| \to 0$.

Consequently, $\mathcal{M}_{\Omega_\varepsilon^*}(w_\varepsilon) \to \mathcal{M}_\Omega(w)$ making use of (1.8) again. □

To state corrector results, we will make use of the averaging operator $\mathcal{U}_\varepsilon^*$, the adjoint of $\mathcal{T}_\varepsilon^*$. Its definition and properties are recalled below.

Definition 1.5. *For p in $[1,+\infty]$, the averaging operator $\mathcal{U}_\varepsilon^* : L^p(\Omega\times Y^*) \mapsto L^p(\Omega_\varepsilon^*)$ is defined as*

$$\mathcal{U}_\varepsilon^*(\Phi)(x) = \begin{cases} \dfrac{1}{|Y|}\displaystyle\int_Y \Phi\Big(\varepsilon\Big[\frac{x}{\varepsilon}\Big]_Y + \varepsilon z, \Big\{\frac{x}{\varepsilon}\Big\}_Y\Big)\,dz & \textit{a.e. for } x\in\widehat{\Omega}_\varepsilon^*, \\ 0 & \textit{a.e. for } x\in\Lambda_\varepsilon^*. \end{cases}$$

Proposition 1.6. *(Properties of $\mathcal{U}_\varepsilon^*$). Suppose that p is in $[1,+\infty[$.*
(i) Let $\{\Phi_\varepsilon\}$ be a bounded sequence in $L^p(\Omega\times Y^*)$ such that $\Phi_\varepsilon \rightharpoonup \Phi$ weakly in $L^p(\Omega\times Y^*)$. Then

$$\widetilde{\mathcal{U}_\varepsilon^*(\Phi_\varepsilon)} \rightharpoonup \frac{|Y^*|}{|Y|}\mathcal{M}_{Y^*}(\Phi) \quad \textit{weakly in } L^p(\Omega).$$

(ii) Let $\{\Phi_\varepsilon\}$ be a sequence such that $\Phi_\varepsilon \to \Phi$ strongly in $L^p(\Omega\times Y^*)$. Then

$$\mathcal{T}_\varepsilon^*(\mathcal{U}_\varepsilon^*(\Phi_\varepsilon)) \to \Phi \qquad \textit{strongly in } L^p(\Omega\times Y^*).$$

(iii) Let w_ε be in $L^p(\Omega_\varepsilon^*)$. Then, the following assertions are equivalent:

$$(c) \quad \mathcal{T}_\varepsilon^*(w_\varepsilon) \to \widehat{w} \quad \textit{strongly in } L^p(\Omega\times Y^*) \quad \textit{and} \quad \int_{\Lambda_\varepsilon^*} |w_\varepsilon|^p \to 0,$$

$$(d) \quad \|w_\varepsilon - \mathcal{U}_\varepsilon^*(\widehat{w})\|_{L^p(\Omega_\varepsilon^*)} \to 0.$$

In [4], the following geometrical hypothesis was introduced for the set Y^*:

Hypothesis ($\mathbf{H_p}$) *The open set Y^* satisfies the Poincaré-Wirtinger inequality for the exponent p ($p\in[1,+\infty]$) and for every vector b_i, $i\in\{1,\dots,n\}$, of the basis of G, the interior of $\overline{Y^*\cup(b_i+Y^*)}$ is connected.*

As a consequence, the following compactness result holds;

Theorem 1.7. *Under Hypothesis ($\mathbf{H_p}$), suppose that w_ε in $W^{1,p}(\Omega_\varepsilon^*)$ satisfies*

$$\|w_\varepsilon\|_{W^{1,p}(\Omega_\varepsilon^*)} \le C.$$

Then, there exist w in $W^{1,p}(\Omega)$ and $\widehat{w}$ in $L^p(\Omega; W^{1,p}_{per}(Y^*))$ with $M_{Y^*}(\widehat{w})\equiv 0$, such that, up to a subsequence,

$$(i) \begin{cases} \mathcal{T}_\varepsilon^*(w_\varepsilon) \to w & \textit{strongly in } L^p_{loc}(\Omega; W^{1,p}(Y^*)), \\ \mathcal{T}_\varepsilon^*(w_\varepsilon) \rightharpoonup w & \textit{weakly in } L^p(\Omega; W^{1,p}(Y^*)), \end{cases}$$

$$(ii) \ \ \mathcal{T}_\varepsilon^*(\nabla w_\varepsilon) \rightharpoonup \nabla w + \nabla_y \widehat{w} \quad \textit{weakly in } L^p(\Omega\times Y^*).$$

This compactness result is essential for homogenization problems. Since the unfolding operator $\mathcal{T}_\varepsilon^*$ transforms functions defined on the oscillating domain Ω_ε^* into functions defined on the fixed domain $\Omega\times Y^*$, there is no need of any extension operator to the whole of Ω. Therefore, regularity hypotheses on the boundary ∂S insuring the existence of such extension operators, are not required (contrary to the "classical" methods, cf. references in [4]).

It follows from the considerations in [4] that under Hypothesis ($\mathbf{H_p}$), there exists a finite $\rho(Y)\ge 2\,\mathrm{diam}(Y)$ such that Proposition 1.12 below holds. To state this Proposition, we introduce the following notations (see Figure 4):

$$\widetilde{\Omega}_\varepsilon \doteq \Big\{x\in\Omega \mid \mathrm{dist}(x,\partial\Omega) > \varepsilon\rho(Y)\Big\}, \qquad \widetilde{\Omega}_\varepsilon^* = \Omega_\varepsilon^* \cap \widetilde{\Omega}_\varepsilon. \tag{1.11}$$

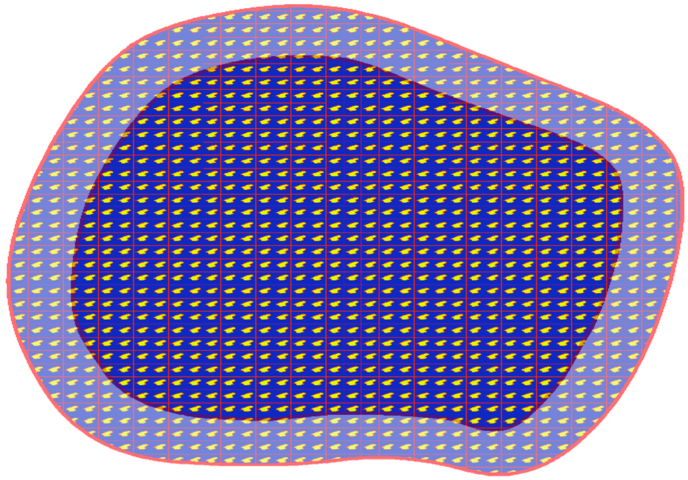


Figure 4. The set $\widetilde{\Omega}^*_\varepsilon$ (in dark blue)

It is easily seen that $\widetilde{\Omega}_\varepsilon \subset \widehat{\Omega}_\varepsilon$,, where $\widehat{\Omega}_\varepsilon$ is defined by (1.4). Moreover, for ε small enough, the set $\widetilde{\Omega}_\varepsilon$ is connected due to the fact that Ω is connected with Lipschitz boundary.

Remark 1.8. *In [4], it can be seen that if Y is a parallelotop, then $\rho(Y) = 2\,diam(Y)$. In general, $\rho(Y)$ can be explicitly computed and is related to the number of* $\mathbf{b}$*-parallelotops needed to cover Y.*

Proposition 1.9. *Suppose that hypothesis* $(\mathbf{H_p})$ *holds. Then, the set $\widetilde{\Omega}^*_\varepsilon$ is included in a single connected component of Ω^*_ε.*

Proof. To show this result, one first notes that $\widetilde{\Omega}^*_\varepsilon$ is included in the set $\widehat{\Omega}^{**}_\varepsilon$ introduced in the appendix of [4]. Furthermore, $\widehat{\Omega}^{**}_\varepsilon$ is connected and is included in Ω^*_ε. Note that if Y is a parallelotop, $\widehat{\Omega}^{**}_\varepsilon$ coincides with $\widehat{\Omega}^*_\varepsilon$. □

Definition 1.10. *The connected component of Ω^*_ε containing $\widetilde{\Omega}^*_\varepsilon$ is denoted $\mathcal{C}^*_\varepsilon$.*

Remark 1.11. *All the connected components of Ω^*_ε, others than $\mathcal{C}^*_\varepsilon$, lie near the boundary of Ω. Therefore, in some sense, $\mathcal{C}^*_\varepsilon$ is the "main" connected component of Ω^*_ε.*

Proposition 1.12. *Suppose that hypothesis* $(\mathbf{H_p})$ *holds.*

(a) For every function w in $W^{1,p}_{loc}(\mathcal{C}^*_\varepsilon)$ with $\nabla w \in L^p(\mathcal{C}^*_\varepsilon)$, its restriction $w\,1_{\widetilde{\Omega}^*_\varepsilon}$ belongs to $L^p(\widetilde{\Omega}^*_\varepsilon)$ (and not only to $L^p_{loc}(\widetilde{\Omega}^*_\varepsilon)$).

(b) For every function φ in $W^{1,p}(\widetilde{\Omega}^*_\varepsilon)$, there exist two functions φ^1 in $W^{1,p}(\widetilde{\Omega}_\varepsilon)$ and $\varphi^2 \in W^{1,p}(\widetilde{\Omega}^*_\varepsilon)$ and a constant C_p independent of ε, such that

$$\begin{aligned}
&(i) \quad \varphi = \varphi^1 + \varphi^2, \quad \text{a.e. in } \widetilde{\Omega}^*_\varepsilon,\\
&(ii) \quad \|\varphi^1\|_{L^p(\widetilde{\Omega}_\varepsilon)} \le C_p \|\varphi\|_{L^p(\widetilde{\Omega}^*_\varepsilon)}, \qquad \|\nabla\varphi^1\|_{L^p(\widetilde{\Omega}_\varepsilon)} \le C_p \|\nabla\varphi\|_{L^p(\widetilde{\Omega}^*_\varepsilon)}, \\
&(iii) \quad \|\varphi^2\|_{L^p(\widetilde{\Omega}^*_\varepsilon)} \le \varepsilon C_p \|\nabla\varphi\|_{L^p(\widetilde{\Omega}^*_\varepsilon)}, \quad \|\nabla\varphi^2\|_{L^p(\widetilde{\Omega}^*_\varepsilon)} \le C_p \|\nabla\varphi\|_{L^p(\widetilde{\Omega}^*_\varepsilon)}.
\end{aligned} \tag{1.12}$$

Proof. (a) This follows from the fact that under $(\mathbf{H_p})$, the set $\widehat{\Omega}_\varepsilon^{**}$ itself satisfies the Poincaré-Wirtinger inequality (but the corresponding constant may not be uniformly bounded with respect to ε). Thus, w belongs to $L^p(\widehat{\Omega}_\varepsilon^{**})$.
(b) This corresponds to the decomposition $w = \mathcal{Q}_\varepsilon^*(w)+\mathcal{R}_\varepsilon^*(w)$, as introduced in Section 3.2.2 of [4] (see Propositions 3.6 – 3.8). □

In the Appendix, the following result is proved:

Lemma 1.13 (Poincaré-Wirtinger inequality)**.** *Assume that Ω is a bounded domain with Lipschitz boundary. Then there exist $\delta_0 > 0$ and a common Poincaré-Wirtinger constant C_p for all the sets $\widetilde{\Omega}_\varepsilon$ for $\varepsilon \in]0, \delta_0]$, i.e.*

$$\forall \phi \in W^{1,p}(\widetilde{\Omega}_\varepsilon), \qquad ||\phi - \mathcal{M}_{\widetilde{\Omega}_\varepsilon}(\phi)||_{L^p(\widetilde{\Omega}_\varepsilon)} \leq C_p||\nabla\phi||_{[L^p(\widetilde{\Omega}_\varepsilon)]},$$

where C is independent of ε.

As a combination of the two previous statements, the sets $\widetilde{\Omega}_\varepsilon^*$ satisfy the Poincaré-Wirtinger inequality with a constant independent of ε.

Theorem 1.14 (Poincaré-Wirtinger inequality for perforated domains)**.** *There is a constant C_p independent of ε such that for every function φ in $W^{1,p}(\widetilde{\Omega}_\varepsilon^*)$,*

$$\|\varphi - \mathcal{M}_{\widetilde{\Omega}_\varepsilon^*}(\varphi)\|_{L^p(\widetilde{\Omega}_\varepsilon^*)} \leq C_p\|\nabla\varphi\|_{L^p(\widetilde{\Omega}_\varepsilon^*)}. \tag{1.13}$$

Proof. Using (b) of Proposition 1.12, we decompose $\varphi = \varphi^1 + \varphi^2$ and apply Lemma 1.13 to φ^1:

$$\|\varphi^1 - \mathcal{M}_{\widetilde{\Omega}_\varepsilon}(\varphi^1)\|_{L^p(\widetilde{\Omega}_\varepsilon)} \leq C_p\|\nabla\varphi^1\|_{L^p(\widetilde{\Omega}_\varepsilon)}.$$

Taking the average over $\widetilde{\Omega}_\varepsilon^*$ gives

$$\|\mathcal{M}_{\widetilde{\Omega}_\varepsilon^*}(\varphi^1) - \mathcal{M}_{\widetilde{\Omega}_\varepsilon}(\varphi^1)\|_{L^p(\widetilde{\Omega}_\varepsilon)} \leq C_p\|\nabla\varphi^1\|_{L^p(\widetilde{\Omega}_\varepsilon)},$$

hence,

$$\|\varphi^1 - \mathcal{M}_{\widetilde{\Omega}_\varepsilon^*}(\varphi^1)\|_{L^p(\widetilde{\Omega}_\varepsilon)} \leq 2C_p\|\nabla\varphi^1\|_{L^p(\widetilde{\Omega}_\varepsilon)} \leq C'_p\|\nabla\varphi\|_{L^p(\widetilde{\Omega}_\varepsilon^*)}.$$

Also by (1.13)(iii),

$$\|\varphi^2\|_{L^p(\widetilde{\Omega}_\varepsilon^*)} + |\mathcal{M}_{\widetilde{\Omega}_\varepsilon^*}(\varphi^2)| \leq \varepsilon C_p\|\nabla\varphi\|_{L^p(\widetilde{\Omega}_\varepsilon^*)},$$

from which the result follows. □

The main convergence result in this section is the following:

Theorem 1.15. *Let p be in $]1,+\infty]$ and w_ε in $W^{1,p}_{loc}(\Omega^*_\varepsilon)$ satisfying*

$$\|\nabla w_\varepsilon\|_{L^p(\mathcal{C}^*_\varepsilon)} \le C, \tag{1.14}$$

and such that $\mathcal{M}_{\widetilde{\Omega}^*_\varepsilon}(w_\varepsilon) = 0$. Then, up to a subsequence (still denoted ε), there are two functions w in $W^{1,p}(\Omega)$ and $\widehat{w}$ in $L^p(\Omega; W^{1,p}_{per}(Y^*))$, such that

$$\begin{aligned}
&\mathcal{T}^*_\varepsilon(w_\varepsilon 1_{\widetilde{\Omega}_\varepsilon}) \rightharpoonup w \quad \textit{weakly in } L^p(\Omega\times Y^*)\cap L^p_{loc}(\Omega; W^{1,p}(Y^*)),\\
&\mathcal{T}^*_\varepsilon\big((\nabla w_\varepsilon)\, 1_{\mathcal{C}^*_\varepsilon}\big) \rightharpoonup \nabla w + \nabla_y \widehat{w} \quad \textit{weakly in } L^p(\Omega\times Y^*).
\end{aligned}$$

For $p=+\infty$, the weak convergences above are replaced by weak-$*$ convergences.

Proof. Note first that according to Proposition 1.12 (a) and (1.14), $\mathcal{M}_{\widetilde{\Omega}^*_\varepsilon}$ exists so that hypothesis $\mathcal{M}_{\widetilde{\Omega}^*_\varepsilon}(w_\varepsilon) = 0$ makes sense. Then, using Theorem 1.14, we deduce that $\|w_\varepsilon\|_{W^{1,p}(\widetilde{\Omega}^*_\varepsilon)}$ is bounded. It follows that, up to a subsequence, there exist W in $L^p(\Omega\times Y^*)$ and F in $[L^p(\Omega\times Y^*)]^n$ such that

$$\begin{aligned}
&\mathcal{T}^*_\varepsilon(w_\varepsilon 1_{\widetilde{\Omega}_\varepsilon}) \rightharpoonup W \quad \text{weakly in } L^p(\Omega\times Y^*),\\
&\mathcal{T}^*_\varepsilon\big((\nabla w_\varepsilon)\, 1_{\mathcal{C}^*_\varepsilon}\big) \rightharpoonup F \quad \text{weakly in } L^p(\Omega\times Y^*).
\end{aligned} \tag{1.15}$$

But Theorem 1.7 applied in every relatively compact open subset ω of Ω implies that, up to a subsequence, there exist w in $W^{1,p}_{loc}(\Omega)$ and $\widehat{w}$ in $L^p_{loc}(\Omega; W^{1,p}_{per}(Y^*))$ with $\mathcal{M}_{Y^*}(\widehat{w}) \equiv 0$, with

$$\begin{aligned}
&\mathcal{T}^*_\varepsilon(w_\varepsilon 1_{\widetilde{\Omega}_\varepsilon}) \rightharpoonup w \quad \text{weakly in } L^p(\omega; W^{1,p}(Y^*)),\\
&\mathcal{T}^*_\varepsilon\big((\nabla w_\varepsilon)\, 1_{\mathcal{C}^*_\varepsilon}\big) \rightharpoonup \nabla w + \nabla_y \widehat{w} \quad \text{weakly in } L^p(\omega; L^p(Y^*)),
\end{aligned} \tag{1.16}$$

for all ω. The above convergences imply $w(x) = W(x,.)$ and $\nabla w = \mathcal{M}_{Y^*}(F)$ in $L^p(\Omega)$. It then follows that w belongs to $W^{1,p}(\Omega)$ and $\widehat{w}$ to $L^p(\Omega; W^{1,p}_{per}(Y^*))$.

For $p=+\infty$, the proof is similar, using the weak-$*$ compactness criterion in L^∞ spaces. $\square$

2 A Neumann problem in perforated domains with Fredholm alternative

In this section, we would like to consider the homogenization of the Neumann problem whose variational formulation is

$$\text{Find } u_\varepsilon \text{ in } V(\Omega^*_\varepsilon) \text{ such that } \int_{\Omega^*_\varepsilon} A^\varepsilon \nabla u_\varepsilon\, \nabla v\, dx = \int_{\Omega^*_\varepsilon} f_\varepsilon v\, dx, \quad \forall v \in V(\Omega^*_\varepsilon), \tag{2.1}$$

where f_ε is given in $L^2(\Omega^*_\varepsilon)$. For any bounded open set $\mathcal{O}$, $V(\mathcal{O})$ denotes the space

$$V(\mathcal{O}) \doteq \{w \in H^1(\mathcal{O}), \mathcal{M}_{\mathcal{O}}(w) = 0\}. \tag{2.2}$$

Here, $A^\varepsilon(x) = (a^\varepsilon_{ij}(x))_{1\le i,j\le n}$ is given in $M(\alpha,\beta,\Omega^*_\varepsilon)$ where the latter denotes the set of the $n\times n$ matrices $B = B(x)$, $B = (b_{ij})_{1\le i,j\le n} \in (L^\infty(\Omega^*_\varepsilon))^{n\times n}$ such that for any $\lambda\in\mathbb{R}^n$ and a.e. on Ω^*_ε,

$$(B(x)\lambda,\lambda) \ge \alpha|\lambda|^2, \qquad |A(x)\lambda| \le \beta|\lambda|.$$

Existence and uniqueness of the solution for (2.1) follows by Lax-Milgram's theorem, provided that Ω^*_ε is connected and that the Poincaré-Wirtinger inequality holds for $H^1(\Omega^*_\varepsilon)$ (which implies that $V(\Omega^*_\varepsilon)$ is a Hilbert space for the norm $|v|_{V(\Omega^*_\varepsilon)} \doteq |\nabla v|_{L^2(\Omega^*_\varepsilon)}$). Under such conditions, without loss of generality, one can assume that $\mathcal{M}_{\Omega^*_\varepsilon}(f_\varepsilon) = 0$ since the test functions have zero average. Indeed, it is standard that this problem corresponds to the following formulation in the sense of distributions:

$$\begin{cases} -\mathrm{div}\,(A^\varepsilon\nabla u_\varepsilon) = f_\varepsilon - \mathcal{M}_{\Omega^*_\varepsilon}(f_\varepsilon) & \text{in } \Omega^*_\varepsilon \\ A^\varepsilon\nabla u_\varepsilon\cdot n_\varepsilon = 0 \quad \text{on } \partial\Omega^*_\varepsilon. \end{cases} \tag{2.3}$$

If $\mathcal{M}_{\Omega^*_\varepsilon}(f_\varepsilon) = 0$, this is expressed in a variational form on the whole of $H^1(\Omega^*_\varepsilon)$ as

$$\text{Find } w_\varepsilon \text{ in } H^1(\Omega^*_\varepsilon) \text{ such that } \int_{\Omega^*_\varepsilon} A^\varepsilon\nabla w_\varepsilon\,\nabla v\,dx = \int_{\Omega^*_\varepsilon} f_\varepsilon v\,dx, \quad \forall v\in H^1(\Omega^*_\varepsilon).$$

This latter problem has many solutions w_ε, all differing by a constant from the unique solution of (2.1), provided it exists. In this framework, the Fredholm alternative applies and the corresponding condition is again $\mathcal{M}_{\Omega^*_\varepsilon}(f_\varepsilon) = 0$.

However, there are multiple reasons why this presentation does not work. It is not true in general that Ω^*_ε is connected, which precludes that it satisfies the Poincaré-Wirtinger inequality. Figure 5 gives an example a disconnected Ω^*_ε for a very smooth Ω.

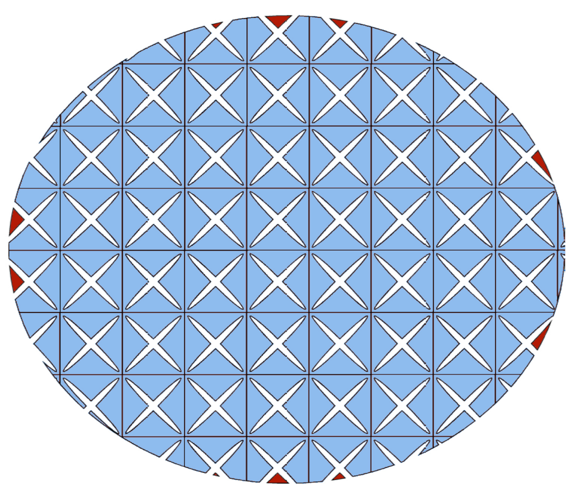


Figure 5. Example of a disconnected set Ω^*_ε ; in blue is the set $\mathcal{C}^*_\varepsilon$

Furthermore, the boundary of Ω^*_ε can be so irregular that there can be functions with gradient in $L^2(\Omega^*_\varepsilon)$ which are not integrable in Ω^*_ε. Consequently, the space $V(\Omega^*_\varepsilon)$ does not make sense. These difficulties are specific to the case of the Fredholm alternative (they do not appear, for example, in [2] and [4]) and the main contribution of this paper is to give a way to overcome them.

Due to Propositions 1.9 and 1.12, a "natural" candidate for domain on which to consider the approximate problem for (2.1) is the set $\mathcal{C}_\varepsilon^*$ introduced in Definition 1.10. With appropriate assumptions on the right hand side f_ε, one can expect that there is a unique solution to the approximate problem (2.7) below in the space

$$W(\mathcal{C}_\varepsilon^*) \doteq \{\varphi \in H^1_{loc}(\mathcal{C}_\varepsilon^*), \nabla\varphi \in L^2(\mathcal{C}_\varepsilon^*), \text{ and } \mathcal{M}_{\widetilde{\Omega}_\varepsilon^*}(\varphi) = 0\}. \tag{2.4}$$

Indeed, due to (a) of Proposition 1.12, there exists a constant C_ε (not necessarily bounded with respect to ε) such that for all φ in $W(\mathcal{C}_\varepsilon^*)$,

$$\|\varphi\|_{L^2(\widetilde{\Omega}_\varepsilon^*)} \leq C_\varepsilon \|\nabla\varphi\|_{L^2(\mathcal{C}_\varepsilon^*)}, \tag{2.5}$$

so that $W(\mathcal{C}_\varepsilon^*)$ is a Hilbert space when endowed with the norm

$$\|\varphi\|_{W(\mathcal{C}_\varepsilon^*)} \doteq \|\nabla\varphi\|_{L^2(\mathcal{C}_\varepsilon^*)}.$$

Uniform estimates can now be obtained under the hypothesis

$$f_\varepsilon \quad \text{belongs to } L^2(\mathcal{C}_\varepsilon^*), \textbf{vanishes outside } \widetilde{\Omega}_\varepsilon^* \text{ and satisfies } \mathcal{M}_{\widetilde{\Omega}_\varepsilon^*}(f_\varepsilon) = 0. \tag{2.6}$$

Theorem 2.1 (Existence and priori estimates for a proper approximation)**.** *Suppose that Hypothesis* $(\mathbf{H_2})$ *is satisfied and* (2.6) *holds. Then, there is a unique function* u_ε *in* $W(\mathcal{C}_\varepsilon^*)$*, solution of*

$$\int_{\mathcal{C}_\varepsilon^*} A^\varepsilon(x)\nabla u_\varepsilon(x)\nabla\varphi(x)\,dx = \int_{\mathcal{C}_\varepsilon^*} f_\varepsilon(x)\varphi(x)dx \tag{2.7}$$

for all φ *in* $W(\mathcal{C}_\varepsilon^*)$.

It satisfies the following estimates with a constant C *independent of* ε*:*

$$\begin{aligned} &\|\nabla u_\varepsilon\|_{L^2(\mathcal{C}_\varepsilon^*)} \leq C\|f_\varepsilon\|_{L^2(\Omega_\varepsilon^*)}, \\ &\|u_\varepsilon\|_{L^2(\widetilde{\Omega}_\varepsilon^*)} \leq C\|f_\varepsilon\|_{L^2(\Omega_\varepsilon^*)}. \end{aligned} \tag{2.8}$$

Proof. Existence and uniqueness of the solution u_ε follow immediately from the Lax-Milgram theorem but this does not give a uniform estimate, unless C_ε in (2.5) is bounded.

To obtain a uniform estimate, we start with the straightforward inequality

$$\alpha\|\nabla u_\varepsilon\|^2_{L^2(\mathcal{C}_\varepsilon^*)} \leq \|f_\varepsilon\|_{L^2(\widetilde{\Omega}_\varepsilon^*)}\|u_\varepsilon\|_{L^2(\widetilde{\Omega}_\varepsilon^*)}.$$

Since $\mathcal{M}_{\widetilde{\Omega}_\varepsilon^*}(u_\varepsilon) = 0$, by Proposition 1.14, $\|u_\varepsilon\|_{L^2(\widetilde{\Omega}_\varepsilon^*)} \leq C_2\|\nabla u_\varepsilon\|_{L^2(\mathcal{C}_\varepsilon^*)}$, from which (2.8) follows. $\square$

3 Homogenization result

Theorem 3.1. *Suppose that hypotheses* $(\mathbf{H_2})$ *and* (2.6) *hold. Assume furthermore that the extension by 0 of* f_ε *to* Ω*, denoted* $\widetilde{f_\varepsilon}$*, satisfies*

$$\widetilde{f_\varepsilon} \rightharpoonup \frac{|Y^*|}{|Y|} f \text{ weakly in } L^2(\Omega)^1. \tag{3.1}$$

Assume that for some matrix A*,*

$$\mathcal{T}^*_\varepsilon\big(A^\varepsilon\big) \to A \qquad a.e. \text{ in } \Omega \times Y^*.$$

Let u_ε *be the solution of Problem* (2.7)*. Then, there exist* u *in* $V(\Omega)$ *and* $\widehat{u}$ *in* $L^2(\Omega; H^1_{per}(Y^*))$*, satisfying*

$$\begin{aligned} &(i) \quad \mathcal{T}^*_\varepsilon((u_\varepsilon)\, 1_{\widetilde{\Omega}^*_\varepsilon}) \rightharpoonup u \quad \text{ weakly in } L^2(\Omega; H^1(Y^*)),\\ &(ii) \quad \mathcal{T}^*_\varepsilon(\nabla u_\varepsilon 1_{\mathcal{C}^*_\varepsilon}) \rightharpoonup \nabla u + \nabla_y \widehat{u} \quad \text{ weakly in } L^2(\Omega \times Y^*), \end{aligned} \tag{3.2}$$

and the pair $(u, \widehat{u})$ *is the unique solution in of the problem*

$$\begin{cases} \mathcal{M}_\Omega(u) = 0, \ \mathcal{M}_{Y^*}(\widehat{u}) = 0\\ \displaystyle\frac{1}{|Y|}\int_{\Omega\times Y^*} A(x,y)\big[\nabla u(x) + \nabla_y \widehat{u}(x,y)\big]\big[\nabla\Psi(x) + \nabla_y\Phi(x,y)\big]\, dxdy\\ \displaystyle\qquad\qquad\qquad = \frac{|Y^*|}{|Y|}\int_\Omega f(x)\,\Psi(x)\, dx,\\ \forall \Psi \in V(\Omega), \ \forall\Phi \in L^2(\Omega;\, H^1_{per}(Y^*)), \end{cases} \tag{3.3}$$

Proof. Using estimates (2.8), convergences (3.2) follow from Theorem 1.15. This is up to a subsequence, but as usual, the uniqueness of the solution of the limit problem implies that the whole sequence converges.

At this point, the proof is somewhat standard when using unfolding. For completeness, we now give its outline.

First, unfold the left-hand side of (2.7) using as test function

$$v_\varepsilon(x) \doteq \Psi(x) + \varepsilon\,\varphi(x)\,\psi\Big(\frac{x}{\varepsilon}\Big) - \mathcal{M}_{\widetilde{\Omega}^*_\varepsilon}\Big(\varepsilon\,\varphi(x)\,\psi\Big(\frac{x}{\varepsilon}\Big)\Big)$$

where Ψ is in $C^\infty(\overline{\Omega})$ with $\mathcal{M}_\Omega(\Psi) = 0$, φ is in $\mathcal{D}(\Omega)$ and $\psi = \psi(y)$ is in $H^1_{per}(Y^*)$. It then follows by Proposition 1.3 (i) and (1.10) that

$$\begin{aligned} &\mathcal{T}^*_\varepsilon(v_\varepsilon) \to \Psi \quad \text{strongly in } L^2(\Omega \times Y^*),\\ &\mathcal{T}^*_\varepsilon\Big(\varphi\psi\Big(\frac{\cdot}{\varepsilon}\Big)\Big) \to \Phi \quad \text{strongly in } L^2(\Omega\times Y^*), \quad \text{with } \Phi(x,y) = \varphi(x)\,\psi(y). \end{aligned} \tag{3.4}$$

Since

$$\nabla v_\varepsilon(x) = \nabla\Psi(x) + \varphi(x)(\nabla_y\psi)\Big(\frac{x}{\varepsilon}\Big) + \varepsilon\nabla\varphi(x)\psi\Big(\frac{x}{\varepsilon}\Big),$$

[1] following the normalization of Proposition 1.3 (iii)

by Proposition 1.3 (i),

$$\mathcal{T}_\varepsilon^*(\nabla v_\varepsilon) \to \nabla \Psi + \nabla_y \Phi \quad \text{ strongly in } L^2(\Omega \times Y^*). \tag{3.5}$$

This gives

$$\begin{aligned}\int_{\mathcal{C}_\varepsilon^*} A^\varepsilon \nabla u_\varepsilon \, \nabla v^\varepsilon \, dx &= \frac{1}{|Y|} \int_{\Omega \times Y^*} \mathcal{T}_\varepsilon^*(A^\varepsilon) \mathcal{T}_\varepsilon^*\big((\nabla u_\varepsilon) 1_{\mathcal{C}_\varepsilon^*}\big)\, \mathcal{T}_\varepsilon^*(\nabla v^\varepsilon) \, dx\, dy \\ &\quad + \int_{\Lambda_\varepsilon^*} A^\varepsilon \nabla u_\varepsilon \, \nabla v^\varepsilon \, dx \\ &\to \frac{1}{|Y|} \int_{\Omega \times Y^*} A(x,y) \big[\nabla u_0(x) + \nabla_y \widehat{u}_0(x,y)\big] \big[\nabla \Psi(x) + \phi(x) \nabla \psi(y)\big] \, dx\, dy,\end{aligned} \tag{3.6}$$

since

$$\Big| \int_{\Lambda_\varepsilon^*} A^\varepsilon \nabla u_\varepsilon \, \nabla v^\varepsilon \, dx \Big| \leq C |\Lambda_\varepsilon^*|^{1/2} \|\nabla v^\varepsilon\|_{L^\infty(\Omega)} \|\nabla u_\varepsilon\|_{L^2(\Omega_\varepsilon^*)} \to 0$$

Concerning the right-hand side of (2.1), we see that

$$\lim_{\varepsilon \to 0} \int_{\Omega_\varepsilon^*} f_\varepsilon \, v_\varepsilon \, dx = \lim_{\varepsilon \to 0} \int_\Omega \widetilde{f}_\varepsilon \, v_\varepsilon \, dx = \frac{|Y^*|}{|Y|} \int_\Omega f \, \Psi \, dx, \qquad \forall \Psi \in C^\infty(\overline{\Omega}). \tag{3.7}$$

Equation (3.3) follows from the density of the functions of $C^\infty(\overline{\Omega})$ with zero average in $V(\Omega)$ and that of the tensor product $\mathcal{D}(\Omega) \otimes H^1_{per}(Y^*)$ in $L^2(\Omega; H^1_{per}(Y^*))$.

Finally, by Corollary 1.4, $\mathcal{M}_\Omega(u) = 0$, which implies the uniqueness for the solution of (3.3). □

The homogenized limit problem can now be made explicit.

Theorem 3.2. *The homogenized formulation associated with Theorem 3.1, under the hypotheses therein, is*

$$\begin{cases} -\text{div } (A^0 \nabla u) = \dfrac{|Y^*|}{|Y|} f \quad \text{in } \Omega, \\ \mathcal{M}_\Omega(u) = 0. \end{cases} \tag{3.8}$$

The homogenized matrix $A^0 = (a^0_{ij})_{1 \leq i,j \leq n}$ *is elliptic and defined by*

$$a^0_{ij} = \mathcal{M}_{Y^*}\Big(a_{ij} - \sum_{k=1}^n a_{ik} \frac{\partial \chi_j}{\partial y_k}\Big) = \mathcal{M}_{Y^*}(a_{ij}) - \mathcal{M}_{Y^*}\Big(\sum_{k=1}^n a_{ik} \frac{\partial \chi_j}{\partial y_k}\Big). \tag{3.9}$$

where the corrector functions $\chi_j \in L^\infty(\Omega; H^1_{per}(Y^*))$ $(j = 1, \ldots, n)$, *are, for a.e.* x *in* Ω, *the solutions of the cell problems*

$$\begin{cases} -\displaystyle\sum_{i,k=1}^n \frac{\partial}{\partial y_i} \Big(a_{ik}(x,y)\big(\frac{\partial \chi_j(x,y)}{\partial y_k} - \delta_{jk}\big)\Big) = 0 \quad \text{in } Y^*, \\ \displaystyle\sum_{i,k=1}^n a_{ik}(x,y)\big(\frac{\partial \chi_j(x,y)}{\partial y_k} - \delta_{jk}\big) n_i = 0 \quad \text{on } \partial S, \\ \mathcal{M}_{Y^*}(\chi_j)(x,\cdot) = 0, \qquad \chi_j(x,\cdot) \quad Y\text{-periodic}, \end{cases} \tag{3.10}$$

Proof. Taking $\Psi = 0$ in problem (3.3) yields

$$\widehat{u}(x,y) = \sum_{i=1}^{n} \frac{\partial u}{\partial x_i}(x)\chi_i(x,y). \tag{3.11}$$

Inserting into (3.3) with $\Phi = 0$ gives the result. $\square$

Under an additional hypothesis on f_ε one also obtains the strong convergence of the energy.

Proposition 3.3. *Suppose that hypotheses of Theorem 3.1 hold. Assume furthermore that*

$$\mathcal{T}^*_\varepsilon(f_\varepsilon) \quad \textit{is compact in } L^2(\Omega \times Y^*). \tag{3.12}$$

Then

$$\lim_{\varepsilon\to 0} \int_{\mathcal{C}^*_\varepsilon} A^\varepsilon \nabla u_\varepsilon \nabla u_\varepsilon \, dx = \frac{1}{|Y|} \int_{\Omega\times Y^*} A\big[\nabla u + \nabla_y \widehat{u}\big] \, \big[\nabla u + \nabla_y \widehat{u}\big] \, dx \, dy \tag{3.13}$$

Consequently,

$$\mathcal{T}^*_\varepsilon(\nabla u_\varepsilon 1_{\mathcal{C}^*_\varepsilon}) \to \nabla u + \nabla_y \widehat{u} \quad \textit{ strongly in } L^2(\Omega \times Y^*). \tag{3.14}$$

Proof. Let F be a strong limit point of $\mathcal{T}^*_\varepsilon(f_\varepsilon)$ in $L^2(\Omega\times Y^*)$ (which exits by assumption (3.12)). According to Proposition 1.3 (iii), $\mathcal{M}_{Y^*}(F) = f$.

Applying Proposition 1.2 (i), it follows that

$$\int_{\widetilde{\Omega}^*_\varepsilon} f_\varepsilon \, u_\varepsilon \, dx = \frac{1}{|Y|} \int_{\Omega\times Y^*} \mathcal{T}^*_\varepsilon(f_\varepsilon)\mathcal{T}^*_\varepsilon(u_\varepsilon) \, dx \, dy.$$

Consequently,

$$\lim_{\varepsilon\to 0} \int_{\widetilde{\Omega}^*_\varepsilon} f_\varepsilon \, u_\varepsilon \, dx = \frac{|Y^*|}{|Y|} \int_\Omega f \, u \, dx$$

for the considered subsequence associated to F. But this result holds independently of F, hence holds for the whole sequence $\{\varepsilon\}$. The proof of (3.13) is now complete, since, on the one hand, by 2.7,

$$\int_{\mathcal{C}^*_\varepsilon} A^\varepsilon \nabla u_\varepsilon \nabla u_\varepsilon \, dx = \int_{\widetilde{\Omega}^*_\varepsilon} f_\varepsilon \, u_\varepsilon \, dx,$$

while on the other, by (3.3),

$$\frac{1}{|Y|} \int_{\Omega\times Y^*} A\big[\nabla u + \nabla_y \widehat{u}\big] \, \big[\nabla u + \nabla_y \widehat{u}\big] \, dx \, dy = \frac{|Y^*|}{|Y|} \int_\Omega f \, u \, dx.$$

The strong convergence (3.14) follows now by Lemma 5.8 of [4], taking into account that $A^\varepsilon \, 1_{\mathcal{C}^*_\varepsilon}$ converges almost everywhere to A. $\square$

Corollary 3.4 (Corrector results)**.** *As $\varepsilon \to 0$,*

$$\Big\|\nabla u_\varepsilon - \nabla u - \sum_{i=1}^n \mathcal{U}^*_\varepsilon\Big(\frac{\partial u}{\partial x_i}\Big)\,\mathcal{U}^*_\varepsilon(\nabla_y \chi_i)\Big\|_{L^2(\mathcal{C}^*_\varepsilon)} \to 0. \tag{3.15}$$

In the case where the matrix field A does not depend on x, the following corrector result holds:

$$\Big\|u_\varepsilon - u - \varepsilon\sum_{i=1}^n \mathcal{Q}_\varepsilon\Big(\frac{\partial u}{\partial x_i}\Big)\chi_i\Big(\Big\{\frac{\cdot}{\varepsilon}\Big\}_Y\Big)\Big\|_{H^1(\widetilde{\Omega}^*_\varepsilon)} \to 0. \tag{3.16}$$

Proof. By construction, for $i = 1, \dots, n$, the function χ_i belongs to $L^\infty(\Omega; H^1(Y^*))$. Due to convergence (3.14) (note that $\Lambda_\varepsilon \cap \widetilde{\Omega}^*_\varepsilon = \emptyset$) , Proposition 1.6 (iii) gives

$$\big\|\nabla u_\varepsilon 1_{\mathcal{C}^*_\varepsilon} - \mathcal{U}^*_\varepsilon(\nabla u + \nabla_y \widehat{u})\big\|_{L^2(\Omega^*_\varepsilon)} \to 0. \tag{3.17}$$

By Proposition 1.3 (i) and (3.11) this implies

$$\Big\|\nabla u_\varepsilon 1_{\mathcal{C}^*_\varepsilon} - \nabla u - \sum_{i=1}^n \mathcal{U}^*_\varepsilon\Big(\frac{\partial u}{\partial x_i}\nabla_y \chi_i\Big)\Big\|_{L^2(\Omega^*_\varepsilon)} \to 0, \tag{3.18}$$

and gives convergence (3.15).

If A does not depend on x, this becomes

$$\Big\|\nabla u_\varepsilon 1_{\mathcal{C}^*_\varepsilon} - \nabla u - \sum_{i=1}^n \mathcal{U}^*_\varepsilon\Big(\frac{\partial u}{\partial x_i}\Big)\nabla_y \chi_i\Big(\Big\{\frac{\cdot}{\varepsilon}\Big\}_Y\Big)\Big\|_{L^2(\Omega^*_\varepsilon)} \to 0, \tag{3.19}$$

From (3.19), the proof of convergence (3.16) follows along the lines of that (5.31) in [4], taking also into account the fact that $\mathcal{M}_{\widetilde{\Omega}^*_\varepsilon}(u_\varepsilon) = 0$ and (1.13). $\square$

4 Appendix

The aim of this section is to prove Lemma 1.13. The notation Ω_ε in this section corresponds to $\widetilde{\Omega}_\varepsilon$ of the previous sections.

Let p in $(1, +\infty)$. From now on, Ω is a bounded open set with Lipschitz boundary in $\mathbb{R}^n$. It is well-known (see [3]) that this is equivalent to the fact that Ω has the uniform cone property which we recall below. In this definition, a cone is the convex envelope of the union of the origin and a closed ball which does not contain the origin.

Definition 4.1. *The bounded open set Ω has the uniform cone property whenever there exist a finite open cover $\{\mathcal{U}_j\}_{1\le j\le N}$ of $\partial\Omega$, and a corresponding family $\{\mathcal{C}_j\}$ of cones, each isometric to some fixed cone $\mathcal{C}$, such that,*

(i) *for some strictly positive constant δ_1,* $\Big\{x \in \Omega \mid dist(x, \partial\Omega) < \delta_1\Big\} \subset \bigcup_{j=1}^N \mathcal{U}_j,$

(ii) *for every j,* $\bigcup_{x\in\Omega\cap\mathcal{U}_j} \Big(x + \mathcal{C}_j\Big) \subset \Omega.$

Let Ω_δ denote the set

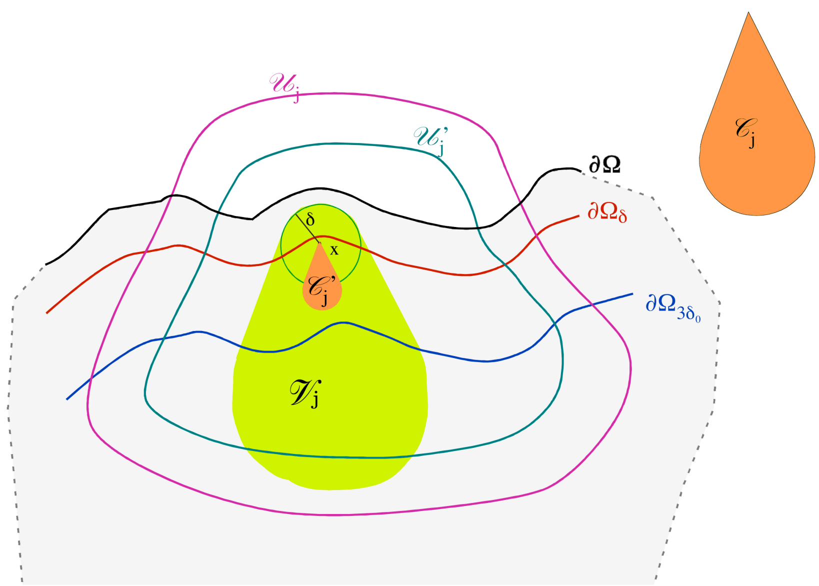
$$\Omega_\delta = \Big\{x \in \Omega \mid dist(x, \partial\Omega) > \delta\Big\}.$$

Lemma 4.2. *There exists $\delta_0 > 0$ such that all the sets Ω_δ for $\delta \in (0, \delta_0]$, satisfy a uniform cone property (with a cone which depends only on δ_0).*

Proof. Since Ω has the uniform cone property, there exist δ_1 and a finite open cover $\{\mathcal{U}_j\}_{1\le j\le N}$ of $\partial\Omega$ satisfying (i) and (ii) of Definition 4.1. Set

$$\delta_2 = \text{diameter } \mathcal{C}, \quad \delta_3 \doteq \min_{x\in\partial\Omega} \max_j dist(x, \partial\mathcal{U}_j) \quad \delta_0 \doteq \frac{1}{3} \min\big\{\delta_1, \delta_2, \delta_3\big\}.$$

Since $\partial\Omega$ is compact, δ_3 is strictly positive. We use the notations of the figure below.



Let now $\mathcal{C}' = \frac{1}{3}\mathcal{C}$ and $\mathcal{C}_j' = \frac{1}{3}\mathcal{C}_j$ (which is isometric to $\mathcal{C}'$) and consider the following open sets:

$$j \in \{1, \dots, N\}, \qquad \mathcal{U}_j' = \Big\{x \in \mathcal{U}_j \mid dist(x, \partial\mathcal{U}_j) > \delta_0\Big\}.$$

By the definition of δ_3, if follows that the family $\{\mathcal{U}_j'\}_{1\le j\le N}$ still covers $\partial\Omega$.

Clearly, for $\delta \in (0, \delta_0]$ the set $\Big\{x \in \Omega_\delta \mid dist(x, \partial\Omega_\delta) < \delta_0\Big\}$ is included in $\displaystyle\bigcup_{j=1}^{N} \mathcal{U}_j'$. This is condition (i) of definition 4.1.

To prove condition (ii), let j be in $\{1, \dots, N\}$ and x in $\Omega_\delta \cap \mathcal{U}_j'$. One can see that

$$\mathcal{V}_j \doteq \{z \in \mathbb{R}^n, \text{dist}(z, x + \mathcal{C}_j') < \delta\} \subset \bigcup_{y\in B(x;\delta)} (y + \mathcal{C}_j).$$

Furthermore, due to the uniform cone property the latter is included in Ω. Thus the set $x+\mathcal{C}_j'$ is included in Ω_δ. This proves the inclusion

$$\bigcup_{x\in\Omega_\delta\cap\mathcal{U}_j'} \big(x+\mathcal{C}_j'\big)\subset\Omega_\delta,$$

and the proof of Lemma 4.2 is complete. $\square$

Corollary 4.3. *From Lemma 4.2, for any $\varepsilon\in(0,\delta_0]$ and each $p\in(1,\infty)$, there exists a continuous extention operator $\mathcal{P}_\varepsilon$ from $W^{1,p}(\Omega_\varepsilon)$ into $W^{1,p}(\Omega)$, i.e. such that for all $\phi\in W^{1,p}(\Omega_\varepsilon)$*

$$\mathcal{P}_\varepsilon(\phi)\in W^{1,p}(\Omega),\qquad \mathcal{P}_\varepsilon(\phi)_{|\Omega_\varepsilon}=\phi.$$

The norm $\|\mathcal{P}_\varepsilon\|$ is bounded uniformly with respect to ε. $\square$

We can now give the proof of Lemma 1.13.

Proof of Lemma 1.13. Since Ω is bounded and connected with Lipschitz boundary, it has a Poincaré-Wirtinger constant C_0. Let ϕ be in $W^{1,p}(\Omega_\varepsilon)$. We apply the Poincaré-Wirtinger inequality to the extension $\mathcal{P}_\varepsilon(\phi)$ of ϕ to get

$$||\mathcal{P}_\varepsilon(\phi)-\mathcal{M}_\Omega\big(\mathcal{P}_\varepsilon(\phi)\big)||_{L^p(\Omega)}\le C_0||\nabla\mathcal{P}_\varepsilon(\phi)||_{[L^p(\Omega)]^n}\le C_0\|\mathcal{P}_\varepsilon\|||\nabla\phi||_{[L^p(\Omega_\varepsilon)]^n}.$$

Here $\mathcal{M}_\Omega\big(\mathcal{P}_\varepsilon(\phi)\big)$ is the mean value of $\mathcal{P}_\varepsilon(\phi)$ over Ω. Let C be a common upper bound for $2C_0\|\mathcal{P}_\varepsilon\|,\varepsilon\in(0,\delta_0]$. Then,

$$||\phi-\mathcal{M}_\Omega\big(\mathcal{P}_\varepsilon(\phi)\big)||_{L^p(\Omega_\varepsilon)}\le||\mathcal{P}_\varepsilon(\phi)-\mathcal{M}_\Omega\big(\mathcal{P}_\varepsilon(\phi)\big)||_{L^p(\Omega)}\le\frac{1}{2}C||\nabla\phi||_{[L^p(\Omega_\varepsilon)]^n}.$$

Taking the mean value over Ω_ε implies $||\mathcal{M}_\Omega(\phi)-\mathcal{M}_\Omega\big(\mathcal{P}_\varepsilon(\phi)\big)||_{L^p(\Omega_\varepsilon)}\le\frac{1}{2}C||\nabla\phi||_{[L^p(\Omega_\varepsilon)]^n}$, which, when added to the previous inequality, ends the proof. $\square$

We end this section by a result which could be of interest by itself.

Lemma 4.4. *Let $\{u_\varepsilon\}$ be a sequence of functions belonging to $W^{1,p}(\Omega_\varepsilon)$ and satisfying*

$$||u_\varepsilon||_{W^{1,p}(\Omega_\varepsilon)}\le C,$$

where the constant does not depend of ε. Then (up to a subsequence) there is a function $u\in W^{1,p}(\Omega)$ with

$$u_\varepsilon 1_{\Omega_\varepsilon}\longrightarrow u\qquad\textit{strongly in }\ L^p(\Omega).$$

Proof. We start by extending u_ε by using the extension operator $\mathcal{P}_\varepsilon$. So there exist a subsequence, still denoted ε, and a function $u\in W^{1,p}(\Omega)$ such that

$$\begin{aligned}\mathcal{P}_\varepsilon(u_\varepsilon)&\rightharpoonup u\quad\text{weakly in }\ W^{1,p}(\Omega),\\ \mathcal{P}_\varepsilon(u_\varepsilon)&\longrightarrow u\quad\text{strongly in }\ L^p(\Omega).\end{aligned}$$

Since the measure $|\Omega \setminus \Omega_\varepsilon|$ goes to zero, the above strong convergence in $L^p(\Omega)$ yields

$$\lim_{\varepsilon\to 0}\int_{\Omega\setminus\Omega_\varepsilon} |\mathcal{P}_\varepsilon(u_\varepsilon)|^p = 0$$

Therefore,

$$\int_\Omega |u|^p = \lim_{\varepsilon\to 0}\int_\Omega |\mathcal{P}_\varepsilon(u_\varepsilon)|^p = \lim_{\varepsilon\to 0}\int_\Omega |u_\varepsilon 1_{\Omega_\varepsilon}|^p + \lim_{\varepsilon\to 0}\int_{\Omega\setminus\Omega_\varepsilon} |\mathcal{P}_\varepsilon(u_\varepsilon)|^p = \lim_{\varepsilon\to 0}\int_\Omega |u_\varepsilon 1_{\Omega_\varepsilon}|^2$$

and this concludes the proof. $\square$

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