



**HAL**  
open science

# Unfolding Method for Diffusion Process in a Rarefied Binary Structure

Georges Griso, Louiza Merzougui

► **To cite this version:**

Georges Griso, Louiza Merzougui. Unfolding Method for Diffusion Process in a Rarefied Binary Structure. *Applicable Analysis*, 2017. <hal-01449413>

**HAL Id: hal-01449413**

**<https://hal.science/hal-01449413v1>**

Submitted on 30 Jan 2017

**HAL** is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.



HAL Authorization

# Unfolding Method for Diffusion Process in a Rarefied Binary Structure.

Georges Griso<sup>a</sup>, Louiza Merzougui<sup>b</sup>

<sup>a</sup> Laboratoire J.-L. Lions–CNRS, Boîte courrier 187, Université Pierre et Marie Curie,  
4 place Jussieu, 75005 Paris, France, Email: griso@ljl.math.upmc.fr

<sup>b</sup> Laboratory of EDP's and their applications, Faculty of mathematics and computer science,  
University of Batna2, Algeria, Email: merzougui\_louiza@yahoo.fr

## Abstract

The aim of this paper is to study the homogenization of a diffusion process which takes place in a binary structure made by an ambient connected phase surrounding the suspensions (very small particles of diameter of order  $\varepsilon\delta$ ) distributed in an  $\varepsilon$ -periodic network. Using the periodic unfolding method introduced in [4], in the critical case, when  $\varepsilon$  and  $\delta$  go to 0 we determine the asymptotic behavior of the solution of an evolution problem.

KEY WORDS: evolution problem, homogenization, strange term, binary structure

Mathematics Subject Classification (2010): 35A01, 74Q10, 74Q15, 76M50.

## 1 Introduction

This article is devoted to describing the asymptotic behavior of an evolution problem which governs the diffusion process of a rarefied binary structure. The problem is given by (4.1) with assumptions (4.2)-(4.3) and (5.7). In this problem, the first constant  $a_{\varepsilon,\delta}$  represents the mass density of the suspensions while the second one  $b_{\varepsilon,\delta}$  is their diffusivity. The parameter  $\delta$  is the size order of the suspensions in the reference cell  $Y$  and  $\varepsilon$  is the periodicity length. The critical case  $\lim_{(\varepsilon,\delta)\rightarrow(0,0)} \delta^{N-2}/\varepsilon^2 \in (0, +\infty)$  is considered.

This problem was first treated in a paper of Bentalha and al. (see [6]) in 2006 . In this article, the authors investigate the case of a diffusivity  $b_{\varepsilon,\delta}$  going to infinity, the suspensions being spheres. They use the control zone method. One year later, Gruais (see [7]) study the stationary problem making the assumption  $\lim_{(\varepsilon,\delta)\rightarrow(0,0)} b_{\varepsilon,\delta} = \lim_{(\varepsilon,\delta)\rightarrow(0,0)} \delta^N a_{\varepsilon,\delta} = 1$ . She use a multiple scale method. The same year, using again the control zone method Gruais and Poliševski find the limit problem in the case  $\lim_{(\varepsilon,\delta)\rightarrow(0,0)} b_{\varepsilon,\delta} < +\infty$  (see [8]).

In the present paper, we revisit the evolution problem (4.1) with the periodic unfolding method introduced in [4], but we focus our attention in the most interesting case: namely the critical case with  $\lim_{(\varepsilon,\delta)\rightarrow(0,0)} b_{\varepsilon,\delta} \in (0, +\infty)$  and  $\lim_{(\varepsilon,\delta)\rightarrow(0,0)} \delta^N a_{\varepsilon,\delta} \in (0, +\infty)$  (see assumptions (5.7)).

The periodic unfolding method is a general tool to study periodic homogenization. Nowadays, this method is used in a lot of new papers since it is a simple way to replace the two-scale convergence by a convergence in a fixed domain: via  $\mathcal{T}_\varepsilon$  in  $\Omega \times Y$  (where  $Y$  is the reference periodic cell) or via  $\mathcal{T}_{\varepsilon,\delta}$  in  $\Omega \times \mathbb{R}^N$ .

The paper is organized as follows. Section 2 contains the main notations regarding the structure. In Section 3, we recall the definitions of some unfolding operators ( $\mathcal{T}_\varepsilon$ ,  $\mathcal{T}_{\varepsilon,\delta}$  and  $\mathcal{M}_\varepsilon$ ) and we also recall the main results concerning  $\mathcal{T}_{\varepsilon,\delta}$  (Lemmas 3.1 and 3.2).

In Section 4, we present the evolution problem in its variational formulation (4.4). Then, the total energy estimate (4.5) allows to introduce the assumptions on the data (4.6)-(4.7)-(5.8). In Section 5, transforming the solution of the evolution problem with the operator  $\mathcal{T}_{\varepsilon,\delta}$  leads to the assumptions on the two small parameters  $\varepsilon$  and  $\delta$  and on  $a_{\varepsilon,\delta}$ ,  $b_{\varepsilon,\delta}$  (see (5.7)).

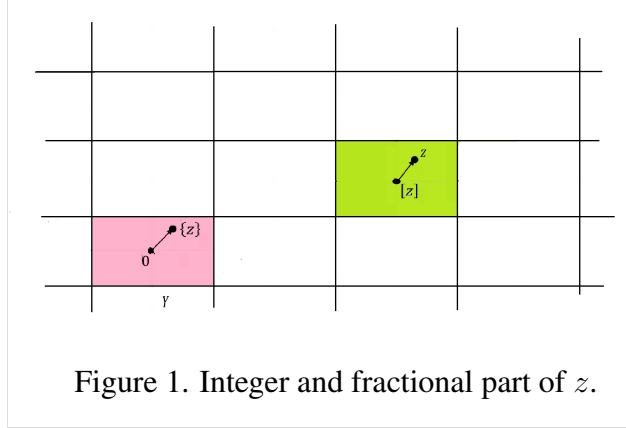
Theorem (6.1) in Section 6 gives the unfolded limit problem. Here, it is worth to note that for the first time the limit evolution problem in its variational form is first posed in  $\Omega \times \mathbb{R}^N$ .

In the last section, using the Laplace transform we give the homogenized limit problem (Theorem 7.2). The limit equation differs from the initial one (4.1) regarding the convolution term, this new term accounting for the memory effects is here in fact a "strange term" (see [2]).

## 2 Notations

Let  $\Omega$  and  $B$  be two bounded domains in  $\mathbb{R}^N$  with Lipschitz boundary. Denote

- $Y = (-1/2, 1/2)^N$  the reference cell,
- $\Omega^T = (0, T) \times \Omega$ ,
- $Y_\delta = Y \setminus \delta\bar{B}$ ,
- $\Xi_\varepsilon = \left\{ \xi \in \mathbb{Z}^N \mid \varepsilon(\xi + Y) \subset \Omega \right\}$ ,
- $\widehat{\Omega}_\varepsilon = \text{interior} \left\{ \bigcup_{\xi \in \Xi_\varepsilon} \varepsilon(\xi + \bar{Y}) \right\}$ ,  $\Lambda_\varepsilon = \Omega \setminus \widehat{\Omega}_\varepsilon$ ,
- $B_{\varepsilon,\delta} = \Omega \cap \bigcup_{\xi \in \Xi_\varepsilon} \varepsilon(\xi + \delta B)$ ,
- $\Omega_{\varepsilon,\delta}^* = \Omega \setminus \overline{B_{\varepsilon,\delta}}$ ,
- $\mathbf{1}_{\Omega_{\varepsilon,\delta}^*}$  is the characteristic function of the set  $\Omega_{\varepsilon,\delta}^*$ .



For a.e  $z \in \mathbb{R}^N$ , one has

$$z = [z] + \{z\}$$

where  $[z] \in \mathbb{Z}^N$  is the integer part of  $z$  while  $\{z\} \in Y$  denotes the fractional part of  $z$ .

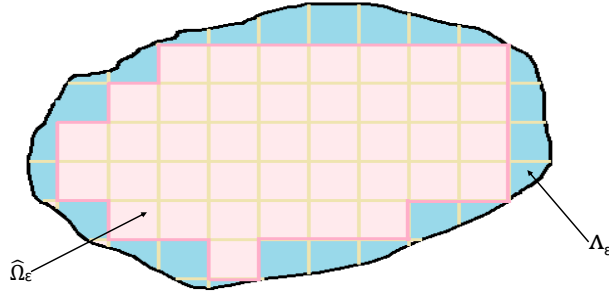


Figure 2. The sets  $\Omega$ ,  $\widehat{\Omega}_\varepsilon$  and  $\Lambda_\varepsilon$ .

### 3 Some recalls on the unfolding operators $\mathcal{T}_\varepsilon$ , $\mathcal{M}_\varepsilon$ and $\mathcal{T}_{\varepsilon,\delta}$

- For  $\phi$  measurable function on  $\Omega$ , the unfolding operator  $\mathcal{T}_\varepsilon$  is defined by

$$\mathcal{T}_\varepsilon(\phi)(x, y) = \begin{cases} \phi\left(\varepsilon \left\lceil \frac{x}{\varepsilon} \right\rceil + \varepsilon y\right) & (x, y) \in \widehat{\Omega}_\varepsilon \times Y, \\ 0 & (x, y) \in \Lambda_\varepsilon \times Y. \end{cases} \quad (3.1)$$

The properties of  $\mathcal{T}_\varepsilon$  are given in [4].

- The local average operator  $\mathcal{M}_\varepsilon : L^1(\Omega) \rightarrow L^1(\Omega)$  is defined as follows:

$$\forall \phi \in L^1(\Omega), \quad \mathcal{M}_\varepsilon(\phi)(x) = \int_Y \mathcal{T}_\varepsilon(\phi)(x, y) dy, \quad \text{for a.e. } x \in \Omega.$$

For any sequence  $\{\phi_\varepsilon\}_\varepsilon$  such that  $\phi_\varepsilon \rightharpoonup \phi$  weakly in  $L^2(\Omega)$ , we have

$$\mathcal{M}_\varepsilon(\phi_\varepsilon) \rightharpoonup \phi \quad \text{weakly in } L^2(\Omega). \quad (3.2)$$

- For  $\phi$  measurable function on  $\Omega$ , the unfolding operator  $\mathcal{T}_{\varepsilon, \delta}$  is defined by

$$\mathcal{T}_{\varepsilon, \delta}(\phi)(x, z) = \begin{cases} \mathcal{T}_\varepsilon(\phi)(x, \delta z) & \text{if } (x, z) \in \widehat{\Omega}_\varepsilon \times \frac{1}{\delta}Y, \\ 0 & \text{otherwise.} \end{cases} \quad (3.3)$$

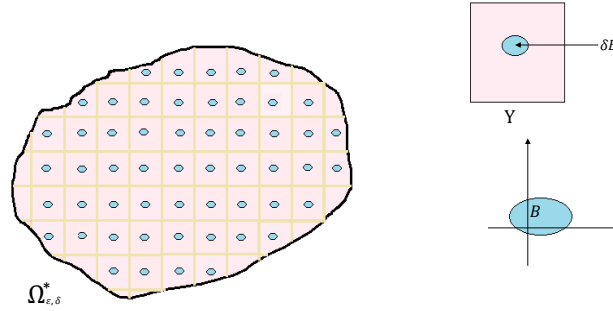


Figure 3. The sets  $\Omega_{\varepsilon, \delta}^*$  and  $B$ .

Below, we recall the main properties of the operator  $\mathcal{T}_{\varepsilon, \delta}$  (see Theorem 2.11 of [5]).

**Lemma 3.1.** *The operator  $\mathcal{T}_{\varepsilon, \delta}$  is linear and continuous from  $L^p(\Omega)$  into  $L^p(\Omega \times \mathbb{R}^N)$ ,  $p \in [1, +\infty]$ .*

1. For any measurable functions  $\psi, \phi$  one has  $\mathcal{T}_{\varepsilon, \delta}(\psi\phi) = \mathcal{T}_{\varepsilon, \delta}(\psi)\mathcal{T}_{\varepsilon, \delta}(\phi)$ .
2. For any  $\phi$  in  $L^1(\Omega)$

$$\begin{aligned} \delta^N \int_{\Omega \times \mathbb{R}^N} \mathcal{T}_{\varepsilon, \delta}(\phi)(x, z) dx dz &= \int_{\Omega \times Y} \mathcal{T}_\varepsilon(\phi)(x, y) dx dy = \int_{\widehat{\Omega}_\varepsilon} \phi dx, \\ \left| \int_{\Omega} \phi(x) dx - \delta^N \int_{\Omega \times \mathbb{R}^N} \mathcal{T}_{\varepsilon, \delta}(\phi)(x, z) dx dz \right| &\leq \int_{\Lambda_\varepsilon} |\phi| dx. \end{aligned}$$

3. For any  $\phi \in L^2(\Omega)$

$$\|\mathcal{T}_{\varepsilon, \delta}(\phi)\|_{L^2(\Omega \times \mathbb{R}^N)}^2 \leq \frac{1}{\delta^N} \|\phi\|_{L^2(\Omega)}^2, \quad \|\mathcal{T}_{\varepsilon, \delta}(\phi)\|_{L^2(\Omega \times B)}^2 \leq \frac{1}{\delta^N} \|\phi\|_{L^2(B_{\varepsilon, \delta})}^2. \quad (3.4)$$

4. For any  $\phi$  in  $H^1(\Omega)$

$$\mathcal{T}_{\varepsilon,\delta}(\nabla\phi) = \frac{1}{\varepsilon\delta}\nabla_z(\mathcal{T}_{\varepsilon,\delta}(\phi)) \quad \text{in } \Omega \times \frac{1}{\delta}Y \quad (3.5)$$

and

$$\begin{aligned} \|\nabla_z(\mathcal{T}_{\varepsilon,\delta}(\phi))\|_{L^2(\Omega \times \mathbb{R}^N)}^2 &\leq \frac{\varepsilon^2}{\delta^{N-2}}\|\nabla\phi\|_{L^2(\Omega)}^2, \\ \|\nabla_z(\mathcal{T}_{\varepsilon,\delta}(\phi))\|_{L^2(\Omega \times B)}^2 &\leq \frac{\varepsilon^2}{\delta^{N-2}}\|\nabla\phi\|_{L^2(B_{\varepsilon,\delta})}^2. \end{aligned} \quad (3.6)$$

5. Denote  $2^* \doteq \frac{2N}{N-2}$ ,  $N \geq 3$ , the Sobolev exponent associated to 2. For any  $\phi$  in  $H^1(\Omega)$

$$\|\mathcal{T}_{\varepsilon,\delta}(\phi - \mathcal{M}_\varepsilon(\phi))\|_{L^2(\Omega; L^{2^*}(\mathbb{R}^N))}^2 \leq C \frac{\varepsilon^2}{\delta^{N-2}}\|\nabla\phi\|_{L^2(\Omega)}^2. \quad (3.7)$$

**Lemma 3.2.** For any  $w \in \mathcal{C}(\overline{\Omega})$

$$\mathcal{T}_{\varepsilon,\delta}(w) \longrightarrow w \quad \text{strongly in } L^2(\Omega; L_{loc}^2(\mathbb{R}^N)). \quad (3.8)$$

*Proof.* Let  $\mathcal{O}$  be a bounded open subset of  $\mathbb{R}^N$ . There exists  $R > 0$  such that  $\mathcal{O} \subset B(O, R)$  where  $B(O, R)$  is the ball of radius  $R$  and center  $O$ . For  $\delta$  small enough  $\mathcal{O} \subset B(0, R) \subset \frac{1}{\delta}Y$ . Hence

$$\begin{aligned} &\int_{\Omega \times \mathcal{O}} |\mathcal{T}_{\varepsilon,\delta}(w)(x, z) - w(x)|^2 dx dz \\ &= \int_{\Lambda_\varepsilon \times \mathcal{O}} |w(x)|^2 dx dz + \int_{\widehat{\Omega}_\varepsilon \times \mathcal{O}} \left| w\left(\varepsilon\left[\frac{x}{\varepsilon}\right] + \varepsilon\delta z\right) - w(x) \right|^2 dx dz \\ &\leq |\mathcal{O}| \int_{\Lambda_\varepsilon} |w(x)|^2 dx + |\Omega \times \mathcal{O}| \sup_{(x,z) \in \Omega \times \mathcal{O}} \left| w\left(\varepsilon\left[\frac{x}{\varepsilon}\right] + \varepsilon\delta z\right) - w(x) \right|^2. \end{aligned}$$

Let  $\varpi$  be the modulus of continuity of  $w$

$$\int_{\Omega \times \mathcal{O}} |\mathcal{T}_{\varepsilon,\delta}(w)(x, z) - w(x)|^2 dx dz \leq C(\|w\|_{L^\infty(\Omega)}^2 |\Lambda_\varepsilon| + \varpi(\varepsilon + \varepsilon\delta R)^2).$$

Passing to the limit gives (3.8). □

## 4 The evolution problem

We consider the evolution problem which governs the diffusion process in our binary mixture

$$\begin{cases} \text{Find } u_{\varepsilon,\delta} \text{ such that} \\ \rho_{\varepsilon,\delta} \frac{du_{\varepsilon,\delta}}{dt} - \operatorname{div}_x(k_{\varepsilon,\delta} \nabla_x u_{\varepsilon,\delta}) = f_{\varepsilon,\delta} & \text{in } \Omega^T, \\ u_{\varepsilon,\delta} = 0 & \text{on } \partial\Omega \times (0, T), \\ u_{\varepsilon,\delta}(0) = u_{\varepsilon,\delta}^0 & \text{on } \Omega \times \{0\}, \end{cases} \quad (4.1)$$

where  $f_{\varepsilon,\delta} \in L^2(\Omega^T)$  and

$$\rho_{\varepsilon,\delta} = \begin{cases} 1 & \text{in } \Omega_{\varepsilon,\delta}^*, \\ a_{\varepsilon,\delta} & \text{in } B_{\varepsilon,\delta}, \end{cases} \quad k_{\varepsilon,\delta} = \begin{cases} 1 & \text{in } \Omega_{\varepsilon,\delta}^*, \\ b_{\varepsilon,\delta} & \text{in } B_{\varepsilon,\delta}. \end{cases} \quad (4.2)$$

Here  $a_{\varepsilon,\delta}$  and  $b_{\varepsilon,\delta}$  are strictly positive constants bounded from below independently of  $\varepsilon$  and  $\delta$ . Hence, there exists  $c_0 > 0$  such that for any  $(\varepsilon, \delta)$

**Assumption  $A_1$ .**

$$a_{\varepsilon,\delta} \geq c_0, \quad b_{\varepsilon,\delta} \geq c_0. \quad (4.3)$$

The first constant  $a_{\varepsilon,\delta}$  represents the density of mass of the suspensions while the second one is the diffusivity of suspensions.

**Lemma 4.1.** *Problem (4.1) has the following variational formulation:*

$$\begin{aligned} & \text{Find } u_{\varepsilon,\delta} \in L^2(0, T; H_0^1(\Omega)) \cap L^\infty(0, T; L^2(\Omega)), \quad \rho_{\varepsilon,\delta} u_{\varepsilon,\delta} \in H^1(0, T; H^{-1}(\Omega)), \\ & \langle \rho_{\varepsilon,\delta} \frac{du_{\varepsilon,\delta}}{dt}(t), w \rangle_{H^{-1}(\Omega), H_0^1(\Omega)} + \int_{\Omega} k_{\varepsilon,\delta}(x) \nabla_x u_{\varepsilon,\delta}(t, x) \nabla_x w(x) dx \\ & = \int_{\Omega} f_{\varepsilon,\delta}(t, x) w(x) dx \quad \text{for a.e. } t \in (0, T), \quad \forall w \in H_0^1(\Omega), \\ & u_{\varepsilon,\delta}(0) = u_{\varepsilon,\delta}^0. \end{aligned} \quad (4.4)$$

Furthermore  $u_{\varepsilon,\delta}$  satisfies

$$\begin{aligned} & \frac{1}{2} \int_{\Omega} \rho_{\varepsilon,\delta}(x) |u_{\varepsilon,\delta}(t, x)|^2 dx + \int_0^t \int_{\Omega} k_{\varepsilon,\delta}(x) |\nabla_x u_{\varepsilon,\delta}(s, x)|^2 ds dx \\ & = \int_0^t \int_{\Omega} f_{\varepsilon,\delta}(s, x) u_{\varepsilon,\delta}(s, x) ds dx + \frac{1}{2} \int_{\Omega} \rho_{\varepsilon,\delta}(x) |u_{\varepsilon,\delta}^0(x)|^2 dx \end{aligned} \quad \text{for all } t \in [0, T]. \quad (4.5)$$

*Proof.* The existence and uniqueness of the solution of Problem (4.4) is given by the Faedo-Galerkin's method (see e.g. [3] or [9]) which also gives the equality (4.5).  $\square$

**Assumption  $A_2$ .** We assume that there exists a constant  $C_0$ , which does not depend on  $\varepsilon$  and  $\delta$ , such that<sup>1</sup>

$$\int_{\Omega} \rho_{\varepsilon,\delta} |u_{\varepsilon,\delta}^0(x)|^2 dx + \int_{\Omega^T} \frac{1}{\rho_{\varepsilon,\delta}} |f_{\varepsilon,\delta}(t, x)|^2 dt dx \leq C_0. \quad (4.6)$$

As a consequence of the above assumptions (4.3)<sub>1</sub> and (4.6) the sequences  $\{u_{\varepsilon,\delta}^0\}$ ,  $\{f_{\varepsilon,\delta} \mathbf{1}_{\Omega_{\varepsilon,\delta}^*}\}$  are bounded respectively in  $L^2(\Omega)$ ,  $L^2(\Omega^T)$ .

**Assumption  $A_3$ .** We assume that there exist  $u^0 \in L^2(\Omega)$  and  $f \in L^2(\Omega^T)$  such that

$$\begin{aligned} u_{\varepsilon,\delta}^0 & \rightharpoonup u^0 \quad \text{weakly in } L^2(\Omega), \\ f_{\varepsilon,\delta} \mathbf{1}_{\Omega_{\varepsilon,\delta}^*} & \rightharpoonup f \quad \text{weakly in } L^2(\Omega^T). \end{aligned} \quad (4.7)$$

<sup>1</sup> We will see in Lemma 4.2 the reason of this assumption.

**Lemma 4.2.** *There exists a positive constant  $C$  independent of  $\varepsilon$  and  $\delta$  such that*

$$\begin{aligned} \int_{\Omega} \rho_{\varepsilon,\delta}(x) |u_{\varepsilon,\delta}(t, x)|^2 dx &\leq C, \quad \forall t \in [0, T], \\ \int_{\Omega T} k_{\varepsilon,\delta}(x) |\nabla_x u_{\varepsilon,\delta}(s, x)|^2 ds dx &\leq C. \end{aligned} \quad (4.8)$$

Moreover

$$\|u_{\varepsilon,\delta}\|_{L^\infty(0,T;L^2(\Omega))} + \|u_{\varepsilon,\delta}\|_{L^2(0,T;H_0^1(\Omega))} \leq C. \quad (4.9)$$

*Proof.* According to the Cauchy-Schwarz inequality we have

$$\begin{aligned} \int_0^t \int_{\Omega} f_{\varepsilon,\delta} u_{\varepsilon,\delta} ds dx &\leq \int_0^t \left\| \frac{1}{\sqrt{\rho_{\varepsilon,\delta}}} f_{\varepsilon,\delta} \right\|_{L^2(\Omega)} \|\sqrt{\rho_{\varepsilon,\delta}} u_{\varepsilon,\delta}\|_{L^2(\Omega)} ds \\ &\leq \frac{1}{2} \left\| \frac{1}{\sqrt{\rho_{\varepsilon,\delta}}} f_{\varepsilon,\delta} \right\|_{L^2(\Omega T)}^2 + \frac{1}{2} \int_0^t \|\sqrt{\rho_{\varepsilon,\delta}} u_{\varepsilon,\delta}\|_{L^2(\Omega)}^2 ds. \end{aligned} \quad (4.10)$$

The above inequality and (4.5) yield

$$\|\sqrt{\rho_{\varepsilon,\delta}} u_{\varepsilon,\delta}(t)\|_{L^2(\Omega)}^2 \leq \|\sqrt{\rho_{\varepsilon,\delta}} u_{\varepsilon,\delta}^0\|_{L^2(\Omega)}^2 + \left\| \frac{1}{\sqrt{\rho_{\varepsilon,\delta}}} f_{\varepsilon,\delta} \right\|_{L^2(\Omega T)}^2 + \int_0^t \|\sqrt{\rho_{\varepsilon,\delta}} u_{\varepsilon,\delta}(s)\|_{L^2(\Omega)}^2 ds.$$

Then, the Gronwall's lemma and (4.6) give

$$\begin{aligned} &\|\sqrt{\rho_{\varepsilon,\delta}} u_{\varepsilon,\delta}(t)\|_{L^2(\Omega)}^2 + \int_0^t \|\sqrt{\rho_{\varepsilon,\delta}} u_{\varepsilon,\delta}(s)\|_{L^2(\Omega)}^2 ds \\ &\leq e^T \left( \|\sqrt{\rho_{\varepsilon,\delta}} u_{\varepsilon,\delta}^0\|_{L^2(\Omega)}^2 + \left\| \frac{1}{\sqrt{\rho_{\varepsilon,\delta}}} f_{\varepsilon,\delta} \right\|_{L^2(\Omega T)}^2 \right) \leq C_0 e^T \end{aligned} \quad (4.11)$$

which in turn leads to (4.8)<sub>1</sub>. Again with (4.6) and (4.10)-(4.11) we get (4.8)<sub>2</sub>. Then, the assumptions (4.3) yield

$$\begin{aligned} \min\{1, c_0\} \int_{\Omega} |u_{\varepsilon,\delta}(t, x)|^2 dx &\leq C \quad \forall t \in [0, T], \\ \min\{1, c_0\} \int_{\Omega T} |\nabla_x u_{\varepsilon,\delta}(s, x)|^2 ds dx &\leq C. \end{aligned}$$

That gives (4.9). □

As a consequence of the above lemma, one has

**Corollary 4.3.** *There exist a subsequence of  $(\varepsilon, \delta)$ , still denoted  $(\varepsilon, \delta)$ , and  $u \in L^\infty(0, T; L^2(\Omega)) \cap L^2(0, T; H_0^1(\Omega))$  such that*

$$\begin{aligned} u_{\varepsilon,\delta} &\rightharpoonup u \quad \text{weakly * in } L^\infty(0, T; L^2(\Omega)), \\ u_{\varepsilon,\delta} &\rightharpoonup u \quad \text{weakly in } L^2(0, T; H_0^1(\Omega)). \end{aligned} \quad (4.12)$$

## 5 Further estimates and convergences

**Lemma 5.1.** *There exists a positive constant  $C$  independent of  $\varepsilon$  and  $\delta$ , such that*

$$\begin{aligned} \|\mathcal{T}_{\varepsilon,\delta}(u_{\varepsilon,\delta} - \mathcal{M}_\varepsilon(u_{\varepsilon,\delta}))\|_{L^2(\Omega^T; L^{2^*}(\mathbb{R}^N))}^2 &\leq C \frac{\varepsilon^2}{\delta^{N-2}}, \\ \|\nabla_z(\mathcal{T}_{\varepsilon,\delta}(u_{\varepsilon,\delta}))\|_{L^2(\Omega^T \times \mathbb{R}^N)}^2 &\leq C \frac{\varepsilon^2}{\delta^{N-2}} \end{aligned} \quad (5.1)$$

and

$$\|\mathcal{T}_{\varepsilon,\delta}(u_{\varepsilon,\delta})\|_{L^2(\Omega^T \times B)}^2 \leq \frac{C}{\delta^N a_{\varepsilon,\delta}}, \quad \|\nabla_z(\mathcal{T}_{\varepsilon,\delta}(u_{\varepsilon,\delta}))\|_{L^2(\Omega^T \times B)}^2 \leq C \frac{\varepsilon^2}{\delta^{N-2} b_{\varepsilon,\delta}}. \quad (5.2)$$

*Proof.* From (3.4) and (4.8) we immediately deduce (5.1). From (3.4)<sub>2</sub> we have

$$\|\mathcal{T}_{\varepsilon,\delta}(u_{\varepsilon,\delta})\|_{L^2(\Omega^T \times B)}^2 \leq \frac{1}{\delta^N} \|u_{\varepsilon,\delta}\|_{L^2(B_{\varepsilon,\delta})}^2 \quad (5.3)$$

and from (4.8)<sub>1</sub>

$$\int_{\Omega_{\varepsilon,\delta}^*} |u_{\varepsilon,\delta}(t, x)|^2 dx + \int_{B_{\varepsilon,\delta}} a_{\varepsilon,\delta} |u_{\varepsilon,\delta}(t, x)|^2 dx \leq C. \quad (5.4)$$

Hence

$$\int_{B_{\varepsilon,\delta}} |u_{\varepsilon,\delta}(t, x)|^2 dx \leq \frac{C}{a_{\varepsilon,\delta}}, \quad (5.5)$$

We substitute (5.5) in (5.3) that gives (5.2)<sub>1</sub>. Then (3.6)<sub>2</sub> yields

$$\|\nabla_z(\mathcal{T}_{\varepsilon,\delta}(u_{\varepsilon,\delta}))\|_{L^2(\Omega^T \times B)}^2 \leq \frac{\varepsilon^2}{\delta^{N-2}} \|\nabla_x u_{\varepsilon,\delta}\|_{L^2(B_{\varepsilon,\delta}^T)}. \quad (5.6)$$

Estimate (4.8)<sub>2</sub> leads to

$$\int_{B_{\varepsilon,\delta} \times (0, T)} |\nabla u_{\varepsilon,\delta}(t, x)|^2 dt dx \leq \frac{C}{b_{\varepsilon,\delta}}.$$

The above inequality and (5.6) give (5.2)<sub>2</sub>. □

So, the above estimates (5.2) highlight several cases. We will focus our attention on the most interesting one.

**Assumption  $A_4$ .** We assume that

$$\begin{aligned} \lim_{(\varepsilon,\delta) \rightarrow (0,0)} \frac{\delta^{N-2}}{\varepsilon^2} &= \gamma_0 \in (0, +\infty), & \lim_{(\varepsilon,\delta) \rightarrow (0,0)} \delta^N a_{\varepsilon,\delta} &= a_0 \in (0, +\infty), \\ \lim_{(\varepsilon,\delta) \rightarrow (0,0)} b_{\varepsilon,\delta} &= b_0 \in (0, +\infty). \end{aligned} \quad (5.7)$$

Now, as a consequence of Assumptions  $A_1$ ,  $A_2$  and the estimate (3.4)<sub>2</sub>, the sequences  $\{\mathcal{T}_{\varepsilon,\delta}(u_{\varepsilon,\delta}^0)\}$ ,  $\{\delta^N \mathcal{T}_{\varepsilon,\delta}(f_{\varepsilon,\delta})\}$  are bounded respectively in  $L^2(\Omega \times B)$ ,  $L^2(\Omega^T \times B)$ .

**Assumption  $A_5$ .** We assume that there exist  $U^0 \in L^2(\Omega \times B)$  and  $F \in L^2(\Omega^T \times B)$  such that

$$\begin{aligned} \mathcal{T}_{\varepsilon,\delta}(u_{\varepsilon,\delta}^0) &\rightharpoonup U_0 && \text{weakly in } L^2(\Omega \times B), \\ \delta^N \mathcal{T}_{\varepsilon,\delta}(f_{\varepsilon,\delta}) &\rightharpoonup F && \text{weakly in } L^2(\Omega^T \times B). \end{aligned} \quad (5.8)$$

**Lemma 5.2.** *There exist a subsequence of  $(\varepsilon, \delta)$ , still denoted  $(\varepsilon, \delta)$ , and  $U \in L^2(\Omega^T; H_{loc}^1(\mathbb{R}^N))$  with  $\nabla_z U \in L^2(\Omega^T \times \mathbb{R}^N)$  such that*

$$\begin{aligned} \mathcal{T}_{\varepsilon,\delta}(u_{\varepsilon,\delta}) &\rightharpoonup U && \text{weakly in } L^2(\Omega^T; H_{loc}^1(\mathbb{R}^N)), \\ \nabla_z \mathcal{T}_{\varepsilon,\delta}(u_{\varepsilon,\delta}) 1_{1/\delta Y} &\rightharpoonup \nabla_z U && \text{weakly in } L^2(\Omega^T \times \mathbb{R}^N). \end{aligned} \quad (5.9)$$

Furthermore, there also exists  $V \in L^2(\Omega^T, L^{2^*}(\mathbb{R}^N))$  with  $\nabla_z V \in L^2(\Omega^T \times \mathbb{R}^N)$  such that

$$\begin{aligned} \mathcal{T}_{\varepsilon,\delta}(u_{\varepsilon,\delta} - \mathcal{M}_\varepsilon(u_{\varepsilon,\delta})) 1_{1/\delta Y} &\rightharpoonup V && \text{weakly in } L^2(\Omega^T; L^{2^*}(\mathbb{R}^N)), \\ \nabla_z \mathcal{T}_{\varepsilon,\delta}(u_{\varepsilon,\delta}) 1_{1/\delta Y} &\rightharpoonup \nabla_z V && \text{weakly in } L^2(\Omega^T \times \mathbb{R}^N) \end{aligned} \quad (5.10)$$

and we have  $U = V + u$ .

*Proof.* From the estimates (5.1)-(5.2), there exist subsequences of  $(\varepsilon, \delta)$  such that the convergences (5.9) and (5.10) hold. Convergences (4.12)<sub>2</sub> and (3.2) yield

$$\begin{aligned} \mathcal{M}_\varepsilon(u_{\varepsilon,\delta}) &\rightharpoonup u && \text{weakly in } L^2(\Omega^T), \\ \text{thus } \mathcal{M}_\varepsilon(u_{\varepsilon,\delta}) &\rightharpoonup u && \text{weakly in } L^2(\Omega^T; L_{loc}^{2^*}(\mathbb{R}^N)). \end{aligned} \quad (5.11)$$

Then, (5.9)<sub>1</sub>, (5.10)<sub>1</sub> and (5.11) lead to  $V = U - u$ . □

Set

$$\begin{aligned} \mathbf{H} &= \{ \Phi \in H_{loc}^1(\mathbb{R}^N) \mid \nabla_z \Phi \in [L^2(\mathbb{R}^N)]^N \} = \dot{H}^1(\mathbb{R}^N) \oplus \mathbb{R}^{(2)}, \\ \mathbf{L} &= \{ \Phi \in L^2(\Omega; \mathbf{H}) \mid \Phi(\cdot, \infty) \in H_0^1(\Omega) \}. \end{aligned} \quad (5.12)$$

For any  $\Phi \in \mathbf{H}$ , we know that there exists a constant denoted  $\Phi(\infty)$  such that

$$\| \Phi - \Phi(\infty) \|_{L^{2^*}(\mathbb{R}^N)} \leq C \| \nabla \Phi \|_{L^2(\mathbb{R}^N)}. \quad (5.13)$$

The constant  $C$  does not depend on  $\Phi$ .

Observe that  $U \in L^2(0, T; \mathbf{L})$  and  $U(t, x, \infty) = u(t, x)$  for a.e.  $(t, x) \in \Omega^T$ .

---

<sup>2</sup> Recall that the space  $\dot{H}^1(\mathbb{R}^N)$  is the completion of  $\mathcal{D}(\mathbb{R}^N)$  for the norm  $\| \nabla \phi \|_{L^2(\mathbb{R}^N)}$ . The Sobolev imbedding theorem implies that for  $N \geq 3$ , it is a subspace of  $L^{2^*}(\mathbb{R}^N)$ , where  $2^*$  is the Sobolev exponent associated to 2. Therefore, all its elements admit 0 as limit at  $\infty$  of  $\mathbb{R}^N$  (in the weak sense of  $L^{2^*}(\mathbb{R}^N)$ ).

## 6 The unfolded limit problem

Denote

$$k_0(z) = \begin{cases} 1 & \text{if } z \in \mathbb{R}^N \setminus \bar{B}, \\ b_0 & \text{if } z \in B. \end{cases} \quad (6.1)$$

**Theorem 6.1.** *Let  $u_{\varepsilon, \delta}$  be the solution of the evolution Problem (4.5). For the whole sequence  $(\varepsilon, \delta)$  we have the convergences (4.12), (5.9) and (5.10).*

*The limit function  $U$  is the solution of the unfolded evolution problem*

$$\begin{aligned} & U \in L^2(0, T; \mathbf{L}), \quad U(\cdot, \cdot, \infty) \in H^1(0, T; H^{-1}(\Omega)), \\ & U \in H^1(0, T; L^2(\Omega; (H^1(B))')), \\ & \left\langle \frac{dU}{dt}(t, \cdot, \infty), \Phi(\cdot, \infty) \right\rangle_{H^{-1}(\Omega), H_0^1(\Omega)} + a_0 \left\langle \frac{dU}{dt}(t), \Phi \right\rangle_{L^2(\Omega; (H^1(B))'), L^2(\Omega; H^1(B))} \\ & + \int_{\Omega} \nabla_x U(t, \cdot, \infty) \nabla_x \Phi(\cdot, \infty) dx + \gamma_0 \int_{\Omega \times \mathbb{R}^N} k_0 \nabla_z U(t) \nabla_z \Phi dx dz \\ & = \int_{\Omega} f(t) \Phi(\cdot, \infty) dx + \int_{\Omega \times B} F(t) \Phi dx dz, \quad \text{for a.e. } t \in (0, T), \quad \forall \Phi \in \mathbf{L}, \\ & U(0, x, \infty) = u^0(x), \quad U(0, x, z) = U^0(x, z) \quad \text{for a.e. } (x, z) \in \Omega \times B. \end{aligned} \quad (6.2)$$

Furthermore  $U$  satisfies

$$\begin{aligned} & \frac{1}{2} \int_{\Omega} |U(t, \cdot, \infty)|^2 dx + \frac{a_0}{2} \int_{\Omega \times B} |U(t)|^2 dx dz \\ & + \int_0^t \int_{\Omega} |\nabla_x U(s, \cdot, \infty)|^2 ds dx + \gamma_0 \int_0^t \int_{\Omega \times \mathbb{R}^N} k_0 |\nabla_z U|^2 ds dx dz \\ & = \int_0^t \int_{\Omega} f(s) U(s, \cdot, \infty) ds dx + \int_0^t \int_{\Omega \times B} F(s) U(s) dx dz \\ & + \frac{1}{2} \int_{\Omega} |u^0|^2 dx + \frac{a_0}{2} \int_{\Omega \times B} |U^0|^2 dx dz. \end{aligned} \quad \text{for all } t \in [0, T]. \quad (6.3)$$

Before proving the unfolded limit problem, we introduce and prove three lemmas. The first one concerns a density result, the second one gives a convergence result for functions vanishing in  $B_{\varepsilon, \delta}$  and the last one introduces a test function.

Since the open set  $B$  is bounded, there exists  $\eta_0 > 0$  such that

$$2\eta_0 B \subset \bar{Y}.$$

**Lemma 6.2.** *The set*

$$\bigcup_{\eta \in (0, \eta_0]} \left\{ \psi \in H_{per}^1(Y) \mid \psi|_{\eta B} = 0 \right\}$$

*is dense in  $H_{per}^1(Y)$ .*

*Proof.* We prove the lemma in two steps. In the first one we choose an element in  $\mathcal{C}_{per}^\infty(\overline{Y})$  and in the second one we investigate the general case.

*Step 1.* We fix a function  $\chi$  in  $\mathcal{C}^\infty(\mathbb{R}^N)$  satisfying

$$\begin{aligned} \chi &= 0 \quad \text{on } B, \\ \chi &= 1 \quad \text{on } \mathbb{R}^N \setminus 2\overline{B}, \\ \chi(x) &\in [0, 1] \quad \text{for all } x \in \mathbb{R}^N. \end{aligned} \tag{6.4}$$

For  $\eta \in (0, \eta_0]$ , set

$$\chi_\eta = \chi\left(\frac{\cdot}{\eta}\right).$$

For any  $\phi \in \mathcal{C}_{per}^\infty(\overline{Y})$ , denote  $\phi_\eta = \chi_\eta \phi$ . Since  $\chi_\eta = 1$  on  $\partial Y$ , the function  $\phi_\eta$  belongs to  $H_{per}^1(Y)$ . We have  $\phi_\eta = \phi$  on  $Y \setminus 2\overline{B}$  and  $\nabla \phi_\eta = \nabla \phi \chi_\eta + \nabla \chi_\eta \phi$ . Hence

$$\|\phi_\eta - \phi\|_{L^2(Y)}^2 + \|\nabla \phi_\eta - \nabla \phi\|_{L^2(Y)}^2 \leq \int_{2\eta B} |\phi|^2 dy + 2 \int_{2\eta B} |\nabla \phi|^2 dy + 2 \int_{2\eta B} |\nabla \chi_\eta|^2 |\phi|^2 dy.$$

One has

$$\int_{2\eta B} |\nabla \chi_\eta|^2 |\phi|^2 dy \leq C \eta^{N-2} \|\nabla \chi\|_{[L^\infty(\mathbb{R}^N)]^N} \|\phi\|_{L^\infty(Y)}^2.$$

Since  $\phi$  belongs to  $\mathcal{C}_{per}^\infty(\overline{Y})$  and the measure of  $\eta B$  goes to zero, one obtains the strong convergence of  $\phi_\eta$  to  $\phi$  in  $H_{per}^1(Y)$ .

*Step 2.* Let  $\psi$  be in  $H_{per}^1(Y)$ . The space  $\mathcal{C}_{per}^\infty(\overline{Y})$  is dense in  $H_{per}^1(Y)$ . Hence for any  $\varepsilon > 0$  there exists  $\varphi \in \mathcal{C}_{per}^\infty(\overline{Y})$  such that

$$\|\psi - \varphi\|_{H^1(Y)} \leq \varepsilon. \tag{6.5}$$

We fix a function  $\varphi$  satisfying (6.5). From Step 1, there exists  $\eta_1 > 0$  such that

$$\forall \eta \in (0, \eta_1] \quad \|\varphi - \varphi_\eta\|_{H^1(Y)} < \varepsilon. \tag{6.6}$$

As a consequence of (6.5) and (6.6) we get

$$\forall \eta \in (0, \eta_1] \quad \|\psi - \varphi_\eta\|_{H^1(Y)} \leq 2\varepsilon.$$

The lemma is proved. □

**Lemma 6.3.** Suppose  $p \in [1, +\infty)$ . For any  $w \in L^p(\Omega)$

$$w \mathbf{1}_{\Omega_{\varepsilon,\delta}^*} \longrightarrow w \quad \text{strongly in } L^p(\Omega). \tag{6.7}$$

*Proof. Step 1.* We prove (6.7) for  $w \in L^\infty(\Omega)$ .

Take  $w$  in  $L^\infty(\Omega)$

$$\begin{aligned} \int_{\Omega} |(w\mathbf{1}_{\Omega_{\varepsilon,\delta}^*} - w)(x)|^p dx &= \int_{B_{\varepsilon,\delta}} |w(x)|^p dx \\ &\leq |B_{\varepsilon,\delta}| \|w\|_{L^\infty(\Omega)}^p \leq C\delta^N \|w\|_{L^\infty(\Omega)}^p. \end{aligned}$$

Passing to the limit gives (6.7) for any  $w \in L^\infty(\Omega)$ .

*Step 2.* We prove (6.7) for  $w \in L^p(\Omega)$ .

Let  $w$  be in  $L^p(\Omega)$ . Since  $\mathcal{C}(\overline{\Omega})$  is dense in  $L^p(\Omega)$ ,  $p \in [1, \infty)$ , for any  $\varepsilon_1 > 0$  there exists  $W \in \mathcal{C}(\overline{\Omega})$  such that

$$\|w - W\|_{L^p(\Omega)} \leq \varepsilon_1. \quad (6.8)$$

We fix  $W$  satisfying (6.8). Then

$$\|W\mathbf{1}_{\Omega_{\varepsilon,\delta}^*} - W\|_{L^p(\Omega)} \leq |B_{\varepsilon,\delta}| \|W\|_{L^\infty(\Omega)} \leq C\delta^N \|W\|_{L^\infty(\Omega)}.$$

Hence, there exists  $\delta_1 > 0$  such that

$$\forall \delta \in (0, \delta_1], \quad \|W\mathbf{1}_{\Omega_{\varepsilon,\delta}^*} - W\|_{L^p(\Omega)} \leq \varepsilon_1.$$

So

$$\begin{aligned} \|w\mathbf{1}_{\Omega_{\varepsilon,\delta}^*} - w\|_{L^p(\Omega)} &\leq \|w\mathbf{1}_{\Omega_{\varepsilon,\delta}^*} - W\mathbf{1}_{\Omega_{\varepsilon,\delta}^*}\|_{L^p(\Omega)} + \|W\mathbf{1}_{\Omega_{\varepsilon,\delta}^*} - W\|_{L^p(\Omega)} + \|W - w\|_{L^p(\Omega)} \\ &\leq \|W\mathbf{1}_{\Omega_{\varepsilon,\delta}^*} - W\|_{L^p(\Omega)} + 2\|W - w\|_{L^p(\Omega)} \leq 3\varepsilon_1. \end{aligned}$$

As a result, the strong convergence (6.7) is proved.  $\square$

**Lemma 6.4.** *Let  $\psi$  be in  $\mathbf{H} \cap C^\infty(\mathbb{R}^N)$  such that the support of  $\nabla_z \psi$  is bounded. Set*

$$\begin{aligned} \psi_{\varepsilon,\delta}(x) &= \psi\left(\frac{1}{\delta}\left\{\frac{x}{\varepsilon}\right\}\right) && \text{if } x \in \widehat{\Omega}_\varepsilon, \\ \psi_{\varepsilon,\delta}(x) &= \psi(\infty) && \text{if } x \in \overline{\Lambda}_\varepsilon. \end{aligned} \quad (6.9)$$

The function  $\psi_{\varepsilon,\delta}$  belongs to  $C^\infty(\overline{\Omega})$  and

$$\psi_{\varepsilon,\delta} \longrightarrow \psi(\infty) \quad \text{strongly in } L^2(\Omega). \quad (6.10)$$

*Proof.* There exists  $R > 0$  such that the support of  $\nabla_z \psi$  is included in the ball  $B(O; R)$ . For  $\delta$  small enough  $B(O; \delta R) \subset Y$ . We perform a change of variables and we use the fact that  $\psi - \psi(\infty)$  belongs to  $H_0^1(B(O; R))$ . That gives

$$\begin{aligned} \int_{\varepsilon\xi + \varepsilon Y} |\psi_{\varepsilon,\delta}(x) - \psi(\infty)|^2 dx &= \int_{\varepsilon\xi + \varepsilon Y} \left| \psi\left(\frac{1}{\delta}\left\{\frac{x}{\varepsilon}\right\}\right) - \psi(\infty) \right|^2 dx \\ &= (\varepsilon\delta)^N \int_{B(O; R)} |\psi(z) - \psi(\infty)|^2 dz \\ &\leq C(\varepsilon\delta)^N R^2 \int_{B(O; R)} |\nabla_z \psi(z)|^2 dz. \end{aligned}$$

The constant does not depend on  $\varepsilon$  and  $\delta$ . Summing over  $\Xi_\varepsilon$  yields

$$\int_{\Omega} |\psi_{\varepsilon,\delta}(x) - \psi(\infty)|^2 dx = \sum_{\xi \in \Xi_\varepsilon} \int_{\varepsilon\xi + \varepsilon Y} |\psi_{\varepsilon,\delta}(x) - \psi(\infty)|^2 dx \leq C\delta^N \|\nabla_z \psi\|_{L^2(\mathbb{R}^N)}^2.$$

Hence, we get the strong convergence (6.10).  $\square$

*Proof of Theorem 6.1.* First observe that Problem (6.2) has a unique solution  $U$  belonging to  $L^2(0, T; \mathbf{L})$ . It is a consequence of the Faedo-Galerkin's method which also gives the equality (6.3). This will imply the convergence of the full sequences in (4.12), (5.9) and (5.10). It is therefore enough to obtain the limit Problem (6.2) for a subsequence, which we do below.

The aim of the three first steps is to obtain the variational formulation (6.20) of the limit problem. Then, in the last step we prove that  $U(\cdot, \cdot, \infty) \in H^1(0, T; H^{-1}(\Omega))$  and  $U \in H^1(0, T; L^2(\Omega; (H^1(B))'))$ .

*Step 1.* Let  $(\phi, w)$  be in  $\mathcal{C}^1([0, T]) \times \mathcal{D}(\Omega)$  such that  $\phi(T) = 0$ . Taking  $\phi(t)w(x)$  as test function in (4.4) and then integrating by parts lead to

$$\begin{aligned} & - \int_{(0,T) \times \Omega_{\varepsilon,\delta}^*} u_{\varepsilon,\delta} \phi' w \, dt dx - \int_{(0,T) \times B_{\varepsilon,\delta}} a_{\varepsilon,\delta} u_{\varepsilon,\delta} \phi' w \, dt dx \\ & + \int_{(0,T) \times \Omega_{\varepsilon,\delta}^*} \phi \nabla u_{\varepsilon,\delta} \nabla w \, dt dx + \int_{(0,T) \times B_{\varepsilon,\delta}} b_{\varepsilon,\delta} \phi \nabla u_{\varepsilon,\delta} \nabla w \, dt dx \quad (6.11) \\ & = \int_{\Omega^T} f_{\varepsilon,\delta} \phi w \, dt dx + \int_{\Omega} \rho_{\varepsilon,\delta} u_{\varepsilon,\delta}^0 \phi(0) w \, dx. \end{aligned}$$

Due to the convergences (4.12) and (6.7) we obtain

$$\begin{aligned} \int_{(0,T) \times \Omega_{\varepsilon,\delta}^*} u_{\varepsilon,\delta} \phi' w \, dt dx &= \int_{\Omega^T} u_{\varepsilon,\delta} \phi' w \mathbf{1}_{\Omega_{\varepsilon,\delta}^*} \, dt dx \longrightarrow \int_{\Omega^T} u \phi' w \, dt dx, \\ \int_{(0,T) \times \Omega_{\varepsilon,\delta}^*} \phi \nabla_x u_{\varepsilon,\delta} \nabla_x w \, dt dx &= \int_{\Omega^T} \phi \nabla_x u_{\varepsilon,\delta} \nabla_x w \mathbf{1}_{\Omega_{\varepsilon,\delta}^*} \, dt dx \longrightarrow \int_{\Omega^T} \phi \nabla_x u \nabla_x w \, dt dx. \end{aligned}$$

Now, unfolding the integral over  $(0, T) \times B_{\varepsilon,\delta}$  and using Lemma 3.2 and convergences (5.9) yield

$$\begin{aligned} \int_{(0,T) \times B_{\varepsilon,\delta}} a_{\varepsilon,\delta} u_{\varepsilon,\delta} \phi' w \, dt dx &= \int_{(0,T) \times \Omega \times B} \delta^N a_{\varepsilon,\delta} \mathcal{T}_{\varepsilon,\delta}(u_{\varepsilon,\delta}) \phi' \mathcal{T}_{\varepsilon,\delta}(w) \, dt dx dz \\ &\longrightarrow \int_{(0,T) \times \Omega \times B} a_0 U \phi' w \, dt dx dz, \\ \int_{(0,T) \times B_{\varepsilon,\delta}} b_{\varepsilon,\delta} \phi \nabla u_{\varepsilon,\delta} \nabla w \, dt dx &= \int_{\Omega^T \times B} \varepsilon \delta \frac{\delta^{N-2}}{\varepsilon^2} b_{\varepsilon,\delta} \phi \nabla_z \mathcal{T}_{\varepsilon,\delta}(u_{\varepsilon,\delta}) \mathcal{T}_{\varepsilon,\delta}(\nabla w) \, dt dx dz \longrightarrow 0. \end{aligned}$$

The convergences (3.8), (5.8), (4.7) and (6.7), lead to

$$\begin{aligned}
\int_{\Omega} \rho_{\varepsilon,\delta} u_{\varepsilon,\delta}^0 \phi(0) w \, dx &= \int_{\Omega} u_{\varepsilon,\delta}^0 \phi(0) w \mathbf{1}_{\Omega^*_{\varepsilon,\delta}} \, dx + \int_{\Omega \times B} \delta^N a_{\varepsilon,\delta} \mathcal{T}_{\varepsilon,\delta}(u_{\varepsilon,\delta}^0) \phi(0) \mathcal{T}_{\varepsilon,\delta}(w) \, dx dz \\
&\longrightarrow \int_{\Omega} u^0 \phi(0) w \, dx + \int_{\Omega \times B} a_0 U^0 \phi(0) w \, dx dz, \\
\int_{\Omega^T} f_{\varepsilon,\delta} \phi w \, dt dx &= \int_{\Omega^T} f_{\varepsilon,\delta} \phi w \mathbf{1}_{\Omega^*_{\varepsilon,\delta}} \, dt dx + \int_{\Omega^T \times B} \delta^N \mathcal{T}_{\varepsilon,\delta}(f_{\varepsilon,\delta}) \phi \mathcal{T}_{\varepsilon,\delta}(w) \, dt dx dz \\
&\longrightarrow \int_{\Omega^T} f \phi w \, dt dx + \int_{\Omega^T \times B} F \phi w \, dt dx dz.
\end{aligned}$$

Finally, summarizing the above limits give

$$\begin{aligned}
&- a_0 \int_{\Omega^T \times B} U(t, x, z) \phi'(t) w(x) \, dt dx dz \\
&- \int_{\Omega^T} u(t, x) \phi'(t) w(x) \, dt dx + \int_{\Omega^T} \phi(t) \nabla_x u(t, x) \nabla w(x) \, dt dx dy \\
&= \int_{\Omega^T} f(t, x) \phi(t) w(x) \, dt dx + \int_{\Omega^T \times B} F(t, x, z) \phi(t) w(x) \, dt dx dz \\
&+ \phi(0) \int_{\Omega} \left( u^0(x) + a_0 \int_B U^0(x, z) \, dz \right) w(x) \, dx.
\end{aligned} \tag{6.12}$$

*Step 2.* Now, let  $(\phi, w)$  be in  $C^1([0, T]) \times \mathcal{D}(\Omega)$  such that  $\phi(T) = 0$  and let  $\psi \in \mathbf{H} \cap C^\infty(\mathbb{R}^N)$  such that the support of  $\nabla_z \psi$  is bounded. In (4.4) we take as test function  $\phi(t) w(x) \psi_{\varepsilon,\delta}(x)$ , where  $\psi_{\varepsilon,\delta}$  is given by (6.9).

If  $\delta$  is small enough the function  $\psi_{\varepsilon,\delta}$  is constant (and equal to  $\psi(\infty)$ ) on the boundary of the cells  $\varepsilon\xi + \varepsilon Y$  ( $\xi \in \Xi_\varepsilon$ ). As a consequence, the function  $w \psi_{\varepsilon,\delta}$  is an admissible test function belonging to  $H_0^1(\Omega)$ . Hence

$$\begin{aligned}
&- \int_{\Omega^T} \rho_{\varepsilon,\delta} u_{\varepsilon,\delta} \phi' w \psi_{\varepsilon,\delta} \, dt dx + \int_{\Omega^T} k_{\varepsilon,\delta} \phi \nabla u_{\varepsilon,\delta} \nabla (w \psi_{\varepsilon,\delta}) \, dt dx \\
&= \int_{\Omega^T} f_{\varepsilon,\delta}(t, x) \phi(t) w(x) \psi_{\varepsilon,\delta}(x) \, dt dx + \int_{\Omega} \rho_{\varepsilon,\delta} u_{\varepsilon,\delta}^0(x) \phi(0) w(x) \psi_{\varepsilon,\delta}(x) \, dx.
\end{aligned} \tag{6.13}$$

Since  $\nabla_z \psi$  has a compact support in  $\mathbb{R}^N$ , from the equality (3.5) and Lemma 3.2 we obtain

$$\varepsilon \delta \mathcal{T}_{\varepsilon,\delta}(w \nabla \psi_{\varepsilon,\delta}) = \mathcal{T}_{\varepsilon,\delta}(w) \nabla_z \psi \longrightarrow w \nabla_z \psi \quad \text{strongly in } L^2(\Omega \times \mathbb{R}^N). \tag{6.14}$$

Now, due to the convergences (4.12)<sub>2</sub>, (5.9)<sub>1</sub>, (6.14) and (6.10) we can write

$$\begin{aligned}
&\int_{\Omega^T} \rho_{\varepsilon,\delta} u_{\varepsilon,\delta} \phi' w \psi_{\varepsilon,\delta} \, dt dx \\
&= \int_{\Omega^T} u_{\varepsilon,\delta} \phi' w \psi_{\varepsilon,\delta} \mathbf{1}_{\Omega^*_{\varepsilon,\delta}} \, dt dx + \int_{\Omega^T \times B} \delta^N a_{\varepsilon,\delta} \mathcal{T}_{\varepsilon,\delta}(u_{\varepsilon,\delta}) \phi' \mathcal{T}_{\varepsilon,\delta}(w) \psi \, dt dx dz \\
&\longrightarrow \int_{\Omega^T} u \phi' w \psi(\infty) \, dt dx + a_0 \int_{\Omega^T \times B} U \phi' w \psi \, dt dx dz.
\end{aligned} \tag{6.15}$$

We have

$$\begin{aligned} & \int_{\Omega^T} k_{\varepsilon,\delta} \phi \nabla_x u_{\varepsilon,\delta} \nabla(w \psi_{\varepsilon,\delta}) dt dx \\ &= \int_{\Omega^T} k_{\varepsilon,\delta} \phi \nabla_x u_{\varepsilon,\delta} \nabla w \psi_{\varepsilon,\delta} dt dx + \int_{\Omega^T} k_{\varepsilon,\delta} \phi \nabla_x u_{\varepsilon,\delta} w \nabla \psi_{\varepsilon,\delta} dt dx. \end{aligned}$$

Convergences (5.9)<sub>1</sub>, (5.10)<sub>2</sub> and (6.10) lead to

$$\begin{aligned} & \int_{\Omega^T} k_{\varepsilon,\delta} \phi \nabla_x u_{\varepsilon,\delta} \nabla w \psi_{\varepsilon,\delta} dt dx = \int_{\Omega^T} \phi \nabla_x u_{\varepsilon,\delta} \nabla w \psi_{\varepsilon,\delta} 1_{\Omega_{\varepsilon,\delta}^*} dt dx \\ & \quad + \int_{\Omega^T \times B} b_{\varepsilon,\delta} \phi \frac{\delta^{N-2}}{\varepsilon^2} \nabla_z \mathcal{T}_{\varepsilon,\delta}(u_{\varepsilon,\delta}) \left( \varepsilon \delta \mathcal{T}_{\varepsilon,\delta}(\nabla w) \right) \psi dt dx dz \quad (6.16) \\ & \quad \longrightarrow \int_{\Omega^T} \phi \nabla_x u \nabla w \psi(\infty) dt dx + 0. \end{aligned}$$

Again from (5.9)<sub>1</sub>, (5.10)<sub>2</sub> and now (6.14) we get

$$\begin{aligned} & \int_{\Omega^T} k_{\varepsilon,\delta} \phi u_{\varepsilon,\delta} w \nabla \psi_{\varepsilon,\delta} dt dx \\ &= \int_{\Omega^T \times \mathbb{R}^N} \phi \mathcal{T}_{\varepsilon,\delta}(k_{\varepsilon,\delta}) \frac{\delta^{N-2}}{\varepsilon^2} \nabla_z \mathcal{T}_{\varepsilon,\delta}(u_{\varepsilon,\delta}) \mathcal{T}_{\varepsilon,\delta}(w) \nabla_z \psi dt dx dz \quad (6.17) \\ & \longrightarrow \gamma_0 \left( \int_{\Omega^T \times (\mathbb{R}^N \setminus \bar{B})} \phi \nabla_z V w \nabla_z \psi dt dx dz + b_0 \int_{\Omega^T \times B} \phi \nabla_z U w \nabla_z \psi dt dx dz \right). \end{aligned}$$

In the same way, using (4.7)<sub>2</sub> and (5.8)<sub>2</sub> give

$$\begin{aligned} & \int_{\Omega^T} f_{\varepsilon,\delta} \phi w \psi_{\varepsilon,\delta} dt dx \longrightarrow \int_{\Omega^T} f \phi w \psi(\infty) dt dx + \int_{\Omega^T \times B} F \phi w \psi dt dx dz, \quad (6.18) \\ & \int_{\Omega} \rho_{\varepsilon,\delta} u_{\varepsilon,\delta}^0 \phi(0) w \psi_{\varepsilon,\delta} dx \longrightarrow \int_{\Omega} u^0 \phi(0) w \psi(\infty) dx + \int_{\Omega \times B} a_0 U^0 \phi(0) w \psi dx dz. \end{aligned}$$

Finally, from the convergences (6.15), (6.16), (6.17) and (6.18) one has

$$\begin{aligned} & - \int_{\Omega^T} u \phi' w \psi(\infty) dt dx - a_0 \int_{\Omega^T \times B} U \phi' w \psi dt dx dz + \int_{\Omega^T} \phi \nabla u \nabla w \psi(\infty) dt dx \\ & + \gamma_0 \int_{\Omega^T \times \mathbb{R}^N} k_0 \phi \nabla_z U w \nabla_z \psi dt dx dz \quad (6.19) \\ &= \int_{\Omega^T} f \phi w \psi(\infty) dt dx + \int_{\Omega^T \times B} F \phi w \psi dt dx dz \\ & + \int_{\Omega} u^0 \phi(0) w \psi(\infty) dx + a_0 \int_{\Omega \times B} U^0 \phi(0) w \psi dx dz. \end{aligned}$$

*Step 3.* The space of functions belonging to  $\mathbf{H} \cap C^\infty(\mathbb{R}^N)$  and whose gradient support is bounded is dense in  $\mathbf{H}$ . Hence, equality (6.19) is satisfied for any  $\psi$  in  $\mathbf{H}$ . Then, the density

of the tensor product  $C^1([0, T]) \otimes \mathcal{D}(\Omega) \otimes \mathbf{H}$  in  $H^1(0, T; \mathbf{L})$  implies

$$\begin{aligned}
& - \int_{\Omega^T} U(t, \cdot, \infty) \frac{\partial \Phi}{\partial t}(t, \cdot, \infty) dt dx - a_0 \int_{\Omega^T \times B} U \frac{\partial \Phi}{\partial t} dt dx dz \\
& + \int_{\Omega^T} \nabla_x U(\cdot, \infty) \nabla_x \Phi(t, \cdot, \infty) dt dx + \gamma_0 \int_{\Omega^T \times \mathbb{R}^N} k_0 \nabla_z U \nabla_z \Phi dt dx dz \\
& = \int_{\Omega^T} f \Phi(t, \cdot, \infty) dt dx + \int_{\Omega^T \times B} F \Phi dt dx dz \\
& + \int_{\Omega} u^0 \Phi(0, \cdot, \infty) dx + a_0 \int_{\Omega \times B} U^0 \Phi(0) dx dz
\end{aligned} \tag{6.20}$$

for any  $\Phi \in H^1(0, T; \mathbf{L})$  such that  $\Phi(T, \cdot, \cdot) = 0$ . Hence (6.2).

*Step 4.* Let  $\Theta$  be in  $\mathbf{L}$  satisfying

$$\Theta(\infty) = 1 \quad \text{and} \quad \Theta(z) = 0 \quad \text{for a.e. } z \in B.$$

For any  $\Psi \in H_0^1(0, T; H_0^1(\Omega))$  we choose  $\Phi = \Psi\Theta$  as test function in (6.20). That gives

$$\begin{aligned}
& - \int_{\Omega^T} U(t, \cdot, \infty) \frac{\partial \Psi}{\partial t} dt dx + \int_{\Omega^T} \nabla_x U(\cdot, \infty) \nabla_x \Psi dt dx \\
& + \gamma_0 \int_{\Omega^T \times \mathbb{R}^N} k_0 \nabla_z U \Psi \nabla_z \Theta dt dx dz = \int_{\Omega^T} f \Psi dt dx.
\end{aligned} \tag{6.21}$$

Hence

$$\begin{aligned}
\left| \int_{\Omega^T} U(t, \cdot, \infty) \frac{\partial \Psi}{\partial t} dt dx \right| & \leq \|\nabla_x U(\cdot, \infty)\|_{L^2(\Omega^T)} \|\nabla_x \Psi\|_{L^2(\Omega^T)} \\
& + C \|\nabla_z U\|_{L^2(\Omega^T \times \mathbb{R}^N)} \|\Psi\|_{L^2(\Omega^T)} \|\nabla_z \Theta\|_{L^2(\mathbb{R}^N)} + \|f\|_{L^2(\Omega^T)} \|\Psi\|_{L^2(\Omega^T)}.
\end{aligned}$$

Since  $U$  belongs to  $L^2(0, T; \mathbf{L})$ , we obtain

$$\left| \int_{\Omega^T} U(t, \cdot, \infty) \frac{\partial \Psi}{\partial t} dt dx \right| \leq C \|\Psi\|_{L^2(0, T; H_0^1(\Omega))}.$$

The constant depends on  $\|U\|_{L^2(0, T; \mathbf{L})}$ ,  $\|\nabla_z \Theta\|_{L^2(\mathbb{R}^N)}$  and  $\|f\|_{L^2(\Omega^T)}$ . As a consequence of the above inequality we get  $U(\cdot, \cdot, \infty) \in H^1(0, T; H^{-1}(\Omega))$ .

Now, let  $\Phi$  be in  $H_0^1(0, T; L^2(\Omega; H^1(B)))$  we extend  $\Phi$  as a function  $\bar{\Phi}$  belonging to  $H_0^1(0, T; L^2(\Omega; H^1(\mathbb{R}^N)))$  satisfying

$$\begin{aligned}
\bar{\Phi}(t, x, z) & = \Phi(t, x, z) \quad \text{for a.e. } (t, x, z) \in \Omega^T \times B, \\
\|\bar{\Phi}\|_{H_0^1(0, T; L^2(\Omega; H^1(\mathbb{R}^N)))} & \leq C \|\Phi\|_{H_0^1(0, T; L^2(\Omega; H^1(B)))}.
\end{aligned}$$

We choose  $\bar{\Phi}$  as test function in (6.20). That leads to

$$-a_0 \int_{\Omega^T \times B} U \frac{\partial \bar{\Phi}}{\partial t} dt dx dz + \gamma_0 \int_{\Omega^T \times \mathbb{R}^N} k_0 \nabla_z U \nabla_z \bar{\Phi} dt dx dz = \int_{\Omega^T \times B} F \bar{\Phi} dt dx dz.$$

Hence

$$\begin{aligned} a_0 \left| \int_{\Omega^T \times B} U \frac{\partial \Phi}{\partial t} dt dx dz \right| &\leq C \|\nabla_z U\|_{L^2(\Omega^T \times B)} \|\nabla_z \Phi\|_{L^2(\Omega^T \times \mathbb{R}^N)} \\ &+ \|F\|_{L^2(\Omega^T \times B)} \|\Phi\|_{L^2(\Omega^T \times \mathbb{R}^N)}. \end{aligned} \quad (6.22)$$

Again, using the fact that  $U$  belongs to  $L^2(0, T; \mathbf{L})$ , we get

$$\left| \int_{\Omega^T \times B} U \frac{\partial \Phi}{\partial t} dt dx dz \right| \leq C \|\Phi\|_{L^2(\Omega^T; H^1(B))}.$$

The constant depends on  $\|U\|_{L^2(0, T; \mathbf{L})}$  and on  $\|F\|_{L^2(\Omega^T \times B)}$ . As a consequence of the above inequality we get  $U \in H^1(0, T; L^2(\Omega; (H^1(B))'))$ .  $\square$

## 7 The homogenized limit problem

From now on, we extend  $f$  and  $F$  by zero on  $[T, +\infty[$ .

For any  $\phi \in L^2(0, +\infty)$  and for any  $p \in \mathbb{R}_+^*$ , the Laplace transform (see e.g. [10], [11] or [12]) is defined by

$$\widehat{\phi}(p) = \int_0^{+\infty} \phi(t) e^{-pt} dt.$$

Hence, the evolution Problem (6.2) becomes

$$\begin{aligned} &p \int_{\Omega} \widehat{U}(p, x, \infty) \Phi(x, \infty) dx + a_0 p \int_{\Omega \times B} \widehat{U}(p, x, z) \Phi(x, z) dx dz \\ &+ \int_{\Omega} \nabla_x \widehat{U}(p, x, \infty) \nabla_x \Phi(x, \infty) dx + \gamma_0 \int_{\Omega \times \mathbb{R}^N} k_0(z) \nabla_z \widehat{U}(p, x, z) \nabla_z \Phi(x, z) dx dz \\ &= \int_{\Omega} \widehat{f}(p, x) \Phi(x, \infty) dx + \int_{\Omega \times B} \widehat{F}(p, x, z) \Phi(x, z) dx dz \\ &+ \int_{\Omega} u^0(x) \Phi(x, \infty) dx + a_0 \int_{\Omega \times B} U^0(x, z) \Phi(x, z) dx dz, \quad \forall p > 0, \quad \forall \Phi \in \mathbf{L}. \end{aligned} \quad (7.1)$$

For any  $p > 0$ , the Lax-Milgram theorem gives the existence and uniqueness of  $\widehat{U}(p) \in \mathbf{L}$ . The following estimates hold:

$$\begin{aligned} &p \|\widehat{U}(p, \cdot, \infty)\|_{L^2(\Omega)}^2 + \|\nabla_x \widehat{U}(p, \cdot, \infty)\|_{L^2(\Omega)}^2 + p \|\widehat{U}(p)\|_{L^2(\Omega \times B)}^2 + \|\nabla_z \widehat{U}(p)\|_{L^2(\Omega \times \mathbb{R}^N)}^2 \\ &\leq \frac{C}{p} \left( \|\widehat{f}(p)\|_{L^2(\Omega)}^2 + \|\widehat{F}(p)\|_{L^2(\Omega \times B)}^2 + \|u^0\|_{L^2(\Omega)}^2 + \|U^0\|_{L^2(\Omega \times B)}^2 \right), \quad \forall p > 0. \end{aligned}$$

The constant does not depend on  $p$ . We recall that

$$\forall p > 0, \quad \|\widehat{f}(p)\|_{L^2(\Omega)}^2 \leq \frac{1}{p} \|f\|_{L^2(\Omega^T)}^2, \quad \|\widehat{F}(p)\|_{L^2(\Omega \times B)}^2 \leq \frac{1}{p} \|F\|_{L^2(\Omega^T \times B)}^2.$$

Now, let us introduce the solutions  $\widehat{\theta}_0$  and  $\widehat{\theta}_1$  of the following cell problems:

$$\begin{aligned}
& \widehat{\theta}_0(p) \in L^2(\Omega; \dot{H}^1(\mathbb{R}^N)), \\
& a_0 p \int_B \widehat{\theta}_0(p, x) \Phi dz + \gamma_0 \int_{\mathbb{R}^N} k_0 \nabla_z \widehat{\theta}_0(p, x) \nabla_z \Phi dz \\
& = \int_B \widehat{F}(p, x) \Phi dz + a_0 \int_B U^0(x, z) \Phi dx dz, \\
& \text{for a.e. } x \in \Omega, \quad \forall \Phi \in \dot{H}^1(\mathbb{R}^N), \quad \forall p > 0,
\end{aligned} \tag{7.2}$$

and

$$\begin{aligned}
& \widehat{\theta}_1(p) \in \mathbf{H}, \quad \widehat{\theta}_1(p, \infty) = 1, \\
& a_0 p \int_B \widehat{\theta}_1(p) \Phi dz + \gamma_0 \int_{\mathbb{R}^N} k_0 \nabla_z \widehat{\theta}_1(p) \nabla_z \Phi dz = 0, \\
& \forall \Phi \in \dot{H}^1(\mathbb{R}^N), \quad \forall p > 0.
\end{aligned} \tag{7.3}$$

Again, the Lax-Milgram theorem gives the existence of  $\widehat{\theta}_0$ ,  $\widehat{\theta}_1$  and

$$p \|\widehat{\theta}_0(p)\|_{L^2(\Omega \times B)}^2 + \|\nabla_z \widehat{\theta}_0(p)\|_{L^2(\Omega \times \mathbb{R}^N)}^2 \leq \frac{C}{p} (\|\widehat{F}(p)\|_{L^2(\Omega \times B)}^2 + \|U^0\|_{L^2(\Omega \times B)}^2) \quad \forall p > 0.$$

Now, in order to estimate  $\widehat{\theta}_1$  we consider a function  $\vartheta \in \mathbf{H} \cap C^\infty(\mathbb{R}^N)$  satisfying

$$\vartheta(\infty) = 1, \quad \vartheta = 0 \quad \text{in } B, \quad \nabla_z \vartheta \text{ has a compact support.} \tag{7.4}$$

The function  $\widehat{\theta}_2 = \widehat{\theta}_1 - \vartheta$  is the solution of

$$\begin{aligned}
& \widehat{\theta}_2(p) \in \dot{H}^1(\mathbb{R}^N), \\
& a_0 p \int_B \widehat{\theta}_2(p) \Phi dz + \gamma_0 \int_{\mathbb{R}^N} k_0 \nabla_z \widehat{\theta}_2(p) \nabla_z \Phi dz \\
& = -\gamma_0 \int_{\mathbb{R}^N} k_0 \nabla_z \vartheta \nabla_z \Phi dz = \gamma_0 \int_{\mathbb{R}^N} k_0 \Delta_z \vartheta \Phi dz, \\
& \forall \Phi \in \dot{H}^1(\mathbb{R}^N), \quad \forall p > 0.
\end{aligned} \tag{7.5}$$

Then, taking  $\widehat{\theta}_2(p)$  has test function in (7.5) leads to

$$p \|\widehat{\theta}_2(p)\|_{L^2(B)}^2 + \|\nabla_z \widehat{\theta}_2(p)\|_{L^2(\mathbb{R}^N)}^2 \leq \frac{C}{p} \|\Delta_z \vartheta\|_{L^2(\mathbb{R}^N)}^2, \quad \forall p > 0.$$

We express  $\widehat{U}(p)$  in terms of  $\widehat{\theta}_0(p)$ ,  $\widehat{\theta}_2(p)$ ,  $\vartheta$  and  $\widehat{u}(p)$ . To do that, observe that the Laplace transform  $\widehat{u}(p)$  of  $u$  is in fact  $\widehat{U}(p, \cdot, \infty)$ . Hence, we have

$$\begin{aligned}
\widehat{U}(p, x, z) &= \widehat{u}(p, x) \widehat{\theta}_1(p, z) + \widehat{\theta}_0(p, x, z) = \widehat{u}(p, x) \widehat{\theta}_2(p, z) + \widehat{\theta}_0(p, x, z) + \widehat{u}(p, x) \vartheta(z) \\
& \text{for a.e. } (x, z) \in \Omega \times \mathbb{R}^N, \quad \forall p > 0.
\end{aligned}$$

Let  $\psi$  be in  $H_0^1(\Omega)$ , we choose  $\psi$  as test-function in (7.1). That gives

$$\begin{aligned}
& p \int_{\Omega} \widehat{u}(p) \psi \, dx + a_0 p \int_{\Omega} \left( \int_B \widehat{\theta}_2(p, z) \, dz \right) \widehat{u}(p) \psi \, dx \\
& + a_0 p \int_{\Omega} \left( \int_B \widehat{\theta}_0(p, \cdot, z) \, dz \right) \psi \, dx + \int_{\Omega} \nabla \widehat{u}(p) \nabla \psi \, dx \\
& = \int_{\Omega} (\widehat{f}(p) + u^0) \psi \, dx + \int_{\Omega} \left( \int_B \widehat{F}(p, \cdot, z) \, dz \right) \psi \, dx \\
& + a_0 \int_{\Omega} \left( \int_B U^0(\cdot, z) \, dz \right) \psi \, dx.
\end{aligned} \tag{7.6}$$

Now, applying the inverse Laplace transform (see [1], [10] and [11]) in Problems (7.2), (7.3) and (7.6) give

$$\begin{aligned}
& \theta_0 \in H^1(0, T; L^2(\Omega; (H^1(B))')), \quad \theta_0 \in L^2(\Omega^T; \dot{H}^1(\mathbb{R}^N)), \\
& a_0 < \frac{d\theta_0}{dt}(t, x), \Phi >_{(H^1(B))', H^1(B)} + \gamma_0 \int_{\mathbb{R}^N} k_0 \nabla_z \theta_0(t, x) \nabla_z \Phi \, dz \\
& = \int_B F(t, x) \Phi \, dz, \quad \text{for a.e. } (t, x) \in (0, +\infty) \times \Omega, \\
& \theta_0(0) = U^0, \quad \forall \Phi \in \dot{H}^1(\mathbb{R}^N),
\end{aligned} \tag{7.7}$$

$\theta_2$  is the solution of the following evolution problem:

$$\begin{aligned}
& \theta_2 \in H^1(0, T; (H^1(B))'), \quad \theta_2 \in L^2(0, T; \dot{H}^1(\mathbb{R}^N)), \\
& a_0 < \frac{d\theta_2}{dt}(t), \Phi >_{(H^1(B))', H^1(B)} + \gamma_0 \int_{\mathbb{R}^N} k_0 \nabla_z \theta_2(t) \nabla_z \Phi \, dz = 0, \\
& \theta_2(0) = \frac{\gamma_0 k_0}{a_0} \Delta_z \vartheta, \quad \text{for a.e. } t \in (0, +\infty), \quad \forall \Phi \in \dot{H}^1(\mathbb{R}^N).
\end{aligned} \tag{7.8}$$

Here again, the Faedo-Galerkin's method gives the existence and uniqueness of the solutions of the two above evolution equations. Proceeding as in Theorem 6.1 we show that  $\theta_0 \in H^1(0, T; L^2(\Omega; (H^1(B))'))$  and  $\theta_2 \in H^1(0, T; (H^1(B))')$ .

**Lemma 7.1.** *Set*

$$\Theta_0 = \int_B \theta_0 \, dz, \quad \Theta_2 = \int_B \theta_2 \, dz.$$

*We have*

$$\Theta_0 \in H^1(0, T; L^2(\Omega)), \quad \Theta_2 \in H^1(0, T).$$

*Moreover, the function  $\Theta_2$  does not depend on the choice of the function  $\vartheta$ .*

*Proof.* Since  $\theta_0 \in L^2(\Omega^T \times B)$  we obtain  $\Theta_0 \in L^2(\Omega^T)$  (as a consequence of the Cauchy-Schwarz inequality and the Fubini's theorem). Choosing  $1 - \vartheta$  as test function in (7.7) (observe

that  $1 - \vartheta \in \dot{H}^1(\mathbb{R}^N)$  gives

$$\begin{aligned}
& a_0 \frac{d}{dt} \left( \langle \theta_0(t, x), 1 \rangle_{(H^1(B))', H^1(B)} \right) = a_0 \langle \frac{d\theta_0}{dt}(t, x), 1 \rangle_{(H^1(B))', H^1(B)} \\
& = a_0 \langle \frac{d\theta_0}{dt}(t, x), 1 - \vartheta \rangle_{(H^1(B))', H^1(B)} \\
& = \gamma_0 \int_{\mathbb{R}^N} k_0 \nabla_z \theta_0(t, x) \nabla_z \vartheta dz + \int_B F(t, x) dz.
\end{aligned} \tag{7.9}$$

Now, the fact that

- $\langle \theta_0(t, x), 1 \rangle_{(H^1(B))', H^1(B)} = \int_B \theta_0(t, x) dz$  for a.e.  $x \in \Omega$ ,
- $\nabla_z \theta_0 \in [L^2(\Omega^T \times \mathbb{R}^N)]^N$ ,
- $\nabla_z \vartheta \in [C^\infty(\mathbb{R}^N)]^N$  (with compact support)
- $F \in L^2(\Omega^T \times B)$

and the above equalities (7.9) yield  $\frac{d\Theta_0}{dt} \in L^2(\Omega^T)$ . As a consequence  $\Theta_0$  belongs to  $H^1(0, T; L^2(\Omega))$ . Proceeding in the same way gives  $\Theta_2 \in H^1(0, T)$ .

Denote  $\widehat{\Theta}_2$  the Laplace transform of  $\Theta_2$ . One has

$$\widehat{\Theta}_2 = \int_B \widehat{\theta}_2 dz = \int_B (\widehat{\theta}_1 - \vartheta) dz = \int_B \widehat{\theta}_1 dz.$$

It follows that  $\Theta_2$  does not depend of the choice of  $\vartheta$ . □

We transform Problem (7.6) by the inverse Laplace transform, that gives

$$\begin{aligned}
& \langle \frac{d}{dt} (u + a_0 \Theta_2 * u), \psi \rangle_{H^{-1}(\Omega), H_0^1(\Omega)} + \int_{\Omega} \nabla u(t) \nabla \psi dx \\
& = -a_0 \langle \frac{d\Theta_0}{dt}(t), \psi \rangle_{H^{-1}(\Omega), H_0^1(\Omega)} + \int_{\Omega} f(t) \psi dx, + a_0 \int_{\Omega} \left( \int_B F(t) dz \right) \psi dx \tag{7.10} \\
& u(0) = u^0, \quad \Theta_0(0) = \int_B U^0 dz, \quad \text{for a.e. } t \in (0, +\infty), \quad \forall \psi \in H_0^1(\Omega).
\end{aligned}$$

Finally, we obtain

**Theorem 7.2.** *The limit field  $u$  is the solution of the following evolution problem:*

$$\begin{aligned}
& u \in L^2(0, T; H_0^1(\Omega)) \cap L^\infty(0, T; L^2(\Omega)) \cap H^1(0, T; H^{-1}(\Omega)), \\
& \frac{d}{dt} (u + a_0 \Theta_2 * u) - \Delta u = -a_0 \frac{d\Theta_0}{dt} + G \quad \text{in } \Omega^T, \\
& u = 0 \quad \text{on } \partial\Omega \times (0, T), \\
& u(0) = u^0 \quad \text{in } \Omega,
\end{aligned} \tag{7.11}$$

where

- $\Theta_2 = \int_B \theta_2 dz$  belongs to  $H^1(0, T)$ , the function  $\theta_2$  being the solution of (7.8),
- $G(t) = f(t) + \int_B F(t, \cdot, z) dz$  belongs to  $L^2(\Omega^T)$ ,
- $\Theta_0 = \int_B \theta_0 dz$  belongs to  $H^1(0, T; L^2(\Omega))$ , the function  $\theta_0$  being the solution of (7.7).

## References

- [1] A. Bensoussan, J.L. Lions, G. Papanicolaou. Asymptotic Analysis for Periodic Structures, North-Holland Publishing Company-Amsterdam. New York. Oxford, 1978.
- [2] D. Cioranescu and F. Murat, Un terme étrange venu d'ailleurs, in *Nonlinear partial differential equations and their applications, Colloge de France Seminar, II & III*, H. Brezis and J.L. Lions (eds), Research Notes in Math. 60-70, Pitman, Boston (1982), 98–138, 154–178.
- [3] D. Cioranescu, P. Donato. An Introduction to Homogenization. Oxford Lecture Ser. in Math. Appl. 17, Oxford University Press, Oxford, 1999.
- [4] D. Cioranescu, A. Damlamian, G. Griso. Periodic unfolding method in homogenization, *SIAM J. Math. Anal.* 40 (2008), (4), 1585-1620.
- [5] D. Cioranescu, A. Damlamian, G. Griso, D. Onofrei. The periodic unfolding method for perforated domains and Neumann sieve models, *J. Math. Pures Appl.* 89 (2008) 248-277.
- [6] F. Bentalha, I. Gruais, D. Poliřevski. A diffusion process in a rarefied binary structure. *Rev. Roum. Math. Pures et Appl.* 52(2), 129-149, 2007.
- [7] I. Gruais. Homogenizing a highly rarefied binary structure of finite diffusivity. *Applicable Analysis* 86 (2007), 633-652.
- [8] I. Gruais, D. Poliřevski. Homogenizing a critical binary structure of finite diffusivities. *Asymptotic Analysis* 55 (2007), 85-101.
- [9] J.L. Lions. Quelques méthodes de résolution des problèmes aux limites non linéaires, Dunod, 1969.
- [10] L. Debnath, D. Bhatta. Integral Transformation and Their Applications. A Chapman & Hall Book /CRC Press, 2007.
- [11] M.R. Spiegel. Schaum's Outline of Theory and Problems of Laplace Transforms. McGRAW-HILL, 1965.

- [12] R. Dautray, J.L. Lions. *Mathematical Analysis and Numerical Methods for Science and Technology, Volume 5, Evolution Problems I*. Springer-Verlag Berlin Heidelberg 1992, 2000.