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# HOMOGENIZATION OF DIFFUSION-DEFORMATION IN DUAL-POROUS MEDIUM WITH DISCONTINUITY INTERFACES*

GEORGES GRISO† AND EDUARD ROHAN‡

**Abstract.** Models of homogenized fluid-saturated dual-porous media with weak, or strong discontinuity interfaces (resembling fissures) are derived using the periodic unfolding method. Stress discontinuities at the interfaces are admitted, requesting further restrictions on the applied external forces. The limit models, obtained by a rigorous asymptotic analysis, reflect some non-local effects inherited from the microstructural interactions. In view of obtaining the a priori estimates, standard approaches based on smooth extensions, well fitted for perforated or high-contrast media, cannot be adopted for fissured domains. Therefore, a new approach is developed which enables to control the norm of some "off-diagonal" terms which in the model equations are generated by the interfaces and are not involved in the energy-related expressions.

**Key words.** porous media, dual porosity, periodic homogenization, high contrast media, discontinuity interfaces, diffusion-deformation.

**AMS subject classifications.** 35B27, 35Q74, 76S05, 74Q05, 74Q15

**1 Introduction** Modeling of porous solids penetrated by fluids is still a challenging issue in continuum mechanics, due to important applications in geology and mining, in civil and environmental engineering, or in tissue biomechanics and material engineering. Nowadays, also new technologies related to the transport of liquids in porous deforming structures are inspired by complex processes in biological tissues characterized by presence of coupled physical fields.

Here we focus on the mechanical aspects of a coupled fluid diffusion and solid deformation, which could serve as a basis for further extension to "multiphysical problems". Biot [6] formulated the basic theory of deformation of a porous isotropic elastic solid subjected to a small strain and saturated by a Newtonian fluid. Later on, this theory was extended to anisotropic elastic fluid-saturated media where all the constituents can be compressible. The detailed description of the poroelastic theory can be found, for example, in the book [14].

The model treated in this paper describes the diffusion and deformation coupled in time at three different scales using the concept of the so-called *dual porosity* [3, 4]. In the context of asymptotic analysis with respect to the scale of heterogeneities, the dual porosity is represented by a scale-dependent permeability [3, 8]. The homogenization of diffusion in such a type of media was discussed broadly in the literature, see e.g. [30, 15, 21], however without considering the deformation. The coupled diffusion and deformation was treated by the asymptotic expansion method of homogenization e.g. in [25, 26] and for the dual porosity distributed in the form of inclusions accounted for by the authors in [16] ; therein the limit behaviour was analyzed using the periodic unfolding method proposed in [10]. A similar problem with connected dual porosity was reported in [29, 28] and [27] in the context of tissue modeling, where the macroscopic and microscopic problems with the fading memory effects were described in detail. Another treatment of flows in double-porous media is based on genuine treatment of interactions between the Darcy flow in a porous material, representing the dual porosity, and the Stokes flow in channels, cf. [23, 22].

In this paper we extend the model of [29] by including pressure discontinuity interfaces (resembling fissures) in a periodic microstructure, where non-standard interface conditions are prescribed. This option is important and interesting from two points of view: firstly, it is a model of *non-local interactions* in the considered type of medium (see Remark 1 below), secondly it requires a special non-standard procedure for obtaining uniform a priori estimates. Indeed, standard approaches based on smooth extensions, well applicable for perforated [9, 2], or high-contrast media

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(cf. [1, 8, 31]) cannot be used in our situation where the discontinuity interfaces induce similar effects to those obtained in fissured domains. That is why we develop a new approach enabling to control the norm of some terms representing the fluid-structure interactions at the interfaces. The major difficulty here is related to the fact that these terms are "off-diagonal" and are not involved in the energy-related expressions. Both these aspects make the problem attractive, since one can think that our approach can be adapted to other applications characterized by the same kind of transmission conditions.

The model we study here is motivated by its possible applications in bone tissue biomechanics, namely to describe the influence of the mechanical loading at the macroscopic scale on the fluid redistribution in the hierarchically porous structure with fractured interfaces , see [20]. The cortical bone is constituted by osteons, see Fig. 2.3 (left), cylindrical units (diameter of $\approx 100\mu m$) containing conducting channels (the Haversian channels) and the tissue matrix which is perforated by very thin channels (diameter of $\approx 0.1 - 1\mu m$) forming the canalicular system of pores. This hierarchical structure is repeated almost periodically. Moreover, each osteon unit is bounded by a cement surface with a reduced permeability supposed to give rise to pressure discontinuities.

The paper is organized as follows. In Section 2 we introduce the Biot-type model of the heterogeneous medium. Because of possible pressure discontinuities, in Section 2.2 we propose a generalized mass conservation equation in the neighborhood of the discontinuity interface $\Gamma$. We explain why this treatment is consistent with the interface transmission conditions admitting discontinuities of the *total stress* and preserving the symmetry of the system constituted by the equilibrium of forces and by the mass conservation. In Section 2.3 we introduce a microstructure decomposition, followed in Section 2.5 by a brief recall of the periodic unfolding method.

In Section 2.6 we distinguish two cases of the discontinuity interfaces. They are characterized by a scale-dependent permeability $\varkappa^\varepsilon \approx \varepsilon$, where $\varepsilon$ is the standard scale parameter, and by the "interface Biot coefficients" $\alpha^{\Gamma,\varepsilon}$. We treat two cases of the "discontinuity effect", depending on hypotheses on the magnitude of the interface Biot coefficients. In Section 3 we treat the "weakly discontinuous case", $\alpha_{ij}^{\Gamma,\varepsilon} \approx \varepsilon$. Under standard model-independent hypotheses on the loading volume forces, a priori estimates (uniform in $\varepsilon$) are obtained in the classical way.

Section 4 is devoted to the "strongly discontinuous case", $\alpha_{ij}^{\Gamma,\varepsilon} \approx 1$. Now, obtaining $\varepsilon$-uniform a priori estimates is a more delicate task. Moreover, a nonvanishing solution can be obtained only for special forms of the volume forces, one of them is represented in the form of interface distributions on the discontinuity interfaces. The convergence result yields vanishing macroscopic displacement field.

For both the cases, keeping the same scheme of development, in Sections 3 and 4, we follow in detail the whole homogenization procedure including a priori estimates, convergence results and description of the scale-decoupled problems for the homogenized media. As the problem is evolutionary, i.e. time-dependent, the scale-decoupling step is more complicated than for stationary problems. Therefore, we present the homogenized models merely in the form involving the Laplace-transformed time variables; the application of the inverse Laplace transformation is omitted here, since it would make the paper excessively long. This step can be found in [29], cf. [28], where an analogical model is treated.

*Notation* We shall use the indicial notation for tensors like $\sigma_{ij}$ and employ the Einstein summation convention for repeated indices. Also boldface italics (like $\boldsymbol{u} = (u_i)_i$) refers to vectorial variables. By $\partial_i^y$ we abbreviate $\frac{\partial}{\partial y_i}$, alternatively we use $\nabla_y = (\partial_i^y)_i$ to deal with the gradient vectors; by default, by $\nabla$ (or $\partial_i$) we mean the gradient (or its component) with respect to $x$. Lebesgue and Sobolev spaces are refered by using standard notation (e.g. $L^2(\Omega)$, $H^1(\Omega) = W^{1,2}(\Omega)$), for vector-valued spaces we use the boldface letters like $\mathbf{L}^2(\Omega)$, $\mathbf{H}^1(\Omega)$. The space of functions with compact support is denoted $C_0^\infty(\Omega)$. The symbol $\lambda$ is the variable of Laplace-transformed functions which are denoted by $*$, like $\mathcal{A}^*$ or $\underset{*}{v}$. Other notation is introduced subsequently in the text. Throughout the paper we adopt much of general notations employed in [10].

**2 Problem formulation** We shall define the set of equations and variables which describe the diffusion and deformation processes in a porous medium. We rely on the Biot model [7, 14] representing the solid-fluid interactions at a "mesoscopic scale" (rather than "microscopic" one)

where the pure solid and pure fluid parts cannot be separated. Thus, the heterogeneities of the medium are given in terms of spatial variations of the material coefficients of the Biot model. In what follows by the *microscopic scale* we mean the scale where these coefficient variations are distinguishable.

**2.1 Porous continuum** The Biot theory of the fluid saturated elastic medium is valid for the linearized material behavior. The stress is decomposed into the effective part $\sigma_{ij}^{\rm eff}$, and the part generated by the pressure of the interstitial fluid, $p$. Thus the total stress $\sigma_{ij}$ is given by

$$\sigma_{ij} = \sigma_{ij}^{\rm eff} - \alpha_{ij} p, \qquad \sigma_{ij}^{\rm eff} = D_{ijkl} e_{ij}(\boldsymbol{u}), \qquad e_{ij}(\boldsymbol{u}) = \frac{1}{2}(\partial_j u_i + \partial_i u_j), \tag{2.1}$$

where $\alpha_{ij}$ is the Biot tensor, $D_{ijkl}$ is the tensor of elastic coefficients and $\boldsymbol{u} = (u_i)$ is the displacement vector field associated to the *porous solid skeleton*. The relative motion of the fluid with respect to the skeleton is described by the filtration (or discharge) velocity $\boldsymbol{w} = (w_i)$, proportional to the pressure gradient by virtue of the Darcy law, $w_i = -K_{ij}\partial_j p$, where $K_{ij}$ is the permeability tensor. In our model of heterogeneous materials, we consider the system of double porosities, see e.g. [3, 21]. Recently in [16] we considered the incompressible Biot medium with the dual porosity distributed in the form of inclusions. Here we treat the compressible Biot model, where both the primary and the dual porosities form connected domains, cf. [29, 28], whereby some semi-permeable interfaces are embedded in the dual porosity. We remark that homogenization of the *parallel flows*, cf. [30], in defermable double-porous medium with two primary mutually disconnected porosities was reported in [27].

We do not take into account any inertia effect in the medium, thus the momentum equation reduces to the balance of forces. According to the Biot theory, the local mass conservation relates the skeleton compression, $\alpha_{ij} e_{ij}(\boldsymbol{u})$, the fluid discharge, $\mathrm{div}\boldsymbol{w}$ and the fluid accumulation due to compressibility of both the skeleton and the fluid, as represented by term $\dot{p}/\mu$, where $1/\mu$ is the Biot bulk compressibility modulus. The model involves force equilibrium and mass conservation equations,

$$\partial_j \sigma_{ij}(\boldsymbol{u}, p) = f_i, \qquad \alpha_{ij} e_{ij}(\boldsymbol{u}) + \mathrm{div}\boldsymbol{w} + \frac{1}{\mu}\dot{p} = 0, \tag{2.2}$$

where $\boldsymbol{f} = (f_i)$ is the field of volume forces. Substituting (2.1) into $(2.2)_1$ and using the Darcy law to express $\boldsymbol{w}$ in $(2.2)_2$, we obtain the reduced system involving just two fields

$$\begin{aligned} -\partial_j D_{ijkl} e_{kl}(\boldsymbol{u}) + \partial_j(\alpha_{ij} p) &= f_i \\ \alpha_{ij} e_{ij}(\dot{\boldsymbol{u}}) - \partial_i K_{ij} \partial_j p + \frac{1}{\mu}\dot{p} &= 0\ . \end{aligned} \tag{2.3}$$

We consider an anisotropic material with the following standard symmetries: $D_{ijkl} = D_{klij} = D_{jikl}$, $\alpha_{ij} = \alpha_{ji}$ and $K_{ij} = K_{ji}$.

**2.2 Generalized formulation for surface discontinuities** We shall treat a problem characterized by pressure discontinuities on the interfaces. In this section we consider the discontinuities distributed on surface $\Gamma \subset \Omega$, where $\Omega \subset \mathbb{R}^3$ is the open bounded domain, occupied by the porous medium.

With regards to the differentiability of material parameters, of the pressure and of displacement fields, the system of equations (2.2) is valid in $\Omega \setminus \Gamma$ where the medium is continuous. On interface $\Gamma$ we impose the following conditions

$$\begin{aligned} &[\boldsymbol{u}]_\Gamma = 0\ , \\ &[D_{ijkl} e_{kl}(\boldsymbol{u}) n_j]_\Gamma = 0\ , \\ &n_i K_{ij} \partial_j p = -w_i n_i = \varkappa [p]_\Gamma, \end{aligned} \tag{2.4}$$

where $[a]_\Gamma$ denotes the jump of a quantity $a$ across $\Gamma$ and $\boldsymbol{n} = (n_i)$ is the unit normal to $\Gamma$ such that $[a]_\Gamma = \lim_{s\to 0_+}(a(x + s\boldsymbol{n}) - a(x - s\boldsymbol{n}))$. The parameter $\varkappa \geq 0$ controls the interface permeability, i.e. the filtration velocity $\boldsymbol{w}$.

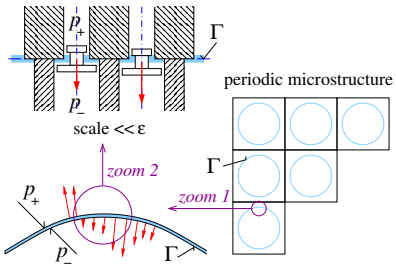


FIG. 2.1. *Hierarchical structure of the porous medium with the discontinuity interface – two zooms illustrated. An example of sub-microscopic arrangement (the 2nd zoom) of the porous interface* Γ *(the grey/blue belt). The pistons, being driven by external forces, may keep the pressure discontinuity* $|p_+ - p_-| > 0$*, whereas the effective stress (related to the solid phase) is continuous.*

There are two extreme cases, $\varkappa \to +\infty$ and $\varkappa \to 0$. The first one yields the perfect pressure bonding and thereby also the bulk stress continuity, $[n_j \sigma_{ij}]_\Gamma = 0$ due to $(2.4)_2$ and $(2.1)_1$, i.e. there is no interface effect. In the second case ($\varkappa \to 0$), the interface is completely impermeable, i.e. $n_i K_{ij} \partial_j p = 0$ in the sense of traces on both sides of Γ.

**Remark 1. Stress discontinuity and a generalized continuum.** We postulated the continuity of the effective stress, $[\sigma_{ij}^{\rm eff}] n_j = 0$ on Γ. This certainly holds at a porous interface when no fluid is present (thereby $p \equiv 0$). If the pores are saturated by the fluid, condition $(2.4)_2$ admits pressure discontinuities, i.e. $[\alpha_{ij} p] n_j \neq 0$, so that the overall (bulk) stress $(2.1)_1$ has a jump on Γ. This *does not conforms physically to standard continua.* However, we have in mind a *generalized continuum* with *non-local effects*: at the microscopic scale the local disbalance is equilibrated (at a larger scale) by means of coupled external forces. The following examples explains relevancy of our ansatz.

- Suppose that the (connected) interface Γ embeds a thin stiff self-supporting structure Σ which can exert local pressure disbalances. Thus the whole medium Ω is decomposed into this structure Σ and the "remainder" $\Omega \setminus \Sigma$. Assuming that Σ has a zero surface measure, we can decouple the two parts of the medium and represent Σ by means of external force fields distributed on Γ.
- Another example is illustrated in Figure 2.1: the pores penetrating the interface are equipped with valves driven by external forces. One may think of the electromagnetic field, but a device based on "micro-cables" (micro-Bowdens) interconnecting couples of distant valves could be constructed.

The above discussion reveals that our generalized continuum can be loaded only by external forces satisfying certain constraints related to the scale of the microstructure. We shall pursue two general cases of heterogeneities in coefficients $\alpha_{ij}$, as specified in Section 2.6, which for vanishing scale parameter leads to different limit models and to different additional constraints relating the acting force and the microstructure scale. □

We need to develop a special form of the mass conservation which takes into account a possible *pressure discontinuity* on the interfaces Γ. In order to obtain a consistent weak formulation of the diffusion-deformation problem, the pressure discontinuities must be well-approximated in the formulation of the mass conservation. Therefore, we consider an integral formulation of this physical law, where the sources and the fluxes are weighted by discontinuous functions.

**2.2.1 Preliminaries** Let Γ be a Lipschitz surface in $\mathbb{R}^3$ and $\omega \subset \mathbb{R}^3$ be an open bounded "control domain" such that $\Gamma_\omega = \Gamma \cap \omega$ is nonempty and divides ω in two subdomains, $\omega = \omega_+ \cup \omega_- \cup \Gamma_\omega$. Set

$$\Gamma^+ = \{x' = x + t\boldsymbol{n}(x) \mid x \in \Gamma,\ t \to 0_+\},$$
$$\Gamma^+ = \{x' = x + t\boldsymbol{n}(x) \mid x \in \Gamma,\ t \to 0_+\}.$$

Now, introduce a test function $q \in L^2(\omega)$ by using the characteristic functions of $\omega_\pm$,

$$q(x) = q^+\chi_{\omega_+}(x) + q^-\chi_{\omega_-}(x) \quad \text{where } \chi_{\omega_\pm}(x) = \begin{cases} 1 & \text{for } x \in \omega_\pm \\ 0 & \text{for } x \in \mathbb{R}^3 \setminus \omega_\pm, \end{cases} , \tag{2.5}$$

where $q^+$ and $q^-$ are real constants. Due to this construction, we can introduce the generalized derivative (gradient) of $q$: for any $\varphi \in C_0^\infty(\omega)$

$$\begin{aligned} \int_\omega \nabla\varphi\, q &= \sum_{k=+,-} \int_{\omega_k} \nabla(q^k\varphi) = \int_{\Gamma^+} q^+\varphi \boldsymbol{n}^+\, dS + \int_{\Gamma^-} q^-\varphi\boldsymbol{n}^-\, dS \\ &= -\int_{\Gamma_\omega} \varphi[q]_\Gamma \boldsymbol{n} = -\int_\omega \varphi\delta_\Gamma[q]_\Gamma\boldsymbol{n}\ , \\ \text{and} \quad \int_\omega \varphi\delta_\Gamma &= \int_{\Gamma_\omega} \varphi\, dS\ , \end{aligned} \tag{2.6}$$

where $\delta_\Gamma$ is the Dirac distribution of $\Gamma$ and $\boldsymbol{n}$ is the unit normal of $\Gamma$, such that $\boldsymbol{n}^+ = -\boldsymbol{n} = -\boldsymbol{n}^-$. Thus, the generalized gradient of $q$ is

$$\nabla q = \delta_\Gamma[q]_\Gamma\, \boldsymbol{n} \quad \text{ on } \Gamma \quad \text{in the sense of distributions.} \tag{2.7}$$

**2.2.2 Local mass conservation for discontinuous pressure fields** Let $\dot{\mathcal{J}}$ be the time rate of the fluid volume drained from an infinitesimal volume of the porous medium; later on we identify $\dot{\mathcal{J}} = \mathrm{div}\boldsymbol{w}$. The mass conservation for the compressible porous medium can be written in the form

$$\int_\omega q\, \left(\frac{1}{\mu}\dot{p} + \dot{\mathcal{J}}\right) + \int_{\partial\omega} q\ \alpha_{ij}\dot{u}_i\ n_j\ dS = 0\ .$$

If all integrands (except $q$) are continuous and $\alpha_{ij}$ are constant in $\omega$, the standard differential form can be obtained, i.e. $\dot{p}/\mu + \dot{\mathcal{J}} + \alpha_{ij}e_{ij}(\dot{\boldsymbol{u}}) = 0$, when taking $q^+ = q^-$. We consider in the sequel more general situations, allowing for discontinuous pressures. Moreover, the Biot coefficients on $\Gamma_\omega$ may be discontinuous in the case of piecewise constant coefficients: $\alpha_{ij}(x) = \alpha_{ij}^k$ in $\omega_k$, $(k = +, -)$, although we shall refrain from such an option later on.

By the Green formula we get formally

$$\begin{aligned} 0 &= \int_\omega q\,(\frac{1}{\mu}\dot{p} + \dot{\mathcal{J}}) + \int_\omega q\,\alpha_{ij}\partial_j\dot{u}_i + \int_\omega \dot{u}_i\,\partial_j(q\,\alpha_{ij}) \\ &= \int_\omega q\, \left(\frac{1}{\mu}\dot{p} + \dot{\mathcal{J}} + \alpha_{ij}e_{ij}(\dot{\boldsymbol{u}})\right) + \int_\omega \dot{u}_i\,\partial_j(q\,\alpha_{ij})\ , \end{aligned}$$

where the last integral can be rewritten using (2.7) to get

$$\begin{aligned} 0 &= \int_\omega q\, \left(\frac{1}{\mu}\dot{p} + \dot{\mathcal{J}} + \alpha_{ij}e_{ij}(\dot{\boldsymbol{u}})\right) + \int_\omega \dot{u}_i\, n_j\delta_\Gamma[q\,\alpha_{ij}]_\Gamma \\ &= \sum_{k=+,-} \int_{\omega_k} q^k \left(\frac{1}{\mu}\dot{p} + \dot{\mathcal{J}} + \alpha_{ij}e_{ij}(\dot{\boldsymbol{u}})\right) + \int_{\Gamma_\omega} \dot{u}_i[q\alpha_{ij}]_\Gamma n_j\, dS\ . \end{aligned} \tag{2.8}$$

Further, recalling the definitions of $\Gamma^+$ and $\Gamma^-$, from(2.8) we derive the following form of *the generalized mass conservation* which is **dual to the pressure discontinuities** at $\Gamma$:

$$\frac{1}{\mu}\dot{p} + \dot{\mathcal{J}} + \alpha_{ij}e_{ij}(\dot{\boldsymbol{u}}) + \dot{u}_i\, n_j\, (\delta_{\Gamma^+}\alpha_{ij}|_{\Gamma^+} - \delta_{\Gamma^-}\alpha_{ij}|_{\Gamma^-}) = 0\ . \tag{2.9}$$

Assuming $\alpha_{ij}|_{\Gamma^+} = \alpha_{ij}|_{\Gamma^-} = \alpha_{ij}^\Gamma$, (2.9) can be rewritten as

$$\frac{1}{\mu}\dot{p} + \dot{\mathcal{J}} + \alpha_{ij}e_{ij}(\dot{\boldsymbol{u}}) + \dot{u}_i n_j\, \alpha_{ij}^\Gamma\, (\delta_{\Gamma^+} - \delta_{\Gamma^-}) = 0\ . \tag{2.10}$$

**2.2.3 Weak formulation of mass conservation for discontinuous pressure fields** Recalling that $\Omega$ is the domain occupied by our heterogeneous material, for any $q$ given by (2.5), equation (2.9) yields

$$\int_\Omega q\left(\frac{1}{\mu}\dot p+\dot{\mathcal{J}}+\alpha_{ij}e_{ij}(\dot{\boldsymbol{u}})\right)+\int_\Omega \dot u_i n_j\,[q\alpha_{ij}]_\Gamma\,\delta_\Gamma=0\,, \tag{2.11}$$

where the last integral can be rewritten in terms of a surface integral by virtue of $(2.6)_2$.

We can now develop the term involving local source/sink of fluid, $\dot{\mathcal{J}}=\mathrm{div}\boldsymbol{w}$. Recalling the impermeability of the outer surface, i.e. $\boldsymbol{n}\cdot\boldsymbol{w}=0$ on $\partial\Omega$, for any $q$ defined by (2.5) with $\nabla q$ defined by (2.7), it follows that

$$\begin{aligned}\int_\Omega q\dot{\mathcal{J}}&=\int_\Omega q\nabla\cdot\boldsymbol{w}=-\int_{\Omega\setminus\Gamma}\boldsymbol{w}\cdot\nabla q-\int_\Omega\boldsymbol{w}\cdot\boldsymbol{n}\,[q]_\Gamma\,\delta_\Gamma\\&=-\int_{\Omega\setminus\Gamma}\boldsymbol{w}\cdot\nabla q-\int_\Gamma\boldsymbol{w}\cdot\boldsymbol{n}\,[q]_\Gamma\,dS=\int_{\Omega\setminus\Gamma}K_{ij}\partial_j p\,\partial_i q+\int_\Gamma\varkappa\,[p]_\Gamma\,[q]_\Gamma\,dS\,,\end{aligned} \tag{2.12}$$

where we used the Darcy law and condition $(2.4)_3$ in order to replace $\boldsymbol{w}$. Thus, from (2.11)-(2.12) it follows that the mass conservation can be written in the form

$$\begin{aligned}&\int_\Omega q\left(\frac{1}{\mu}\dot p+\alpha_{ij}e_{ij}(\dot{\boldsymbol{u}})\right)+\int_{\Omega\setminus\Gamma}K_{ij}\partial_j p\,\partial_i q\\&\quad+\int_\Gamma\varkappa\,[p]_\Gamma\,[q]_\Gamma\,dS+\int_\Gamma[q\alpha_{ij}]_\Gamma\,\dot u_i n_j\,dS=0\quad\forall q\,.\end{aligned} \tag{2.13}$$

**2.2.4 Balance of forces with pressure discontinuities** A consequence of the continuity of the effective stress $\sigma_{ij}^{\mathrm{eff}}=D_{ijkl}e_{kl}(\boldsymbol{u})$ and of the discontinuity of the pressure as defined in (2.17), the overall stress $\sigma_{ij}=\sigma_{ij}^{\mathrm{eff}}-\alpha_{ij}p$ is *discontinuous*, see Remark 1 below. Therefore, the standard differential form $(2.2)_1$ of the balance of forces holds in $\Omega\setminus\Gamma$. Taking a continuous test displacement field $\boldsymbol{v}\in\mathbf{H}^1_0(\Omega)$, we can integrate by parts in $\Omega\setminus\Gamma$ to get

$$\begin{aligned}&-\int_{\Omega\setminus\Gamma}v_i\partial_j\sigma_{ij}=\int_\Omega\boldsymbol{f}\cdot\boldsymbol{v},\\&\int_{\Omega\setminus\Gamma}\sigma_{ij}e_{ij}(\boldsymbol{v})+\int_\Gamma v_i\,[\sigma_{ij}]_\Gamma\,n_j\,dS=\int_\Omega\boldsymbol{f}\cdot\boldsymbol{v},\\&\int_{\Omega\setminus\Gamma}\left(D_{ijkl}e_{kl}(\boldsymbol{u})-\alpha_{ij}p\right)e_{ij}(\boldsymbol{v})-\int_\Gamma v_i\,[\alpha_{ij}p]_\Gamma\,n_j\,dS=\int_\Omega\boldsymbol{f}\cdot\boldsymbol{v},\end{aligned} \tag{2.14}$$

where the volume integral over $\Omega\setminus\Gamma$ can be replaced by an integral over $\Omega$, since all integrands are "sufficiently smooth" in $\Omega$.

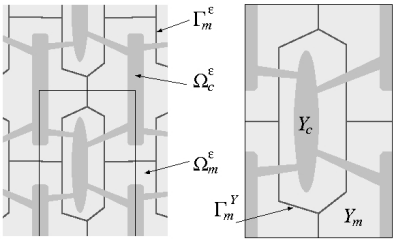


FIG. 2.2. *An example of a periodic microstructure generated by a representative cell $Y$. A two-dimensional section of a 3D structure is displayed. Both the matrix part $Y_m$ and the channels $Y_c$ generate the connected domains $\Omega_m^\varepsilon$ and $\Omega_m^\varepsilon$. The discontinuity interface $\Gamma_m^\varepsilon$ is illustrated by thick lines, it may also form a connected hypersurface in $\Omega$.*

**2.3 A periodic microstructure with two compartments** The heterogeneous porous medium consists of two distinct parts with different magnitudes of the respective hydraulic permeabilities. We consider an open bounded domain $\Omega \subset \mathbb{R}^3$, which is decomposed into two parts $\Omega_m$ (matrix) and $\Omega_c$ (channels), so that

$$\Omega = \Omega_m \cup \Omega_c \cup \Gamma_{mc}, \quad \text{with} \quad \Omega_m \cap \Omega_c = \emptyset \,.$$

The microstructure is generated by periodic unit cubes $Y = ]0,1[^3$; this choice of the cell $Y$ is made for the sake of simplicity (one can consider general parallelotops as periodicity cells via some technicalities). Let $Y_c$ and $Y_m$ be connected, disjoint subdomains of $Y$ with Lipschitz boundaries, so that $\partial Y_c$ has common measurable sets with all faces of $\overline{Y}$ and

$$\begin{aligned}
&Y_m = Y \setminus \overline{Y_c}, \quad \partial_m Y_c = \partial_c Y_m = \overline{Y_c} \cap \overline{Y_m},\\
&\partial_c Y_c = \overline{Y_c} \cap \partial Y, \quad \partial_m Y_m = \overline{Y_m} \cap \partial Y,\\
&Y = Y_m \cup Y_c \cup \partial_m Y_c, \quad \text{with } Y_m \cap Y_c = \emptyset,\\
&\partial Y = \overline{\partial_c Y_c \cup \partial_m Y_m}\,, \quad \partial Y_c = \overline{\partial_c Y_c \cup \partial_m Y_c}\,.
\end{aligned}$$

Denote by $\varepsilon > 0$ the scale parameter defining the size of microstructures. The channel $Y_c$ is such that $\varepsilon Y_c$ generates a connected $\varepsilon-$periodic domain $\Omega_c^\varepsilon$; let us introduce

$$(\mathbb{R}^n)_c = \text{Interior}\Big(\bigcup_{\zeta\in\mathbb{Z}^3} (\zeta + \overline{Y_c})\Big)\,,$$

then the open set $\Omega_c^\varepsilon$ is defined by

$$\Omega_c^\varepsilon = \varepsilon(\mathbb{R}^n)_c \cap \Omega\,.$$

The "matrix" $\Omega_m^\varepsilon$ is then obtained by removing the "channel network" from the whole domain: $\Omega_m^\varepsilon = \Omega \setminus \overline{\Omega_c}^\varepsilon$. We assume that $\Omega_m^\varepsilon$ is connected.

Further, we introduce the interface $\Gamma_m^Y$ embedded in the matrix part, $\Gamma_m^Y \subset \overline{Y_m}$. As discussed in Section 2.2, pressure discontinuities may be expected on $\Gamma_m^Y$. An example of the surface location is illustrated in Figure 2.3. Due to the interface $\Gamma_m^Y$, the matrix compartment generated by $\varepsilon Y_m$ is periodically subdivided into the subdomains $\Omega_{m\circ,k}^\varepsilon$, $k \in J_m^\varepsilon$, separated by the interface $\Gamma_m^\varepsilon$ so that

$$\Gamma_m^\varepsilon = \Omega_m^\varepsilon \setminus \bigcup_{k\in J_m^\varepsilon} \Omega_{m\circ,k}^\varepsilon\,, \quad \Omega_{m\circ,k}^\varepsilon \cap \Omega_{m\circ,l}^\varepsilon = \emptyset \quad \text{for } k \neq l\,, \quad \Omega_m^\varepsilon = \text{Interior}\Big(\bigcup_{k\in J_m^\varepsilon} \overline{\Omega_{m\circ,k}}^\varepsilon\Big)\,. \tag{2.15}$$

Note that the *diameter* of each $\Omega_{m\circ,k}^\varepsilon$ is proportional to $\varepsilon$. Obviously, if $\Omega \subset \mathbb{R}^3$, then $|\Gamma_m^\varepsilon| \approx \varepsilon^2$.

In order to define an extension operator (from the channels to the matrix, or briefly an "off-channels" extension), we introduce the domain containing the "entire" periods $\varepsilon Y$:

$$\begin{aligned}
&\hat{\Omega}^\varepsilon = \text{interior} \bigcup_{\zeta\in\Xi^\varepsilon} Y_\zeta^\varepsilon\,, \quad Y_\zeta^\varepsilon = \varepsilon(\overline{Y} + \zeta)\\
&\text{where } \Xi^\varepsilon = \{\zeta \in \mathbb{Z}^3 \,|\, \varepsilon(\overline{Y} + \zeta) \subset \Omega\}\,.
\end{aligned} \tag{2.16}$$

**2.4 Weak formulation of the problem** Having developed a suitable weak forms of both the balance of forces and mass conservation laws, respectively (2.13) and (2.14), we are ready to define the *diffusion-deformation problem with interface pressure discontinuities* in the periodic heterogeneous structure. We consider the following boundary and initial conditions:

$$\begin{aligned}
\boldsymbol{u}^\varepsilon(t,\cdot) &= \boldsymbol{u}_0(t,\cdot) \quad \text{on } \partial\Omega, \text{ for } t \in ]0,T[\,,\\
n_i K_{ij}^\varepsilon \partial_j p^\varepsilon(t,\cdot) &= 0 \quad \text{on } \partial\Omega, \text{ for } t \in ]0,T[\,,\\
\boldsymbol{u}^\varepsilon(0,\cdot) &= 0 \quad \text{in } \Omega,\\
p^\varepsilon(0,\cdot) &= 0 \quad \text{in } \Omega.
\end{aligned}$$

Obviously, we assume the consistency constraint $\boldsymbol{u}_0(0,x)=0$ for $x\in\partial\Omega$. Moreover, in Section 4, we impose the homogeneous Dirichlet condition $\boldsymbol{u}_0\equiv 0$.

Since the pressure field can be discontinuous on $\Gamma^\varepsilon$, we need the following space of discontinuous functions:

$$\begin{aligned} H^1(\Omega^\varepsilon_m\setminus\Gamma^\varepsilon_m)&=\{q\in L^2(\Omega^\varepsilon_m):\ \nabla q\in L^2(\Omega^\varepsilon_m\setminus\Gamma^\varepsilon_m)\}\,,\\ H^1(\Omega\setminus\Gamma^\varepsilon_m)&=\{q\in L^2(\Omega):\ \nabla q\in L^2(\Omega\setminus\Gamma^\varepsilon_m)\}\,. \end{aligned} \tag{2.17}$$

As in [16], we integrate (2.13) in time and introduce the integrated pressure

$$P^\varepsilon(t,x)=\int_0^t p^\varepsilon(t,x)\,dt\,. \tag{2.18}$$

Clearly, $P^\varepsilon(0)=0$.

Our aim is to study the asymptotic behavior as $\varepsilon\to 0$, of the following problem: Find $\boldsymbol{u}^\varepsilon\in H^1(0,T;\mathbf{H}^1_0(\Omega))+\boldsymbol{u}_0$ and $P^\varepsilon\in H^1_0(0,T;H^1(\Omega\setminus\Gamma^\varepsilon_m))$ such that for a.e. $t\in]0,T[$

$$\left\{\begin{aligned} &\int_\Omega D^\varepsilon_{ijkl}e_{kl}(\boldsymbol{u}^\varepsilon)e_{ij}(\boldsymbol{v})-\int_\Omega\frac{\mathrm{d}\,P^\varepsilon}{\mathrm{d}\,t}\,\alpha^\varepsilon_{ij}e_{ij}(\boldsymbol{v})-\int_{\Gamma^\varepsilon_m}\alpha^{\Gamma,\varepsilon}_{ij}v_in^\varepsilon_j\left[\frac{\mathrm{d}\,P^\varepsilon}{\mathrm{d}\,t}\right]_{\Gamma^\varepsilon_m}dS\\ &\qquad\qquad=\int_\Omega\boldsymbol{f}\cdot\boldsymbol{v},\quad\forall\boldsymbol{v}\in\mathbf{H}^1_0(\Omega),\\ &\int_\Omega q\,\alpha^\varepsilon_{ij}e_{ij}(\boldsymbol{u}^\varepsilon)+\int_{\Gamma^\varepsilon_m}\alpha^{\Gamma,\varepsilon}_{ij}u^\varepsilon_in^\varepsilon_j\,[q]_{\Gamma^\varepsilon_m}\,dS+\int_{\Omega\setminus\Gamma^\varepsilon_m}K^\varepsilon_{ij}\partial_jP^\varepsilon\,\partial_jq+\int_\Omega\frac{1}{\mu^\varepsilon}\frac{\mathrm{d}\,P^\varepsilon}{\mathrm{d}\,t}\,q\\ &\qquad\qquad+\int_{\Gamma^\varepsilon_m}\varkappa^\varepsilon\,[P^\varepsilon]_{\Gamma^\varepsilon_m}\,[q]_{\Gamma^\varepsilon_m}\,dS=0,\quad\forall q\in H^1(\Omega\setminus\Gamma^\varepsilon_m). \end{aligned}\right. \tag{2.19}$$

The material coefficients $D^\varepsilon_{ijkl}$, $\alpha^\varepsilon_{ij}$, $\alpha^{\Gamma,\varepsilon}_{ij}$, $K^\varepsilon_{ij}$ and $\varkappa^\varepsilon$ are oscillating and $\varepsilon$-periodic, as specified in Section 2.6. We claim that there exists a unique solution of (2.19). For the proof, we refer to [16], Section 3, where a similar model was treated.

**2.5 The periodic unfolding method** In this paper we apply the *unfolding method of homogenization*, cf. [10, 11], to derive the homogenized model. For the reader's convenience we recall the notion of the *periodic unfolding method* and of the *periodic unfolding operator*, in particular. We shall use the convergence results in the unfolded domain $\Omega\times Y$ which can be found in [10].

For all $z\in\mathbb{R}^3$, let $[z]$ be the unique integer such that $z-[z]\in Y$. We may write $z=[z]+\{z\}$ for all $z\in\mathbb{R}^3$, so that for all $\varepsilon>0$, we get the unique decomposition

$$x=\varepsilon\left(\left[\frac{x}{\varepsilon}\right]+\left\{\frac{x}{\varepsilon}\right\}\right)\quad\forall x\in\mathbb{R}^3\,.$$

Based on this decomposition, the periodic unfolding operator $\mathcal{T}_\varepsilon:L^2(\Omega;\mathbb{R})\to L^2(\Omega\times Y;\mathbb{R})$ is defined as follows: for any function $v\in L^1(\Omega;\mathbb{R})$, extended to $L^1(\mathbb{R}^3;\mathbb{R})$ by zero outside $\Omega$, i.e. $v=0$ in $\mathbb{R}^3\setminus\Omega$,

$$\mathcal{T}_\varepsilon(v)(x,y)=\begin{cases} v\left(\varepsilon\left[\dfrac{x}{\varepsilon}\right]+\varepsilon y\right)\,, & x\in\hat\Omega^\varepsilon, y\in Y\,,\\ 0 & \text{otherwise}\,. \end{cases}$$

The following integration formula holds:

$$\int_{\hat\Omega^\varepsilon}v\,dx=\frac{1}{|Y|}\int_{\Omega\times Y}\mathcal{T}_\varepsilon(v)\,dy\,dx\quad\forall v\in L^1(\Omega)\,.$$

Analogously, when integrating on a surface $\hat\Gamma^\varepsilon$ generated by a surface $\Gamma^Y\subset\overline{Y}$ (see (2.15)), the formula reads

$$\int_{\hat\Gamma^\varepsilon}v\,dS=\frac{1}{\varepsilon|Y|}\int_{\Omega\times\Gamma^Y}\mathcal{T}_\varepsilon(v)\,dS_y\,dx,\quad\forall v\in L^1(\Omega).$$

These formulae will be used in the sequel to evaluate integrals over $\Omega$, which typically is approved upon satisfying so-called *unfolding criterion for integrals*. For more details and convergence results, we refer the reader to [10], for error estimates see [17, 18, 19].

In what follows we use the following abbreviations (for any $Y_a \subset Y$)

$$\frac{1}{|Y|}\int_{Y_a} = \fint_{Y_a}, \quad \text{and also } \frac{1}{|Y|}\int_{\Omega\times Y_a} = \fint_{\Omega\times Y_a}.$$

**2.6 Oscillating material coefficients** The porous medium distributed in $\Omega$ is featured by material heterogeneities characterized at the length scale $\varepsilon$, thereby the coefficients in equations (2.3) are highly oscillating. We treat strongly heterogeneous permeability coefficients, where the heterogeneity is related to the domain decomposition into the "matrix" part and the "channel" part.

The setting of the problem is based on (2.13) and (2.14) where the material parameters are defined piecewise with respect to the domain decomposition introduced above. In general, for any material parameter $c^\varepsilon = c^\varepsilon(x)$ identified with $D^\varepsilon_{ijkl}, \alpha^\varepsilon_{ij}$ or $\mu^\varepsilon$, we assume the following:

$$c^\varepsilon \in L^\infty(\Omega), \quad \mathcal{T}_\varepsilon(c^\varepsilon(x)) \to \tilde{c}(x,y) \quad \text{a.e. in } \Omega\times Y, \tag{2.20}$$

where $\tilde{c}(x,y)$ is identified respectively, as $\tilde{D}_{ijkl}, \tilde{\alpha}_{ij}$ or $\tilde{\mu}$. For the permeability coefficients $K^\varepsilon_{ij} \in L^\infty(\Omega)$, instead of (2.20), we assume that

$$\mathcal{T}_\varepsilon\big(\chi^\varepsilon_c K^\varepsilon_{ij}(x)\big) \to \chi_c(y)K^c_{ij}(x,y) \quad \text{a.e. in } \Omega\times Y,$$
$$\frac{1}{\varepsilon^2}\mathcal{T}_\varepsilon\big(\chi^\varepsilon_m K^\varepsilon_{ij}(x)\big) \to \chi_m(y)K^m_{ij}(x,y) \quad \text{a.e. in } \Omega\times Y,$$

where $\chi^\varepsilon_d(x) = \chi_d\Big(\Big\{\frac{x}{\varepsilon}\Big\}\Big)$ is the characteristic function of domain $\Omega^\varepsilon_d$, $d = c, m$. Thus we assume that the permeability coefficients depend strongly on the scale parameter. In particular, due to the $\varepsilon^2$-scaling of the permeability in $\Omega^\varepsilon_m$, the matrix part presents a dual porosity. All the other material coefficients in their unfolded form can also be refered to by superscripts $c, m$ in the domains $Y_c$ and $Y_m$, respectively. Hence, by (2.20),

$$\tilde{c}(x,y) = \chi_c(y)c^c(x,y) + \chi_m(y)c^m(x,y),$$

so that $D^d_{ijkl}, \alpha^d_{ij}$ and $\mu^d$ have a meaning.

Further, we assume the existence of positive constants $c_D, C_D, c_K, C_K, c_\mu, C_\mu$, independent of $\varepsilon$ and such that for a.e. $x \in \Omega$,

$$\begin{aligned}
&c_D|\xi|^2 \le D^\varepsilon_{ijkl}(x)\xi_{kl}\xi_{ij}, \quad |D^\varepsilon_{ijkl}(x)| \le C_D \quad \text{for any (symmetric) } \xi \in M_2,\\
&c_\mu \le 1/\mu^\varepsilon(x) \le C_\mu,\\
&\text{for a.e. } x \in \Omega^\varepsilon_m,\ \forall \zeta \in \mathbb{R}^2, \quad |K^\varepsilon_{ij}(x)| \le \varepsilon^2 C_K, \quad \varepsilon^2 c_K|\zeta|^2 \le K^\varepsilon_{ij}(x)\zeta_i\zeta_j,\\
&\text{for a.e. } x \in \Omega^\varepsilon_c,\ \forall \zeta \in \mathbb{R}^2, \quad |K^\varepsilon_{ij}(x)| \le C_K, \quad c_K|\zeta|^2 \le K^\varepsilon_{ij}(x)\zeta_i\zeta_j.
\end{aligned} \tag{2.21}$$

Because of the pressure discontinuities on $\Gamma^\varepsilon_m$ we need to specify the definitions of $\varkappa^\varepsilon$ and $\alpha^\varepsilon_{ij}$ on $\Gamma^\varepsilon_m$. To do so, we assume that (2.10) holds with $\alpha^{\Gamma,\varepsilon}_{ij}$ as the interface Biot coefficients.

Below we shall employ the boundary unfolding opreator $\mathcal{T}^b_\varepsilon()$ which is introduced as $\mathcal{T}_\varepsilon()$ operating on interface $\Gamma^\varepsilon_m$, i.e. for any $v \in L^1(\Gamma^\varepsilon_m;\mathbb{R})$, $\mathcal{T}^b_\varepsilon(v(x)) = v\left(\varepsilon\left[\frac{x}{\varepsilon}\right] + \varepsilon y\right)$ whenever $x \in \Gamma^\varepsilon_m \cap \hat{\Omega}^\varepsilon$, so that $y \in \Gamma_Y$, and $\mathcal{T}^b_\varepsilon(v(x)) = 0$ otherwise.

*The interface permeability* $\varkappa^\varepsilon \in L^\infty(\Gamma^\varepsilon_m)$. We assume that

$$\mathcal{T}^b_\varepsilon(\varkappa^\varepsilon(x)) = \varepsilon\tilde{\varkappa}(\bar{x}^\varepsilon, y), \quad \text{where } c_\varkappa \le \tilde{\varkappa} \le C_\varkappa, \tag{2.22}$$

for some given $c_\varkappa, C_\varkappa > 0$, where $\bar{x}^\varepsilon = \varepsilon[x/\varepsilon]$ is the lattice restriction of $x$ (using the unfolding operation) and $y \in \Gamma^Y_m$.

*Biot coefficients* $\alpha_{ij}^\varepsilon \in L^\infty(\Gamma_m^\varepsilon)$ *on the interface.* As we shall see, the limit model depends strongly on the uniform estimates of $\alpha_{ij}^\varepsilon$ with respect to $\varepsilon$. We consider the following two particular situations:

$$\textbf{weakly discontinuous data (WD)} \quad \mathcal{T}_\varepsilon^b\Big(\alpha_{ij}^{\Gamma,\varepsilon}\Big) = \varepsilon\tilde\alpha_{ij}^\Gamma(\bar x^\varepsilon, y)\,, \tag{2.23}$$

$$\textbf{strongly discontinuous data (SD)} \quad \mathcal{T}_\varepsilon^b\Big(\alpha_{ij}^{\Gamma,\varepsilon}\Big) = \tilde\alpha_{ij}^\Gamma(\bar x^\varepsilon, y)\,, \tag{2.24}$$

where $\bar x^\varepsilon = \varepsilon[x/\varepsilon]$ and $y \in \Gamma_m^Y$. In the first case $\mathcal{T}_\varepsilon^b\,\alpha_{ij}^{\Gamma,\varepsilon} \to 0$ a.e. in $\Omega\times\Gamma_m^Y$. Anyway, in $\Omega\setminus\Gamma_m^\varepsilon$ we suppose that $\mathcal{T}_\epsilon(\alpha_{ij}^\varepsilon) \to \tilde\alpha_{ij}(x,y)$ a.e. in $\Omega\times(Y\setminus\Gamma_m^Y)$.

**3 Homogenization of the model with WD data** Throughout this section we assume we are given standard "moderate" forces $\boldsymbol{f}^\varepsilon$, i.e., such that

$$\|\boldsymbol{f}^\varepsilon\|_{L^2(0,T;\mathbf{L}^2(\Omega))} \le C\,. \tag{3.1}$$

The plan of this section is as follows. In Section 3.1 we give a priori estimates which allow us to pass to the limit in Section 3.2. The procedure of the scale-decoupling for the "Laplace-transformed in time" limit equations is explained in Section 3.3.

**3.1 A priori estimates** Let us recall some basic inequalities that will be used also in the case of *strongly discontinuous data* (2.24).

$$\begin{aligned} &\textit{The Young inequality:}\quad ab \le \frac{a^2}{2\nu} + \frac{\nu b^2}{2} \qquad \text{for all } a,b,\nu\in\mathbb{R}, \nu>0\,,\\ &\textit{The Korn inequality:}\quad \|\nabla\boldsymbol{u}^\varepsilon\|^2_{\mathbf{L}^2(\Omega)} \le C_1\sum_{i,j}\|e_{ij}(\boldsymbol{u}^\varepsilon)\|^2_{\mathbf{L}^2(\Omega)}. \end{aligned} \tag{3.2}$$

Using appropriate test functions, we can eliminate in (2.19) the "mixed terms", so that only quadratic forms appear in the resulting identity,

$$\begin{aligned} &\int_\Omega D_{ijkl}^\varepsilon e_{kl}(\boldsymbol{u}^\varepsilon)e_{ij}(\boldsymbol{u}^\varepsilon) - \int_\Omega \frac{\mathrm{d}\,P^\varepsilon}{\mathrm{d}\,t}\,\alpha_{ij}^\varepsilon e_{ij}(\boldsymbol{u}^\varepsilon) - \int_{\Gamma_m^\varepsilon}\alpha_{ij}^{\Gamma,\varepsilon}u_i^\varepsilon n_j^\varepsilon\left[\frac{\mathrm{d}\,P^\varepsilon}{\mathrm{d}\,t}\right]_{\Gamma_m^\varepsilon} dS = \int_\Omega \boldsymbol{f}\cdot\boldsymbol{u}^\varepsilon\,,\\ &\int_\Omega \frac{\mathrm{d}\,P^\varepsilon}{\mathrm{d}\,t}\,\alpha_{ij}^\varepsilon e_{ij}(\boldsymbol{u}^\varepsilon) + \int_{\Gamma_m^\varepsilon}\alpha_{ij}^{\Gamma,\varepsilon}u_i^\varepsilon n_j^\varepsilon\left[\frac{\mathrm{d}\,P^\varepsilon}{\mathrm{d}\,t}\right]_{\Gamma_m^\varepsilon} dS + \int_{\Omega\setminus\Gamma_m^\varepsilon} K_{ij}^\varepsilon\partial_j P^\varepsilon\,\partial_j\frac{\mathrm{d}\,P^\varepsilon}{\mathrm{d}\,t}\\ &\qquad + \int_\Omega \frac{1}{\mu^\varepsilon}\frac{\mathrm{d}\,P^\varepsilon}{\mathrm{d}\,t}\frac{\mathrm{d}\,P^\varepsilon}{\mathrm{d}\,t} + \int_{\Gamma_m^\varepsilon}\varkappa^\varepsilon\,[P^\varepsilon]_{\Gamma_m^\varepsilon}\left[\frac{\mathrm{d}\,P^\varepsilon}{\mathrm{d}\,t}\right]_{\Gamma_m^\varepsilon} dS = 0. \end{aligned} \tag{3.3}$$

Upon summation we obtain

$$\begin{aligned} &\int_\Omega D_{ijkl}^\varepsilon e_{kl}(\boldsymbol{u}^\varepsilon)e_{ij}(\boldsymbol{u}^\varepsilon) + \frac12\frac{\mathrm{d}}{\mathrm{d}\,t}\int_{\Omega\setminus\Gamma_m^\varepsilon} K_{ij}^\varepsilon\partial_j P^\varepsilon\,\partial_j P^\varepsilon\\ &\qquad + \int_\Omega\frac{1}{\mu^\varepsilon}\left|\frac{\mathrm{d}\,P^\varepsilon}{\mathrm{d}\,t}\right|^2 + \frac12\frac{\mathrm{d}}{\mathrm{d}\,t}\int_{\Gamma_m^\varepsilon}\varkappa^\varepsilon\,[P^\varepsilon]^2_{\Gamma_m^\varepsilon}\,dS = \int_\Omega \boldsymbol{f}\cdot\boldsymbol{u}^\varepsilon. \end{aligned} \tag{3.4}$$

Then we integrate in time $t\in[0,\bar t]$, $\bar t\le T$, recalling $P^\varepsilon(0)=0$, see (2.18), and use the lower boundedness of all the material coefficients, see (2.21) and (2.22); this yields

$$\begin{aligned} &c_D\int_0^{\bar t}\int_\Omega\sum_{i,j}|e_{ij}(\boldsymbol{u}^\varepsilon)|^2\,dt + c_\mu\int_0^{\bar t}\int_\Omega\left|\frac{\mathrm{d}\,P^\varepsilon}{\mathrm{d}\,t}\right|^2 dt\\ &+ c_K\int_{\Omega_c^\varepsilon}|\nabla P^\varepsilon(\bar t)|^2 + c_K\varepsilon^2\int_{\Omega_m^\varepsilon\setminus\Gamma_m^\varepsilon}|\nabla P^\varepsilon(\bar t)|^2 + c_\varkappa\varepsilon\int_{\Gamma_m^\varepsilon}[P^\varepsilon(\bar t)]^2_{\Gamma_m^\varepsilon} \le \int_0^{\bar t}\int_\Omega \boldsymbol{f}^\varepsilon\cdot\boldsymbol{u}^\varepsilon\,dt\,. \end{aligned} \tag{3.5}$$

Due to the definition of the volume forces and using (3.1), we have

$$\int_0^{\bar t}\int_\Omega \boldsymbol{f}^\varepsilon\cdot\boldsymbol{u}^\varepsilon\,dt \le \frac{1}{2\nu}\|\boldsymbol{f}^\varepsilon\|^2_{L^2(0,T;\mathbf{L}^2(\Omega))}+\frac{\nu C_1}{2}\|\nabla\boldsymbol{u}^\varepsilon\|^2_{L^2(0,T;\mathbf{L}^2(\Omega))}\,,\quad \text{for a.a. } \bar t\le T\,, \tag{3.6}$$

where we used the Poincaré inequality and $(3.2)_1$, $\nu>0$ being an arbitrary constant. Then, using the Korn inequality $(3.2)_2$ to deal with $|e_{ij}(\boldsymbol{u}^\varepsilon)|^2$, combining (3.6) with (3.5), and choosing $\nu$ appropriately, the following estimates are obtained:

$$\begin{aligned}
&\|\boldsymbol{u}^\varepsilon\|_{L^2(0,T;\mathbf{H}^1(\Omega))}\le C, \qquad \|\nabla P^\varepsilon\|_{L^\infty(0,T;\mathbf{L}^2(\Omega_c^\varepsilon))}\le C,\\
&\|\nabla P^\varepsilon\|_{L^\infty(0,T;\mathbf{L}^2(\Omega_m^\varepsilon\setminus\Gamma_m^\varepsilon))}\le\frac{C}{\varepsilon}, \qquad \Big\|\frac{\mathrm{d}\,P^\varepsilon}{\mathrm{d}\,t}\Big\|_{L^2(0,T;L^2(\Omega))}\le C,\\
&\|[P^\varepsilon]_{\Gamma_m^\varepsilon}\|_{L^\infty(0,T;L^2(\Gamma_m^\varepsilon))}\le\frac{C}{\sqrt{\varepsilon}}.
\end{aligned}\tag{3.7}$$

From $(3.7)_4$ we deduce immediately that $\|P^\varepsilon\|_{H^1(0,T;L^2(\Omega))}\le C$, which follows due to $P^\varepsilon(0)=0$, see (2.18).

**3.2 Limit problems** In this section we give the limit representation of the model (2.19) with assumption (2.23). First, however, we obtain some results which are applicable in both the *strongly* and *weakly* discontinuous cases. It is worth to emphasize that we do not use extension operators which otherwise have been used commonly when dealing with problems in perforated domains – instead we employ the convergence theorems developed recently in [12].

We shall need the *space of discontinuous unfolded functions*. Let $\Gamma_m^Y\subset\overline{Y}$ be the representative discontinuity interface, i.e. $\Gamma_m^\varepsilon=\bigcup_{\zeta\in\Xi^\varepsilon}\varepsilon\Gamma_m^Y+\varepsilon\zeta$, see (2.16), and set

$$H_{\#c0}(Y,\Gamma_m^Y)=\{q\in H^1_\#(Y\setminus\Gamma_m^Y)|\ q=0 \text{ in } \overline{Y_c}\}\,. \tag{3.8}$$

Below we rely on the following two Theorems which were introduced as Theorem 3.1 and Theorem 3.12 in [12]. Here we adapted these results according to our situation with a changed notation. We recall the decomposition (2.15) and the space $H^1(\Omega_m^\varepsilon\setminus\Gamma_m^\varepsilon)$.

THEOREM 3.1. *Let $(w_\varepsilon)_\varepsilon$ be a sequence belonging to $H^1(\Omega_m^\varepsilon\setminus\Gamma_m^\varepsilon)$ and satisfy*

$$\|w_\varepsilon\|_{L^2(\Omega_m^\varepsilon)}+\varepsilon\|\nabla w_\varepsilon\|_{\mathbf{L}^2(\Omega_m^\varepsilon\setminus\Gamma_m^\varepsilon)}\le C\,.$$

*Then there exists some $\hat w\in L^2(\Omega;H^1_\#(Y_m\setminus\Gamma_m^Y))$, such that, up to a subsequence,*

$$\begin{aligned}
&\mathcal{T}_\varepsilon(w_\varepsilon)\rightharpoonup\hat w \quad \textit{weakly in } L^2(\Omega;H^1(Y_m))\,,\\
&\varepsilon\mathcal{T}_\varepsilon(\nabla w_\varepsilon)\rightharpoonup\nabla_y\hat w \quad \textit{weakly in } \mathbf{L}^2(\Omega\times Y_m\setminus\Gamma_m^Y)\,.
\end{aligned}$$

THEOREM 3.2. *Suppose that $w_\varepsilon\in H^1(\Omega_c^\varepsilon)$ satisfies $\|w_\varepsilon\|_{H^1(\Omega_c^\varepsilon)}\le C$. Then there exists $w\in H^1(\Omega)$ and $\hat w\in L^2(\Omega;H^1_\#(Y_c))$, such that, up to a subsequence,*

$$\begin{aligned}
&\mathcal{T}_\varepsilon(w_\varepsilon)\rightharpoonup\hat w \quad \textit{weakly in } L^2(\Omega;H^1(Y_c))\,,\\
&\mathcal{T}_\varepsilon(\nabla w_\varepsilon)\rightharpoonup\nabla w+\nabla_y\hat w \quad \textit{weakly in } \mathbf{L}^2(\Omega\times Y_c)\,.
\end{aligned}$$

LEMMA 3.3. *Due to estimates (3.7), the following limit fields exist:*

$$\begin{aligned}
&\boldsymbol{u}\in\mathbf{L}^2(0,T;\mathbf{H}^1_0(\Omega)), \quad \boldsymbol{u}^1\in\mathbf{L}^2(0,T;\mathbf{H}^1_\#(Y)),\\
&P\in H^1(0,T;L^2(\Omega))\cap L^\infty(0,T;H^1(\Omega)), \quad P^1\in L^\infty(0,T;L^2(\Omega;H^1_\#(Y_c))),\\
&\widehat{P}_m\in H^1(0,T;L^2(\Omega\times Y_m))\cap L^\infty(0,T;L^2(\Omega;H_{\#c0}(Y,\Gamma_m^Y))),
\end{aligned}\tag{3.9}$$

*such that the following convergences hold, up to subsequences:*

$$\begin{aligned}
&\boldsymbol{u}^\varepsilon\rightharpoonup\boldsymbol{u} \quad \textit{weakly in } \mathbf{L}^2(0,T;\mathbf{H}^1_0(\Omega)),\\
&\mathcal{T}_\varepsilon(\boldsymbol{u}^\varepsilon)\rightharpoonup\boldsymbol{u} \quad \textit{weakly in } \mathbf{L}^2(0,T;\mathbf{L}^2(\Omega\times Y)),\\
&\mathcal{T}_\varepsilon(\nabla\boldsymbol{u}^\varepsilon)\rightharpoonup\nabla_x\boldsymbol{u}+\nabla_y\boldsymbol{u}^1 \quad \textit{weakly in } \mathbf{L}^2(0,T;\mathbf{L}^2(\Omega\times Y)),
\end{aligned}\tag{3.10}$$

*and*

$$
\begin{aligned}
&\mathcal{T}_\varepsilon(P^\varepsilon) \rightharpoonup P \quad \text{weakly in } H^1(0,T;L^2(\Omega\times Y_c)),\\
&\mathcal{T}_\varepsilon(\nabla P^\varepsilon) \rightharpoonup \nabla_x P + \nabla_y P^1 \quad \text{weakly} * \text{ in } L^\infty(0,T;L^2(\Omega\times Y_c)),\\
&\mathcal{T}_\varepsilon(P^\varepsilon) \rightharpoonup \widehat{P}_m \quad \text{weakly in } H^1(0,T;L^2(\Omega\times Y_m)),\\
&\varepsilon\mathcal{T}_\varepsilon(\nabla P^\varepsilon) \rightharpoonup \nabla_y \widehat{P}_m \quad \text{weakly} * \text{ in } L^\infty(0,T;L^2(\Omega\times Y_m\setminus\Gamma_m^Y))\\
&\mathcal{T}_\varepsilon\big([P^\varepsilon]_{\Gamma_m^\varepsilon}\big) \rightharpoonup \Big[\widehat{P}_m\Big]_{\Gamma_m^Y} \quad \text{weakly} * \text{ in } L^\infty(0,T;L^2(\Omega;L^2(\Gamma_m^Y))).
\end{aligned}
\tag{3.11}
$$

*Proof.* The proof is the direct consequence of estimates (3.7) and Theorems 3.1 and 3.2, whereby some standard convegence results from [10] are applied. □

Yet we need to establish a relationship between the limit pressure in $\Omega\times Y_c$ and in $\Omega\times Y_m$, cf. [30].

LEMMA 3.4. *The limit fields $P$ and $\widehat{P}_m$ satisfy the following condition:*

$$
\widehat{P}_m(\cdot,y) = P \quad \text{for a.a. } y\in\partial Y_c\cap\partial Y_m \text{ and a.e. in } ]0,T[\times\Omega\,. \tag{3.12}
$$

*Proof.* Let us consider $\varphi\in C_0^\infty(\Omega)$ and $\boldsymbol{\psi}\in[H^1_\#(Y)]^3$. If $\varepsilon$ is small enough, due to the convergence result (3.11) we obtain the limit

$$
\begin{aligned}
\int_{\Omega\setminus\Gamma_m^\varepsilon}\varepsilon\nabla P^\varepsilon(x)\cdot\varphi(x)\boldsymbol{\psi}(x/\varepsilon) &= \int_\Omega\fint_{Y_m\setminus\Gamma_m^Y}\varepsilon\mathcal{T}_\varepsilon(\nabla P^\varepsilon)(x,y)\mathcal{T}_\varepsilon(\varphi)(x,y)\cdot\boldsymbol{\psi}(y)\\
&\to \int_\Omega\fint_{Y_m\setminus\Gamma_m^Y}\nabla_y\widehat{P}_m\cdot\varphi\boldsymbol{\psi}\,.
\end{aligned}
\tag{3.13}
$$

Above the left hand side integral can be rewritten on integrationg by parts:

$$
\begin{aligned}
&-\int_{\Omega\setminus\Gamma_m^\varepsilon}\varepsilon P^\varepsilon(x)\left(\nabla\varphi(x)\cdot\boldsymbol{\psi}(x,x/\varepsilon)+\varphi(x)\nabla\cdot\boldsymbol{\psi}(x/\varepsilon)\right)+\int_{\Gamma_m^\varepsilon}[P^\varepsilon(x)]_{\Gamma_m^\varepsilon}\varepsilon\varphi(x)\boldsymbol{n}(x/\varepsilon)\cdot\boldsymbol{\psi}(x,x/\varepsilon)\,\mathrm{dS}_x\\
&=-\int_\Omega\fint_{Y\setminus\Gamma_m^Y}\mathcal{T}_\varepsilon(P^\varepsilon)\left(\mathcal{T}_\varepsilon(\varepsilon\nabla_x\varphi)\cdot\boldsymbol{\psi}+\mathcal{T}_\varepsilon(\varphi)\nabla_y\cdot\boldsymbol{\psi}\right)+\int_\Omega\frac{1}{\varepsilon}\fint_{\Gamma_m^Y}\mathcal{T}_\varepsilon\big([P^\varepsilon]_{\Gamma_m^\varepsilon}\big)\varepsilon\mathcal{T}_\varepsilon(\varphi)\boldsymbol{n}\cdot\boldsymbol{\psi}\,\mathrm{dS}_y\,.
\end{aligned}
$$

Then we can pass to the limit and integrate by parts again:

$$
\begin{aligned}
&\to -\int_\Omega\fint_{Y_c}P(x)\varphi(x)\nabla_y\cdot\boldsymbol{\psi}(y)-\int_\Omega\fint_{Y_m\setminus\Gamma_m^Y}\widehat{P}_m(x,y)\varphi(x)\nabla_y\cdot\boldsymbol{\psi}(y)\\
&\quad+\int_\Omega\fint_{\Gamma_m^Y}\Big[\widehat{P}_m(x,y)\Big]_{\Gamma_m^Y}\varphi(x)\boldsymbol{n}(y)\cdot\boldsymbol{\psi}(y)\,\mathrm{dS}_y\\
&=\int_\Omega\varphi\fint_{Y_m\setminus\Gamma_m^Y}\nabla_y\widehat{P}_m\cdot\boldsymbol{\psi}+\int_\Omega P\varphi\fint_{\partial Y_c}\boldsymbol{n}\cdot\boldsymbol{\psi}\,\mathrm{dS}_y+\int_\Omega\varphi\fint_{\partial Y_m}\widehat{P}_m\boldsymbol{n}\cdot\boldsymbol{\psi}\,\mathrm{dS}_y\,,
\end{aligned}
\tag{3.14}
$$

where the last two boundary integrals vanish on $\partial Y$ due to the Y-periodicity of the integrands, so that the only nonvanishing parts are evaluated on $\partial Y_m\cap\partial Y_c$. Condition (3.12) now follows by comparing both the limit expressions at (3.13) and (3.14), since $\boldsymbol{\psi}$ and $\varphi$ can be chosen arbitrarily. □

We may now introduce the bubble function by setting $\widehat{P}=\widehat{P}_m-P$ whereby $\widehat{P}\in H^1(0,T;L^2(\Omega\times Y_m))\cap L^\infty(0,T;L^2(\Omega;H_{\#c0}(Y,\Gamma_m^Y)))$ as the consequence of Lemma 3.4. Thus, since $\widehat{P}(x,\cdot)=0$ on the channel-matrix interface $\partial Y_m\cap\partial Y_c$, $\widehat{P}$ can be extended continuously by zero to all $\Omega$. Then convergences (3.11) yield

$$
\begin{aligned}
&\mathcal{T}_\varepsilon(P^\varepsilon) \rightharpoonup P+\widehat{P} \quad \text{weakly in } H^1(0,T;L^2(\Omega\times Y)),\\
&\varepsilon\mathcal{T}_\varepsilon(\nabla P^\varepsilon) \rightharpoonup \nabla_y\widehat{P} \quad \text{weakly} * \text{ in } L^\infty(0,T;L^2(\Omega\times Y_m\setminus\Gamma_m^Y)),\\
&\mathcal{T}_\varepsilon\big([P^\varepsilon]_{\Gamma_m^\varepsilon}\big) \rightharpoonup \Big[\widehat{P}\Big]_{\Gamma_m^Y} \quad \text{weakly} * \text{ in } L^\infty(0,T;L^2(\Omega;L^2(\Gamma_m^Y))).
\end{aligned}
\tag{3.15}
$$

The limit fields satisfy the limit homogenized problem.

THEOREM 3.5. *Let $(\boldsymbol{u}^\varepsilon, P^\varepsilon)$ be solution of problem (2.19) where $\alpha^\varepsilon$ on $\Gamma_m^\varepsilon$ is given by (2.23). Then, the limit fields* (3.9) *with convergences (3.10)-(3.11) are such that $(\boldsymbol{u}, P)$ and $(\boldsymbol{u}^1, P^1, \widehat{P})$ satisfy the coupled* unfolded problem *(in the sense of time distributions):*

$$\begin{aligned}
&\fint_{\Omega\times Y} \tilde{D}_{ijkl}[e^x_{kl}(\boldsymbol{u}) + e^y_{kl}(\boldsymbol{u}^1)]\,[e^x_{ij}(\boldsymbol{v}^0) + e^y_{ij}(\tilde{\boldsymbol{v}}^1)] \\
&\qquad - \fint_{\Omega\times Y} \tilde{\alpha}_{ij}[e^x_{ij}(\boldsymbol{v}^0) + e^y_{ij}(\tilde{\boldsymbol{v}}^1)] \left(\frac{\mathrm{d}\,P}{\mathrm{d}\,t} + \chi_m \frac{\mathrm{d}\,\widehat{P}}{\mathrm{d}\,t}\right) \\
&\qquad - \int_\Omega v_i^0 \fint_{\Gamma_m^Y} n_j \tilde{\alpha}^\Gamma_{ij} \left[\frac{\mathrm{d}}{\mathrm{d}\,t}\widehat{P}\right]_{\Gamma_m^Y} dS_y = \int_\Omega \boldsymbol{f}\cdot\,\boldsymbol{v}^0,
\end{aligned} \tag{3.16}$$

*and*

$$\begin{aligned}
&\fint_{\Omega\times Y_c} K^c_{ij}(\partial^x_j P + \partial^y_j P^1)\,(\partial^x_j q^0 + \partial^y_j \tilde{q}^1) + \fint_{\Omega\times Y_m} K^m_{ij}\partial_j\widehat{P}\partial_i\hat{q}^0 \\
&\quad + \int_\Omega \fint_{\Gamma_m^Y} n_j\tilde{\alpha}^\Gamma_{ij} u_i \left[\hat{q}^0\right]_{\Gamma_m^Y} dS_y + \fint_{\Omega\times Y} \tilde{\alpha}_{ij}[e^x_{ij}(\boldsymbol{u}) + e^y_{ij}(\boldsymbol{u}^1)]\,[q^0 + \chi_m\hat{q}^0] \\
&\quad + \int_\Omega \fint_{\Gamma_m^Y} \tilde{\varkappa}\left[\widehat{P}\right]_{\Gamma_m^Y} \left[\hat{q}^0\right]_{\Gamma_m^Y} dS_y + \fint_{\Omega\times Y_c} \frac{1}{\mu^c}\frac{\mathrm{d}\,P}{\mathrm{d}\,t} q^0 \\
&\quad + \fint_{\Omega\times Y_m} \frac{1}{\mu^m}\left(\frac{\mathrm{d}\,P}{\mathrm{d}\,t} + \frac{\mathrm{d}\,\widehat{P}}{\mathrm{d}\,t}\right)(q^0 + \hat{q}^0) = 0\,,
\end{aligned} \tag{3.17}$$

*for all $\boldsymbol{v}^0 \in V_0 = \mathbf{H}^1_0(\Omega)$, $q^0 \in H^1(\Omega)$, $\tilde{\boldsymbol{v}}^1 \in L^2(\Omega; \mathbf{H}^1_\#(Y))$, $\tilde{q}^1 \in L^2(\Omega; H^1_\#(Y))$ and $\hat{q}^0 \in L^2(\Omega; H_{\#c0}(Y, \Gamma_m^Y))$.*

**Remark 2.** The "two-scale" problem (3.16)-(3.17) retains the symmetry of (2.19). The existence and uniqueness of weak solutions defined in $\Omega \times Y$ can be proved using similar technique, as for the original problem (2.19), see [16], Section 3, however, extended for the two-scale functions and appropriate spaces employed in (3.16)-(3.17). □

*Proof.* of Theorem 3.5. We shall derive limit expressions for all unfolded integrals involved in (2.19). For this we need to introduce suitable test functions $\boldsymbol{v}^\varepsilon$ and $q^\varepsilon$. Due to (3.10)-(3.11), the following test displacements are considered:

$$\boldsymbol{v}^\varepsilon(x) = \boldsymbol{v}(x) + \varepsilon\boldsymbol{v}^1(\{x/\varepsilon\}_Y)\theta(x)\,, \quad \boldsymbol{v} \in \mathbf{H}^1_0(\Omega), \quad \boldsymbol{v}^1 \in \mathbf{H}^1_\#(Y)/\mathbb{R}, \quad \theta \in C^\infty_0(\Omega).$$

Further, we introduce the pressure test functions $q^\varepsilon$ of the following form

$$q^\varepsilon(x) = q^0(x) + \varepsilon q^1(y)\vartheta(x) + \hat{q}(y)\hat{\vartheta}(x)\,, \quad \text{where } \hat{q} = 0 \quad \text{in } \overline{Y_c}, \tag{3.18}$$

Passing to the limit in the interface integrals from (2.19), we get

$$\begin{aligned}
&\int_{\Gamma_m^\varepsilon} n_j\alpha^\varepsilon_{ij} v_i^\varepsilon [\frac{\mathrm{d}}{\mathrm{d}\,t}\widehat{P}^\varepsilon]_{\Gamma_m^\varepsilon}\, dS \\
&= \int_\Omega \frac{1}{\varepsilon|Y|}\Big(v_i \int_{\Gamma_m^Y} n_j\varepsilon\tilde{\alpha}^\Gamma_{ij}\left[\frac{\mathrm{d}}{\mathrm{d}\,t}\mathcal{T}^b_\varepsilon\Big(\widehat{P}^\varepsilon\Big)\right]_{\Gamma_m^Y} dS_y + \varepsilon^2\theta\int_{\Gamma_m^Y} n_j\tilde{\alpha}^\Gamma_{ij}v_i^1\left[\frac{\mathrm{d}}{\mathrm{d}\,t}\mathcal{T}^b_\varepsilon\Big(\widehat{P}^\varepsilon\Big)\right]_{\Gamma_m^Y} dS_y\Big) \\
&\qquad \to \int_\Omega v_i \fint_{\Gamma_m^Y} n_j\tilde{\alpha}^\Gamma_{ij}\left[\frac{\mathrm{d}}{\mathrm{d}\,t}\widehat{P}\right]_{\Gamma_m^Y} dS_y\,,
\end{aligned} \tag{3.19}$$

and

$$\begin{aligned}\int_{\Gamma_m^\varepsilon} n_j\alpha_{ij}^\varepsilon u_i^\varepsilon[\hat q(x/\varepsilon)]_{\Gamma_m^\varepsilon}\hat\vartheta\,dS &= \int_\Omega \frac{1}{|Y|}\int_{\Gamma_m^Y} n_j\tilde\alpha_{ij}^\Gamma \mathcal{T}_\varepsilon^b(u_i^\varepsilon)\,[\hat q]_{\Gamma_m^Y}\;\mathcal{T}_\varepsilon^b\Big(\hat\vartheta\Big)dS_y\\ &\to \int_\Omega \hat\vartheta u_i \fint_{\Gamma_m^Y} n_j\tilde\alpha_{ij}^\Gamma\,[\hat q]_{\Gamma_m^Y}\,dS_y\,.\end{aligned}\tag{3.20}$$

For the other integrals in (2.19), we have the following limits (recalling that $\alpha_{ij}^\varepsilon$ is not proportional to $\varepsilon$ in $\Omega\setminus\Gamma_m^\varepsilon$):

$$\begin{aligned}&\int_\Omega D_{ijkl}^\varepsilon e_{kl}(\boldsymbol{u}^\varepsilon)e_{ij}(\boldsymbol{v}^\varepsilon)\to\int_\Omega\fint_Y \tilde D_{ijkl}(e_{kl}^x(\boldsymbol{u})+e_{kl}^y(\boldsymbol{u}^1))\,(e_{ij}^x(\boldsymbol{v})+e_{ij}^y(\boldsymbol{v}^1)\theta),\\ &\int_\Omega \alpha_{ij}^\varepsilon e_{ij}(\boldsymbol{v}^\varepsilon)\frac{\mathrm{d}}{\mathrm{d}\,t}P^\varepsilon\to\int_\Omega\fint_Y\tilde\alpha_{ij}(e_{ij}^x(\boldsymbol{v})+e_{ij}^y(\boldsymbol{v}^1)\theta)\frac{\mathrm{d}}{\mathrm{d}\,t}\left(P(x)+\chi_m\widehat P\right),\\ &\int_\Omega \boldsymbol{f}^\varepsilon\cdot\boldsymbol{v}^\varepsilon\to\int_\Omega\boldsymbol{f}\cdot\boldsymbol{v},\end{aligned}\tag{3.21}$$

where $\boldsymbol{f}^\varepsilon\rightharpoonup\boldsymbol{f}$ weakly in $L^2(0,T;\mathbf{L}^2(\Omega))$, and

$$\begin{aligned}&\int_\Omega K_{ij}^\varepsilon\partial_j P^\varepsilon\partial_i q^\varepsilon\to\int_\Omega\fint_{Y_c}K_{ij}^c(\partial_j^x P+\partial_j^y P^1)\,(\partial_j^x q^0+\vartheta\partial_j^y q^1)+\int_\Omega\hat\vartheta\fint_{Y_m}K_{ij}^m\partial_j\widehat P\partial_i\hat q,\\ &\int_\Omega\alpha_{ij}^\varepsilon e_{ij}(\boldsymbol{u}^\varepsilon)q^\varepsilon\to\int_\Omega\fint_Y\tilde\alpha_{ij}(e_{ij}^x(\boldsymbol{u})+e_{ij}^y(\boldsymbol{u}^1))(q^0+\chi_m\hat q\hat\vartheta),\\ &\int_\Omega\varkappa^\varepsilon[P^\varepsilon]_{\Gamma_m^\varepsilon}[q^\varepsilon]_{\Gamma_m^\varepsilon}\to\int_\Omega\fint_{\Gamma_m^Y}\tilde\varkappa\left[\widehat P\right]_{\Gamma_m^Y}[\hat q]_{\Gamma_m^Y}\,dS_y.\end{aligned}\tag{3.22}$$

The limit expressions in (3.20)-(3.22) are valid (by density arguments) for any test functions of the form

$$\begin{aligned}&\boldsymbol{v}^\varepsilon(x)=\boldsymbol{v}^0(x)+\varepsilon\tilde{\boldsymbol{v}}^1(x,y)\,,\quad \boldsymbol{v}\in\mathbf{H}_0^1(\Omega),\quad \tilde{\boldsymbol{v}}\in L^2(\Omega;\mathbf{H}_\#^1(Y))\,,\\ &q^\varepsilon=q^0(x)+\varepsilon\tilde q^1(x,y)+\chi_m(y)\hat q^0(x,y),\\ &q^0\in H^1(\Omega),\quad \tilde q^1\in L^2(\Omega;H_\#^1(Y)),\quad \hat q^0\in L^2(\Omega;H_{\#c0}(Y,\Gamma_m^Y)).\end{aligned}\tag{3.23}$$

So, instead of $\boldsymbol{v}^1\theta$, we can take $\tilde{\boldsymbol{v}}^1\in L^2(\Omega;\mathbf{H}_\#^1(Y))$; instead of $q^1\vartheta$, $\tilde q^1\in L^2(\Omega;H_\#^1(Y))$ and instead of $\hat\vartheta\hat q$ we take $\hat q^0\in L^2(\Omega;H_{\#c0}(Y,\Gamma_m^Y))$. It is now possible to write down the limit form of (2.19). The *two-scale problem* is obtained from (3.19)-(3.22), making use of the generalized test functions (3.23), and taking $q^1=0$ in (3.22). □

**3.3 Scale decoupling and the homogenized constitutive laws** In (3.16) and (3.17) we can take suitable combinations of vanishing and non-vanishing parts of the test functions defined in (3.23), so that the "local" and the "global" problems can be identified.

The local problem describes the diffusion-deformation driven by $e^x(\boldsymbol{u})$, $\boldsymbol{u}$ and $P$. For all $\boldsymbol{v}^1\in\mathbf{H}_\#^1(Y)$ and $\hat q\in H_{\#c0}(Y,\Gamma_m^Y)$, one has

$$\begin{aligned}&\fint_Y\tilde D_{ijkl}[e_{kl}^x(\boldsymbol{u})+e_{kl}^y(\boldsymbol{u}^1)]\,e_{ij}^y(\boldsymbol{v}^1)-\fint_Y\tilde\alpha_{ij}e_{ij}^y(\boldsymbol{v}^1)\frac{\mathrm{d}\,P}{\mathrm{d}\,t}-\fint_{Y_m}\alpha_{ij}^m e_{ij}^y(\boldsymbol{v}^1)\frac{\mathrm{d}\,\widehat P}{\mathrm{d}\,t}=0\,,\\ &\fint_{Y_m}\alpha_{ij}^m[e_{ij}^x(\boldsymbol{u})+e_{ij}^y(\boldsymbol{u}^1)]\,\hat q+u_i\fint_{\Gamma_m^Y}n_j\tilde\alpha_{ij}^\Gamma\,[\hat q]_{\Gamma_m^Y}\,dS_y+\fint_{Y_m}K_{ij}^m\,\partial_j^y\widehat P\,\partial_i^y\hat q\\ &\qquad+\fint_{\Gamma_m^Y}\tilde\varkappa\left[\widehat P\right]_{\Gamma_m^Y}[\hat q]_{\Gamma_m^Y}\,dS_y+\fint_{Y_m}\frac{1}{\mu^m}\left(\frac{\mathrm{d}\,P}{\mathrm{d}\,t}+\frac{\mathrm{d}\,\widehat P}{\mathrm{d}\,t}\right)\hat q=0\,.\end{aligned}\tag{3.24}$$

The global problem is the diffusion-deformation problem described in terms of $e^x(\boldsymbol{u})$ and $P$, involving the local perturbations $\widehat{P}, \boldsymbol{u}^1, P^1$. For all $\boldsymbol{v}^0 \in V_0 = \mathbf{H}^1_0(\Omega)$ and $q^0 \in H^1(\Omega)$,

$$\begin{aligned}
&\fint_{\Omega\times Y} \tilde{D}_{ijkl}[e^x_{kl}(\boldsymbol{u}) + e^y_{kl}(\boldsymbol{u}^1)]\, e^x_{ij}(\boldsymbol{v}^0) - \fint_{\Omega\times Y} \tilde{\alpha}_{ij} e^x_{ij}(\boldsymbol{v}^0) \left(\frac{\mathrm{d}\,P}{\mathrm{d}\,t} + \chi_m \frac{\mathrm{d}\,\widehat{P}}{\mathrm{d}\,t}\right) \\
&\qquad - \int_\Omega v^0_i \fint_{\Gamma^Y_m} n_j \tilde{\alpha}^\Gamma_{ij} \left[\frac{\mathrm{d}}{\mathrm{d}\,t}\widehat{P}\right]_{\Gamma^Y_m} dS_y = \int_\Omega \boldsymbol{f}\cdot\boldsymbol{v}^0\,, \\
&\int_\Omega \mathcal{C}_{ij}\partial^x_j P\, \partial^x_i q^0 + \fint_{\Omega\times Y} \tilde{\alpha}_{ij}[e^x_{ij}(\boldsymbol{u}) + e^y_{ij}(\boldsymbol{u}^1)]\, q^0 + \fint_{\Omega\times Y_c} \frac{1}{\mu^c}\frac{\mathrm{d}\,P}{\mathrm{d}\,t} q^0 \\
&\qquad + \fint_{\Omega\times Y_m} \frac{1}{\mu^m}\left(\frac{\mathrm{d}\,P}{\mathrm{d}\,t} + \frac{\mathrm{d}\,\widehat{P}^0}{\mathrm{d}\,t}\right) q^0 = 0.
\end{aligned} \tag{3.25}$$

Homogenized permeability. By selecting $q^0 \equiv 0$ and $\hat{q} \equiv 0$, due to (3.22) the limit equation (3.17) reduces to

$$\fint_{Y_c} K^c_{ij}(\partial^x_j P + \partial^y_j P^1)\, \partial^y_j \tilde{q}^1 = 0\,, \quad \forall \tilde{q}^1 \in L^2(\Omega; H^1_\#(Y)) \quad \text{a.e. in } \Omega. \tag{3.26}$$

Due to the linearity, we can define corrector basis functions $\eta^k$ such that $P^1(t,x,y) = \eta^k(y)\partial^x_k P(t,x)$. Therefore, (3.26) is equivalent to the following problem (set in channels $Y_c$):

$$\begin{cases} \text{Find } \eta^k \in H^1_\#(Y)/\mathbb{R}\ (k=1,2,3) \text{ such that} \\ \fint_{Y_c} K^c_{ij}\partial^y_j(\eta^k + y_k)\, \partial^y_i \psi = 0 \quad \forall \psi \in H^1_\#(Y)\,. \end{cases} \tag{3.27}$$

It is now easy to replace in (3.17) the only integral involving $P^1$ using the *homogenized permeability* $\mathcal{C}_{ij}$ defined as follows:

$$\int_\Omega \mathcal{C}_{ij}\partial^x_j P \partial^x_i q^0 := \fint_{\Omega\times Y_c} K^c_{ij}\,[\partial^x_j P + \partial^y_j P^1]\, \partial^x_i q^0 = \fint_{Y_c} K^c_{il}\partial^y_l(\eta^j + y_j)\, \partial^x_j P\, \partial^x_i q^0\,.$$

We can identify $\mathcal{C}_{ij}$ as follows:

$$\mathcal{C}_{ij} = \fint_{Y_c} K^c_{il}\partial^y_l(\eta^j + y_j) = \fint_{Y_c} K^c_{kl}\partial^y_l(\eta^j + y_j)\, \partial^y_k y_i = \fint_{Y_c} K^c_{kl}\partial^y_l(\eta^j + y_j)\, \partial^y_k(\eta^i + y_i). \tag{3.28}$$

The last symmetric expression is a simple consequence of identity (3.27) evaluated for $\psi = \eta^i$, where other indices have been changed appropriately.

Let us point out that the effective medium permeability $\mathcal{C}_{ij}$ (relevant to the macroscopic scale) depends exclusively on the geometry and permeability of the primary porosity in the channels represented by $Y_c$.

**3.3.1 Auxiliary local problems and corrector basis functions** Throughout this section and in Section 4.3 bellow, we use the following notation:

$$\begin{aligned}
&a_Y(\boldsymbol{u},\,\boldsymbol{v}) = \fint_Y \tilde{D}_{ijkl} e^y_{kl}(\boldsymbol{u})\, e^y_{ij}(\boldsymbol{v}), \qquad b_Y(\varphi,\,\boldsymbol{v}) = \fint_Y \varphi\, \tilde{\alpha}_{ij} e^y_{ij}(\boldsymbol{v})\,, \\
&b_{Y_m}(\varphi,\,\boldsymbol{v}) = \fint_{Y_m} \varphi\, \alpha^m_{ij} e^y_{ij}(\boldsymbol{v})\,, \\
&\hat{c}_{Y_m,\Gamma_m}(\varphi,\,\psi) = \fint_{Y_m\setminus\Gamma^Y_m} K^m_{ij}\partial^y_j\varphi\partial^y_i\psi + \fint_{\Gamma^Y_m} \tilde{\varkappa}[\varphi]_{\Gamma^Y_m}[\psi]_{\Gamma^Y_m} dS_y\,, \\
&d_{Y_m}(\varphi,\,\psi) = \fint_{Y_m} \frac{1}{\mu^m}\varphi\psi, \qquad \gamma_{\Gamma,k}(\varphi) = \fint_{\Gamma^Y_m} n_j \tilde{\alpha}^\Gamma_{kj}[\varphi]_{\Gamma^Y_m}\, dS_y\,.
\end{aligned} \tag{3.29}$$

In order to decouple the microscopic and macroscopic *evolutionary* problems, we can apply the usual method of the scale separation based on the Laplace transformation $v(t) \to \mathcal{L}\{v\}(\lambda)$, where $\lambda$ is the variable in the Laplace domain. For brevity we denote all functions depending on $\lambda$ as follows: $\mathcal{L}\{v\}(\lambda) = \underset{*}{v}$. In the transformed space we define a suitable multiplicative decomposition to arrive at autonomous local problems arising from (3.24) allowing to compute the local corrector functions and, consequently to compute also the homogenized coefficients. We use *zero* initial conditions, namely $\boldsymbol{u}(0,\cdot) = 0$, whereas $P(0,\cdot) \equiv 0$ by definition. By the Laplace transformation, the local problem becomes

$$
\begin{aligned}
&a_Y\left(\underset{*}{\boldsymbol{u}}^1,\, \boldsymbol{v}\right) - b_{Y_m}\left(\lambda\widehat{\underset{*}{P}},\, \boldsymbol{v}\right) = -a_Y\left(\boldsymbol{\Pi}^{ij},\, \boldsymbol{v}\right) e_{ij}(\underset{*}{\boldsymbol{u}}) + \lambda b_Y\left(1,\, \boldsymbol{v}\right) \underset{*}{P}\,,\\
&b_{Y_m}\left(\underset{*}{\boldsymbol{u}}^1,\, \phi\right) + \hat{c}_{Y_m,\Gamma_m}\left(\widehat{\underset{*}{P}}^0,\, \phi\right) + \lambda d_{Y_m}\left(\widehat{\underset{*}{P}},\, \phi\right)\\
&\qquad\qquad = -b_{Y_m}\left(\phi,\, \boldsymbol{\Pi}\right) e_{ij}(\underset{*}{\boldsymbol{u}}) - d_{Y_m}\left(1,\, \phi\right) \lambda \underset{*}{P} - \gamma_{\Gamma,k}(\phi)\underset{*}{u}_k.
\end{aligned}
$$

Due to the linearity, we define the multiplicative decomposition by introducing the corrector functions in the $\mathcal{L}$-transformed time domain, in the form

$$
\begin{aligned}
\underset{*}{\boldsymbol{u}}^1(\lambda,x,y) &= \lambda\underset{*}{\boldsymbol{\omega}}^{rs}(\lambda,y)\, e^x_{rs}(\underset{*}{\boldsymbol{u}})(\lambda,x) + \lambda\underset{*}{\boldsymbol{\omega}}^{P}(\lambda,y)\underset{*}{P}(\lambda,x) + \lambda\underset{*}{\boldsymbol{\omega}}^{k}(\lambda,y)\underset{*}{u}_k(\lambda,x)\,,\\
\widehat{\underset{*}{P}}^0(\lambda,x,y) &= \lambda\underset{*}{\pi}^{rs}(\lambda,y)\, e^x_{rs}(\underset{*}{\boldsymbol{u}})(\lambda,x) + \lambda\underset{*}{\pi}^{P}(\lambda,y)\underset{*}{P}(\lambda,x) + \lambda\underset{*}{\pi}^{k}(\lambda,y)\underset{*}{u}_k(\lambda,x)\,.
\end{aligned}
$$

The functions $\underset{*}{\boldsymbol{\omega}}^{rs}, \underset{*}{\boldsymbol{\omega}}^{P}, \underset{*}{\boldsymbol{\omega}}^{k}$ and $\underset{*}{\pi}^{rs}, \underset{*}{\pi}^{P}, \underset{*}{\pi}^{k}$ satisfy the following local auxiliary problems.

*Strain corrector problem:* Find $(\underset{*}{\boldsymbol{\omega}}^{rs}, \underset{*}{\pi}^{rs}) \in \mathbf{H}^1_{\#}(Y) \times H_{\#c0}(Y,\Gamma^Y_m)$ such that

$$
\begin{cases}
a_Y\left(\underset{*}{\boldsymbol{\omega}}^{rs},\, \boldsymbol{v}\right) - \lambda b_{Y_m}\left(\underset{*}{\pi}^{rs},\, \boldsymbol{v}\right) = -\dfrac{1}{\lambda}\, a_Y\left(\boldsymbol{\Pi}^{rs},\, \boldsymbol{v}\right), \quad \forall \boldsymbol{v} \in \mathbf{H}^1_{\#}(Y)\,,\\
b_{Y_m}\left(\psi,\, \underset{*}{\boldsymbol{\omega}}^{rs}\right) + \hat{c}_{Y_m,\Gamma_m}\left(\underset{*}{\pi}^{rs},\, \psi\right) + \lambda d_{Y_m}\left(\underset{*}{\pi}^{rs},\, \psi\right) = -\dfrac{1}{\lambda} b_{Y_m}\left(\boldsymbol{\Pi}^{rs},\, \psi\right),\\
\forall \psi \in H_{\#c0}(Y,\Gamma^Y_m),
\end{cases}
\tag{3.30}
$$

where $\boldsymbol{\Pi}^{rs} = (\Pi^{rs}_i) = (y_s\delta_{ir})$.

*Pressure corrector problem:* Find $(\underset{*}{\boldsymbol{\omega}}^{rs}, \underset{*}{\pi}^{rs}) \in \mathbf{H}^1_{\#}(Y) \times H_{\#c0}(Y,\Gamma^Y_m)$ such that

$$
\begin{cases}
a_Y\left(\underset{*}{\boldsymbol{\omega}}^{P},\, \boldsymbol{v}\right) - \lambda b_{Y_m}\left(\underset{*}{\pi}^{P},\, \boldsymbol{v}\right) = b_Y\left(1,\, \boldsymbol{v}\right) \quad \forall \boldsymbol{v} \in \mathbf{H}^1_{\#}(Y)\,,\\
b_{Y_m}\left(\psi,\, \underset{*}{\boldsymbol{\omega}}^{P}\right) + \hat{c}_{Y_m,\Gamma_m}\left(\underset{*}{\pi}^{P},\, \psi\right) + \lambda d_{Y_m}\left(\underset{*}{\pi}^{P},\, \psi\right) = -d_{Y_m}\left(1,\, \psi\right)\\
\forall \psi \in H_{\#c0}(Y,\Gamma^Y_m).
\end{cases}
\tag{3.31}
$$

*Displacement corrector problem:* Find $(\underset{*}{\boldsymbol{\omega}}^{k}, \underset{*}{\pi}^{k}) \in \mathbf{H}^1_{\#}(Y) \times H_{\#c0}(Y,\Gamma^Y_m)$ such that

$$
\begin{cases}
a_Y\left(\underset{*}{\boldsymbol{\omega}}^{k},\, \boldsymbol{v}\right) - \lambda b_{Y_m}\left(\underset{*}{\pi}^{k},\, \boldsymbol{v}\right) = 0 \quad \forall \boldsymbol{v} \in \mathbf{H}^1_{\#}(Y),\\
b_{Y_m}\left(\psi,\, \underset{*}{\boldsymbol{\omega}}^{k}\right) + \hat{c}_{Y_m,\Gamma_m}\left(\underset{*}{\pi}^{k},\, \psi\right) + \lambda d_{Y_m}\left(\underset{*}{\pi}^{k},\, \psi\right) = -\dfrac{1}{\lambda}\gamma_{\Gamma,k}(\psi),\\
\forall \psi \in H_{\#c0}(Y,\Gamma^Y_m).
\end{cases}
\tag{3.32}
$$

**3.3.2 Homogenized coefficients and the macroscopic problem** We now study the main homogenization result for weakly discontinuous data, namely (2.23). Application of the Laplace transformation to (3.25) yields

$$
\begin{aligned}
&\fint_{\Omega\times Y} e^x_{ij}(\boldsymbol{v}^0)\, \tilde{D}_{ijkl}[e^y_{kl}(\boldsymbol{\Pi}^{rs}) + \lambda e^y_{kl}(\underset{*}{\boldsymbol{\omega}}^{rs})]\, e^x_{rs}(\underset{*}{\boldsymbol{u}})\\
&\quad + \fint_{\Omega\times Y} e^x_{ij}(\boldsymbol{v}^0)\, \tilde{D}_{ijkl} e^y_{kl}(\underset{*}{\boldsymbol{\omega}}^{P})\, \lambda \underset{*}{P} + \fint_{\Omega\times Y} e^x_{ij}(\boldsymbol{v}^0)\, \tilde{D}_{ijkl} e^y_{kl}(\underset{*}{\boldsymbol{\omega}}^{n})\, \lambda \underset{*}{u}_n\\
&\quad - \fint_{\Omega\times Y} e^x_{ij}(\boldsymbol{v}^0)\tilde{\alpha}_{ij}\lambda\left(\underset{*}{P} + \chi_m\lambda\underset{*}{\pi}^{rs} e^x_{rs}(\underset{*}{\boldsymbol{u}}) + \chi_m\lambda\underset{*}{\pi}^{P}\underset{*}{P} + \chi_m\lambda\underset{*}{\pi}^{n}\underset{*}{u}_n\right)\\
&\quad - \int_{\Omega} v^0_i \fint_{\Gamma^Y_m} n_j\tilde{\alpha}^{\Gamma}_{ij}\lambda^2\left(\left[\underset{*}{\pi}^{rs}\right]_{\Gamma^Y_m} e^x_{rs}(\underset{*}{\boldsymbol{u}}) + \left[\underset{*}{\pi}^{P}\right]_{\Gamma^Y_m}\underset{*}{P} + \left[\underset{*}{\pi}^{n}\right]_{\Gamma^Y_m}\underset{*}{u}_n\right) dS_y = \int_{\Omega} \underset{*}{\boldsymbol{f}}\cdot\boldsymbol{v}^0,
\end{aligned}
\tag{3.33}
$$

and

$$\begin{aligned}&\fint_{\Omega\times Y} q^0\tilde{\alpha}_{kl}\Big(\left[e^y_{kl}(\mathbf{\Pi}^{rs})+\lambda e^y_{kl}(\boldsymbol{\omega}^{rs}_*)\right]e^x_{rs}(\underset{*}{\boldsymbol{u}})+\lambda e^y_{kl}(\boldsymbol{\omega}^P_*)\underset{*}{P}+\lambda e^y_{kl}(\boldsymbol{\omega}^n_*)\underset{*}{u}_n\Big)\\&\quad+\int_\Omega \mathcal{C}_{ij}\partial_j\underset{*}{P}\partial_i q^0+\sim\!\!\!\!\!\!\int_{\Omega\times Y}\frac{q^0}{\mu}\lambda\underset{*}{P}+\sim\!\!\!\!\!\!\int_{\Omega\times Y_m}\frac{q^0}{\mu^m}\Big(\lambda^2\underset{*}{\pi}^{ij}e^x_{ij}(\underset{*}{\boldsymbol{u}})+\lambda^2\underset{*}{\pi}^P\underset{*}{P}+\lambda^2\underset{*}{\pi}^k\underset{*}{u}_k\Big)=0.\end{aligned}\tag{3.34}$$

In these equations we can identify the homogenized coefficients, as explained bellow.

*Homogenized viscoelasticity.* This tensor is obtained by collecting in (3.34) all the terms which contain $\lambda e^x_{ij}(\underset{*}{\boldsymbol{u}})$,

$$\begin{aligned}\mathcal{A}^*_{ijkl}(\lambda)&=\lambda\Big[a_Y\left(\frac{1}{\lambda}\mathbf{\Pi}^{kl}+\boldsymbol{\omega}^{kl}_*,\frac{1}{\lambda}\mathbf{\Pi}^{ij}\right)-b_{Y_m}\left(\underset{*}{\pi}^{kl},\mathbf{\Pi}^{ij}\right)\Big]\\&=\lambda\Big[a_Y\left(\frac{1}{\lambda}\mathbf{\Pi}^{kl}+\boldsymbol{\omega}^{kl}_*,\frac{1}{\lambda}\mathbf{\Pi}^{ij}+\boldsymbol{\omega}^{ij}_*\right)+\lambda\hat{c}_{Y_m,\Gamma_m}\left(\underset{*}{\pi}^{kl},\underset{*}{\pi}^{ij}\right)+\lambda^2 d_{Y_m}\left(\underset{*}{\pi}^{kl},\underset{*}{\pi}^{ij}\right)\Big],\end{aligned}$$

where the symmetric expression is a consequence of (3.30).

*The homogenized Biot modulus.* This tensor is obtained by collecting in (3.33) all the terms containing $\lambda\underset{*}{P}$,

$$\mathcal{M}^*(\lambda)=\fint_Y\frac{1}{\mu}+\lambda\fint_{Y_m}\frac{1}{\mu^m}\underset{*}{\pi}^P+\fint_Y\tilde{\alpha}_{ij}e^y_{ij}(\boldsymbol{\omega}^P_*)=\fint_Y\frac{1}{\mu}+\lambda d_{Y_m}\left(\underset{*}{\pi}^P,1\right)+b_{Y_m}\left(1,\boldsymbol{\omega}^P_*\right).\tag{3.35}$$

*The homogenized Biot coefficients.* They can be obtained independently from both equations (3.33)-(3.34). On collecting in (3.33) all the terms involving $\lambda\underset{*}{P}$, one obtains

$$\alpha^*_{ij}(\lambda)=\fint_Y\tilde{\alpha}_{ij}-a_Y\left(\boldsymbol{\omega}^P_*,\mathbf{\Pi}^{ij}\right)-\lambda b_{Y_m}\left(\underset{*}{\pi}^P,\mathbf{\Pi}^{ij}\right).\tag{3.36}$$

By collecting in (3.34) all the terms involving $\lambda e^x_{ij}(\underset{*}{\boldsymbol{u}})$), one gets

$$\beta^*_{ij}(\lambda)=\frac{1}{\lambda}\fint_Y\tilde{\alpha}_{ij}+b_Y\left(1,\boldsymbol{\omega}^{ij}_*\right)+\lambda d_{Y_m}\left(\underset{*}{\pi}^{ij},1\right).\tag{3.37}$$

It is easily seen that the following result holds true:

LEMMA 3.6. *The homogenized Biot coefficients defined in (3.36) and (3.37) satisfy*

$$\alpha^*_{ij}(\lambda)=\lambda\beta^*_{ij}(\lambda).\tag{3.38}$$

*Proof.* Relation (3.38) can be obtained using the microscopic local problems, (3.30) and (3.31), where we use special forms of test functions. First $(3.30)_2$ yields

$$\frac{1}{\lambda}b_{Y_m}\left(\underset{*}{\pi}^P,\mathbf{\Pi}^{ij}\right)=-b_{Y_m}\left(\underset{*}{\pi}^P,\boldsymbol{\omega}^{ij}_*\right)-\lambda d_{Y_m}\left(\underset{*}{\pi}^{ij},\underset{*}{\pi}^P\right)-\hat{c}_{Y_m,\Gamma_m}\left(\underset{*}{\pi}^{ij},\underset{*}{\pi}^P\right).$$

Then the first and the last terms can be expressed using $(3.31)_1$ and $(3.31)_2$, respectively, so that

$$\begin{aligned}\frac{1}{\lambda}b_{Y_m}\left(\underset{*}{\pi}^P,\mathbf{\Pi}^{ij}\right)&=\frac{1}{\lambda}b_Y\left(1,\boldsymbol{\omega}^{ij}_*\right)-\frac{1}{\lambda}a_Y\left(\boldsymbol{\omega}^P_*,\boldsymbol{\omega}^{ij}_*\right)-\lambda d_{Y_m}\left(\underset{*}{\pi}^{ij},\underset{*}{\pi}^P\right)\\&\quad+b_{Y_m}\left(\underset{*}{\pi}^{ij},\boldsymbol{\omega}^P_*\right)+d_{Y_m}\left(1+\lambda\underset{*}{\pi}^P,\underset{*}{\pi}^{ij}\right)\\&=\frac{1}{\lambda}b_Y\left(1,\boldsymbol{\omega}^{ij}_*\right)-\frac{1}{\lambda}a_Y\left(\boldsymbol{\omega}^P_*,\boldsymbol{\omega}^{ij}_*\right)\quad+b_{Y_m}\left(\underset{*}{\pi}^{ij},\boldsymbol{\omega}^P_*\right)+d_{Y_m}\left(1,\underset{*}{\pi}^{ij}\right).\end{aligned}\tag{3.39}$$

Now $(3.30)_1$ yields

$$\frac{1}{\lambda}a_Y\left(\boldsymbol{\omega}^P_*,\mathbf{\Pi}^{ij}\right)=-a_Y\left(\boldsymbol{\omega}^{ij}_*,\boldsymbol{\omega}^P_*\right)+\lambda b_{Y_m}\left(\underset{*}{\pi}^P,\boldsymbol{\omega}^P_*\right),\tag{3.40}$$

so that on substituting (3.39) and (3.40) in (3.36),

$$\begin{aligned}\alpha^*_{ij}(\lambda) &= \lambda a_Y\left(\boldsymbol{\omega}^{ij}_*, \boldsymbol{\omega}^P_*\right) - \lambda^2 b_{Y_m}\left(\pi^P_*, \boldsymbol{\omega}^P_*\right) + \lambda b_Y\left(1, \boldsymbol{\omega}^{ij}_*\right) - \lambda a_Y\left(\boldsymbol{\omega}^P_*, \boldsymbol{\omega}^{ij}_*\right)\\ &\quad + \lambda^2 b_{Y_m}\left(\pi^{ij}_*, \boldsymbol{\omega}^P_*\right) + \lambda^2 d_{Y_m}\left(1, \pi^{ij}_*\right) + \fint_Y \tilde{\alpha}_{ij}\\ &= \lambda b_Y\left(1, \boldsymbol{\omega}^{ij}_*\right) + \lambda^2 d_{Y_m}\left(1, \pi^{ij}_*\right) + \fint_Y \tilde{\alpha}_{ij} = \lambda\beta^*_{ij}(\lambda)\,.\end{aligned}$$

□

*Coefficients due to the interface terms on "micro" and "macro".* We introduce coefficients $g^{III*}_{kij}, g^{II*}_{ij}, g^{I*}_k$ and $h^{III*}_{kij}, h^{I*}_k$ to express the following integrals appearing in (3.33), (3.34):

$$\begin{aligned}&\int_\Omega v_i \fint_{\Gamma^Y_m} n_j\tilde{\alpha}^\Gamma_{ij}\lambda^2\left[\pi^{rs}_*\right]_{\Gamma^Y_m} e^x_{rs}(\underset{*}{\boldsymbol{u}}) = \int_\Omega v_i g^{III*}_{irs}(\lambda)\,\lambda e^x_{rs}(\underset{*}{\boldsymbol{u}}),\\ &\int_\Omega v_i \fint_{\Gamma^Y_m} n_j\tilde{\alpha}^\Gamma_{ij}\lambda^2\left[\pi^{n}_*\right]_{\Gamma^Y_m} \underset{*}{u}_n = \int_\Omega v_i g^{II*}_{in}(\lambda)\,\lambda\underset{*}{u}_n,\\ &\int_\Omega v_i \fint_{\Gamma^Y_m} n_j\tilde{\alpha}^\Gamma_{ij}\lambda^2\left[\pi^{P}_*\right]_{\Gamma^Y_m} \underset{*}{P} = \int_\Omega v_i g^{I*}_{i}(\lambda)\,\lambda\underset{*}{P},\\ &\fint_{\Omega\times Y} e^x_{ij}(\boldsymbol{v})\,\tilde{D}_{ijkl}e^y_{kl}(\boldsymbol{\omega}^k_*)\,\lambda\underset{*}{u}_k - \fint_{\Omega\times Y_m} e^x_{ij}(\boldsymbol{v})\alpha^m_{ij}\lambda^2\pi^k_*\underset{*}{u}_k = \int_\Omega e^x_{ij}(\boldsymbol{v})h^{III*}_{kij}(\lambda)\,\lambda\underset{*}{u}_k\end{aligned}$$

and

$$\fint_{\Omega\times Y} q\tilde{\alpha}_{ij}\lambda e^y_{ij}(\boldsymbol{\omega}^k_*)\underset{*}{u}_k + \fint_{\Omega\times Y_m}\frac{q}{\mu^m}\lambda^2\pi^k_*\underset{*}{u}_k = \int_\Omega q h^{I*}_k(\lambda)\,\lambda u_k,$$

where

$$\begin{aligned}&g^{III*}_{kij}(\lambda) = \lambda\gamma_{\Gamma,k}(\pi^{ij}_*), \qquad g^{II*}_{kj}(\lambda) = \lambda\gamma_{\Gamma,k}(\pi^j_*), \qquad g^{I*}_k(\lambda) = \lambda\gamma_{\Gamma,k}(\pi^P_*),\\ &h^{III*}_{kij}(\lambda) = a_Y\left(\boldsymbol{\omega}^k_*, \boldsymbol{\Pi}^{ij}\right) - \lambda b_{Y_m}\left(\pi^k_*, \boldsymbol{\Pi}^{ij}\right),\\ &h^{I*}_k(\lambda) = b_Y\left(1, \boldsymbol{\omega}^k_*\right) + \lambda d_{Y_m}\left(\pi^k_*, 1\right)\,.\end{aligned} \tag{3.41}$$

For the well-posedness of the macroscopic problem, its symmetry is important. It is a consequence of the symmetries of the coefficients (3.41) stated bellow.

LEMMA 3.7. *The following relationships hold:*

$$g^{II*}_{kj} = g^{II*}_{jk}, \quad g^{I*}_k = \lambda h^{I*}_k, \quad h^{III*}_{kij} = -g^{III*}_{kij}, \quad g^{III*}_{kij} = g^{III*}_{kji}. \tag{3.42}$$

*Proof.* The first symmetry follows easily from (3.32) which leads to

$$\gamma_{\Gamma,k}(\pi^j_*) = d_{Y_m}\left(\lambda\pi^k_*, \lambda\pi^j_*\right) + \lambda\hat{c}_{Y_m,\Gamma_m}\left(\pi^k_*, \pi^j_*\right) + a_Y\left(\boldsymbol{\omega}^k_*, \boldsymbol{\omega}^j_*\right) = \gamma_{\Gamma,j}(\pi^k_*).$$

To show the second symmetry, we use (3.31) and rewrite both the terms involved in definition $(3.41)_5$; the first one combined with $(3.32)_2$ and $(3.32)_1$, yields

$$\begin{aligned}b_Y\left(1, \boldsymbol{\omega}^k_*\right) &= a_Y\left(\boldsymbol{\omega}^k_*, \boldsymbol{\omega}^P_*\right) - \lambda b_{Y_m}\left(\pi^P_*, \boldsymbol{\omega}^k_*\right)\\ &= \lambda b_{Y_m}\left(\pi^k_*, \boldsymbol{\omega}^P_*\right) + \gamma_{\Gamma,k}(\pi^P_*) + \lambda\hat{c}_{Y_m,\Gamma_m}\left(\pi^k_*, \pi^P_*\right) + \lambda^2 d_{Y_m}\left(\pi^k_*, \pi^P_*\right).\end{aligned}$$

Then, combining the second term in $(3.41)_5$ with $(3.31)_2$, gives

$$\lambda d_{Y_m}\left(\pi^k_*, 1\right) = -\lambda^2 d_{Y_m}\left(\pi^P_*, \pi^k_*\right) - \lambda\hat{c}_{Y_m,\Gamma_m}\left(\pi^P_*, \pi^k_*\right) - \lambda b_{Y_m}\left(\pi^k_*, \boldsymbol{\omega}^P_*\right)\,,$$

so that

$$h^{I*}_k(\lambda) = b_Y\left(1, \boldsymbol{\omega}^k_*\right) + \lambda d_{Y_m}\left(\pi^k_*, 1\right) = \frac{1}{\lambda}\gamma_{\Gamma,k}(\lambda\pi^P_*) = \frac{1}{\lambda}g^{I*}_k(\lambda)\,.$$

For the third relationship in (3.42), using (3.30) and due to (3.32), we have

$$\begin{aligned} h^{III*}_{kij} &= a_Y\left(\underset{*}{\boldsymbol{\omega}}^k,\, \boldsymbol{\Pi}^{ij}\right) - \lambda b_{Y_m}\left(\underset{*}{\pi}^k,\, \boldsymbol{\Pi}^{ij}\right) \\ &= \lambda^2 \hat{c}_{Y_m,\Gamma_m}\left(\underset{*}{\pi}^{ij},\, \underset{*}{\pi}^k\right) + \lambda^3 d_{Y_m}\left(\underset{*}{\pi}^{ij},\, \underset{*}{\pi}^k\right) + \lambda^2 b_{Y_m}\left(\underset{*}{\pi}^k,\, \underset{*}{\boldsymbol{\omega}}^{ij}\right) \\ &\quad - \lambda a_Y\left(\underset{*}{\boldsymbol{\omega}}^k,\, \underset{*}{\boldsymbol{\omega}}^{ij}\right) + \lambda^2 b_{Y_m}\left(\underset{*}{\pi}^{ij},\, \underset{*}{\boldsymbol{\omega}}^k\right) \\ &= -\lambda \gamma_{\Gamma,k}(\underset{*}{\pi}^{ij}) = -g^{III*}_{kij}\ . \end{aligned}$$

Obviously $g^{III*}_{kij} = g^{III*}_{kji}$, due to the symmetry $\pi^{ij} = \pi^{ji}$. □

The main result of this section is the *macroscopic* problem, obtained from (3.33)-(3.34) by replacing the integrals over $Y$, $Y_m$ and $\Gamma_m^Y$ by expressions involving the associated homogenized coefficients (the symmetries (3.38) and (3.42) playing an essential role).

*The macroscopic homogeneized problem.* Given $\lambda \in \mathbb{R}_+$, find $\underset{*}{\boldsymbol{u}}$ in $\mathbf{H}^1_0(\Omega) + \underset{*}{\boldsymbol{u}}_0$ and $\underset{*}{P}$ in $H^1(\Omega)$ such that

$$\left\{ \begin{aligned} &\int_\Omega \mathcal{A}^*_{ijkl}(\lambda) e^x_{kl}(\lambda \underset{*}{\boldsymbol{u}})\, e^x_{ij}(\boldsymbol{v}) - \int_\Omega \beta^*_{ij}(\lambda) e^x_{ij}(\boldsymbol{v}) \lambda^2 \underset{*}{P} - \int_\Omega g^{I*}_k(\lambda) \lambda \underset{*}{P} v_k - \int_\Omega g^{II*}_{ik}(\lambda) \lambda \underset{*}{u}_k\, v_i \\ &\qquad\qquad - \int_\Omega v_k g^{III*}_{kji} e^x_{ij}(\lambda \underset{*}{\boldsymbol{u}}) - \int_\Omega e^x_{ij}(\boldsymbol{v}) g^{III*}_{kji} \lambda \underset{*}{u}_k = \int_\Omega \underset{*}{\boldsymbol{f}} \cdot \boldsymbol{v}, \\ &\int_\Omega \mathcal{C}_{ij} \partial_j \underset{*}{P} \partial_i q + \int_\Omega \beta^*_{ij}(\lambda) e^x_{ij}(\lambda \underset{*}{\boldsymbol{u}}) q + \int_\Omega \mathcal{M}^*(\lambda) \lambda \underset{*}{P} q + \int_\Omega \lambda^{-1} g^{I*}_k(\lambda) \underset{*}{u}_k q = 0, \end{aligned} \right. \tag{3.43}$$

for all $\boldsymbol{v} \in \mathbf{H}^1_0(\Omega)$ and $q \in H^1(\Omega)$.

PROPOSITION 3.8. *There exists $\lambda_0 > 0$ such that for every $\lambda \in \Lambda_0 \equiv ]0, \lambda_0]$,*

*(i)* $\mathcal{M}^*(\lambda) > 0$,

*(ii) problem (3.43) is coercive.*

*Proof.* (i) Using appropriate test functions in (3.31), we obtain

$$b_Y\left(1,\, \underset{*}{\boldsymbol{\omega}}^P\right) - \lambda d_{Y_m}\left(1,\, \underset{*}{\pi}^P\right) = a_Y\left(\underset{*}{\boldsymbol{\omega}}^P,\, \underset{*}{\boldsymbol{\omega}}^P\right) + \lambda \hat{c}_{Y_m,\Gamma_m}\left(\underset{*}{\pi}^P,\, \underset{*}{\pi}^P\right) + \lambda^2 d_{Y_m}\left(\underset{*}{\pi}^P,\, \underset{*}{\pi}^P\right) \geq 0\ .$$

Hence problem (3.31) is coercive, so that for any $\lambda > 0$ its solution is unique and bounded. Thus, there exists $\lambda_0 > 0$ such that

$$d_{Y_m}(1,\, 1) - 2\lambda_0 |d_{Y_m}\left(1,\, \underset{*}{\pi}^P(\lambda_0)\right)| > 0\ .$$

This implies that $\mathcal{M}^*(\lambda) > 0$ for $\lambda \in \Lambda_0$. Indeed, $(3.35)_2$ now yields

$$\begin{aligned} \mathcal{M}^*(\lambda) =& \fint_{Y_c} \frac{1}{\mu^c} + d_{Y_m}(1,\, 1) + \lambda d_{Y_m}\left(1,\, \underset{*}{\pi}^P\right) + b_Y\left(1,\, \underset{*}{\boldsymbol{\omega}}^P\right) \\ =& \fint_{Y_c} \frac{1}{\mu^c} + d_{Y_m}(1,\, 1) + 2\lambda d_{Y_m}\left(1,\, \underset{*}{\pi}^P\right) + b_Y\left(1,\, \underset{*}{\boldsymbol{\omega}}^P\right) - \lambda d_{Y_m}\left(1,\, \underset{*}{\pi}^P\right) > 0. \end{aligned}$$

(ii) Upon substituting in (3.43) $\boldsymbol{v} := \underset{*}{\boldsymbol{u}} - \underset{*}{\boldsymbol{u}}_0$ and $q := \underset{*}{P}$ and summing the two identities, the proof is based on the following symmetries, $g^{II*}_{kl} = g^{II*}_{lk}$, $\mathcal{A}^*_{ijkl} = \mathcal{A}^*_{klij}$ and $\mathcal{C}_{ij} = \mathcal{C}_{ji}$, and on positive definiteness of $\mathcal{C}_{ji}$, on the positivity of $\mathcal{M}^*(\lambda)$, as shown above, and positive definiteness of

$$\mathbb{A} = \begin{pmatrix} \mathcal{A}^*_{ijkl} & -g^{III*}_{sij} \\ -g^{III*}_{rkl} & -g^{II*}_{rs} \end{pmatrix}\ .$$

To see it, we rewrite $g^{II*}_{rs}$ and $g^{III*}_{sij}$ using the corrector problems, as follows:

$$\begin{aligned} g^{II*}_{kl} &= \lambda a_Y\left(\underset{*}{\boldsymbol{\omega}}^k,\, \underset{*}{\boldsymbol{\omega}}^l\right) + \lambda^2 \left[\hat{c}_{Y_m,\Gamma_m}\left(\underset{*}{\pi}^k,\, \underset{*}{\pi}^l\right) + d_{Y_m}\left(\underset{*}{\pi}^k,\, \underset{*}{\pi}^l\right)\right], \\ g^{III*}_{kij} &= \lambda a_Y\left(\lambda^{-1}\boldsymbol{\Pi}^{ij} + \underset{*}{\boldsymbol{\omega}}^{ij},\, \underset{*}{\boldsymbol{\omega}}^k\right) + \lambda^2 \left[\hat{c}_{Y_m,\Gamma_m}\left(\underset{*}{\pi}^{ij},\, \underset{*}{\pi}^k\right) + d_{Y_m}\left(\underset{*}{\pi}^{ij},\, \underset{*}{\pi}^k\right)\right]. \end{aligned}$$

Further, let us introduce $\boldsymbol{W}(\underset{*}{\boldsymbol{u}}) := (\underset{*}{\boldsymbol{\omega}}^{ij} + \lambda^{-1}\boldsymbol{\Pi}^{ij})e_{ij}(\underset{*}{\boldsymbol{u}}) + \underset{*}{\boldsymbol{\omega}}^{k}\underset{*}{u}_{k}$ and $Q(\underset{*}{\boldsymbol{u}}) := \underset{*}{\pi}^{ij}e_{ij}(\underset{*}{\boldsymbol{u}}) + \underset{*}{\pi}^{k}\underset{*}{u}_{k}$. Now we can see that the positive definiteness of $\mathbb{A}$ results form the ellipticity of $a_Y(\cdot,\cdot)$, $d_{Y_m}(\cdot,\cdot)$ and $\hat{c}_{Y_m,\Gamma_m}(\cdot,\cdot)$. Indeed, there are $m', m > 0$ such that for any $\underset{*}{\boldsymbol{u}}$ and a.e. $x \in \Omega$,

$$\begin{aligned}[e(\underset{*}{\boldsymbol{u}}), \boldsymbol{u}]^T \mathbb{A}[e(\underset{*}{\boldsymbol{u}}), \boldsymbol{u}] &= \lambda a_Y(\boldsymbol{W}, \boldsymbol{W}) + \lambda^2\Big[\hat{c}_{Y_m,\Gamma_m}(Q, Q) + d_{Y_m}(Q, Q)\Big] \\ &\geq m'(\|e^y_{ij}(\boldsymbol{W})\|^2_{L^2(Y)} + \|\nabla_y(Q)\|^2_{\mathbf{L}^2(Y)}) \\ &\geq m(|e^x_{ij}(\underset{*}{\boldsymbol{u}})|^2 + |\underset{*}{\boldsymbol{u}}|^2)\,.\end{aligned}$$

To see the second inequality, due to the uniqueness of solutions to (3.30) and (3.32), for nonvanishing $e^x_{ij}(\underset{*}{\boldsymbol{u}})$ or $\underset{*}{\boldsymbol{u}}$, $\boldsymbol{W}$ and $Q$ cannot be identically zero. □

**Remark 3.** The macroscopic problem (3.43) can rewritten in a form involving the standard pressure. On multiplying $(3.43)_2$ by $\lambda$ and recalling (2.18) (i.e. $\underset{*}{p} = \lambda \underset{*}{P}$), the macroscopic $\mathcal{L}$-transformed problem reads as follows: given $\lambda \in \mathbb{C}$, find $\underset{*}{\boldsymbol{u}} \in \mathbf{H}^1_0(\Omega)$ and $\underset{*}{p} \in H^1(\Omega)$ such that

$$\begin{cases} \displaystyle\int_\Omega [\lambda \mathcal{A}^*_{ijkl}(\lambda)]e^x_{kl}(\underset{*}{\boldsymbol{u}})\, e^x_{ij}(\boldsymbol{v}) - \int_\Omega \alpha^*_{ij}(\lambda)e^x_{ij}(\boldsymbol{v})\underset{*}{p} - \int_\Omega g^{I*}_k(\lambda)\underset{*}{p}v_k - \int_\Omega g^{II*}_{ik}(\lambda)\,\lambda\underset{*}{u}_k\, v_i \\ \qquad\qquad - \displaystyle\int_\Omega v_k g^{III*}_{kji} e^x_{ij}(\lambda\underset{*}{\boldsymbol{u}}) - \int_\Omega e^x_{ij}(\boldsymbol{v})g^{III*}_{kji}\lambda\underset{*}{u}_k = \int_\Omega \underset{*}{\boldsymbol{f}}\cdot\boldsymbol{v}, \\ \displaystyle\int_\Omega \mathcal{C}_{ij}\partial_j\underset{*}{p}\partial_i q + \int_\Omega \alpha^*_{ij}(\lambda)e^x_{ij}(\lambda\underset{*}{\boldsymbol{u}})q + \int_\Omega \mathcal{M}^*(\lambda)\lambda\underset{*}{p}q + \int_\Omega g^{I*}_k(\lambda)\lambda\underset{*}{u}_k q = 0, \end{cases}$$

for all $\boldsymbol{v} \in \mathbf{H}^1_0(\Omega)$ and $q \in H^1(\Omega)$. □

**4 Homogenization of the model with strongly discontinuous (SD) data** In this section we consider coefficients $\alpha^\varepsilon_{ij}$ defined according to (2.24). We show, see Remark 4, that in this case the standard form of the volume forces treated in the *weakly discontinuous case* is not relevant and leads to a *vanishing solution*. Correspondingly to the jump of the pressure and thereby also in the total stress on $\Gamma^\varepsilon_m$, the local equilibrium can be preserved, if the forces are scaled w.r.t. the heterogeneities. The following forms of the scale-dependent forces $\boldsymbol{f}^\varepsilon = (f^\varepsilon_i)$ will be considered:

- Case AF – progressively increasing magnitude of imposed forces as $\varepsilon \to 0$

$$\|\boldsymbol{f}^\varepsilon\|_{L^2(0,T;\mathbf{H}^1(\Omega))} \leq \frac{C}{\varepsilon}\,, \quad \boldsymbol{f}^\varepsilon(t,x) = \frac{1}{\varepsilon}\bar{\boldsymbol{f}}(t,x), \quad \|\bar{\boldsymbol{f}}\|_{L^2(0,T;\mathbf{H}^1(\Omega))} \leq C. \tag{4.1}$$

- Case BF – forces containing the Dirac distribution $\delta_{\Gamma^\varepsilon_m}(x)$ on $\Gamma^\varepsilon_m$ and such that

$$\begin{aligned} &\boldsymbol{f}^\varepsilon(t,x) = \bar{\boldsymbol{f}}(t,x) + \delta_{\Gamma^\varepsilon_m}(x)\tilde{\boldsymbol{f}}^\varepsilon(t,x)\,, \quad \bar{\boldsymbol{f}} \in L^2(0,T;\mathbf{L}^2(\Omega)), \\ &\mathcal{T}_\varepsilon\Big(\tilde{\boldsymbol{f}}^\varepsilon\Big) \to \tilde{\boldsymbol{f}} \quad \text{strongly in } L^2((0,T)\times\Omega;\mathbf{L}^2(\Gamma^Y_m)), \\ &\tilde{\boldsymbol{f}} \in L^2((0,T)\times\Omega;\mathbf{L}^2(\Gamma^Y_m)), \quad \|\tilde{\boldsymbol{f}}\|_{L^2((0,T)\times\Omega;\mathbf{L}^2(\Gamma^Y_m))} \leq C\,, \\ &\int_{\Gamma^Y_m}\tilde{\boldsymbol{f}}(t,x,y)\,dS_y = \hat{\boldsymbol{F}}(t,x)\,, \quad \|\hat{\boldsymbol{F}}\|_{L^2((0,T);\mathbf{H}^1(\Omega))} \leq C. \end{aligned} \tag{4.2}$$

- Case CF – forces acting on $\Gamma^\varepsilon_m$ with "zero average", satisfying (4.2) and such that

$$\hat{\boldsymbol{F}}(t,x) \equiv 0 \quad \text{for a.a. } x \in \Omega\,, t \in ]0,T[. \tag{4.3}$$

- Case DF – progressively increasing gradients of imposed forces with $\varepsilon \to 0$

$$\begin{aligned} &\|\boldsymbol{f}^\varepsilon\|_{L^2(0,T;\mathbf{H}^1(\Omega))} \leq \frac{C}{\varepsilon}\,, \quad \|\boldsymbol{f}^\varepsilon\|_{L^2(0,T;\mathbf{L}^2(\Omega))} \leq C\,, \\ &\mathcal{T}_\varepsilon(\boldsymbol{f}^\varepsilon) \to \tilde{\boldsymbol{f}} \quad \text{strongly in } L^2((0,T)\times\Omega;\mathbf{L}^2(Y)). \end{aligned} \tag{4.4}$$

**4.1 A priori estimates** We now give a priori estimates for all the cases of external forces (4.1)–(4.4). The strong discontinuity affects the "off-diagonal" interface integrals which, by making use of the standard treatment (3.3)-(3.5), disappear from the principal inequality. The crucial role is played by Proposition 4.1 which allows to involve the interface integral forms in the estimates. The results obtained in this section are summarized in Proposition 4.2.

PROPOSITION 4.1. *Let $(\boldsymbol{u}^\varepsilon, P^\varepsilon)$ be solution to (2.19), where $\alpha^\varepsilon$ is defined according to (2.24). Then there exists a constant $C_{\hat{Q}}$ such that*

$$\|\mathcal{M}^\varepsilon_Y(\boldsymbol{u}^\varepsilon)\|_{\mathbf{H}^{-1}(\Omega)} \leq \varepsilon C_{\hat{Q}} \left( \|\nabla \boldsymbol{u}^\varepsilon\|_{\mathbf{L}^2(\Omega)} + \|\frac{\mathrm{d}\, P^\varepsilon}{\mathrm{d}\, t}\|_{L^2(\Omega)} + \varepsilon \|\nabla P^\varepsilon\|_{\mathbf{L}^2(\Omega^\varepsilon_m)} + \sqrt{\varepsilon}\|[P^\varepsilon]_{\Gamma^\varepsilon_m}\|_{L^2(\Gamma^\varepsilon_m)} \right) . \tag{4.5}$$

*Proof.* There exists $\hat{Q} \in W^{1,\infty}(Y \setminus \Gamma^Y_m)$ such that

$$\hat{Q}(y) = 0 \quad \text{for } y \in \overline{Y}_c, \quad \text{and} \quad \fint_{\Gamma^Y_m} \tilde{\alpha}^\Gamma_{ij} n_j \left[\hat{Q}\right]_{\Gamma^Y_m} = 1. \tag{4.6}$$

Assertion (4.5) follows from $(2.19)_2$ written for the test function $q^\varepsilon(x) = \hat{Q}(\{x/\varepsilon\})\theta(x)$ with $\theta \in H^1_0(\Omega)$. Using the decomposition $\mathcal{T}_\varepsilon(\boldsymbol{u}^\varepsilon) = (\mathcal{T}_\varepsilon(\boldsymbol{u}^\varepsilon) - \mathcal{M}^\varepsilon_Y(\boldsymbol{u}^\varepsilon)) + \mathcal{M}^\varepsilon_Y(\boldsymbol{u}^\varepsilon)$ in the interface integral $(2.19)_2$, yields

$$\begin{aligned} \Big| \int_\Omega \mathcal{M}^\varepsilon_Y(u^\varepsilon_i) \frac{\theta}{\varepsilon|Y|} \int_{\Gamma^Y_m} n_j \tilde{\alpha}^\Gamma_{ij} \left[\hat{Q}\right]_{\Gamma^Y_m} dS_y \Big| \\ \leq \Big| \int_\Omega \frac{\theta}{\varepsilon|Y|} \int_{\Gamma^Y_m} n_j \tilde{\alpha}^\Gamma_{ij} (\mathcal{T}^b_\varepsilon(u^\varepsilon_i) - \mathcal{M}^\varepsilon_Y(u^\varepsilon_i)) \left[\hat{Q}\right]_{\Gamma^Y_m} dS_y \Big| \\ + \Big| \int_\Omega q^\varepsilon\, \alpha^\varepsilon_{ij} e_{ij}(\boldsymbol{u}^\varepsilon) \Big| + \Big| \int_{\Omega\setminus\Gamma^\varepsilon_m} K^\varepsilon_{ij} \partial_j P^\varepsilon\, \partial_j q^\varepsilon \Big| \\ + \Big| \int_\Omega \frac{1}{\mu^\varepsilon} \frac{\mathrm{d}\, P^\varepsilon}{\mathrm{d}\, t}\, q^\varepsilon \Big| + \Big| \int_{\Gamma^\varepsilon_m} \varkappa^\varepsilon\, [P^\varepsilon]_{\Gamma^\varepsilon_m} \left[q^\varepsilon\right]_{\Gamma^\varepsilon_m} dS \Big|. \end{aligned} \tag{4.7}$$

We now estimate all the right-hand side integrals in this inequality. Due to the Poincaré–Wirtinger inequality and since $\nabla_y(\mathcal{T}_\varepsilon(\boldsymbol{u}^\varepsilon) - \mathcal{M}^\varepsilon_Y(\boldsymbol{u}^\varepsilon)) = \nabla_y \mathcal{T}_\varepsilon(\boldsymbol{u}^\varepsilon) = \varepsilon \mathcal{T}_\varepsilon(\nabla_x \boldsymbol{u}^\varepsilon)$,

$$\|\mathcal{T}^b_\varepsilon(\boldsymbol{u}^\varepsilon) - \mathcal{M}^\varepsilon_Y(\boldsymbol{u}^\varepsilon)\|_{\mathbf{L}^2(\Omega;\mathbf{H}^1(Y))} \leq \varepsilon C \|\nabla \boldsymbol{u}^\varepsilon\|_{\mathbf{L}^2(\Omega)}. \tag{4.8}$$

Then, by using the trace theorem to estimate $\left[\hat{Q}\right]_{\Gamma^Y_m}$ and $(\mathcal{T}^b_\varepsilon(\boldsymbol{u}^\varepsilon) - \mathcal{M}^\varepsilon_Y(\boldsymbol{u}^\varepsilon))$ on $\Gamma^Y_m$, we obtain

$$\begin{aligned} \Big| \int_\Omega \frac{\theta}{\varepsilon|Y|} \int_{\Gamma^Y_m} n_j \tilde{\alpha}^\Gamma_{ij} (\mathcal{T}^b_\varepsilon(u^\varepsilon_i) - \mathcal{M}^\varepsilon_Y(u^\varepsilon_i)) \left[\hat{Q}\right]_{\Gamma^Y_m} dS_y \Big| \\ \leq C \|\nabla \boldsymbol{u}^\varepsilon\|_{\mathbf{L}^2(\Omega)} \|\theta\|_{L^2(\Omega)} \|\hat{Q}\|_{W^{1,\infty}(Y_m)}. \end{aligned} \tag{4.9}$$

Since

$$\nabla(\theta(x)\hat{Q}(\{x/\varepsilon\})) = \hat{Q}(y)\nabla_x\theta(x) + \theta(x)\varepsilon^{-1}\nabla_y \hat{Q}(y), \ \text{ for } y = \left\{\frac{x}{\varepsilon}\right\}$$

where $\hat{Q}(y) = 0$ outside $\Omega^\varepsilon_m$, as follows due to $(4.6)_1$, we get

$$\Big| \int_{\Omega\setminus\Gamma^\varepsilon_m} K^\varepsilon_{ij} \partial_j P^\varepsilon\, \partial_j q^\varepsilon \Big| \leq \varepsilon C \|\nabla P^\varepsilon\|_{\mathbf{L}^2(\Omega^\varepsilon_m)} \|\hat{Q}\|_{W^{1,\infty}(Y_m)} \|\theta\|_{H^1_0(\Omega)}\, , \tag{4.10}$$

where we used the fact that $K^\varepsilon_{ij} \approx \varepsilon^2$ in $\Omega^\varepsilon_m$. In order to estimate the last integral in (4.7), we employ the standard inequality

$$\varepsilon \|\theta\|^2_{L^2(\Gamma^\varepsilon_m)} \leq C \|\theta\|^2_{L^2(\Omega^\varepsilon_m)} + C\, \varepsilon^2 \|\nabla\theta\|^2_{\mathbf{L}^2(\Omega^\varepsilon_m)},$$

thus, we get

$$\begin{aligned}\int_{\Gamma^\varepsilon_m}\varepsilon([q^\varepsilon]_{\Gamma^\varepsilon_m})^2dS&=\int_{\Gamma^\varepsilon_m}\varepsilon(\theta(x)[\hat Q(x/\varepsilon)]_{\Gamma^\varepsilon_m})^2dS\le C\|\hat Q\|^2_{W^{1,\infty}(Y_m)}\int_{\Gamma^\varepsilon_m}\varepsilon|\theta(x)|^2\\&\le C\|\hat Q\|^2_{W^{1,\infty}(Y_m)}\Big(\|\theta\|^2_{L^2(\Omega^\varepsilon_m)}+\varepsilon^2\|\nabla\theta\|^2_{\mathbf{L}^2(\Omega^\varepsilon_m)}\Big)\\&\le C'\|\hat Q\|^2_{W^{1,\infty}(Y_m)}\|\nabla\theta\|^2_{\mathbf{L}^2(\Omega^\varepsilon_m)}\,.\end{aligned}$$

Recalling (2.22), i.e. $\varkappa^\varepsilon\approx\varepsilon$, this inequality yields the estimate

$$\int_{\Gamma^\varepsilon_m}\varkappa^\varepsilon[P^\varepsilon]_{\Gamma^\varepsilon_m}[q^\varepsilon]_{\Gamma^\varepsilon_m}dS\le\sqrt{\varepsilon}C'_{\hat Q}\|[P^\varepsilon]_{\Gamma^\varepsilon_m}\|_{L^2(\Gamma^\varepsilon_m)}\|\nabla\theta\|_{\mathbf{L}^2(\Omega^\varepsilon_m)}\,,\tag{4.11}$$

where $C'_{\hat Q}$ depends on $\|\hat Q\|_{W^{1,\infty}(Y_m)}$.

The estimates of the other integrals in the right-hand side of (4.7) are straightforward. Finally, using (4.9),(4.10) and (4.11) we obtain

$$\begin{aligned}&\Big|\int_\Omega\mathcal{M}^\varepsilon_Y(u^\varepsilon_i)\frac{\theta}{\varepsilon|Y|}\int_{\Gamma^Y_m}n_j\tilde\alpha^\Gamma_{ij}\left[\hat Q\right]_{\Gamma^Y_m}dS_y\Big|\\&\le C_{\hat Q}\|\theta\|_{H^1_0(\Omega)}\left(\|\nabla\boldsymbol{u}^\varepsilon\|_{\mathbf{L}^2(\Omega)}+\|\frac{\mathrm{d}\,P^\varepsilon}{\mathrm{d}\,t}\|_{L^2(\Omega)}+\varepsilon\|\nabla P^\varepsilon\|_{\mathbf{L}^2(\Omega^\varepsilon_m)}+\sqrt{\varepsilon}\|[P^\varepsilon]_{\Gamma^\varepsilon_m}\|_{L^2(\Gamma^\varepsilon_m)}\right)\,,\end{aligned}\tag{4.12}$$

from where we deduce boundedness of $\mathcal{M}^\varepsilon_Y(\boldsymbol{u}^\varepsilon)$ in the dual space $\mathbf{H}^{-1}(\Omega)$. Indeed, $(4.6)_3$ used in (4.7), hence in (4.12), yields assertion (4.5). □ Below we shall consider the load Cases AF to BF defined in (4.1)-(4.4). In Cases AF and DF, we will make use of the following preliminary estimate obtained for $\boldsymbol{f}^\varepsilon$ is in $L^2(0,T;\mathbf{H}^1(\Omega))$,

$$\begin{aligned}\int_\Omega\boldsymbol{f}^\varepsilon\cdot\boldsymbol{u}^\varepsilon&=\int_\Omega\boldsymbol{f}^\varepsilon\cdot\mathcal{M}^\varepsilon_Y(\boldsymbol{u}^\varepsilon)+\int_\Omega\boldsymbol{f}^\varepsilon\cdot(\boldsymbol{u}^\varepsilon-\mathcal{M}^\varepsilon_Y(\boldsymbol{u}^\varepsilon))\\&\le\|\boldsymbol{f}^\varepsilon\|_{\mathbf{H}^1_0(\Omega)}\|\mathcal{M}^\varepsilon_Y(\boldsymbol{u}^\varepsilon)\|_{\mathbf{H}^{-1}(\Omega)}+\varepsilon C\|\boldsymbol{f}^\varepsilon\|_{\mathbf{L}^2(\Omega)}\|\nabla\boldsymbol{u}^\varepsilon\|_{\mathbf{L}^2(\Omega)}\,,\end{aligned}\tag{4.13}$$

for a.e. $t\in]0,T[$. We employed (4.8) to derive (4.13).

**4.1.1 Case AF** Due to Proposition 4.1, recalling (4.1), we substitute (4.5) into (4.13) and integrate in time to get

$$\begin{aligned}\int_0^T\int_\Omega\boldsymbol{f}^\varepsilon\cdot\boldsymbol{u}^\varepsilon\,dt&\le\varepsilon C\|\boldsymbol{f}^\varepsilon\|_{L^2(0,T;\mathbf{H}^1(\Omega))}\Big(\|\nabla\boldsymbol{u}^\varepsilon\|_{L^2(0,T;\mathbf{L}^2(\Omega))}+\Big\|\frac{\mathrm{d}\,P^\varepsilon}{\mathrm{d}\,t}\Big\|_{L^2(0,T;L^2(\Omega))}\Big)\\&\quad+\varepsilon C\|\boldsymbol{f}^\varepsilon\|_{L^2(0,T;\mathbf{H}^1(\Omega))}\Big(\varepsilon\|\nabla P^\varepsilon\|_{L^2(0,T;\mathbf{L}^2(\Omega^\varepsilon_m))}+\sqrt{\varepsilon}\|[P^\varepsilon]_{\Gamma^\varepsilon_m}\|_{L^2(0,T;L^2(\Gamma^\varepsilon_m))}\Big)\\&\le\frac{C}{2\nu}\varepsilon^2\|\boldsymbol{f}^\varepsilon\|^2_{L^2(0,T;\mathbf{H}^1(\Omega))}+\frac{\nu C_1}{2}\Big(\|\nabla\boldsymbol{u}^\varepsilon\|^2_{L^2(0,T;\mathbf{L}^2(\Omega))}+\Big\|\frac{\mathrm{d}\,P^\varepsilon}{\mathrm{d}\,t}\Big\|^2_{L^2(0,T;L^2(\Omega))}\Big)\\&\quad+\frac{\nu C_1}{2}\Big(\varepsilon^2\|\nabla P^\varepsilon\|^2_{L^2(0,T;\mathbf{L}^2(\Omega^\varepsilon_m))}+\varepsilon\|[P^\varepsilon]_{\Gamma^\varepsilon_m}\|^2_{L^2(0,T;L^2(\Gamma^\varepsilon_m))}\Big).\end{aligned}\tag{4.14}$$

Then the estimates of $(\boldsymbol{u}^\varepsilon,P^\varepsilon)$ can be obtained from (4.14) with the force defined in (4.1). For this, we proceed formally as in the *weakly discontinuous case*, using the Korn inequality to deal with $|e_{ij}(\boldsymbol{u}^\varepsilon)|^2$ and combining (4.14) with (3.5), with a $\nu$ chosen appropriately. This leads to estimates (3.7).

In addition, due to (4.5), we obtain another important estimate, namely

$$\|\mathcal{M}^\varepsilon_Y(\boldsymbol{u}^\varepsilon)\|_{L^2(0,T;\mathbf{H}^{-1}(\Omega))}\le\varepsilon C\,.\tag{4.15}$$

**4.1.2 Case BF** In this case we cannot use directly (4.13). However, the following analogous inequality can be obtained by virtue of definition (4.2),

$$\begin{aligned}
&\int_\Omega \boldsymbol{f}^\varepsilon \cdot \boldsymbol{u}^\varepsilon = \int_\Omega \bar{\boldsymbol{f}} \cdot \boldsymbol{u}^\varepsilon + \int_{\Gamma_m^\varepsilon} \tilde{\boldsymbol{f}}^\varepsilon \cdot \boldsymbol{u}^\varepsilon \\
&\le \|\bar{\boldsymbol{f}}\|_{\mathbf{L}^2(\Omega)} \|\boldsymbol{u}^\varepsilon\|_{\mathbf{L}^2(\Omega)} \\
&\quad + \frac{1}{\varepsilon}\|\hat{\boldsymbol{F}}\|_{\mathbf{H}^1(\Omega)} \|\mathcal{M}_Y^\varepsilon(\boldsymbol{u}^\varepsilon)\|_{\mathbf{H}^{-1}(\Omega)} + \frac{1}{\varepsilon}\|\tilde{\boldsymbol{f}}\|_{\mathbf{L}^2(\Omega\times\Gamma_m^Y)} \|\mathcal{T}_\varepsilon(\boldsymbol{u}^\varepsilon) - \mathcal{M}_Y^\varepsilon(\boldsymbol{u}^\varepsilon)\|_{\mathbf{L}^2(\Omega\times\Gamma_m^Y)} \\
&\le C_f \left( \frac{1}{\varepsilon}\|\mathcal{M}_Y^\varepsilon(\boldsymbol{u}^\varepsilon)\|_{\mathbf{H}^{-1}(\Omega)} + \|\nabla \boldsymbol{u}^\varepsilon\|_{\mathbf{L}^2(\Omega)} \right) ,
\end{aligned} \tag{4.16}$$

where we used the Poincaré and the Poincaré–Wirtinger inequalities, and (4.8).

We now proceed as in (4.14). By (4.5) one obtains

$$\begin{aligned}
\int_0^T \int_\Omega \boldsymbol{f}^\varepsilon \cdot \boldsymbol{u}^\varepsilon \, dt \le\ & \sqrt{T} C_f \Big( \|\nabla \boldsymbol{u}^\varepsilon\|_{L^2(0,T;\mathbf{L}^2(\Omega))} + \Big\| \frac{\mathrm{d}\, P^\varepsilon}{\mathrm{d}\, t} \Big\|_{L^2(0,T;L^2(\Omega))} \Big) \\
&+ \sqrt{T} C_f \Big( \varepsilon \|\nabla P^\varepsilon\|_{L^2(0,T;\mathbf{L}^2(\Omega_m^\varepsilon))} + \sqrt{\varepsilon} \|[P^\varepsilon]_{\Gamma_m^\varepsilon}\|_{L^2(0,T;L^2(\Gamma_m^\varepsilon))} \Big) \\
\le\ & \frac{T C_f^2}{2\nu} + \frac{\nu C_1}{2} \Big( \|\nabla \boldsymbol{u}^\varepsilon\|^2_{L^2(0,T;\mathbf{L}^2(\Omega))} + \| \frac{\mathrm{d}\, P^\varepsilon}{\mathrm{d}\, t} \|^2_{L^2(0,T;L^2(\Omega))} \Big) \\
&+ \frac{\nu C_1}{2} \Big( \varepsilon^2 \|\nabla P^\varepsilon\|^2_{L^2(0,T;\mathbf{L}^2(\Omega_m^\varepsilon))} + \varepsilon \|[P^\varepsilon]_{\Gamma_m^\varepsilon}\|^2_{L^2(0,T;L^2(\Gamma_m^\varepsilon))} \Big) .
\end{aligned}$$

Thus, we get estimates (3.7) and (4.15), as in the Case AF.

**4.1.3 Case CF** Since (4.3) holds, (4.16) becomes simply

$$\begin{aligned}
\int_\Omega \boldsymbol{f}^\varepsilon \cdot \boldsymbol{u}^\varepsilon &\le \|\bar{\boldsymbol{f}}\|_{\mathbf{L}^2(\Omega)} \|\boldsymbol{u}^\varepsilon\|_{\mathbf{L}^2(\Omega)} + \frac{1}{\varepsilon}\|\tilde{\boldsymbol{f}}\|_{\mathbf{L}^2(\Omega\times\Gamma_m^Y)} \|\boldsymbol{u}^\varepsilon - \mathcal{M}_Y^\varepsilon(\boldsymbol{u}^\varepsilon)\|_{\mathbf{L}^2(\Omega\times\Gamma_m^Y)} \\
&\le C_f \|\nabla \boldsymbol{u}^\varepsilon\|_{\mathbf{L}^2(\Omega)} .
\end{aligned}$$

Hence,

$$\int_0^T \int_\Omega \boldsymbol{f}^\varepsilon \cdot \boldsymbol{u}^\varepsilon \, dt \le \frac{T C_f^2}{2\nu} + \frac{\nu C_1}{2} \|\nabla \boldsymbol{u}^\varepsilon\|^2_{L^2(0,T;\mathbf{L}^2(\Omega))} ,$$

which again yields estimates (3.7) and (4.15).

**4.1.4 Case DF** Invoking directly (4.13), (4.14) is satisfied and, consequently, estimates (3.7) and (4.15) as well.

**4.1.5 Main result on the a priori estimates** Here we summarize the results obtained for all the cases of volume forces considered above.

PROPOSITION 4.2. *Let $(\boldsymbol{u}^\varepsilon, P^\varepsilon)$ be solution to (2.19), where $\alpha^\varepsilon$ is defined by (2.24). Then, for all volume forces specified attaining one of the form (4.1)–(4.4), estimates (3.7) hold and*

$$\|\mathcal{M}_Y^\varepsilon(\boldsymbol{u}^\varepsilon)\|_{\mathbf{H}^{-1}(\Omega)} \le \varepsilon C . \tag{4.17}$$

*Moreover, if $\|\boldsymbol{f}^\varepsilon\|_{L^2(0,T;\mathbf{H}^1(\Omega))} \le C_f$ (e.g. if in Case AF, in (3.1)$_2$ $\boldsymbol{f}^\varepsilon = \bar{\boldsymbol{f}}$), then in estimates (3.7) the generic constant $C$ is proportional to $\varepsilon$, i.e. $O_\varepsilon(C) = \varepsilon$.*

**4.2 Convergence result and limit problems** We shall obtain the limit representation of model (2.19) for the *strongly discontinuous case*. For this we follow the procedure explained in Section 3.2, namely we use the pressure extension $\widetilde{P^\varepsilon}$ and its consequences from the proof of Lemma 3.3. We shall also need the *space of discontinuous unfolded functions* given in (3.8).

LEMMA 4.3. *Due to estimates (3.7), there exist the limit fields $\boldsymbol{u}^1 \in L^2(]0,T[\times\Omega; \mathbf{H}^1_\#(Y))$, $\bar{\boldsymbol{u}} \in L^2(0,T;\mathbf{H}^{-1}(\Omega))$, $P \in L^\infty(0,T;L^2(\Omega))$, $P^1 \in L^\infty(0,T;L^2(\Omega; H^1_\#(Y_c)))$, and $\widehat{P} \in L^\infty(0,T;L^2(\Omega; H_{\#c0}(Y,\Gamma_m^Y)))$ such that*

*(i) (3.9)$_{3,4,5}$ is satisfied and convergences (3.11) of $P^\varepsilon$ hold,*
*(ii) (3.9)$_{1,2}$ is satisfied and the displacement $\boldsymbol{u}^\varepsilon$ converges (in the sense of subsequences), as follows:*

$$\begin{aligned}
&\boldsymbol{u}^\varepsilon \rightharpoonup 0 \quad \text{weakly in } \mathbf{L}^2(0,T;\mathbf{H}^1_0(\Omega))\,,\\
&\frac{1}{\varepsilon}\left(\mathcal{T}_\varepsilon(\boldsymbol{u}^\varepsilon)-\mathcal{M}^\varepsilon_Y(\boldsymbol{u}^\varepsilon)\right)\rightharpoonup \boldsymbol{u}^1(t,x,y) \quad \text{weakly in } \mathbf{L}^2(0,T;\mathbf{L}^2(\Omega\times Y)),\\
&\mathcal{T}_\varepsilon(\nabla\boldsymbol{u}^\varepsilon)\rightharpoonup \nabla_y\boldsymbol{u}^1(t,x,y) \quad \text{weakly in } \mathbf{L}^2(0,T;\mathbf{L}^2(\Omega\times Y)),\\
&\frac{1}{\varepsilon}\mathcal{M}^\varepsilon_Y(\boldsymbol{u}^\varepsilon)\rightharpoonup \bar{\boldsymbol{u}} \quad \text{weakly in } \mathbf{L}^2(0,T;\mathbf{H}^{-1}(\Omega))\,,
\end{aligned}\tag{4.18}$$

*where* $\fint_Y \boldsymbol{u}^1(t,x,y)\,dy=0$.

**Remark 4.** When standard form of the loading forces is considered, all the limit fields considered above *vanish a priory* ( consequence of the last assertion in Proposition 4.2). □

*Proof.* of Lemma 4.3. Due to estimates (3.7), convergences (3.11) hold also for the case of strongly discontinuous data. Therefore, the point (i) of the lemma follows when using the same arguments as those from the proof of Lemma 3.3. As a simple consequence of Proposition 4.2, (4.17) yields (4.18)$_5$. Moreover, since $\|\boldsymbol{u}^\varepsilon(t,\cdot)\|_{\mathbf{L}^2(\Omega)}\le C$, the macroscopic displacements must vanish, i.e.

$$\mathcal{M}^\varepsilon_Y(\boldsymbol{u}^\varepsilon)\rightharpoonup 0 \quad \text{weakly in } L^2(0,T;\mathbf{L}^2(\Omega))\,.\tag{4.19}$$

The remainder of assertion (i) is the standard result of the periodic unfolding, cf. [10]. □

The main result of this section is summarized in Theorem 4.4 bellow, where all Cases AF–DF are considered. To state it, we need to introduce the linear form $G(x):\mathbf{H}^1(Y)\to\mathbb{R}$ defined for a.e. $x\in\Omega$, as follows:

$$\begin{aligned}
&\text{Case AF:}\quad G(\boldsymbol{v})(x)=\bar{\boldsymbol{f}}(x)\cdot\fint_Y \boldsymbol{v}(y)\,,\\
&\text{Cases BF and CF:}\quad G(\boldsymbol{v})(x)=\fint_{\Gamma^Y_m}\tilde{\boldsymbol{f}}(x,y)\cdot\boldsymbol{v}(y)\,dS_y\,,\\
&\text{Case DF:}\quad G(\boldsymbol{v})\equiv 0\,.
\end{aligned}$$

THEOREM 4.4. *Let $(\boldsymbol{u}^\varepsilon,P^\varepsilon)$ be solution of problem (2.19) where $\alpha^{\Gamma,\varepsilon}$ on $\Gamma^\varepsilon_m$ is given by (2.24). Then there exist the limit fields defined in* (3.9)*, such that convergences (3.10)-(3.11) hold and the limit fields $(\bar{\boldsymbol{u}},P)$ and $(\boldsymbol{u}^1,\widehat{P},P^1)$ satisfy for a.e. $x\in\Omega$ the following two identities (in the sense of time distributions):*

$$\begin{aligned}
&\left\langle\bar{\boldsymbol{v}},\left(\bar{g}_i+\fint_{\Gamma^Y_m}\tilde{\alpha}^\Gamma_{ij}n_j\left[\frac{\mathrm{d}}{\mathrm{d}\,t}\widehat{P}\right]_{\Gamma^Y_m}dS_y\right)\right\rangle_{\langle\mathbf{H}^{-1}(\Omega),\mathbf{H}^1(\Omega)\rangle}+\fint_Y\tilde{D}_{ijkl}e^y_{kl}(\boldsymbol{u}^1)\,e^y_{ij}(\tilde{\boldsymbol{v}}^1)\\
&\quad-\fint_Y\tilde{\alpha}_{ij}e^y_{ij}(\tilde{\boldsymbol{v}}^1)\frac{\mathrm{d}\,P}{\mathrm{d}\,t}-\fint_{Y_m}\alpha^m_{ij}e^y_{ij}(\tilde{\boldsymbol{v}}^1)\frac{\mathrm{d}\,\widehat{P}}{\mathrm{d}\,t}-\fint_{\Gamma^Y_m}n_j\tilde{\alpha}^\Gamma_{ij}\tilde{v}^1_i\left[\frac{\mathrm{d}}{\mathrm{d}\,t}\widehat{P}\right]_{\Gamma^Y_m}dS_y=G(\tilde{\boldsymbol{v}}^1)\,,
\end{aligned}\tag{4.20}$$

*for all $\tilde{\boldsymbol{v}}^1\in L^2(\Omega;\mathbf{H}^1_\#(Y))$, $\bar{\boldsymbol{v}}\in\mathbf{H}^{-1}(\Omega)$, and*

$$\begin{aligned}
&\fint_{\Omega\times Y_c}K^c_{ij}(\partial^x_jP+\partial^y_jP^1)\,(\partial^x_jq^0+\partial^y_j\tilde{q}^1)+\fint_{\Omega\times Y_m}K^m_{ij}\,\partial^y_j\widehat{P}\,\partial^y_i\hat{q}^0+\fint_{\Gamma^Y_m}\tilde{\varkappa}\left[\widehat{P}\right]_{\Gamma^Y_m}\left[\hat{q}^0\right]_{\Gamma^Y_m}dS_y\\
&\qquad+\fint_{\Omega\times Y}(q^0+\chi_m\hat{q}^0)\tilde{\alpha}_{ij}e^y_{ij}(\boldsymbol{u}^1)+\int_\Omega\fint_{\Gamma^Y_m}n_j\tilde{\alpha}^\Gamma_{ij}u^1_i\left[\hat{q}^0\right]_{\Gamma^Y_m}dS_y\\
&\qquad+\left\langle\bar{\boldsymbol{u}},\fint_{\Gamma^Y_m}\tilde{\alpha}^\Gamma_{ij}n_j\left[\hat{q}^0\right]_{\Gamma^Y_m}dS_y\right\rangle_{\langle\mathbf{H}^{-1}(\Omega),\mathbf{H}^1(\Omega)\rangle}\\
&\qquad+\fint_{\Omega\times Y}\frac{1}{\tilde{\mu}}\left(\frac{\mathrm{d}\,P}{\mathrm{d}\,t}+\chi_m\frac{\mathrm{d}\,\widehat{P}}{\mathrm{d}\,t}\right)(q^0+\chi_m\hat{q}^0)=0\,,
\end{aligned}\tag{4.21}$$

*for all* $q^0 \in H^1(\Omega)$, $\tilde{q}^1 \in L^2(\Omega; H^1_{\#}(y))$ *and* $\hat{q}^0 \in L^2(\Omega; H_{\#c0}(Y, \Gamma_m^Y))$. *In (4.20), the force* $\bar{\boldsymbol{g}}$ *is defined in Cases AF and BF, by* $\bar{\boldsymbol{f}}$ *and* $\hat{\boldsymbol{F}}$ *respectively, while in Cases CF and DF,* $\bar{\boldsymbol{g}} \equiv 0$.

*Proof.* We start by deriving the limits of all bilinear forms in the left-hand side of problem (2.19). Then, for each particular case of loading forces (4.1)–(4.4), we examine the right-hand side integral in $(2.19)_1$.

Due to (4.18) and (4.19), for $\theta \in C_0^\infty(\Omega)$, the following test displacements can be used in $(2.19)_1$:

$$\boldsymbol{v}^\varepsilon(x) = \varepsilon \bar{\boldsymbol{v}}(x) + \varepsilon \boldsymbol{v}^1(\{x/\varepsilon\}_Y)\theta(x), \qquad \bar{\boldsymbol{v}} \in \mathbf{H}^1_0(\Omega)\,, \quad \boldsymbol{v}^1 \in \mathbf{H}^1_{\#}(Y), \quad \int_Y \boldsymbol{v}^1 = 0. \tag{4.22}$$

The pressure test functions $q^\varepsilon$ are chosen according to (3.18).

First, we pass to the limit in the interface integrals to get

$$\begin{aligned}
&\int_{\Gamma^\varepsilon_m} n_j \alpha^\varepsilon_{ij} v^\varepsilon_i [\frac{\mathrm{d}}{\mathrm{d}\,t}\widehat{P}^\varepsilon]_{\Gamma^\varepsilon_m}\, dS \\
&= \int_\Omega \frac{1}{\varepsilon|Y|}\left(\varepsilon \bar{v}_i \int_{\Gamma^Y_m} n_j \tilde{\alpha}^\Gamma_{ij} \left[\frac{\mathrm{d}}{\mathrm{d}\,t}\mathcal{T}^b_\varepsilon\Big(\widehat{P}^\varepsilon\Big)\right]_{\Gamma^Y_m} dS_y + \varepsilon\theta \int_{\Gamma^Y_m} n_j \tilde{\alpha}^\Gamma_{ij} v^1_i \left[\frac{\mathrm{d}}{\mathrm{d}\,t}\mathcal{T}^b_\varepsilon\Big(\widehat{P}^\varepsilon\Big)\right]_{\Gamma^Y_m} dS_y\right) \\
&\to \int_\Omega \bar{v}_i \fint_{\Gamma^Y_m} n_j \tilde{\alpha}^\Gamma_{ij} \left[\frac{\mathrm{d}}{\mathrm{d}\,t}\widehat{P}\right]_{\Gamma^Y_m} dS_y + \int_\Omega \theta \fint_{\Gamma^Y_m} n_j \tilde{\alpha}^\Gamma_{ij} v^1_i \left[\frac{\mathrm{d}}{\mathrm{d}\,t}\widehat{P}\right]_{\Gamma^Y_m} dS_y\,.
\end{aligned} \tag{4.23}$$

and

$$\begin{aligned}
\int_{\Gamma^\varepsilon_m} n_j \alpha^\varepsilon_{ij} u^\varepsilon_i [\hat{q}(x/\varepsilon)]_{\Gamma^\varepsilon_m} \hat{\vartheta}\, dS &= \int_\Omega \frac{\hat{\vartheta}\mathcal{M}^\varepsilon_Y(u^\varepsilon_i)}{\varepsilon|Y|} \int_{\Gamma^Y_m} n_j \tilde{\alpha}^\Gamma_{ij}\, [\hat{q}]_{\Gamma^Y_m}\, dS_y \\
&\quad + \int_\Omega \frac{\hat{\vartheta}}{|Y|} \int_{\Gamma^Y_m} n_j \tilde{\alpha}^\Gamma_{ij} \frac{(\mathcal{T}^b_\varepsilon(u^\varepsilon_i) - \mathcal{M}^\varepsilon_Y(u^\varepsilon_i))}{\varepsilon}\, [\hat{q}]_{\Gamma^Y_m}\, dS_y \\
&\to \int_\Omega \hat{\vartheta} \left(\bar{u}_i \fint_{\Gamma^Y_m} n_j \tilde{\alpha}^\Gamma_{ij}\, [\hat{q}]_{\Gamma^Y_m}\, dS_y + \fint_{\Gamma^Y_m} n_j \tilde{\alpha}^\Gamma_{ij} u^1_i\, [\hat{q}]_{\Gamma^Y_m}\, dS_y\right)\,.
\end{aligned} \tag{4.24}$$

Due to Lemma 4.3, one has the following convergences:

$$\begin{aligned}
&\int_\Omega D^\varepsilon_{ijkl} e_{kl}(\boldsymbol{u}^\varepsilon) e_{kl}(\boldsymbol{v}^\varepsilon) \to \int_\Omega \fint_Y \tilde{D}_{ijkl} e^y_{kl}(\boldsymbol{u}^1) e^y_{kl}(\boldsymbol{v}^1)\theta\,, \\
&\int_\Omega \alpha^\varepsilon_{ij} e_{ij}(\boldsymbol{v}^\varepsilon) \frac{\mathrm{d}}{\mathrm{d}\,t} P^\varepsilon \to \int_\Omega \theta \fint_Y \tilde{\alpha}_{ij} e^y_{ij}(\boldsymbol{v}^1) \frac{\mathrm{d}}{\mathrm{d}\,t}\left(P(x) + \widehat{P}\right),
\end{aligned} \tag{4.25}$$

and also

$$\begin{aligned}
&\int_\Omega K^\varepsilon_{ij} \partial_j P^\varepsilon \partial_i q^\varepsilon \to \int_\Omega \fint_{Y_c} K^c_{ij} (\partial^x_j P + \partial^y_j P^1)\,(\partial^x_j q^0 + \vartheta \partial^y_j q^1) + \int_\Omega \hat{\vartheta} \fint_{Y_m} K^m_{ij} \partial_j \widehat{P} \partial_i \hat{q}\,, \\
&\int_\Omega \alpha^\varepsilon_{ij} e_{ij}(\boldsymbol{u}^\varepsilon) q^\varepsilon \to \int_\Omega q^0 \fint_Y \tilde{\alpha}_{ij} e^y_{ij}(\boldsymbol{u}^1) + \int_\Omega \hat{\vartheta} \fint_{Y_m} \hat{q} \alpha^m_{ij} e^y_{ij}(\boldsymbol{u}^1)\,, \\
&\int_\Omega \varkappa^\varepsilon [P^\varepsilon]_{\Gamma^\varepsilon_m} [q^\varepsilon]_{\Gamma^\varepsilon_m} \to \int_\Omega \fint_{\Gamma^Y_m} \tilde{\varkappa} \left[\widehat{P}\right]_{\Gamma^Y_m} [\hat{q}]_{\Gamma^Y_m}\, dS_y\,.
\end{aligned} \tag{4.26}$$

It remains to compute the limits in the external force integral.

* Case AF, see (4.1),

$$\int_\Omega \boldsymbol{f}^\varepsilon \cdot \boldsymbol{v}^\varepsilon \to \int_\Omega \bar{\boldsymbol{f}} \cdot \bar{\boldsymbol{v}} + \int_\Omega \bar{\boldsymbol{f}}\theta \cdot \fint_Y \boldsymbol{v}^1\,, \tag{4.27}$$

* Case BF, see (4.2),

$$\int_\Omega \boldsymbol{f}^\varepsilon \cdot \boldsymbol{v}^\varepsilon \to \int_\Omega \bar{\boldsymbol{v}} \cdot \hat{\boldsymbol{F}} + \int_\Omega \theta \fint_{\Gamma_m^Y} \tilde{\boldsymbol{f}} \cdot \boldsymbol{v}^1 \,, \tag{4.28}$$

* Case CF, see (4.3),

$$\int_\Omega \boldsymbol{f}^\varepsilon \cdot \boldsymbol{v}^\varepsilon \to \int_\Omega \theta \fint_{\Gamma_m^Y} \tilde{\boldsymbol{f}} \cdot \boldsymbol{v}^1 \,, \tag{4.29}$$

* Case DF, see (4.4),

$$\int_\Omega \boldsymbol{f}^\varepsilon \cdot \boldsymbol{v}^\varepsilon \to 0 \,. \tag{4.30}$$

The limit expressions (4.23)-(4.30) substituted in (2.19) now yield (4.20) and (4.21).
□

**4.3 Scale decoupling and homogenized constitutive laws** In this section we present the final result on the strongly discontinuous case (2.24). The homogenized model describes a complex Darcy flow with embedded microstructural effects of the diffusion-deformation.

**4.3.1 Global and local problems** We proceed as in the "weakly" discontinuous case. Choosing different combinations of vanishing and non-vanishing test functions, we derive from (4.20) and (4.21) the local and global problems.

1. The *global* problem: the limit fields $P$ and $(\boldsymbol{u}^1, \widehat{P})$ satisfy, for all $q \in H^1(\Omega)$,

$$\int_\Omega \mathcal{C}_{ij} \partial_j^x P \, \partial_i^x q^0 + \int_\Omega q^0 \fint_Y \tilde{\alpha}_{ij} e_{ij}^y(\boldsymbol{u}^1) + \int_\Omega q^0 \fint_Y \frac{1}{\tilde{\mu}} \left( \frac{\mathrm{d}}{\mathrm{d}\, t} P + \chi_m \frac{\mathrm{d}}{\mathrm{d}\, t} \widehat{P} \right) = 0, \tag{4.31}$$

   where $\mathcal{C}_{ij}$ are defined as in (3.27) and (3.28). Equation (4.31) is obtained from (4.21) by the following choice of the test functions: $q^0 \not\equiv 0$, whereas $\tilde{q}^1 \equiv 0$ and $\widehat{p}^0 \equiv 0$. Since the macroscopic part of the test displacement field vanishes, there is *no global balance-of-forces* in the standard sense, see Remark 1.

2. The *local* problems: the limit fields $(\boldsymbol{u}^1, \widehat{P})$ and $(\bar{\boldsymbol{u}}(\cdot, x), P(\cdot, x))$ satisfy for a.e. $x \in \Omega$,

$$\begin{aligned} \fint_Y \tilde{D}_{ijkl} e_{kl}^y(\boldsymbol{u}^1)\, e_{ij}^y(\boldsymbol{v}^1) - \fint_Y \tilde{\alpha}_{ij} e_{ij}^y(\boldsymbol{v}^1) \frac{\mathrm{d}\, P}{\mathrm{d}\, t} - \fint_{Y_m} \alpha_{ij}^m e_{ij}^y(\boldsymbol{v}^1) \frac{\mathrm{d}\, \widehat{P}}{\mathrm{d}\, t} \\ - \fint_{\Gamma_m^Y} n_j \tilde{\alpha}_{ij}^\Gamma v_i^1 \left[ \frac{\mathrm{d}}{\mathrm{d}\, t} \widehat{P} \right]_{\Gamma_m^Y} dS_y = G(\boldsymbol{v}^1), \\ \fint_{Y_m} K_{ij}^m \, \partial_j^y \widehat{P} \, \partial_i^y \hat{q} + \fint_{\Gamma_m^Y} \tilde{\varkappa} \left[ \widehat{P} \right]_{\Gamma_m^Y} [\hat{q}]_{\Gamma_m^Y} \, dS_y + \fint_{Y_m} \alpha_{ij}^m e_{ij}^y(\boldsymbol{u}^1)\, \hat{q} \\ + \fint_{\Gamma_m^Y} n_j \tilde{\alpha}_{ij}^\Gamma u_i^1 \, [\hat{q}]_{\Gamma_m^Y} \, dS_y + \bar{u}_i \fint_{\Gamma_m^Y} \tilde{\alpha}_{ij}^\Gamma n_j \, [\hat{q}]_{\Gamma_m^Y} \, dS_y \\ + \fint_{Y_m} \frac{1}{\mu^m} \left( \frac{\mathrm{d}\, P}{\mathrm{d}\, t} + \frac{\mathrm{d}\, \widehat{P}}{\mathrm{d}\, t} \right) \hat{q} = 0, \end{aligned} \tag{4.32}$$

   for all $\boldsymbol{v}^1 \in \mathbf{H}^1_\#(Y)$ and $\hat{q} \in H_{\#c0}(Y, \Gamma_m^Y)$. The identities follow from (4.20) and (4.21) with $q^0, \tilde{q}^1, \bar{\boldsymbol{v}} \equiv 0$.

3. The *force-equilibrium constraint* is obtained from (4.20), with $\bar{\boldsymbol{v}} \neq 0$ and $\tilde{\boldsymbol{v}}^1 \equiv 0$

$$\bar{g}_i + \fint_{\Gamma_m^Y} \tilde{\alpha}_{ij}^\Gamma n_j \left[ \frac{\mathrm{d}}{\mathrm{d}\, t} \widehat{P} \right]_{\Gamma_m^Y} dS_y = 0 \,, \quad \text{a.e. in } \Omega, \tag{4.33}$$

   where the force $\bar{\boldsymbol{g}}$ corresponds to the cases AF and BF, see Theorem 4.4.

*The global problem* describes the diffusion flow with embedded effects of the fluid–structure microscopic interaction. Let us point out that the form of (4.31) is independent of the particular definition of forces. *The local problem* involving the interface integrals, describes the coupled diffusion-deformation processes relevant to the microscopic scale and involves the external forces according to the specific definition of $G(\cdot)$. Equation (4.33) presents an *interface pressure constraint*; according to (4.22), the macroscopic part of the test displacement field vanishes, so that there is *no global balance-of-forces* in the standard sense, see Remark 1.

**4.3.2 Laplace transformation and local correctors** We consider here only the Case BF, the other cases can be treated in a similar manner. Moreover, we shall consider a special form of the force $\tilde{\boldsymbol{f}}$ introduced in (4.2), in order to allow for the scale decoupling. Let us assume that at the microscale represented by $Y$, the "reference" interface forces $\hat{f}_i(y)\mathbf{1}^i$ are given at $\Gamma_m^Y$ ($\mathbf{1}^i$ is the unit vector in the $i$-th direction). We introduce the tensor $\Phi_{kl}(t,x)$ satisfying

$$\tilde{f}_k(t,x,y) = \Phi_{kl}(t,x)\hat{f}_l(y)\ , \quad \Phi_{kl} \in L^2((0,T)\times\Omega)\ , \quad \hat{\boldsymbol{f}} \in \mathbf{L}^2(\Gamma_m^Y)\ . \tag{4.34}$$

With the notation introduced in (3.29), we set

$$\hat{b}_{Y_m,\Gamma_m}(\varphi,\ \boldsymbol{v}) = b_{Y_m}(\varphi,\ \boldsymbol{v}) + \fint_{\Gamma_m^Y} \tilde{\alpha}_{ij}^{\Gamma} n_j v_i [\varphi]_{\Gamma_m^Y} dS_y,$$

$$g_{\Gamma_m^Y}(\psi,\phi) = \fint_{\Gamma_m^Y} \psi\phi\, dS_y.$$

Now the local problem (4.32) can be written as follows (for a.e. $x \in \Omega$):

$$\begin{aligned} &a_Y\left(\boldsymbol{u}^1,\ \boldsymbol{v}\right) - \hat{b}_{Y_m,\Gamma_m}\left(\frac{\mathrm{d}}{\mathrm{d}\,t}\widehat{P},\ \boldsymbol{v}\right) = b_Y\left(1,\ \boldsymbol{v}\right)\frac{\mathrm{d}}{\mathrm{d}\,t}P(t,x) + \Phi_{lk}(t,x)\, g_{\Gamma_m^Y}(\hat{f}_k, v_l),\\ &\gamma_{\Gamma_m,i}(\psi)\,\bar{u}_i + \hat{b}_{Y_m,\Gamma_m}\left(\psi,\ \boldsymbol{u}^1\right) + \hat{c}_{Y_m,\Gamma_m}\left(\widehat{P},\ \psi\right) + d_{Y_m}\left(\frac{\mathrm{d}}{\mathrm{d}\,t}\widehat{P},\ \psi\right)\\ &\qquad\qquad = -d_{Y_m}\left(1,\ \psi\right)\frac{\mathrm{d}}{\mathrm{d}\,t}P(t,x)\ , \end{aligned} \tag{4.35}$$

for all $\boldsymbol{v} \in \mathbf{H}^1_{\#}(Y)$ and $\psi \in H_{\#c0}(Y,\Gamma_m^Y)$. Moreover, the constraint (4.33) must be satisfied, i.e.

$$\gamma_{\Gamma_m,i}\left(\frac{\mathrm{d}}{\mathrm{d}\,t}\widehat{P}\right) = -\Phi_{ik}(t,x) g_{\Gamma_m^Y}(f_k, 1). \tag{4.36}$$

Once $\boldsymbol{u}$ and $P$ are given, it is possible to solve problem (4.35)-(4.36) to obtain $(\bar{\boldsymbol{u}}, \boldsymbol{u}^1, \widehat{P})$. As a consequence, we conclude that $\bar{\boldsymbol{u}}(t,\cdot) \in \mathbf{L}^2(\Omega)$.

We proceed by decomposing the microscopic response using the multiplicative split into the local auxiliary response (corrector basis functions) and the macroscopic response. For this, in analogy with the treatment in Section 3.3, we apply the Laplace transformation to (4.35)-(4.36), hence

$$\begin{aligned} &a_Y\left(\underset{*}{\boldsymbol{u}}^1,\ \boldsymbol{v}\right) - \hat{b}_{Y_m,\Gamma_m}\left(\lambda\underset{*}{\widehat{P}},\ \boldsymbol{v}\right) = b_Y\left(1,\ \boldsymbol{v}\right)\lambda\underset{*}{P} + \lambda\underset{*}{\Phi}_{kl} g_{\Gamma_m^Y}(\hat{f}_k, v_l),\\ &\gamma_{\Gamma_m,i}(\psi)\,\underset{*}{\bar{u}}_i + \hat{b}_{Y_m,\Gamma_m}\left(\psi,\ \underset{*}{\boldsymbol{u}}^1\right) + \hat{c}_{Y_m,\Gamma_m}\left(\underset{*}{\widehat{P}},\ \psi\right) + d_{Y_m}\left(\lambda\underset{*}{\widehat{P}},\ \psi\right) = -d_{Y_m}\left(1,\ \psi\right)\lambda\underset{*}{P},\\ &\gamma_{\Gamma_m,i}(\lambda\underset{*}{\widehat{P}}) = -\underset{*}{\Phi}_{ik} g_{\Gamma_m^Y}(f_k, 1). \end{aligned} \tag{4.37}$$

Introducing the multiplicative decomposition, by the linearity of our problem we can define $\mathcal{L}$-transformed corrector functions $(\underset{*}{\boldsymbol{\omega}}^P, \underset{*}{\pi}^P, \underset{*}{\zeta}^P)$ and $(\underset{*}{\boldsymbol{\omega}}^{kl}, \underset{*}{\pi}^{kl}, \underset{*}{\zeta}^{kl})$ such that

$$\begin{aligned} \underset{*}{\boldsymbol{u}}^1 &= \lambda\underset{*}{\boldsymbol{\omega}}^P\underset{*}{P} + \lambda\underset{*}{\boldsymbol{\omega}}^{kl}\underset{*}{\Phi}_{kl},\\ \underset{*}{\widehat{P}} &= \lambda\underset{*}{\pi}^P\underset{*}{P} + \lambda\underset{*}{\pi}^{kl}\underset{*}{\Phi}_{kl},\\ \underset{*}{\bar{\boldsymbol{u}}} &= \lambda\underset{*}{\zeta}^P\underset{*}{P} + \lambda\underset{*}{\zeta}^{kl}\underset{*}{\Phi}_{kl}. \end{aligned} \tag{4.38}$$

Substituting (4.38) in (4.37), one can formulate the following auxiliary problems where $\lambda$ is the parameter:

1. Find $(\underset{*}{\boldsymbol{\omega}}^P, \underset{*}{\pi}^P, \underset{*}{\zeta}^P) \in \mathbf{H}^1_{\#}(Y) \times H_{\#c0}(Y, \Gamma^Y_m) \times \mathbb{R}^3$ such that

$$
\begin{aligned}
&a_Y\left(\underset{*}{\boldsymbol{\omega}}^P,\, \boldsymbol{v}\right) - \hat{b}_{Y_m,\Gamma_m}\left(\lambda\underset{*}{\pi}^P,\, \boldsymbol{v}\right) = b_Y\left(1,\, \boldsymbol{v}\right),\\
&\gamma_{\Gamma_m,i}(\psi)\underset{*}{\zeta}^P + \hat{b}_{Y_m,\Gamma_m}\left(\psi,\, \underset{*}{\boldsymbol{\omega}}^P\right) + \hat{c}_{Y_m,\Gamma_m}\left(\underset{*}{\pi}^P,\, \psi\right) + d_{Y_m}\left(\lambda\underset{*}{\pi}^P,\, \psi\right) = -d_{Y_m}\left(1,\, \psi\right),\\
&\gamma_{\Gamma_m,i}(\underset{*}{\pi}^P) = 0\,, \quad i = 1,2,3,
\end{aligned}
$$

   for all $\boldsymbol{v} \in \mathbf{H}^1_{\#}(Y)$ and $\psi \in H_{\#c0}(Y, \Gamma^Y_m)$.

2. Find $(\underset{*}{\boldsymbol{\omega}}^{kl}, \underset{*}{\pi}^{kl}, \underset{*}{\zeta}^{kl}) \in \mathbf{H}^1_{\#}(Y) \times H_{\#c0}(Y, \Gamma^Y_m) \times \mathbb{R}^3$ such that

$$
\begin{aligned}
&a_Y\left(\underset{*}{\boldsymbol{\omega}}^{kl},\, \boldsymbol{v}\right) - \hat{b}_{Y_m,\Gamma_m}\left(\lambda\underset{*}{\pi}^{kl},\, \boldsymbol{v}\right) = \frac{1}{\lambda} g_{\Gamma^Y_m}(\hat{f}_k, v_l),\\
&\gamma_{\Gamma_m,i}(\psi)\underset{*}{\zeta}^{kl} + \hat{b}_{Y_m,\Gamma_m}\left(\psi,\, \underset{*}{\boldsymbol{\omega}}^{kl}\right) + \hat{c}_{Y_m,\Gamma_m}\left(\underset{*}{\pi}^{kl},\, \psi\right) + d_{Y_m}\left(\lambda\underset{*}{\pi}^{kl},\, \psi\right) = 0,\\
&\gamma_{\Gamma_m,i}(\underset{*}{\pi}^{kl}) = \frac{1}{\lambda} g_{\Gamma^Y_m}(\hat{f}_k, \delta_{il})\,, \quad i = 1,2,3,
\end{aligned}
\tag{4.39}
$$

   for all $\boldsymbol{v} \in \mathbf{H}^1_{\#}(Y)$ and $\psi \in H_{\#c0}(Y, \Gamma^Y_m)$.

**4.3.3 The macroscopic problem** We now apply the $\mathcal{L}$-transformation to the global problem (4.31). On substituting there the decomposed form of $\underset{*}{\boldsymbol{u}}^1$ and $\widehat{\underset{*}{P}}$, we get

$$
\begin{aligned}
\int_\Omega \mathcal{C}_{ij}\partial_j \underset{*}{P}\, \partial_i q + \int_\Omega q \fint_Y \tilde{\alpha}_{ij}\lambda\left(e^y_{ij}(\underset{*}{\boldsymbol{\omega}}^P)\underset{*}{P} + e^y_{ij}(\underset{*}{\boldsymbol{\omega}}^{kl})\underset{*}{\Phi}_{kl}\right)\\
+ \int_\Omega q \fint_Y \frac{1}{\tilde{\mu}}\lambda\underset{*}{P} + \int_\Omega q \fint_{Y_m} \frac{1}{\mu^m}\lambda^2\left(\underset{*}{\pi}^P\underset{*}{P} + \underset{*}{\pi}^{kl}\underset{*}{\Phi}_{kl}\right) = 0\,.
\end{aligned}
\tag{4.40}
$$

It is now possible to collect all the terms involving $\underset{*}{P}$ and integrations in $Y$; this leads to the homogenized coefficient associated physically to the term $1/\mu^\varepsilon$ of the original model,

$$
\mathcal{M}^*(\lambda) = \fint_Y \frac{1}{\tilde{\mu}} + \fint_Y \tilde{\alpha}_{ij} e^y_{ij}(\underset{*}{\boldsymbol{\omega}}^P) + \fint_{Y_m} \frac{1}{\mu^m}\lambda\underset{*}{\pi}^P\,. \tag{4.41}
$$

Proceeding analogously for $\underset{*}{\Phi}_{kl}$ leads to the following homogenized tensorial coefficient:

$$
\mathcal{F}^*_{kl}(\lambda) = \fint_Y \tilde{\alpha}_{ij} e^y_{ij}(\underset{*}{\boldsymbol{\omega}}^{kl}) + \fint_{Y_m} \frac{1}{\mu^m}\lambda\underset{*}{\pi}^{kl}\,, \tag{4.42}
$$

which is associated with the applied forces distributed on $\Gamma^Y_m$. On substituting now (4.41) and (4.42) in (4.40), one obtains the homogenized macroscopic problem.

*The homogenized macroscopic problem.* Given $\underset{*}{\Phi}_{kl} \in L^2(\Omega)$,see (4.34), compute $\underset{*}{P} \in H^1(\Omega)$ such that

$$
\int_\Omega \mathcal{C}_{ij}\partial_j \underset{*}{P}\, \partial_i q + \int_\Omega q\mathcal{M}^*(\lambda)\lambda\underset{*}{P} + \int_\Omega q\mathcal{F}^*_{kl}(\lambda)\lambda\underset{*}{\Phi}_{kl} = 0\,, \quad \forall q \in H^1(\Omega). \tag{4.43}
$$

Coercivness of (4.43) is conditioned by the positivity of $\mathcal{M}^*(\lambda)$; this is subject, as stated in Proposition 3.8, to the inequality $\mathcal{M}^*(\lambda) > 0$ for $\lambda$ positive but small enough.

**Remark 5.** Note that corrector basis functions $\boldsymbol{\omega}^{kl}$ and $\pi^{kl}$ are not symmetric in $kl$ (in contrast with the strain-associated correctors in Section 3.3), since the right-hand side terms in (4.39) are not symmetric with respect to $kl$, in general. Moreover, $\boldsymbol{\omega}^{kl}$ and $\pi^{kl}$ are related to the "stress-like" quantity $\Phi_{kl}$ introduced in (4.34) which is not subject to any symmetry. □

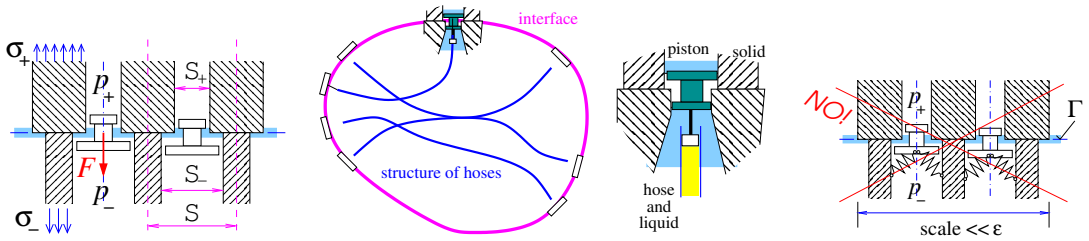


FIG. 5.1. *An example of a device which can ensure the interface conditions (2.4). Left: illustration to equations (6.1); Middle: mechanical realization of* locally *"external forces" F which drive the pistons; Right: example of a structure for which (6.2) fail – the pistons must not be fitted "locally" to the solid skeleton.*

**5 Concluding remarks** We developed a homogenized model of the Biot continuum for a dual-porous, bi-phase heterogeneous medium which embeds discontinuity interfaces. Such a model is motivated by its potential applications in biomechanical research of bone poroelastic properties [20] inherited from the *osteons* which constitute the basic unit generating the bone (almost periodic) structure.

Recently developed model [29, 16] is extended here for more complicated forms of the dual porosity embedding interfaces. Discontinuities in pressure may appear due to transmission conditions involving permeability and Biot coefficients

We studied two possible situations leading to different homogenization results.

In the "weakly discontinuous case" the homogenized macroscopic model (3.43) is non-degenerate, has a standard structure including the stress equilibrium and the mass conservation equations. It involves homogenized coefficients which arise from the discontinuity interfaces in the dual porosity of the microstructure.

In the "strongly discontinuous case", special forms of the volume forces lead to non-trivial limit models. One of them is represented in the form of interface distributions on $\Gamma_m$, which has clear mechanical interpretation. The homogenized macroscopic model reduces to a single equation describing the fluid redistribution driven by the macroscopic stress representing the *given* surface-distributed forces in the microstructure.

For both homogenized models, we derived the Laplace-transformed forms. The inverse transformation in the time domain can be performed as it was done in detail in [29].

In both cases, the homogenized (inverse) Biot modulus, $\mathcal{M}^*(\lambda)$ is positive for $\lambda$ small enough, thus, restricting the well-posedness of the limit problem. This result is consistent with the standard observation that the quasi-static problem is coercive only for small frequencies of loading, i.e. for small $\lambda$.

The approach used in the asymptotic analysis for obtaining a priori estimates can be adapted in other problems where for example, the heterogeneous medium is featured by fissures, cracks or other kinds of discontinuities.

**6 Appendix** We shall explain a mechanical justification of the interface condition (2.4). For this we consider the ultra-structure of the interface as depicted in Fig. 5.1 (a). The pistons allow the pressure discontinuity $p_+ \neq p_-$, while a small leakage of fluid is possible (drilled pistons) – this explains the "semipermeable" interface condition $(2.4)_3$.

To explain the overall stress discontinuity, we write the balance of forces. Denoting by $S$ the reference surface on the interface $\Gamma$, by $\phi_\pm = S_\pm/S$ the effective porosities on the two sides and by $A_\pm = S(1-\phi_\pm)$ the corresponding effective crossections of the solid phase, the balance of forces in the normal direction with respect to the interface yields

$$\begin{aligned} \text{in the fluid:} \quad & F + p_+ S_+ - p_- S_- = 0 \,, \\ \text{in the solid:} \quad & \sigma_+^{\text{solid}} A_+ - \sigma_-^{\text{solid}} A_- = 0 \,. \end{aligned} \tag{6.1}$$

From there the discontinuity of the total stress is obvious: let us introduce

$$\sigma_\pm^{\text{eff}} := \phi_\pm \sigma_\pm^{\text{solid}} \,, \quad \text{and} \quad \sigma_\pm := \sigma_\pm^{\text{eff}} - \phi_\pm p_\pm \,, \tag{6.2}$$

then (6.1) yields

$$\sigma_+^{\rm eff} - \sigma_-^{\rm eff} = 0 \ , \quad \text{and} \quad \sigma_+ - \sigma_- = \phi_- p_- - \phi_+ p_+ = F/S \ . \tag{6.3}$$

Hence $[\sigma]_\Gamma \neq 0$ if $[\phi p]_\Gamma \neq 0$, but the continuity of the effective stress is ensured, $[\sigma^{\rm eff}]_\Gamma = 0$.

This simple example illustrates that conditions (2.4) are realistic. The "external" forces, here referred to by $F$, can have different meanings:

- effects of other physical fields (e.g. electric field acting on the pistons),
- effects of separated self-supporting mechanical structure, as depicted in Fig. 5.1 (b), e.g. a hydraulic system, or a similar system of bowdens connecting couples of pistons,
- inertia effects — in our study we consider a quasistatic events, however, for the dynamical model the same interface conditions are relevant, where the inertia of the pistons is associated with the force $F$, as long as the piston "vibrates near the inteface".

The pistons cannot be bonded to the solid skeleton, like in Fig. 5.1 (b), which does not adhere to the assumed long-distance interaction — in such a case the "standard" continuity of the total stress would hold, i.e. $[\sigma]_\Gamma = 0$, while $[\sigma^{\rm eff}]_\Gamma \neq 0$, in general.

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