Exact convergence rates in central limit theorems for a branching random walk with a random environment in time

Zhiqiang Gao, Quansheng Liu

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Abstract

Chen [Ann. Appl. Probab. **11** (2001), 1242–1262] derived exact convergence rates in a central limit theorem and a local limit theorem for a supercritical branching Wiener process. We extend Chen's results to a branching random walk under weaker moment conditions. For the branching Wiener process, our results sharpen Chen's by relaxing the second moment condition used by Chen to a moment condition of the form $\mathbb{E}X(\ln^+ X)^{1+\lambda} < \infty$. In the rate functions that we find for a branching random walk, we figure out some new terms which didn't appear in Chen's work. The results are established in the more general framework, i.e. for a branching random walk with a random environment in time. The lack of the second moment condition for the offspring distribution and the fact that the exponential moment does not exist necessarily for the displacements make the proof delicate; the difficulty is overcome by a careful analysis of martingale convergence using a truncating argument. The analysis is significantly more awkward due to the appearance of the random environment.

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1 Introduction

The theory of branching random walk has been studied by many authors. It plays an important role, and is closely related to many problems arising in a variety of applied probability setting, including branching processes, multiplicative cascades, infinite particle systems, Quicksort algorithms and random fractals (see e.g. [30, 31]). For recent developments of the subject, see e.g. Hu and Shi [23], Shi [37], Hu [22], Attia and Barral [4] and the references therein.

In the classical branching random walk, the point processes indexed by the particles u, formulated by the number of its children and their displacements, have a fixed constant distribution for all particles u. In reality this distributions may vary from generation to generation according to a random environment, just as in the case of a branching process in random environment introduced in [2, 3, 38]. In other words, the distributions themselves may be realizations of a stochastic process, rather than being fixed. This property makes the model be closer to the reality compared to the classical branching random walk. In this paper, we shall consider such a model, called *a branching random walk with a random environment in time* .

Different kinds of branching random walks in random environments have been introduced and studied in the literature. Baillon, Clément, Greven and den Hollander [6, 19] considered the case where the offspring distribution of a particle situated at $z \in \mathbb{Z}^d$ depends on a random environment indexed by the location z, while the moving mechanism is controlled by a fixed deterministic law. Comets and Popov [12, 13] studied the case where both the offspring distributions and the moving laws depend on a random environment

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indexed by the location. In the model studied in [9, 14, 24, 33, 40], the offspring distribution of a particle of generation n situated at $z \in \mathbb{Z}^d (d \geq 1)$ depends on a random space-time environment indexed by $\{(z, n)\}$, while each particle performs a simple symmetric random walk on d-dimensional integer lattice $\mathbb{Z}^d (d \geq 1)$. The model that we study in this paper is different from those mentioned above. It should also be mentioned that recently another different kind of branching random walks in time inhomogeneous environments has been considered extensively, see e.g. Fang and Zeitouni (2012, [16]), Zeitouni (2012, [42]) and Bovier and Hartung(2014, [10]). The readers may refer to these articles and references therein for more information.

Denote by $Z_n(\cdot)$ the counting measure which counts the number of particles of generation n situated in a given set. For the classical branching random walk, a central limit theorem on $Z_n(\cdot)$, first conjectured by Harris (1963, [21]), was shown by Asmussen and Kaplan (1976, [1, 26]), and then extended to a general case by Klebaner (1982, [27]) and Biggins (1990, [7]); for a branching Wiener process, Révész (1994,[35]) studied the convergence rates in the central limit theorems and conjectured the exact convergence rates, which were confirmed by Chen (2001,[11]). Kabluchko (2012,[41]) generalized Chen's partial results using a different method. Révész, Rosen and Shi (2005,[36]) obtained a large time asymptotic expansion in the local limit theorem for branching Wiener processes, generalizing Chen's result.

The first objective of our present paper is to extend Chen's results to the branching random walk under weaker moment conditions. In our results about the exact convergence rate in the central limit theorem and the local limit theorem, the rate functions that we find include some new terms which didn't appear in Chen's paper [11]. In Chen's work, the second moment condition was assumed for the offspring distribution. Although the setting we consider now is much more general, in our results the second moment condition will be relaxed to a moment condition of the form $\mathbb{E}X(\ln^+ X)^{1+\lambda} < \infty$. It has been well known that in branching random walks, such a relaxation is quite delicate. Another interesting aspect is that we do not assume the existence of exponential moments for the moving law, which holds automatically in the case of the branching Wiener process. The lack of the second moment condition (resp. the exponential moment condition) for the offspring distribution (resp. the moving law) makes the proof delicate. The difficulty will be overcome via a careful analysis of the convergence of some associated martingales using truncating arguments.

The second objective of our paper is to extend the results to the branching random walk with a random environment in time. This model first appeared in Biggins and Kyprianou (2004, [8, Section 6]), where a criterion was given for the non-degeneration of the limit of the natural martingale; see also Kuhlbusch (2004, [28]) for the equivalent form of the criterion on weighted branching processes in random environment. For $Z_n(\cdot)$ and related quantities on this model, Liu (2007,[32]) surveyed several limit theorems, including large deviations theorems and a law of large numbers on the rightmost particle. In [18], Gao, Liu and Wang showed a central limit theorem on the counting measure $Z_n(\cdot)$ with appropriate norming. Here we study the convergence rate in the central limit theorem and a local limit theorem for $Z_n(\cdot)$. Compared with the classical branching random walk, the approach is significantly more difficult due to the appearance of the random environment.

The article is organized as follows. In Section 2, we give a rigorous description of the model and introduce the basic assumptions and notation, then we formulate our main results as Theorems 2.3 and 2.4. In Section 3, we introduce some notation and recall a theorem on the Edgeworth expansions for sums of independent random variables used in our proofs. We give the proofs of the main theorems in Section 5 and 6, respectively. Whilst Section 4 will be devoted to the proofs of the reminders.

2 Description of the model and the main results

2.1 Description of the model

We describe the model as follows ([18, 32]). *A random environment in time* $\xi = (\xi_n)$ is formulated as a sequence of random variables independent and identically distributed with values in some measurable space $(\Theta, \mathcal{F})$. Each realization of ξ_n corresponds to two probability distributions: the offspring distribution $p(\xi_n) = (p_0(\xi_n), p_1(\xi_n), \cdots)$ on $\mathbb{N} = \{0, 1, \cdots\}$, and the moving distribution $G(\xi_n)$ on $\mathbb{R}$. Without loss of generality, we can take ξ_n as coordinate functions defined on the product space $(\Theta^{\mathbb{N}}, \mathcal{F}^{\otimes\mathbb{N}})$ equipped with the product law τ of some probability law τ_0 on $(\Theta, \mathcal{F})$, which is invariant and ergodic under the usual shift transformation θ on $\Theta^{\mathbb{N}}$: $\theta(\xi_0, \xi_1, \cdots) = (\xi_1, \xi_2, \cdots)$.

When the environment $\xi = (\xi_n)$ is given, the process can be described as follows. It begins at time 0 with one initial particle $\varnothing$ of generation 0 located at $S_\varnothing = 0 \in \mathbb{R}$; at time 1, it is replaced by $N = N_\varnothing$

new particles $\varnothing i = i (1 \le i \le N)$ of generation 1, located at $S_i = L_{\varnothing i} (1 \le i \le N)$, where $N, L_1, L_2, \cdots$ are mutually independent, N has the law $p(\xi_0)$, and each L_i has the law $G(\xi_0)$. In general, each particle $u = u_1 ... u_n$ of generation n is replaced at time $n+1$ by N_u new particles $ui (1 \le i \le N_u)$ of generation $n+1$, with displacements $L_{ui} (1 \le i \le N_u)$, so that the i-th child ui is located at

$$S_{ui} = S_u + L_{ui},$$

where $N_u, L_{u1}, L_{u2}, \cdots$ are mutually independent, N_u has the law $p(\xi_n)$, and each L_{ui} has the same law $G(\xi_n)$. By definition, given the environment ξ, the random variables N_u and L_u, indexed by all the finite sequences u of positive integers, are independent of each other.

For each realization $\xi \in \Theta^{\mathbb{N}}$ of the environment sequence, let $(\Gamma, \mathcal{G}, \mathbb{P}_\xi)$ be the probability space under which the process is defined (when the environment ξ is fixed to the given realization). The probability $\mathbb{P}_\xi$ is usually called *quenched law*. The total probability space can be formulated as the product space $(\Theta^{\mathbb{N}} \times \Gamma, \mathcal{E}^{\mathbb{N}} \otimes \mathcal{G}, \mathbb{P})$, where $\mathbb{P} = \mathbb{E}(\delta_\xi \otimes \mathbb{P}_\xi)$ with δ_ξ the Dirac measure at ξ and $\mathbb{E}$ the expectation with respect to the random variable ξ, so that for all measurable and positive g defined on $\Theta^{\mathbb{N}} \times \Gamma$, we have

$$\int_{\Theta^{\mathbb{N}} \times \Gamma} g(x,y) d\mathbb{P}(x,y) = \mathbb{E} \int_\Gamma g(\xi, y) d\mathbb{P}_\xi(y).$$

The total probability $\mathbb{P}$ is usually called *annealed law*. The quenched law $\mathbb{P}_\xi$ may be considered to be the conditional probability of $\mathbb{P}$ given ξ. The expectation with respect to $\mathbb{P}$ will still be denoted by $\mathbb{E}$; there will be no confusion for reason of consistence. The expectation with respect to $\mathbb{P}_\xi$ will be denoted by $\mathbb{E}_\xi$.

Let $\mathbb{T}$ be the genealogical tree with $\{N_u\}$ as defining elements. By definition, we have: (a) $\varnothing \in \mathbb{T}$; (b) $ui \in \mathbb{T}$ implies $u \in \mathbb{T}$; (c) if $u \in \mathbb{T}$, then $ui \in \mathbb{T}$ if and only if $1 \le i \le N_u$. Let

$$\mathbb{T}_n = \{u \in \mathbb{T} : |u| = n\}$$

be the set of particles of generation n, where $|u|$ denotes the length of the sequence u and represents the number of generation to which u belongs.

2.2 Main results

Let $Z_n(\cdot)$ be the counting measure of particles of generation n: for $B \subset \mathbb{R}$,

$$Z_n(B) = \sum_{u \in \mathbb{T}_n} \mathbf{1}_B(S_u).$$

Then $\{Z_n(\mathbb{R})\}$ constitutes a branching process in a random environment (see e.g. [2, 3, 38]). For $n \ge 0$, let $\widehat{N}_n$ (resp. $\widehat{L}_n$) be a random variable with distribution $p(\xi_n)$ (resp. $G(\xi_n)$) under the law $\mathbb{P}_\xi$, and define

$$m_n = m(\xi_n) = \mathbb{E}_\xi \widehat{N}_n, \quad \Pi_n = m_0 \cdots m_{n-1}, \quad \Pi_0 = 1.$$

It is well known that the normalized sequence

$$W_n = \frac{1}{\Pi_n} Z_n(\mathbb{R}), \quad n \ge 1$$

constitutes a martingale with respect to the filtration $(\mathscr{F}_n)$ defined by

$$\mathscr{F}_0 = \{\emptyset, \Omega\}, \quad \mathscr{F}_n = \sigma(\xi, N_u : |u| < n), \text{ for } n \ge 1.$$

Throughout the paper, we shall always assume the following conditions:

$$\mathbb{E} \ln m_0 > 0 \quad \text{and} \quad \mathbb{E}\left[\frac{1}{m_0} \widehat{N}_0 \Big(\ln^+ \widehat{N}_0 \Big)^{1+\lambda}\right] < \infty, \tag{2.1}$$

where the value of $\lambda > 0$ is to be specified in the hypothesis of the theorems. Under these conditions, the underlying branching process $\{Z_n(\mathbb{R})\}$ is *supercritical*, $Z_n(\mathbb{R}) \to \infty$ with positive probability, and the limit

$$W = \lim_n W_n$$

verifies $\mathbb{E}W = 1$ and $W > 0$ almost surely (a.s.) on the explosion event $\{Z_\infty \to \infty\}$ (cf. e.g. [3, 39]).

For $n \geq 0$, define

$$l_n = \mathbb{E}_\xi \widehat{L}_n, \;\; \sigma_n^{(\nu)} = \mathbb{E}_\xi \big(\widehat{L}_n - l_n\big)^\nu, \text{ for } \nu \geq 2;$$
$$\ell_n = \sum_{k=0}^{n-1} l_k, \quad s_n^{(\nu)} = \sum_{k=0}^{n-1} \sigma_k^{(\nu)}, \text{ for } \nu \geq 2, \quad s_n = \big(s_n^{(2)}\big)^{\frac{1}{2}}.$$

We will need the following conditions on the motion of particles:

$$\mathbb{P}\Big(\limsup_{|t|\to\infty} \big|\mathbb{E}_\xi e^{it\widehat{L}_0}\big| < 1\Big) > 0 \quad \text{and} \quad \mathbb{E}\big(|\widehat{L}_0|^\eta\big) < \infty, \tag{2.2}$$

where the value of $\eta > 1$ is to be specified in the hypothesis of the theorems. The first hypothesis means that Cramér's condition about the characteristic function of $\widehat{L}_0$ holds with positive probability.

Let $\{N_{1,n}\}$ and $\{N_{2,n}\}$ be two sequences of random variables, defined respectively by

$$N_{1,n} = \frac{1}{\Pi_n} \sum_{u\in\mathbb{T}_n} (S_u - \ell_n) \quad \text{and} \quad N_{2,n} = \frac{1}{\Pi_n} \sum_{u\in\mathbb{T}_n} \Big((S_u - \ell_n)^2 - s_n^2\Big).$$

We shall prove that they are martingales with respect to the filtration $(\mathscr{D}_n)$ defined by

$$\mathscr{D}_0 = \{\emptyset, \Omega\}, \quad \mathscr{D}_n = \sigma(\xi, N_u, L_{ui} : i \geq 1, |u| < n), \text{ for } n \geq 1.$$

More precisely,we have the following propositions.

Proposition 2.1. *Assume* (2.1) *and* $\mathbb{E}\big(\ln^- m_0\big)^{1+\lambda} < \infty$ *for some* $\lambda > 1$, *and* $\mathbb{E}\big(|\widehat{L}_0|^\eta\big) < \infty$ *for some* $\eta > 2$. *Then the sequence* $\{(N_{1,n}, \mathscr{D}_n)\}$ *is a martingale and converges a.s.:*

$$V_1 := \lim_{n\to\infty} N_{1,n} \text{ exists a.s. in } \mathbb{R}.$$

Proposition 2.2. *Assume* (2.1) *and* $\mathbb{E}\big(\ln^- m_0\big)^{1+\lambda} < \infty$ *for some* $\lambda > 2$, *and* $\mathbb{E}\big(|\widehat{L}_0|^\eta\big) < \infty$ *for some* $\eta > 4$. *Then the sequence* $\{(N_{2,n}, \mathscr{D}_n)\}$ *is a martingale and converges a.s.:*

$$V_2 := \lim_{n\to\infty} N_{2,n} \text{ exists a.s. in } \mathbb{R}.$$

Our main results are the following two theorems. The first theorem concerns the exact convergence rate in the central limit theorem about the counting measure Z_n, while the second one is a local limit theorem. We shall use the notation

$$Z_n(t) = Z_n((-\infty, t]), \;\; \phi(t) = \frac{1}{\sqrt{2\pi}} e^{-t^2/2}, \;\; \Phi(t) = \int_{-\infty}^t \phi(x)\mathrm{d}x, \quad t \in \mathbb{R}.$$

Theorem 2.3. *Assume* (2.1) *for some* $\lambda > 8$, (2.2) *for some* $\eta > 12$ *and* $\mathbb{E}m_0^{-\delta} < \infty$ *for some* $\delta > 0$. *Then for all* $t \in \mathbb{R}$,

$$\sqrt{n}\Big[\frac{1}{\Pi_n} Z_n(\ell_n + s_n t) - \Phi(t)W\Big] \xrightarrow{n\to\infty} \mathcal{V}(t) \quad a.s., \tag{2.3}$$

where

$$\mathcal{V}(t) = -\frac{\phi(t)\, V_1}{(\mathbb{E}\sigma_0^{(2)})^{1/2}} + \frac{(\mathbb{E}\sigma_0^{(3)})\,(1-t^2)\,\phi(t)\, W}{6(\mathbb{E}\sigma_0^{(2)})^{3/2}}.$$

Theorem 2.4. *Assume* (2.1)*for some* $\lambda > 16$, (2.2) *for some* $\eta > 16$ *and* $\mathbb{E}m_0^{-\delta} < \infty$ *for some* $\delta > 0$. *Then for any bounded measurable set* $A \subset \mathbb{R}$ *with Lebesgue measure* $|A| > 0$,

$$n\left[\sqrt{2\pi} s_n \Pi_n^{-1} Z_n(A + \ell_n) - W \int_A e^{-\frac{x^2}{2s_n^2}} dx\right] \xrightarrow{n\to\infty} \mu(A) \quad a.s., \tag{2.4}$$

where

$$\mu(A) = \frac{|A|}{2\mathbb{E}\sigma_0^{(2)}}\Big(- V_2 + 2\,\overline{x}_A V_1\Big) + \frac{|A|\ c(A)}{8(\mathbb{E}\sigma_0^{(2)})^2}$$

with $\overline{x}_A = \dfrac{1}{|A|}\displaystyle\int_A x dx$ *and*

$$c(A) = W\ \mathbb{E}\big(\sigma_0^{(4)} - 3\big(\sigma_0^{(2)}\big)^2\big)\ + 4\ (\mathbb{E}\sigma_0^{(3)})(V_1 - \overline{x}_A W) - \frac{5(\mathbb{E}\sigma_0^{(3)})^2}{3\ \mathbb{E}\sigma_0^{(2)}} W.$$

Remark 2.5. For a branching Wiener process, Theorems 2.3 and 2.4 improve Theorems 3.1 and 3.2 of Chen (2001,[11]) by relaxing the second moment condition used by Chen to the moment condition of the form $\mathbb{E}X(\ln^+ X)^{1+\lambda} < \infty$ (cf. (2.1)). For a branching random walk with a constant or random environment, the second terms in $\mathcal{V}(\cdot)$ and $\mu(\cdot)$ are new: they did not appear in Chen's results [11] for a branching Wiener process; the reason is that in the case of a Brownian motion, we have $\sigma_0^{(3)} = \sigma_0^{(4)} - 3\big(\sigma_0^{(2)}\big)^2 = 0$.

Remark 2.6. As will be seen in the proof, the numbers 8, 12, 16, 16 listed in the conditions are due to technical reasons. It is interesting to find the best bounds for λ and η, which seems to be delicate.

Remark 2.7. In the deterministic case (with constant environment), Theorem 2.3 reduces to [17, Theorem 1.2] which itself improves a result by Kabluchko [41, Theorem 5 and Remark 2] obtained under the stronger second moment condition $\mathbb{E}Z_1^2(\mathbb{R}) < \infty$ on the branching mechanism.

Remark 2.8. When the Cramér condition $\mathbb{P}\Big(\limsup_{|t|\to\infty} \big|\mathbb{E}_\xi e^{it\widehat{L}_0}\big| < 1\Big) > 0$ fails, the situation is different. Actually, while revising our manuscript we find that for a branching random walk on $\mathbb{Z}$ in a constant environment, a different lattice version (while the Cramér condition fails) of Theorems 2.3 and 2.4 has been established very recently in [20].

For simplicity and without loss of generality, hereafter we always assume that $l_n = 0$ (otherwise, we only need to replace L_{ui} by $L_{ui} - l_n$) and hence $\ell_n = 0$. In the following, we will write K_ξ for a constant depending on the environment, whose value may vary from lines to lines.

3 Notation and Preliminary results

In this section, we introduce some notation and important lemmas which will be used in the sequel.

3.1 Notation

In addition to the $\sigma-$fields $\mathscr{F}_n$ and $\mathscr{D}_n$, the following σ-fields will also be used:

$$\mathscr{I}_0 = \{\emptyset, \Omega\}, \quad \mathscr{I}_n \ = \ \sigma(\xi_k, N_u, L_{ui} : k < n, i \geq 1, |u| < n) \text{ for } n \geq 1.$$

For conditional probabilities and expectations, we write:

$$\mathbb{P}_{\xi,n}(\cdot) = \mathbb{P}_\xi(\cdot|\mathscr{D}_n), \quad \mathbb{E}_{\xi,n}(\cdot) = \mathbb{E}_\xi(\cdot|\mathscr{D}_n); \quad \mathbb{P}_n(\cdot) = \mathbb{P}(\cdot|\mathscr{I}_n), \quad \mathbb{E}_n(\cdot) = \mathbb{E}(\cdot|\mathscr{I}_n).$$

As usual, we set $\mathbb{N}^* = \{1, 2, 3, \cdots\}$ and denote by

$$U = \bigcup_{n=0}^{\infty} (\mathbb{N}^*)^n$$

the set of all finite sequences, where $(\mathbb{N}^*)^0 = \{\varnothing\}$ contains the null sequence $\varnothing$.

For all $u \in U$, let $\mathbb{T}(u)$ be the shifted tree of $\mathbb{T}$ at u with defining elements $\{N_{uv}\}$: we have 1) $\varnothing \in \mathbb{T}(u)$, 2) $vi \in \mathbb{T}(u) \Rightarrow v \in \mathbb{T}(u)$ and 3) if $v \in \mathbb{T}(u)$, then $vi \in \mathbb{T}(u)$ if and only if $1 \leq i \leq N_{uv}$. Define $\mathbb{T}_n(u) = \{v \in \mathbb{T}(u) : |v| = n\}$. Then $\mathbb{T} = \mathbb{T}(\varnothing)$ and $\mathbb{T}_n = \mathbb{T}_n(\varnothing)$.

For every integer $m \geq 0$, let H_m be the Chebyshev-Hermite polynomial of degree m ([34]):

$$H_m(x) = m! \sum_{k=0}^{\lfloor \frac{m}{2} \rfloor} \frac{(-1)^k x^{m-2k}}{k!(m-2k)!2^k}. \tag{3.1}$$

The first few Chebyshev-Hermite polynomials relevant to us are:

$$H_2(x) = x^2 - 1, \quad H_3(x) = x^3 - 3x, \quad H_4(x) = x^4 - 6x^2 + 3,$$
$$H_5(x) = x^5 - 10x^3 + 15x, \quad H_6(x) = x^6 - 15x^4 + 45x^2 - 15,$$
$$H_7(x) = x^7 - 21x^5 + 105x^3 - 105x, \quad H_8(x) = x^8 - 28x^6 + 210x^4 - 420x^2 + 105.$$

It is known that ([34]) : for every integer $m \geq 0$

$$\Phi^{(m+1)}(x) = \frac{d^{m+1}}{dx^{m+1}}\Phi(x) = (-1)^m \phi(x) H_m(x).$$

3.2 Two preliminary lemmas

We first give an elementary lemma which will be often used in Section 4.

Lemma 3.1. *(a) For $x, y \geq 0$,*

$$\ln^+(x+y) \leq 1 + \ln^+ x + \ln^+ y, \qquad \ln(1+x) \leq 1 + \ln^+ x. \tag{3.2}$$

(b) For each $\lambda > 0$, there exists a constant $K_\lambda > 0$, such that

$$(\ln^+ x)^{1+\lambda} \leq K_\lambda x, \ \ x > 0, \tag{3.3}$$

(c) For each $\lambda > 0$, the function

$$(\ln(e^\lambda + x))^{1+\lambda} \quad \textit{is concave for} \ \ x > 0. \tag{3.4}$$

Proof. Part (a) holds since $\ln^+(x+y) \leq \ln^+(2\max\{x,y\}) \leq 1 + \ln^+ x + \ln^+ y$. Parts (b) and (c) can be verified easily. □

We next present the Edgeworth expansion for sums of independent random variables, that we shall need in Sections 5 and 6 to prove the main theorems. Let us recall the theorem used in this paper obtained by Bai and Zhao(1986, [5]), that generalizing the case for i.i.d random variables (cf. [34, P.159, Theorem 1]).

Let $\{X_j\}$ be independent random variables, s atisfying for each $j \geq 1$

$$\mathbb{E}X_j = 0, \mathbb{E}|X_j|^k < \infty \text{ with some integer } k \geq 3. \tag{3.5}$$

We write $B_n^2 = \sum_{j=1}^n \mathbb{E}X_j^2$ and only consider the nontrivial case $B_n > 0$. Let $\gamma_{\nu j}$ be the ν-order cumulant of X_j for each $j \geq 1$. Write

$$\lambda_{\nu,n} = n^{(\nu-2)/2} B_n^{-\nu} \sum_{j=1}^n \gamma_{\nu j}, \quad \nu = 3, 4 \cdots, k;$$
$$Q_{\nu,n}(x) = \sum{}' (-1)^{\nu+2s} \Phi^{(\nu+2s)}(x) \prod_{m=1}^{\nu} \frac{1}{k_m!}\left(\frac{\lambda_{m+2,n}}{(m+2)!}\right)^{k_m}$$
$$= -\phi(x) \sum{}' H_{\nu+2s-1}(x) \prod_{m=1}^{\nu} \frac{1}{k_m!}\left(\frac{\lambda_{m+2,n}}{(m+2)!}\right)^{k_m},$$

where the summation $\sum{}'$ is carried out over all nonnegative integer solutions $(k_1, \ldots, k_\nu)$ of the equation $k_1 + 2k_2 + \cdots + \nu k_\nu = \nu$, and $s = k_1 + \cdots + k_\nu$.

For $1 \leq j \leq n$ and $x \in \mathbb{R}$, define

$$F_n(x) = \mathbb{P}\Big(B_n{}^{-1} \sum_{j=1}^n X_j \leq x\Big), \quad v_j(t) = \mathbb{E}e^{itX_j};$$
$$Y_{nj} = X_j \mathbf{1}_{\{|X_j| \leq B_n\}}, \quad Z_{nj}^{(x)} = X_j \mathbf{1}_{\{|X_j| \leq B_n(1+|x|)\}}, \quad W_{nj}^{(x)} = X_j \mathbf{1}_{\{|X_j| > B_n(1+|x|)\}}.$$

The Edgeworth expansion theorem can be stated as follows.

Lemma 3.2 ([5]). *Let $n \geq 1$ and $X_1, \cdots, X_n$ be a sequence of independent random variables satisfying* (3.5) *and $B_n > 0$. Then for the integer $k \geq 3$,*

$$|F_n(x) - \Phi(x) - \sum_{\nu=1}^{k-2} Q_{\nu n}(x) n^{-\nu/2}| \leq C(k)\Bigg\{(1+|x|)^{-k} B_n^{-k} \sum_{j=1}^{n} \mathbb{E}|W_{nj}^{(x)}|^k + (1+|x|)^{-k-1} B_n^{-k-1} \sum_{j=1}^{n} \mathbb{E}|Z_{nj}^{(x)}|^{k+1} + (1+|x|)^{-k-1} n^{k(k+1)/2} \Big(\sup_{|t|\geq \delta_n} \frac{1}{n} \sum_{j=1}^{n} |v_j(t)| + \frac{1}{2n} \Big)^n \Bigg\},$$

where $\delta_n = \frac{1}{12} B_n^2 (\sum_{j=1}^{n} \mathbb{E}|Y_{nj}|^3)^{-1}$, $C(k) > 0$ is a constant depending only on k.

4 Convergence of the martingales $\{(N_{1,n}, \mathscr{D}_n)\}$ and $\{(N_{2,n}, \mathscr{D}_n)\}$

Now we can proceed to prove the convergence of the two martingales defined in Section 2.

4.1 Convergence of the martingale $\{(N_{1,n}, \mathscr{D}_n)\}$

The fact that $\{(N_{1,n}, \mathscr{D}_n)\}$ is a martingale can be easily shown: it suffices to notice that

$$\begin{aligned} \mathbb{E}_{\xi,n} N_{1,n+1} &= \mathbb{E}_{\xi,n}\Big(\frac{1}{\Pi_{n+1}} \sum_{u \in \mathbb{T}_{n+1}} S_u \Big) = \frac{1}{\Pi_{n+1}} \mathbb{E}_{\xi,n}\Big(\sum_{u \in \mathbb{T}_n} \sum_{i=1}^{N_u} (S_u + L_{ui}) \Big) \\ &= \frac{1}{\Pi_{n+1}} \sum_{u \in \mathbb{T}_n} \mathbb{E}_{\xi,n}\Big(\sum_{i=1}^{N_u} (S_u + L_{ui}) \Big) \\ &= \frac{1}{\Pi_{n+1}} \sum_{u \in \mathbb{T}_n} m_n S_u = N_{1,n}. \end{aligned}$$

We shall prove the a.s. convergence of the martingale by showing that the series

$$\sum_{n=1}^{\infty} I_n \quad \text{converges a.s., with} \quad I_n = N_{1,n+1} - N_{1,n}. \tag{4.1}$$

To this end, we first establish a lemma. For $n \geq 1$ and $|u| = n$, set

$$X_u = S_u \Big(\frac{N_u}{m_{|u|}} - 1 \Big) + \sum_{i=1}^{N_u} \frac{L_{ui}}{m_{|u|}}, \tag{4.2}$$

and let $\widehat{X}_n$ be a generic random variable of X_u, i.e. $\widehat{X}_n$ has the same distribution with X_u (for $|u| = n$). Recall that $\widehat{N}_n$ has the same distribution as N_u, $|u| = n$.

We proceed the proof by proving the following lemma:

Lemma 4.1. *Under the conditions of Proposition 2.1, we have*

$$\mathbb{E}_\xi |\widehat{X}_n| (\ln^+ |\widehat{X}_n|)^{1+\lambda} \leq K_\xi n \left((\ln n)^{1+\lambda} + \mathbb{E}_\xi \frac{\widehat{N}_n}{m_n} (\ln^+ \widehat{N}_n)^{1+\lambda} + (\ln^- m_n)^{1+\lambda} \right), \tag{4.3}$$

where K_ξ is a constant.

Proof. For $u \in \mathbb{T}_n$,

$$|X_u| \leq |S_u| \left(1 + \frac{N_u}{m_n} \right) + \frac{\left| \sum_{i=1}^{N_u} L_{ui} \right|}{m_n},$$

$$\ln^+|X_u| \le 2 + \ln^+|S_u| + \ln(1+N_u/m_n) + \ln^+\left|\frac{1}{m_n}\sum_{i=1}^{N_u} L_{ui}\right|,$$

$$4^{-\lambda}(\ln^+|X_u|)^{1+\lambda} \le 2^{1+\lambda} + (\ln^+|S_u|)^{1+\lambda} + \left(\ln\left(1+\frac{N_u}{m_n}\right)\right)^{1+\lambda} + \left(\ln^+\left|\frac{1}{m_n}\sum_{i=1}^{N_u} L_{ui}\right|\right)^{1+\lambda}.$$

Hence we get that

$$4^{-\lambda}|X_u|(\ln^+|X_u|)^{1+\lambda} \le \sum_{i=1}^{8} \mathbb{J}_i,$$

with

$$\mathbb{J}_1 = 2^{1+\lambda}|S_u|\left(1+\frac{N_u}{m_n}\right), \qquad \mathbb{J}_2 = |S_u|(\ln^+|S_u|)^{1+\lambda}\left(1+\frac{N_u}{m_n}\right),$$

$$\mathbb{J}_3 = |S_u|\left(1+\frac{N_u}{m_n}\right)\left(\ln\left(1+\frac{N_u}{m_n}\right)\right)^{1+\lambda}, \qquad \mathbb{J}_4 = |S_u|\left(1+\frac{N_u}{m_n}\right)\left(\ln^+\left|\frac{1}{m_n}\sum_{i=1}^{N_u} L_{ui}\right|\right)^{1+\lambda},$$

$$\mathbb{J}_5 = \frac{2^{1+\lambda}}{m_n}\left|\sum_{i=1}^{N_u} L_{ui}\right|, \quad \mathbb{J}_6 = \frac{(\ln^+|S_u|)^{1+\lambda}}{m_n}\left|\sum_{i=1}^{N_u} L_{ui}\right|, \quad \mathbb{J}_7 = \left(\ln\left(1+\frac{N_u}{m_n}\right)\right)^{1+\lambda}\left|\frac{1}{m_n}\sum_{i=1}^{N_u} L_{ui}\right|,$$

$$\mathbb{J}_8 = \frac{1}{m_n}\left|\sum_{i=1}^{N_u} L_{ui}\right|\left(\ln^+\left|\frac{1}{m_n}\sum_{i=1}^{N_u} L_{ui}\right|\right)^{1+\lambda}.$$

Since

$$\lim_{n\to\infty}\frac{1}{n}\sum_{j=1}^{n}\mathbb{E}_\xi|\widehat{L}_j|^q = \mathbb{E}|\widehat{L}_1|^q < \infty, \quad q=1,2,$$

there exists a constant $K_\xi < \infty$ depending only on ξ such that for $n \ge 1$ and $|u| = n$,

$$\mathbb{E}_\xi|\widehat{L}_n| \le K_\xi n, \quad \mathbb{E}_\xi|S_u| \le \sum_{j=1}^{n}\mathbb{E}_\xi|\widehat{L}_j| \le K_\xi n, \quad \mathbb{E}_\xi|S_u|^2 = \sum_{j=1}^{n}\mathbb{E}_\xi|\widehat{L}_j|^2 \le K_\xi n. \tag{4.4}$$

By the definition of the model, S_u, N_u and L_{ui} are mutually independent under $\mathbb{P}_\xi$. On the basis of the above estimates, we have the following inequalities, where K_ξ is a constant depending on ξ, whose value may be different from lines to lines: for $n \ge 1$ and $|u| = n$,

$$\mathbb{E}_\xi\mathbb{J}_1 = 2^{1+\lambda}\mathbb{E}_\xi|S_u|\mathbb{E}_\xi\left(1+\frac{N_u}{m_{|u|}}\right) \le K_\xi n;$$

$$\mathbb{E}_\xi\mathbb{J}_2 \le_{(3.3)} K_\lambda\mathbb{E}_\xi(|S_u|^2 + |S_u|) \le K_\xi n \quad ;$$

$$\begin{aligned}\mathbb{E}_\xi\mathbb{J}_3 &\le \mathbb{E}_\xi|S_u|\mathbb{E}_\xi\left(1+\frac{N_u}{m_{|u|}}\right)\left(\ln\left(1+\frac{N_u}{m_n}\right)\right)^{1+\lambda}\\ &\le K_\xi n\left(K_\xi + \mathbb{E}_\xi\frac{\widehat{N}_n}{m_n}(\ln^+\widehat{N}_n)^{1+\lambda} + \big(\ln^- m_n\big)^{1+\lambda}\right);\end{aligned}$$

$$\begin{aligned}\mathbb{E}_\xi\mathbb{J}_4 &\le \mathbb{E}_\xi|S_u|\,\mathbb{E}_\xi\left[\left(1+\frac{N_u}{m_{|u|}}\right)\left(\ln\Big(e^\lambda + \frac{1}{m_{|u|}}\left|\sum_{i=1}^{N_u} L_{ui}\right|\Big)\right)^{1+\lambda}\right]\\ &\le (K_\xi n)\mathbb{E}_\xi\left[\left(1+\frac{N_u}{m_{|u|}}\right)\left(\ln\mathbb{E}_\xi\Big(e^\lambda + \frac{1}{m_{|u|}}\sum_{i=1}^{N_u}|L_{ui}| \;\Big|\; N_u\Big)\right)^{1+\lambda}\right]\\ &\qquad \text{(by Jensen's inequality under } \mathbb{E}_\xi(\cdot|N_u) \text{ using the concavity of } (\ln(e^\lambda+x))^{1+\lambda})\\ &= (K_\xi n)\mathbb{E}_\xi\left(1+\frac{N_u}{m_{|u|}}\right)\left(\ln\Big(e^\lambda + \frac{1}{m_{|u|}}\sum_{i=1}^{N_u}\mathbb{E}_\xi|L_{ui}|\Big)\right)^{1+\lambda}\\ &\le K_\xi n\left(K_\xi(\ln n)^{1+\lambda} + \mathbb{E}_\xi\Big(\frac{1}{m_{|u|}}N_u\big(\ln^+N_u\big)^{1+\lambda}\Big) + 2\big(\ln^- m_n\big)^{1+\lambda}\right)\end{aligned}$$

$$\leq K_\xi n(\ln n)^{1+\lambda} + K_\xi n\mathbb{E}_\xi \frac{1}{m_n}\widehat{N}_n\big(\ln^+ \widehat{N}_n\big)^{1+\lambda} + K_\xi n\big(\ln^- m_n\big)^{1+\lambda};$$

$$\mathbb{E}_\xi \mathbb{J}_5 \leq 2^{1+\lambda}\mathbb{E}_\xi|\widehat{L}_n| \leq K_\xi n;$$

$$\mathbb{E}_\xi \mathbb{J}_6 = \mathbb{E}_\xi(\ln^+|S_u|)^{1+\lambda}\mathbb{E}_\xi \frac{1}{m_{|u|}}\left|\sum_{i=1}^{N_u} L_{ui}\right| \leq \mathbb{E}_\xi(\ln(e^\lambda + |S_u|))^{1+\lambda}\mathbb{E}_\xi \frac{1}{m_{|u|}}\left|\sum_{i=1}^{N_u} L_{ui}\right|$$

$$\leq (\ln(e^\lambda + \mathbb{E}_\xi|S_u|))^{1+\lambda}\mathbb{E}_\xi|\widehat{L}_n| \leq (\ln(K_\xi n))^{1+\lambda}K_\xi n \leq K_\xi n(\ln n)^{1+\lambda};$$

$$\mathbb{E}_\xi \mathbb{J}_7 \leq \mathbb{E}_\xi\left[\frac{1}{m_n}\sum_{i=1}^{N_u}(\mathbb{E}_\xi|L_{ui}|)\Big(\ln\big(1+\frac{N_u}{m_n}\big)\Big)^{1+\lambda}\right] \text{ (by the independence between } N_u \text{ and } L_{ui})$$

$$\leq K_\xi n\mathbb{E}_\xi\left[\frac{1}{m_n}N_u 3^\lambda\Big(1+(\ln^+ N_u)^{1+\lambda} + (\ln^- m_n)^{1+\lambda}\Big)\right]$$

$$\leq K_\xi n + K_\xi n\mathbb{E}_\xi \frac{1}{m_n}\widehat{N}_n\big(\ln^+ \widehat{N}_n\big)^{1+\lambda} + K_\xi n\big(\ln^- m_n\big)^{1+\lambda};$$

$$\mathbb{E}_\xi \mathbb{J}_8 \leq \mathbb{E}_\xi\left[\frac{1}{m_n}\Big|\sum_{i=1}^{N_u} L_{ui}\Big|\left(\ln^+\Big|\sum_{i=1}^{N_u} L_{ui}\Big| + \ln^- m_n\right)^{1+\lambda}\right]$$

$$\leq \mathbb{E}_\xi\left[\frac{1}{m_n}\Big|\sum_{i=1}^{N_u} L_{ui}\Big|2^\lambda\left(\Big(\ln^+\Big|\sum_{i=1}^{N_u} L_{ui}\Big|\Big)^{1+\lambda} + (\ln^- m_n)^{1+\lambda}\right)\right]$$

$$\leq_{(3.3)} K_\lambda \frac{1}{m_n}\mathbb{E}_\xi\Big|\sum_{i=1}^{N_u} L_{ui}\Big|^2 + 2^\lambda(\ln^- m_n)^{1+\lambda}\frac{1}{m_n}\mathbb{E}_\xi\Big|\sum_{i=1}^{N_u} L_{ui}\Big|$$

$$\leq K_\lambda \frac{1}{m_n}\mathbb{E}_\xi\sum_{i=1}^{N_u}\mathbb{E}_\xi|L_{ui}|^2 + 2^\lambda(\ln^- m_n)^{1+\lambda}\frac{1}{m_n}\mathbb{E}_\xi\sum_{i=1}^{N_u}\mathbb{E}_\xi\Big|L_{ui}\Big|$$

$$\leq K_\xi n + K_\xi n(\ln^- m_n)^{1+\lambda}.$$

Hence we get that for $n \geq 1$ and $|u| = n$,

$$\mathbb{E}_\xi|X_u|(\ln^+|X_u|)^{1+\lambda} \leq K_\xi n\left((\ln n)^{1+\lambda} + \mathbb{E}_\xi\frac{\widehat{N}_n}{m_n}\Big(\ln^+\widehat{N}_n\Big)^{1+\lambda} + (\ln^- m_n)^{1+\lambda}\right). \tag{4.5}$$

This gives (4.3).

□

Proof of Proposition 2.1. We have already seen that $\{(N_{1,n}, \mathscr{D}_n)\}$ is a martingale. We now prove its convergence by showing the a.s. convergence of $\sum I_n$ (cf. (4.1)). Notice that

$$I_n = N_{1,n+1} - N_{1,n} = \frac{1}{\Pi_n}\sum_{u\in\mathbb{T}_n} X_u.$$

We shall use a truncating argument to prove the convergence. Let

$$X'_u = X_u\mathbf{1}_{\{|X_u|\leq\Pi_{|u|}\}} \quad \text{and } I'_n = \frac{1}{\Pi_n}\sum_{u\in\mathbb{T}_n} X'_u.$$

The following decomposition will play an important role:

$$\sum_{n=0}^{\infty} I_n = \sum_{n=0}^{\infty}(I_n - I'_n) + \sum_{n=0}^{\infty}(I'_n - \mathbb{E}_{\xi,n}I'_n) + \sum_{n=0}^{\infty}\mathbb{E}_{\xi,n}I'_n. \tag{4.6}$$

We shall prove that each of the three series on the right hand side converges a.s. To this end, let us first prove that

$$\sum_{n=1}^{\infty}\frac{1}{(\ln\Pi_n)^{1+\lambda}}\mathbb{E}_\xi|\widehat{X}_n|(\ln^+|\widehat{X}_n|)^{1+\lambda} < \infty \quad \text{a.s.} \tag{4.7}$$

Since $\lim_{n\to\infty} \ln \Pi_n/n = \mathbb{E} \ln m_0 > 0$ a.s., for a given constant $0 < \delta_1 < \mathbb{E} \ln m_0$ and for n large enough,

$$\ln \Pi_n > \delta_1 n,$$

so that, by Lemma 4.1,

$$\frac{1}{(\ln \Pi_n)^{1+\lambda}} \mathbb{E}_\xi |\widehat{X}_n| (\ln^+ |\widehat{X}_n|)^{1+\lambda} \le \frac{K_\xi}{\delta_1^{1+\lambda}} \frac{1}{n^\lambda} \left[(\ln n)^{1+\lambda} + \mathbb{E}_\xi \frac{\widehat{N}_n}{m_n} (\ln^+ \widehat{N}_n)^{1+\lambda} + (\ln^- m_n)^{1+\lambda} \right].$$

Observe that for $\lambda > 1$,

$$\begin{aligned} &\mathbb{E} \sum_{n=1}^\infty \frac{1}{n^\lambda} \left[\mathbb{E}_\xi \frac{\widehat{N}_n}{m_n} (\ln^+ \widehat{N}_n)^{1+\lambda} + (\ln^- m_n)^{1+\lambda} \right] \\ &= \sum_{n=1}^\infty \frac{1}{n^\lambda} \left[\mathbb{E} \frac{\widehat{N}_0}{m_0} (\ln^+ \widehat{N}_0)^{1+\lambda} + \mathbb{E} (\ln^- m_0)^{1+\lambda} \right] < \infty, \end{aligned}$$

which implies that

$$\sum_{n=1}^\infty \frac{1}{n^\lambda} \left[\mathbb{E}_\xi \frac{\widehat{N}_n}{m_n} (\ln^+ \widehat{N}_n)^{1+\lambda} + (\ln^- m_n)^{1+\lambda} \right] < \infty \quad \text{a.s.}$$

Therefore (4.7) holds.

For the first series $\sum_{n=0}^\infty (I_n - I'_n)$ in (4.6), we observe that

$$\begin{aligned} \mathbb{E}_\xi |I_n - I'_n| &= \mathbb{E}_\xi \left| \frac{1}{\Pi_n} \sum_{u \in \mathbb{T}_n} X_u \mathbf{1}_{\{|X_u| > \Pi_n\}} \right| \\ &\le \mathbb{E}_\xi \left\{ \frac{1}{\Pi_n} \sum_{u \in \mathbb{T}_n} \mathbb{E}_{\xi,n} (|X_u| \mathbf{1}_{\{|X_u| > \Pi_n\}}) \right\} \\ &= \mathbb{E}_\xi \big(|\widehat{X}_n| \mathbf{1}_{\{|\widehat{X}_n| > \Pi_n\}} \big) \\ &\le \frac{1}{(\ln \Pi_n)^{1+\lambda}} \mathbb{E}_\xi |\widehat{X}_n| (\ln^+ |\widehat{X}_n|)^{1+\lambda}. \end{aligned}$$

From this and (4.7),

$$\mathbb{E}_\xi \sum_{n=0}^\infty \Big| I_n - I'_n \Big| \le \sum_{n=0}^\infty \mathbb{E}_\xi |I_n - I'_n| < \infty,$$

whence $\sum_{n=0}^\infty (I_n - I'_n)$ converges a.s.

For the third series $\sum_{n=0}^\infty \mathbb{E}_{\xi,n} I'_n$, as $\mathbb{E}_{\xi,n} I_n = 0$, we have

$$\mathbb{E}_\xi \sum_{n=0}^\infty |\mathbb{E}_{\xi,n} I'_n| = E_\xi \sum_{n=0}^\infty |\mathbb{E}_{\xi,n} (I_n - I'_n)| \le \sum_{n=0}^\infty \mathbb{E}_\xi |I_n - I'_n| < \infty,$$

so that $\sum_{n=0}^\infty \mathbb{E}_{\xi,n} I'_n$ converges a.s. It remains to prove that the second series

$$\sum_{n=0}^\infty (I'_n - \mathbb{E}_{\xi,n} I'_n) \text{ converges a.s.} \tag{4.8}$$

By using the fact that $\sum_{k=1}^n (I'_k - \mathbb{E}_{\xi,k} I'_k)$ is a martingale w.r.t. $\{\mathscr{D}_{n+1}\}$ and by the a.s. convergence of an L^2 bounded martingale (see e.g. [15, P. 251, Ex. 4.9]), we only need to show that the series $\sum_{n=0}^\infty \mathbb{E}_\xi (I'_n - \mathbb{E}_{\xi,n} I'_n)^2$ converges a.s. Notice

$$\begin{aligned} \mathbb{E}_\xi (I'_n - \mathbb{E}_{\xi,n} I'_n)^2 &= \mathbb{E}_\xi \left(\frac{1}{\Pi_n} \sum_{u \in \mathbb{T}_n} (X'_u - \mathbb{E}_{\xi,n} X'_u) \right)^2 = \mathbb{E}_\xi \left(\frac{1}{\Pi_n^2} \sum_{u \in \mathbb{T}_n} \mathbb{E}_{\xi,n} (X'_u - \mathbb{E}_{\xi,n} X'_u)^2 \right) \\ &\le \mathbb{E}_\xi \frac{1}{\Pi_n^2} \sum_{u \in \mathbb{T}_n} \mathbb{E}_{\xi,n} X'^2_u = \frac{1}{\Pi_n} \mathbb{E}_\xi (\widehat{X}_n^2 \mathbf{1}_{\{|\widehat{X}_n| \le \Pi_n\}}) \end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{\Pi_n}\mathbb{E}_\xi\Big(\widehat{X}_n^2\mathbf{1}_{\{|\widehat{X}_n|\le\Pi_n\}}\mathbf{1}_{\{|\widehat{X}_n|\le e^{2\lambda}\}} + \widehat{X}_n^2\mathbf{1}_{\{|\widehat{X}_n|\le\Pi_n\}}\mathbf{1}_{\{|\widehat{X}_n|> e^{2\lambda}\}}\Big)\\
&\le \frac{e^{4\lambda}}{\Pi_n} + \frac{1}{\Pi_n}\mathbb{E}_\xi\frac{\widehat{X}_n^2\Pi_n(\ln\Pi_n)^{-(1+\lambda)}}{|\widehat{X}_n|(\ln^+|\widehat{X}_n|)^{-(1+\lambda)}}\\
&\quad(\text{ because } x(\ln x)^{-1-\lambda} \text{ is increasing for } x > e^{2\lambda})\\
&= \frac{e^{4\lambda}}{\Pi_n} + \frac{1}{(\ln\Pi_n)^{1+\lambda}}\mathbb{E}_\xi|\widehat{X}_n|(\ln^+|\widehat{X}_n|)^{1+\lambda}.
\end{aligned}
$$

Therefore by (4.7), we see that $\sum_{n=0}^{\infty}\mathbb{E}_\xi(I'_n - \mathbb{E}_{\xi,n}I'_n)^2 < \infty$ a.s.. This implies (4.8).

Combining the above results, we see that the series $\sum I_n$ converges a.s., so that $N_{1,n}$ converges a.s. to

$$V_1 = \sum_{n=1}^{\infty}(N_{1,n+1} - N_{1,n}) + N_{1,1}.$$

□

4.2 Convergence of the martingale $\{(N_{2,n}, \mathscr{D}_n)\}$

To see that $\{(N_{2,n}, \mathscr{D}_n)\}$ is a martingale, it suffices to notice that (remind that we have assumed $\ell_n = 0$)

$$
\begin{aligned}
\mathbb{E}_{\xi,n}N_{2,n+1} &= \mathbb{E}_{\xi,n}\Big(\frac{1}{\Pi_{n+1}}\sum_{u\in\mathbb{T}_{n+1}}(S_u^2 - s_{n+1}^2)\Big)\\
&= \frac{1}{\Pi_{n+1}}\sum_{u\in\mathbb{T}_n}\mathbb{E}_{\xi,n}\Big(\sum_{i=1}^{N_u}\Big((S_u + L_{ui})^2 - s_{n+1}^2\Big)\Big)\\
&= \frac{1}{\Pi_{n+1}}\sum_{u\in\mathbb{T}_n}\mathbb{E}_{\xi,n}\Big(\sum_{i=1}^{N_u}(S_u^2 + 2S_uL_{ui} + L_{ui}^2 - s_{n+1}^2)\Big)\\
&= \frac{1}{\Pi_{n+1}}\sum_{u\in\mathbb{T}_n}\mathbb{E}_{\xi,n}\Big(\sum_{i=1}^{N_u}\mathbb{E}_{\xi,n}\Big\{(S_u^2 + 2S_uL_{ui} + L_{ui}^2 - s_{n+1}^2)\Big|N_u\Big\}\Big)\\
&= \frac{1}{\Pi_{n+1}}\sum_{u\in\mathbb{T}_n}m_n(S_u^2 + \sigma_n^{(2)} - s_{n+1}^2) = \frac{1}{\Pi_n}\sum_{u\in\mathbb{T}_n}(S_u^2 - s_n^2) = N_{2,n}.
\end{aligned}
$$

As in the case of $\{(N_{1,n}, \mathscr{D}_n)\}$, we will prove the convergence of the martingale $\{(N_{2,n}, \mathscr{D}_n)\}$ by showing that

$$\sum_{n=1}^{\infty}(N_{2,n+1} - N_{2,n}) \text{ converges a.s.,}$$

following the same lines as before. For $n \ge 1$ and $|u| = n$, we will still use the notation X_u and I_n, but this time they are defined by:

$$X_u = (S_u^2 - s_n^2)(\frac{N_u}{m_n} - 1) + \frac{1}{m_n}\sum_{i=1}^{N_u}(L_{ui}^2 - \sigma_n^{(2)}) + \frac{2}{m_n}S_u\sum_{i=1}^{N_u}L_{ui}, \tag{4.9}$$

$$I_n = N_{2,n+1} - N_{2,n} = \frac{1}{\Pi_n}\sum_{u\in\mathbb{T}_n}X_u. \tag{4.10}$$

Instead of Lemma 4.1, we have:

Lemma 4.2. *For $n \ge 1$ and $|u| = n$, let $\widehat{X}_n$ be a random variable with the common distribution of X_u defined by (4.9), under the law $\mathbb{P}_\xi$. If the conditions of Proposition 2.2 holds, then*

$$\mathbb{E}_\xi|\widehat{X}_n|(\ln^+|\widehat{X}_n|)^{1+\lambda} \le K_\xi n^2\Big[\mathbb{E}_\xi\frac{\widehat{N}_n}{m_n}(\ln^+\widehat{N}_n)^{1+\lambda} + (\ln^- m_n)^{1+\lambda} + 1\Big]. \tag{4.11}$$

Proof. Observe that for $|u| = n$,

$$|X_u| \le |s_n^2 - S_u^2|(1 + \frac{N_u}{m_n}) + \left|\frac{1}{m_n}\sum_{i=1}^{N_u}(\sigma_n^{(2)} - L_{ui}^2)\right| + |S_u|\left|\frac{2}{m_n}\sum_{i=1}^{N_u} L_{ui}\right|,$$

$$\ln^+|X_u| \le 2 + \ln^+|s_n^2 - S_u^2| + \ln(1 + \frac{N_u}{m_n}) + \ln^+\left|\frac{1}{m_n}\sum_{i=1}^{N_u}(\sigma_n^{(2)} - L_{ui}^2)\right| + \ln^+\left|\frac{2}{m_n}\sum_{i=1}^{N_u} L_{ui}\right| + \ln^+|S_u|,$$

$$6^{-\lambda}(\ln^+|X_u|)^{1+\lambda} \le 2^{1+\lambda} + (\ln^+|s_n^2 - S_u^2|)^{1+\lambda} + (\ln(1 + \frac{N_u}{m_n}))^{1+\lambda} + \left(\ln^+\left|\frac{1}{m_n}\sum_{i=1}^{N_u}(\sigma_n^{(2)} - L_{ui}^2)\right|\right)^{1+\lambda} + \left(\ln^+\left|\frac{2}{m_n}\sum_{i=1}^{N_u} L_{ui}\right|\right)^{1+\lambda} + (\ln^+|S_u|)^{1+\lambda}.$$

Therefore

$$6^{-\lambda}|X_u|(\ln^+|X_u|)^{1+\lambda} \le \sum_{i=1}^{8}\mathbb{K}_i$$

with

$$\mathbb{K}_1 = |s_n^2 - S_u^2|(1 + \frac{N_u}{m_n})\left[2^{1+\lambda} + \Big(\ln(1 + \frac{N_u}{m_n})\Big)^{1+\lambda} + \left(\ln^+\left|\frac{1}{m_n}\sum_{i=1}^{N_u}(\sigma_n^{(2)} - L_{ui}^2)\right|\right)^{1+\lambda} + \left(\ln^+\left|\frac{2}{m_n}\sum_{i=1}^{N_u} L_{ui}\right|\right)^{1+\lambda}\right],$$

$$\mathbb{K}_2 = |s_n^2 - S_u^2|(1 + \frac{N_u}{m_n})\Big[(\ln^+|s_n^2 - S_u^2|)^{1+\lambda} + (\ln^+|S_u|)^{1+\lambda}\Big],$$

$$\mathbb{K}_3 = \left|\frac{1}{m_n}\sum_{i=1}^{N_u}(\sigma_n^{(2)} - L_{ui}^2)\right|\Big[2^{1+\lambda} + (\ln^+|s_n^2 - S_u^2|)^{1+\lambda} + (\ln^+|S_u|)^{1+\lambda}\Big],$$

$$\mathbb{K}_4 = \left|\frac{1}{m_n}\sum_{i=1}^{N_u}(\sigma_n^{(2)} - L_{ui}^2)\right|\Big(\ln(1 + \frac{N_u}{m_n})\Big)^{1+\lambda},$$

$$\mathbb{K}_5 = \left|\frac{1}{m_n}\sum_{i=1}^{N_u}(\sigma_n^{(2)} - L_{ui}^2)\right|\left[\left(\ln^+\left|\frac{2}{m_n}\sum_{i=1}^{N_u} L_{ui}\right|\right)^{1+\lambda} + \left(\ln^+\left|\frac{1}{m_n}\sum_{i=1}^{N_u}(\sigma_n^{(2)} - L_{ui}^2)\right|\right)^{1+\lambda}\right],$$

$$\mathbb{K}_6 = \left|\frac{2}{m_n}\sum_{i=1}^{N_u} L_{ui}\right||S_u|\Big[2^{1+\lambda} + (\ln^+|s_n^2 - S_u^2|)^{1+\lambda} + (\ln^+|S_u|)^{1+\lambda}\Big],$$

$$\mathbb{K}_7 = \left|\frac{2}{m_n}\sum_{i=1}^{N_u} L_{ui}\right||S_u|\Big(\ln(1 + \frac{N_u}{m_n})\Big)^{1+\lambda},$$

$$\mathbb{K}_8 = \left|\frac{2}{m_n}\sum_{i=1}^{N_u} L_{ui}\right||S_u|\left[\left(\ln^+\left|\frac{2}{m_n}\sum_{i=1}^{N_u} L_{ui}\right|\right)^{1+\lambda} + \left(\ln^+\left|\frac{1}{m_n}\sum_{i=1}^{N_u}(\sigma_n^{(2)} - L_{ui}^2)\right|\right)^{1+\lambda}\right].$$

It is clear that (4.4) remains valid here; similarly, we get

$$\mathbb{E}_\xi|\sigma_n^{(2)} - L_{ui}^2| = \mathbb{E}_\xi|\sigma_n^{(2)} - \widehat{L}_n^2|^2 \le K_\xi n$$

(recall that $\widehat{L}_n$ is a random variable with the same distribution as L_{ui} for any $|u| = n$ and $i \ge 1$). By the definition of the model, S_u, N_u and L_{ui} are mutually independent under $\mathbb{P}_\xi$. On the basis of the above estimates, we have the following inequalities: for $|u| = n$,

$$\mathbb{E}_\xi\mathbb{K}_1 \le \mathbb{E}_\xi|S_u^2 + s_n^2|\mathbb{E}_\xi(1 + \frac{N_u}{m_n})\left[2^{1+\lambda} + \Big(\ln(1 + \frac{N_u}{m_n})\Big)^{1+\lambda} + \left(\ln\Big(e^\lambda + \frac{1}{m_n}\sum_{i=1}^{N_u}\mathbb{E}_\xi|\sigma_n^{(2)} - L_{ui}^2|\Big)\right)^{1+\lambda}\right.$$

$$+\left(\ln\Big(e^{\lambda}+\frac{2}{m_n}\sum_{i=1}^{N_u}\mathbb{E}_\xi|L_{ui}|\Big)\right)^{1+\lambda}\Bigg] \quad \text{(by Jensen's inequality under } \mathbb{E}_\xi(\cdot|N_u))$$

$$\le K_\xi n\left[K_\xi+\mathbb{E}_\xi\frac{N_u}{m_n}\big(\ln^+ N_u\big)^{1+\lambda}+\big(\ln^- m_n\big)^{1+\lambda}+(\ln n)^{1+\lambda}\right];$$

$$\mathbb{E}_\xi\mathbb{K}_2 \le 2(\mathbb{E}_\xi|S_u|^{2+\varepsilon}+|s_n|^{2+\varepsilon}) \le K_\xi n^2;$$

$$\mathbb{E}_\xi\mathbb{K}_3 \le \mathbb{E}_\xi\Big(\frac{1}{m_n}\sum_{i=1}^{N_u}\mathbb{E}_\xi|\sigma_n^{(2)}-L_{ui}^2|\Big)\Big(2^{1+\lambda}+(\ln(e^\lambda+\mathbb{E}_\xi|s_n^2-S_u^2|))^{1+\lambda}+(\ln(e^\lambda+\mathbb{E}_\xi|S_u|))^{1+\lambda}\Big)$$

$$\le K_\xi n(\ln n)^{1+\lambda};$$

$$\mathbb{E}_\xi\mathbb{K}_4 \le K_\xi n+K_\xi n\mathbb{E}_\xi\frac{N_u}{m_n}\Big(\ln(1+\frac{N_u}{m_n})\Big)^{1+\lambda};$$

$$\mathbb{E}_\xi\mathbb{K}_5 \le 3^\lambda\mathbb{E}_\xi\frac{1}{m_n}\Big|\sum_{i=1}^{N_u}(\sigma_n^{(2)}-L_{ui}^2)\Big|\Bigg[\Big(\ln^+\Big|\sum_{i=1}^{N_u}L_{ui}\Big|\Big)^{1+\lambda}+\Big(\ln^+\Big|\sum_{i=1}^{N_u}(\sigma_n^{(2)}-L_{ui}^2)\Big|\Big)^{1+\lambda}+$$

$$2(\ln^- m_n)^{1+\lambda}+1\Bigg]$$

$$\le K_\lambda\frac{1}{m_n}\mathbb{E}_\xi\left[\Big|\sum_{i=1}^{N_u}(\sigma_n^{(2)}-L_{ui}^2)\Big|^2+\Big(\ln^+\Big|\sum_{i=1}^{N_u}L_{ui}\Big|\Big)^{2+2\lambda}\right]+$$

$$K_\lambda\mathbb{E}_\xi\frac{1}{m_n}\Big|\sum_{i=1}^{N_u}(\sigma_n^{(2)}-L_{ui}^2)\Big|^2+K_\xi n((\ln^- m_n)^{1+\lambda}+1) \qquad \text{(by (3.3) and } 2ab\le a^2+b^2)$$

$$\le_{(3.3)} K_\lambda\frac{1}{m_n}\mathbb{E}_\xi\Big[\sum_{i=1}^{N_u}\mathbb{E}_\xi(\sigma_n^{(2)}-L_{ui}^2)^2\Big]+K_\lambda\frac{1}{m_n}\mathbb{E}_\xi\Big[\sum_{i=1}^{N_u}\mathbb{E}_\xi\Big|L_{ui}\Big|\Big]+K_\xi n((\ln^- m_n)^{1+\lambda}+1)$$

$$\le K_\xi n\big((\ln^- m_n)^{1+\lambda}+1\big);$$

$$\mathbb{E}_\xi\mathbb{K}_6 \le_{(3.2)} \mathbb{E}_\xi\Big(\frac{2}{m_n}\sum_{i=1}^{N_u}\mathbb{E}_\xi|L_{ui}|\Big)\mathbb{E}_\xi\Big[K_\lambda|S_u|\big(1+(\ln^+|S_u|)^{1+\lambda}+(\ln s_n^2)^{1+\lambda}\big)\Big]$$

$$\le_{(3.3)} K_\xi n\mathbb{E}_\xi\Big[|S_u|^2+|S_u|+s_n^2\big)\Big] \le K_\xi n^2;$$

$$\mathbb{E}_\xi\mathbb{K}_7 \le \mathbb{E}_\xi\left(\Big(\ln(1+\frac{N_u}{m_n})\Big)^{1+\lambda}\frac{2}{m_n}\sum_{i=1}^{N_u}\mathbb{E}_\xi|L_{ui}|\right)\mathbb{E}_\xi|S_u|$$

$$\le K_\xi n^2\Big[\mathbb{E}_\xi\frac{N_u}{m_n}\Big(\ln^+ N_u\Big)^{1+\lambda}+(\ln^- m_n)^{1+\lambda}\Big];$$

$$\mathbb{E}_\xi\mathbb{K}_8 \le K_\xi n^2\big((\ln^- m_n)^{1+\lambda}+1\big) \qquad \text{(similar reason as in the estimation for } \mathbb{E}_\xi\mathbb{K}_5).$$

Combining the above estimates, we get that

$$\mathbb{E}_\xi|\widehat{X}_n|(\ln^+|\widehat{X}_n|)^{1+\lambda} \le K_\xi n^2\left(\mathbb{E}_\xi\frac{\widehat{N}_n}{m_n}\Big(\ln^+\widehat{N}_n\Big)^{1+\lambda}+(\ln^- m_n)^{1+\lambda}+1\right) \tag{4.12}$$

This ends the proof of Lemma 4.2. □

Proof of Proposition 2.2 . The proof is almost the same as that of Proposition 2.1: we still use the decomposition (4.6), but with I_n and X_u defined by (4.10) and (4.9), and Lemma 4.2 instead of Lemma 4.1, to prove that the series $\sum_{n=0}^{\infty}(N_{2,n+1}-N_{2,n})$ converges a.s., yielding that $\{N_{2,n}\}$ converges a.s. to

$$V_2=\sum_{n=1}^{\infty}(N_{2,n+1}-N_{2,n})+N_{2,1}.$$

□

5 Proof of Theorem 2.3

5.1 A key decomposition

For $u \in (\mathbb{N}^*)^k (k \geq 0)$ and $n \geq 1$, write for $B \subset \mathbb{R}$,

$$Z_n(u, B) = \sum_{v \in \mathbb{T}_n(u)} \mathbf{1}_B(S_{uv} - S_u).$$

It can be easily seen that the law of $Z_n(u, B)$ under $\mathbb{P}_\xi$ is the same as that of $Z_n(B)$ under $P_{\theta^k \xi}$. Define

$$\begin{aligned} &W_n(u, B) = Z_n(u, B)/\Pi_n(\theta^k \xi), \quad W_n(u, t) = W_n(u, (-\infty, t]), \\ &W_n(B) = Z_n(B)/\Pi_n, \quad W_n(t) = W_n((-\infty, t]). \end{aligned}$$

By definition, we have $\Pi_n(\theta^k \xi) = m_k \cdots m_{k+n-1}$, $Z_n(B) = Z_n(\varnothing, B)$, $W_n(B) = W_n(\varnothing, B)$, $W_n = W_n(\mathbb{R})$. The following decomposition will play a key role in our approach: for $k \leq n$,

$$Z_n(B) = \sum_{u \in \mathbb{T}_k} Z_{n-k}(u, B - S_u). \tag{5.1}$$

Remark that by our definition, for $u \in \mathbb{T}_k$,

$$Z_{n-k}(u, B - S_u) = \sum_{v_1 \cdots v_{n-k} \in \mathbb{T}_{n-k}(u)} \mathbf{1}_B(S_{uv_1 \cdots v_{n-k}})$$

represents number of the descendants of u at time n situated in B.

For each n, we choose an integer $k_n < n$ as follows. Let β be a real number such that $\max\{\frac{2}{\lambda}, \frac{3}{\eta}\} < \beta < \frac{1}{4}$ and set $k_n = \lfloor n^\beta \rfloor$, the integral part of n^β. Then on the basis of (5.1), the following decomposition will hold:

$$\Pi_n^{-1} Z_n(s_n t) - \Phi(t) W = \mathbb{A}_n + \mathbb{B}_n + \mathbb{C}_n, \tag{5.2}$$

where

$$\begin{aligned} \mathbb{A}_n &= \frac{1}{\Pi_{k_n}} \sum_{u \in \mathbb{T}_{k_n}} \left[W_{n-k_n}(u, s_n t - S_u) - \mathbb{E}_{\xi, k_n} W_{n-k_n}(u, s_n t - S_u) \right], \\ \mathbb{B}_n &= \frac{1}{\Pi_{k_n}} \sum_{u \in \mathbb{T}_{k_n}} \left[\mathbb{E}_{\xi, k_n} W_{n-k_n}(u, s_n t - S_u) - \Phi(t) \right], \\ \mathbb{C}_n &= (W_{k_n} - W)\Phi(t). \end{aligned}$$

Here we remind that the random variables $W_{n-k_n}(u, s_n t - S_u)$ are independent of each other under the conditional probability $\mathbb{P}_{\xi, k_n}$.

5.2 Proof of Theorem 2.3

First, observe that the condition $\mathbb{E} m_0^{-\delta} < \infty$ implies that $\mathbb{E}\big(\ln^- m_0\big)^\kappa < \infty$ for all $\kappa > 0$. So the hypotheses of Propositions 2.1 and 2.2 are satisfied under the conditions of Theorem 2.3.

By virtue of the decomposition (5.2), we shall divide the proof into three lemmas.

Lemma 5.1. *Under the hypothesis of Theorem 2.3,*

$$\sqrt{n} \mathbb{A}_n \xrightarrow{n \to \infty} 0 \text{ a.s.} \tag{5.3}$$

Lemma 5.2. *Under the hypothesis of Theorem 2.3,*

$$\sqrt{n} \mathbb{B}_n \xrightarrow{n \to \infty} \frac{1}{6} \mathbb{E}\sigma_0^{(3)} (\mathbb{E}\sigma_0^{(2)})^{-\frac{3}{2}} (1 - t^2) \phi(t) W - (\mathbb{E}\sigma_0^{(2)})^{-\frac{1}{2}} \phi(t)\, V_1 \text{ a.s.} \tag{5.4}$$

Lemma 5.3. *Under the hypothesis of Theorem 2.3,*

$$\sqrt{n} \mathbb{C}_n \xrightarrow{n \to \infty} 0 \text{ a.s.} \tag{5.5}$$

Now we go to prove the lemmas subsequently.

Proof of Lemma 5.1. For ease of notation, we define for $|u| = k_n$,

$$X_{n,u} = W_{n-k_n}(u, s_n t - S_u) - \mathbb{E}_{\xi,k_n} W_{n-k_n}(u, s_n t - S_u), \;\; \bar{X}_{n,u} = X_{n,u}\mathbf{1}_{\{|X_{n,u}|<\Pi_{k_n}\}},$$
$$\bar{A}_n = \frac{1}{\Pi_{k_n}} \sum_{u\in\mathbb{T}_{k_n}} \bar{X}_{n,u}.$$

Then we see that $|X_{n,u}| \le W_{n-k_n}(u) + 1$.

To prove Lemma 5.1, we will use the extended Borel-Cantelli Lemma. We can obtain the required result once we prove that $\forall \varepsilon > 0$,

$$\sum_{n=1}^{\infty} \mathbb{P}_{k_n}(|\sqrt{n}A_n| > 2\varepsilon) < \infty. \tag{5.6}$$

Notice that

$$\begin{aligned} & \mathbb{P}_{k_n}(|A_n| > 2\frac{\varepsilon}{\sqrt{n}}) \\ \le \;\; & \mathbb{P}_{k_n}(A_n \neq \bar{A}_n) + \mathbb{P}_{k_n}(|\bar{A}_n - \mathbb{E}_{\xi,k_n}\bar{A}_n| > \frac{\varepsilon}{\sqrt{n}}) + \mathbb{P}_{k_n}(|\mathbb{E}_{\xi,k_n}\bar{A}_n| > \frac{\varepsilon}{\sqrt{n}}). \end{aligned}$$

We will proceed the proof in 3 steps.

Step 1 We first prove that

$$\sum_{n=1}^{\infty} \mathbb{P}_{k_n}(A_n \neq \overline{A}_n) < \infty. \tag{5.7}$$

To this end, define

$$W^* = \sup_n W_n,$$

and we need the following result :

Lemma 5.4. *([29, Th. 1.2]) Assume* (2.1) *for some $\lambda > 0$ and $\mathbb{E}m_0^{-\delta} < \infty$ for some $\delta > 0$. Then*

$$\mathbb{E}(W^* + 1)(\ln(W^* + 1))^{\lambda} < \infty. \tag{5.8}$$

We observe that

$$\begin{aligned} \mathbb{P}_{k_n}(A_n \neq \overline{A}_n) &\le \sum_{u\in\mathbb{T}_{k_n}} \mathbb{P}_{k_n}(X_{n,u} \neq \overline{X}_{n,u}) = \sum_{u\in\mathbb{T}_{k_n}} \mathbb{P}_{k_n}(|X_{n,u}| \ge \Pi_{k_n}) \\ &\le \sum_{u\in\mathbb{T}_{k_n}} \mathbb{P}_{k_n}(W_{n-k_n}(u) + 1 \ge \Pi_{k_n}) \\ &= W_{k_n}\Big[r_n \mathbb{P}(W_{n-k_n} + 1 \ge r_n)\Big]_{r_n=\Pi_{k_n}} \\ &\le W_{k_n}\Big[\mathbb{E}\big((W_{n-k_n} + 1)\mathbf{1}_{\{W_{n-k_n}+1\ge r_n\}}\big)\Big]_{r_n=\Pi_{k_n}} \\ &\le W_{k_n}\Big[\mathbb{E}\big((W^* + 1)\mathbf{1}_{\{W^*+1\ge r_n\}}\big)\Big]_{r_n=\Pi_{k_n}} \\ &\le W^*(\ln \Pi_{k_n})^{-\lambda}\mathbb{E}(W^* + 1)(\ln(W^* + 1))^{\lambda} \\ &\le K_\xi W^* n^{-\lambda\beta}\mathbb{E}(W^* + 1)(\ln(W^* + 1))^{\lambda}, \end{aligned}$$

where the last inequality holds since

$$\frac{1}{n}\ln \Pi_n \to \mathbb{E}\ln m_0 > 0 \text{ a.s. }, \tag{5.9}$$

and $k_n \sim n^{\beta}$. By the choice of β and Lemma 5.4, we obtain (5.7).

Step 2. We next prove that $\forall \varepsilon > 0$,

$$\sum_{n=1}^{\infty} \mathbb{P}_{k_n}(|\overline{A}_n - \mathbb{E}_{\xi,k_n}\overline{A}_n| > \frac{\varepsilon}{\sqrt{n}}) < \infty. \tag{5.10}$$

Take a constant $b \in (1, e^{\mathbb{E}\ln m_0})$. Observe that $\forall u \in \mathbb{T}_{k_n}, n \geq 1$,

$$\begin{aligned}
\mathbb{E}_{k_n}\bar{X}_{n,u}^2 &= \int_0^\infty 2x\mathbb{P}_{k_n}(|\bar{X}_{n,u}| > x)dx = 2\int_0^\infty x\mathbb{P}_{k_n}(|X_{n,u}|\mathbf{1}_{\{|X_{n,u}|<\Pi_{k_n}\}} > x)dx \\
&\leq 2\int_0^{\Pi_{k_n}} x\mathbb{P}_{k_n}(|W_{n-k_n}(u)+1| > x)dx = 2\int_0^{\Pi_{k_n}} x\mathbb{P}(|W_{n-k_n}+1| > x)dx \\
&\leq 2\int_0^{\Pi_{k_n}} x\mathbb{P}(W^*+1 > x)dx \\
&\leq 2\int_e^{\Pi_{k_n}} (\ln x)^{-\lambda}\mathbb{E}(W^*+1)(\ln(W^*+1))^\lambda dx + 9 \\
&\leq 2\mathbb{E}(W^*+1)(\ln(W^*+1))^\lambda\left(\int_e^{b^{k_n}} (\ln x)^{-\lambda}dx + \int_{b^{k_n}}^{\Pi_{k_n}} (\ln x)^{-\lambda}dx\right) + 9 \\
&\leq 2\mathbb{E}(W^*+1)(\ln(W^*+1))^\lambda(b^{k_n} + (\Pi_{k_n} - b^{k_n})(k_n\ln b)^{-\lambda}) + 9.
\end{aligned}$$

Then we have that

$$\begin{aligned}
&\sum_{n=1}^\infty \mathbb{P}_{k_n}(|\overline{A}_n - \mathbb{E}_{\xi,k_n}\overline{A}_n| > \frac{\varepsilon}{\sqrt{n}}) \\
&= \sum_{n=1}^\infty \mathbb{E}_{k_n}\mathbb{P}_{\xi,k_n}(|\overline{A}_n - \mathbb{E}_{\xi,k_n}\overline{A}_n| > \frac{\varepsilon}{\sqrt{n}}) \\
&\leq \varepsilon^{-2}\sum_{n=1}^\infty n\mathbb{E}_{k_n}\left(\Pi_{k_n}^{-2}\sum_{u\in\mathbb{T}_{k_n}}\mathbb{E}_{\xi,k_n}\overline{X}_{n,u}^2\right) = \varepsilon^{-2}\sum_{n=1}^\infty n\left(\Pi_{k_n}^{-2}\sum_{u\in\mathbb{T}_{k_n}}\mathbb{E}_{k_n}\overline{X}_{n,u}^2\right) \\
&\leq \varepsilon^{-2}\sum_{n=1}^\infty \frac{nW_{k_n}}{\Pi_{k_n}}\left[2\mathbb{E}(W^*+1)(\ln(W^*+1)^\lambda)(b^{k_n} + (\Pi_{k_n} - b^{k_n})(k_n\ln b)^{-\lambda}) + 9\right] \\
&\leq 2\varepsilon^{-2}W^*\mathbb{E}(W^*+1)(\ln(W^*+1)^\lambda)\left(\sum_{n=1}^\infty \frac{n}{\Pi_{k_n}}b^{k_n} + \sum_{n=1}^\infty n(k_n\ln b)^{-\lambda}\right) + 9\varepsilon^{-2}W^*\sum_{n=1}^\infty \frac{n}{\Pi_{k_n}}.
\end{aligned}$$

By (5.9) and $\lambda\beta > 2$, the three series in the last expression above converge under our hypothesis and hence (5.10) is proved.

Step 3. Observe

$$\begin{aligned}
&\mathbb{P}_{k_n}\left(|\mathbb{E}_{\xi,k_n}\bar{A}_n| > \frac{\varepsilon}{\sqrt{n}}\right) \\
\leq\ & \frac{\sqrt{n}}{\varepsilon}\mathbb{E}_{k_n}|\mathbb{E}_{\xi,k_n}\bar{A}_n| = \frac{\sqrt{n}}{\varepsilon}\mathbb{E}_{k_n}\Big|\frac{1}{\Pi_{k_n}}\sum_{u\in\mathbb{T}_{k_n}}\mathbb{E}_{\xi,k_n}\bar{X}_{n,u}\Big| \\
=\ & \frac{\sqrt{n}}{\varepsilon}\mathbb{E}_{k_n}\Big|\frac{1}{\Pi_{k_n}}\sum_{u\in\mathbb{T}_{k_n}}(-\mathbb{E}_{\xi,k_n}X_{n,u}\mathbf{1}_{\{|X_{n,u}|\geq\Pi_{k_n}\}})\Big| \\
\leq\ & \frac{\sqrt{n}}{\varepsilon}\frac{1}{\Pi_{k_n}}\sum_{u\in\mathbb{T}_{k_n}}\mathbb{E}_{k_n}(W_{n-k_n}(u)+1)\mathbf{1}_{\{W_{n-k_n}(u)+1\geq\Pi_{k_n}\}} \\
=\ & \frac{\sqrt{n}W_{k_n}}{\varepsilon}\Big[\mathbb{E}(W_{n-k_n}+1)\mathbf{1}_{\{W_{n-k_n}+1\geq r_n\}}\Big]_{r_n=\Pi_{k_n}} \\
\leq\ & \frac{W^*}{\varepsilon}\sqrt{n}\Big[\mathbb{E}(W^*+1)\mathbf{1}_{\{W^*+1\geq r_n\}}\Big]_{r_n=\Pi_{k_n}} \\
\leq\ & \frac{W^*}{\varepsilon}\frac{\sqrt{n}}{(\ln\Pi_{k_n})^\lambda}\mathbb{E}(W^*+1)\ln^\lambda(W^*+1) \\
\leq\ & \frac{W^*}{\varepsilon}K_\xi n^{\frac{1}{2}-\lambda\beta}\mathbb{E}(W^*+1)\ln^\lambda(W^*+1).
\end{aligned}$$

Then by (5.9) and $\lambda\beta > 2$, it follows that

$$\sum_{n=1}^{\infty} \mathbb{P}_{k_n}\left(|\mathbb{E}_{\xi,k_n} \bar{A}_n| > \frac{\varepsilon}{\sqrt{n}} \right) < \infty.$$

Combining Steps 1-3, we obtain (5.6). Hence the lemma is proved. □

Proof of Lemma 5.2 . For ease of notation, set

$$D_1(t) = (1-t^2)\phi(t), \qquad \kappa_{1,n} = \frac{s_n^{(3)} - s_{k_n}^{(3)}}{6(s_n^2 - s_{k_n}^2)^{3/2}}.$$

Observe that

$$\mathbb{B}_n = \mathbb{B}_{n1} + \mathbb{B}_{n2} + \mathbb{B}_{n3} + \mathbb{B}_{n4}, \tag{5.11}$$

where

$$\mathbb{B}_{n1} = \frac{1}{\Pi_{k_n}} \sum_{u \in \mathbb{T}_{k_n}} \left(\mathbb{E}_{\xi,k_n} W_{n-k_n}(u, s_n t - S_u) - \Phi\left(\frac{s_n t - S_u}{(s_n^2 - s_{k_n}^2)^{1/2}} \right) - \kappa_{1,n} D_1\left(\frac{s_n t - S_u}{(s_n^2 - s_{k_n}^2)^{1/2}} \right) \right);$$

$$\mathbb{B}_{n2} = \frac{1}{\Pi_{k_n}} \sum_{u \in \mathbb{T}_{k_n}} \left(\Phi\left(\frac{s_n t - S_u}{(s_n^2 - s_{k_n}^2)^{1/2}} \right) - \Phi(t) \right);$$

$$\mathbb{B}_{n3} = \kappa_{1,n} \frac{1}{\Pi_{k_n}} \sum_{u \in \mathbb{T}_{k_n}} \left(D_1\left(\frac{s_n t - S_u}{(s_n^2 - s_{k_n}^2)^{1/2}} \right) - D_1(t) \right);$$

$$\mathbb{B}_{n4} = \kappa_{1,n} D_1(t) W_{k_n}.$$

Then the lemma will be proved once we show that

$$\sqrt{n}\mathbb{B}_{n1} \xrightarrow{n\to\infty} 0; \tag{5.12}$$

$$\sqrt{n}\mathbb{B}_{n2} \xrightarrow{n\to\infty} -(\mathbb{E}\sigma_0^{(2)})^{-\frac{1}{2}} \phi(t) V_1; \tag{5.13}$$

$$\sqrt{n}\mathbb{B}_{n3} \xrightarrow{n\to\infty} 0; \tag{5.14}$$

$$\sqrt{n}\mathbb{B}_{n4} \xrightarrow{n\to\infty} \frac{1}{6}\mathbb{E}\sigma_0^{(3)} (\mathbb{E}\sigma_0^{(2)})^{-\frac{3}{2}} D_1(t) W. \tag{5.15}$$

We will prove these results subsequently.

We first prove (5.12). The proof will mainly be based on the following result about asymptotic expansion of the distribution of the sum of independent random variables:

Proposition 5.5. *Under the hypothesis of Theorem 2.3, for a.e. ξ,*

$$\epsilon_n = n^{1/2} \sup_{x\in\mathbb{R}} \left| \mathbb{P}_\xi\left(\frac{\sum_{k=k_n}^{n-1} \widehat{L}_k}{(s_n^2 - s_{k_n}^2)^{1/2}} \le x \right) - \Phi(x) - \kappa_{1,n} D_1(x) \right| \xrightarrow{n\to\infty} 0.$$

Proof. Let $X_k = 0$ for $0 \le k \le k_n - 1$ and $X_k = \widehat{L}_k$ for $k_n \le k \le n-1$. Then the random variables $\{X_k\}$ are independent under $\mathbb{P}_\xi$. Denote by $v_k(\cdot)$ the characteristic function of X_k: $v_k(t) := \mathbb{E}_\xi e^{itX_k}$. Using the Markov inequality and Lemma 3.2, we obtain the following result:

$$\begin{aligned}
&\sup_{x\in\mathbb{R}} \left| \mathbb{P}_\xi\left(\frac{\sum_{k=k_n}^{n-1} \widehat{L}_k}{(s_n^2 - s_{k_n}^2)^{1/2}} \le x \right) - \Phi(x) - \kappa_{1,n} D_1(x) \right| \\
\le & K_\xi \left\{ (s_n^2 - s_{k_n}^2)^{-2} \sum_{j=k_n}^{n-1} \mathbb{E}_\xi |\widehat{L}_j|^4 + n^6 \left(\sup_{|t|>T} \frac{1}{n} \left(k_n + \sum_{j=k_n}^{n-1} |v_j(t)| \right) + \frac{1}{2n} \right)^n \right\}.
\end{aligned}$$

By our conditions on the environment, we know that

$$\lim_{n\to\infty} n(s_n^2 - s_{k_n}^2)^{-2} \sum_{j=k_n}^{n-1} \mathbb{E}_\xi |\widehat{L}_k|^4 = \mathbb{E}|\widehat{L}_0|^4 / (\mathbb{E}\sigma_0^{(2)})^2. \tag{5.16}$$

By (2.2), $\widehat{L}_n$ satisfies

$$\mathbb{P}\Big(\limsup_{|t|\to\infty}|v_n(t)|<1\Big)>0.$$

So there exists a constant $c_n\le 1$ depending on ξ_n such that

$$\sup_{|t|>T}|v_n(t)|\le c_n \quad \text{and} \quad \mathbb{P}(c_n<1)>0.$$

Then $\mathbb{E}c_0<1$. By the Birkhoff ergodic theorem, we have

$$\sup_{|t|>T}\left(\frac{1}{n}\sum_{j=k_n}^{n-1}|v_j(t)|\right)\le\frac{1}{n}\sum_{j=1}^{n-1}c_j\to\mathbb{E}c_0<1.$$

Then for n large enough,

$$\left(\sup_{|t|>T}\frac{1}{n}\left(k_n+\sum_{j=k_n}^{n-1}|v_j(t)|\right)+\frac{1}{2n}\right)^n=o(n^{-m}),\quad\forall m>0. \tag{5.17}$$

From (5.16) and (5.17), we get the conclusion of the proposition. □

From Proposition 5.5, it is easy to see that

$$\sqrt{n}|\mathbb{B}_{n1}|\le W_{k_n}\epsilon_n\xrightarrow{n\to\infty}0.$$

Hence (5.12) is proved.

We next prove (5.13). Observe that

$$\mathbb{B}_{n2}=\mathbb{B}_{n21}+\mathbb{B}_{n22}+\mathbb{B}_{n23}+\mathbb{B}_{n24}+\mathbb{B}_{n25},$$

$$\text{with }\ \mathbb{B}_{n21}=\frac{1}{\Pi_{k_n}}\sum_{u\in\mathbb{T}_{k_n}}\left[\Phi\Big(\frac{s_nt-S_u}{(s_n^2-s_{k_n}^2)^{1/2}}\Big)-\Phi(t)-\phi(t)\Big(\frac{s_nt-S_u}{(s_n^2-s_{k_n}^2)^{1/2}}-t\Big)\right]\mathbf{1}_{\{|S_u|\le k_n\}},$$

$$\mathbb{B}_{n22}=\frac{1}{\Pi_{k_n}}\sum_{u\in\mathbb{T}_{k_n}}\left[\Phi\Big(\frac{s_nt-S_u}{(s_n^2-s_{k_n}^2)^{1/2}}\Big)-\Phi(t)\right]\mathbf{1}_{\{|S_u|>k_n\}},$$

$$\mathbb{B}_{n23}=-\frac{1}{\Pi_{k_n}}\sum_{u\in\mathbb{T}_{k_n}}\Big(\frac{s_nt-S_u}{(s_n^2-s_{k_n}^2)^{1/2}}-t\Big)\phi(t)\mathbf{1}_{\{|S_u|>k_n\}},$$

$$\mathbb{B}_{n24}=\frac{1}{(s_n^2-s_{k_n}^2)^{1/2}}\big(s_n-(s_n^2-s_{k_n}^2)^{1/2}\big)W_{k_n}\phi(t)t,$$

$$\mathbb{B}_{n25}=-\frac{1}{(s_n^2-s_{k_n}^2)^{1/2}}\phi(t)N_{1,k_n}.$$

By Taylor's formula and the choice of β and k_n, we get

$$\begin{aligned}\widetilde{\epsilon}_n=&\sqrt{n}\sup_{|y|\le k_n}\left|\Phi\Big(\frac{s_nt-y}{(s_n^2-s_{k_n}^2)^{1/2}}\Big)-\Phi(t)-\phi(t)\Big(\frac{s_nt-y}{(s_n^2-s_{k_n}^2)^{1/2}}-t\Big)\right|\\ \le&\sqrt{n}\sup_{|y|\le k_n}\left|\frac{s_nt-y}{(s_n^2-s_{k_n}^2)^{1/2}}-t\right|^2\xrightarrow{n\to\infty}0.\end{aligned}$$

Thus

$$|\sqrt{n}\mathbb{B}_{n21}|\le W_{k_n}\widetilde{\epsilon}_n\xrightarrow{n\to\infty}0. \tag{5.18}$$

We continue to prove that

$$\sqrt{n}\mathbb{B}_{n22}\xrightarrow{n\to\infty}0;\qquad\sqrt{n}\mathbb{B}_{n23}\xrightarrow{n\to\infty}0. \tag{5.19}$$

This will follow from the facts:

$$\frac{1}{\Pi_{k_n}}\sum_{u\in\mathbb{T}_{k_n}}|S_u|\mathbf{1}_{\{|S_u|>k_n\}}\xrightarrow{n\to\infty}0\ \text{ a.s.};\ \ \sqrt{n}\frac{1}{\Pi_{k_n}}\sum_{u\in\mathbb{T}_{k_n}}\mathbf{1}_{\{S_u|>k_n\}}\xrightarrow{n\to\infty}0\ \text{a.s.} \tag{5.20}$$

In order to prove (5.20), we firstly observe that

$$\begin{aligned}
&\mathbb{E}\left(\sum_{n=1}^{\infty}\frac{1}{\Pi_{k_n}}\sum_{u\in\mathbb{T}_{k_n}}|S_u|\mathbf{1}_{\{|S_u|>k_n\}}\right)\\
=&\sum_{n=1}^{\infty}\mathbb{E}|\widehat{S}_{k_n}|\mathbf{1}_{\{|\widehat{S}_{k_n}|>k_n\}}\le\sum_{n=1}^{\infty}k_n^{1-\eta}\mathbb{E}|\widehat{S}_{k_n}|^{\eta}\le\sum_{n=1}^{\infty}k_n^{-\frac{\eta}{2}}\sum_{j=0}^{k_n-1}\mathbb{E}|\widehat{L}_j|^{\eta}=\sum_{n=1}^{\infty}k_n^{1-\frac{\eta}{2}}\mathbb{E}|\widehat{L}_0|^{\eta},\\
&\mathbb{E}\left(\sum_{n=1}^{\infty}\sqrt{n}\frac{1}{\Pi_{k_n}}\sum_{u\in\mathbb{T}_{k_n}}\mathbf{1}_{\{|S_u|>k_n\}}\right)\\
=&\sum_{n=1}^{\infty}\sqrt{n}\mathbb{E}\mathbf{1}_{\{|\widehat{S}_{k_n}|>k_n\}}\le\sum_{n=1}^{\infty}\sqrt{n}k_n^{-\eta}\mathbb{E}|\widehat{S}_{k_n}|^{\eta}\le\sum_{n=1}^{\infty}\sqrt{n}k_n^{-\frac{\eta}{2}-1}\sum_{j=0}^{k_n-1}\mathbb{E}|\widehat{L}_j|^{\eta}=\sum_{n=1}^{\infty}n^{\frac{1}{2}}k_n^{-\frac{\eta}{2}}\mathbb{E}|\widehat{L}_0|^{\eta}.
\end{aligned}$$

The assumptions on β, k_n and η ensure that the series in the right hand side of the above two expressions converge. Hence

$$\sum_{n=1}^{\infty}\frac{1}{\Pi_{k_n}}\sum_{u\in\mathbb{T}_{k_n}}|S_u|\mathbf{1}_{\{|S_u|>k_n\}}<\infty,\quad\sum_{n=1}^{\infty}\sqrt{n}\frac{1}{\Pi_{k_n}}\sum_{u\in\mathbb{T}_{k_n}}\mathbf{1}_{\{|S_u|>k_n\}}<\infty\ \text{ a.s.},$$

which deduce (5.20), and consequently, (5.19) is proved.

By the Birkhoff ergodic theorem, we have

$$\lim_{n\to\infty}\frac{s_n^2}{n}=\mathbb{E}\sigma_0^{(2)},\tag{5.21}$$

whence by the choice of $\beta<1/4$ and the conditions on the environment,

$$\sqrt{n}\mathbb{B}_{n24}=\frac{\sqrt{n}}{(s_n^2-s_{k_n}^2)^{1/2}}\frac{s_{k_n}^2}{s_n+(s_n^2-s_{k_n}^2)^{1/2}}W_{k_n}\phi(t)t\xrightarrow{n\to\infty}0.\tag{5.22}$$

Due to Proposition 2.1 and (5.21), we conclude that

$$\sqrt{n}\mathbb{B}_{n25}\xrightarrow{n\to\infty}-(\mathbb{E}\sigma_0^{(2)})^{-\frac{1}{2}}\phi(t)V_1\quad\text{a.s.}\tag{5.23}$$

From (5.18), (5.19), (5.22) and (5.23), we derive (5.13).

Now we turn to the proof of (5.14).

According to the hypothesis of Theorem 2.3, it follows from the Birkhoff ergodic theorem that

$$\lim_{n\to\infty}\sqrt{n}\kappa_{1,n}=\frac{1}{6}(\mathbb{E}\sigma_0^{(2)})^{-3/2}\mathbb{E}\sigma_0^{(3)}.\tag{5.24}$$

Notice that

$$\begin{aligned}
&\left|\frac{1}{\Pi_{k_n}}\sum_{u\in\mathbb{T}_{k_n}}\left(D_1\left(\frac{s_nt-S_u}{(s_n^2-s_{k_n}^2)^{1/2}}\right)-D_1(t)\right)\right|\\
\le\ &\frac{2}{\Pi_{k_n}}\sum_{u\in\mathbb{T}_{k_n}}\mathbf{1}_{\{|S_u|>k_n\}}+\frac{1}{\Pi_{k_n}}\sum_{u\in\mathbb{T}_{k_n}}\left|D_1\left(\frac{s_nt-S_u}{(s_n^2-s_{k_n}^2)^{1/2}}\right)-D_1(t)\right|\mathbf{1}_{\{|S_u|\le k_n\}}.
\end{aligned}$$

The first term in the last expression above tends to 0 a.s. by (5.20), and the second one tends to 0 a.s. because the martingale $\{W_n\}$ converges and

$$\sup_{|y|\le k_n}\left|D_1\left(\frac{s_nt-y}{(s_n^2-s_{k_n}^2)^{1/2}}\right)-D_1(t)\right|\xrightarrow{n\to\infty}0.$$

Combining the above results, we obtain (5.14).

It remains to prove (5.15), which is immediate from (5.24) and the fact $W_n\xrightarrow{n\to\infty}W$.

So Lemma 5.2 has been proved. □

Proof of Lemma 5.3. This lemma follows from the following result given in [25].

Proposition 5.6 ([25])**.** *Assume the condition* (2.1)*. Then*

$$W - W_n = o(n^{-\lambda}) \qquad a.s.$$

By the choice of β and k_n, we see that

$$\sqrt{n}(W - W_{k_n}) = o(n^{\frac{1}{2}-\lambda\beta}) \xrightarrow{n\to\infty} 0.$$

□

Now Theorem 2.3 follows from the decomposition (5.2) and Lemmas 5.1 – 5.3.

6 Proof of Theorem 2.4

We will follow the similar procedure as in the proof of Theorem 2.3.

We remind that $\lambda, \eta > 16$ in the current setting. Hereafter we will choose $\max\{\frac{4}{\lambda}, \frac{4}{\eta}\} < \beta < \frac{1}{4}$ and let $k_n = \lfloor n^\beta \rfloor$ (the integral part of n^β).

By (5.1), we have

$$\sqrt{2\pi}s_n\Pi_n^{-1}Z_n(A) - W\int_A \exp\{-\frac{x^2}{2s_n^2}\}dx = \Lambda_{1,n} + \Lambda_{2,n} + \Lambda_{3,n}, \tag{6.1}$$

$$\begin{aligned}
\text{with} \quad \Lambda_{1,n} &= \sqrt{2\pi}s_n\Pi_{k_n}^{-1}\sum_{u\in\mathbb{T}_{k_n}}\Big(W_{n-k_n}(u, A-S_u) - \mathbb{E}_{\xi,k_n}W_{n-k_n}(u, A-S_u)\Big);\\
\Lambda_{2,n} &= \Pi_{k_n}^{-1}\sum_{u\in\mathbb{T}_{k_n}}\Big(\sqrt{2\pi}s_n\mathbb{E}_{\xi,k_n}W_{n-k_n}(u, A-S_u) - \int_A \exp\{-\frac{x^2}{2s_n^2}\}dx\Big);\\
\Lambda_{3,n} &= (W_{k_n} - W)\int_A \exp\{-\frac{x^2}{2s_n^2}\}dx.
\end{aligned}$$

On basis of this decomposition, we shall divide the proof of Theorem 2.4 into the following lemmas.

Lemma 6.1. *Under the hypothesis of Theorem 2.4, a.s.*

$$n\Lambda_{1,n} \xrightarrow{n\to\infty} 0. \tag{6.2}$$

Lemma 6.2. *Under the hypothesis of Theorem 2.4, a.s.*

$$\begin{aligned}
n\Lambda_{2,n} \xrightarrow{n\to\infty} (\mathbb{E}\sigma_0^{(2)})^{-1}(-\frac{1}{2}V_2 + \overline{x}_A V_1)|A| + \frac{1}{2}\mathbb{E}\sigma_0^{(3)}(\mathbb{E}\sigma_0^{(2)})^{-2}(V_1 - \overline{x}_A W)|A|\\
+ \frac{1}{8}(\mathbb{E}\sigma_0^{(2)})^{-2}\mathbb{E}(\sigma_0^{(4)} - 3(\sigma_0^{(2)})^2)W|A| - \frac{5}{24}(\mathbb{E}\sigma_0^{(2)})^{-3}(\mathbb{E}\sigma_0^{(3)})^2 W|A|.
\end{aligned} \tag{6.3}$$

Lemma 6.3. *Under the hypothesis of Theorem 2.4, a.s.*

$$n\Lambda_{3,n} \xrightarrow{n\to\infty} 0. \tag{6.4}$$

Now we go to prove the lemmas subsequently.

Proof of Lemma 6.1. The proof of Lemma 6.1 follows the same procedure as that of Lemma 5.1 with minor changes in scaling. We omit the details. □

Proof of Lemma 6.2. We start the proof by introducing some notation: set

$$\begin{aligned}
\kappa_{1,n} &= \frac{1}{6}(s_n^2 - s_{k_n}^2)^{-3/2}(s_n^{(3)} - s_{k_n}^{(3)}), \quad \kappa_{2,n} = \frac{1}{72}(s_n^2 - s_{k_n}^2)^{-3}(s_n^{(3)} - s_{k_n}^{(3)})^2,\\
\kappa_{3,n} &= \frac{1}{24}(s_n^2 - s_{k_n}^2)^{-2}\sum_{j=k_n}^{n-1}\Big(\sigma_j^{(4)} - 3\big(\sigma_j^{(2)}\big)^2\Big).
\end{aligned}$$

Define for $x \in \mathbb{R}$,

$$
\begin{aligned}
D_1(x) =& - H_2(x)\phi(x),\ D_2(x) = -H_5(x)\phi(x),\ D_3(x) = -H_3(x)\phi(x),\\
R_n(x) =& - \frac{\left(s_n^{(3)} - s_{k_n}^{(3)}\right)^3}{1296(s_n^2 - s_{k_n}^2)^{9/2}} H_8(x)\phi(x) - \frac{\sum_{j=k_n}^{n-1}\left(\sigma_j^{(5)} - 10\sigma_j^{(3)}\sigma_j^{(2)}\right)}{120(s_n^2 - s_{k_n}^2)^{5/2}} H_4(x)\phi(x)\\
& - \frac{\left(s_n^{(3)} - s_{k_n}^{(3)}\right)\sum_{j=k_n}^{n-1}\left(\sigma_j^{(4)} - 3\big(\sigma_j^{(2)}\big)^2\right)}{144(s_n^2 - s_{k_n}^2)^{7/2}} H_6(x)\phi(x),
\end{aligned}
$$

where H_m are Chebyshev-Hermite polynomials defined in (3.1). We decompose $\Lambda_{2,n}$ into 7 terms:

$$
\Lambda_{2,n} = \Lambda_{2,n1} + \Lambda_{2,n2} + \Lambda_{2,n3} + \Lambda_{2,n4} + \Lambda_{2,n5} + \Lambda_{2,n6} + \Lambda_{2,n7}, \tag{6.5}
$$

where

$$
\begin{aligned}
\Lambda_{2,n1} =& \sqrt{2\pi} s_n \Pi_{k_n}^{-1} \sum_{u\in\mathbb{T}_{k_n}} \Bigg[\mathbb{E}_{\xi,k_n} W_{n-k_n}(u, A - S_u) - \int_A \Bigg(\phi\Big(\frac{x - S_u}{(s_n^2 - s_{k_n}^2)^{1/2}}\Big)\\
& + \sum_{\nu=1}^{3} \kappa_{\nu,n} D_\nu'\Big(\frac{x - S_u}{(s_n^2 - s_{k_n}^2)^{1/2}}\Big) + R_n'\Big(\frac{x - S_u}{(s_n^2 - s_{k_n}^2)^{1/2}}\Big)\Bigg) \frac{dx}{(s_n^2 - s_{k_n}^2)^{1/2}} \Bigg],\\
\Lambda_{2,n2} =& \Pi_{k_n}^{-1} \sum_{u\in\mathbb{T}_{k_n}} \mathbf{1}_{\{|S_u|\le k_n\}} \int_A \Big[\frac{s_n}{(s_n^2 - s_{k_n}^2)^{1/2}} \exp\{-\frac{(x - S_u)^2}{2(s_n^2 - s_{k_n}^2)}\} - \exp\{-\frac{x^2}{2s_n^2}\}\Big] dx,\\
\Lambda_{2,n3} =& \frac{\sqrt{2\pi}\kappa_{1,n} s_n}{(s_n^2 - s_{k_n}^2)^{1/2}} \Pi_{k_n}^{-1} \sum_{u\in\mathbb{T}_{k_n}} \mathbf{1}_{\{|S_u|\le k_n\}} \int_A D_1'\Big(\frac{x - S_u}{(s_n^2 - s_{k_n}^2)^{1/2}}\Big) dx,\\
\Lambda_{2,n4} =& \frac{\sqrt{2\pi}\kappa_{2,n} s_n}{(s_n^2 - s_{k_n}^2)^{1/2}} \Pi_{k_n}^{-1} \sum_{u\in\mathbb{T}_{k_n}} \mathbf{1}_{\{|S_u|\le k_n\}} \int_A D_2'\Big(\frac{x - S_u}{(s_n^2 - s_{k_n}^2)^{1/2}}\Big) dx,\\
\Lambda_{2,n5} =& \frac{\sqrt{2\pi}\kappa_{3,n} s_n}{(s_n^2 - s_{k_n}^2)^{1/2}} \Pi_{k_n}^{-1} \sum_{u\in\mathbb{T}_{k_n}} \mathbf{1}_{\{|S_u|\le k_n\}} \int_A D_3'\Big(\frac{x - S_u}{(s_n^2 - s_{k_n}^2)^{1/2}}\Big) dx,\\
\Lambda_{2,n6} =& \frac{\sqrt{2\pi} s_n}{(s_n^2 - s_{k_n}^2)^{1/2}} \Pi_{k_n}^{-1} \sum_{u\in\mathbb{T}_{k_n}} \mathbf{1}_{\{|S_u|\le k_n\}} \int_A R_n'\Big(\frac{x - S_u}{(s_n^2 - s_{k_n}^2)^{1/2}}\Big) dx,\\
\Lambda_{2,n7} =& \frac{\sqrt{2\pi} s_n}{(s_n^2 - s_{k_n}^2)^{1/2}} \Pi_{k_n}^{-1} \sum_{u\in\mathbb{T}_{k_n}} \Bigg(\int_A \Bigg(\phi\Big(\frac{x - S_u}{(s_n^2 - s_{k_n}^2)^{1/2}}\Big) + R_n\Big(\frac{x - S_u}{(s_n^2 - s_{k_n}^2)^{1/2}}\Big)\\
& + \sum_{\nu=1}^{3} \kappa_{\nu,n} D_\nu'\Big(\frac{x - S_u}{(s_n^2 - s_{k_n}^2)^{1/2}}\Big) - \Big(1 - \frac{s_{k_n}^2}{s_n^2}\Big)^{1/2} \phi(x/s_n)\Big) dx \Bigg) \mathbf{1}_{\{|S_u| > k_n\}}.
\end{aligned}
$$

The lemma will follow once we prove that a.s.

$$
n\Lambda_{2,n1} \xrightarrow{n\to\infty} 0, \tag{6.6}
$$

$$
n\Lambda_{2,n2} \xrightarrow{n\to\infty} (\mathbb{E}\sigma_0^{(2)})^{-1}(-\frac{1}{2}V_2 + \overline{x}_A V_1)|A|, \tag{6.7}
$$

$$
n\Lambda_{2,n3} \xrightarrow{n\to\infty} \frac{1}{2}\mathbb{E}\sigma_0^{(3)}(\mathbb{E}\sigma_0^{(2)})^{-2}(V_1 - \overline{x}_A W)|A|, \tag{6.8}
$$

$$
n\Lambda_{2,n4} \xrightarrow{n\to\infty} -\frac{5}{24}(\mathbb{E}\sigma_0^{(2)})^{-3}(\mathbb{E}\sigma_0^{(3)})^2 W|A|, \tag{6.9}
$$

$$
n\Lambda_{2,n5} \xrightarrow{n\to\infty} \frac{1}{8}(\mathbb{E}\sigma_0^{(2)})^{-2}\mathbb{E}(\sigma_0^{(4)} - 3(\sigma_0^{(2)})^2) W|A|, \tag{6.10}
$$

$$
n\Lambda_{2,n6} \xrightarrow{n\to\infty} 0, \tag{6.11}
$$

$$
n\Lambda_{2,n7} \xrightarrow{n\to\infty} 0. \tag{6.12}
$$

The proof of (6.6) is based on the following result on the asymptotic expansion of the distribution of the sum of independent random variables:

Proposition 6.4. *Under the hypothesis of Theorem 2.4, for a.e. ξ,*

$$\epsilon_n = n^{3/2} \sup_{x\in\mathbb{R}} \left| \mathbb{P}_\xi \left(\frac{\sum_{k=k_n}^{n-1} \widehat{L}_k}{(s_n^2 - s_{k_n}^2)^{1/2}} \le x \right) - \Phi(x) - \sum_{\nu=1}^{3} \kappa_{\nu,n} D_\nu(x) - R_n(x) \right| \xrightarrow{n\to\infty} 0.$$

Proof. Let $X_k = 0$ for $0 \le k \le k_n - 1$ and $X_k = \widehat{L}_k$ for $k_n \le k \le n-1$. Then the random variables $\{X_k\}$ are independent under P_ξ. By Markov's inequality and Lemma 3.2 we obtain the following result:

$$\begin{aligned}
&\sup_{x\in\mathbb{R}} \left| \mathbb{P}_\xi \left(\frac{\sum_{k=k_n}^{n-1} \widehat{L}_k}{(s_n^2 - s_{k_n}^2)^{1/2}} \le x \right) - \Phi(x) - \sum_{\nu=1}^{3} \kappa_{\nu,n} D_\nu(x) - R_n(x) \right| \\
\le & K_\xi \left\{ (s_n^2 - s_{k_n}^2)^{-3} \sum_{j=k_n}^{n-1} \mathbb{E}_\xi |L_j|^6 + n^{15} \left(\sup_{|t|>T} \frac{1}{n} \left(k_n + \sum_{j=k_n}^{n-1} |v_j(t)| \right) + \frac{1}{2n} \right)^n \right\}.
\end{aligned}$$

By our conditions on the environment, we know that

$$\lim_{n\to\infty} n^2 (s_n^2 - s_{k_n}^2)^{-3} \sum_{j=k_n}^{n-1} \mathbb{E}_\xi |\widehat{L}_k|^6 = \mathbb{E}|\widehat{L}_0|^6 / (\mathbb{E}\sigma_0^{(2)})^3. \tag{6.13}$$

The required proposition concludes from (6.13) and (5.17). □

Using Proposition 6.4, we deduce that

$$|n\Lambda_{2,n1}| \le \sqrt{2\pi} s_n n^{-\frac{1}{2}} W_{k_n} \epsilon_n \xrightarrow{n\to\infty} 0,$$

and (6.6) is proved.

Next we turn to the proof of (6.7). Using Taylor's expansion and the boundedness of the set A, together with the choice of β and k_n, we get that

$$\frac{s_n}{(s_n^2 - s_{k_n}^2)^{1/2}} \exp\{-\frac{(x-y)^2}{2(s_n^2 - s_{k_n}^2)}\} - \exp\{-\frac{x^2}{2s_n^2}\} = \frac{1}{2(s_n^2 - s_{k_n}^2)} \big(s_{k_n}^2 - y^2 + 2xy + o(1) \big),$$

uniformly for all $|y| \le k_n$ and $x \in A$ as $n \to \infty$. By the same arguments as in the proof of (5.20), we can show that for $\eta > 16$, with β, k_n chosen above,

$$n\Pi_{k_n}^{-1} \sum_{u\in\mathbb{T}_{k_n}} \mathbf{1}_{\{|S_u|>k_n\}} \xrightarrow{n\to\infty} 0 \quad \text{and} \qquad \Pi_{k_n}^{-1} \sum_{u\in\mathbb{T}_{k_n}} S_u^2 \mathbf{1}_{\{|S_u|>k_n\}} \xrightarrow{n\to\infty} 0 \quad \text{a.s.} \tag{6.14}$$

Therefore as n tends to infinity, we have a.s.

$$\begin{aligned}
n\Lambda_{2,n2} = & \; n \frac{1}{2(s_n^2 - s_{k_n}^2)} \Big(- |A| \Pi_{k_n}^{-1} \sum_{u\in\mathbb{T}_{k_n}} (S_u^2 - s_{k_n}^2) \mathbf{1}_{\{|S_u|\le k_n\}} \\
& + 2 \int_A x dx \Pi_{k_n}^{-1} \sum_{u\in\mathbb{T}_{k_n}} S_u \mathbf{1}_{\{|S_u|\le k_n\}} + o(1) \Big) \\
= & \; \frac{n}{2(s_n^2 - s_{k_n}^2)} \big(- N_{2,k_n} |A| + 2|A| \overline{x}_A N_{1,k_n} + o(1) \big) \\
= & \; (2\mathbb{E}\sigma_0^{(2)})^{-1} (-V_2 + 2\overline{x}_A V_1) |A| + o(1),
\end{aligned}$$

which proves (6.7).

To prove (6.8), we observe that

$$\begin{aligned}
\Lambda_{2,n3} = & \frac{\kappa_{1,n} s_n}{(s_n^2 - s_{k_n}^2)^{1/2}} \Pi_{k_n}^{-1} \sum_{u\in\mathbb{T}_{k_n}} \mathbf{1}_{\{|S_u|\le k_n\}} \int_A \left(\frac{(x - S_u)^3}{(s_n^2 - s_{k_n}^2)^{3/2}} - \frac{3(x - S_u)}{(s_n^2 - s_{k_n}^2)^{1/2}} \right) e^{-\frac{(x-S_u)^2}{2(s_n^2 - s_{k_n}^2)}} dx \\
= & \; \Lambda_{2,n31} + \Lambda_{2,n32} + \Lambda_{2,n33} + \Lambda_{2,n34},
\end{aligned}$$

with

$$\Lambda_{2,n31} = \frac{\kappa_{1,n}s_n}{(s_n^2-s_{k_n}^2)^{1/2}}\Pi_{k_n}^{-1}\sum_{u\in\mathbb{T}_{k_n}}\mathbf{1}_{\{|S_u|\le k_n\}}\int_A \frac{(x-S_u)^3}{(s_n^2-s_{k_n}^2)^{3/2}}e^{-\frac{(x-S_u)^2}{2(s_n^2-s_{k_n}^2)}}dx;$$

$$\Lambda_{2,n32} = \frac{\kappa_{1,n}s_n}{(s_n^2-s_{k_n}^2)^{1/2}}\Pi_{k_n}^{-1}\sum_{u\in\mathbb{T}_{k_n}}\mathbf{1}_{\{|S_u|\le k_n\}}\int_A \frac{3(x-S_u)}{(s_n^2-s_{k_n}^2)^{1/2}}\left(1-e^{-\frac{(x-S_u)^2}{2(s_n^2-s_{k_n}^2)}}\right)dx;$$

$$\Lambda_{2,n33} = -\frac{\kappa_{1,n}s_n}{(s_n^2-s_{k_n}^2)^{1/2}}\Pi_{k_n}^{-1}\sum_{u\in\mathbb{T}_{k_n}}\int_A \frac{3(x-S_u)}{(s_n^2-s_{k_n}^2)^{1/2}}dx;$$

$$\Lambda_{2,n34} = \frac{\kappa_{1,n}s_n}{(s_n^2-s_{k_n}^2)^{1/2}}\Pi_{k_n}^{-1}\sum_{u\in\mathbb{T}_{k_n}}\mathbf{1}_{\{|S_u|>k_n\}}\int_A \frac{3(x-S_u)}{(s_n^2-s_{k_n}^2)^{1/2}}dx.$$

It is clear that

$$n|\Lambda_{2,n31}| \le \frac{n\kappa_{1,n}s_n}{(s_n^2-s_{k_n}^2)^2}\int_A(|x|+k_n)^3dxW_{k_n} \xrightarrow{n\to\infty} 0 \text{ a.s.},$$

$$n|\Lambda_{2,n32}| \le \frac{n\kappa_{1,n}s_n}{(s_n^2-s_{k_n}^2)^2}\int_A\frac{3}{2}(|x|+k_n)^3dxW_{k_n} \xrightarrow{n\to\infty} 0 \text{ a.s.} \quad (1-e^{-x}\le x, \text{ for } x>0),$$

$$\begin{aligned} n\Lambda_{2,n33} &= \frac{n(s_n^{(3)}-s_{k_n}^{(3)})s_n}{6(s_n^2-s_{k_n}^2)^{5/2}}\cdot 3|A|(N_{1,k_n}-\overline{x}_AW_{k_n}) \\ &\xrightarrow{n\to\infty} \frac{1}{2}\mathbb{E}\sigma_0^{(3)}(\mathbb{E}\sigma_0)^{-2}(V_1-\overline{x}_AW)|A| \text{ a.s.}, \end{aligned}$$

$$\begin{aligned} n|\Lambda_{2,n34}| &\le \frac{3n\kappa_{1,n}s_n}{(s_n^2-s_{k_n}^2)}\left(\int_A|x|dx\Pi_{k_n}^{-1}\sum_{u\in\mathbb{T}_{k_n}}\mathbf{1}_{\{|S_u|>k_n\}}+|A|\Pi_{k_n}^{-1}\sum_{u\in\mathbb{T}_{k_n}}|S_u|\mathbf{1}_{\{|S_u|>k_n\}}\right) \\ &\xrightarrow{n\to\infty} 0 \text{ a.s. (by (5.20))}, \end{aligned}$$

whence (6.8) follows.

By the Birkhoff ergodic theorem, we see that

$$\lim_{n\to\infty}\frac{n\kappa_{2,n}s_n}{(s_n^2-s_{k_n}^2)^{1/2}} = \frac{(\mathbb{E}\sigma_0^{(3)})^2}{72(\mathbb{E}\sigma_0^{(2)})^3}, \quad \lim_{n\to\infty}\frac{n\kappa_{3,n}s_n}{(s_n^2-s_{k_n}^2)^{1/2}} = \frac{\mathbb{E}(\sigma_0^{(3)}-3(\sigma_0^{(2)})^2)}{24(\mathbb{E}\sigma_0^{(2)})^2}. \tag{6.15}$$

Elementary calculus shows that, uniformly for $|y|\le k_n$

$$\text{if } \nu\ge 1, \quad \int_A\left(\frac{x-y}{(s_n^2-s_{k_n}^2)^{1/2}}\right)^\nu\exp\left(-\frac{(x-y)^2}{2(s_n^2-s_{k_n}^2)}\right)dx \xrightarrow{n\to\infty} 0 \text{ a.s. }, \tag{6.16}$$

$$\text{and} \quad \int_A\exp\left(-\frac{(x-y)^2}{2(s_n^2-s_{k_n}^2)}\right)dx \xrightarrow{n\to\infty} 1 \text{ a.s.} \tag{6.17}$$

Combining (6.14),(6.15), (6.16) and (6.17), we deduce (6.9) and (6.10).

By the Birkhoff ergodic theorem and the definition of $H_m(x)$ and $\phi(x)$, we see that

$$\sup_{x\in\mathbb{R}}|R_n'(x)| = O(\frac{1}{n^{3/2}}),$$

whence (6.11) follows.

Finally because $|\Lambda_{2,n7}|$ is bounded by $K_\xi\cdot\Pi_{k_n}^{-1}\sum_{u\in\mathbb{T}_{k_n}}\mathbf{1}_{\{|S_u|>k_n\}}$, (6.14) implies (6.12).

So the required result (6.3) follows from (6.6) – (6.12). □

Proof of Lemma 6.3. By Proposition 5.6 , under our assumption, we have

$$W-W_n = o(n^{-\lambda}) \qquad a.s.$$

By the choice of β and k_n, we see that

$$n^{\frac{3}{2}}(W-W_{k_n}) = o(n^{\frac{3}{2}-\lambda\beta}) \xrightarrow{n\to\infty} 0.$$

□

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