Wave Equation With Cone-Bounded Control Laws

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***Abstract*—This paper deals with a wave equation with a one-dimensional space variable, which describes the dynamics of string deflection. Two kinds of control are considered: a distributed action and a boundary control. It is supposed that the control signal is subject to a cone-bounded nonlinearity. This kind of feedback laws includes (but is not restricted to) saturating inputs. By closing the loop with such a nonlinear control, it is thus obtained a nonlinear partial differential equation, which is the generalization of the classical 1D wave equation. The well-posedness is proven by using nonlinear semigroups techniques. Considering a sector condition to tackle the control nonlinearity and assuming that a tuning parameter has a suitable sign, the asymptotic stability of the closed-loop system is proven by Lyapunov techniques. Some numerical simulations illustrate the asymptotic stability of the closed-loop nonlinear partial differential equations.**

I. Introduction

The general problem in this paper is the study of the wave in a one-dimensional media, as considered e.g. when modeling the dynamics of an elastic slope vibrating around its rest position. To be more specific, it is considered the wave equation describing the dynamics of the deformation denoted by $z(x,t)$. The control is either defined by an external force $f(x,t)$, or by a boundary action $g(t)$, where the force and the deformation may depend on the space and the time variables. A scheme of the considered problem is depicted in Figure 1.

Depending on the control action, two classes of partial differential equations (PDEs) are obtained. In the presence of an external distributed force f when the slope is attached at both extremities, the dynamic of the vibration is described by the following (see e.g. [14, Chap. 5.3]) for all $t \geq 0$, $x \in (0,1)$,

$$z_{tt}(x,t) \quad = \quad z_{xx}(x,t) + f(x,t) \tag{1}$$

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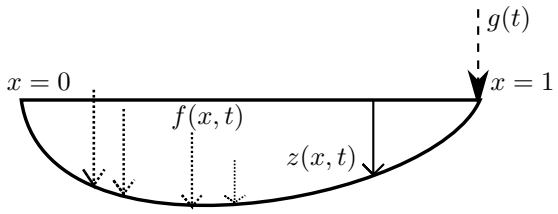

Fig. 1. Vibrating slope subject to a external distributed action $f(x,t)$ and to a boundary action $g(t)$.

where z stands for the state (the length of the string and other physical parameters are normalized), and $f(x,t) \in \mathbb{R}$ is the control. The control f is distributed (in contrast to boundary control), and is given by a bounded control operator. Let us equip this system with the following boundary conditions, for all $t \geq 0$,

$$z(0,t) = 0 \ , \tag{2a}$$

$$z(1,t) = 0 \ , \tag{2b}$$

and with the following initial condition, for all x in $(0,1)$,

$$\begin{array}{rcl} z(x,0) & = & z^0(x) \ , \\ z_t(x,0) & = & z^1(x) \ , \end{array} \tag{3}$$

where z^0 and z^1 stand respectively for the initial deflection of the slope and the initial deflection speed.

When the control action is only at the boundary, it is necessary to consider the following string equation, for all $t \geq 0$, $x \in (0,1)$,

$$z_{tt}(x,t) \quad = \quad z_{xx}(x,t) \tag{4}$$

with the boundary conditions, for all $t \geq 0$,

$$z(0,t) = 0 \tag{5a}$$

$$z_x(1,t) = g(t) \tag{5b}$$

where $g(t)$ is the boundary action at time t.

When closing the loop with a linear state feedback law, the control problem of such a 1D wave equation is considered in many works, see e.g. [8] where, in particular, stabilizing linear controllers and optimal linear feedback laws are computed respectively by an application of linear semigroup theory and LQR techniques. The aim of this paper is to investigate the well-posedness and the asymptotic stability of these classes of PDEs by means of nonlinear control laws, and more precisely of cone-bounded nonlinear control laws.

Neglecting the presence of nonlinearity in the input can be source of undesirable and even catastrophic behaviors for the closed-loop system. See e.g., [3], where it is shown that in presence of magnitude actuator limitation, even for finite-dimensional systems, the fact of disregarding the nonlinearity in the control loop may yield an unstable system. For an introduction of such nonlinear finite-dimensional systems and techniques on how to estimate the basin of attraction for locally asymptotically stable equilibrium, see [25], [28] among other references. Similarly, in the context of systems with cone-bounded nonlinearities, a quite natural approach to tackle the stability problem consists in combining Lyapunov theory with cone-bounded sector conditions (see e.g., [13], [4]). This allows to provide an estimate of the basin of attraction of the nonlinear systems in appropriate Sobolev spaces. This estimate can be either a neighborhood of the equilibrium in the local case or the whole space in the global case.

To our best knowledge, the well-posedness and the asymptotic stability of PDE by means of a cone-bounded input signal is less investigated than for corresponding finite-dimensional systems. This class of feedback laws includes classical saturation and deadzone nonlinearities as well as more general nonlinear maps. The aim of this work is to study the wave equation in presence of such nonlinear control laws. To prove first the existence and uniqueness of solutions to the PDE (1) (respectively (4)) with the boundary conditions (2) (resp. (5)) and initial conditions (3), when the loop is closed with an odd, Lipschitz and cone-bounded control law (see Theorems 1 and 2 below for precise statements), the nonlinear semigroup theory in appropriate Sobolev spaces is rigorously applied. The second contribution is to prove the global (resp. local) asymptotic stability of the corresponding closed-loop system by exploiting a cone-boundedness assumption on the control as introduced in [13], [25] for finite-dimensional systems, see Theorem 1 (resp. Theorem 2) below for a precise statement. In other words, this paper combines techniques that are usual for finite dimensional systems in closed loop with cone-bounded nonlinear control laws (see e.g. [12], [26]) with Lyapunov theory for PDEs (see e.g. [10], [5], [21]).

It is worth noticing that for both PDEs (1)-(2) and (4)-(5), Lyapunov techniques are applied, but the stability proofs are quite different. To be more specific, the proof of the stability of the PDE (1) with the boundary conditions (2) when closing the loop with a nonlinear feedback law is done by using the LaSalle invariance principle, which needs to state a precompactness property of the solutions. On the other hand, since a strict decreasing Lyapunov function is computed, we do not need to use the LaSalle invariance principle for the PDE (4) with a nonlinear boundary control.

To our best knowledge, the first work considering infinite dimensional systems with bounded control is [24, Thm 5.1] where only compact and bounded control operators with an a priori constraint are considered. In [23] only the case of a distributed saturating control has been considered. On the other hand, the paper [17] suggests to use an observability assumption for the study of PDE in closed loop with saturating controllers. In particular the contraction semigroup obtained from the saturating closed-loop system is compared to the corresponding semigroup without saturation. In the present paper, more general cone-bounded input nonlinearities are considered and different techniques are used, in particular the LaSalle invariance principle. Then, the paper can be considered as complementary to [17] by extending not only the class of nonlinear control laws but also the nature of the used tools. Note finally that both papers [24], [17] do not consider the case of saturation of the *value* of the solution, but rather saturation of the *norm* of the solution (compare [24, Eq. (2.8)] and [17, Eq. (1.6)] with the definition of nonlinear controller (8) below). Dealing with saturation of the value of the solution is more complex and is more relevant for applications, since it yields to a locally defined PDE. However it needs further developments (in particular more regularity is required to ensure that the nonlinear map is well-defined).

The rest of the paper is organized as follows. In Section II, the nonlinear PDE (1) is introduced, and the first main result is stated, namely the well-posedness of the Cauchy problem, along with the global asymptotic stability, when closing the loop with a nonlinear distributed control law. In Section III, the main result is stated for the PDE (4), where a nonlinear boundary action is considered. The proof of the two main results are given respectively in Sections IV and V. Section VI presents numerical simulations to illustrate both main results. Some concluding remarks and possible further research lines are presented in Section VII.

Notation: z_t (resp. z_x) stands for the partial derivative of the function z with respect to t (resp. x) (this is a shortcut for $\frac{\partial z}{\partial t}$, resp. $\frac{\partial z}{\partial x}$). When there is only one independent variable, $\dot{z}$ and z' stand respectively for the time and the space derivative. For a matrix A, $A^\top$ denotes the transpose, and for a partitioned symmetric matrix, the symbol $\bigstar$ stands for symmetric blocks. $\Re(s)$ and $\Im(s)$ stand respectively for the real and imaginary part of a complex value s in $\mathbb{C}$, $\overline{s}$ is the conjugate of s, and $|s|$ its modulus. $\|\ \|_{L^2}$ denotes the norm in $L^2(0,1)$ space, defined by $\|u\|^2_{L^2(0,1)} = \int_0^1 |u|^2 dx$ for all functions $u \in L^2(0,1)$. Similarly, $H^2(0,1)$ is the set of all functions $u \in H^2(0,1)$ such that $\int_0^1 (|u|^2 + |u_x|^2 + |u_{xx}|^2) dx$ is finite. Finally $H^1_0(0,1)$ is the closure in $L^2(0,1)$ of the set of smooth functions

that are vanishing at $x = 0$ and at $x = 1$. It is equipped with the norm $\|u\|^2_{H^1_0(0,1)} = \int_0^1 |u_x|^2 dx$. The associate inner products are denoted $\langle \cdot, \cdot \rangle_{L^2(0,1)}$ and $\langle \cdot, \cdot \rangle_{H^1_0(0,1)}$.

II. Wave equation with a nonlinear distributed control

Consider the PDE (1), with the boundary conditions (2) and the initial condition (3).

Letting for the control, for all $t \geq 0$ and all $x \in (0,1)$,

$$f(x,t) = -az_t(x,t), \tag{6}$$

where a is a constant value, and exploiting properties of the following energy function:

$$V_1 = \frac{1}{2}\int (z_x^2 + z_t^2) dx, \tag{7}$$

for any solution z to (1) and (2), when closing the loop with the linear controller (6), allow to show that the closed-loop system is (globally) exponentially stable in $H^1_0(0,1) \times L^2(0,1)$.

This can be formally shown by considering the time derivative of V along the solutions to (1)-(2), which yields

$$\begin{array}{rcl} \dot{V}_1 & = & \int_0^1 (z_x z_{xt} - a z_t^2 + z_t z_{xx}) dx \\ & = & -\int_0^1 a z_t^2 dx + [z_t z_x]_{x=0}^{x=1} \\ & = & -\int_0^1 a z_t^2 dx \end{array}$$

where an integration by parts is performed to get the second line from the first one, and the boundary conditions (2) are applied for the last line. Therefore, for any positive value a, it is formally obtained that the energy is decreasing at long as z_t is non vanishing in $[0,1]$. In other words, V_1 is a (non strict) Lyapunov function.

Due to actuator limitations or imperfections, the actual control law applied to the system, instead of (6), can be modeled as follows

$$f(x,t) = -\sigma_1(a z_t(x,t)) \tag{8}$$

with $\sigma_1 : \mathbb{R} \to \mathbb{R}$ being a bounded and continuous nonlinear function satisfying, for a constant value $L > 0$ and for all $(s, \widetilde{s}) \in \mathbb{R}^2$,

$$(\sigma_1(s) - \sigma_1(\widetilde{s}))(s - \widetilde{s}) \geq 0 \ , \tag{9a}$$

$$|\sigma_1(s)| \leq L|s| \ . \tag{9b}$$

Note that (9a) generalizes the odd property. Examples of such functions σ_1 include the saturation maps, and are considered in Section VI-A below.

Equation (1) in closed loop with the control (8) becomes

$$z_{tt} \ = \ z_{xx} - \sigma_1(a z_t) \ . \tag{10}$$

A formal computation gives, along the solutions to (10) and (2),

$$\dot{V}_1 = -\int_0^1 z_t \sigma_1(a z_t) dx \tag{11}$$

which asks to handle the nonlinearity $z_t \sigma_1(a z_t)$. A convergence result is stated below, where the well-posedness is separate from the asymptotic stability property.

Theorem 1. *For all positive values a, and for all bounded and continuous functions σ_1 satisfying (9), the model (10) with the boundary conditions (2) is globally asymptotically stable. More precisely the following properties hold:*

- [Well-posedness] *For all (z^0, z^1) in $\left(H^2(0,1) \cap H^1_0(0,1)\right) \times H^1_0(0,1)$, there exists a unique solution z: $[0,\infty) \to H^2(0,1) \cap H^1_0(0,1)$ to (10), with the boundary conditions (2) and the initial condition (3), that is differentiable from $[0,\infty)$ on $H^1_0(0,1)$.*
- [Global asymptotic stability] *Moreover, for all initial conditions (z^0, z^1) in $\left(H^2(0,1) \cap H^1_0(0,1)\right) \times H^1_0(0,1)$, the solution to (10), with the boundary conditions (2) and the initial condition (3), satisfies the following stability property*

$$\begin{array}{l} \|z(.,t)\|_{H^1_0(0,1)} + \|z_t(.,t)\|_{L^2(0,1)} \\ \quad \leq \|z^0\|_{H^1_0(0,1)} + \|z^1\|_{L^2(0,1)}, \ \forall t \geq 0 \ , \end{array} \tag{12}$$

together with the attractivity property

$$\|z(.,t)\|_{H^1_0(0,1)} + \|z_t(.,t)\|_{L^2(0,1)} \to_{t\to\infty} 0 \ . \tag{13}$$

The proof of Theorem 1 is provided in Section IV.

III. Wave equation with a nonlinear boundary action

Consider the PDE (4), with the boundary conditions (5) and the initial condition (3).

Letting for the control

$$g(t) = -b z_t(1,t), \tag{14}$$

where b is a positive tuning parameter, inspired by [16], the following Lyapunov function candidate:

$$\begin{array}{rcl} V_2 & = & \frac{1}{2}\left(\int_0^1 e^{\mu x}(z_t + z_x)^2 dx \right. \\ & & \left. + \int_0^1 e^{-\mu x}(z_t - z_x)^2 dx\right) \ , \end{array} \tag{15}$$

where $\mu > 0$ will be prescribed below, is considered. It may be proven that we have asymptotic stability of (4) and (5). With this aim, let us formally compute the time-derivative of V_2 along the solutions of (4) and (5)

as follows

$$\begin{aligned}\dot{V}_2 &= \int_0^1 e^{\mu x}(z_t+z_x)(z_{tt}+z_{xt})dx \\ &\quad + \int_0^1 e^{-\mu x}(z_t-z_x)(z_{tt}-z_{xt})dx \\ &= \int_0^1 e^{\mu x}(z_t+z_x)(z_{xx}+z_{xt})dx \\ &\quad - \int_0^1 e^{-\mu x}(z_t-z_x)(z_{xt}-z_{xx})dx \\ &= -\tfrac{\mu}{2}\int_0^1 e^{\mu x}(z_t+z_x)^2dx + \tfrac{1}{2}[e^{\mu x}(z_t+z_x)^2]_{x=0}^{x=1} \\ &\quad -\tfrac{\mu}{2}\int_0^1 e^{-\mu x}(z_t-z_x)^2dx - \tfrac{1}{2}[e^{-\mu x}(z_t-z_x)^2]_{x=0}^{x=1}\end{aligned}$$

where the partial differential equation (4) has been used in the first equality and two integrations by parts have been performed in the second equality.

Now, note that the boundary condition (5a) implies that $z_t(0,t)=0$ and thus, for all $t\geq 0$,

$$\begin{aligned}[e^{\mu x}(z_t+z_x)^2](0,t) - [e^{-\mu x}(z_t-z_x)^2](0,t) \\ = z_x^2(0,t) - z_x^2(0,t) = 0\end{aligned}$$

Therefore, with (5b), it is deduced

$$\begin{aligned}\dot{V}_2 &= -\mu V_2 + \frac{e^{\mu}}{2}(z_t(1,t)+z_x(1,t))^2 \\ &\quad -\frac{e^{-\mu}}{2}(z_t(1,t)-z_x(1,t))^2 \end{aligned} \tag{16}$$

and thus

$$\begin{aligned}\dot{V}_2 &= -\mu V_2 + \frac{e^{\mu}}{2}(z_t(1,t)-bz_t(1,t))^2 \\ &\quad -\frac{e^{-\mu}}{2}(z_t(1,t)+bz_t(1,t))^2 \\ &= -\mu V_2 + \frac{1}{2}\left(e^{\mu}(1-b)^2 - e^{-\mu}(1+b)^2\right)z_t^2(1,t)\end{aligned}$$

For any positive value b, it holds $|1-b|<|1+b|$. Now, pick any $\mu>0$ such that

$$e^{\mu}(1-b)^2 \leq e^{-\mu}(1+b)^2 \tag{17}$$

holds.

With such a value b, we get $\dot{V}_2 \leq -\mu V_2$ and thus the partial differential equation (4), with the boundary condition (5), is exponentially stable.

Now instead of the boundary condition (5) in closed loop with the linear controller (14), consider the boundary conditions, for all $t\geq 0$,

$$\begin{aligned}z(0,t) &= 0 \\ z_x(1,t) &= -\sigma_2(bz_t(1,t))\end{aligned} \tag{18}$$

resulting from the boundary condition (5) in closed loop with a bounded and continuous map $\sigma_2 : \mathbb{R}\to\mathbb{R}$ satisfying, for all $(s,\widetilde{s})\in\mathbb{R}$,

$$(\sigma_2(s)-\sigma_2(\widetilde{s}))(s-\widetilde{s})\geq 0\ , \tag{19a}$$

$$|\sigma_2(s)|\leq u_2\ , \tag{19b}$$

with $u_2>0$. Assume moreover that, for all c in $\mathbb{R}$, and for all s in $\mathbb{R}$, such that $|(b-c)s|\leq u_2$, it holds

$$\varphi_2(bs)(\varphi_2(bs)+cs)\leq 0\ , \tag{19c}$$

where $\varphi_2(s)=\sigma_2(s)-s$. Such a function σ_2 includes the nonlinear functions satisfying some sector bounded condition, as the saturation maps of level u_2 (see [25, Chap. 1] or [13, Chap. 7]). Since σ_2 is a function of $z_t(1,t)$, it is needed in the next result a stronger regularity on the initial condition than the one imposed in e.g. [17], so that (18) makes sense.

The stability analysis of the corresponding nonlinear partial differential equation (4) and (18) asks for special care. It is done in our second main result, given below, where, following the notation in [5, Sec. 2.4], it is denoted $H^1_{(0)}(0,1)=\{u\in H^1(0,1),\ u(0)=0\}$, and $\|u\|_{H^1_{(0)}(0,1)}=\sqrt{\int_0^1|u'|^2(x)dx}$, for all $u\in H^1_{(0)}$.

Theorem 2. *For all positive values b, and for all continuous functions σ_2 satisfying (19), the model (4) with the boundary conditions (18) is globally asymptotically stable. More precisely the following properties hold:*

- [Well-posedness] *For all (z^0,z^1) in $\{(u,v),\ (u,v)\in H^2(0,1)\times H^1_{(0)}(0,1),\ u'(1)+bv(1)=0,\ u(0)=0\}$, there exists a unique continuous solution z: $[0,\infty)\to H^2(0,1)\cap H^1_{(0)}(0,1)$ to (4), with the boundary conditions (18) and the initial condition (3), that is differentiable from $[0,\infty)$ to $H^1_{(0)}(0,1)$.*
- [Global asymptotic stability] *For all initial conditions (z^0,z^1) in $\{(u,v),\ (u,v)\in H^2(0,1)\times H^1_{(0)}(0,1),\ u'(1)+bv(1)=0,\ u(0)=0\}$, the solution to (4), with the boundary conditions (18) and the initial condition (3), satisfies the following* global stability property

$$\begin{aligned}&\|z(.,t)\|_{H^1_{(0)}(0,1)} + \|z_t(.,t)\|_{L^2(0,1)} \\ &\quad \leq \|z^0\|_{H^1_{(0)}(0,1)} + \|z^1\|_{L^2(0,1)},\ \forall t\geq 0\ ,\end{aligned} \tag{20}$$

together with the attractivity property*,*

$$\|z(.,t)\|_{H^1_{(0)}(0,1)} + \|z_t(.,t)\|_{L^2(0,1)} \to_{t\to\infty} 0, \tag{21}$$

holds.

Remark 1. *As for many nonlinear control systems, in particular the finite-dimensional ones subject to input saturation (see e.g., [25]), only the local exponential stability can sometimes be obtained, requiring to prove the exponential stability of the system only for a set of admissible initial conditions. Regarding a similar case for an infinite dimensional system, Theorem 2 does not state the global exponential stability, however we are able to prove the global asymptotic stability.*

The proof of Theorem 2 is provided in Section V. Note that perturbation arguments (as considered in e.g., [20, Chap. 3]) may be used to study (10) in closed loop with a saturating controller instead of the nonlinear function σ_2. It yields a local asymptotic stability property without exhibiting any estimate of the basin of attraction,

in contrast to the results given in Theorem 2 where an explicit estimate of the basin of attraction is provided.

IV. Proof of Theorem 1

The proof of Theorem 1 is split into two parts: 1) the Cauchy problem has a unique solution, 2) the system is globally asymptotically stable.

Part 1: Well-posedness of the Cauchy problem (10), (2), (3).

Let us first prove the existence and unicity of solution to the nonlinear equation (10) with the boundary conditions (2) and the initial condition (3). To do that, let us introduce the following nonlinear operator

$$A_1 \begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} v \\ u'' - \sigma_1(a\, v) \end{pmatrix}$$

with the domain $D(A_1) = (H^2(0,1) \cap H_0^1(0,1)) \times H_0^1(0,1)$.

To prove the well-posedness of the Cauchy problem, we shall state that A_1 generates a semigroup of contractions, and thus we need to prove that A_1 is closed, dissipative, and satisfies a range condition (see (25) below). Let us prove these properties successively.

Note that, using the terminology of [19, Def. 2.6], A_1 is the sum of a closed operator and of the following continuous operator

$$\begin{pmatrix} u \\ v \end{pmatrix} \mapsto \begin{pmatrix} 0 \\ -\sigma_1(a\, v) \end{pmatrix} ,$$

and it is closed as proved in the following.

Claim 1. *The sum of a closed operator and a continuous operator is closed.*

Proof of Claim 1: Inspired from [15, Page 296, Ex. 12], consider T_1 a closed operator from $D(T_1) \subset X$ to Y, where X and Y are two Banach spaces. Let T_2 be a continuous operator from X to Y. Let $(x_n)_{n\in\mathbb{N}}$ be a sequence in X and (x, y) in $X \times Y$ such that $x_n \to x$ and $(T_1 + T_2)x_n \to y$ as $n \to \infty$. We have to prove that $(T_1 + T_2)x = y$. To do this, note first that $T_2x_n \to T_2x$ (since T_2 is continuous). Moreover $\|T_1x_n + T_2x - y\| \leq \|T_1x_n + T_2x_n - y\| + \|T_2x_n - T_2x\| \to 0$ as $n \to \infty$. Thus $T_1x_n \to y - T_2x$ as $n \to \infty$, and (by using the closedness of T_1) $T_1x = y - T_2x$. Therefore $y = T_1x + T_2x$. □

Moreover, following the terminology of [19, Def. 2.4], and using the nonnegativity of a, we may prove the following lemma.

Lemma 1. *A_1 is a dissipative operator.*

Proof of Lemma 1: Let us first denote by H_1 the space $H_0^1(0,1) \times L^2(0,1)$. It is a Hilbert space equipped with the inner product

$$\begin{aligned} \langle \begin{pmatrix} u \\ v \end{pmatrix}, \begin{pmatrix} \widetilde{u} \\ \widetilde{v} \end{pmatrix} \rangle &= \int_0^1 u'(x)\overline{\widetilde{u}'(x)}dx \\ &\quad + \int_0^1 v(x)\overline{\widetilde{v}(x)}dx \ , \end{aligned} \tag{22}$$

and the norm

$$\left\| \begin{pmatrix} u \\ v \end{pmatrix} \right\| = \sqrt{\int_0^1 |u'(x)|^2\, dx + \int_0^1 |v(x)|^2\, dx} \ .$$

Let us enlarge the domain of definition of the function σ_1 to the complex numbers, by letting, for all s in $\mathbb{C}$,

$$\sigma_{1\mathbb{C}}(s) := \sigma_1(\Re(s)) + i\sigma_1(\Im(s)) \ .$$

To ease the notation, we still use σ_1 instead of $\sigma_{1\mathbb{C}}$. We define φ_1 for complex numbers in a similar way.

To check that A_1 is dissipative, using first the definition of A_1 and then recalling the definition of the inner product (22), let us compute the following, for all $\begin{pmatrix} u \\ v \end{pmatrix}, \begin{pmatrix} \widetilde{u} \\ \widetilde{v} \end{pmatrix}$ in $D(A_1)$,

$$\begin{aligned} &\langle A_1 \begin{pmatrix} u \\ v \end{pmatrix} - A_1 \begin{pmatrix} \widetilde{u} \\ \widetilde{v} \end{pmatrix}, \begin{pmatrix} u \\ v \end{pmatrix} - \begin{pmatrix} \widetilde{u} \\ \widetilde{v} \end{pmatrix} \rangle \\ &= \int_0^1 (v - \widetilde{v})'(x)\overline{(u - \widetilde{u})'(x)}dx \\ &+ \int_0^1 \left[(u'' - \sigma_1(a\, v)) - (\widetilde{u}'' - \sigma_1(a\, \widetilde{v})) \right](x) \\ &\qquad \times \overline{(v - \widetilde{v})(x)}dx \ , \\ &= \int_0^1 (v - \widetilde{v})'(x)\overline{(u - \widetilde{u})'(x)}dx \\ &+ \int_0^1 (u'' - \widetilde{u}'')(x)\overline{(v - \widetilde{v})(x)}dx \\ &- \int_0^1 (\sigma_1(a\, v) - \sigma_1(a\, \widetilde{v}))(x)\overline{(v - \widetilde{v})(x)}dx \ . \end{aligned} \tag{23}$$

Consider the second integral in the last equation. Performing an integration by parts and using the definition of $D(A_1)$, it gives

$$\begin{aligned} &\int_0^1 (u'' - \widetilde{u}'')(x)\overline{(v - \widetilde{v})(x)}dx \\ &= -\int_0^1 (u' - \widetilde{u}')(x)\overline{(v' - \widetilde{v}')(x)}dx \\ &\quad + [(u' - \widetilde{u}')(x)\overline{(v - \widetilde{v})(x)}]_{x=0}^{x=1} \\ &= -\int_0^1 (u' - \widetilde{u}')(x)\overline{(v' - \widetilde{v}')(x)}dx \end{aligned}$$

Thus, with (23), it follows

$$\begin{aligned} &\Re\left(\langle A_1 \begin{pmatrix} u \\ v \end{pmatrix} - A_1 \begin{pmatrix} \widetilde{u} \\ \widetilde{v} \end{pmatrix}, \begin{pmatrix} u \\ v \end{pmatrix} - \begin{pmatrix} \widetilde{u} \\ \widetilde{v} \end{pmatrix} \rangle \right) \\ &= \Re\left(\int_0^1 (v - \widetilde{v})'(x)\overline{(u - \widetilde{u})'(x)}dx \right) \\ &-\Re\left(\int_0^1 (u' - \widetilde{u}')(x)\overline{(v' - \widetilde{v}')(x)}dx \right) \\ &\quad -\Re\left(\int_0^1 (\sigma_1(a\, v) - \sigma_1(a\, \widetilde{v}))(x) \right. \\ &\qquad \left. \times \overline{(v - \widetilde{v})(x)}dx \right) \ , \\ &= -\Re\left(\int_0^1 (\sigma_1(a\, v) - \sigma_1(a\, \widetilde{v}))(x) \right. \\ &\qquad \left. \times \overline{(v - \widetilde{v})(x)}dx \right) \ . \end{aligned} \tag{24}$$

Note that, due to (9a), it holds, for all $(s,\widetilde{s}) \in \mathbb{C}$,

$$\Re\left((\sigma_1(s)-\sigma_1(\widetilde{s}))\overline{(s-\widetilde{s})}\right) \geq 0 \ .$$

Therefore it follows, from (24) and the nonnegativity of a,

$$\Re\left(\langle A_1 \left(\begin{smallmatrix} u \\ v \end{smallmatrix}\right) - A_1 \left(\begin{smallmatrix} \widetilde{u} \\ \widetilde{v} \end{smallmatrix}\right), \left(\begin{smallmatrix} u \\ v \end{smallmatrix}\right) - \left(\begin{smallmatrix} \widetilde{u} \\ \widetilde{v} \end{smallmatrix}\right)\rangle\right) \leq 0 \ ,$$

and thus A_1 is dissipative. □

Let us now show that the operator A_1 generates a semigroup of contractions. To do that, we apply [2, Thm 1.3, Page 104] (or [19, Thm 4.2, Page 77]) and we need to prove that

$$D(A_1) \subset \texttt{Ran}(I-\lambda A_1) \tag{25}$$

for all $\lambda > 0$ sufficiently small, where $\texttt{Ran}(I-\lambda A_1)$ is the range of the operator $I-\lambda A_1$. To prove (25), let us pick $\begin{pmatrix} u \\ v \end{pmatrix}$ in $D(A_1)$ and let us prove that there exists $\begin{pmatrix} \widetilde{u} \\ \widetilde{v} \end{pmatrix}$ in $D(A_1)$ such that $(I-\lambda A_1)\begin{pmatrix} \widetilde{u} \\ \widetilde{v} \end{pmatrix} = \begin{pmatrix} u \\ v \end{pmatrix}$. Let us first note that this latter equation is equivalent to

$$\left\{ \begin{array}{c} \widetilde{u} - \lambda \widetilde{v} = u \ , \\ \widetilde{v} - \lambda(\widetilde{u}'' - \sigma_1(a\,\widetilde{v})) = v \ , \end{array} \right.$$

which may be rewritten as

$$\left\{ \begin{array}{c} \widetilde{v} = \frac{1}{\lambda}(\widetilde{u}-u) \\ \widetilde{u}'' - \frac{1}{\lambda^2}\widetilde{u} - \sigma_1(\frac{a}{\lambda}(\widetilde{u}-u)) = -\frac{1}{\lambda}v - \frac{1}{\lambda^2}u \end{array} \right. \tag{26}$$

To check that there exists $\widetilde{u} \in H^2(0,1) \cap H_0^1(0,1)$ such that the second line of (26) holds, let us first note that this is a nonhomogeneous nonlinear differential equation in the $\widetilde{u}$-variable with two boundary conditions (at $x=0$ and at $x=1$), as considered in the following:

Lemma 2. *If a is nonnegative and λ is positive, then there exists $\widetilde{u} \in H^2(0,1) \cap H_0^1(0,1)$ solution to*

$$\begin{array}{c} \widetilde{u}'' - \frac{1}{\lambda^2}\widetilde{u} - \sigma_1(\frac{a}{\lambda}(\widetilde{u}-u)) = -\frac{1}{\lambda}v - \frac{1}{\lambda^2}u \\ \widetilde{u}(0) = \widetilde{u}(1) = 0 \end{array} \tag{27}$$

Proof of Lemma 2: The proof of this lemma follows from classical techniques (see e.g., [19, Page 113], or [5, Page 179]) and uses the Schauder fixed-point theorem (see e.g., [5, Thm B.19]).

To prove this lemma, let us introduce the following map

$$\begin{array}{rccl} \mathcal{T}_1: & L^2(0,1) & \to & L^2(0,1) \ , \\ & y & \mapsto & z = \mathcal{T}_1(y) \ , \end{array}$$

where $z = \mathcal{T}_1(y)$ is the unique solution to

$$\begin{array}{c} z'' - \frac{1}{\lambda^2}z = -\frac{1}{\lambda}v - \frac{1}{\lambda^2}u + \sigma_1(\frac{a}{\lambda}(y-u)) \ , \\ z(0) = z(1) = 0 \ . \end{array} \tag{28}$$

This map $\mathcal{T}_1$ is well-defined as soon as $-\frac{1}{\lambda^2} \leq 0$, i.e. as soon as $\lambda > 0$. The well-posedness of $\mathcal{T}_1$ can be seen as the well-posedness of the associate Sturm Liouville problem[1].

Let us prove the following intermediate result.

Claim 2. *There exists $M > 0$ such that $\mathcal{T}_1(L^2(0,1)) \subset K$, where K is the set of functions w that are continuously differentiable on $[0,1]$ and such that $\|w\|_{C^0([0,1])} \leq M$ and $\|w'\|_{C^0([0,1])} \leq M$.*

Proof of Claim 2: To prove this claim, let us first note that each solution to (28) is a solution to

$$\begin{array}{c} z'' - \frac{1}{\lambda^2}z = -\frac{1}{\lambda}v - \frac{1}{\lambda^2}u + \sigma_1(\frac{a}{\lambda}(y-u)) \ , \\ z(0) = 0, \ z'(0) = C_1 \ . \end{array} \tag{29}$$

for a suitable $C_1 \in \mathbb{R}$. To be more specific, the solution to (29) is given by

$$\begin{array}{rcl} z(x) & = & (1\ 0)e^{\mathcal{A}x}(0\ C_1)^\top \\ & & + \int_0^x e^{\mathcal{A}(x-s)}(-\frac{1}{\lambda}v(s) - \frac{1}{\lambda^2}u(s) \\ & & +\sigma_1(\frac{a}{\lambda}(y(s)-u(s))))ds \end{array}$$

where $\mathcal{A} = \begin{pmatrix} 0 & 1 \\ \frac{1}{\lambda^2} & 0 \end{pmatrix}$. It holds $\mathcal{A}^2 = \frac{1}{\lambda^2}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$. Thus, by using a recurrence argument, we may prove that, for all x in $[0,1]$,

$$e^{\mathcal{A}x} = \begin{pmatrix} \cosh(\frac{\sqrt{x}}{\lambda}) & \frac{\lambda\sinh(\frac{\sqrt{x}}{\lambda})}{\sqrt{x}} \\ \frac{\sqrt{x}\sinh(\frac{\sqrt{x}}{\lambda})}{\lambda} & \cosh(\frac{\sqrt{x}}{\lambda}) \end{pmatrix} \ . \tag{30}$$

Recall that σ_1 is assumed to be bounded, and consider a bound $u_1 > 0$ such that, for all $s \in \mathbb{R}$, $|\sigma_1(s)| \leq u_1$. By inspecting $(1\ 0)e^{\mathcal{A}}(0\ C_1)^\top$, since $\lambda\sinh(\frac{1}{\lambda}) \neq 0$ and since, for all $y \in L^2(0,1)$, $|\sigma_1(\frac{a}{\lambda}(y-u))| \leq u_1$, we get that the value C_1 lies in a bounded set of $\mathbb{R}$ (which does not depend on $y \in L^2(0,1)$) and thus the existence of $M > 0$ follows, as stated in Claim 2. □

Moreover $\mathcal{T}_1$ is a continuous operator. Finally the set K is convex and compact (by the Ascoli-Arzela theorem), as a subset of $L^2(0,1)$.

Therefore, by the Schauder fixed-point theorem (see e.g., [5, Thm B.19]), there exists $\widetilde{u} \in K$ such that $\mathcal{T}_1(\widetilde{u}) = \widetilde{u}$. This concludes the proof of Lemma 2. □

Now from the existence of $\widetilde{u} \in H^2(0,1) \cap H^1_{(0)}(0,1)$ such that the second line of (26) holds, let us remark that the first line of (26) defines a unique $\widetilde{v}$ in $H_0^1(0,1)$. Therefore $(I-\lambda A_1)\begin{pmatrix} \widetilde{u} \\ \widetilde{v} \end{pmatrix} = \begin{pmatrix} u \\ v \end{pmatrix}$ and (25) hold.

Since A_1 is dissipative (due to Lemma 1), it follows, from [2, Thm 1.3, Page 104] (or [19, Thm 4.2]), that A_1 generates a semigroup of contractions $T_1(t)$. Moreover, by [2, Thm 1.2, Page 102] (or [19, Thm 4.5]), for all $\begin{pmatrix} \widetilde{u} \\ \widetilde{v} \end{pmatrix}$ in $D(A_1)$, $T_1(t)\begin{pmatrix} \widetilde{u} \\ \widetilde{v} \end{pmatrix}$ is differentiable for all

[1]Note that the homogeneous Sturm Liouville problem (i.e. with $v = u = y$) has only the trivial function as the solution, since $-\frac{1}{\lambda^2} < 0$. Therefore the Sturm Liouville problem described by $\mathcal{T}_1$ is well-posed, as proven e.g. in [1, Chap. 14].

$t > 0$ and is a solution to the Cauchy Problem (10), (2) and (3). Moreover due to [19, Thm 4.10], it is the unique solution to this Cauchy problem.

Part 2: Global asymptotic stability of the nonlinear equation (10) with the boundary conditions (2).

Let us consider a solution to (10) and (2), for a given initial condition in $D(A_1)$. The formal computation yielding (11) makes sense. Therefore with the nonnegativity of a and with (9a), we get $\dot{V}_1 \leq 0$, along the solutions to (10) and (2), for any initial condition in $D(A_1)$.

To be able to apply LaSalle's Invariance Principle, we have to check that the trajectories are precompact (see e.g. [9]). This precompactness is a corollary of the following lemma (which is similar to [10, Lem. 2] where different boundary conditions are considered).

Lemma 3. *The canonical embedding from $D(A_1)$, equipped with the graph norm, into H_1 is compact.*

Proof of Lemma 3: Before proving this lemma, recall that its statement is equivalent to prove, for each sequence in $D(A_1)$, which is bounded with the graph norm, that it exists a subsequence that (strongly) converges in H_1.

Recalling the definition of the graph norm, it holds, for all $\left(\begin{smallmatrix} u \\ v \end{smallmatrix}\right)$ in $D(A_1)$,

$$\begin{aligned} \left\|\left(\begin{smallmatrix} u \\ v \end{smallmatrix}\right)\right\|^2_{D(A_1)} &:= \left\|\left(\begin{smallmatrix} u \\ v \end{smallmatrix}\right)\right\|^2 + \left\|A_1 \left(\begin{smallmatrix} u \\ v \end{smallmatrix}\right)\right\|^2 \\ &= \int_0^1 (|u'|^2 + |v|^2 + |v'|^2 \\ &\quad + |u'' - \sigma_1(av)|^2) dx \ . \end{aligned}$$

Therefore, on the one hand, one gets

$$\left\|\left(\begin{smallmatrix} u \\ v \end{smallmatrix}\right)\right\|^2_{D(A_1)} \geq \int_0^1 (|v|^2 + |v'|^2) dx, \qquad (31)$$

and on the other hand, due to (9b), it holds $|v| \geq \min(1, \frac{1}{La}) |\sigma_1(av)|$ and $|u'' - \sigma_1(av)| \geq \min(1, \frac{1}{La}) |u'' - \sigma_1(av)|$, it holds

$$\begin{aligned} \left\|\left(\begin{smallmatrix} u \\ v \end{smallmatrix}\right)\right\|^2_{D(A_1)} \geq \int_0^1 \Big(|u'|^2 + \min(1, \tfrac{1}{L^2a^2}) \, |\sigma_1(av)|^2 \\ + \min(1, \tfrac{1}{L^2a^2}) \, |u'' - \sigma_1(av)|^2 \Big) \, dx. \end{aligned}$$

Since, for all $(s, \widetilde{s}) \in \mathbb{C}^2$, it holds $|s+\widetilde{s}|^2 \leq 2|s|^2 + 2|\widetilde{s}|^2$, it follows $2 \, |u'' - \sigma_1(av)|^2 + 2 \, |\sigma_1(av)|^2 \geq |u''|^2$ and thus

$$\begin{aligned} \left\|\left(\begin{smallmatrix} u \\ v \end{smallmatrix}\right)\right\|^2_{D(A_1)} &\geq \int_0^1 (|u'|^2 + \tfrac{\min(1, \frac{1}{L^2a^2})}{2} |u''|^2) dx, \\ &\geq \tfrac{\min(1, \frac{1}{L^2a^2})}{2} \int_0^1 (|u'|^2 + |u''|^2) \ . \end{aligned} \qquad (32)$$

Consider now a sequence $\left(\begin{smallmatrix} u_n \\ v_n \end{smallmatrix}\right)_{n \in \mathbb{N}}$ in $D(A_1)$ bounded for the graph norm of $D(A_1)$. From (31) and (32), it follows that this sequence is bounded in the product space $(H^2(0,1) \cap H_0^1(0,1)) \times H^1(0,1)$. Since the canonical embedding from $H^2(0,1)$ to $H_0^1(0,1)$ (resp. from $H^1(0,1)$ to $L^2(0,1)$) is compact, there exists a subsequence still denoted $\left(\begin{smallmatrix} u_n \\ v_n \end{smallmatrix}\right)_{n \in \mathbb{N}}$ such that

$$u_n \to u \text{ in } H_0^1(0,1) \ , \quad v_n \to v \text{ in } L^2(0,1)$$

and thus $\left(\begin{smallmatrix} u \\ v \end{smallmatrix}\right)$ belongs to H_1, which proves the lemma. □

Inspired by [11, Thm 1, (iii)], using the dissipativity (see Lemma 1) it follows that the function

$$t \mapsto \left\| \begin{pmatrix} z(.,t) \\ z_t(.,t) \end{pmatrix} \right\| \qquad (33)$$

is a nonincreasing. Moreover, since A_1 is single valued, the canonical restriction of A_1 equals A_1 (see [19, Def. 2.7] for the introduction of this notion), and thus with [19, Cor. 3.7], we get that

$$t \mapsto \left\| A_1 \begin{pmatrix} z(.,t) \\ z_t(.,t) \end{pmatrix} \right\| \qquad (34)$$

is also a nonincreasing function. Therefore, with Lemma 3, the trajectory $\begin{pmatrix} z(.,t) \\ z_t(.,t) \end{pmatrix}$ is precompact in H_1. Moreover the ω-limit set $\omega \left[\begin{pmatrix} z(.,0) \\ z_t(.,0) \end{pmatrix} \right] \subset D(A_1)$, is not empty and invariant with respect to the nonlinear semigroup $T_1(t)$ (see [24, Thm 3.1]).

We now use LaSalle's invariance principle to show that $\omega \left[\begin{pmatrix} z(.,0) \\ z_t(.,0) \end{pmatrix} \right] = \{0\}$. To do that, consider a solution such that $\dot{V}_1(t) = 0$, for all $t \geq 0$. It follows from (11) that $z_t(x,t) = 0$ for almost all x in $(0,1)$ and for all $t \geq 0$. Due to (9b) and the continuity of σ_1, it follows $\sigma_1(0) = 0$. Therefore z is a solution to the linear equation (1) with the boundary conditions (2), such that $z_{tt}(x,t) = z_{xx}(x,t) = 0$ which implies that $z = 0$. Therefore $\omega \left[\begin{pmatrix} z(.,0) \\ z_t(.,0) \end{pmatrix} \right] = \{0\}$, and the convergence property (13) holds along the solutions to the nonlinear equation (10) with the boundary conditions (2).

This concludes the proof of Theorem 1. □

V. Proof of Theorem 2

Part 1: Well-posedness of the Cauchy problem (4), (3), (18).

Let us first prove the existence and unicity of solution to the nonlinear equation (4), with the boundary conditions (18) and the initial condition (3). To do that, let us

introduce the following nonlinear operator

$$A_2 \begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} v \\ u'' \end{pmatrix}$$

with the domain $D(A_2) = \{(u,v),\ (u,v) \in H^2(0,1) \times H^1(0,1),\ u_x(1)+\sigma_2(bv(1)) = 0,\ u(0) = 0,\ v(0) = 0\}$.

To prove the well-posedness of the Cauchy problem, we shall state that A_2 generates a semigroup of contractions by applying [2, Thm 1.3, Page 104], and thus we need to prove that A_2 is closed, dissipative, and satisfies a range condition (see (25) below). Let us prove these properties successively.

The nonlinear operator A_2 is closed, and using the nonnegativity of b, we may prove the following lemma.

Lemma 4. *A_2 is a dissipative operator.*

Proof of Lemma 4: Recall that $H^1_{(0)}(0,1)$ is a Hilbert space with the inner product

$$\langle u, \widetilde{u} \rangle_{H^1_{(0)}(0,1)} = \int_0^1 u'(x)\overline{\widetilde{u}'(x)}dx.$$

Now denote by H_2 the space $H^1_{(0)}(0,1) \times L^2(0,1)$. It is a Hilbert space equipped with the same inner product as for H_1, that is (22).

To check that A_2 is dissipative, using first the definition of A_2 and then recalling the definition of the inner product in H_2, let us compute the following, for all $\begin{pmatrix} u \\ v \end{pmatrix}$, $\begin{pmatrix} \widetilde{u} \\ \widetilde{v} \end{pmatrix}$ in $D(A_2)$,

$$\begin{aligned} &\langle A_2 \begin{pmatrix} u \\ v \end{pmatrix} - A_2 \begin{pmatrix} \widetilde{u} \\ \widetilde{v} \end{pmatrix}, \begin{pmatrix} u \\ v \end{pmatrix} - \begin{pmatrix} \widetilde{u} \\ \widetilde{v} \end{pmatrix} \rangle \\ &= \langle \begin{pmatrix} v - \widetilde{v} \\ u'' - \widetilde{u}'' \end{pmatrix}, \begin{pmatrix} u - \widetilde{u} \\ v - \widetilde{v} \end{pmatrix} \rangle , \\ &= \int_0^1 (v-\widetilde{v})'(x)\overline{(u-\widetilde{u})'(x)}dx \\ &+ \int_0^1 (u''-\widetilde{u}'')(x)\overline{(v-\widetilde{v})(x)}dx \ . \end{aligned} \tag{35}$$

Consider the second integral in the last equation. Performing an integration by parts and using the definition of $D(A_2)$, it gives

$$\begin{aligned} &\int_0^1 (u''-\widetilde{u}'')(x)\overline{(v-\widetilde{v})(x)}dx \\ &= -\int_0^1 (u'-\widetilde{u}')(x)\overline{(v'-\widetilde{v}')(x)}dx \\ &+[(u'-\widetilde{u}')(x)\overline{(v-\widetilde{v})(x)}]_{x=0}^{x=1} \\ &= -\int_0^1 (u'-\widetilde{u}')(x)\overline{(v'-\widetilde{v}')(x)}dx \\ &+(u'(1)-\widetilde{u}'(1))\overline{(v(1)-\widetilde{v}(1))} \\ &= -\int_0^1 (u'-\widetilde{u}')(x)\overline{(v'-\widetilde{v}')(x)}dx \\ &+(-\sigma_2(bv(1))+\sigma_2(b\widetilde{v}(1)))\overline{(v(1)-\widetilde{v}(1))} \ . \end{aligned}$$

By combining the previous equation with (35), we conclude the proof of Lemma 4 using (19a) as in the end of Lemma 1. □

Let us now show that the operator A_2 generates a semigroup of contractions. To do that, we apply [2, Thm 1.3, Page 104] (or [19, Thm 4.2, Page 77]) and we need to prove that

$$D(A_2) \subset \mathtt{Ran}(I - \lambda A_2) \tag{36}$$

for all $\lambda > 0$ sufficiently small, where $\mathtt{Ran}(I-\lambda A_2)$ is the range of the operator $I - \lambda A_2$. To prove (36), let us pick $\begin{pmatrix} u \\ v \end{pmatrix}$ in $D(A_2)$ and let us prove that there exists $\begin{pmatrix} \widetilde{u} \\ \widetilde{v} \end{pmatrix}$ in $D(A_2)$ such that $(I-\lambda A_2)\begin{pmatrix} \widetilde{u} \\ \widetilde{v} \end{pmatrix} = \begin{pmatrix} u \\ v \end{pmatrix}$. Let us first note that this latter equation is equivalent to

$$\begin{cases} \widetilde{u} - \lambda\widetilde{v} = u \ , \\ \widetilde{v} - \lambda\widetilde{u}'' = v \ , \end{cases}$$

which may be rewritten as

$$\begin{cases} \widetilde{v} = \frac{1}{\lambda}(\widetilde{u} - u) \ , \\ \widetilde{u}'' - \frac{1}{\lambda^2}\widetilde{u} = -\frac{1}{\lambda}v - \frac{1}{\lambda^2}u \ . \end{cases} \tag{37}$$

To check that there exists $\widetilde{u} \in H^2(0,1)$ such that $\widetilde{u}(0) = 0$, $\widetilde{u}'(1) = -\sigma_2(\frac{b}{\lambda}\widetilde{u}(1) - bu(1))$, and such that the second line of (37) holds, let us first note that this is a nonhomogeneous nonlinear differential equation in the $\widetilde{u}$-variable with two boundary conditions (at $x = 0$ and at $x = 1$), as considered below:

Lemma 5. *If λ is positive, then there exists $\widetilde{u} \in H^2(0,1)$ solution to*

$$\begin{aligned} \widetilde{u}'' - \tfrac{1}{\lambda^2}\widetilde{u} = -\tfrac{1}{\lambda}v - \tfrac{1}{\lambda^2}u \\ \widetilde{u}(0) = 0 \ ,\ \widetilde{u}'(1) = -\sigma_2(\tfrac{b}{\lambda}\widetilde{u}(1) - bu(1)) \end{aligned} \tag{38}$$

Proof of Lemma 5: To prove this lemma, let us introduce the following map

$$\begin{aligned} \mathcal{T}_2 : \quad H^1(0,1) &\to L^2(0,1) \ , \\ y &\mapsto z = \mathcal{T}_2(y) \ , \end{aligned}$$

where $z = \mathcal{T}_2(y)$ is the unique solution to

$$\begin{aligned} z'' - \tfrac{1}{\lambda^2}z = -\tfrac{1}{\lambda}v - \tfrac{1}{\lambda^2}u \ , \\ z(0) = 0 \ ,\ z'(1) = -\sigma_2(\tfrac{b}{\lambda}y(1) - bu(1)) \ . \end{aligned} \tag{39}$$

This map $\mathcal{T}_2$ can be seen as a boundary value problem, and it is well-defined for all $\lambda > 0$. In a similar way as in the proof of Claim 2, we have the following claim.

Claim 3. *There exists $M > 0$ such that $\mathcal{T}_2(L^2(0,1)) \subset K$, where K is the set of functions w that are continuously differentiable on $[0,1]$ and such that $\|w\|_{C^0([0,1])} \leq M$ and $\|w'\|_{C^0([0,1])} \leq M$.*

Proof of Claim 3: As for the proof of Claim 2, let us first note that each solution to (39) is such that

$$\begin{aligned} z'(x) \quad = \quad & (0\ 1)e^{\mathcal{A}x}(0\ C_2)^\top \\ & + \int_0^x e^{\mathcal{A}(x-s)}(-\tfrac{1}{\lambda}v - \tfrac{1}{\lambda^2}u)ds \end{aligned}$$

for a suitable value of C_2, where the matrix $\mathcal{A}$ is defined in the proof of Claim 2 and $e^{\mathcal{A}x}$ is given by (30). The

value of C_2 is so that $z'(1) = -\sigma_2(\frac{b}{\lambda}y(1) - bu(1))$. Recall (19b). By inspecting $(0\ \ 1)e^{\mathcal{A}}(0\ \ C_2)^\top$, since $\cosh(\frac{1}{\lambda}) \neq 0$ and since, for all $y \in L^2(0,1)$, $|\sigma_2(\frac{b}{\lambda}y(1) - bu(1))| \leq u_2$, we get that the value C_2 lies in a bounded set of $\mathbb{R}$ (which does not depend on $y \in L^2(0,1)$) and thus the existence of $M > 0$ follows, as stated in Claim 3. □

Moreover $\mathcal{T}_2$ is a continuous operator. Finally the set K is convex and compact (by the Ascoli-Arzela theorem), as a subset of $L^2(0,1)$.

Therefore, by the Schauder fixed-point theorem (see e.g., [5, Thm B.19]), there exists $\widetilde{u} \in K$ such that $\mathcal{T}_2(\widetilde{u}) = \widetilde{u}$. This concludes the proof of Lemma 5. □

Now from the existence of $\widetilde{u} \in H^2(0,1) \cap H^1_{(0)}(0,1)$ such that the second line of (37) holds, let us remark that the first line of (37) defines a unique $\widetilde{v}$ in $H^1(0,1)$. Therefore $(I - \lambda A_2)\begin{pmatrix} \widetilde{u} \\ \widetilde{v} \end{pmatrix} = \begin{pmatrix} u \\ v \end{pmatrix}$ and (36) hold.

Since A_2 is dissipative (due to Lemma 4), it follows, from [2, Thm 1.3, Page 104] (or [19, Thm 4.2]), that A_2 generates a semigroup of contractions $T_2(t)$. Moreover, by [2, Thm 1.2, Page 102] (or [19, Thm 4.5]), for all $\begin{pmatrix} \widetilde{u} \\ \widetilde{v} \end{pmatrix}$ in $D(A_2)$, $T_2(t)\begin{pmatrix} \widetilde{u} \\ \widetilde{v} \end{pmatrix}$ is strongly differentiable for all $t > 0$ and is a solution to the Cauchy problem (4), (18), and (3). Moreover due to [19, Thm 4.10], it is the unique solution to this Cauchy problem.

Part 2: Global asymptotic stability of the nonlinear equation (4) with the boundary conditions (18).

Let us consider a solution to (4) and (18) for a given initial condition in $D(A_2)$. Now, as in the proof of Theorem 1, using the dissipativity (see Lemma 4) and [19, Cor. 3.7], we get

$$t \mapsto \left\| \begin{pmatrix} z(.,t) \\ z_t(.,t) \end{pmatrix} \right\| \ , \ t \mapsto \left\| A_2 \begin{pmatrix} z(.,t) \\ z_t(.,t) \end{pmatrix} \right\| \ ,$$

are two non-increasing functions. The decreasing property of the first function yields the global stability property as written in Theorem 2. Now, by focusing on the second function, it holds, for all $t \geq 0$,

$$\|z_t(.,t)\|_{H^1_{(0)}(0,1)} \leq \left\| A_2 \begin{pmatrix} z(.,0) \\ z_t(.,0) \end{pmatrix} \right\| \ .$$

Moreover, on the one hand, by definition of A_2 and of H_2, it holds

$$\left\| A_2 \begin{pmatrix} z^0 \\ z^1 \end{pmatrix} \right\|^2 = \|z^{0\prime\prime}\|^2_{L^2(0,1)} + \|z^1\|^2_{H^1_{(0)}(0,1)} \ . \quad (40)$$

On the other hand, since $z_t(0,t) = 0$, it holds $|z_t(1,t)|^2 = |\int_0^1 z_{xt}(.,t)dx|^2 \leq \int_0^1 |z_{xt}(.,t)|^2 dx = \|z_t(.,t)\|^2_{H^1_{(0)}(0,1)}$. Thus, with (40), for all $t \geq 0$,

$$|z_t(1,t)| \leq \left\| A_2 \begin{pmatrix} z(.,0) \\ z_t(.,0) \end{pmatrix} \right\| \ . \quad (41)$$

Let us now prove the following

Lemma 6. *For each $b > 0$ and for all $\epsilon > 0$, there exist $\lambda > 0$, $\mu > 0$ and c in $\mathbb{R}$ such that*

$$\mathcal{M} = \begin{pmatrix} (1-b)^2e^{\mu} - (1+b)^2e^{-\mu} & \star \\ (b-1)e^{\mu} - (1+b)e^{-\mu} - \lambda c & e^{\mu} - e^{-\mu} - 2\lambda \end{pmatrix} \leq 0 \quad (42)$$

and $|b - c| \leq \epsilon$ hold.

Proof of Lemma 6: Let $b > 0$ and $\epsilon > 0$. First consider the following matrix

$$\widetilde{\mathcal{M}} = \begin{pmatrix} -4b & \star \\ -2 - \lambda c & -2\lambda \end{pmatrix}$$

Since the trace of $\widetilde{\mathcal{M}}$ is $-4b - 2\lambda$, the sum of the eigenvalues is negative as soon as $\lambda > 0$ and $b > 0$. The determinant is the product of the eigenvalues: $P = 8b\lambda - (2 + \lambda c)^2$. Let us denote $\widetilde{\epsilon} = b - c$, and make a Taylor expansion of P with respect to $\widetilde{\epsilon}$ at $\widetilde{\epsilon} = 0$. It holds $P = 8b\lambda - 4 - 4\lambda b + 4\lambda\widetilde{\epsilon} - \lambda^2 b^2 + 2\lambda^2 b\widetilde{\epsilon} + o(\widetilde{\epsilon}) = -(\lambda b - 2)^2 + 2\widetilde{\epsilon}\lambda(2 + \lambda b) + o(\widetilde{\epsilon})$. Therefore letting $\lambda = 2/b$, it holds $P = 16\widetilde{\epsilon}/b + o(\widetilde{\epsilon})$ which is positive as soon as $\widetilde{\epsilon}$ is sufficiently small and positive. This implies that both eigenvalues of $\widetilde{\mathcal{M}}$ are negative and we conclude that $\widetilde{\mathcal{M}} < 0$. Note now that for $\mu \to 0$ we get that matrix $\widetilde{\mathcal{M}}$ approaches $\mathcal{M}$. Hence, by the continuity of the eigenvalues of matrix $\mathcal{M}$ with respect to parameters, we get the existence of $\lambda > 0$, c in $\mathbb{R}$ (close to b) and $\mu > 0$ (close to 0) such that (42) and $|b - c| \leq \epsilon$ hold. This concludes the proof of Lemma 6. □

Pick $r > 0$ and an initial condition satisfying $\|z^{0\prime\prime}\|^2_{L^2(0,1)} + \|z^1\|^2_{H^1_{(0)}(0,1)} \leq r^2$. Apply Lemma 6 with $b > 0$ and $\epsilon = u_2/r$. We obtain that the initial condition satisfies $|b-c|^2(\|z^{0\prime\prime}\|^2_{L^2(0,1)} + \|z^1\|^2_{H^1_{(0)}(0,1)}) \leq u_2^2$. With (40) and (41), we obtain, along the solutions of (4) and (18) starting from such an initial condition,

$$|(b - c)z_t(1,t)| \leq u_2 \ .$$

Using (19c) with $s = z_t(1,t)$, it follows, along the solutions to (4) and (18) starting from such an initial condition,

$$\varphi_2(bz_t(1,t))(\varphi_2(bz_t(1,t)) + cz_t(1,t)) \leq 0 \ . \quad (43)$$

The formal computation (16) of the Lyapunov function candidate V_2 along the solutions to (4) and (18) makes sense and it holds, along the solutions to (4) and (18),

$$\begin{aligned} &\dot{V}_2 + \mu V_2 \\ &= e^{\mu}(z_t(1,t) + z_x(1,t))^2 \\ &\quad - e^{-\mu}(z_t(1,t) - z_x(1,t))^2 \\ &= e^{\mu}(z_t(1,t) - \sigma_2(bz_t(1,t)))^2 \\ &\quad - e^{-\mu}(z_t(1,t) + \sigma_2(bz_t(1,t)))^2 \\ &= e^{\mu}(z_t(1,t) - \varphi_2(bz_t(1,t)) - bz_t(1,t))^2 \\ &\quad - e^{-\mu}(z_t(1,t) + \varphi_2(bz_t(1,t)) + bz_t(1,t))^2 \ . \end{aligned}$$

Let us denote φ_2 instead of $\varphi_2(bz_t(1,t))$. For $\lambda > 0$ given by Lemma 6, and using (43), it follows

$$\begin{aligned}\dot{V}_2 \ &\leq -\mu V_2 + e^{\mu}(z_t(1,t) - \varphi_2 - bz_t(1,t))^2 \\ &\quad -e^{-\mu}(z_t(1,t) + \varphi_2 + bz_t(1,t))^2 \\ &\quad -2\lambda\varphi_2(\varphi_2 + cz_t(1,t)) \\ &\leq -\mu V_2 + \begin{pmatrix} z_t(1,t) \\ \varphi_2 \end{pmatrix}^{\top} \mathcal{M} \begin{pmatrix} z_t(1,t) \\ \varphi_2 \end{pmatrix}\end{aligned}$$

where $\mathcal{M}$ is defined in (42). With (42), we get $\dot{V}_2 \leq -\mu V_2$, along the solutions to (4), (18) and (3) starting from an initial condition satisfying

$$\|z^{0\prime\prime}\|^2_{L^2(0,1)} + \|z^1\|^2_{H^1_{(0)}(0,1)} \leq r^2 \ . \tag{44}$$

As a conclusion, for all $b > 0$, and for all $r > 0$, for all initial conditions in $\{(u,v),\ (u,v) \in H^2(0,1) \times H^1_{(0)}(0,1),\ u'(1) + bv(1) = 0,\ u(0) = 0\}$ satisfying (44), there exists $\mu > 0$ such that $\dot{V}_2(t) \leq -\mu V_2(t)$, and thus with the expression of V_2 in (15), it implies (21). This implies the global attractivity as stated in Theorem 2. □

VI. Numerical simulations

A. Illustrating Theorem 1

Let us illustrate Theorem 1 by discretizing the PDE (10) with the boundary conditions (2) and the initial condition (3) by means of finite difference method. To do that we compute the values of z at the next time step by using the values known at the previous two time steps (see e.g. [18] for an introduction on the numerical implementation). It is chosen the time and the space steps so that the stability condition of the numerical scheme is satisfied. Due to the presence of the nonlinear map, an implicit equation has to be solved when discretizing the dynamics. [2]

Consirer $a = 1$, and the following nonlinear function $\sigma_1(s) = \mathtt{sat}_1(\sin(s) + s)$, for $s \in \mathbb{R}$, where $\mathtt{sat}_1$ is the saturation map of level $u_1 = 1.5$. This function satisfies (9).

Let us consider the initial condition (3) with $z^0(x) = \sin(\pi x)$ and $z^1(x) = 0$, for $x \in [0,1]$. The time evolution of the numerically computed z-component of the solution is given in Figure 2 where it is checked that it converges to the equilibrium. Moreover on Figure 3 it can be observed that the control law saturates for small positive time.

[2]The simulation code for both examples can be downloaded from `http://www.gipsa-lab.fr/~christophe.prieur/Codes/code-tac16.zip`

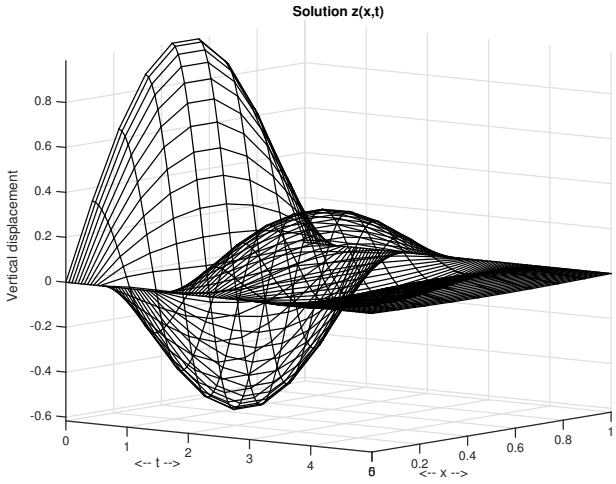

Fig. 2. Time evolution of the z-component of the solution to (10), (2) and (3) with σ_1.

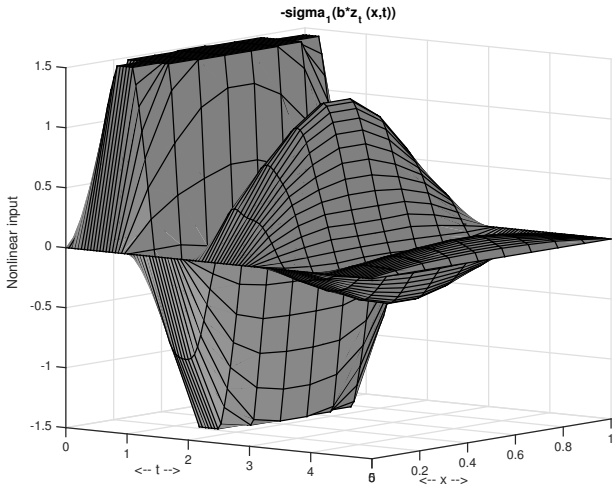

Fig. 3. Time evolution of the nonlinear control law with σ_1.

B. Illustrating Theorem 2

Let us illustrate now Theorem 2 by discretizing the PDE (4) with the boundary conditions (18) and the initial condition (3).

Consider $b = 1$, and the following nonlinear function $\sigma_2(s) = \mathtt{sat}_2(s)$, for $s \in \mathbb{R}$, where $\mathtt{sat}_2$ is the saturation map of level $u_2 = 0.05$. This function satisfies (19).

Let us consider the same initial condition as in the previous subsection.

The time evolution of the numerically computed z-component of the solution is given in Figure 4 where it is observed that it converges to the equilibrium $z = 0$. In Figure 5 it is observed that the control law saturates around 30 times within 20s. Therefore, in spite of the saturation, note that the convergent behavior can be observed as predicted by Theorem 2.

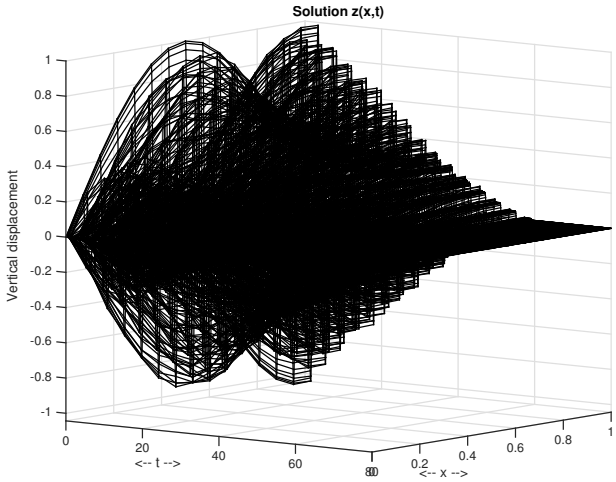

Fig. 4. Time evolution of the z-component of the solution to (4), (18) and (3) with σ_2.

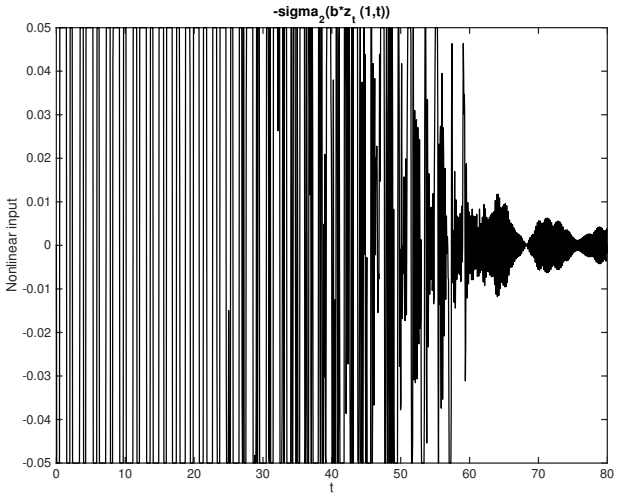

Fig. 5. Time evolution of the nonlinear control law with σ_2.

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VII. CONCLUSION

The well-posedness and the asymptotic stability of a class of 1D wave equations have been studied. The PDE under consideration resulted from the feedback connection of a classical wave equation and a cone bounded nonlinear control law. The controller is either applied in the space domain (distributed input) or at one boundary (boundary action). The well-posedness issue has been tackled by using nonlinear semigroup techniques and the stability has been proven by Lyapunov theory for infinite dimensional systems.

This work lets some questions open. In particular, it could be interesting to use other classes of Lyapunov functions, as those considered in [27] and to compare the obtained domain of attraction with the one estimated in Theorem 2. It would also be interesting to study other PDEs appearing in vibration control theory, such as the beam equation (as considered in [7]). Other hyperbolic systems as the one considered in this paper may also be considered as the conservation laws that are useful for the flow control (see [6], [22]). For such a class of PDEs, Lyapunov theory is an useful tool when designing stabilizing linear controllers, and may be also the key when computing saturating stabilizing feedback laws. Generalization to the design of output feedback laws, instead of state feedback controls as considered in particular in Theorem 1, is also a natural research line.

REFERENCES

[1] R.P. Agarwal and D. O'Regan. *Ordinary and partial differential equations, with special functions, Fourier series, and boundary value problems*. Universitext. Springer, New York, 2009.
[2] V. Barbu. *Nonlinear semigroups and differential equations in Banach spaces*. Editura Academiei, 1976.
[3] P.J. Campo and M. Morari. Robust control of processes subject to saturation nonlinearities. *Computers & Chemical Engineering*, 14(4–5):343 – 358, 1990.
[4] E.B. Castelan, S. Tarbouriech, and I. Queinnec. Control design for a class of nonlinear continuous-time systems. *Automatica*, 44(8):2034 – 2039, 2008.
[5] J.-M. Coron. *Control and Nonlinearity*, volume 136 of *Mathematical Surveys and Monographs*. American Mathematical Society, Providence, RI, 2007.
[6] J.-M. Coron, B. d'Andréa Novel, and G. Bastin. A strict Lyapunov function for boundary control of hyperbolic systems of conservation laws. *IEEE Transactions on Automatic Control*, 52(1):2–11, 2007.
[7] E. Crépeau and C. Prieur. Control of a clamped-free beam by a piezoelectric actuator. *ESAIM: Control, Optimisation and Calculus of Variations*, 12:545–563, 2006.
[8] R. Curtain and H. Zwart. *An introduction to infinite-dimensional linear systems theory*. Springer-Verlag, 1995.
[9] C.M. Dafermos and M. Slemrod. Asymptotic behavior of nonlinear contraction semigroups. *Journal of Functional Analysis*, 13(1):97–106, 1973.
[10] B. d'Andréa Novel, F. Boustany, F. Conrad, and B.P. Rao. Feedback stabilization of a hybrid PDE-ODE system: Application to an overhead crane. *Mathematics of Control, Signals, and Systems*, 7(1):1–22, 1994.
[11] B. d'Andréa Novel and J.-M. Coron. Exponential stabilization of an overhead crane with flexible cable via the cascade approach. *Automatica*, 36(4):587–593, 2000.
[12] T. Hu, A. R. Teel, and L. Zaccarian. Stability and performance for saturated systems via quadratic and nonquadratic Lyapunov functions. *IEEE Transactions on Automatic Control*, 51(11):1770–1786, 2006.
[13] H.K. Khalil. *Nonlinear Systems*. Prentice-Hall, 3rd edition, 2002.
[14] G.C. King. *Vibrations and Waves*. Manchester Physics Series. Wiley, 2013.
[15] E. Kreyszig. *Introductory Functional Analysis with Applications*. Wiley, New York, 1978.
[16] M. Krstic. Dead-time compensation for wave/string PDEs. *Journal of Dynamic Systems, Measurement, and Control*, 133(3):Paper 031004 (13 pages), 2011.

[17] I. Lasiecka and T.I. Seidman. Strong stability of elastic control systems with dissipative saturating feedback. *Systems & Control Letters*, 48(3-4):243–252, 2003.

[18] G.R. Lindfield and J.E.T. Penny. *Numerical Methods: Using MATLAB*. Academic Press, 2012.

[19] I. Miyadera. *Nonlinear Semigroups*. Translations of mathematical monographs. American Mathematical Society, 1992.

[20] A. Pazy. *Semigroups of linear operators and applications to partial differential equations*. Applied mathematical sciences. Springer-Verlag, 1983.

[21] C. Prieur and J. de Halleux. Stabilization of a 1-D tank containing a fluid modeled by the shallow water equations. *Systems & Control Letters*, 52(3-4):167–178, 2004.

[22] C. Prieur and F. Mazenc. ISS-Lyapunov functions for time-varying hyperbolic systems of balance laws. *Mathematics of Control, Signals, and Systems*, 24(1):111–134, 2012.

[23] C. Prieur, S. Tarbouriech, and J. M. Gomes da Silva Jr. Well-posedness and stability of a 1D wave equation with saturating distributed input. In *Proceedings of the 53rd IEEE Conference on Decision and Control*, Los Angeles, CA, 2014.

[24] M. Slemrod. Feedback stabilization of a linear control system in Hilbert space with an a priori bounded control. *Mathematics of Control, Signals, and Systems*, 2(3):265–285, 1989.

[25] S. Tarbouriech, G. Garcia, J.-M. Gomes da Silva Jr, and I. Queinnec. *Stability and Stabilization of Linear Systems with Saturating Actuators*. Springer, London, 2011.

[26] S. Tarbouriech, C. Prieur, and J.M. Gomes da Silva Jr. Stability analysis and stabilization of systems presenting nested saturations. *IEEE Transactions on Automatic Control*, 51(8):1364–1371, 2006.

[27] G. Valmórbida, M. Ahmadi, and A. Papachristodoulou. Semi-definite programming and functional inequalities for distributed parameter systems. In *Proceedings of the 53rd IEEE Conference on Decision and Control*, Los Angeles, CA, 2014.

[28] L. Zaccarian and A.R. Teel. *Modern Anti-windup Synthesis: Control Augmentation for Actuator Saturation*. Princeton Series in Applied Mathematics. Princeton University Press, 2011.

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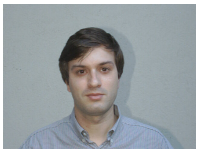

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