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# CRITICAL BEHAVIOR OF THE ANNEALED ISING MODEL ON RANDOM REGULAR GRAPHS

VAN HAO CAN

ABSTRACT. In [15], the authors have defined an annealed Ising model on random graphs and proved limit theorems for the magnetization of this model on some random graphs including random 2-regular graphs. Then in [9], we generalized their results to the class of all random regular graphs. In this paper, we study the critical behavior of this model. In particular, we determine the critical exponents and prove a non standard limit theorem stating that the magnetization scaled by  $n^{3/4}$  converges to a specific random variable, with  $n$  the number of vertices of random regular graphs.

## 1. INTRODUCTION

Ising model is one of the most well-known model in the field of statistical physics that exhibits phase transitions. This model has been investigated fruitfully for integer lattices, see e.g. [13]. Recently, Ising model has been studied in random graphs as a model of the cooperative interaction of spins in random networks, see for instance [1, 2, 4, 6, 17]. As for other models in random environments, probabilists study this model in both quenched setting and annealed setting. In the quenched one, the Ising model is defined accordingly to typical samples of graphs. On the other hand, in the annealed one, the Ising model is defined by taking information of all realizations of graphs. In contrast of the well-development of studies on quenched setting (see e.g. [2, 4, 14, 17]), there are few contributions in the annealed one. In two recent papers [15, 5], the authors defined an annealed Ising model as follows.

Let  $G_n = (V_n, E_n)$  be a random multigraph (i.e. a random graph possibly having self-loops and multiple edges between vertices) with the set of vertices  $V_n = \{v_1, \dots, v_n\}$  and the set of edges  $E_n$ . A spin  $\sigma_i$  is assigned to each vertex  $v_i$ . Then for any configuration  $\sigma \in \Omega_n := \{+1, -1\}^n$ , the Halmintonian is given by

$$H(\sigma) = -\beta \sum_{i \leq j} k_{i,j} \sigma_i \sigma_j - B \sum_{i=1}^n \sigma_i,$$

where  $k_{i,j}$  is the number of edges between  $v_i$  and  $v_j$ , where  $\beta \geq 0$  is the inverse temperature and  $B \in \mathbb{R}$  is the uniform external magnetic field.

Then the configuration probability is given by the *annealed measure*: for all  $\sigma \in \Omega_n$ ,

$$\mu_n(\sigma) = \frac{\mathbb{E}(\exp(-H(\sigma)))}{\mathbb{E}(Z_n(\beta, B))},$$

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where  $\mathbb{E}$  denotes the expectation with respect to the random graph, and  $Z_n(\beta, B)$  is the partition function:

$$Z_n(\beta, B) = \sum_{\sigma \in \Omega_n} \exp(-H(\sigma)).$$

In [15], Giardinà, Giberti, van der Hofstad and Prioriello study this annealed Ising model on the rank-one inhomogeneous random graph, the random regular graph with degree 2 and the configuration model with degrees 1 and 2. After determining limits of thermodynamic quantities and the critical inverse temperature, they prove laws of large numbers and central limit theorems for the magnetization. Continuing this work, the authors of [15] and Dommers investigate the critical behaviors of the Ising model on inhomogeneous random graphs in [5].

In [9], we generalize the result in [15] for all random regular graphs, and show that the thermodynamic limits in quenched and annealed models are actually the same. In this paper, we are going to study critical behaviors of the annealed model. More precisely, we aim to determine critical exponents of thermodynamics limits and prove a non-classical scaling limit theorem for the magnetization.

Before stating our main results, we first give some definitions following [15, 9] of the thermodynamic quantities in finite volume.

- (i) The annealed pressure is given by

$$\psi_n(\beta, B) = \frac{1}{n} \log \mathbb{E}(Z_n(\beta, B)).$$

- (ii) The annealed magnetization is given by

$$M_n(\beta, B) = \frac{\partial}{\partial B} \psi_n(\beta, B).$$

An interpretation of the magnetization is

$$M_n(\beta, B) = \mathbb{E}_{\mu_n} \left( \frac{S_n}{n} \right),$$

with  $S_n$  the total spin, i.e.  $S_n = \sigma_1 + \dots + \sigma_n$ .

- (iii) The annealed susceptibility is given by

$$\chi_n(\beta, B) = \frac{\partial}{\partial B} M_n(\beta, B) = \frac{\partial^2}{\partial B^2} \psi_n(\beta, B).$$

We also have

$$\chi_n(\beta, B) = \text{Var}_{\mu_n} \left( \frac{S_n}{\sqrt{n}} \right).$$

- (iv) The annealed specific heat is given by

$$\mathcal{C}_n(\beta, B) = \frac{\partial^2}{\partial \beta^2} \psi_n(\beta, B).$$

When the sequence  $(M_n(\beta, B))_n$  converges to a limit, say  $\mathcal{M}(\beta, B)$ , we define the spontaneous magnetization as  $\mathcal{M}(\beta, 0^+) = \lim_{B \searrow 0} \mathcal{M}(\beta, B)$ . Then the critical inverse temperature is defined as

$$\beta_c = \inf\{\beta > 0 : \mathcal{M}(\beta, 0^+) > 0\}.$$

The uniqueness region of the existence of the limit magnetization is defined as

$$\mathcal{U} = \{(\beta, B) : \beta \geq 0, B \neq 0 \text{ or } 0 < \beta < \beta_c, B = 0\}.$$

In [9], we have proved the existence of the limit of thermodynamic quantities.

**Theorem 1.1.** [9, Theorem 1.1]. *Let us consider the Ising model on the random  $d$ -regular graph with  $d \geq 2$ . Then the following assertions hold.*

(i) *For all  $\beta \geq 0$  and  $B \in \mathbb{R}$ , the annealed pressure converges*

$$\lim_{n \rightarrow \infty} \psi_n(\beta, B) = \psi(\beta, B) = \frac{\beta d}{2} - B + \max_{0 \leq t \leq 1} [H_\beta(t) + 2Bt],$$

where

$$H_\beta(t) = (t - 1) \log(1 - t) - t \log t + dF_\beta(t),$$

with

$$F_\beta(t) = \int_0^{t \wedge (1-t)} \log f_\beta(s) ds,$$

and  $t \wedge (1 - t) = \min\{t, 1 - t\}$ ,

$$f_\beta(s) = \frac{e^{-2\beta}(1 - 2s) + \sqrt{1 + (e^{-4\beta} - 1)(1 - 2s)^2}}{2(1 - s)}.$$

(ii) *For all  $(\beta, B) \in \mathcal{U}$ , the magnetization converges*

$$\lim_{n \rightarrow \infty} M_n(\beta, B) = \mathcal{M}(\beta, B) = \frac{\partial}{\partial B} \psi(\beta, B).$$

Moreover, the critical inverse temperature is

$$\beta_c = \operatorname{atanh}(1/(d - 1)) = \begin{cases} \frac{1}{2} \log \left( \frac{d}{d-2} \right) & \text{if } d \geq 3 \\ \infty & \text{if } d = 2. \end{cases}$$

(iii) *For all  $(\beta, B) \in \mathcal{U}$ , the annealed susceptibility converges*

$$\lim_{n \rightarrow \infty} \chi_n(\beta, B) = \chi(\beta, B) = \frac{\partial^2}{\partial B^2} \psi(\beta, B).$$

The convergence of annealed pressure has been first proved by Dembo, Montanari, Sly and Sun in [3]. By showing the replica symmetry of the partition function, the authors prove that annealed and quenched pressures converge to a common limit, which has been established in [2]. Our proof of the convergence of annealed pressure in [9] is based on the direct relation between the Hamiltonian and the number of disagreeing edges (i.e. edges with different spins) in random regular graphs. To characterize the law of the disagreeing edges, we combine the exchangeability of the model and many combinatorial computations. The convergences of magnetization and susceptibility follow from the one of pressure and standard arguments introduced in [10, 15].

Unfortunately, we are not able to show the convergence of specific heat, though it is very natural to expect that  $\mathcal{C}_n(\beta, B)$  tends to the second derivative of  $\psi(\beta, B)$  w.r.t  $\beta$ . Hence, we study an "artificial" specific heat limit defined as

$$\mathcal{C}(\beta, B) = \frac{\partial^2}{\partial \beta^2} \psi(\beta, B).$$

Following [5], we give a definition of critical exponents of thermodynamic limits.

**Definition.** The annealed critical exponents  $\beta, \delta, \gamma, \gamma', \alpha, \alpha'$  are defined by:

$$\begin{aligned} \mathcal{M}(\beta, 0^+) &\asymp (\beta - \beta_c)^\beta && \text{for } \beta \searrow \beta_c, \\ \mathcal{M}(\beta_c, B) &\asymp B^{1/\delta} && \text{for } B \searrow 0, \\ \chi(\beta, 0^+) &\asymp (\beta_c - \beta)^{-\gamma} && \text{for } \beta \nearrow \beta_c, \\ \chi(\beta, 0^+) &\asymp (\beta - \beta_c)^{-\gamma'} && \text{for } \beta \searrow \beta_c, \\ \mathcal{C}(\beta, 0^+) &\asymp (\beta_c - \beta)^{-\alpha} && \text{for } \beta \nearrow \beta_c, \\ \mathcal{C}(\beta, 0^+) &\asymp (\beta - \beta_c)^{-\alpha'} && \text{for } \beta \searrow \beta_c, \end{aligned}$$

where we write  $f(x) \asymp g(x)$  if the ratio  $f(x)/g(x)$  is bounded from 0 and infinity for the specified limit.

**1.1. Main results.** Our first result aims at determining the critical exponents defined above.

**Theorem 1.2.** (*Annealed critical exponents*). *Let us consider the annealed Ising model on random  $d$ -regular graph with  $d \geq 3$ . Then the critical exponents satisfy*

$$\begin{aligned} \beta &= \frac{1}{2} \\ \delta &= 3 \\ \gamma = \gamma' &= 1 \\ \alpha = \alpha' &= 0. \end{aligned}$$

In [4], the authors settle the quenched critical exponents for a large class of random graphs, so-called locally-tree like graphs. In particular, for the random regular graphs, the quenched critical exponents satisfy  $\beta = \frac{1}{2}$ ,  $\delta = 3$ ,  $\gamma = 1$ . Additionally, we have proved in [9] that for the case of random regular graphs, the annealed and quenched thermodynamic quantities are equal. Therefore, the values of  $\beta, \delta, \gamma$  can be directly deduced from the result in [4]. On the other hand, the values of other critical exponents  $\gamma', \alpha, \alpha'$  are new and are the main contribution of Theorem 1.2.

Our second result is on the asymptotic behavior of the total spin  $S_n$  as  $n$  tends to infinity. In [9, Theorem 1.3 and Proposition 1.4], we have proved that if  $(\beta > 0, B \neq 0)$  or  $(0 < \beta < \beta_c, B = 0)$  then  $S_n$  satisfies a central limit theorem, and if  $(\beta > \beta_c, B = 0)$  then  $S_n/n$  is concentrated at two opposite values. In the following result, we study the scaling limit of  $S_n$  for the remained case when  $\beta = \beta_c$  and  $B = 0$ .

**Theorem 1.3.** (*Scaling limit theorem at criticality*). *Consider the annealed Ising model on random  $d$ -regular graphs with  $d \geq 3$ . Suppose that  $\beta = \beta_c$  and  $B = 0$ . Then*

$$\frac{\sigma_1 + \dots + \sigma_n}{n^{3/4}} \xrightarrow{(\mathcal{D})} X \quad \text{w.r.t. } \mu_n,$$

where  $X$  is a random variable with density proportional to

$$\exp\left(\frac{-(d-1)(d-2)x^4}{12d^2}\right).$$

Different from classical central limit theorems, the scaling limit theorem at criticality has non-Gaussian limit distribution. This phenomena has been observed for some spin models, such as for Curie-Weiss model, or Ising model on  $\mathbb{Z}^2$  and inhomogeneous random graphs, see [10, 11, 12, 7, 8, 5]. In fact, some authors believe that the critical nature of the

total spin has universal scaling limit, see for example [10, 5]. Indeed, they guess that when  $\beta = \beta_c$  and  $B = 0$ ,  $S_n$  scaled by  $n^{\delta/(\delta+1)}$ , with  $\delta$  the exponent of magnetization, converges in law to a random variable whose the tail of the density behaves like  $\exp(-cx^{\delta+1})$  for  $x$  large enough. Our results confirm this belief for the class of random regular graphs.

**1.2. Discussion.** We now make some further remarks on our results.

- (i) Since  $\beta_c$  is finite if and only if  $d \geq 3$ , in our results, we always assume that  $d \geq 3$ .
- (ii) A simple interpretation of the specific heat is as follows

$$\mathcal{C}_n(\beta, B) = \text{Var}_{\hat{\mu}_n} \left( \frac{\sum_{i \leq j} k_{i,j} \sigma_i \sigma_j}{\sqrt{n}} \right),$$

where  $\hat{\mu}_n$  is a probability measure on  $\mathcal{G}_n \times \Omega_n$ , with  $\mathcal{G}_n$  the sample space of the random  $d$ -regular graph, given by

$$\int_{\mathcal{G}_n \times \Omega_n} f d\hat{\mu}_n = \frac{\sum_{\sigma \in \Omega_n} \mathbb{E}(f(G, \sigma) e^{-H_n(\sigma)})}{\sum_{\sigma \in \Omega_n} \mathbb{E}(e^{-H_n(\sigma)})}.$$

We notice that  $\mu_n$  is a marginal measure of  $\hat{\mu}_n$ ,

$$\mu_n(\sigma) = \mathbb{E}(\hat{\mu}_n(G, \sigma)).$$

Studying the measure  $\hat{\mu}_n$  might give some ideas to derive the convergence of  $(\mathcal{C}_n(\beta, B))$ .

(iii) A natural and interesting question is to generalize our results for the configuration model random graphs with general degree distributions (see [16] for a definition). Comparing with the case of random regular graphs, we have additionally a source of randomness coming from the sequence of degrees. This randomness makes the problem much more difficult. In particular, we have proved in [9, Proposition 7.3] that the annealed pressure converges to a limit given by

$$\psi(\beta, B) = -B + \max_{0 \leq t \leq 1} [(t-1) \log(1-t) - t \log t + 2Bt + G_\beta(t)],$$

where  $G_\beta(t)$  is a Lipschitz function concerning with a large deviation result on the degree distribution of configuration model. Due to the complexity of  $G_\beta(t)$ , we are not able to show the differentiability of  $\psi(\beta, B)$ . Without the differentiability, we can not go further to other thermodynamic limits or critical exponents. We also remark that when the degrees of vertices fluctuates, the authors of [14] conjecture that annealed and quenched Ising models behaves differently. In particular, they guess that the critical inverse temperatures are different. It would be very interesting to know whether the annealed and quenched critical exponents are equal or not. Notice that in the case of inhomogeneous random graphs, though the annealed and quenched models have different critical inverse temperatures, they have the same critical exponents, see [5].

(iv) On the proof of Theorems 1.2 and 1.3, we largely use techniques and results in [9, 5]. In particular, to achieve the critical exponents, we exploit the representation of the annealed pressure  $\psi(\beta, B)$  in Theorem 1.1 and use Taylor expansion to study the partial derivatives of  $\psi$  when variables  $\beta, B$  tend to critical values. On the other hand, to prove Theorem 1.3, we show the convergence of the generating function of  $S_n/n^{3/4}$  as  $n$  tends to infinity, by using Laplace method as in [9]. Previously, the same strategy of proof has been applied by the authors in [5] to identify critical exponents and prove scaling limit theorems for the case of inhomogeneous random graphs.

Finally, the paper is organized as follows. In Section 2, we give a definition of random regular graphs and prove some useful preliminary results. Then, we prove Theorems 1.2 and 1.3 in Sections 3 and 4 respectively.

## 2. PRELIMINARIES

**2.1. Random regular graphs.** For each  $n$ , we start with a vertex set  $V_n$  of cardinality  $n$  and construct the edge set as follows. For each vertex  $v_i$ , start with  $d$  half-edges incident to  $v_i$ . Then we denote by  $\mathcal{H}$  the set of all the half-edges. Select one of them  $h_1$  arbitrarily and then choose a half-edge  $h_2$  uniformly from  $\mathcal{H} \setminus \{h_1\}$ , and match  $h_1$  and  $h_2$  to form an edge. Next, select arbitrarily another half-edge  $h_3$  from  $\mathcal{H} \setminus \{h_1, h_2\}$  and match it to another  $h_4$  uniformly chosen from  $\mathcal{H} \setminus \{h_1, h_2, h_3\}$ . Then continue this procedure until there are no more half-edges. We finally get a multiple random graph that may have self-loops and multiple edges between vertices satisfying all vertices have degree  $d$ . We denote the obtained graph by  $G_{n,d}$  and call it *random  $d$ -regular graph*.

**2.2. Preliminary results.** Following the notation in [9], we denote by  $G_{m,1}$  the random 1-regular graph with the vertex set  $\bar{V}_m = \{w_1, \dots, w_m\}$ . For any  $k \leq m$ ,  $X(k, m)$  is the number of edges between  $\bar{U}_k = \{w_1, \dots, w_k\}$  and  $\bar{U}_k^c = \bar{V}_m \setminus \bar{U}_k$  in  $G_{m,1}$ . Then for all  $0 \leq k \leq m$ , we define

$$g_\beta(k, m) = \mathbb{E} \left( e^{-2\beta X(k, m)} \right).$$

We have already proved in [9, Section 2] that

$$(1) \quad \mu_n(\sigma) = \frac{\mathbb{E} \left( e^{-H_n(\sigma)} \right)}{\mathbb{E}(Z_n(\beta, B))} = \frac{e^{\left(\frac{\beta d}{2} - B\right)n} g_\beta(d|\sigma_+|, dn) e^{2B|\sigma_+|}}{\mathbb{E}(Z_n(\beta, B))},$$

where

$$\sigma_+ = \{v_j : \sigma_j = 1\},$$

and

$$(2) \quad \mathbb{E}(Z_n(\beta, B)) = e^{\left(\frac{\beta d}{2} - B\right)n} \times \sum_{j=0}^n \binom{n}{j} e^{2Bj} g_\beta(dj, dn).$$

In [9], by deriving recursive formulas for the number of disagreeing edges ( $X(k, m)$ ), we obtain the following result on the asymptotic behavior of the sequence ( $g_\beta(dj, dn)$ ).

**Lemma 2.1.** [9, Lemma 3.1] *Suppose that  $\beta \geq 0$ . Then there exists a positive constant  $C$ , such that for all  $0 \leq i \leq j \leq n$ ,*

$$(3) \quad \left| \left[ \log g_\beta(dj, dn) - ndF_\beta \left( \frac{j}{n} \right) \right] - \left[ \log g_\beta(di, dn) - ndF_\beta \left( \frac{i}{n} \right) \right] \right| \leq \frac{C(j-i)}{n},$$

where  $F_\beta(t)$  is defined in Theorem 1.1.

In the following lemma, we summarize some properties of critical points of the function  $H_\beta(t) + 2Bt$ , which plays a key role in the formula of  $\psi(\beta, B)$ .

**Lemma 2.2.** *Let  $H_\beta(t)$  be the function defined in Theorem 1.1 (i). The following statements hold.*

- (i) *For  $\beta \geq 0$  and  $B > 0$ , the equation  $\partial_t H_\beta(t) + 2B = 0$  has a unique solution  $t_* = t_*(\beta, B) \in (\frac{1}{2}, 1)$ .*
- (ii) *For  $\beta > \beta_c$ , the equation  $\partial_t H_\beta(t) = 0$  has a unique solution  $t_+ = t(\beta) \in (\frac{1}{2}, 1)$ . Moreover, as  $B \searrow 0$ , we have  $t_* \rightarrow t_+$ .*

- (iii) As  $\beta \searrow \beta_c$ , we have  $t_+ \rightarrow \frac{1}{2}$ .  
 (iv) For  $\beta < \beta_c$ , as  $B \searrow 0$ , we have  $t_* \rightarrow \frac{1}{2}$ .

*Proof.* Part (i) is proved in Claim 1\* in [9, Section 4]. Parts (ii) and (iv) are Claims 2a and 2b in [9, Section 4]. We now prove (iii) by contradiction. Suppose that  $t_+(\beta)$  does not converges to  $\frac{1}{2}$  as  $\beta \searrow \beta_c$ . Then there exist  $\varepsilon > 0$  and a sequence  $(\beta_i) \searrow \beta_c$ , such that  $|t_+(\beta_i) - \frac{1}{2}| \geq \varepsilon$ . We observe that the sequence  $(t_+(\beta_i))$  is bounded in  $(\frac{1}{2}, 1)$ . Hence there exists a subsequence  $(\beta_{i_k}) \searrow \beta_c$ , such that the sequence  $(t_+(\beta_{i_k}))$  converges to a point  $x \in [\frac{1}{2}, 1]$ . By the assumption on the value of  $(t_+(\beta_i))$ , we have  $x \geq \frac{1}{2} + \varepsilon$ . Moreover,

$$\partial_t H_\beta(t) = \log\left(\frac{1-t}{t}\right) + d\partial_t F_\beta(t) = \begin{cases} \log\left(\frac{1-t}{t}\right) + \log f_\beta(t) & \text{if } t \in [0, \frac{1}{2}) \\ \log\left(\frac{1-t}{t}\right) - \log f_\beta(1-t) & \text{if } t \in (\frac{1}{2}, 1]. \end{cases}$$

Since  $\log f_\beta(\frac{1}{2}) = 0$ , we have  $\partial_t H_\beta(\frac{1}{2}^+) = \partial_t H_\beta(\frac{1}{2}^-)$ . Hence, the function  $H_\beta(\cdot)$  is differentiable at the point  $\frac{1}{2}$ . In addition, the function  $f_\beta(t)$  is jointly continuous at every point  $(t, \beta)$  with  $t \leq \frac{1}{2}$ . Hence, the function  $\partial_t H_\beta(t)$  is jointly continuous. Therefore,

$$0 = \lim_{k \rightarrow \infty} \partial_t H_{\beta_{i_k}}(t_+(\beta_{i_k})) = \partial_t H_{\beta_c}(x).$$

This leads to a contradiction, since by Lemma 2.3 below the equation  $\partial_t H_{\beta_c}(t) = 0$  has a unique solution  $t = \frac{1}{2}$ .  $\square$

The behavior of the function  $H_{\beta_c}(t)$  around the extreme point  $t = \frac{1}{2}$  is described in the following result, by using Taylor expansion.

**Lemma 2.3.** *Let us consider  $H(t) = H_{\beta_c}(t)$  with  $H_\beta(t)$  as in Theorem 1.1. Then we have*

$$(4) \quad \max_{0 \leq t \leq 1} H(t) = H(\frac{1}{2}).$$

Moreover,

$$(5) \quad H'(\frac{1}{2}) = H''(\frac{1}{2}) = H'''(\frac{1}{2}) = 0,$$

and

$$(6) \quad H^{(4)}(\frac{1}{2}) = \frac{-32(d-1)(d-2)}{d^2} < 0.$$

*Proof.* Using the same arguments for Claim 2b in [9, Section 4], we have  $H''(t) \leq 0$  is a consequence of the following

$$(7) \quad e^{-4\beta}[(d-2)^2(t-t^2) + d-1] \geq t(1-t)(d-2)^2.$$

Since  $\beta = \beta_c$ ,

$$(8) \quad c := e^{-2\beta} = \frac{d-2}{d}.$$

Hence (7) is equivalent to

$$(d-2)^2(t-t^2) + d-1 \geq d^2(t-t^2),$$

or equivalently,

$$1 \geq 4t(1-t)$$

which holds for all  $t \in [0, 1]$ . Hence the function  $H(t)$  is concave. Moreover, by a simple computation we have  $H'(\frac{1}{2}) = 0$ . Therefore  $H(t)$  gets the maximum at  $t = \frac{1}{2}$ . Now we prove (5) and (6). We observe that

$$(9) \quad H(t) = I(t) + dF(t),$$



where

$$I(t) = (t-1)\log(1-t) - t\log t,$$

and  $F(t) = F_{\beta_c}(t)$  is defined in Theorem 1.1. We have

$$\begin{aligned} I'(t) &= \log\left(\frac{1-t}{t}\right), & I''(t) &= \frac{-1}{t(1-t)}, \\ I'''(t) &= \frac{1}{t^2} - \frac{1}{(1-t)^2}, & I^{(4)}(t) &= \frac{2}{(t-1)^3} - \frac{2}{t^3}. \end{aligned}$$

Hence

$$(10) \quad I'(\tfrac{1}{2}) = I'''(\tfrac{1}{2}) = 0, \quad I''(\tfrac{1}{2}) = -4, \quad I^{(4)}(\tfrac{1}{2}) = -32.$$

On the other hand,

$$F'(t) = \begin{cases} \log f(t) & \text{if } t \in [0, \frac{1}{2}) \\ -\log f(1-t) & \text{if } t \in (\frac{1}{2}, 1], \end{cases}$$

with

$$f(t) = f_{\beta_c}(t).$$

In addition,  $f(\frac{1}{2}) = 1$ , so  $F'(\frac{1}{2}^+) = F'(\frac{1}{2}^-) = 0$ . Hence  $F'(\frac{1}{2}) = 0$  and  $F$  is a  $C^1$  function on  $(0, 1)$ . Furthermore,

$$F''(t) = \begin{cases} \frac{f'(t)}{f(t)} & \text{if } t \in [0, \frac{1}{2}) \\ \frac{f'(1-t)}{f(1-t)} & \text{if } t \in (\frac{1}{2}, 1]. \end{cases}$$

Therefore,  $F''(\frac{1}{2}^+) = F''(\frac{1}{2}^-)$ . Hence,  $F''(\frac{1}{2})$  exists and  $F$  is a  $C^2$  function on  $(0, 1)$ . Similarly,

$$F'''(t) = \begin{cases} \frac{f''(t)f(t) - (f'(t))^2}{f^2(t)} & \text{if } t \in [0, \frac{1}{2}) \\ \frac{(f'(1-t))^2 - f''(1-t)f(1-t)}{f^2(1-t)} & \text{if } t \in (\frac{1}{2}, 1]. \end{cases}$$

Thus  $F'''(\frac{1}{2}^+) = F'''(\frac{1}{2}^-) = 0$ , so  $F'''(\frac{1}{2}) = 0$  and  $F$  is a  $C^3$  function on  $(0, 1)$ . Moreover,

$$F^{(4)}(t) = \begin{cases} \frac{\frac{f'''(t)f(t) - f'(t)f''(t)}{f^2(t)} - \frac{2f'(t)[f''(t)f(t) - (f'(t))^2]}{f^3(t)}}{f^2(t)} & \text{if } t \in [0, \frac{1}{2}) \\ \frac{\frac{f'''(1-t)f(1-t) - f'(1-t)f''(1-t)}{f^2(1-t)} - \frac{2f'(1-t)[f''(1-t)f(1-t) - (f'(1-t))^2]}{f^3(1-t)}}{f^2(1-t)} & \text{if } t \in (\frac{1}{2}, 1]. \end{cases}$$

Hence,  $F^{(4)}(\frac{1}{2}^+) = F^{(4)}(\frac{1}{2}^-)$ , so  $F^{(4)}(\frac{1}{2})$  exists and  $F$  is a  $C^4$  function on  $(0, 1)$ . We now compute the values  $F''(\frac{1}{2})$  and  $F^{(4)}(\frac{1}{2})$ . Observe that

$$f(t) = \frac{A(t)}{B(t)},$$

where

$$A(t) = c(1-2t) + \sqrt{1 + (c^2 - 1)(2t-1)^2} \quad \text{and} \quad B(t) = 2(1-t),$$

with  $c$  as in (8). Hence

$$\begin{aligned} f'(t) &= \frac{A'(t)}{B(t)} - \frac{A(t)B'(t)}{B^2(t)}, \\ f''(t) &= \frac{A''(t)}{B(t)} - \frac{A(t)B''(t) + 2A'(t)B'(t)}{B^2(t)} + \frac{2A(t)(B'(t))^2}{B^3(t)}, \\ f'''(t) &= \frac{A'''(t)}{B(t)} - \frac{A(t)B'''(t) + 3A'(t)B''(t) + 3A''(t)B'(t)}{B^2(t)} \\ &\quad + \frac{6A(t)B'(t)B''(t) + 6A'(t)(B'(t))^2}{B^3(t)} - \frac{6A(t)(B'(t))^3}{B^4(t)}. \end{aligned}$$

After some computations, we get

$$A(\tfrac{1}{2}) = 1, \quad A'(\tfrac{1}{2}) = -2c, \quad A''(\tfrac{1}{2}) = 4(c^2 - 1), \quad A'''(\tfrac{1}{2}) = 0,$$

and

$$B(\tfrac{1}{2}) = 1, \quad B'(\tfrac{1}{2}) = -2, \quad B''(\tfrac{1}{2}) = B'''(\tfrac{1}{2}) = 0.$$

Thus

$$f(\tfrac{1}{2}) = 1, \quad f'(\tfrac{1}{2}) = 2(1 - c), \quad f''(\tfrac{1}{2}) = 4(1 - c)^2, \quad f'''(\tfrac{1}{2}) = 24(1 - c)^2.$$

Therefore

$$(11) \quad F'(\tfrac{1}{2}) = F'''(\tfrac{1}{2}) = 0, \quad F''(\tfrac{1}{2}) = 2(1 - c), \quad F^{(4)}(\tfrac{1}{2}) = 24(1 - c)^2 - 8(1 - c)^3.$$

Combining (8), (9), (10) and (11), we obtain desired results.  $\square$

### 3. PROOF OF THEOREM 1.2

We have proved in [9, Section 4, Claim 1\*] that for all  $\beta \geq 0$  and  $B > 0$ ,

$$\psi(\beta, B) = \beta d/2 - B + L(t_*, \beta, B),$$

where

$$L(t, \beta, B) = H_\beta(t) + 2Bt,$$

and  $t_* = t_*(\beta, B) \in (\frac{1}{2}, 1)$  is the unique zero of the function  $\partial_t L(t, \beta, B)$ , i.e.

$$(12) \quad \partial_t L(t_*, \beta, B) = \partial_t H_\beta(t_*) + 2B = 0.$$

**3.1. Proof of  $\delta = 3$ .** We have shown in [9, Section 4] that for all  $\beta \geq 0$  and  $B > 0$ ,

$$\mathcal{M}(\beta, B) = \frac{\partial}{\partial B} \psi(\beta, B) = -1 + 2t_*,$$

where  $t_*$  is the solution of (12). By Lemma 2.2 (i) and (iii) we have  $t_* \searrow \frac{1}{2}$  as  $B \searrow 0$ . We set

$$s_* = t_* - \frac{1}{2}.$$

Hence

$$s_* \searrow 0 \quad \text{as} \quad B \searrow 0.$$

We notice also that for  $t > \frac{1}{2}$ ,

$$(13) \quad \partial_t H_\beta(t) = \log\left(\frac{1-t}{t}\right) - d \log f_\beta(1-t),$$

with

$$f_\beta(1-t) = \frac{e^{-2\beta}(2t-1) + \sqrt{1 + (e^{-4\beta} - 1)(2t-1)^2}}{2t}.$$

Therefore the equation (12) is equivalent to the following

$$(14) \quad 2B = -\log \theta_1(s_*) + d \log \theta_2(s_*),$$

where

$$(15) \quad \theta_1(s) = \frac{1-2s}{1+2s},$$

and

$$(16) \quad \theta_2(s) = f_\beta(1-s) = \frac{2e^{-2\beta}s + \sqrt{1+4(e^{-4\beta}-1)s^2}}{1+2s}.$$

We have

$$\theta_1(s_*) - 1 = \frac{-4s_*}{1+2s_*},$$

and

$$\theta_2(s_*) - 1 = \frac{2s_*(e^{-2\beta}-1) + \sqrt{1+4(e^{-4\beta}-1)s_*^2} - 1}{1+2s_*}.$$

Using Taylor expansion, we get

$$\frac{1}{1+2s_*} = 1 - 2s_* + 4s_*^2 - 8s_*^3 + \mathcal{O}(s_*^4)$$

and

$$\sqrt{1+4(e^{-4\beta}-1)s_*^2} - 1 = 2(e^{-4\beta}-1)s_*^2 + \mathcal{O}(s_*^4).$$

Therefore

$$\begin{aligned} \theta_1(s_*) - 1 &= -4s_* + 8s_*^2 - 16s_*^3 + \mathcal{O}(s_*^4), \\ (\theta_1(s_*) - 1)^2 &= 16s_*^2 - 64s_*^3 + \mathcal{O}(s_*^4), \\ (\theta_1(s_*) - 1)^3 &= -64s_*^3 + \mathcal{O}(s_*^4), \end{aligned}$$

and

$$\begin{aligned} \theta_2(s_*) - 1 &= 2(e^{-2\beta}-1)s_* + 2(e^{-2\beta}-1)^2s_*^2 - 4(e^{-2\beta}-1)^2s_*^3 + \mathcal{O}(s_*^4), \\ (\theta_2(s_*) - 1)^2 &= 4(e^{-2\beta}-1)^2s_*^2 + 8(e^{-2\beta}-1)^3s_*^3 + \mathcal{O}(s_*^4), \\ (\theta_2(s_*) - 1)^3 &= 8(e^{-2\beta}-1)^3s_*^3 + \mathcal{O}(s_*^4). \end{aligned}$$

Combining these equations and Taylor expansion, we have

$$\begin{aligned} \log \theta_1(s_*) &= \theta_1(s_*) - 1 + \frac{-(\theta_1(s_*) - 1)^2}{2} + \frac{(\theta_1(s_*) - 1)^3}{3} + \mathcal{O}((\theta_1(s_*) - 1)^4) \\ (17) \quad &= -4s_* - \frac{16}{3}s_*^3 + \mathcal{O}(s_*^4), \end{aligned}$$

and

$$\begin{aligned} \log \theta_2(s_*) &= \theta_2(s_*) - 1 + \frac{-(\theta_2(s_*) - 1)^2}{2} + \frac{(\theta_2(s_*) - 1)^3}{3} + \mathcal{O}(s_*^4) \\ (18) \quad &= 2(e^{-2\beta}-1)s_* - \frac{4}{3}(e^{-2\beta}-1)^2(e^{-2\beta}+2)s_*^3 + \mathcal{O}(s_*^4). \end{aligned}$$

In this subsection, we consider

$$\beta = \beta_c = \frac{1}{2} \log \left( \frac{d}{d-2} \right) \quad \text{or} \quad e^{-2\beta} = \frac{d-2}{d}.$$

Hence

$$\log \theta_2(s_*) = -\frac{4}{d}s_* + \left( \frac{32}{3d^3} - \frac{16}{d^2} \right) s_*^3 + \mathcal{O}(s_*^4).$$

Therefore,

$$-\log \theta_1(s_*) + d \log \theta_2(s_*) = \frac{16(d-1)(d-2)}{3d^2} s_*^3 + \mathcal{O}(s_*^4).$$

Combining this with (14), we get

$$B = \frac{8(d-1)(d-2)}{3d^2} s_*^3 + \mathcal{O}(s_*^4).$$

Thus as  $B \searrow 0$ ,

$$\mathcal{M}(\beta, B) = 2s_* \asymp B^{1/3}.$$

Therefore

$$\delta = 3.$$

**3.2. Proof of  $\beta = \frac{1}{2}$ .** Suppose that  $\beta > \beta_c$ . We have proved in [9, Claim 2a] that

$$\mathcal{M}(\beta, 0^+) = -1 + 2t_+,$$

where  $t_+ = t_+(\beta) \in (\frac{1}{2}, 1)$  is the root of  $\partial_t H_\beta(t)$ . Moreover, by Lemma 2.2 (ii) and (iii) we have  $t_+ \searrow \frac{1}{2}$  as  $\beta \searrow \beta_c$ . We set

$$s_+ = t_+ - \frac{1}{2}.$$

Then

$$(19) \quad \mathcal{M}(\beta, 0^+) = 2s_+,$$

and  $s_+$  is the positive solution of the equation  $\partial_t H_\beta(s_+ + \frac{1}{2}) = 0$ . From (13), we can write this equation

$$(20) \quad \log \theta_1(s_+) = d \log \theta_2(s_+),$$

with  $\theta_1(s)$  and  $\theta_2(s)$  as in (15) and (16). Using similar arguments and calculations for (17) and (18), we get

$$\log \theta_1(s_+) = -4s_+ - \frac{16}{3}s_+^3 + \mathcal{O}(s_+^4),$$

and

$$\log \theta_2(s_+) = 2(e^{-2\beta} - 1)s_+ - \frac{4}{3}(e^{-2\beta} - 1)^2(e^{-2\beta} + 2)s_+^3 + \mathcal{O}(s_+^4).$$

Hence, for any  $\varepsilon > 0$ , there exists  $\delta > 0$ , such that for all  $\beta_c < \beta < \beta_c + \delta$ ,

$$(21) \quad -4s_+ - \left( \frac{16}{3} + \varepsilon \right) s_+^3 \leq \log \theta_1(s_+) \leq -4s_+ - \left( \frac{16}{3} - \varepsilon \right) s_+^3,$$

and

$$(22) \quad \log \theta_2(s_+) \leq 2(e^{-2\beta} - 1)s_+ + \left( \varepsilon - \frac{4}{3}(e^{-2\beta} - 1)^2(e^{-2\beta} + 2) \right) s_+^3,$$

and

$$(23) \quad \log \theta_2(s_+) \geq 2(e^{-2\beta} - 1)s_+ - \left( \varepsilon + \frac{4}{3}(e^{-2\beta} - 1)^2(e^{-2\beta} + 2) \right) s_+^3.$$

Using (20), (21) and (22), we get

$$-4s_+ - \left(\frac{16}{3} + \varepsilon\right) s_+^3 \leq 2d(e^{-2\beta} - 1)s_+ + d \left(\varepsilon - \frac{4}{3}(e^{-2\beta} - 1)^2(e^{-2\beta} + 2)\right) s_+^3.$$

Therefore

$$-4 - 2d(e^{-2\beta} - 1) \leq \left(\frac{16}{3} - \frac{4d}{3}(e^{-2\beta} - 1)^2(e^{-2\beta} + 2) + (d+1)\varepsilon\right) s_+^2.$$

We observe that as  $\beta \searrow \beta_c$ ,

$$\frac{16}{3} - \frac{4d}{3}(e^{-2\beta} - 1)^2(e^{-2\beta} + 2) \rightarrow \frac{16}{3} - \frac{16}{3} \frac{(3d-2)}{d^2} = \frac{16(d-1)(d-2)}{3d^2}.$$

Moreover,

$$\begin{aligned} \frac{d-2}{d} - e^{-2\beta} &= e^{-2\beta_c} - e^{-2\beta} = 2e^{-2\beta_c}(\beta - \beta_c) + \mathcal{O}((\beta - \beta_c)^2) \\ (24) \qquad \qquad \qquad &= 2 \left(\frac{d-2}{d}\right) (\beta - \beta_c) + \mathcal{O}((\beta - \beta_c)^2). \end{aligned}$$

Thus

$$-4 - 2d(e^{-2\beta} - 1) = 2d \left(\frac{d-2}{d} - e^{-2\beta}\right) = 4(d-2)(\beta - \beta_c) + o((\beta - \beta_c))$$

Hence for  $\beta$  close enough to  $\beta_c$ ,

$$(4(d-2) - \varepsilon)(\beta - \beta_c) \leq \left(\frac{16(d-1)(d-2)}{3d^2} + (d+2)\varepsilon\right) s_+^2.$$

Similarly, using (20), (21) and (23) we can also prove that for  $\beta$  close to  $\beta_c$ ,

$$(4(d-2) + \varepsilon)(\beta - \beta_c) \geq \left(\frac{16(d-1)(d-2)}{3d^2} - (d+2)\varepsilon\right) s_+^2.$$

It follows from last two inequalities that

$$(25) \qquad \qquad \qquad s_+^2 = \frac{3d^2}{4(d-1)}(\beta - \beta_c) + o((\beta - \beta_c)).$$

Combining (19) and (25), we get

$$\beta = \frac{1}{2}.$$

**3.3. Proof of  $\gamma = \gamma' = 1$ .** We have shown in [9, Section 5] that for  $B > 0$

$$(26) \qquad \qquad \qquad \chi(\beta, B) = \frac{-4}{\partial_{tt}H_\beta(t_*)}.$$

In the proof of Claim 1\* in [9, Section 4], it is shown that for all  $t \in (0, 1)$

$$(27) \qquad \qquad \qquad \partial_{tt}H_\beta(t) = \frac{-P(t)}{Q(t)},$$

where

$$P(t) = d(1-t)(e^{-2\beta}\theta_2(t) - 1) + e^{-2\beta}(2t-1)\theta_2(t) + 2 - 2t,$$

and

$$Q(t) = t(1-t)(e^{-2\beta}(2t-1)\theta_2(t) + 2 - 2t),$$

with

$$\theta_2(t) = f_\beta(1-t).$$

**Case  $\beta > \beta_c$ .** By Lemma 2.2 (ii), we have  $t_* \rightarrow t_+$  as  $B \rightarrow 0^+$ , with  $t_+ = t_+(\beta)$  the root of  $\partial_t H_\beta$ . Therefore, by (26)

$$(28) \quad \chi(\beta, 0^+) = \frac{-4}{\partial_{tt} H_\beta(t_+)}.$$

Using (13), the equation  $\partial_t H_\beta(t_+) = 0$  is equivalent to

$$d \log f_\beta(1 - t_+) = \log \left( \frac{1 - t_+}{t_+} \right),$$

or equivalently

$$\theta_2(t_+) = \left( \frac{1 - t_+}{t_+} \right)^{1/d} = \left( \frac{1 - 2s_+}{1 + 2s_+} \right)^{1/d},$$

where

$$s_+ = t_+ - \frac{1}{2}.$$

Notice that as  $\beta \searrow \beta_c$ , we have  $t_+ \searrow \frac{1}{2}$ , thus  $s_+ \searrow 0$ . By Taylor expansion,

$$\theta_2(t_+) = \left( \frac{1 - 2s_+}{1 + 2s_+} \right)^{1/d} = 1 - \frac{4s_+}{d} + \frac{16s_+^2}{d^2} + \mathcal{O}(s_+^3).$$

Hence

$$(29) \quad \begin{aligned} Q(t_+) &= \left( \frac{1}{4} - s_+^2 \right) (2e^{-2\beta} \theta_2(t_+) s_+ + 1 - 2s_+) = \frac{1}{4} + \mathcal{O}(s_+) \\ &\quad \text{(by (25))} = \frac{1}{4} + \mathcal{O}(\sqrt{\beta - \beta_c}). \end{aligned}$$

Similarly,

$$\begin{aligned} P(t_+) &= d \left( \frac{1}{2} - s_+ \right) (e^{-2\beta} \theta_2(t_+) - 1) + 2e^{-2\beta} \theta_2(t_+) s_+ + 1 - 2s_+ \\ &= d \left( \frac{1}{2} - s_+ \right) \left( e^{-2\beta} \left( 1 - \frac{4s_+}{d} + \frac{16s_+^2}{d^2} \right) - 1 + \mathcal{O}(s_+^3) \right) + 1 - 2s_+ \\ &\quad + 2e^{-2\beta} \left( 1 - \frac{4s_+}{d} \right) s_+ + \mathcal{O}(s_+^3) \\ &= \frac{d}{2} \left( e^{-2\beta} - \frac{d-2}{d} \right) + d \left( \frac{d-2}{d} - e^{-2\beta} \right) s_+ + 4e^{-2\beta} s_+^2 + \mathcal{O}(s_+^3). \end{aligned}$$

Combining this with (24) and (25), we get

$$(30) \quad P(t_+) = \frac{(d-2)(2d+1)}{(d-1)} (\beta - \beta_c) + o((\beta - \beta_c)).$$

It follows from (27), (29), (30) that

$$(31) \quad \partial_{tt} H_\beta(t_+) = \frac{-4(d-2)(2d+1)}{(d-1)} (\beta - \beta_c) + o((\beta - \beta_c)).$$

This together with (28) imply that as  $\beta \searrow \beta_c$ ,

$$\chi(\beta, 0^+) \asymp (\beta - \beta_c)^{-1},$$

or equivalently

$$\gamma' = 1.$$

**Case  $\beta < \beta_c$ .** By Lemma 2.2 (iv), we have  $t_* \rightarrow \frac{1}{2}$  as  $B \rightarrow 0^+$ . Therefore, by (26)

$$(32) \quad \chi(\beta, 0^+) = \frac{-4}{\partial_{tt}H_\beta(\frac{1}{2})}.$$

We have  $\theta_2(\frac{1}{2}) = f_\beta(\frac{1}{2}) = 1$ . Hence

$$Q(\frac{1}{2}) = \frac{1}{4},$$

and

$$P(\frac{1}{2}) = \frac{d}{2}(e^{-2\beta} - 1) + 1 = \frac{d}{2} \left( e^{-2\beta} - \frac{d-2}{d} \right).$$

Thus

$$(33) \quad \partial_{tt}H_\beta(\frac{1}{2}) = -2d \left( e^{-2\beta} - \frac{d-2}{d} \right).$$

Using (24) and (33), we have as  $\beta \nearrow \beta_c$ ,

$$\partial_{tt}H_\beta(\frac{1}{2}) = -4(d-2)(\beta_c - \beta) + \mathcal{O}((\beta_c - \beta)^2),$$

so we get

$$\chi(\beta, 0^+) \asymp (\beta_c - \beta)^{-1},$$

and thus

$$\gamma = 1.$$

**3.4. Proof of  $\alpha = \alpha' = 0$ .** We recall that

$$(34) \quad \psi(\beta, B) = \beta d/2 - B + L(t_*, \beta, B),$$

where  $t_* = t_*(\beta, B) \in (\frac{1}{2}, 1)$  is the solution of the following equation

$$(35) \quad \partial_t L(t_*, \beta, B) = 0.$$

In addition, by Claim 1\* in [9],  $\partial_{tt}L(t_*, \beta, B) \neq 0$ . Hence,  $t_*$  is a differentiable function by the implicit function theorem. Taking derivative in  $\beta$  of (35), we get

$$\partial_{t\beta}L(t_*, \beta, B) + \partial_{tt}L(t_*, \beta, B)\partial_\beta t_* = 0.$$

Thus

$$\partial_\beta t_* = -\frac{\partial_{t\beta}L(t_*, \beta, B)}{\partial_{tt}L(t_*, \beta, B)}.$$

Using (34) and (35), we get

$$\partial_\beta \psi(\beta, B) = \frac{d}{2} + \partial_t L(t_*, \beta, B)\partial_\beta t_* + \partial_\beta L(t_*, \beta, B) = \frac{d}{2} + \partial_\beta L(t_*, \beta, B).$$

It follows from the last two equations that

$$(36) \quad \begin{aligned} \partial_{\beta\beta}\psi(\beta, B) &= \partial_{\beta\beta}L(t_*, \beta, B) + \partial_{t\beta}L(t_*, \beta, B)\partial_\beta t_* \\ &= \partial_{\beta\beta}L(t_*, \beta, B) - \frac{(\partial_{t\beta}L(t_*, \beta, B))^2}{\partial_{tt}L(t_*, \beta, B)}. \end{aligned}$$

For  $t > \frac{1}{2}$ , we have

$$(37) \quad \partial_{t\beta}L(t, \beta, B) = d\partial_{t\beta}F_\beta(t) = -d\partial_\beta \log f_\beta(1-t) = -dp_\beta(1-t)$$

and

$$(38) \quad \partial_{\beta\beta}L(t, \beta, B) = d\partial_{\beta\beta}F_\beta(t) = d \int_0^{1-t} \partial_{\beta\beta}p_\beta(s)ds,$$

where

$$p_\beta(s) = \frac{\partial_\beta f_\beta(s)}{f_\beta(s)}.$$

**Case**  $\beta > \beta_c$ . By Lemma 2.2 (ii),  $t_* \rightarrow t_+$  as  $B \rightarrow 0^+$ . Hence using (36), we have

$$(39) \quad \partial_{\beta\beta}\psi(\beta, 0^+) = \partial_{\beta\beta}L(t_+, \beta, 0) - \frac{(\partial_{t\beta}L(t_+, \beta, 0))^2}{\partial_{tt}L(t_+, \beta, 0)}.$$

By (31), as  $\beta \searrow \beta_c$

$$(40) \quad \partial_{tt}L(t_+, \beta, 0) = \partial_{tt}H_\beta(t_+) = -\frac{4(d-2)(2d+1)}{d-1}(\beta - \beta_c) + o(\beta - \beta_c).$$

By direct calculations, we can show that

$$(41) \quad p_\beta(s) = \frac{\partial_\beta f_\beta(s)}{f_\beta(s)} = \frac{-2e^{-2\beta}(1-2s)}{\sqrt{1+(e^{-4\beta}-1)(1-2s)^2}}.$$

Using (25) and (24), we get as  $\beta \searrow \beta_c$ ,

$$(42) \quad e^{-2\beta} = \frac{d-2}{d} + \mathcal{O}(\beta - \beta_c),$$

and

$$(43) \quad 2t_+ - 1 = \sqrt{\frac{3d^2}{d-1}}\sqrt{\beta - \beta_c} + o\left(\sqrt{\beta - \beta_c}\right).$$

Using (37), (41), (42), (43) we have as  $\beta \searrow \beta_c$ ,

$$\partial_{t\beta}L(t_+, \beta, 0) = -dp_\beta(1-t_+) = -\frac{2\sqrt{3}d(d-2)}{\sqrt{d-1}}\sqrt{\beta - \beta_c} + o\left(\sqrt{\beta - \beta_c}\right).$$

Hence

$$(44) \quad (\partial_{t\beta}L(t_+, \beta, 0))^2 = \frac{12d^2(d-2)^2}{d-1}(\beta - \beta_c) + o(\beta - \beta_c).$$

We have

$$(45) \quad \partial_\beta p_\beta(s) = \frac{4e^{-2\beta}(1-2s)(1-(1-2s)^2)}{\left(\sqrt{1+(e^{-4\beta}-1)(1-2s)^2}\right)^3}.$$

Therefore using (38), we obtain

$$\begin{aligned} \partial_{\beta\beta}L(t_+, \beta, 0) &= 4d \int_0^{1-t_+} \frac{e^{-2\beta}(1-2s)(1-(1-2s)^2)}{\left(\sqrt{1+(e^{-4\beta}-1)(1-2s)^2}\right)^3} ds \\ &= 2d \int_{2t_+-1}^1 \frac{e^{-2\beta}u(1-u^2)}{\left(\sqrt{1+(e^{-4\beta}-1)u^2}\right)^3} du \\ &= 2d \int_0^1 \frac{e^{-2\beta}u(1-u^2)}{\left(\sqrt{1+(e^{-4\beta}-1)u^2}\right)^3} du - 2d \int_0^{2t_+-1} \frac{e^{-2\beta}u(1-u^2)}{\left(\sqrt{1+(e^{-4\beta}-1)u^2}\right)^3} du \\ (46) \quad &= J_1 - J_2. \end{aligned}$$



We observe that

$$0 \leq \frac{e^{-2\beta}u(1-u^2)}{\left(\sqrt{1+(e^{-4\beta}-1)u^2}\right)^3} \leq e^{4\beta}.$$

Hence

$$0 \leq J_2 \leq 2de^{4\beta}(2t_+ - 1).$$

Combining this inequality with (43) yields that as  $\beta \searrow \beta_c$

$$J_2 = \mathcal{O}(\sqrt{\beta - \beta_c}).$$

On the other hand, by using (42) we have

$$J_1 = \int_0^1 \frac{2d^3(d-2)u(1-u^2)}{\left(\sqrt{d^2+(4-4d)u^2}\right)^3} du + \mathcal{O}(\beta - \beta_c).$$

Combining the last two equations and (46) gives that

$$(47) \quad \partial_{\beta\beta}L(t_+, \beta, 0) = \int_0^1 \frac{2d^3(d-2)u(1-u^2)}{\left(\sqrt{d^2+(4-4d)u^2}\right)^3} du + \mathcal{O}(\beta - \beta_c).$$

It follows from (39), (40), (44) and (47) that as  $\beta \searrow \beta_c$

$$\begin{aligned} \mathcal{C}(\beta, 0^+) = \partial_{\beta\beta}\psi(\beta, 0^+) &= \int_0^1 \frac{2d^3(d-2)u(1-u^2)}{\left(\sqrt{d^2+(4-4d)u^2}\right)^3} du + \frac{\frac{12d^2(d-2)^2}{d-1}(\beta - \beta_c) + o(\beta - \beta_c)}{\frac{4(d-2)(2d+1)}{d-1}(\beta - \beta_c) + o(\beta - \beta_c)} \\ &= \int_0^1 \frac{2d^3(d-2)u(1-u^2)}{\left(\sqrt{d^2+(4-4d)u^2}\right)^3} du + \frac{3d^2(d-2)}{2d+1} + o(1). \end{aligned}$$

Thus

$$\boldsymbol{\alpha}' = 0.$$

**Case  $\beta < \beta_c$ .** By Lemma 2.2 (iv), we have  $t_* \rightarrow \frac{1}{2}$  as  $B \rightarrow 0^+$ . Therefore, by (36)

$$\partial_{\beta\beta}\psi(\beta, 0^+) = \partial_{\beta\beta}L\left(\frac{1}{2}, \beta, 0\right) - \frac{(\partial_{t\beta}L\left(\frac{1}{2}, \beta, 0\right))^2}{\partial_{tt}L\left(\frac{1}{2}, \beta, 0\right)}.$$

Using (33), we obtain

$$\partial_{tt}L\left(\frac{1}{2}, \beta, 0\right) = \partial_{tt}H_\beta\left(\frac{1}{2}\right) = -2d \left( e^{-2\beta} - \frac{d-2}{d} \right).$$

Moreover, by (37) and (41)

$$\partial_{t\beta}L\left(\frac{1}{2}, \beta, 0\right) = -dp_\beta\left(\frac{1}{2}\right) = 0.$$

We have

$$\begin{aligned} \partial_{\beta\beta}L\left(\frac{1}{2}, \beta, 0\right) &= 4d \int_0^{1/2} \frac{e^{-2\beta}(1-2s)(1-(1-2s)^2)}{\left(\sqrt{1+(e^{-4\beta}-1)(1-2s)^2}\right)^3} ds \\ &= 2d \int_0^1 \frac{e^{-2\beta}u(1-u^2)}{\left(\sqrt{1+(e^{-4\beta}-1)u^2}\right)^3} du. \end{aligned}$$

Combining the last four equations yields that

$$\partial_{\beta\beta}\psi(\beta, 0^+) = 2d \int_0^1 \frac{e^{-2\beta}u(1-u^2)}{\left(\sqrt{1+(e^{-4\beta}-1)u^2}\right)^3} du.$$

Now, by using (42), we get as  $\beta \nearrow \beta_c$ ,

$$\mathcal{C}(\beta, 0^+) = \partial_{\beta\beta}\psi(\beta, 0^+) = \int_0^1 \frac{2d^3(d-2)u(1-u^2)}{\left(\sqrt{d^2+(4-4d)u^2}\right)^3} du + o(1).$$

Hence

$$\alpha = 0.$$

#### 4. PROOF OF THEOREM 1.3

In this section, we use the same strategy as in the proof of [9, Theorem 1.3] to prove our result. In particular, we show that the generating function of  $(\sigma_1 + \dots + \sigma_n)/n^{3/4}$  converges to the one of a specific random variable. In fact, Theorem 1.3 is a direct consequence of the following proposition.

**Proposition 4.1.** *Suppose that  $\beta = \beta_c$  and  $B = 0$ . Then for all  $r \in \mathbb{R}$ , we have*

$$(48) \quad \mathbb{E}_{\mu_n} \left( \exp \left( r \frac{\sigma_1 + \dots + \sigma_n}{n^{3/4}} \right) \right) \longrightarrow \mathbb{E} (e^{rX}),$$

where  $X$  is the random variable defined in Theorem 1.3.

*Proof.* Using (1) and (2), we have

$$(49) \quad \begin{aligned} \mathbb{E}_{\mu_n} \left( \exp \left( r \frac{\sigma_1 + \dots + \sigma_n}{n^{3/4}} \right) \right) &= \sum_{\sigma \in \Omega_n} \exp \left( \frac{r \sum \sigma_j}{n^{3/4}} \right) \mu_n(\sigma) \\ &= \sum_{\sigma \in \Omega_n} \exp \left( \frac{r \sum \sigma_j}{n^{3/4}} \right) \frac{g(d|\sigma_+|, dn)}{B_n}, \end{aligned}$$

where

$$\sigma_+ = \{v_j : \sigma_j = 1\},$$

and

$$(50) \quad B_n = \sum_{j=0}^n \binom{n}{j} g(dj, dn),$$

and

$$g(dj, dn) = g_{\beta_c}(dj, dn)$$

with the sequence  $(g_{\beta_c}(k, m))$  as in Lemma 2.1. Therefore,

$$(51) \quad \mathbb{E}_{\mu_n} \left( \exp \left( r \frac{\sigma_1 + \dots + \sigma_n}{n^{3/4}} \right) \right) = \frac{A_n}{B_n},$$

with

$$(52) \quad \begin{aligned} A_n &= \sum_{\sigma \in \Omega_n} \exp \left( \frac{r \sum \sigma_j}{n^{3/4}} \right) g(d|\sigma_+|, dn) \\ &= \sum_{j=0}^n \binom{n}{j} \exp \left( \frac{r(2j-n)}{n^{3/4}} \right) g(dj, dn). \end{aligned}$$

We set

$$j_* = [n/2],$$

where  $[x]$  stands for the integer part of  $x$ . Define for  $0 \leq j \leq n$ ,

$$(53) \quad x_j(n) = \binom{n}{j} g(dj, dn).$$

Using the same arguments as in [9, Section 5], we prove in Appendix that

$$(54) \quad \frac{x_j(n)}{x_{j_*}(n)} = (1 + o(1)) \sqrt{\frac{j_*(n - j_*)}{j(n - j)}} \exp \left( n [H(j/n) - H(j_*/n)] \right. \\ \left. + [\log g(dj, dn) - ndF(j/n)] - [\log g(dj_*, dn) - ndF(j_*/n)] \right),$$

and

$$(55) \quad \frac{A_n}{B_n} = \frac{\hat{A}_n}{\hat{B}_n} + o(1/n^2),$$

where

$$\hat{A}_n = \sum_{|j - j_*| \leq n^{5/6}} x_j(n) \\ \hat{B}_n = \sum_{|j - j_*| \leq n^{5/6}} \exp \left( \frac{r(2j - n)}{n^{3/4}} \right) x_j(n).$$

Observe that when  $|j - j_*| \leq n^{5/6}$ ,

$$(56) \quad \sqrt{\frac{j_*(n - j_*)}{j(n - j)}} = 1 + \mathcal{O}(|j - j_*|/n) = 1 + \mathcal{O}(n^{-1/6}).$$

Lemma 2.1 implies that for all  $j$ ,

$$(57) \quad \left| [\log g(dj, dn) - ndF(j/n)] - [\log g(dj_*, dn) - ndF(j_*/n)] \right| = \mathcal{O}(|j - j_*|/n).$$

Using Taylor expansion and Lemma 2.3, we have

$$H'(j_*/n) = H'(\frac{1}{2}) + H''(\frac{1}{2}) \left( \frac{j_*}{n} - \frac{1}{2} \right) + H'''(\frac{1}{2}) \left( \frac{j_*}{n} - \frac{1}{2} \right)^2 \\ + \mathcal{O} \left( \left( \frac{j_*}{n} - \frac{1}{2} \right)^3 \right) = \mathcal{O}(n^{-3}).$$

Similarly,

$$H''(j_*/n) = \mathcal{O}(n^{-2}), \quad H'''(j_*/n) = \mathcal{O}(n^{-1}), \quad H^{(4)}(j_*/n) = H^{(4)}(\frac{1}{2}) + \mathcal{O}(n^{-1}).$$

Hence for all  $|j - j_*| \leq n^{5/6}$ ,

$$\begin{aligned}
 H(j/n) - H(j_*/n) &= H' \left( \frac{j_*}{n} \right) \left( \frac{j - j_*}{n} \right) + H'' \left( \frac{j_*}{n} \right) \frac{(j - j_*)^2}{2n^2} + H''' \left( \frac{j_*}{n} \right) \frac{(j - j_*)^3}{6n^3} \\
 &\quad + H^{(4)} \left( \frac{j_*}{n} \right) \frac{(j - j_*)^4}{24n^4} + \mathcal{O} \left( \left( \frac{j - j_*}{n} \right)^5 \right) \\
 &= \mathcal{O} \left( n^{-(3+1/6)} \right) + \mathcal{O} \left( n^{-(2+1/3)} \right) + \mathcal{O} \left( n^{-(1+1/2)} \right) \\
 &\quad + \left( H^{(4)} \left( \frac{1}{2} \right) + \mathcal{O}(n^{-1}) \right) \frac{(j - j_*)^4}{24n^4} + \mathcal{O}(n^{-1/6}) \frac{(j - j_*)^4}{24n^4} \\
 (58) \quad &= \left( 1 + \mathcal{O}(n^{-1/6}) \right) \frac{H^{(4)} \left( \frac{1}{2} \right) (j - j_*)^4}{24 n^4} + \mathcal{O}(n^{-3/2}).
 \end{aligned}$$

We observe that by Lemma 2.3

$$\alpha_* := \frac{H^{(4)} \left( \frac{1}{2} \right)}{24} = -\frac{4(d-1)(d-2)}{3d^2} < 0.$$

Let  $\varepsilon > 0$  be any given positive real number. Using (54), (56), (57) and (58), we get that for all  $n$  large enough and  $|j - j_*| \leq n^{5/6}$ ,

$$(59) \quad x_j(n) \leq (1 + \varepsilon) \exp \left( (\alpha_* + \varepsilon) \frac{(j - j_*)^4}{n^3} \right) x_{j_*}(n),$$

and

$$(60) \quad x_j(n) \geq (1 - \varepsilon) \exp \left( (\alpha_* - \varepsilon) \frac{(j - j_*)^4}{n^3} \right) x_{j_*}(n).$$

Using (59) and similar arguments as in [9, Section 5], we can show that

$$\begin{aligned}
 \hat{A}_n &= \sum_{|j-j_*| \leq n^{5/6}} x_j(n) \exp \left( \frac{2r(j - j_*)}{n^{3/4}} \right) \\
 &\leq (1 + \varepsilon) x_{j_*}(n) \sum_{|k| \leq n^{5/6}} \exp \left( \frac{(\alpha_* + \varepsilon)k^4}{n^3} + \frac{2rk}{n^{3/4}} \right) \\
 &\leq (1 + \varepsilon) x_{j_*}(n) \sum_{k=-\infty}^{\infty} \exp \left( \frac{(\alpha_* + \varepsilon)k^4}{n^3} + \frac{2rk}{n^{3/4}} \right) \\
 &\leq (1 + 2\varepsilon) x_{j_*}(n) n^{3/4} \int_{-\infty}^{\infty} \exp \left( (\alpha_* + \varepsilon)x^4 + 2rx \right) dx \\
 (61) \quad (y = 2x) \quad &= (1 + 2\varepsilon) \frac{x_{j_*}(n) n^{3/4}}{2} \int_{-\infty}^{\infty} \exp \left( \frac{(\alpha_* + \varepsilon)y^4}{16} + ry \right) dy.
 \end{aligned}$$

Similarly, using (60) we have

$$(62) \quad \hat{A}_n \geq (1 - 2\varepsilon) \frac{x_{j_*}(n) n^{3/4}}{2} \int_{-\infty}^{\infty} \exp \left( \frac{(\alpha_* - \varepsilon)y^4}{16} + ry \right) dy.$$

Using the same arguments for (61) and (62), we can also prove that

$$(63) \quad \hat{B}_n = \sum_{|j-j_*| \leq n^{5/6}} x_j(n) \leq (1 + 2\varepsilon) \frac{x_{j_*}(n) n^{3/4}}{2} \int_{-\infty}^{\infty} \exp \left( \frac{(\alpha_* + \varepsilon)y^4}{16} \right) dy,$$

and

$$(64) \quad \hat{B}_n \geq (1 - 2\varepsilon) \frac{x_{j^*}(n)n^{3/4}}{2} \int_{-\infty}^{\infty} \exp\left(\frac{(\alpha_* - \varepsilon)y^4}{16}\right) dy.$$

Combining (61), (62), (63) and (64), we obtain

$$(65) \quad \left(\frac{1 - 2\varepsilon}{1 + 2\varepsilon}\right) \frac{\mathcal{A}(\alpha_* - \varepsilon, r)}{\mathcal{B}(\alpha_* + \varepsilon)} \leq \frac{\hat{A}_n}{\hat{B}_n} \leq \left(\frac{1 + 2\varepsilon}{1 - 2\varepsilon}\right) \frac{\mathcal{A}(\alpha_* + \varepsilon, r)}{\mathcal{B}(\alpha_* - \varepsilon)},$$

where

$$\mathcal{A}(\alpha, r) = \int_{-\infty}^{\infty} \exp\left(\frac{\alpha y^4}{16} + ry\right) dy \quad \text{and} \quad \mathcal{B}(\alpha, r) = \int_{-\infty}^{\infty} \exp\left(\frac{\alpha y^4}{16}\right) dy.$$

We observe that the derivatives with respect to  $\alpha$  at  $\alpha_*$  of the functions  $\mathcal{A}(\alpha, r)$  and  $\mathcal{B}(\alpha)$  are bounded. Hence, there exists a constant  $C$ , such that

$$(66) \quad \left| \frac{\mathcal{A}(\alpha_* \pm \varepsilon, r)}{\mathcal{B}(\alpha_* \pm \varepsilon)} - \frac{\mathcal{A}(\alpha_*, r)}{\mathcal{B}(\alpha_*)} \right| \leq C\varepsilon.$$

On the other hand,

$$(67) \quad \frac{\mathcal{A}(\alpha_*, r)}{\mathcal{B}(\alpha_*)} = \mathbb{E}(e^{rX}),$$

where  $X$  is a random variable with density proportional to

$$\exp\left(\frac{\alpha_* x^4}{16}\right) = \exp\left(\frac{-(d-1)(d-2)x^4}{12d^2}\right).$$

Combining (65), (66) and (67), and letting  $n$  tends to infinity and  $\varepsilon$  tend to 0, we have

$$\frac{\hat{A}_n}{\hat{B}_n} \longrightarrow \mathbb{E}(e^{rX}).$$

From this convergence and (51), we can deduce the desired result.  $\square$

## 5. APPENDIX:PROOF OF (54) AND (55)

We will repeat some computations in [9] and use Lemma 2.1 to prove these claims.

**5.1. Proof of (54).** Using Stirling's formula, we have

$$\binom{n}{j} = \left(\frac{1}{\sqrt{2\pi}} + o(1)\right) \sqrt{\frac{n}{j(n-j)}} \exp\left(nI\left(\frac{j}{n}\right)\right),$$

with

$$I(t) = (t-1)\log(1-t) - t\log t.$$

Therefore using (53), we get

$$\begin{aligned}
 \frac{x_j(n)}{x_{j_*}(n)} &= (1 + o(1)) \sqrt{\frac{j_*(n - j_*)}{j(n - j)}} \exp \left( n \left[ I \left( \frac{j}{n} \right) - I \left( \frac{j_*}{n} \right) \right] + \log g(dj, dn) - \log g(dj_*, dn) \right) \\
 &= (1 + o(1)) \sqrt{\frac{j_*(n - j_*)}{j(n - j)}} \exp \left( n \left[ I \left( \frac{j}{n} \right) - dF \left( \frac{j}{n} \right) \right] - n \left[ I \left( \frac{j_*}{n} \right) - dF \left( \frac{j_*}{n} \right) \right] \right. \\
 &\quad \left. + \left[ \log g(dj, dn) - ndF \left( \frac{j}{n} \right) \right] - \left[ \log g(dj_*, dn) - ndF \left( \frac{j_*}{n} \right) \right] \right) \\
 &= (1 + o(1)) \sqrt{\frac{j_*(n - j_*)}{j(n - j)}} \exp \left( n \left[ H \left( \frac{j}{n} \right) - H \left( \frac{j_*}{n} \right) \right] \right. \\
 &\quad \left. + \left[ \log g(dj, dn) - ndF \left( \frac{j}{n} \right) \right] - \left[ \log g(dj_*, dn) - ndF \left( \frac{j_*}{n} \right) \right] \right),
 \end{aligned}$$

which yields (54).

**5.2. Proof of (55).** Since  $H(t)$  attains the maximum at a unique point  $\frac{1}{2}$ , there exists a positive constant  $\varepsilon$ , such that for all  $\delta \leq \varepsilon$ ,

$$\max_{|t - \frac{1}{2}| \geq \delta} H(t) = \max\{H(\frac{1}{2} \pm \delta)\}.$$

Hence for  $n$  large enough (such that  $n^{-1/6} \leq \varepsilon$ ), we have for all  $|j - j_*| > n^{5/6}$ ,

$$(68) \quad H(j/n) - H(j_*/n) \leq \max\{H(j_*/n \pm n^{-1/6}) - H(j_*/n)\}.$$

Using the same arguments for (58), we can prove that

$$H(j_*/n \pm n^{-1/6}) - H(j_*/n) = \alpha_* n^{-2/3} + o(n^{-2/3}).$$

Therefore

$$(69) \quad n(H(j_*/n \pm n^{-1/6}) - H(j_*/n)) = \alpha_* n^{1/3} + o(n^{1/3}).$$

Using  $\alpha_*$  and (68), (69), we have for  $n$  large enough and  $|j - j_*| > n^{5/6}$ ,

$$(70) \quad n(H(j/n) - H(j_*/n)) \leq \frac{\alpha_* n^{1/3}}{2}.$$

On the other hand, for all  $j$

$$(71) \quad \sqrt{\frac{j_*(n - j_*)}{j(n - j)}} \leq \sqrt{n}.$$

It follows from (54), (57), (70) and (71) that for  $n$  large enough and  $|j - j_*| > n^{5/6}$ ,

$$x_j(n) \leq x_{j_*}(n) \sqrt{n} \exp \left( \frac{\alpha_* n^{1/3}}{2} \right) \leq x_{j_*}(n) n^{-6},$$

since  $\alpha_* < 0$ . Therefore

$$(72) \quad \bar{A}_n := \sum_{|j - j_*| > n^{5/6}} x_j(n) \leq x_{j_*}(n) n^{-5},$$

here we recall that  $x_j(n) = 0$  for all  $j < 0$  or  $j > n$ . Similarly, for  $n$  large enough and  $|j - j_*| > n^{5/6}$ ,

$$\begin{aligned} \exp\left(\frac{r(2j-n)}{n^{3/4}}\right) x_j(n) &\leq x_{j_*}(n) \sqrt{n} \exp\left(\frac{|r(2j-n)|}{n^{3/4}}\right) \exp\left(\frac{\alpha_* n^{1/3}}{2}\right) \\ &\leq x_{j_*}(n) \sqrt{n} \exp\left(|r| n^{1/4} + \frac{\alpha_* n^{1/3}}{2}\right) \\ &\leq x_{j_*}(n) n^{-6}. \end{aligned}$$

Hence

$$(73) \quad \bar{B}_n := \sum_{|j-j_*| > n^{5/6}} \exp\left(\frac{r(2j-n)}{n^{3/4}}\right) x_j(n) \leq x_{j_*}(n) n^{-5}.$$

Since all the terms  $(x_j(n))$  are non negative,

$$(74) \quad \hat{A}_n = \sum_{|j-j_*| \leq n^{5/6}} x_j(n) \geq x_{j_*}(n),$$

and

$$(75) \quad \begin{aligned} \hat{B}_n = \sum_{|j-j_*| \leq n^{5/6}} \exp\left(\frac{r(2j-n)}{n^{3/4}}\right) x_j(n) &\geq \exp\left(\frac{r(2j_*-n)}{n^{3/4}}\right) x_{j_*}(n) \\ &\geq \exp\left(-\frac{|r|}{n^{3/4}}\right) x_{j_*}(n) \geq \frac{x_{j_*}(n)}{2}, \end{aligned}$$

for  $n$  large enough and  $r$  fixed. Finally, combining (72), (73), (74) and (75) yields that

$$\frac{A_n}{B_n} = \frac{\hat{A}_n + \bar{A}_n}{\hat{B}_n + \bar{B}_n} = \frac{\hat{A}_n}{\hat{B}_n} + o(n^{-2}).$$

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## REFERENCES

- [1] R. Albert, A. L. Barabási. *Statistical mechanics of complex networks*. Rev. Modern Phys. 74, no. 1, 47–97 (2002).
- [2] A. Dembo, A. Montanari. *Ising models on locally tree-like graphs*. Ann. Appl. Probab. 20, no. 2, 565–592 (2010).
- [3] A. Dembo, A. Montanari, A. Sly and N. Sun. *The replica symmetric solution for Potts models on  $d$ -regular graphs*. Comm. Math. Phys. 327, 551–575 (2014)
- [4] S. Dommers, C. Giardinà, R. van der Hofstad. *Ising critical exponents on random trees and graphs*. Comm. Math. Phys. 328, no. 1, 355–395 (2014).
- [5] S. Dommers, C. Giardinà, C. Gilberti, R. van der Hofstad, M. L. Prioriello. *Ising critical behavior of inhomogeneous Curie-Weiss models and annealed random graphs*, Comm. Math. Phys. 38, 221–263 (2016).
- [6] S. N. Dorogovtsev, A. V. Goltsev, J. F. F. Mendes. *Ising model on networks with an arbitrary distribution of connections*. Phys. Rev. E 66, 016104 (2002).
- [7] F. Camia, C. M. Newman, C. Garban. *The Ising magnetization exponent is  $\frac{1}{15}$* , Prob. Theory Rel. Fields, 160, 175–187 (2014).
- [8] F. Camia, C. M. Newman, C. Garban. *Planar Ising magnetization field I. Uniqueness of the scaling limit*, Anna. Prob, 43, 528–571 (2015).

- [9] V. H. Can. *Annealed limit theorems for the Ising model on random regular graphs*, arXiv:1701.08639.
- [10] R. Ellis. *Entropy, large deviations, and statistical mechanics*. Grundlehren der Mathematischen Wissenschaften, 271. Springer-Verlag, New York, 1985.
- [11] R. Ellis, C. Newman. *The statistics of Curie-Weiss models*, J. Stat. Phys. 19, 149–161 (1978).
- [12] R. Ellis, C. Newman. *Limit theorems for sums of dependent random variables occurring in statistical mechanics*, Z. Wahrsch. Verw. Gebiete. 44, 117–139 (1978).
- [13] G. Grimmett. *The random-cluster model*. Springer, Berlin (2006).
- [14] C. Giardinà, C. Giberti, R. van der Hofstad, M. L. Prioriello. *Quenched central limit theorems for the Ising model on random graphs*, J. Stat. Phys. 160, 1623–1657 (2015).
- [15] C. Giardinà, C. Giberti, R. van der Hofstad, M. L. Prioriello. *Annealed central limit theorems for the Ising model on random graphs*, ALEA, Lat. Am. J. Probab. Math. Stat. 13 (1), 121–161 (2016).
- [16] R. van der Hofstad. *Random graphs and complex networks*. Available at <http://www.win.tue.nl/~rhofstad/NotesRGCN.html>.
- [17] E. Mossel, A. Sly. *Exact thresholds for Ising-Gibbs samplers on general graphs*. Ann. Prob. 41(1), 294–328 (2013).

INSTITUTE OF MATHEMATICS, VIETNAM ACADEMY OF SCIENCE AND TECHNOLOGY, 18 HOANG QUOC VIET, 10307 HA NOI, VIET NAM  
*E-mail address:* cvhao89@gmail.com