



HAL
open science

Recursive computation of the invariant distribution of Markov and Feller processes

Gilles Pagès, Clément Rey

► **To cite this version:**

Gilles Pagès, Clément Rey. Recursive computation of the invariant distribution of Markov and Feller processes. 2017. hal-01447257

HAL Id: hal-01447257

<https://hal.science/hal-01447257>

Preprint submitted on 26 Jan 2017

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

Recursive computation of the invariant distribution of Markov and Feller processes

Gilles Pagès¹ and Clément Rey¹

¹Université Pierre et Marie Curie, LPMA, 4 Place Jussieu, 75005 Paris, France

January 26, 2017

Abstract

This paper provides a general and abstract approach to approximate ergodic regimes of Markov and Feller processes. More precisely, we show that the recursive algorithm presented in [7] and based on simulation algorithms of stochastic schemes with decreasing step can be used to build invariant measures for general Markov and Feller processes. We also propose applications in three different configurations: Approximation of Markov switching Brownian diffusion ergodic regimes using Euler scheme, approximation of Markov Brownian diffusion ergodic regimes with Milstein scheme and approximation of general diffusions with jump components ergodic regimes.

Keywords : Ergodic theory, Markov processes, Invariant measures, Limit theorem, Stochastic approximation.

AMS MSC 2010: 60G10, 47A35, 60F05, 60J25, 60J35, 65C20.

1 Introduction

In this paper, we propose a method for the computation of invariant measures of Markov processes (denoted ν). In particular, we study a sequence of empirical stochastic measures $(\nu_n)_{n \in \mathbb{N}^*}$ which can be recursively computed using a discrete process simulated with a sequence of vanishing step $\gamma = (\gamma_n)_{n \in \mathbb{N}}$ and transition semigroups $(Q_n^\gamma)_{n \in \mathbb{N}^*}$. We show that $\lim_{n \rightarrow \infty} \nu_n f = \nu f$ *a.s.*, for a class of test functions f . The recursive algorithm which is employed to build $(\nu_n)_{n \in \mathbb{N}^*}$ considering that $(Q_n^\gamma)_{n \in \mathbb{N}^*}$ is given, has been introduced in the seminal paper [7].

Invariant measures are crucial in the study of the long term behavior of stochastic differential systems. We invite the reader to refer to [5] and [2] for an overview of the subject. The construction of invariant measure for stochastic systems has already been widely explored in the literature. In [16], the author provides a computation of the invariant distribution for some solutions of Stochastic Differential Equations but in many cases there is no explicit formula for ν . A first approach consists in studying the convergence of the semigroup of the Markov process (denoted $(P_t)_{t \geq 0}$) with infinitesimal generator A towards the invariant measure ν as it is done in [4] for the variation topology. If $(P_t)_{t \geq 0}$ can be computed, one can approximate ν controlling only the error between $(P_t)_{t \geq 0}$ and ν . If the process with semigroup $(P_t)_{t \geq 0}$ can be simulated, we can use a Monte Carlo method to estimate $(P_t)_{t \geq 0}$ producing a second term in the error analysis. When the process with semigroup $(P_t)_{t \geq 0}$ can not be simulated with reasonable time, a solution consists in simulating an approximation for the semigroup $(P_t)_{t \geq 0}$, using $(Q_n^\gamma)_{n \in \mathbb{N}^*}$ (given a step sequence $(\gamma_n)_{n \in \mathbb{N}}$). The semigroup $(Q_n^\gamma)_{n \in \mathbb{N}^*}$ is supposed to weakly converge towards $(P_t)_{t \geq 0}$. A natural construction rely on numerical homogeneous schemes $((\gamma_n)_{n \in \mathbb{N}}$ is constant and equal to the time step γ_0). This approach induced two more terms to control in the approximation

e-mails : gilles.pages@upmc.fr, clement.rey@upmc.fr This research benefited from the support of the "Chaire Risques Financiers".

of ν in addition to the error between $(P_t)_{t \geq 0}$ and ν : The first one is due to the approximation of $(P_t)_{t \geq 0}$ by $(Q_n^\gamma)_{n \in \mathbb{N}^*}$ and the second one is due to the Monte Carlo error involved in the computation of the law of the process simulated with $(Q_n^\gamma)_{n \in \mathbb{N}^*}$.

Nevertheless, for Brownian diffusions, many efforts have been done in order to simplify this problem. In [17], the author suggests an elegant procedure to simplify this last approach. He considers the case where the process simulated with $(Q_n^\gamma)_{n \in \mathbb{N}^*}$ (where $(\gamma_n)_{n \in \mathbb{N}}$ is still constant) has an invariant measure ν^{γ_0} . In a first step, he shows that $\lim_{n \rightarrow \infty} \nu_n f = \nu^{\gamma_0} f$, and then he proves that $\lim_{\gamma_0 \rightarrow 0} \nu^{\gamma_0} = \nu$. Consequently, he gets rid of the Monte Carlo approximation (since there is no estimation procedure for the computation of $(P_t)_{t \geq 0}$ or $(Q_n^\gamma)_{n \in \mathbb{N}^*}$), and there are only two terms to treat in the error. He manages to control this error under a uniform ellipticity condition that is not necessary in our work. He also extended these results in [18].

Another approach has been proposed in [1] and avoid asymptotic analysis with respect to the size of the time step. In this paper, the authors prove directly that the random discrete process simulated with $(Q_n^\gamma)_{n \in \mathbb{N}^*}$, with $(\gamma_n)_{n \in \mathbb{N}}$ vanishing to 0, converges weakly toward ν . Therefore, there are two terms to treat in the error: The first one is due to this convergence and the second one to the Monte Carlo error involved in the computation of the law of the process simulated with $(Q_n^\gamma)_{n \in \mathbb{N}^*}$. The reader may notice that in all those cases, strong ergodicity assumptions are required for the process with infinitesimal generator A .

Inspired among others by the ideas from [17] and [1], in [7], the authors designed a recursive algorithm with decreasing step and showed that the sequence $(\nu_n)_{n \in \mathbb{N}^*}$ built with a discrete process that can be simulated using a sequence of vanishing step $\gamma = (\gamma_n)_{n \in \mathbb{N}}$ and transition semigroups $(Q_n^\gamma)_{n \in \mathbb{N}^*}$ directly converges towards ν . This initial paper treated the case where $(Q_n^\gamma)_{n \in \mathbb{N}^*}$ is the transition semigroup of an inhomogeneous Euler scheme with decreasing step associated to a strongly mean reverting ergodic Brownian diffusion process. In this paper, they introduce the recursive algorithm to build the sequence of random measures $(\nu_n)_{n \in \mathbb{N}^*}$ given $(Q_n^\gamma)_{n \in \mathbb{N}^*}$ (which is the procedure that is used in every work we mention from now and is also the one we use in this paper). Moreover, they prove that $\lim_{n \rightarrow \infty} \nu_n f = \nu f$ *a.s.* for a class of test functions which is larger than the domain (denoted $\mathcal{D}(A)$) of A and contains test functions with polynomial growth. They also obtained rates and limit gaussian laws for the convergence of $(\nu_n(f))_{n \in \mathbb{N}^*}$ for test functions f which can be written $f = A\varphi$. Finally they do not require that the invariant measure ν is unique controversially to the results obtained in [17] and [1] for instance. In the case where ν is an invariant distribution of a stochastic diffusion, many complementary works to [7] have been led. The authors extended their first results in [8], where they achieve convergence towards invariant measures for Euler scheme of Brownian diffusions using weak mean reverting assumptions for the dynamical stochastic system. Thereafter, in its thesis [9], the author extended the class of function for which we have $\lim_{n \rightarrow \infty} \nu_n f = \nu f$ *a.s.* from test functions f with polynomial growth to test functions with exponential growth. Finally, in [13], the author generalized those results to the construction of invariant measures for Levy diffusion processes still using the algorithm from [7]. He thus opened the door to treat not only approximation of Brownian diffusions' ergodic regime but also a larger class of processes.

The aim of this paper is to show that the algorithm presented in [7] enables to approximate invariant measures (when there exists without being necessarily unique) for general Markov and Feller processes. We present a general framework adapted to the construction of invariant measures for Markov processes under general mean reverting assumption (which includes weak mean reverting assumptions). Then, we provide some applications for three different configurations always under weak mean reverting assumptions. The first one treats the case of the Euler scheme for Markov Switching diffusions for test functions with polynomial growth. This particular case has already been studied in [10] under strong ergodicity assumptions (which includes among others strong mean reverting assumption). Then, we prove convergence for the Milstein scheme for test functions with polynomial or exponential growth. Finally, we consider the Euler scheme for general diffusion processes with jump and test functions with polynomial growth. In particular, this last results involves Levy processes (as in [13]) but also piecewise deterministic Markov processes or diffusions processes with censored jump.

In a first step we present some general useful results in this study. Then we present the general and abstract framework in order to obtain convergence toward an invariant measure of a general Markov and Feller processes. The end of this paper is devoted to the miscellaneous examples mentioned above.

2 Preliminary results

In this paper, we propose a general approach to compute invariant measures for Markov and Feller processes. In this section, we give some well known general results that we employ to show that convergence. We begin with some notations. For E a locally compact separable metric space, we denote $\mathcal{C}_0(E)$ the set of continuous functions that vanish at infinity. We equip this space with the sup norm $\|f\|_\infty = \sup_{x \in E} |f(x)|$ and then $(\mathcal{C}_0(E), \|\cdot\|_\infty)$ is a Banach space. We will denote $\mathcal{B}(E)$ the σ -algebra of Borel subsets of E and $\mathcal{P}(E)$ the family of Borel probability measures on E .

To be a bit more specific, we consider the generator of a Markov and Feller process denoted by A . In this paper, our main purpose is to build a measure $\tilde{\nu}$ and show that there exists a stationary solution to the Martingale problem $(A, \tilde{\nu})$. In this case we also say that $\tilde{\nu}$ is an invariant measure for the Feller process with generator A . We recall the definition of the martingale problem.

Definition 2.1. *Let $\tilde{\nu} \in \mathcal{P}(E)$ with E a locally compact and separable metric space and let A be a linear operator defined on a subset $\mathcal{D}(A)$ of $\mathcal{C}_0(E)$. We say that a process $(X_t)_{t \geq 0}$ is a solution of the martingale problem $(A, \tilde{\nu})$ if X is progressive and we have $\mathbb{P}(X_0)^{-1} = \tilde{\nu}$ and for every $f \in \mathcal{D}(A)$, the process $(Y_t)_{t \geq 0}$ such that $Y_t = f(X_t) - \int_0^t Af(X_s)ds$, for every $t \geq 0$, is a martingale.*

However, we will not stick to that definition to prove the existence of a stationary solution for the martingale problem $(A, \tilde{\nu})$. Instead, we use the Echeverria Weiss theorem which provides a way to obtain the existence of a solution for the martingale problem $(A, \tilde{\nu})$ under a practical property for our approach. Now, we give this theorem which can be found in [2] (Theorem 9.17).

Theorem 2.1. (Echeverria Weiss). *Let E be a locally compact and separable metric space and let A be a linear operator of $\mathcal{C}_0(E)$ satisfying the positive maximum principle¹ and such that $\mathcal{D}(A)$ is dense in $\mathcal{C}_0(E)$. If $\tilde{\nu} \in \mathcal{P}(E)$ satisfies*

$$\forall f \in \mathcal{D}(A), \quad \int_E Af d\tilde{\nu} = 0, \quad (1)$$

then there exists a stationary solution to the martingale problem $(A, \tilde{\nu})$.

For our approach, one main advantage of this result is that the only property we will have to prove to obtain the existence of a stationary solution to the martingale problem $(A, \tilde{\nu})$ is (1). Indeed, for Feller processes, we have the following useful result

Proposition 2.1. *Let A be the generator of a Feller semigroup. The space $\mathcal{D}(A)$ is dense in $\mathcal{C}_0(E)$. Moreover, A satisfies the positive maximum principle.*

The proof of this result can be found in [15] (Chapter VII, Proposition 1.3 and Proposition 1.5) or [2] (Chapter IV, Theorem 2.2). Consequently, this paper will be devoted to the construction of a measure $\tilde{\nu}$, and then to the proof of (1) with this measure. Using the results mentioned in this section, property (1) is sufficient to prove that $\tilde{\nu}$ is an invariant measure for the process with infinitesimal generator A . To be more concrete, in this paper, the measure $\tilde{\nu}$ will be built as the limit of a sequence of random measure $(\nu_n)_{n \in \mathbb{N}^*}$ that we specify in the sequel. When (1) holds for this limit, we say that the sequence $(\nu_n)_{n \in \mathbb{N}^*}$ converges towards an invariant measure of the Feller process with generator A . In order to obtain (1) for this measure, we will employ the following well known results.

Lemma 2.1. (Kronecker). *Let $(a_n)_{n \in \mathbb{N}^*}$ and $(b_n)_{n \in \mathbb{N}^*}$ be two sequences of real numbers. If $(b_n)_{n \in \mathbb{N}^*}$ is non-decreasing, positive, with $\lim_{n \rightarrow \infty} b_n = \infty$ and $\sum_{n \geq 1} a_n/b_n$ converges in \mathbb{R} , then*

$$\lim_{n \rightarrow \infty} \frac{1}{b_n} \sum_{k=1}^n a_k = 0.$$

Theorem 2.2. (Chow). *Let $(M_n)_{n \in \mathbb{N}^*}$ be a real valued martingale with respect to some filtration $\mathcal{F} = (\mathcal{F}_n)_{n \in \mathbb{N}}$. Then*

$$\forall r \in (0, 1], \quad \lim_{n \rightarrow \infty} M_n = M_\infty \in \mathbb{R} \quad a.s.$$

on the event

$$\left\{ \sum_{n=1}^{\infty} \mathbb{E}[|M_n - M_{n-1}|^{1+r} | \mathcal{F}_{n-1}] < \infty \right\}.$$

¹ $\forall f \in \mathcal{D}(A), f(x_0) = \sup\{f(x), x \in E\} \geq 0, x_0 \in E \Rightarrow Af(x_0) \leq 0.$

3 Convergence to invariant distribution - A general approach

This section presents a general approach inspired from the seminal work in [7] to construct $(\nu_n)_{n \in \mathbb{N}^*}$ and prove that it converges towards an invariant measure of a Markov and Feller with infinitesimal generator A as soon as it is built with a sequence of approximating semigroup of that Markov and Feller process.

3.1 Presentation of the framework

In this part, we present the recursive algorithm in order to build $(\nu_n)_{n \in \mathbb{N}^*}$ and also the general hypothesis that are required to obtain convergence towards an invariant distribution of a Markov and Feller process. In other words, we give some general assumptions on $(\nu_n)_{n \in \mathbb{N}^*}$ in order to obtain (1) for $\lim_{n \rightarrow \infty} \nu_n$.

3.1.1 Construction of the random measures

In this paper we consider a locally compact and separable metric space E . We introduce a sequence of finite transition measures $\mathcal{P}_n(x, dy)$, $n \in \mathbb{N}^*$ from E to itself. This means that for each fixed x and n , $\mathcal{P}_n(x, dy)$ is a probability measure on $(E, \mathcal{B}(E))$ with the Borel σ -field and, for each bounded measurable function f , the mapping

$$x \mapsto \mathcal{P}_n f(x) := \int_E f(y) \mathcal{P}_n(x, dy)$$

is Borel measurable. We also suppose that $\mathcal{P}_n f \in \mathcal{C}_0(E)$ for every measurable function $f \in \mathcal{C}_0(E)$ and $n \in \mathbb{N}^*$. Now we associate the sequence \mathcal{P} to a time grid with vanishing steps. Let $\gamma := (\gamma_n)_{n \in \mathbb{N}^*}$ such that

$$\forall n \in \mathbb{N}^*, \quad 0 \leq \gamma_n \leq \bar{\gamma} < \infty, \quad \lim_{n \rightarrow \infty} \gamma_n = 0 \text{ and } \lim_{n \rightarrow \infty} \Gamma_n = +\infty \quad (2)$$

with the notations $\Gamma_0 = 0$ and $\Gamma_n = \sum_{k=1}^n \gamma_k$. From now, we will use the notation $\mathcal{P}_n^\gamma = \mathcal{P}_n$.

Definition 3.1. We define the family of discrete linear operator $(P_n^\gamma)_{n \in \mathbb{N}^*}$ from $\mathcal{C}_0(E)$ to itself in the following way.

$$P_0^\gamma f(x) = f(x), \quad P_{n+1}^\gamma f(x) = P_n^\gamma \mathcal{P}_{n+1}^\gamma f(x) = P_n^\gamma \int_{\mathbb{R}^d} f(y) \mathcal{P}_{n+1}^\gamma(x, dy).$$

Remark 3.1. If we define more generally $(P_{n,m}^\gamma)_{n,m \in \mathbb{N}; n \leq m}$ by

$$P_{n,n}^\gamma f(x) = f(x), \quad \forall n, m \in \mathbb{N}^*, n \leq m, \quad P_{n,m+1}^\gamma f(x) = P_{n,m}^\gamma \mathcal{P}_{m+1}^\gamma f(x),$$

we have the following semigroup property: for $n, m, k \in \mathbb{N}$, $n \leq m \leq k$, $P_{n,k}^\gamma f = P_{n,m}^\gamma P_{m,k}^\gamma f$.

We consider now a second sequence of finite transition probability measures $\mathcal{Q}_n^\gamma(x, dy)$, $n \in \mathbb{N}^*$. Moreover, we introduce the corresponding semigroup Q^γ defined in a similar way as P^γ with \mathcal{P}^γ replaced by \mathcal{Q}^γ . Finally, we assume that there exists a continuous Feller semigroup $(P_t)_{t \geq 0}$ such that for every $t \in \{\Gamma_n, n \in \mathbb{N}\}$, then $P_{\Gamma_n} = P_n^\gamma$. We do not make such assumption for $(Q_n^\gamma)_{n \in \mathbb{N}}$. In our framework, Q^γ is thus considered as the approximation discrete semigroup of $(P_t)_{t \geq 0}$ on the time grid $\{\Gamma_n, n \in \mathbb{N}\}$. In the sequel we will denote by A the infinitesimal generator of P and ν will denote an invariant measure for A . We now propose the construction of $(\nu_n)_{n \in \mathbb{N}^*}$ which inspired from [7]. We define the Markov process $X := (X_n)_{n \in \mathbb{N}}$ in the following way

$$X_0 \in E, \quad \mathbb{P}(X_{n+1} \in dy | X_n) = \mathcal{Q}_{n+1}^\gamma(X_n, dy) \quad (3)$$

The main difference with [7] is that we do not suppose that Q^γ is the semigroup associated to the Euler scheme of a Brownian diffusion process. In this study, we simply consider approximations of Markov processes that can be simulated. At this point, we are going to define a weighted empirical measure with X . This construction is totally similar to the one in [7] but with the Euler scheme replaced by $(X_n)_{n \in \mathbb{N}}$. First, we introduce the weights. Let $\eta := (\eta_n)_{n \in \mathbb{N}^*}$ such that

$$\forall n \in \mathbb{N}^*, \quad \eta_n \geq 0, \quad \lim_{n \rightarrow \infty} H_n = \infty, \quad (4)$$

with the notation $H_n = H_{\eta,n} = \sum_{k=1}^n \eta_k$. Now we present the algorithm initially introduced in [7]. First, for $x \in E$, let δ_x denote the Dirac mass at point x . For every $n \in \mathbb{N}^*$, we define the random weighted empirical random measures as follows

$$\nu_n^\eta(dx) = \frac{1}{H_n} \sum_{k=1}^n \eta_k \delta_{X_{k-1}}(dx). \quad (5)$$

The aim of this paper is to show that $\lim_{n \rightarrow \infty} \nu_n^\eta f = \nu f$ *a.s.*, for a class of test functions f . This will hold as soon as $(\nu_n)_{n \in \mathbb{N}^*}$ is tense, (1) is satisfied with $\tilde{\nu}$ replaced by $\lim_{n \rightarrow \infty} \nu_n$ and $f(x) = \begin{matrix} o \\ |x| \rightarrow \infty \end{matrix} g(x)$ with $\sup_{n \in \mathbb{N}^*} \nu_n(g) < \infty$.

3.1.2 Assumptions on the random measures

In this section, we present the hypothesis that we require in order to prove that convergence. Those assumptions are related to the increments of the approximation semigroup Q^γ , also called pseudo generator of Q^γ . More particularly a first assumption concerns the recursive control of this pseudo generator while the second describe its connection to the infinitesimal generator A . We begin with some definitions.

Let us define the family of linear operators $\tilde{A}^\gamma := (\tilde{A}_n^\gamma)_{n \in \mathbb{N}^*}$ from $\mathcal{C}_0(E)$ to itself, in the following way

$$\forall f \in \mathcal{C}_0(E), x \in E, \quad n \in \mathbb{N}^*, \quad \tilde{A}_n^\gamma f = \frac{Q_n^\gamma f - f}{\gamma_n}. \quad (6)$$

The reader may notice that \tilde{A}^γ is also called the pseudo generator of the semigroup Q^γ . In order to obtain our results, it is necessary to introduce some hypothesis concerning the stability of the semigroup Q^γ . A key point in our approach, as it is the case for most studies concerning invariant distributions, is the existence of a Lyapunov function. We say that V is a Lyapunov function if

$$\mathcal{L}_V \quad V : E \rightarrow [v_*, +\infty), v_* > 0 \quad \text{and} \quad \lim_{|x| \rightarrow \infty} V(x) = \infty. \quad (7)$$

A classical interest of Lyapunov functions is to show the existence and sometimes uniqueness of the invariant measure for the process with infinitesimal generator A . We invite the reader to refer to the large literature on the subject for more details: See for instance [5], [2] or [12].

Recursive control

In our framework, we introduce a well suited stability assumption for the pseudo generator in order to obtain existence (in weak sense) of the limit of the sequence of random measures $(\nu_n)_{n \in \mathbb{N}}$. This will be done using a tightness property. We now give this assumption that will be mentioned from now as the recursive control of the pseudo generator \tilde{A}^γ :

Let $v_* > 0$, $V : E \rightarrow [v_*, \infty)$, $\psi, \phi \in [v_*, \infty) \rightarrow \mathbb{R}_+$, such that $\tilde{A}_n^\gamma \psi \circ V$ exists for every $n \in \mathbb{N}^*$. Then we assume that there exists $\alpha > 0$ and $\beta \in \mathbb{R}_+$, such that

$$\mathcal{I}_{Q,V}(\psi, \phi) \left\{ \begin{array}{l} \exists n_0 \in \mathbb{N}^*, \forall n \geq n_0, x \in E, \quad \tilde{A}_n^\gamma \psi \circ V(x) \leq V^{-1}(x) \psi \circ V(x) (\beta - \alpha \phi \circ V(x)). \\ \exists \lambda \in [0, 1), C_\lambda \geq 0, \forall y \geq v_*, \quad \psi(y) (\beta - \lambda \alpha \phi(y)) \leq C_\lambda y. \end{array} \right. \quad (8)$$

Let us notice that the second part of the assumption $\mathcal{I}_{Q,V}(\psi, \phi)$ is satisfied as soon as $\lim_{y \rightarrow \infty} \phi(y) = \infty$. The function ϕ controls the mean reverting property. In particular, we say that we have strong mean reverting property if $\phi = I_d$ and that we have weak mean reverting property when $\phi(y) = y^a$, $a \in (0, 1)$ for every $y \in [v_*, \infty)$. The function ψ is referred in this paper as the test function and is related to the set of functions f for which we have $\lim_{n \rightarrow \infty} \nu_n^\eta(f) = \nu(f)$, when ν is the unique invariant measure of the process with infinitesimal generator A . This assumption is crucial to prove the tightness of the sequence $(\nu_n^\eta)_{n \in \mathbb{N}^*}$ and consequently to obtain the existence of a limit point (not necessarily unique) for this sequence.

Infinitesimal approximation

This part presents the assumption that enable to show that any limit point of the sequence $(\nu_n^\eta)_{n \in \mathbb{N}^*}$ is an invariant measure for the Markov and Feller process with infinitesimal generator A . We aim to estimate the distance between an invariant measure of P and ν_n^η (see (5)) for n large enough. In order to do it, we introduce an additional hypothesis concerning the distance between \tilde{A}^γ , the pseudo generator of Q^γ , and A , the infinitesimal generator of P .

We assume that

$$\mathcal{E}(\tilde{A}^\gamma, A) \quad \exists n_0 \in \mathbb{N}^*, \forall n \geq n_0, \forall f \in \mathcal{D}(A), \forall x \in E \quad |\tilde{A}_n^\gamma f(x) - Af(x)| \leq \Lambda_f(x, \gamma_n), \quad (9)$$

where $\Lambda_f : \mathbb{R}^d \times \mathbb{R}_+ \rightarrow \mathbb{R}_+$ can be decomposed in the following way:

Let $q \in \mathbb{N}$. We introduce $g = (g_1, \dots, g_q)$ and $\tilde{\Lambda}_f = (\tilde{\Lambda}_{f,1}, \dots, \tilde{\Lambda}_{f,q})$ two families of functions with $g_i : \mathbb{R}^d \rightarrow \mathbb{R}_+$ and $\tilde{\Lambda}_{f,i} : E \times \mathbb{R}_+ \times \mathbb{R}^{N_i} \times G_i \rightarrow \mathbb{R}_+$. Moreover, for every $i \in \{1, \dots, q\}$, we introduce a positive finite measure π_i defined on a measurable space (G_i, \mathcal{G}_i) , and a family of random processes $(U_i(x, \Theta_i))_{\Theta_i \in G_i}$ taking values in \mathbb{R}^{N_i} , $N_i \in \mathbb{N}^*$. We suppose that we have

$$\forall x \in \mathbb{R}^d, \forall t \in [0, \gamma_{n_0}], \quad \Lambda_f(x, t) = \sum_{i=1}^q \int_{G_i} \mathbb{E}[\tilde{\Lambda}_{f,i}(x, t, U_i(x, \Theta_i)_t, \Theta_i)] \pi_i(d\Theta_i) g_i(x)$$

with $\sup_{x \in E, t \in [0, \gamma_{n_0}], i \in \{1, \dots, q\}} \int_{G_i} \mathbb{E}[\tilde{\Lambda}_{f,i}(x, t, U_i(x, \Theta_i)_t, \Theta_i)] \pi_i(d\Theta_i) < \infty$. Using this decomposition, we assume that for every couple of functions $(\tilde{\Lambda}_{f,i}, g_i)$, $i \in \{1, \dots, q\}$, one the following assumption holds, that is $\mathcal{E}_{loc}(\tilde{\Lambda}_{f,i}, g_i)$ or $\mathcal{E}_{ergo}(\tilde{\Lambda}_{f,i}, g_i)$.

I) Locally compact case We say that $\mathcal{E}_{loc}(\tilde{\Lambda}_{f,i}, g_i)$ holds if g_i is locally compact and: for every $u \in \mathbb{R}^{N_i}$, for every $\Theta_i \in G_i$ and every compact subset K of E we have

$$\limsup_{t \rightarrow 0} \sup_{x \in K} \tilde{\Lambda}_{f,i}(x, t, u, \Theta_i) = 0. \quad (10)$$

Moreover, we assume that there exists $t_0 > 0$ and a compact subset K_0 of E such that

$$\forall x \in E \setminus K_0, \forall t \in [0, t_0], \forall u \in \mathbb{R}^{N_i}, \forall \Theta_i \in G_i, \quad \tilde{\Lambda}_{f,i}(x, t, u, \Theta_i) = 0. \quad (11)$$

II) Case $\sup_{n \in \mathbb{N}^*} \nu_n^\eta(\mathbf{g}_i) < \infty$ a.s. We say that $\mathcal{E}_{ergo}(\tilde{\Lambda}_{f,i}, g_i)$ holds if g_i is locally compact, $\sup_{n \in \mathbb{N}^*} \nu_n^\eta(g_i) < \infty$ and one of the following properties holds:

For every compact subset K of E and all $\Theta_i \in G_i$, we have

$$\begin{aligned} \limsup_{t \rightarrow 0} \sup_{x \in K} \tilde{\Lambda}_{f,i}(x, t, U_i(x, \Theta_i)_t, \Theta_i) &= 0 \text{ a.s.}, \text{ and} \\ \lim_{|x| \rightarrow \infty} \sup_{t \in [0, \gamma_{n_0}]} \tilde{\Lambda}_{f,i}(x, t, U_i(x, \Theta_i)_t, \Theta_i) &= 0 \text{ a.s.} \end{aligned} \quad (12)$$

or the following holds instead: For every $\Theta_i \in G_i$,

$$\limsup_{t \rightarrow 0} \sup_{x \in E} \tilde{\Lambda}_{f,i}(x, t, U_i(x, \Theta_i)_t, \Theta_i) g_i(x) = 0 \text{ a.s.}, \quad (13)$$

The reader may notice that the measures π_i , $i \in \{1, \dots, q\}$ are not supposed to be probability measures. However, in many cases, those measures are built using some probability measures. This representation assumption is related to the fact that the transition functions $\mathcal{Q}_n^\gamma(x, dy)$, $x \in E$ can be represented using random variables (which does not depend from γ) through the variable Θ_i and using random processes through $(U_i(x, \Theta_i)_t)_{t \geq 0}$. This approach is well adapted to stochastic approximations that can be associated to a time grid such as numerical schemes for stochastic differential equation with a Brownian part or/and a Jump part.

This concludes the part concerning the assumption and we can focus on the main results concerning this abstract approach.

3.2 Convergence

3.2.1 Almost sure tightness

From the recursive control assumption, we obtain the tightness of $(\nu_n^\eta)_{n \in \mathbb{N}^*}$. This is one of the purpose of the following Theorem. We recall that tightness implies that the sequence has at least one limit point. Another interest of this result is that the functions ϕ and ψ are not specified explicitly and then this framework apply to many diverse configurations.

Theorem 3.1. *Let $s \in (1, 2]$, $v_* > 0$, $V : E \rightarrow [v_*, \infty)$, $\psi, \phi : [v_*, \infty) \rightarrow \mathbb{R}_+$. We assume that ψ is lower bounded, that $\mathcal{I}_{Q,V}(\psi, \phi)$ (see (8)) hold and that*

$$\mathbb{P}\text{-a.s.} \quad \sup_{n \in \mathbb{N}^*} -\frac{1}{H_n} \sum_{k=1}^n \eta_k \tilde{A}_k^\gamma (\psi \circ V)^{1/s}(X_{k-1}) < \infty. \quad (14)$$

Then

$$\mathbb{P}\text{-a.s.} \quad \sup_{n \in \mathbb{N}^*} \nu_n^\eta (V^{-1} \cdot \phi \circ V \cdot (\psi \circ V)^{1/s}) < \infty. \quad (15)$$

Finally, if \mathcal{L}_V (see (7)) holds, and the function $x \mapsto x^{-1} \phi(x) \psi(x)^{1/s}$ tends to infinity as x goes to infinity, then the sequence $(\nu_n^\eta)_{n \in \mathbb{N}^*}$ is tight. Consequently, if the sequence $(\nu_n^\eta)_{n \in \mathbb{N}^*}$ has a unique weak limit ν then for every continuous function f satisfying $f = o(V^{-1} \cdot \phi \circ V \cdot (\psi \circ V)^{1/s})$, we have $\lim_{n \rightarrow \infty} \nu_n^\eta(f) = \nu(f)$.

Proof. Using (8), there exists $n_0 \in \mathbb{N}$ such that for every $n \geq n_0$, we have

$$\mathbb{E} \left[\frac{\psi \circ V(X_{n+1})}{\psi \circ V(X_n)} \middle| \mathcal{F}_n \right] \leq 1 + \gamma_{n+1} \frac{\beta - \alpha \phi \circ V(X_n)}{V(X_n)}.$$

Since the function defined on \mathbb{R}_+^* by $y \mapsto y^{1/s}$ is concave, we use the Jensen's inequality and obtain

$$\begin{aligned} \mathbb{E} \left[\frac{\psi \circ V(X_{n+1})^{1/s}}{\psi \circ V(X_n)^{1/s}} \middle| \mathcal{F}_n \right] &\leq \left(1 + \gamma_{n+1} \frac{\beta - \alpha \phi \circ V(X_n)}{V(X_n)} \right)^{1/s} \\ &\leq 1 + \frac{\gamma_{n+1} (\beta - \alpha \phi \circ V(X_n))}{s V(X_n)}. \end{aligned}$$

Now, we use (8) and it follows that there exists $\lambda \in [0, 1)$, $C_\lambda \geq 0$, such that

$$\begin{aligned} Q_{n+1}^\gamma (\psi \circ V)^{1/s}(X_n) &\leq (\psi \circ V)^{1/s}(X_n) + \frac{\gamma_{n+1}}{s} (\psi \circ V)^{1/s}(X_n) V^{-1}(X_n) (\beta - \alpha \phi \circ V(X_n)) \\ &\leq (\psi \circ V)^{1/s}(X_n) + \gamma_{n+1} \psi \circ V(X_n)^{1/s-1} \left(\frac{C_\lambda}{s} - \frac{\alpha(1-\lambda)}{s} V^{-1}(X_n) \phi \circ V(X_n) \right). \end{aligned}$$

or equivalently,

$$V^{-1}(X_n) \phi \circ V(X_n) \psi \circ V(X_n)^{1/s} \leq -\frac{s}{\alpha(1-\lambda)} \tilde{A}_{n+1}^\gamma (\psi \circ V)^{1/s}(X_n) + \frac{C_\lambda}{\alpha(1-\lambda)} \left(\inf_{x \geq v_*} |\psi(x)| \right)^{1/s-1}.$$

Consequently, the result follows from (14), $\mathbb{E}[\psi \circ V(X_{n_0})] < \infty$, and the fact that ψ is lower bounded on $[v_*, \infty)$. \square

3.2.2 Identification of the limit

In Theorem 3.1, we obtained tightness of $(\nu_n^\eta)_{n \in \mathbb{N}^*}$. It remains to show that any limit point of this sequence is an invariant measure for the process with infinitesimal generator A . This is the interest of the following Theorem which uses the infinitesimal approximation.

Theorem 3.2. *Let $n_0 \in \mathbb{N}^*$. We assume that for every $f \in \mathcal{D}(A)$, we have*

$$\mathbb{P}\text{-a.s.} \quad \lim_{n \rightarrow \infty} \nu_n^\eta (\tilde{A}_k^\gamma f) = 0. \quad (16)$$

We also assume that $\mathcal{E}(\tilde{A}^\gamma, A)$ (see (9)), holds. Then

$$\mathbb{P}\text{-a.s.} \quad \lim_{n \rightarrow \infty} \nu_n^\eta (Af) = 0 \quad (17)$$

It follows that, \mathbb{P} -a.s., every (weak) limiting distribution ν_∞^η of the sequence $(\nu_n^\eta)_{n \in \mathbb{N}^*}$ is an invariant distribution for the semigroup $(P_t)_{t \geq 0}$ with infinitesimal generator A .

A direct consequence of this result is that if $(\nu_n^\eta)_{n \in \mathbb{N}^*}$ is almost surely tight and the semigroup $(P_t)_{t \geq 0}$ with infinitesimal generator A admits a unique invariant measure ν , then almost surely $(\nu_n^\eta)_{n \in \mathbb{N}^*}$ converges to ν .

Proof. First we write

$$\nu_n^\eta(\tilde{A}_k^\gamma f) - \nu_n^\eta(Af) = \frac{1}{H_n} \sum_{k=1}^n \eta_k \tilde{A}_k^\gamma f(X_{k-1}) - \eta_k Af(X_{k-1}).$$

Now we use the short time approximation $\mathcal{E}(\tilde{A}^\gamma, A)$ (see (9)) and it follows that there exists $n_0 \in \mathbb{N}^*$ such that

$$\frac{1}{H_n} \sum_{k=n_0}^n \eta_k \tilde{A}_k^\gamma f(X_{k-1}) - \eta_k Af(X_{k-1}) = \frac{C}{H_n} \sum_{k=n_0}^n \eta_k \Lambda_f(X_{k-1}, \gamma_k).$$

Moreover, we have the following decomposition

$$\forall x \in \mathbb{R}^d, \forall t \in [0, \gamma_{n_0}], \quad \Lambda_f(x, t) = \sum_{i=1}^q \int_{G_i} \mathbb{E}[\tilde{\Lambda}_{f,i}(x, \gamma, U_i(x, \Theta_i)_t, \Theta_i)] \pi_i(d\Theta_i) g_i(x)$$

with for every $\Theta_i \in G_i$, $U_i(x, \Theta_i) = (U_i(x, \Theta_i)_t)_{t \geq 0}$ a \mathbb{R}^N -valued random process and

$$\sup_{x \in E, t \in [0, \gamma_{n_0}], i \in \{1, \dots, q\}} \int_{G_i} \mathbb{E}[\tilde{\Lambda}_{f,i}(x, t, U_i(x, \Theta_i)_t, \Theta_i)] \pi_i(d\Theta_i) < \infty.$$

Now we assume that $\mathcal{E}_{ergo}(\tilde{\Lambda}_{f,i}, g_i)$ (see (12) and (13)) holds for $\Lambda_{f,i}$. If $\mathcal{E}_{loc}(\tilde{\Lambda}_{f,i}, g_i)$ (see (10) and (11)) holds instead of $\mathcal{E}_{ergo}(\tilde{\Lambda}_{f,i}, g_i)$, the proof is similar by simpler so we leave it out. In order to obtain the desired convergence, we first fix $\Theta_i \in G_i$ and study

$$\frac{1}{H_n} \sum_{k=n_0}^n \eta_k \tilde{\Lambda}_{f,i}(X_{k-1}, \gamma_k, U_i(X_{k-1}, \Theta_i)_{\gamma_k}, \Theta_i) g_i(X_{k-1}).$$

We assume that (12) holds. If instead (13) is satisfied, the proof is similar but simpler so we leave it to the reader. For $R > 0$, we denote $\bar{B}_R = \{x \in E, |x| \leq R\}$. Using $\mathcal{E}_{ergo}(\tilde{\Lambda}_{f,i}, g_i)$ (see (12)), we have immediately $\lim_{n \rightarrow \infty} \tilde{\Lambda}_{f,i}(X_{n-1}, \gamma_n, U_i(X_{n-1}, \Theta_i)_{\gamma_n}, \Theta_i) \mathbb{1}_{|X_{n-1}| \leq R} = 0$ a.s. Then, since g_i is a continuous function, as an immediate consequence of the Cesaro's lemma, we obtain

$$\lim_{n \rightarrow \infty} \frac{1}{H_n} \sum_{k=n_0}^n \eta_k \tilde{\Lambda}_{f,i}(X_{k-1}, \gamma_k, U_i(X_{k-1}, \Theta_i)_{\gamma_k}, \Theta_i) g_i(X_{k-1}) \mathbb{1}_{\{|X_{k-1}| \leq R\}} = 0 \quad a.s.$$

Moreover, using (12), for every $n \geq n_0$, we have $\lim_{|x| \rightarrow \infty} \tilde{\Lambda}_{f,i}(x, \gamma_n, U_i(x, \Theta_i)_{\gamma_n}, \Theta_i) = 0$ a.s.. Then, almost surely, we obtain

$$\begin{aligned} \frac{1}{H_n} \sum_{k=n_0}^n \eta_k \tilde{\Lambda}_{f,i}(X_{k-1}, \gamma_k, U_i(X_{k-1}, \Theta_i)_{\gamma_k}, \Theta_i) g_i(X_{k-1}) \mathbb{1}_{\{|X_{k-1}| > R\}} \\ \leq \sup_{|x| > R, t \in [0, \gamma_{n_0}]} |\tilde{\Lambda}_{f,i}(x, t, U_i(x, \Theta_i)_t, \Theta_i)| \sup_{n \in \mathbb{N}^*} \nu_n^\eta(g_i). \end{aligned}$$

We let R tends to infinity and since $\sup_{n \in \mathbb{N}^*} \nu_n^\eta(g_i) < \infty$, the left hand side of the above equation converges almost surely to 0. It remains to obtain the hypothesis of the Dominated Convergence Theorem. We have, for every $n \in \mathbb{N}^*$, $n \geq n_0$,

$$\begin{aligned} \mathbb{E} \left[\int_{G_i} \frac{1}{H_n} \left| \sum_{k=n_0}^n \eta_k \tilde{\Lambda}_{f,i}(X_{k-1}, \gamma_k, U_i(x, \Theta_i)_{\gamma_k}, \Theta_i) g_i(X_{k-1}) \right| \pi_i(d\Theta_i) \right] \\ \leq \sup_{x \in E, t \in [0, \gamma_{n_0}]} \mathbb{E} \left[\int_{G_i} |\tilde{\Lambda}_{f,i}(x, t, U_i(x, \Theta_i)_t, \Theta_i)| \pi_i(d\Theta_i) \right] \sup_{n \in \mathbb{N}^*} \nu_n^\eta(g_i). \end{aligned}$$

with $\sup_{x \in E, t \in [0, t_0]} \mathbb{E} \left[\int_{G_i} |\tilde{\Lambda}_{f,i}(x, t, U_i(x, \Theta_i)_t, \Theta_i)| \pi_i(d\Theta_i) \right] < \infty$. Consequently, it follows from the Dominated Convergence Theorem that

$$\lim_{n \rightarrow \infty} \frac{1}{H_n} \sum_{k=1}^n \eta_k \tilde{A}_k^\gamma f(X_{k-1}) - \nu_n^\eta A f = 0.$$

It follows that almost surely $\lim_{n \rightarrow \infty} \nu_n^\eta A f = 0$ for every $f \in \mathcal{D}(A)$. Since Af belongs to $\mathcal{C}_0(E)$, we obtain that $\nu_\infty^\eta(Af) = 0$ a.s. for any limit point ν_∞^η of the sequence ν_n^η and the conclusion follows from the Echeverria Weiss theorem (see Theorem 2.1). \square

3.2.3 General approach to prove (14) and (16)

Through Theorem 3.1 and Theorem 3.2, we showed how to obtain existence of limit points for $(\nu_n^\eta)_{n \in \mathbb{N}^*}$ and how to identify these limit points to invariant measures for A . This simply requires a recursive control assumption, an infinitesimal approximation hypothesis and also (14) and (16). This section is dedicated to the study of (14) and (16). These assumptions are not trivial to verify in most cases but a general approach can be used. We present it from here.

Let $\rho \in (1, 2]$ and an increasing function $\epsilon_{\mathcal{I}} : \mathbb{R}_+ \rightarrow \mathbb{R}_+$. For $f, g : E \rightarrow \mathbb{R}$ and $(\mathfrak{X}_n)_{n \in \mathbb{N}}$ a sequence of random variables with $\mathfrak{X}_n \in \sigma(X_i, i \in \{0, \dots, n\})$ for every $n \in \mathbb{N}$, we assume that there exists $n_0 \in \mathbb{N}$, and $C > 0$ such that, for every $n \geq n_0$,

$$\tilde{\mathcal{I}}_X(f, g, \mathfrak{X}, \rho, \epsilon_{\mathcal{I}}) \quad \mathbb{E}[|f(X_{n+1}) - \mathfrak{X}_n|^\rho | X_n] \leq C \epsilon_{\mathcal{I}}(\gamma_{n+1}) |g(X_n)|. \quad (18)$$

and

$$\mathcal{S}W_{\mathcal{I}, \gamma, \eta}(g, \rho, \epsilon_{\mathcal{I}}) \quad \sum_{n=1}^{\infty} \left| \frac{\eta_n}{H_n \gamma_n} \right|^\rho \epsilon_{\mathcal{I}}(\gamma_n) \mathbb{E}[|g(X_n)|] < \infty. \quad (19)$$

and we will also use the notation $\mathcal{S}W_{\mathcal{I}, \gamma, \eta}(g, \rho, \epsilon_{\mathcal{I}})$. We will also use the hypothesis

$$\mathcal{S}W_{\mathcal{I}\mathcal{I}, \gamma, \eta}(f) \quad \sum_{n=0}^{\infty} \frac{(\eta_{n+1}/\gamma_{n+1} - \eta_n/\gamma_n)_+}{H_{n+1}} \mathbb{E}[|f(X_n)|] < \infty. \quad (20)$$

with the convention $\eta_0/\gamma_0 = 1$. One notices that this last assumption holds as soon as the sequence $(\eta_n/\gamma_n)_{n \in \mathbb{N}^*}$ is non increasing. We propose a first result which enlightens the interest of these hypothesis and will be of particular interest when $f = (\psi \circ V)^{1/s}$ in the study of the tightness and when $f \in \mathcal{D}(A)$ for the identification part.

Lemma 3.1. *Let $\rho \in (1, 2]$, $g : E \rightarrow \mathbb{R}_+$, $f : E \rightarrow \mathbb{R}$, such that $\tilde{A}_n^\gamma f$ exists for every $n \in \mathbb{N}^*$, $\epsilon_{\mathcal{I}} : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ an increasing function and $(\mathfrak{X}_n)_{n \in \mathbb{N}}$ a sequence of random variables with $\mathfrak{X}_n \in \sigma(X_i, i \in \{0, \dots, n\})$ for every $n \in \mathbb{N}$*

We assume $\tilde{\mathcal{I}}_X(f, g, \mathfrak{X}, \rho, \epsilon_{\mathcal{I}})$ (see (18)) and $\mathcal{S}W_{\mathcal{I}, \gamma, \eta}(g, \rho, \epsilon_{\mathcal{I}})$ (see (19)) hold. We have the following properties

A. *If $f : E \rightarrow \mathbb{R}_+$ and $\mathcal{S}W_{\mathcal{I}\mathcal{I}, \gamma, \eta}(f)$ (see (20)) holds, then*

$$\mathbb{P}\text{-a.s.} \quad \sup_{n \in \mathbb{N}^*} -\frac{1}{H_n} \sum_{k=1}^n \eta_k \tilde{A}_k^\gamma f(X_{k-1}) < \infty. \quad (21)$$

B. *If f is bounded and*

$$\lim_{n \rightarrow \infty} \frac{1}{H_n} \sum_{k=1}^n |\eta_{k+1}/\gamma_{k+1} - \eta_k/\gamma_k| = 0, \quad (22)$$

Then

$$\mathbb{P}\text{-a.s.} \quad \lim_{n \rightarrow \infty} \frac{1}{H_n} \sum_{k=1}^n \eta_k \tilde{A}_k^\gamma f(X_{k-1}) = 0 \quad (23)$$

Proof. We write

$$-\sum_{k=1}^n \eta_k \tilde{A}_k^\gamma f(X_{k-1}) = -\sum_{k=1}^n \frac{\eta_k}{\gamma_k} (f(X_k) - f(X_{k-1})) + \sum_{k=1}^n \frac{\eta_k}{\gamma_k} (f(X_k) - \mathcal{Q}_k^\gamma f(X_{k-1}))$$

We study the first term of the right hand side. First we write

$$-\frac{1}{H_n} \sum_{k=1}^n \frac{\eta_k}{\gamma_k} (f(X_k) - f(X_{k-1})) = \frac{\eta_1}{H_n \gamma_1} f(X_0) - \frac{\eta_n}{H_n \gamma_n} f(X_n) + \frac{1}{H_n} \sum_{k=2}^n \left(\frac{\eta_k}{\gamma_k} - \frac{\eta_{k-1}}{\gamma_{k-1}} \right) f(X_{k-1}).$$

First, we assume that $f : E \rightarrow \mathbb{R}_+$ and $\mathcal{SW}_{\mathcal{I}, \gamma, \eta}(f)$ (see (20)) holds. From $\mathcal{SW}_{\mathcal{I}, \gamma, \eta}(f)$ (see (20)) together with Kronecker's lemma, we obtain

$$\lim_{n \rightarrow \infty} \frac{1}{H_n} \sum_{k=2}^n \left(\frac{\eta_k}{\gamma_k} - \frac{\eta_{k-1}}{\gamma_{k-1}} \right)_+ \mathbb{E}[f(X_{k-1})] = 0,$$

and since f is positive, we deduce that

$$\sup_{n \in \mathbb{N}} -\frac{1}{H_n} \sum_{k=1}^n \frac{\eta_k}{\gamma_k} (f(X_k) - f(X_{k-1})) < \infty \quad a.s.$$

Now, when f is bounded, we deduce from (22), that $\lim_{n \rightarrow \infty} \eta_n / (H_n \gamma_n) = 0$ and

$$\lim_{n \rightarrow \infty} \frac{1}{H_n} \sum_{k=1}^n \frac{\eta_k}{\gamma_k} (f(X_k) - f(X_{k-1})) = 0 \quad a.s.$$

This concludes the study of the first term and now we focus on the second one. From Kronecker lemma, it remains to prove the almost sure convergence towards zero of the martingale $(M_n)_{n \in \mathbb{N}^*}$ such that $M_0 := 0$ and for every $n \in \mathbb{N}^*$,

$$M_n := \sum_{k=1}^n \frac{\eta_k}{\gamma_k H_k} (f(X_k) - \mathcal{Q}_k^\gamma f(X_{k-1})).$$

From the Chow's theorem, this convergence will be a consequence of the finiteness of the series

$$\sum_{k=1}^n \left(\frac{\eta_k}{\gamma_k H_k} \right)^\rho \mathbb{E}[|f(X_k) - \mathcal{Q}_k^\gamma f(X_{k-1})|^\rho].$$

Moreover

$$\begin{aligned} \mathbb{E}[|f(X_k) - \mathcal{Q}_k^\gamma f(X_{k-1})|^\rho | X_{k-1}]^{1/\rho} &\leq \mathbb{E}[|f(X_k) - \mathfrak{X}_{k-1}|^\rho | X_{k-1}]^{1/\rho} \\ &\quad + \mathbb{E}[|\mathfrak{X}_{k-1} - \mathcal{Q}_k^\gamma f(X_{k-1})|^\rho | X_{k-1}]^{1/\rho} \end{aligned}$$

with

$$\begin{aligned} \mathbb{E}[|\mathfrak{X}_{k-1} - \mathcal{Q}_k^\gamma f(X_{k-1})|^\rho | X_{k-1}] &\leq \mathbb{E}[\mathbb{E}[|\mathfrak{X}_{k-1} - f(X_k)|^\rho | \mathcal{F}_{k-1}] | X_{k-1}] \\ &\leq \mathbb{E}[|f(X_k) - \mathfrak{X}_{k-1}|^\rho | X_{k-1}] \end{aligned}$$

We conclude using $\mathcal{I}_X(f, g, \mathfrak{X}, s, \epsilon_{\mathcal{I}})$ (see (18)) with $\mathcal{SW}_{\mathcal{I}, \gamma, \eta}(g, s, \epsilon_{\mathcal{I}})$ (see (19)). \square

The following Lemma presents a L_1 -finiteness property that we can obtain under recursive control hypothesis and strong mean reverting assumption ($\phi = I_d$). This result is thus useful to prove $\mathcal{SW}_{\mathcal{I}, \gamma, \eta}(g, \rho, \epsilon_{\mathcal{I}})$ (see (19)) or $\mathcal{SW}_{\mathcal{I}, \gamma, \eta}(f)$ (see (20)) for well chosen functions f and g in this particular situation.

Lemma 3.2. *Let $v_* > 0$, $V : E \rightarrow [v_*, \infty)$, $\psi : [v_*, \infty) \rightarrow \mathbb{R}_+$, such that $\tilde{A}_n^\gamma \psi \circ V$ exists for every $n \in \mathbb{N}$. We assume that $\mathcal{I}_{Q,V}(\psi, I_d)$ (see (19)) hold and that $\mathbb{E}[\psi \circ V(X_{n_0})] < \infty$ for every $n_0 \in \mathbb{N}^*$. Then*

$$\sup_{n \in \mathbb{N}} \mathbb{E}[\psi \circ V(X_n)] < \infty \quad (24)$$

Proof. First, we deduce from (8) that there exists $n_0 \in \mathbb{N}$ such that for $n \geq n_0$, $\mathcal{I}_{Q,V}(\psi, I_d)$ can be rewritten

$$\begin{aligned} \mathbb{E}[\psi \circ V(X_{n+1})|X_n] &\leq \psi \circ V(X_n)(1 - \gamma_{n+1}\alpha V^{-1}(X_n)V(X_n)) + \gamma_{n+1}\beta\psi \circ V(X_n)V^{-1}(X_n) \\ &\leq \psi \circ V(X_n)(1 - \gamma_{n+1}(1 - \lambda)\alpha) + \gamma_{n+1}C\lambda. \end{aligned}$$

Applying a simple induction we deduce that $\mathbb{E}[\psi \circ V(X_n)] \leq \mathbb{E}[\psi \circ V(X_{n_0})] \vee \frac{C\lambda}{(1-\lambda)\alpha}$. \square

The same results holds if with assume only that $\mathbb{E}[\psi \circ V(X_{n_0})] < \infty$ for the same n_0 as in $\mathcal{I}_{Q,V}(\psi, I_d)$ (see (19)). Now, we provide a general way to obtain $\mathcal{SW}_{\mathcal{I},\gamma,\eta}(g, \rho, \epsilon_{\mathcal{I}})$ and $\mathcal{SW}_{\mathcal{II},\gamma,\eta}(f)$ for some specific g and f as soon as a recursive control hypothesis hold but without making strong mean reversion assumptions.

Lemma 3.3. *Let $v_* > 0$, $V : E \rightarrow [v_*, \infty)$, $\psi, \phi : [v_*, \infty) \rightarrow \mathbb{R}_+$, such that $\tilde{A}_n^\gamma \psi \circ V$ exists for every $n \in \mathbb{N}$. We also introduce the non increasing sequence $(\Theta_n)_{n \in \mathbb{N}^*}$ such that $\sum_{n \geq 1} \Theta_n \gamma_n < \infty$. We assume that $\mathcal{I}_{Q,V}(\psi, \phi)$ (see (8)) hold. Then*

$$\sum_{n=1}^{\infty} \Theta_n \gamma_n \mathbb{E}[V^{-1}(X_{n-1})\phi \circ V(X_{n-1})\psi \circ V(X_{n-1})] < \infty \quad (25)$$

In particular, let $\rho \in (1, 2]$ and an increasing function $\epsilon_{\mathcal{I}} : \mathbb{R}_+ \rightarrow \mathbb{R}_+$. If, we also assume

$$\mathcal{SW}_{\mathcal{I},\gamma,\eta}(\rho, \epsilon_{\mathcal{I}}) \quad \left(\gamma_n^{-1} \epsilon_{\mathcal{I}}(\gamma_n) \left(\frac{\eta_n}{H_n \gamma_n} \right)^\rho \right)_{n \in \mathbb{N}^*} \text{ is non increasing and } \sum_{n=1}^{\infty} \left(\frac{\eta_n}{H_n \gamma_n} \right)^\rho \epsilon_{\mathcal{I}}(\gamma_n) < \infty, \quad (26)$$

then we have $\mathcal{SW}_{\mathcal{I},\gamma,\eta}(V^{-1}\phi \circ V\psi \circ V, \rho, \epsilon_{\mathcal{I}})$ (see (19)). Moreover, if

$$\mathcal{SW}_{\mathcal{II},\gamma,\eta} \quad \left(\gamma_n^{-1} \frac{(\eta_{n+1}/\gamma_{n+1} - \eta_n/\gamma_n)_+}{H_n} \right)_{n \in \mathbb{N}^*} \text{ is non increasing and } \sum_{n=1}^{\infty} \frac{(\eta_{n+1}/\gamma_{n+1} - \eta_n/\gamma_n)_+}{H_n} < \infty, \quad (27)$$

then we have $\mathcal{SW}_{\mathcal{II},\gamma,\eta}(V^{-1}\phi \circ V\psi \circ V)$ (see (20)).

Proof. There exists $n_0 \in \mathbb{N}$ such that for $n \geq n_0$, $\mathcal{I}_{Q,V}(\psi, \phi)$ can be rewritten

$$\gamma_{n+1}V^{-1}(X_n)\psi \circ V(X_n)(\phi \circ V(X_n) - \beta/\alpha) \leq \frac{\psi \circ V(X_n) - \mathbb{E}[\psi \circ V(X_{n+1})|X_n]}{\alpha}.$$

Using (8), and since the sequence $(\Theta_n)_{n \in \mathbb{N}^*}$ is non increasing, we obtain a telescopic decomposition as follows

$$\begin{aligned} \Theta_{n+1}\gamma_{n+1}V^{-1}(X_n)\psi \circ V(X_n)\phi \circ V(X_n) &\leq \Theta_{n+1} \frac{\psi \circ V(X_n) - \mathbb{E}[\psi \circ V(X_{n+1})|X_n]}{\alpha(1-\lambda)} + \gamma_{n+1}\Theta_{n+1}C\lambda/\alpha \\ &\leq \frac{\Theta_n\psi \circ V(X_n) - \Theta_{n+1}\mathbb{E}[\psi \circ V(X_{n+1})|X_n]}{\alpha(1-\lambda)} + \gamma_{n+1}\Theta_{n+1} \frac{C\lambda}{\alpha(1-\lambda)}. \end{aligned}$$

Taking expectancy and summing for every $n \geq n_0$ yields the result as ψ takes positive values and $\mathbb{E}[\psi \circ V(X_{n_0})] < \infty$. The second part of the result is a consequence of Chow's theorem. \square

This result concludes the general approach to prove convergence. The next part of this paper is dedicated to show how the approach we propose is well adapted to many diverse and classical applications.

4 Applications

In this section, we apply the general approach presented above to practical cases. Before doing it, we give some standard notations and properties that will be used extensively in the sequel. First, for $\alpha \in (0, 1]$ and f a α -Hölder function we denote $[f]_\alpha = \sup_{x \neq y} |f(y) - f(x)|/|y - x|^\alpha$.

Now, let $d \in \mathbb{N}$. For any $\mathbb{R}^{d \times d}$ -valued symmetric matrix S , we define $\lambda_S := \sup\{\lambda_{S,1}, \dots, \lambda_{S,d}, 0\}$, with $\lambda_{S,i}$ the i -th eigenvalue of S .

We follow with some useful polynomial inequalities. Let $u, v \in \mathbb{R}_+$, then

$$\forall \alpha \in (0, 1), \quad (u + v)^\alpha \leq u^\alpha + v^\alpha. \quad (28)$$

$$\forall \alpha \geq 1, \quad (u + v)^\alpha \leq u^\alpha + \alpha 2^{\alpha-1} (u^{\alpha-1} v + v^\alpha). \quad (29)$$

Let $l \in \mathbb{N}^*$. We have also

$$\forall \alpha > 0, u_i \in \mathbb{R}^d, i = 1, \dots, l, \quad \left| \sum_{i=1}^l u_i \right|^\alpha \leq l^{(\alpha-1)_+} \sum_{i=1}^l |u_i|^\alpha. \quad (30)$$

We also recall the Burkholder Davies Gundy (BDG) inequality for discrete martingales. Let $p \geq 1$ and $(\tilde{M}_n)_{n \in \mathbb{N}}$ a \mathbb{R}^d -valued martingale and define $\mathcal{F}_M^n = \sigma(\tilde{M}_k, k \in \{0, \dots, n\})$. Then, there exists $C_p \geq 0$ such that

$$\mathbb{E}[|\tilde{M}_n|^p] \leq C_p \sum_{k=0}^{n-1} \mathbb{E}[|\tilde{M}_{k+1} - \tilde{M}_k|^2 | \mathcal{F}_M^k]^{p/2} \quad (31)$$

In the following, we propose some applications for three different configurations always under weak mean reverting assumptions. The first one treats the case of the Euler scheme for Markov Switching diffusions for test functions with polynomial growth. The second, we prove convergence for the Milstein scheme for test functions with polynomial or exponential growth. Finally, we consider the Euler scheme for general diffusion processes with jump and test functions with polynomial growth. For each of the three applications, we give the proof of the recursive control assumption, of the infinitesimal approximation hypothesis and also of (14) and (16). We invite the reader to refer to the previous section to see how this assumptions interact together in order to obtain convergence and to identify the limit. The reader may notice that each of these three applications are treated independently from one another and then can be read in any desired order.

4.1 The Euler scheme for a Markov Switching diffusion

In this part of the paper we study ergodic regimes for Markov switching Brownian diffusions. This study is a complement to the study made in [10]. More particularly they treat the convergence $(\nu_n^\eta)_{n \in \mathbb{N}^*}$ under strong mean reverting assumption that is $\phi = I_d$. In this paper, we do not restrict to that case and consider weak mean reverting assumption that is $\phi(y) = y^a$, $a \in (0, 1]$ for every $y \in [v_*, \infty)$. Similarly as in their study we consider polynomial test functions ψ such that $\psi(y) = y^p$, $p \geq 1$ for every $y \in [v_*, \infty)$. Nevertheless, a slight difference with this paper is that they consider only $p \geq 4$.

Now, we present the Markov switching model, its decreasing step Euler approximation and the hypothesis necessary to obtain the convergence of $(\nu_n^\eta)_{n \in \mathbb{N}^*}$ built with this Euler scheme. We consider a d -dimension Brownian motion $(W_t)_{t \geq 0}$ and $(\zeta_t)_{t \geq 0}$ a continuous time Markov chain taking values in the finite state space $\{1, \dots, M_0\}$, $M_0 \in \mathbb{N}^*$ with generator $\mathcal{Q} = (q_{z,w})_{z,w \in \{1, \dots, M_0\}}$ and independent from W . We are interested in the solution of the d dimensional stochastic equation

$$X_t = x + \int_0^t b(X_s, \zeta_s) ds + \int_0^t \sigma(X_s, \zeta_s) dW_s \quad (32)$$

where for every $z \in \{1, \dots, M_0\}$, $b(\cdot, z) : \mathbb{R}^d \rightarrow \mathbb{R}^d$ and $\sigma(\cdot, z) \rightarrow \mathbb{R}^{d \times d}$, $l \in \{1, \dots, d\}$, are locally bounded and continuous functions. We recall that $q_{z,w} \geq 0$ for $z \neq w$ and $q_{z,1} + q(z,1)$ for $z, w \in \{1, \dots, M_0\}$. The infinitesimal generator of this process is given by

$$Af(x, z) = \langle b(x, z), \nabla_x f(x, z) \rangle + \sum_{i,j=1}^d (\sigma\sigma^*)_{i,j}(x, z) \frac{\partial^2 f}{\partial x_i \partial x_j}(x, z) + \sum_{w=1}^{M_0} q_{z,w} f(x, w) \quad (33)$$

for every $(x, z) \in \mathbb{R}^d \times \{1, \dots, M_0\}$. We study the Euler scheme for this process such that for every $n \in \mathbb{N}$ and $t \in [\Gamma_n, \Gamma_{n+1}]$, we have

$$\bar{X}_t = \bar{X}_{\Gamma_n} + (t - \Gamma_n)b(\bar{X}_{\Gamma_n}, \zeta_t) + \sigma(\bar{X}_{\Gamma_n}, \zeta_t)(W_t - W_{\Gamma_n}) \quad (34)$$

We will also denote $\Delta \bar{X}_{n+1} = \bar{X}_{\Gamma_{n+1}} - \bar{X}_{\Gamma_n}$ and

$$\begin{aligned} \Delta \bar{X}_{n+1}^1 &= \gamma_{n+1} b(\bar{X}_{\Gamma_n}, \zeta_{\Gamma_n}), \\ \Delta \bar{X}_{n+1}^2 &= \sigma(\bar{X}_{\Gamma_n}, \zeta_{\Gamma_n})(W_{\Gamma_{n+1}} - W_{\Gamma_n}). \end{aligned} \quad (35)$$

and $\bar{X}_{\Gamma_{n+1}}^i = \bar{X}_{\Gamma_n} + \sum_{j=1}^i \Delta \bar{X}_{n+1}^j$. In the sequel we will use the notation $U_{n+1} = \gamma_{n+1}^{-1/2}(W_{\Gamma_{n+1}} - W_{\Gamma_n})$. Actually, we introduce a weaker assumption than Gaussian distribution for the sequence $(U_n)_{n \in \mathbb{N}^*}$. Let $q \in \mathbb{N}^*$, $p \geq 0$. We suppose that $(U_n)_{n \in \mathbb{N}^*}$ is a sequence of independent random variables such that

$$M_{\mathcal{N},q}(U) \quad \forall n \in \mathbb{N}^*, \forall \tilde{q} \in \{1, \dots, q\}, \quad \mathbb{E}[(U_n)^{\otimes \tilde{q}}] = \mathbb{E}[(\mathcal{N}(0, I_d))^{\otimes \tilde{q}}] \quad (36)$$

$$M_p(U) \quad \sup_{n \in \mathbb{N}^*} \mathbb{E}[|U_n|^p] < \infty \quad (37)$$

Now, we assume that the Lyapunov function $V : \mathbb{R}^d \times \{1, \dots, M_0\} \rightarrow [v_*, \infty)$, $v_* > 0$, satisfies \mathcal{L}_V (see (7)) with $E = \mathbb{R}^d \times \{1, \dots, M_0\}$, and

$$|\nabla_x V|^2 \leq C_V V, \quad \sup_{x \in \mathbb{R}^d, z \in \{1, \dots, M_0\}} |D_x^2 V(x, z)| < \infty \quad (38)$$

and

$$\forall z \in \{1, \dots, M_0\}, \exists c_{V,z} \geq 0, \forall x \in \mathbb{R}^d, \quad V(x, z) \leq c_{V,z} \inf_{w \in \{1, \dots, M_0\}} V(x, w) \quad (39)$$

We also define

$$\forall x \in \mathbb{R}^d, z \in \{1, \dots, M_0\}, \quad \lambda_\psi(x, z) := \frac{1}{2} \lambda_{D_x^2 V(x, z) + \nabla_x V(x, z) \otimes \psi'' \circ V(x, z) \psi' \circ V(x, z)^{-1}}. \quad (40)$$

When $\psi(y) = y^p$, we will also use the notation λ_p instead of λ_ψ . We suppose that there exists $C > 0$ such that, for every $x \in \mathbb{R}^d$, $z \in \{1, \dots, M_0\}$,

$$\mathfrak{B}(\phi) \quad |b(x, z)|^2 + |\sigma\sigma^*(x, z)| \leq C\phi \circ V(x, z) \quad (41)$$

We now introduce the key hypothesis in order to obtain recursive control for the polynomial case, that is for $p \geq 1$, we have $\psi(y) = y^p$ for every $y \in [v_*, \infty)$. We assume that there exists $\beta \in \mathbb{R}_+$, $\alpha > 0$ and $\epsilon > 0$, such that for every $x \in \mathbb{R}^d$, $z \in \{1, \dots, M_0\}$, we have

$$\mathcal{R}_p \quad \langle \nabla V(x, z), b(x, z) \rangle + \chi_p(x, z) \leq \beta - \alpha\phi \circ V(x, z), \quad (42)$$

with

$$\chi_p(x, z) = \|\lambda_p\|_\infty 2^{(2p-3)+} \text{Tr}[\sigma\sigma^*(x, z)] + V^{1-p}(x, z) \sum_{w=1}^{M_0} (q_{z,w} + \epsilon) V^p(x, w) \quad (43)$$

4.1.1 Recursive control

Proposition 4.1. *Let $v_* > 0$, and $\phi : [v_*, \infty) \rightarrow \mathbb{R}_+^*$ a continuous function such that $\phi(y) \leq Cy$ with $C > 0$. Now let $p \geq 1$ and define $\psi : [v_*, \infty) \rightarrow \mathbb{R}_+$ such that $\psi(y) = y^p$.*

We assume that the sequence $(U_n)_{n \in \mathbb{N}^}$ satisfies $M_{\mathcal{N},2}(U)$ (see (36)) and $M_{2p}(U)$ (see (37)).*

We suppose that (38), $\mathfrak{B}(\phi)$ (see (41)), \mathcal{R}_p (see (42)), are satisfied. Then, there exists $\alpha > 0$, $\beta \in \mathbb{R}_+$ and $n_0 \in \mathbb{N}^$, such that*

$$\forall n \geq n_0, x \in \mathbb{R}^d, \forall z \in \{1, \dots, M_0\}, \quad \tilde{A}_n^\gamma \psi \circ V(x, z) \leq V^{-1}(x, z) \psi \circ V(x, z) (\beta - \alpha \phi \circ V(x)). \quad (44)$$

Moreover, when $\phi = Id$ we have

$$\sup_{n \in \mathbb{N}} \mathbb{E}[\psi \circ V(\bar{X}_{\Gamma_n}, \zeta_{\Gamma_n})] < \infty. \quad (45)$$

Proof. First we write

$$\begin{aligned} V^p(\bar{X}_{\Gamma_{n+1}}, \zeta_{\Gamma_{n+1}}) - V^p(\bar{X}_{\Gamma_n}, \zeta_{\Gamma_n}) &= V^p(\bar{X}_{\Gamma_{n+1}}, \zeta_{\Gamma_n}) - V^p(\bar{X}_{\Gamma_n}, \zeta_{\Gamma_n}) \\ &\quad + V^p(\bar{X}_{\Gamma_{n+1}}, \zeta_{\Gamma_{n+1}}) - V^p(\bar{X}_{\Gamma_{n+1}}, \zeta_{\Gamma_n}) \end{aligned}$$

We study the first term. From the Taylor's formula and the definition of λ_ψ (see (40)), we have

$$\begin{aligned} \psi \circ V(\bar{X}_{\Gamma_{n+1}}, \zeta_{\Gamma_n}) &= \psi \circ V(\bar{X}_{\Gamma_n}, \zeta_{\Gamma_n}) + \langle \bar{X}_{\Gamma_{n+1}} - \bar{X}_{\Gamma_n}, \nabla_x V(\bar{X}_{\Gamma_n}, \zeta_{\Gamma_n}) \rangle \psi' \circ V(\bar{X}_{\Gamma_n}, \zeta_{\Gamma_n}) \\ &\quad + \frac{1}{2} (D^2 V(\xi_{n+1}, \zeta_{\Gamma_n}) \psi' \circ V(\xi_{n+1}, \zeta_{\Gamma_n}) + \nabla V(\xi_{n+1}, \zeta_{\Gamma_n})^2 \psi'' \circ V(\xi_{n+1}, \zeta_{\Gamma_n})) (\bar{X}_{\Gamma_{n+1}} - \bar{X}_{\Gamma_n})^{\otimes 2}. \\ &\leq \psi \circ V(\bar{X}_{\Gamma_n}, \zeta_{\Gamma_n}) + \langle \bar{X}_{\Gamma_{n+1}} - \bar{X}_{\Gamma_n}, \nabla_x V(\bar{X}_{\Gamma_n}, \zeta_{\Gamma_n}) \rangle \psi' \circ V(\bar{X}_{\Gamma_n}, \zeta_{\Gamma_n}) \\ &\quad + \lambda_\psi(\xi_{n+1}) |\psi' \circ V(\xi_{n+1}, \zeta_{\Gamma_n})| (\bar{X}_{\Gamma_{n+1}} - \bar{X}_{\Gamma_n})^2. \end{aligned}$$

with $\xi_{n+1} \in (\bar{X}_{\Gamma_n}, \bar{X}_{\Gamma_{n+1}})$. First, from (38), we have $\sup_{x \in \mathbb{R}^d} \lambda_p(x) < \infty$.

Now, since $(U_n)_{n \in \mathbb{N}^*}$ is a sequence of independent random variables satisfying $M_{\mathcal{N},1}(U)$ (see (36)), we have

$$\begin{aligned} \mathbb{E}[\bar{X}_{\Gamma_{n+1}} - \bar{X}_{\Gamma_n} | \bar{X}_{\Gamma_n}, \zeta_{\Gamma_n}] &= \gamma_{n+1} b(\bar{X}_{\Gamma_n}, \zeta_{\Gamma_n}) \\ \mathbb{E}[|\bar{X}_{\Gamma_{n+1}} - \bar{X}_{\Gamma_n}|^2 | \bar{X}_{\Gamma_n}, \zeta_{\Gamma_n}] &= \gamma_{n+1} \text{Tr}[\sigma \sigma^*(\bar{X}_{\Gamma_n}, \zeta_{\Gamma_n})] + \gamma_{n+1}^2 |b(\bar{X}_{\Gamma_n}, \zeta_{\Gamma_n})|^2 \end{aligned}$$

Assume first that $p = 1$. Using (41), for every $\tilde{\alpha} \in (0, \alpha)$, there exists $n_0(\tilde{\alpha})$ such that for every $n \geq n_0(\tilde{\alpha})$,

$$\|\lambda_1\|_\infty \gamma_{n+1}^2 |b(\bar{X}_{\Gamma_n}, \zeta_{\Gamma_n})|^2 \leq \gamma_{n+1} (\alpha - \tilde{\alpha}) \phi \circ V(\bar{X}_{\Gamma_n}, \zeta_{\Gamma_n}). \quad (46)$$

From assumption (43), we conclude that

$$\gamma_{n+1}^{-1} \mathbb{E}[V(\bar{X}_{\Gamma_{n+1}}, \zeta_{\Gamma_n}) - V(\bar{X}_{\Gamma_n}, \zeta_{\Gamma_n}) | \bar{X}_{\Gamma_n}] + \sum_{z=1}^2 (q_{\zeta_{\Gamma_n}, w} + \epsilon) V(\bar{X}_{\Gamma_n}, z) \leq \beta - \tilde{\alpha} \phi \circ V(\bar{X}_{\Gamma_n}, \zeta_{\Gamma_n})$$

Assume now that $p > 1$. Since $|\nabla V| \leq C_V V$ (see (38)), then \sqrt{V} is Lipschitz. Using (30), it follows that

$$\begin{aligned} V^{p-1}(\xi_{n+1}, \zeta_{\Gamma_n}) &\leq (\sqrt{V}(\bar{X}_{\Gamma_n}, \zeta_{\Gamma_n}) + [\sqrt{V}]_1 |\bar{X}_{\Gamma_{n+1}} - \bar{X}_{\Gamma_n}|)^{2p-2} \\ &\leq 2^{(2p-3)+} (V^{p-1}(\bar{X}_{\Gamma_n}, \zeta_{\Gamma_n}) + [\sqrt{V}]_1^{2p-2} |\bar{X}_{\Gamma_{n+1}} - \bar{X}_{\Gamma_n}|^{2p-2}) \end{aligned}$$

We focus on the study of the second term of the remainder. First, using $\mathfrak{B}(\phi)$ (see (41)), for any $p \geq 1$,

$$|\bar{X}_{\Gamma_{n+1}} - \bar{X}_{\Gamma_n}|^{2p} \leq c_p \gamma_{n+1}^p \phi \circ V(\bar{X}_{\Gamma_n}, \zeta_{\Gamma_n})^p (1 + |U_{n+1}|^{2p}).$$

Let $\hat{\alpha} \in (0, \alpha)$. Therefore, we deduce from $M_{2p}(U)$ (see (37)) that there exists $n_0(\hat{\alpha}) \in \mathbb{N}$ such that for any $n \geq n_0(\hat{\alpha})$, we have

$$\mathbb{E}[|\bar{X}_{\Gamma_{n+1}} - \bar{X}_{\Gamma_n}|^{2p} | \bar{X}_{\Gamma_n}, \zeta_{\Gamma_n}] \leq \gamma_{n+1} \phi \circ V(\bar{X}_{\Gamma_n}, \zeta_{\Gamma_n})^p \frac{\alpha - \hat{\alpha}}{\|\phi/I_d\|_\infty^{p-1} \|\lambda_p\|_\infty 2^{(2p-3)+} [\sqrt{V}]_1^{2p-2}}$$

To treat the other term we proceed as in (69) with $\|\lambda_1\|_\infty$ replaced by $\|\lambda_p\|_\infty 2^{2p-3} [\sqrt{V}]_1^{2p-2}$, α replace by $\hat{\alpha}$ and $\tilde{\alpha} \in (0, \hat{\alpha})$. We gather all the terms together and using \mathcal{R}_p (see (42) and (43)), for every $n \geq n_0(\tilde{\alpha}) \vee n_0(\hat{\alpha})$, we obtain

$$\begin{aligned} \mathbb{E}[V^p(\bar{X}_{\Gamma_{n+1}}, \zeta_{\Gamma_n}) - V^p(\bar{X}_{\Gamma_n}, \zeta_{\Gamma_n}) | \bar{X}_{\Gamma_n}, \zeta_{\Gamma_n}] + V^{1-p}(x, z) \sum_{w=1}^{M_0} (q_{z,w} + \epsilon) V^p(x, w) \\ \leq \gamma_{n+1} p V^{p-1}(\bar{X}_{\Gamma_n}, \zeta_{\Gamma_n}) (\beta - \alpha \phi \circ V(\bar{X}_{\Gamma_n}, \zeta_{\Gamma_n})) \\ + \gamma_{n+1} p V^{p-1}(\bar{X}_{\Gamma_n}, \zeta_{\Gamma_n}) (\phi \circ V(\bar{X}_{\Gamma_n}, \zeta_{\Gamma_n})) (\hat{\alpha} - \tilde{\alpha}) + (\alpha - \hat{\alpha}) \frac{V^{1-p}(\bar{X}_{\Gamma_n}, \zeta_{\Gamma_n}) \phi \circ V(\bar{X}_{\Gamma_n}, \zeta_{\Gamma_n})^p}{\|\phi/I_d\|_\infty^{p-1}} \\ \leq \gamma_{n+1} V^{p-1}(\bar{X}_{\Gamma_n}, \zeta_{\Gamma_n}) (\beta p - \tilde{\alpha} p \phi \circ V(\bar{X}_{\Gamma_n}, \zeta_{\Gamma_n})). \end{aligned}$$

Now, we focus on the second term. First, since ζ and W are independent, it follows that

$$\mathbb{E}[V^p(\bar{X}_{\Gamma_{n+1}}, \zeta_{\Gamma_{n+1}}) - V^p(\bar{X}_{\Gamma_{n+1}}, \zeta_{\Gamma_n}) | \bar{X}_{\Gamma_n}, \zeta_{\Gamma_n}, \Delta \bar{X}_{n+1}] = \gamma_{n+1} \sum_{z=1}^2 (q_{\zeta_{\Gamma_n}, z} + \underset{n \rightarrow \infty}{o}(\gamma_{n+1})) V^p(\bar{X}_{\Gamma_{n+1}}, z)$$

Now, we use the same reasoning as in the study of the first term and for every $z \in \{1, \dots, M_0\}$, we obtain

$$\begin{aligned} \mathbb{E}[V^p(\bar{X}_{\Gamma_{n+1}}, z) - V^p(\bar{X}_{\Gamma_n}, z) | \bar{X}_{\Gamma_n}, \zeta_{\Gamma_n}] \leq C (\gamma_{n+1}^{1/2} V^{p-1}(\bar{X}_{\Gamma_n}, z) \phi \circ V(\bar{X}_{\Gamma_n}, z) + \gamma_{n+1}^p \phi \circ V(\bar{X}_{\Gamma_n}, z)^p) \\ \leq C \gamma_{n+1}^{1/2} V^p(\bar{X}_{\Gamma_n}, z) \end{aligned}$$

where $C > 0$ is a constant which can change from line to line. It follows that there exists $\varepsilon : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ satisfying $\lim_{t \rightarrow 0} \varepsilon(t) = 0$, such that we have

$$\begin{aligned} \mathbb{E}[V^p(\bar{X}_{\Gamma_{n+1}}, \zeta_{\Gamma_{n+1}}) - V^p(\bar{X}_{\Gamma_{n+1}}, \zeta_{\Gamma_n}) | \bar{X}_{\Gamma_n}, \zeta_{\Gamma_n}] = \gamma_{n+1} \sum_{z=1}^{M_0} (q_{\zeta_{\Gamma_n}, z} + \underset{n \rightarrow \infty}{o}(\gamma_{n+1})) \mathbb{E}[V^p(\bar{X}_{\Gamma_{n+1}}, z) | \bar{X}_{\Gamma_n}, \zeta_{\Gamma_n}] \\ \leq \gamma_{n+1} \sum_{z=1}^{M_0} (q_{\zeta_{\Gamma_n}, z} + \varepsilon(\gamma_{n+1})) V^p(\bar{X}_{\Gamma_n}, z) \end{aligned}$$

and (44) is a direct consequence of \mathcal{R}_p (see (42) and (43)). The proof of (45) is immediate application of Lemma 3.2 as soon as we notice that the increments of the Euler scheme (for Markov Switching diffusions) have finite polynomial moments which implies (24). \square

4.1.2 Infinitesimal control

Proposition 4.2. *We suppose that the sequences $(U_n)_{n \in \mathbb{N}^*}$ satisfies $M_{\mathcal{N}, 2}(U)$ (see (36)), $M_2(U)$ (see (37)). We also assume that b and σ are locally bounded functions, that ϕ has sublinear growth, that $\mathfrak{B}(\phi)$ (see (41)) holds and that $\sup_{n \in \mathbb{N}^*} \nu_n^n(|\sigma|^2) < \infty$. Then, we have $\mathcal{E}(\tilde{A}^\gamma, A)$ (see (9)).*

Proof. First we recall that $\mathcal{D}(A) = \mathcal{C}_K^2(\mathbb{R}^d)$ and we write

$$\begin{aligned} f(\bar{X}_{\Gamma_{n+1}}, \zeta_{\Gamma_{n+1}}) - f(\bar{X}_{\Gamma_n}, \zeta_{\Gamma_n}) = f(\bar{X}_{\Gamma_{n+1}}, \zeta_{\Gamma_n}) - f(\bar{X}_{\Gamma_n}, \zeta_{\Gamma_n}) \\ + f(\bar{X}_{\Gamma_{n+1}}, \zeta_{\Gamma_{n+1}}) - f(\bar{X}_{\Gamma_{n+1}}, \zeta_{\Gamma_n}). \end{aligned}$$

Since W and ζ are independent, we have

$$\mathbb{E}[f(\bar{X}_{\Gamma_{n+1}}, \zeta_{\Gamma_{n+1}}) - f(\bar{X}_{\Gamma_{n+1}}, \zeta_{\Gamma_n}) | \bar{X}_{\Gamma_n}, \zeta_{\Gamma_n}, \Delta \bar{X}_{n+1}] = \gamma_{n+1} \sum_{z=1}^{M_0} (q_{\zeta_{\Gamma_n}, z} + \underset{n \rightarrow \infty}{o}(\gamma_{n+1})) f(\bar{X}_{\Gamma_{n+1}}, z)$$

Using Taylor expansions of order one and two, for every $z \in \{1, \dots, M_0\}$ and the fact that U_{n+1} is centered, we obtain

$$\begin{aligned} & \mathbb{E}[f(\bar{X}_{\Gamma_{n+1}}, z) - f(\bar{X}_{\Gamma_n}, z) | \bar{X}_{\Gamma_n} = x, \zeta_{\Gamma_n}] \\ &= \mathbb{E}[f(\bar{X}_{\Gamma_n} + \Delta \bar{X}_{n+1}^1, z) - f(\bar{X}_{\Gamma_n}, z) | \bar{X}_{\Gamma_n} = x, \zeta_{\Gamma_n}] \\ &+ \mathbb{E}[f(\bar{X}_{\Gamma_{n+1}}, z) - f(\bar{X}_{\Gamma_n} + \Delta \bar{X}_{n+1}^1, z) | \bar{X}_{\Gamma_n} = x, \zeta_{\Gamma_n}] \\ &\leq \int_0^1 |\nabla_x f(x + \theta b(x, \zeta_{\Gamma_n}) \gamma_{n+1}, z)| |b(x, \zeta_{\Gamma_n}) \gamma_{n+1}| d\theta \\ &+ \int_0^1 |D_x^2 f(x + b(x, \zeta_{\Gamma_n}) \gamma_{n+1} + \theta \sigma(x, \zeta_{\Gamma_n}) \sqrt{\gamma_{n+1}} v, z)| |\sigma(x, \zeta_{\Gamma_n}) \sqrt{\gamma_{n+1}} v|^2 d\theta p_U(dv). \end{aligned}$$

where p_N denotes the density of the centered Gaussian random variable taking values in \mathbb{R}^d of which covariance matrix is the identity matrix. Combining the two last inequalities, we obtain,

$$\begin{aligned} \gamma_{n+1}^{-1} \mathbb{E}[f(\bar{X}_{\Gamma_{n+1}}, \zeta_{\Gamma_{n+1}}) - f(\bar{X}_{\Gamma_{n+1}}, \zeta_{\Gamma_n}) | \bar{X}_{\Gamma_n}, \zeta_{\Gamma_n}] &\leq \sum_{z=1}^{M_0} q_{\zeta_{\Gamma_n}, z} f(\bar{X}_{\Gamma_n}, z) + o_{n \rightarrow \infty}(\gamma_{n+1}) \|f\|_\infty \\ &+ \sum_{z=1}^{M_0} (|q_{\zeta_{\Gamma_n}, z}| + o_{n \rightarrow \infty}(\gamma_{n+1})) (\Lambda_{f,1}(\bar{X}_{\Gamma_n}, \zeta_{\Gamma_n}, \gamma_{n+1}) |b(\bar{X}_{\Gamma_n}, \zeta_{\Gamma_n})| + \Lambda_{f,2}(\bar{X}_{\Gamma_n}, \zeta_{\Gamma_n}, \gamma_{n+1}) |\sigma \sigma^*(\bar{X}_{\Gamma_n}, \zeta_{\Gamma_n})|). \end{aligned}$$

Now we define $E = \mathbb{R}^d \times \{1, \dots, M_0\}$, $G_1 = [0, 1]$, $\Theta_1 = \theta$, π_1 the measure defined on $(G_1, \mathcal{B}(G_1))$ (with $\mathcal{B}(G_1)$ the sigma fields endowed by the Borelians of G_1) by $\pi(d\Theta_1) = d\Theta_1$ (that is the Lebesgue measure), and for every $(x, z) \in \mathbb{R}^d \times \{1, \dots, M_0\} = E$, we have $\Lambda_{f,1}(x, z, \gamma) = \int_{G_1} \tilde{\Lambda}_{f,1}(x, z, \gamma, \Theta_1) \pi_1(d\Theta_1)$,

with

$$\begin{aligned} \tilde{\Lambda}_{f,1} : \mathbb{R}^d \times \{1, \dots, M_0\} \times \mathbb{R}_+ \times [0, 1] &\rightarrow \mathbb{R}_+ \\ (x, z, \gamma, \theta) &\mapsto \sum_{w=1}^{M_0} \nabla_x f(x + \theta b(x, z) \gamma, w) |\gamma|, \end{aligned}$$

and $g_1(x, z) = |b(x, z)|$. We are going to prove that $\mathcal{E}_{loc}(\tilde{\Lambda}_{f,1}, g_1)$ (see (10) and (11)) holds. Using (41) and the fact that $\phi(x) \leq C|x|$, the functions b have sublinear growth: there exists $C_b \geq 0$ such that $|b(x, z)| \leq C_b(1 + |x|)$ for every $x \in \mathbb{R}^d$ and $z \in \{1, \dots, M_0\}$. Therefore, since f has compact support, it follows that there exists $t_0 > 0$ and $R > 0$ such that $\sup_{|x| > R, z \in \{1, \dots, M_0\}} \sup_{\gamma \leq t_0} |\tilde{\Lambda}_{f,1}(x, z, \gamma, \theta)| = 0$ for every $\theta \in [0, 1]$. Moreover since $\nabla_x f$ is bounded and b is locally bounded, we conclude that $\tilde{\Lambda}_{f,1}$, satisfies $\mathcal{E}_{loc}(\tilde{\Lambda}_{f,1}, g_1)$ (see (10) and (11)).

We focus on the other term. We define $G_2 = \mathbb{R}^d \times [0, 1]$, $\Theta_2 = (v, \theta)$, π_2 the measure defined on $(G_2, \mathcal{B}(G_2))$ (with $\mathcal{B}(G_2)$ the sigma fields endowed by the Borelians of G_2) by $\pi_2(d\Theta_2) = d\theta p_U(dv)$, and for every $(x, z) \in \mathbb{R}^d \times \{1, \dots, M_0\} = E$, we have $\Lambda_{f,2}(x, z, \gamma) = \int_{G_2} \tilde{\Lambda}_{f,2}(x, z, \gamma, \Theta_2) \pi_2(d\Theta_2)$, with

$$\begin{aligned} \tilde{\Lambda}_{f,2} : \mathbb{R}^d \times \{1, \dots, M_0\} \times \mathbb{R}_+ \times \mathbb{R}^N \times [0, 1] &\rightarrow \mathbb{R}_+ \\ (x, z, \gamma, v, \theta) &\mapsto \sum_{w=1}^{M_0} D_x^2 f(x + b(x, z) \gamma + \theta \sigma(x, z) \sqrt{\gamma} v, w) |\sqrt{\gamma} v|^2, \end{aligned}$$

and $g_2(x, z) = |\sigma(x, z)|^2$. We are going to prove that $\mathcal{E}_{ergo}(\tilde{\Lambda}_{f,2}, g_2)$ (see (12)) holds. We fix $v \in \mathbb{R}^N$ and $\theta \in [0, 1]$. Now using (41) and the fact that $\phi(x) \leq C|x|$, the functions b and σ , have sublinear growth: there exists $C_{b,\sigma} \geq 0$ such that $|b(x, z)| + |\sigma(x, z)| \leq C_{b,\sigma}(1 + |x|)$ for every $x \in \mathbb{R}^d$ and $z \in \{1, \dots, M_0\}$. Therefore, since f has compact support, it follows that there exists $t_0 > 0$ and $R > 0$ such that $\sup_{|x| > R, z \in \{1, \dots, M_0\}} \sup_{\gamma \leq t_0} |\tilde{\Lambda}_{f,2}(x, z, \gamma, v, \theta)| = 0$. Moreover since $D_x^2 f$ is bounded and b and σ are locally bounded, we conclude that we have $\mathcal{E}_{ergo}(\tilde{\Lambda}_{f,2}, g_2)$ (see (12)).

Besides, it is immediate to show that $\mathcal{E}_{ergo}(o_{n \rightarrow \infty}(\gamma_{n+1}) \|f\|_\infty, 1)$ (see (13)) holds.

Finally, it remains to study $\mathbb{E}[f(\bar{X}_{\Gamma_{n+1}}, \zeta_{\Gamma_n}) - f(\bar{X}_{\Gamma_n}, \zeta_{\Gamma_n}) | \bar{X}_{\Gamma_n}, \zeta_{\Gamma_n}]$. Using once again Taylor expansions of order one and two, we have

$$\begin{aligned} \gamma_{n+1}^{-1} \mathbb{E}[f(\bar{X}_{\Gamma_{n+1}}, \zeta_{\Gamma_n}) - f(\bar{X}_{\Gamma_n}, \zeta_{\Gamma_n}) | \bar{X}_{\Gamma_n} = x, \zeta_{\Gamma_n} = z] &= \langle \nabla_x f(x, z), b(x, z) \rangle - \sum_{i,j=1}^d (\sigma \sigma^*)_{i,j}(x, z) \frac{\partial^2 f}{\partial x_i \partial x_j}(x, z) \\ &\leq \int_0^1 |\nabla_x f(x + \theta b(x, z) \gamma_{n+1}, z) - \nabla_x f(x)| |b(x, z)| d\theta \\ &\quad + \int_0^1 |D_x^2 f(x + b(x, z) \gamma_{n+1} + \theta \sigma(x, z) \sqrt{\gamma_{n+1}} v, z) - D_x^2 f(x)| |\sigma(x, z) v|^2 d\theta p_U(dv). \end{aligned}$$

Using the same reasoning as before, one can show that $\mathcal{E}_{loc}(\tilde{\Lambda}_{f,3}, g_1)$ (see (10) and (11)) and $\mathcal{E}_{ergo}(\tilde{\Lambda}_{f,4}, g_2)$ (see (12)) hold with

$$\begin{aligned} \tilde{\Lambda}_{f,3} : \mathbb{R}^d \times \{1, \dots, M_0\} \times \mathbb{R}_+ \times [0, 1] &\rightarrow \mathbb{R}_+ \\ (x, z, \gamma, \theta) &\mapsto |\nabla_x f(x + \theta b(x, z) \gamma, z) - \nabla_x f(x, z)|, \end{aligned}$$

and

$$\begin{aligned} \tilde{\Lambda}_{f,4} : \mathbb{R}^d \times \{1, \dots, M_0\} \times \mathbb{R}_+ \times \mathbb{R}^N \times [0, 1] &\rightarrow \mathbb{R}_+ \\ (x, z, \gamma, v, \theta) &\mapsto |D_x^2 f(x + b(x, z) \gamma + \theta \sigma(x, z) \sqrt{\gamma} v, z) - D_x^2 f(x)| |v|^2 \end{aligned}$$

We gather all the terms together and the result follows. \square

4.1.3 Proof of (14) and (16)

Proposition 4.3. *Let $p \geq 1, a \in (0, 1], \rho, s \in (1, 2]$ and, $\psi(y) = y^p, \phi(y) = y^a$ and $\epsilon_{\mathcal{I}}(t) = t^{\rho/2}$. We suppose that the sequence $(U_n)_{n \in \mathbb{N}^*}$ satisfies $M_{N,2}(U)$ (see (36)) and $M_{2p}(U)$ (see (37)).*

We also assume that (38), $\mathfrak{B}(\phi)$ (see (41)) and \mathcal{R}_p (see (42)) and $\mathcal{SW}_{\mathcal{I}, \gamma, \eta}(\rho, \epsilon_{\mathcal{I}})$ (see (26)) hold. Then $\mathcal{SW}_{\mathcal{I}, \gamma, \eta}(V^{p+a-1}, \rho, \epsilon_{\mathcal{I}})$ (see (19)) hold and we have the following properties

A. *If $\mathcal{SW}_{\mathcal{II}, \gamma, \eta}$ (see (27)) and (39) also hold and $\rho \leq s(1 - (1 - a)/p)$, then we have $\mathcal{SW}_{\mathcal{II}, \gamma, \eta}(V^{p/s})$ (see (20)) and*

$$\mathbb{P}\text{-a.s.} \quad \sup_{n \in \mathbb{N}^*} -\frac{1}{H_n} \sum_{k=1}^n \eta_k \tilde{A}_k^\gamma (\psi \circ V)^s(\bar{X}_{\Gamma_{k-1}}, \zeta_{\Gamma_{k-1}}) < \infty. \quad (47)$$

Moreover,

$$\mathbb{P}\text{-a.s.} \quad \sup_{n \in \mathbb{N}^*} \nu_n^\eta(V^{p/s+a-1}) < \infty, \quad (48)$$

and that if \mathcal{L}_V (see (7)) holds, then $(\nu_n^\eta)_{n \in \mathbb{N}^*}$ is tight

B. *If $f \in \mathcal{D}(A)$ and (22) also holds, then*

$$\mathbb{P}\text{-a.s.} \quad \lim_{n \rightarrow \infty} \frac{1}{H_n} \sum_{k=1}^n \eta_k \tilde{A}_k^\gamma f(\bar{X}_{\Gamma_{k-1}}, \zeta_{\Gamma_{k-1}}) = 0 \quad (49)$$

Proof. The result is an immediate consequence of Lemma 3.1. It remains to check the assumption of this Lemma. First, we show $\mathcal{SW}_{\mathcal{I}, \gamma, \eta}(V^{p+a-1}, \rho, \epsilon_{\mathcal{I}})$ (see (19)). Since (38), $\mathfrak{B}(\phi)$ (see (41)) and \mathcal{R}_p (see (42)) hold, it follows from Proposition 4.1 that $\mathcal{I}_{Q,V}(\psi, \phi)$ (see (8)) holds. Then, using $\mathcal{SW}_{\mathcal{I}, \gamma, \eta}(\rho, \epsilon_{\mathcal{I}})$ (see (26)) with Lemma 3.3 gives $\mathcal{SW}_{\mathcal{I}, \gamma, \eta}(V^{p+a-1}, \rho, \epsilon_{\mathcal{I}})$ (see (19)). In the same way, since $\rho \leq s(1 - (1 - a)/p)$, we deduce from $\mathcal{SW}_{\mathcal{II}, \gamma, \eta}$ (see (27)) and Lemma 3.3 that $\mathcal{SW}_{\mathcal{II}, \gamma, \eta}(V^{p/s})$ (see (20)) holds. Now, we are going to prove $\tilde{\mathcal{I}}_X(f, V^{a+p-1}, \mathfrak{X}, \rho, \epsilon_{\mathcal{I}})$ (see (18)) for $f \in \mathcal{D}(A)$ and $f = V^{p/s}$ and the proof of (47) and (49) will be completed. Notice that (48) will follow from $\mathcal{I}_{Q,V}(\psi, \phi)$ (see (8)) and Theorem 3.1. This is a consequence of Lemma 4.1 which is given below. We notice indeed that Lemma 4.1 and the fact that under $\mathfrak{B}(\phi)$ (see (41)) and $p \geq 1$, we have $|\sigma \sigma^*| \leq CV^{p+a-1}$, imply that for every $f \in \mathcal{D}(A)$ and $f = V^{p/s}$, there exists a sequence \mathfrak{X} , such that $\tilde{\mathcal{I}}_X(f, V^{a+p-1}, \mathfrak{X}, \rho, \epsilon_{\mathcal{I}})$ (see (18)) holds and the proof is completed. \square

Lemma 4.1. *Let $p \geq 1, a \in (0, 1], \rho \in (1, 2]$ and, $\psi(y) = y^p$ and $\phi(y) = y^a$. We suppose that the sequence $(U_n)_{n \in \mathbb{N}^*}$ satisfies $M_{2pp/s}(U)$ (see (37)). Then, for every $n \in \mathbb{N}$, we have*

$$\forall f \in \mathcal{D}(A), \quad \mathbb{E}[|f(\bar{X}_{\Gamma_{n+1}}, \zeta_{\Gamma_{n+1}}) - f(\bar{X}_{\Gamma_n}^1, \zeta_{\Gamma_n})|^\rho | \bar{X}_{\Gamma_n}, \zeta_{\Gamma_n}] \leq C \gamma_{n+1}^{\rho/2} 1_V |\sigma \sigma^*(\bar{X}_{\Gamma_n}, \zeta_{\Gamma_n})|^{\rho/2}. \quad (50)$$

with $\mathcal{D}(A) = \{f : \mathbb{R}^d \times \{1, \dots, M_0\}, \forall z \in \{1, \dots, M_0\}, f(\cdot, z) \in \mathcal{C}_K^2(\mathbb{R}^d)\}$. In other words, for every $f \in \mathcal{D}(A)$, we have $\tilde{\mathcal{I}}_X(f, |\sigma \sigma^*|^{\rho/2}, \mathfrak{X}, \rho, \epsilon_{\mathcal{I}})$ (see (18)) with $\mathfrak{X}_n = f(\bar{X}_{\Gamma_n}^1, \zeta_{\Gamma_n})$ for every $n \in \mathbb{N}$ and $\epsilon_{\mathcal{I}}(t) = t^{\rho/2}$ for every $t \in \mathbb{R}_+$.

Moreover, if (38), (39) and $\mathfrak{B}(\phi)$ (see (41)) hold and $\rho \leq s(1 - (1 - a)/p)$, then, for every $n \in \mathbb{N}$, we have

$$\mathbb{E}[|V^{p/s}(\bar{X}_{\Gamma_{n+1}}, \zeta_{\Gamma_{n+1}}) - V^{p/s}(\bar{X}_{\Gamma_n}, \zeta_{\Gamma_n})|^\rho | \bar{X}_{\Gamma_n}, \zeta_{\Gamma_n}] \leq C \gamma_{n+1}^{\rho/2} V^{p+a-1}(\bar{X}_{\Gamma_n}, \zeta_{\Gamma_n}), \quad (51)$$

In other words, we have $\tilde{\mathcal{I}}_X(V^{p/s}, V^{p+a-1}, \mathfrak{X}, \rho, \epsilon_{\mathcal{I}})$ (see (18)) with $\mathfrak{X}_n = (\psi \circ V)^{1/s}(\bar{X}_{\Gamma_n}, \zeta_{\Gamma_n})$ for every $n \in \mathbb{N}$ and $\epsilon_{\mathcal{I}}(t) = t^{\rho/2}$ for every $t \in \mathbb{R}_+$.

Proof. We begin by noticing that

$$|\bar{X}_{\Gamma_{n+1}} - \bar{X}_{\Gamma_n}^1| \leq C \gamma_{n+1}^{1/2} |\sigma \sigma^*(\bar{X}_{\Gamma_n}, \zeta_{\Gamma_n})|^{1/2} |U_{n+1}|$$

Let $f \in \mathcal{D}(A)$. We employ this estimation and since for $f \in \mathcal{D}(A)$ then, for every $z \in \{1, \dots, M_0\}$, $f(\cdot, z)$ is Lipschitz, and it follows,

$$\begin{aligned} \forall f \in \mathcal{D}(A), \quad \mathbb{E}[|f(\bar{X}_{\Gamma_{n+1}}, \zeta_{\Gamma_n}) - f(\bar{X}_{\Gamma_n}^1, \zeta_{\Gamma_n})|^\rho | \bar{X}_{\Gamma_n}, \zeta_{\Gamma_n}] &\leq C \gamma_{n+1}^{\rho/2} |\sigma \sigma^*(\bar{X}_{\Gamma_n}, \zeta_{\Gamma_n})|^{\rho/2} \\ &\leq C \gamma_{n+1}^{\rho/2} V^{a\rho/2}(\bar{X}_{\Gamma_n}, \zeta_{\Gamma_n}) \\ &\leq C \gamma_{n+1}^{\rho/2} V^{a+p-1}(\bar{X}_{\Gamma_n}, \zeta_{\Gamma_n}). \end{aligned}$$

Moreover,

$$\begin{aligned} &\mathbb{E}[|f(\bar{X}_{\Gamma_{n+1}}, \zeta_{\Gamma_{n+1}}) - f(\bar{X}_{\Gamma_{n+1}}, \zeta_{\Gamma_n})|^\rho | \bar{X}_{\Gamma_n}, \zeta_{\Gamma_n}] \\ &= \gamma_{n+1} \sum_{z=1}^2 (q_{\zeta_{\Gamma_n}, z} + o_{n \rightarrow \infty}) \mathbb{E}[|f(\bar{X}_{\Gamma_{n+1}}, z) - f(\bar{X}_{\Gamma_{n+1}}, \zeta_{\Gamma_n})|^\rho | \bar{X}_{\Gamma_n}, \zeta_{\Gamma_n}] \\ &\leq C \gamma_{n+1} \|f\|_\infty^\rho, \end{aligned}$$

which concludes the study for $f \in \mathcal{D}(A)$. We focus now on the case $f = V^{p/s}$. We notice that (41) implies that for any $n \in \mathbb{N}$,

$$|\bar{X}_{\Gamma_{n+1}} - \bar{X}_{\Gamma_n}| \leq C \gamma_{n+1}^{1/2} \sqrt{\phi \circ V(\bar{X}_{\Gamma_n}, \zeta_{\Gamma_n})} (1 + |U_{n+1}|)$$

Once again we rewrite the term that we study as follows

$$\begin{aligned} V^{p/s}(\bar{X}_{\Gamma_{n+1}}, \zeta_{\Gamma_{n+1}}) - V^{p/s}(\bar{X}_{\Gamma_n}, \zeta_{\Gamma_n}) &= V^{p/s}(\bar{X}_{\Gamma_{n+1}}, \zeta_{\Gamma_n}) - V^{p/s}(\bar{X}_{\Gamma_n}, \zeta_{\Gamma_n}) \\ &\quad + V^{p/s}(\bar{X}_{\Gamma_{n+1}}, \zeta_{\Gamma_{n+1}}) - V^{p/s}(\bar{X}_{\Gamma_{n+1}}, \zeta_{\Gamma_n}) \end{aligned}$$

We study the first term. Using (29) with $\alpha = 2p/s$, it follows from (38) that for any $z \in \{1, \dots, M_0\}$, $\sqrt{V(\cdot, z)}$ is Lipschitz and we have

$$\begin{aligned} V^{p/s}(\bar{X}_{\Gamma_{n+1}}, z) - V^{p/s}(\bar{X}_{\Gamma_n}, z) &\leq 2^{2p/s} p/s (V^{p/s-1/2}(\bar{X}_{\Gamma_n}, z) |\sqrt{V}(\bar{X}_{\Gamma_{n+1}}, z) - \sqrt{V}(\bar{X}_{\Gamma_n}, z)| \\ &\quad + |\sqrt{V}(\bar{X}_{\Gamma_{n+1}}, z) - \sqrt{V}(\bar{X}_{\Gamma_n}, z)|^{2p/s}) \\ &\leq 2^{2p/s} p/s ([\sqrt{V}]_1 V^{p/s-1/2}(\bar{X}_{\Gamma_n}, z) |\bar{X}_{\Gamma_{n+1}} - \bar{X}_{\Gamma_n}| \\ &\quad + [\sqrt{V}]_1^{2p/s} |\bar{X}_{\Gamma_{n+1}} - \bar{X}_{\Gamma_n}|^{2p/s}). \end{aligned}$$

We use the assumption $\rho \leq s(1 - (1 - a)/p)$ and it follows from $\mathfrak{B}(\phi)$ (see (41)) that

$$\mathbb{E}[|V^{p/s}(\bar{X}_{\Gamma_{n+1}}, z) - V^{p/s}(\bar{X}_{\Gamma_n}, z)|^\rho | \bar{X}_{\Gamma_n}, z] \leq C\gamma_{n+1}^{\rho/2} V^{p+a-1}(\bar{X}_{\Gamma_n}, z).$$

In order to treat the first term, we put $z = \zeta_{\Gamma_n}$ in this estimation. It remains to study the second term. We notice that since $\rho \leq s(1 - (1 - a)/p)$, it is immediate from the previous inequality that for every $z \in \{1, \dots, M_0\}$, we have

$$\mathbb{E}[V^{p\rho/s}(\bar{X}_{\Gamma_{n+1}}, z) | \bar{X}_{\Gamma_n}, z] \leq CV^{p+a-1}(\bar{X}_{\Gamma_n}, z).$$

We focus on the term to estimate and using this inequality, we obtain

$$\begin{aligned} & \mathbb{E}[|V^{p/s}(\bar{X}_{\Gamma_{n+1}}, \zeta_{\Gamma_{n+1}}) - V^{p/s}(\bar{X}_{\Gamma_{n+1}}, \zeta_{\Gamma_n})|^\rho | \bar{X}_{\Gamma_n}, \zeta_{\Gamma_n}] \\ &= \gamma_{n+1} \sum_{z=1}^{M_0} (q_{\zeta_{\Gamma_n}, z} + \underset{n \rightarrow \infty}{o}(\gamma_{n+1})) \mathbb{E}[|V^{p/s}(\bar{X}_{\Gamma_{n+1}}, z) - V^{p/s}(\bar{X}_{\Gamma_{n+1}}, \zeta_{\Gamma_n})|^\rho | \bar{X}_{\Gamma_n}, \zeta_{\Gamma_n}] \\ &\leq C\gamma_{n+1} \sum_{z=1}^{M_0} (|q_{\zeta_{\Gamma_n}, z}| + \underset{n \rightarrow \infty}{o}(\gamma_{n+1})) (V^{p+a-1}(\bar{X}_{\Gamma_n}, z) + V^{p+a-1}(\bar{X}_{\Gamma_n}, \zeta_{\Gamma_n})) \\ &\leq C\gamma_{n+1} V^{p+a-1}(\bar{X}_{\Gamma_n}, \zeta_{\Gamma_n}), \end{aligned}$$

where the last inequality follows from (39). We rearrange the terms and the proof is completed. \square

4.2 The Milstein scheme

In this part we treat the case of a Milstein scheme (introduced in [11]) with decreasing step for a Brownian diffusion process. As far as we know, there is no study concerning that scheme for the algorithm we use as for high weak or strong order numerical scheme. We propose two approaches under weak mean reverting assumption. The first one relies on polynomial and the second one relies on exponential test functions. More particularly we use an approach with test functions ψ such that $\psi(y) = y^p$, $p \geq 0$ for every $y \in [v_*, \infty)$. The other approach is based on test functions $\psi(y) = \exp(\lambda y^p)$, $p \in [0, 1/2]$, $\lambda \geq 0$, for every $y \in [v_*, \infty)$.

We consider a d -dimension Brownian motion $(W_t)_{t \geq 0}$. We are interested in the solution of the d dimensional stochastic equation

$$X_t = x + \int_0^t b(X_s) ds + \int_0^t \sigma(X_s) dW_s \quad (52)$$

where $b : \mathbb{R}^d \rightarrow \mathbb{R}^d$ and $\sigma, \partial_{x_l} \sigma : \mathbb{R}^d \rightarrow \mathbb{R}^{d \times d}$, $l \in \{1, \dots, d\}$, are locally bounded and continuous functions. The infinitesimal generator of this process is given by

$$Af(x) = \langle b(x), \nabla f(x) \rangle + \sum_{i,j=1}^d (\sigma \sigma^*)_{i,j}(x) \frac{\partial^2 f}{\partial x_i \partial x_j}(x) \quad (53)$$

Now, we introduce the Milstein scheme for $(X_t)_{t \geq 0}$ such that for every $n \in \mathbb{N}$ and $t \in [\Gamma_n, \Gamma_{n+1}]$, we have

$$\begin{aligned} \bar{X}_t &= \bar{X}_{\Gamma_n} + (t - \Gamma_n) b(\bar{X}_{\Gamma_n}) + \sigma(\bar{X}_{\Gamma_n})(W_t - W_{\Gamma_n}) \\ &+ \sum_{i,j=1}^d \sum_{l=1}^d \partial_{x_l} \sigma_i(\bar{X}_{\Gamma_n}) \sigma_{l,j}(\bar{X}_{\Gamma_n}) \int_{\Gamma_n}^t \int_{\Gamma_n}^s dW_u^j dW_s^i \end{aligned} \quad (54)$$

with $\sigma_l : \mathbb{R}^d \rightarrow \mathbb{R}^d$, $x \mapsto \sigma_l(x) = (\sigma_{1,l}(x), \dots, \sigma_{d,l}(x))$.

$$\begin{aligned}
\Delta \bar{X}_{n+1}^1 &= \gamma_{n+1} b(\bar{X}_{\Gamma_n}), \\
\Delta \bar{X}_{n+1}^2 &= \sum_{i,j=1}^d \sum_{l=1}^d \partial_{x_l} \sigma_i(\bar{X}_{\Gamma_n}) \sigma_{l,j}(\bar{X}_{\Gamma_n}) \int_{\Gamma_n}^{\Gamma_{n+1}} \int_{\Gamma_n}^s dW_u^j dW_s^i, \\
\Delta \bar{X}_{n+1}^3 &= \sigma(\bar{X}_{\Gamma_n})(W_{\Gamma_{n+1}} - W_{\Gamma_n}).
\end{aligned} \tag{55}$$

and $\bar{X}_{\Gamma_{n+1}}^i = \bar{X}_{\Gamma_n} + \sum_{j=1}^i \Delta \bar{X}_{n+1}^j$. In the sequel we will use the notation $U_{n+1} = \gamma_{n+1}^{-1/2}(W_{\Gamma_{n+1}} - W_{\Gamma_n})$ and $\mathcal{W}_{n+1} = (\mathcal{W}_{n+1}^{i,j})_{i,j \in \{1, \dots, d\}}$ with $\mathcal{W}_{n+1}^{i,j} = \gamma_{n+1}^{-1} \int_{\Gamma_n}^{\Gamma_{n+1}} \int_{\Gamma_n}^s dW_u^j dW_s^i$. Actually, for the polynomial case, we introduce a weaker assumption for the sequence $(U_n)_{n \in \mathbb{N}^*}$ and $(\mathcal{W}_n)_{n \in \mathbb{N}^*}$. Let $q \in \mathbb{N}^*$, $p \geq 0$. We suppose that $(U_n)_{n \in \mathbb{N}^*}$ is a sequence of independent random variables such that U satisfies

$$M_{\mathcal{N},q}(U) \quad \forall n \in \mathbb{N}^*, \forall \tilde{q} \in \{1, \dots, q\}, \quad \mathbb{E}[(U_n)^{\otimes \tilde{q}}] = \mathbb{E}[(\mathcal{N}(0, I_d))^{\otimes \tilde{q}}] \tag{56}$$

$$M_p(U) \quad \sup_{n \in \mathbb{N}^*} \mathbb{E}[|U_n|^p] < \infty \tag{57}$$

Moreover, we assume that $(\mathcal{W}_n)_{n \in \mathbb{N}^*}$ is a sequence of independent and centered random variables such that

$$M_p(\mathcal{W}) \quad \sup_{n \in \mathbb{N}^*} \mathbb{E}[|\mathcal{W}_n|^p] < \infty \tag{58}$$

Now, we assume that the Lyapunov function $V : \mathbb{R}^d \rightarrow [v_*, \infty)$, $v_* > 0$, satisfies \mathcal{L}_V (see (7)) and

$$|\nabla V|^2 \leq C_V V, \quad \sup_{x \in \mathbb{R}^d} |D^2 V(x)| < \infty \tag{59}$$

We also define

$$\forall x \in \mathbb{R}^d, \quad \lambda_\psi(x) := \frac{1}{2} \lambda_{D^2 V(x) + \nabla V(x) \otimes \psi'' \circ V(x) \psi' \circ V(x)^{-1}}. \tag{60}$$

When $\psi(y) = y^p$, we will also use the notation λ_p instead of λ_ψ . We will suppose that, for every $x \in \mathbb{R}^d$,

$$\mathfrak{B}(\phi) \quad |b(x)|^2 + |\sigma \sigma^*(x)| + \sum_{i,j,l=1}^d |\partial_{x_l} \sigma_i(x) \sigma_{l,j}(x)|^2 \leq C \phi \circ V(x) \tag{61}$$

We now introduce the key hypothesis in order to obtain recursive control for polynomial and exponential form for ψ .

Polynomial case . First for the polynomial case, let $p \geq 0$. We assume that there exists $\beta \in \mathbb{R}_+$, $\alpha > 0$, such that for every $x \in \mathbb{R}^d$, we have

$$\mathcal{R}_p \quad \langle \nabla V(x), b(x) \rangle + \chi_p(x) \leq \beta - \alpha \phi \circ V(x), \tag{62}$$

with

$$\chi_p(x) = \begin{cases} \|\lambda_1\|_\infty \text{Tr}[\sigma \sigma^*(x)] & \text{if } p \leq 1 \\ \|\lambda_p\|_\infty 2^{(2p-3)+} \text{Tr}[\sigma \sigma^*(x)] & \text{if } p > 1. \end{cases} \tag{63}$$

Exponential case . For the exponential case we modify this assumption in the following way. Let $p \leq 1/2$. We assume that there exists $\beta \in \mathbb{R}_+$, $\alpha > 0$, such that for every $x \in \mathbb{R}^d$, we have

$$\mathcal{R}_{p,\lambda} \quad \langle \nabla V(x), b(x) + \kappa(x) \rangle + \chi_p(x) \leq \beta - \alpha \phi \circ V(x), \tag{64}$$

with

$$\kappa(x) = \frac{1}{2} \sum_{i=1}^d \sum_{l=1}^d \partial_{x_l} \sigma_i(x) \sigma_{l,i}(x) + \lambda p V^{p-1}(x) \frac{1}{\phi \circ V(x)} \operatorname{Tr}[\sigma \sigma^*(x)] \nabla V(x) \quad (65)$$

and

$$\chi_p(x) = \frac{V^{1-p}(x)}{\phi \circ V(x)} C_\sigma(x)^{-1} \ln(2^d / \sqrt{\det(\Sigma(x))}) \quad (66)$$

for $C_\sigma(x) : \mathbb{R}^d \rightarrow \mathbb{R}_+^*$ a continuous function such that $\inf_{x \in \mathbb{R}^d} C_\sigma(x) > 0$ and that for every $x \in \mathbb{R}^d$, the matrix $\Sigma(x) \in \mathbb{R}^{d \times d}$ defined by

$$\begin{cases} \Sigma(x)_{i,j} = -2C_\sigma(x) [\sqrt{V}]_1 v_*^{p-1/2} \frac{1}{2} \sum_{l=1}^d |\partial_{x_l} \sigma_i(x) \sigma_{l,j}(x)| & \forall i, j \in \{1, \dots, d\}, i \neq j, \\ \Sigma(x)_{i,i} = 1 - 2C_\sigma(x) (\|D^2 V\|_\infty V^{p-1}(x) \operatorname{Tr}[\sigma \sigma^*(x)] + [\sqrt{V}]_1 v_*^{p-1/2} \frac{1}{2} \sum_{l=1}^d |\partial_{x_l} \sigma_i(x) \sigma_{l,i}(x)|) & \forall i \in \{1, \dots, d\} \end{cases}$$

is a positive definite matrix.

4.2.1 Recursive control

Polynomial case

Proposition 4.4. *Let $v_* > 0$, and $\psi, \phi : [v_*, \infty) \rightarrow \mathbb{R}_+^*$ a continuous function such that $\phi(y) \leq Cy$ with $C > 0$. Now let $p \geq 0$ and define $\psi(y) = y^p$.*

We suppose that $(U_n)_{n \in \mathbb{N}^}$ is a sequence of independent random variables such that U satisfies $M_{\mathcal{N},2}(U)$ (see (56)) and $M_{2p \vee 2}(U)$ (see (57)). Moreover, we assume that $(W_n)_{n \in \mathbb{N}^*}$ is a sequence of independent and centered random variables such that $M_{2p \vee 2}(\mathcal{W})$ (see (58)) holds.*

We suppose that (59), $\mathfrak{B}(\phi)$ (see (61)), \mathcal{R}_p (see (62)), are satisfied. Then, there exists $\alpha > 0$, $\beta \in \mathbb{R}_+$ and $n_0 \in \mathbb{N}^$, such that*

$$\forall n \geq n_0, x \in \mathbb{R}^d, \quad \tilde{A}_n^\gamma \psi \circ V(x) \leq V^{-1}(x) \psi \circ V(x) (\beta - \alpha \phi \circ V(x)). \quad (67)$$

Moreover, when $\phi = Id$ we have

$$\sup_{n \in \mathbb{N}} \mathbb{E}[\psi \circ V(\bar{X}_{\Gamma_n})] < \infty. \quad (68)$$

Proof. First, we focus on the case $p \geq 1$. From the Taylor's formula and the definition of λ_ψ (see (60)), we have

$$\begin{aligned} \psi \circ V(\bar{X}_{\Gamma_{n+1}}) &= \psi \circ V(\bar{X}_{\Gamma_n}) + \langle \bar{X}_{\Gamma_{n+1}} - \bar{X}_{\Gamma_n}, \nabla V(\bar{X}_{\Gamma_n}) \rangle \psi' \circ V(\bar{X}_{\Gamma_n}) \\ &\quad + \frac{1}{2} (D^2 V(\xi_{n+1}) \psi' \circ V(\xi_{n+1}) + \nabla V(\xi_{n+1})^2 \psi'' \circ V(\xi_{n+1})) (\bar{X}_{\Gamma_{n+1}} - \bar{X}_{\Gamma_n})^{\otimes 2}. \\ &\leq \psi \circ V(\bar{X}_{\Gamma_n}) + \langle \bar{X}_{\Gamma_{n+1}} - \bar{X}_{\Gamma_n}, \nabla V(\bar{X}_{\Gamma_n}) \rangle \psi' \circ V(\bar{X}_{\Gamma_n}) \\ &\quad + \lambda_\psi(\xi_{n+1}) |\psi' \circ V(\xi_{n+1})| (\bar{X}_{\Gamma_{n+1}} - \bar{X}_{\Gamma_n})^2. \end{aligned}$$

with $\xi_{n+1} \in (\bar{X}_{\Gamma_n}, \bar{X}_{\Gamma_{n+1}})$. First, from (59), we have $\sup_{x \in \mathbb{R}^d} \lambda_p(x) < \infty$.

Since \mathcal{W} is made of centered random variables, we deduce from $M_{\mathcal{N},2}(U)$ (see (56)), $M_2(U)$ (see (58)) and $M_2(\mathcal{W})$ (see (57)), that

$$\begin{aligned} \mathbb{E}[\bar{X}_{\Gamma_{n+1}} - \bar{X}_{\Gamma_n} | \bar{X}_{\Gamma_n}] &= \gamma_{n+1} b(\bar{X}_{\Gamma_n}) \\ \mathbb{E}[|\bar{X}_{\Gamma_{n+1}} - \bar{X}_{\Gamma_n}|^2 | \bar{X}_{\Gamma_n}] &\leq \gamma_{n+1} \operatorname{Tr}[\sigma \sigma^*(\bar{X}_{\Gamma_n})] + \gamma_{n+1}^2 |b(\bar{X}_{\Gamma_n})|^2 + c_d \gamma_{n+1}^2 \sum_{i,j,l=1}^d |\partial_{x_l} \sigma_i(\bar{X}_{\Gamma_n}) \sigma_{l,j}(\bar{X}_{\Gamma_n})|^2 \\ &\quad + c_d \gamma_{n+1}^{3/2} \sum_{i,j,l=1}^d |\partial_{x_l} \sigma_i(\bar{X}_{\Gamma_n}) \sigma_{l,j}(\bar{X}_{\Gamma_n})| |\sigma(\bar{X}_{\Gamma_n})| \end{aligned}$$

with c_d a positive constant. Assume first that $p = 1$. Using $\mathfrak{B}(\phi)$ (see (61)), for every $\tilde{\alpha} \in (0, \alpha)$, there exists $n_0(\tilde{\alpha})$ such that for every $n \geq n_0(\tilde{\alpha})$,

$$\begin{aligned} \|\lambda_1\|_\infty \gamma_{n+1}^2 (|b(\bar{X}_{\Gamma_n})|^2 + c_d) \sum_{i,j,l=1}^d \partial_{x_l} \sigma_i(\bar{X}_{\Gamma_n}) \sigma_{l,j}(\bar{X}_{\Gamma_n})|^2 \\ + \|\lambda_1 c_d \gamma_{n+1}^{3/2}\|_\infty \sum_{i,j,l=1}^d |\partial_{x_l} \sigma_i(\bar{X}_{\Gamma_n}) \sigma_{l,j}(\bar{X}_{\Gamma_n})| |\sigma(\bar{X}_{\Gamma_n})| \leq \gamma_{n+1} (\alpha - \tilde{\alpha}) \phi \circ V(\bar{X}_{\Gamma_n}). \end{aligned} \quad (69)$$

From assumption \mathcal{R}_p (see (62) and (63)), we conclude that

$$\tilde{A}_n^\gamma \psi \circ V(x) \leq \beta - \tilde{\alpha} \phi \circ V(x)$$

Assume now that $p > 1$. Since $|\nabla V| \leq C_V V$ (see (59)), then \sqrt{V} is Lipschitz. Using (30), it follows that

$$\begin{aligned} V^{p-1}(\xi_{n+1}) &\leq (\sqrt{V}(\bar{X}_{\Gamma_n}) + [\sqrt{V}]_1 |\bar{X}_{\Gamma_{n+1}} - \bar{X}_{\Gamma_n}|)^{2p-2} \\ &\leq 2^{(2p-3)+} (V^{p-1}(\bar{X}_{\Gamma_n}) + [\sqrt{V}]_1^{2p-2} |\bar{X}_{\Gamma_{n+1}} - \bar{X}_{\Gamma_n}|^{2p-2}) \end{aligned}$$

We focus on the study of the second term of the remainder. First, using $\mathfrak{B}(\phi)$ (see (61)), for any $p \geq 1$,

$$|\bar{X}_{\Gamma_{n+1}} - \bar{X}_{\Gamma_n}|^{2p} \leq c_p \gamma_{n+1}^p \phi \circ V(\bar{X}_{\Gamma_n})^p (1 + |U_{n+1}|^{2p} + |\mathcal{W}_{n+1}|^{2p}).$$

Let $\hat{\alpha} \in (0, \alpha)$. Then, we deduce from $M_{2p}(U)$ (see (58)), $M_{2p}(\mathcal{W})$ (see (57)), that there exists $n_0(\hat{\alpha}) \in \mathbb{N}$ such that for any $n \geq n_0(\hat{\alpha})$, we have

$$\mathbb{E}[|\bar{X}_{\Gamma_{n+1}} - \bar{X}_{\Gamma_n}|^{2p} | \bar{X}_{\Gamma_n}] \leq \gamma_{n+1} \phi \circ V(\bar{X}_{\Gamma_n})^p \frac{\alpha - \hat{\alpha}}{\|\phi/I_d\|_\infty^{p-1} \|\lambda_p\|_\infty 2^{(2p-3)+} [\sqrt{V}]_1^{2p-2}}$$

To treat the other term we proceed as in (69) with $\|\lambda_1\|_\infty$ replaced by $\|\lambda_p\|_\infty 2^{2p-3} [\sqrt{V}]_1^{2p-2}$, α replace by $\hat{\alpha}$ and $\tilde{\alpha} \in (0, \hat{\alpha})$. We gather all the terms together and using (63), for every $n \geq n_0(\tilde{\alpha}) \vee n_0(\hat{\alpha})$, we obtain

$$\begin{aligned} \mathbb{E}[V^p(\bar{X}_{\Gamma_{n+1}}) - V^p(\bar{X}_{\Gamma_n}) | \bar{X}_{\Gamma_n}] &\leq \gamma_{n+1} p V^{p-1}(\bar{X}_{\Gamma_n}) (\beta - \alpha \phi \circ V(\bar{X}_{\Gamma_n})) \\ &\quad + \gamma_{n+1} p V^{p-1}(\bar{X}_{\Gamma_n}) (\phi \circ V(\bar{X}_{\Gamma_n})) (\hat{\alpha} - \tilde{\alpha}) + (\alpha - \hat{\alpha}) \frac{V^{1-p}(\bar{X}_{\Gamma_n}) \phi \circ V(\bar{X}_{\Gamma_n})^p}{\|\phi/I_d\|_\infty^{p-1}} \\ &\leq \gamma_{n+1} V^{p-1}(\bar{X}_{\Gamma_n}) (\beta p - \tilde{\alpha} p \phi \circ V(\bar{X}_{\Gamma_n})). \end{aligned}$$

which is exactly the recursive control for $p > 1$. Now, we treat the case $p < 1$. Since $x \mapsto x^p$ is concave, we have

$$V^p(\bar{X}_{\Gamma_{n+1}}) - V^p(\bar{X}_{\Gamma_n}) \leq p V^{p-1}(\bar{X}_{\Gamma_n}) (V(\bar{X}_{\Gamma_{n+1}}) - V(\bar{X}_{\Gamma_n}))$$

We have just proved that we have the recursive control $\mathcal{I}_{Q,V}(\psi, \phi)$ holds for $\psi = I_d$ (with some constants $\beta \in \mathbb{R}_+$ and $\alpha > 0$), and since V takes positive values, we obtain

$$\begin{aligned} \mathbb{E}[V^p(\bar{X}_{\Gamma_{n+1}}) - V^p(\bar{X}_{\Gamma_n}) | \bar{X}_{\Gamma_n}] &\leq p V^{p-1}(\bar{X}_{\Gamma_n}) \mathbb{E}[V(\bar{X}_{\Gamma_{n+1}}) - V(\bar{X}_{\Gamma_n}) | \bar{X}_{\Gamma_n}] \\ &\leq V^{p-1}(\bar{X}_{\Gamma_n}) (p\beta - p\alpha \phi \circ V(\bar{X}_{\Gamma_n})), \end{aligned}$$

which completes the proof of (67). The proof of (68) is immediate application of Lemma 3.2 as soon as we notice that the increments of the Milstein scheme have finite polynomial moments which implies (24). \square

Exponential case

In this section we will not relax the assumption on the Gaussian structure of the increment as we do in the polynomial case with hypothesis (see (56), (57) and (58)). In order to obtain our result, we introduce a supplementary assumption in order to express the iterated stochastic integrals in terms of products of the increments of the Brownian motion. The so called commutative noise assumption is the following

$$\forall x \in \mathbb{R}^d, \forall i, j \in \{1, \dots, d\}, \quad \sum_{l=1}^d \partial_{x_l} \sigma_i(x) \sigma_{l,j}(x) = \sum_{l=1}^d \partial_{x_l} \sigma_j(x) \sigma_{l,i}(x). \quad (70)$$

In this case, with the notation from (55), we have

$$\begin{aligned} \Delta \bar{X}_{n+1}^2 &= \frac{1}{2} \sum_{i,j=1}^d \sum_{l=1}^d \partial_{x_l} \sigma_i(\bar{X}_{\Gamma_n}) \sigma_{l,j}(\bar{X}_{\Gamma_n}) (W_{\Gamma_{n+1}}^j - W_{\Gamma_n}^j) (W_{\Gamma_{n+1}}^i - W_{\Gamma_n}^i) \\ &\quad - \frac{1}{2} \gamma_{n+1} \sum_{i=1}^d \sum_{l=1}^d \partial_{x_l} \sigma_i(\bar{X}_{\Gamma_n}) \sigma_{l,i}(\bar{X}_{\Gamma_n}). \end{aligned} \quad (71)$$

In the sequel we will adopt the following notation

$$\begin{aligned} \tilde{\Delta} \bar{X}_{n+1}^1 &= \gamma_{n+1} b(\bar{X}_{\Gamma_n}) - \frac{1}{2} \gamma_{n+1} \sum_{i=1}^d \sum_{l=1}^d \partial_{x_l} \sigma_i(\bar{X}_{\Gamma_n}) \sigma_{l,i}(\bar{X}_{\Gamma_n}), \\ \tilde{\Delta} \bar{X}_{n+1}^2 &= \frac{1}{2} \sum_{i,j=1}^d \sum_{l=1}^d \partial_{x_l} \sigma_i(\bar{X}_{\Gamma_n}) \sigma_{l,j}(\bar{X}_{\Gamma_n}) (W_{\Gamma_{n+1}}^j - W_{\Gamma_n}^j) (W_{\Gamma_{n+1}}^i - W_{\Gamma_n}^i) \end{aligned} \quad (72)$$

Lemma 4.2. *Let $\Lambda_{i,j} \in \mathbb{R}, i, j \in \{1, \dots, d\}$ and U a \mathbb{R}^d -valued random variable made with d independent and identically distributed standard normal random variables $U = (U_i)_{i \in \{1, \dots, d\}}, U_i \sim \mathcal{N}(0, 1)$. We define $\Sigma \in \mathbb{R}^{d \times d}$ such that $\Sigma_{i,i} = 1 - 2\Lambda_{i,i}, i \in \{1, \dots, d\}$ and $\Sigma_{i,j} = -2\Lambda_{i,j}, i, j \in \{1, \dots, d\}, i \neq j$. We assume that Σ is a positive definite matrix. Then*

$$\mathbb{E}[\exp(\sum_{i,j=1}^d \Lambda_{i,j} |U_i U_j|)] \leq 2^d \det(\Sigma)^{-1/2}. \quad (73)$$

Proof. A direct computation yields

$$\begin{aligned} \mathbb{E}[\exp(\sum_{i,j=1}^d \Lambda_{i,j} |U_i U_j|)] &= \int_{\mathbb{R}^d} (2\pi)^{-d/2} \exp(\sum_{i,j=1}^d \Lambda_{i,j} |u_i u_j| - 1/2 \sum_{i=1}^d |u_i|^2) du \\ &\leq \sum_{\zeta \in \{-1, 1\}^d} \int_{\mathbb{R}^d} (2\pi)^{-d/2} \exp(\sum_{i,j=1}^d \Lambda_{i,j} \zeta_i u_i \zeta_j u_j - 1/2 \sum_{i=1}^d |\zeta_i u_i|^2) du \\ &= 2^d \int_{\mathbb{R}^d} (2\pi)^{-d/2} \exp(\sum_{i,j=1}^d \Lambda_{i,j} u_i u_j - 1/2 \sum_{i=1}^d |\zeta_i u_i|^2) du = 2^d \det(\Sigma)^{-1/2}. \end{aligned}$$

□

Lemma 4.3. *Let U a \mathbb{R}^d -valued random variable made with d independent and identically distributed standard normal random variables $U = (U_i)_{i \in \{1, \dots, d\}}, U_i \sim \mathcal{N}(0, 1)$. for every $h \in (0, 1)$, we have*

$$\forall v \in \mathbb{R}^d, \quad \mathbb{E}[\exp(\sqrt{h} \langle v, U \rangle + h \sum_{i,j} \Lambda_{i,j} |U_i U_j|)] \leq \exp(\frac{h}{2(1-h)} |v|^2) 2^{hd} \det(\Sigma)^{-h/2} \quad (74)$$

Proof. Using the Hölder inequality we have

$$\mathbb{E}[\exp(\sqrt{h} \langle v, U \rangle + h \sum_{i,j=1}^d \Lambda_{i,j} |U_i U_j|)] \leq \mathbb{E}[\exp(\frac{\sqrt{h}}{1-h} \langle v, U \rangle)]^{1-h} \mathbb{E}[\exp(h \sum_{i,j=1}^d \Lambda_{i,j} |U_i U_j|)]$$

The results follows from Lemma 4.2. □

Using those results, we deduce the recursive control for exponential test functions.

Proposition 4.5. *Let $v_* > 0$, and $\phi : [v_*, \infty) \rightarrow \mathbb{R}_+$ a continuous function such that $\phi(y) \leq Cy$ with $C > 0$ and $\lim_{y \rightarrow \infty} \phi(y) = \infty$. Now let $p \in [0, 1/2]$, $\lambda \geq 0$ and define $\psi : [v_*, \infty) \rightarrow \mathbb{R}_+$ such that $\psi(y) = \exp(\lambda y^p)$. We suppose that (59), $\mathfrak{B}(\phi)$ (see (61)), $\mathcal{R}_{p,\lambda}$ (see (64)), are satisfied. We also assume that*

$$\forall x \in \mathbb{R}^d, \quad \text{Tr}[\sigma\sigma^*(x)]|b(x)|(\sqrt{V}(x) + |b(x)|) \leq CV^{1-p}(x)\phi \circ V(x) \quad (75)$$

Then, there exists $\alpha > 0$, $\beta \in \mathbb{R}_+$ and $n_0 \in \mathbb{N}^*$, such that

$$\forall n \geq n_0, x \in \mathbb{R}^d, \quad \tilde{A}_n^\gamma \psi \circ V(x) \leq V^{-1}(x)\psi \circ V(x)(\beta - \alpha\phi \circ V(x)). \quad (76)$$

Moreover, when $\phi = Id$ we have

$$\sup_{n \in \mathbb{N}} \mathbb{E}[\psi \circ V(\bar{X}_{\Gamma_n})] < \infty. \quad (77)$$

Proof. First, with notations (72), we rewrite

$$\begin{aligned} V^p(\bar{X}_{\Gamma_{n+1}}) - V^p(\bar{X}_{\Gamma_n}) &= V^p(\bar{X}_{\Gamma_n} + \tilde{\Delta}\bar{X}_{n+1}^1 + \Delta\bar{X}_{n+1}^3) - V^p(\bar{X}_{\Gamma_n}) \\ &\quad + V^p(\bar{X}_{\Gamma_n} + \tilde{\Delta}\bar{X}_{n+1}^1 + \Delta\bar{X}_{n+1}^3) - V^p(\bar{X}_{\Gamma_n}) \end{aligned}$$

an we study each term separately. Since $p \leq 1$, the function defined on $[v_*, \infty)$ by $y \mapsto y^p$ is concave. Using then the Taylor expansion of order 2 of the function V , for every $x, y \in \mathbb{R}^d$, there exists $\theta \in [0, 1]$ such that

$$\begin{aligned} V^p(y) - V^p(x) &\leq pV^{p-1}(x)(V(y) - V(x)) \\ &= pV^{p-1}(x)(\langle \nabla V(x), y - x \rangle + \frac{1}{2} \text{Tr}[D^2V(\theta x + (1 - \theta)y)(y - x)^{\otimes 2}]) \end{aligned}$$

and then,

$$V^p(y) - V^p(x) \leq pV^{p-1}(x)(\langle \nabla V(x), y - x \rangle + \frac{1}{2} \|D^2V\|_\infty |y - x|^2). \quad (78)$$

Using this inequality with $x = \bar{X}_{\Gamma_n}$ and $y = \bar{X}_{\Gamma_n} + \tilde{\Delta}\bar{X}_{n+1}^1 + \Delta\bar{X}_{n+1}^3$, it follows that

$$\begin{aligned} V^p(\bar{X}_{\Gamma_n} + \tilde{\Delta}\bar{X}_{n+1}^1 + \Delta\bar{X}_{n+1}^3) - V^p(\bar{X}_{\Gamma_n}) &\leq pV^{p-1}(\bar{X}_{\Gamma_n})(\langle \nabla V(\bar{X}_{\Gamma_n}), \tilde{\Delta}\bar{X}_{n+1}^1 + \Delta\bar{X}_{n+1}^3 \rangle \\ &\quad + \frac{1}{2} pV^{p-1}(\bar{X}_{\Gamma_n}) \|D^2V\|_\infty |\tilde{\Delta}\bar{X}_{n+1}^1 + \Delta\bar{X}_{n+1}^3|^2). \end{aligned}$$

Now, we study the other term. Since $p \leq 1/2$, then the function defined on $[v_*, \infty)$ by $y \mapsto y^{2p}$ is concave and we obtain

$$\begin{aligned} V^p(\bar{X}_{\Gamma_{n+1}}) - V^p(\bar{X}_{\Gamma_n} + \tilde{\Delta}\bar{X}_{n+1}^1 + \Delta\bar{X}_{n+1}^3) &\leq pV^{p-1/2}(\tilde{\Delta}\bar{X}_{n+1}^1 + \Delta\bar{X}_{n+1}^3)(\sqrt{V}(\bar{X}_{\Gamma_{n+1}}) - \sqrt{V}(\bar{X}_{\Gamma_n} + \tilde{\Delta}\bar{X}_{n+1}^1 + \Delta\bar{X}_{n+1}^3)) \\ &\leq p[\sqrt{V}]_1 v_*^{p-1/2} |\tilde{\Delta}\bar{X}_{n+1}^2| \end{aligned}$$

In the sequel, we will use the notation

$$\forall x \in \mathbb{R}^d, \quad \tilde{b}(x) = b(x) + \frac{1}{2} \sum_{i=1}^d \sum_{l=1}^d \partial_{x_i} \sigma_i(x) \sigma_{l,i}(x).$$

It follows that

$$\mathbb{E}[\exp(\lambda V^p(\bar{X}_{\Gamma_{n+1}})) - \exp(\lambda V^p(\bar{X}_{\Gamma_n})) | \bar{X}_{\Gamma_n}] \leq H_{\gamma_{n+1}}(\bar{X}_{\Gamma_n}) L_{\gamma_{n+1}}(\bar{X}_{\Gamma_n})$$

with, for every $x \in \mathbb{R}^d$, $t \in \mathbb{R}_+^*$,

$$H_t(x) = \exp(\lambda V^p(x) + t\lambda p V^{p-1}(x) \langle \nabla V(x), \tilde{b}(x) \rangle + t^2 \frac{1}{2} \lambda p \|D^2 V\|_\infty V^{p-1}(x) |\tilde{b}(x)|^2)$$

and

$$\begin{aligned} L_t(x) = & \mathbb{E}[\exp(\sqrt{t}\lambda p V^{p-1}(x) \langle \nabla V(x), \sigma(x)U \rangle + t\lambda p \|D^2 V\|_\infty V^{p-1}(x) \text{Tr}[\sigma\sigma^*(x)]|U|^2 \\ & + t\lambda p [\sqrt{V}]_1 v_*^{p-1/2} \frac{1}{2} \sum_{i,j=1}^d \sum_{l=1}^d |\partial_{x_l} \sigma_i(x) \sigma_{l,j}(x)| |U_i U_j| + t^{3/2} \lambda p V^{p-1}(x) \|D^2 V\|_\infty 2 \langle \tilde{b}(x), \sigma(x)U \rangle)] \end{aligned}$$

where $U = (U_1, \dots, U_d)$, with U_i , $i \in \{1, \dots, d\}$, some independent and identically distributed standard normal random variables. In order to compute $L_t(x)$, we use Lemma 4.3 (see (74)) with $h = C_\sigma(x)^{-1} t\lambda p$, $v = \sqrt{C_\sigma(x) \lambda p V^{p-1}(x) \sigma^*(x) (\nabla V(x) + t \|D^2 V\|_\infty 2b(x))}$ and $\Sigma(x)$ the matrix such that for every $i, j \in \{1, \dots, d\}$,

$$\begin{cases} \Sigma(x)_{i,j} = -2C_\sigma(x) [\sqrt{V}]_1 v_*^{p-1/2} \frac{1}{2} \sum_{l=1}^d |\partial_{x_l} \sigma_i(x) \sigma_{l,j}(x)| & \forall i, j \in \{1, \dots, d\}, i \neq j, \\ \Sigma(x)_{i,i} = 1 - 2C_\sigma(x) (\|D^2 V\|_\infty V^{p-1}(x) \text{Tr}[\sigma\sigma^*(x)] + [\sqrt{V}]_1 v_*^{p-1/2} \frac{1}{2} \sum_{l=1}^d |\partial_{x_l} \sigma_i(x) \sigma_{l,i}(x)|) & \forall i \in \{1, \dots, d\} \end{cases}$$

where $C_\sigma : \mathbb{R}^d \rightarrow \mathbb{R}_+^*$ is such that $\inf_{x \in \mathbb{R}^d} C_\sigma(x) > 0$ and for every $x \in \mathbb{R}^d$ then $\Sigma(x)$ is a positive definite matrix. We apply Lemma 4.3 and it follows that for $t \leq \inf_{x \in \mathbb{R}^d} C_\sigma(x) / (2\lambda p)$

$$\begin{aligned} L_t(x) & \leq \exp\left(\frac{t\lambda p C_\sigma(x)^{-1}}{2(1 - \lambda p C_\sigma(x)^{-1} t)} |v|^2 + t\lambda p C_\sigma(x)^{-1} (\ln(2^d / \sqrt{\det(\Sigma(x))}))\right) \\ & \leq \exp(t\lambda p C_\sigma(x)^{-1} |v|^2 + t\lambda p C_\sigma(x)^{-1} (\ln(2^d / \sqrt{\det(\Sigma(x))})) \end{aligned}$$

At this point, we focus on the first term of the exponential. We have

$$|v|^2 \leq C_\sigma(x) \lambda p \text{Tr}[\sigma\sigma^*(x)] V^{2p-2}(x) (|\nabla V(x)|^2 + t \|D^2 V\|_\infty 4 \langle \nabla V(x), b(x) \rangle + t^2 4 \|D^2 V\|_\infty^2 |\tilde{b}(x)|^2)$$

Using $\mathcal{B}(\phi)$, (75) and $\mathcal{R}_{p,\lambda}$ (see (64)), it follows there exists $\bar{C} > 0$ such that

$$H_t(x) L_t(x) \leq \exp(\lambda V^p(x) + t\lambda p V^{p-1}(x) (\beta - \alpha \phi \circ V(x)) + \bar{C} t^2 V^{p-1}(x) \phi \circ V(x))$$

Then, we have

$$\begin{aligned} H_t(x) L_t(x) & \leq \exp((1 - t p \alpha V^{-1}(x) \phi \circ V(x)) \lambda V^p(x) \\ & \quad + t p \alpha V^{-1}(x) \phi \circ V(x) V^p(x) \left(\frac{\beta}{\alpha \phi \circ V(x)} + t \bar{C} / (\alpha p)\right)). \end{aligned}$$

Using the convexity of the exponential function, we have for $t p \alpha V^{-1}(x) \phi \circ V(x) < 1$,

$$\begin{aligned} H_t(x) L_t(x) & \leq \exp(\lambda V^p(x) - t p \alpha V^{-1}(x) \phi \circ V(x)) \exp(\lambda V^p(x)) \\ & \quad + t p \alpha V^{-1}(x) \phi \circ V(x) \exp(V^p(x) \left(\frac{\beta}{\alpha \phi \circ V(x)} + t \bar{C} / (\alpha p)\right)). \end{aligned}$$

At this point we notice that (24) holds with this ψ which will be useful in order to obtain (77). Moreover and independently from that, the function defined on \mathbb{R}^d by $x \mapsto \exp(V^p(x) \left(\frac{\beta}{\alpha \phi \circ V(x)} + t \bar{C} / (\alpha p)\right))$ is continuous and bounded on any compact set. Moreover ϕ tends to infinity at the infinity and then we have

$$\phi \circ V(x) \exp(V^p(x) \left(\frac{\beta}{\alpha \phi \circ V(x)} + t \bar{C} / (\alpha p)\right)) = \underset{x \rightarrow \infty}{O} \exp(\lambda V^p(x))$$

for every $t < \lambda \alpha p / \bar{C}$, and the proof of the recursive control (76) is completed. Combining it with (24) (which is obtained above) and applying Lemma 3.2 gives (77). \square

4.2.2 Proof of the infinitesimal estimation

Proposition 4.6. *We suppose that the sequence $(U_n)_{n \in \mathbb{N}^*}$ satisfies $M_{\mathcal{N},2}(U)$ (see (36)) and $M_2(U)$ (see (57)) and that the sequence $(W_n)_{n \in \mathbb{N}^*}$ is centered and satisfies $M_2(W)$ (see (58)).*

We also assume that b and σ are locally bounded functions, that ϕ has sublinear growth, that $\mathfrak{B}(\phi)$ (see (61)) holds and that we have $\sup_{n \in \mathbb{N}^} \nu_n^\eta(|\sigma|^2) < \infty$ and $\sup_{n \in \mathbb{N}^*} \nu_n^\eta(\sum_{i,j,l=1}^d |\partial_{x_l} \sigma_i \sigma_{l,j}|) < \infty$.*

Then, we have $\mathcal{E}(\tilde{A}^\gamma, A)$ (see (9)).

Proof. First, we recall that $\mathcal{D}(A) \subset \mathcal{C}_K^2(E)$. Using the Taylor expansion, we have

$$f(\bar{X}_{\Gamma_n}^1) - f(\bar{X}_{\Gamma_{n-1}}) = \langle \nabla f(x), \Delta \bar{X}_n^1 \rangle + R_1(\bar{X}_{\Gamma_{n-1}}, \bar{X}_{\Gamma_{n-1}}^1)$$

with $R_1(x, y) = f(y) - f(x) - \langle \nabla f(x), y - x \rangle$. First we notice that $\mathbb{E}[\Delta \bar{X}_n^1] = \gamma_n b(\bar{X}_{\Gamma_{n-1}})$. Now, we focus on the expectation of $R_1(\bar{X}_{\Gamma_{n-1}}, \bar{X}_{\Gamma_{n-1}}^1)$. First, we define the function from $\mathbb{R}^d \times \mathbb{R}^d$ to \mathbb{R}_+ as follows

$$r_1(x, y) = \sup_{\theta \in [0,1]} |\nabla f(x + \theta(y - x)) - \nabla f(x)|. \quad (79)$$

Then r_1 is a bounded continuous function such that $r_1(x, x) = 0$. Moreover, it follows immediately that

$$R_1(x, y) \leq r_1(x, y)|y - x|.$$

Therefore, we deduce that

$$\begin{aligned} \mathbb{E}[R_1(\bar{X}_{\Gamma_{n-1}}, \bar{X}_{\Gamma_{n-1}}^1) | \bar{X}_{\Gamma_{n-1}}] &\leq \mathbb{E}[|\Delta \bar{X}_n^1| r_1(\bar{X}_{\Gamma_{n-1}}, \bar{X}_{\Gamma_{n-1}}^1) | \bar{X}_{\Gamma_{n-1}}] \\ &\leq C \gamma_n |b(\bar{X}_{\Gamma_{n-1}})| r_1(\bar{X}_{\Gamma_{n-1}}, \bar{X}_{\Gamma_{n-1}}^1) \end{aligned}$$

Now, we notice that hypothesis (61) and the fact that $\phi(x) \leq C|x|$, imply that b has a sublinear growth: there exists $C_b \geq 0$ such that $b(x) \leq C_b(1 + |x|)$. Now since f has a compact support, there exists $R > 0$ such that $f(x) = 0$ for every $x \in \mathbb{R}^d$ such that $|x| > R$. As a consequence if $|x| > 2R$ and $\gamma \leq t_0 = R/(C_b(1 + 2R))$ then for $y = x + \gamma b(x)$,

$$|y| > |x| - \gamma C_b(1 + |x|) > R,$$

and then $r_1(x, y) = 0$. It follows that the function $\tilde{\Lambda}_{f,1} : (x, \gamma) \mapsto \gamma r_1(x, x + \gamma b(x))$ is uniformly continuous on $\mathbb{R}^d \times [0, t_0]$ and then we obtain $\mathcal{E}_{loc}(\tilde{\Lambda}_{f,1}, |b|)$ (see (10) and (11)) with $t_0 = R/(C_b(1 + 2R))$. In the same way we have

$$f(\bar{X}_{\Gamma_n}^2) - f(\bar{X}_{\Gamma_n}^1) = \langle \nabla f(\bar{X}_{\Gamma_n}^1), \Delta \bar{X}_n^2 \rangle + R_1(\bar{X}_{\Gamma_{n-1}}^1, \bar{X}_{\Gamma_{n-1}}^2).$$

The first term of the right hand side of the above equation is a centered random variable and we obtain

$$\begin{aligned} \mathbb{E}[R_1(\bar{X}_{\Gamma_{n-1}}^1, \bar{X}_{\Gamma_{n-1}}^2) | \bar{X}_{\Gamma_{n-1}}] \\ \leq \sum_{i,j,l=1}^d |\partial_{x_l} \sigma_i(\bar{X}_{\Gamma_{n-1}}) \sigma_{l,j}(\bar{X}_{\Gamma_{n-1}})| \mathbb{E}[|\int_{\gamma_{n-1}}^{\gamma_n} \int_{\gamma_{n-1}}^s dW_u^j dW_s^i| r_1(\bar{X}_{\Gamma_n}^1, \bar{X}_{\Gamma_{n-1}}^2) | \bar{X}_{\Gamma_{n-1}}]. \end{aligned}$$

Now we define $G_2 = \mathbb{R}^{d \times d}$, $\Theta_2 = v$, π_2 the measure defined on $(G_2, \mathcal{B}(G_2))$ (with $\mathcal{B}(G_2)$ the sigma fields endowed by the Borelians of G_2) by $\pi_2(d\Theta_2) = \mathfrak{W}(dv)$ where \mathfrak{W} denotes the law of the $\mathbb{R}^{d \times d}$ -valued random variable with components $\int_0^1 \int_0^s dW_u^j dW_s^i$ for $i, j \in \{1, \dots, d\}$. Then for every $x \in \mathbb{R}^d = E$, we have

$$\begin{aligned} \mathbb{E}[|\int_{\gamma_{n-1}}^{\gamma_n} \int_{\gamma_{n-1}}^s dW_u^j dW_s^i| r_1(\bar{X}_{\Gamma_n}^1, \bar{X}_{\Gamma_{n-1}}^2) | \bar{X}_{\Gamma_{n-1}} = x] &= \int_{\mathbb{R}^{d \times d}} \tilde{\Lambda}_{f,2}^{i,j}(x, \gamma_n, v) \mathfrak{W}(dv) \\ &= \int_{G_2} \tilde{\Lambda}_{f,2}^{i,j}(x, \gamma_n, \Theta_2) \pi_2(d\Theta_2) \end{aligned}$$

with

$$\begin{aligned} \tilde{\Lambda}_{f,2}^{i,j} : \mathbb{R}^d \times \mathbb{R}_+ \times \mathbb{R}^{d \times d} &\rightarrow \mathbb{R}_+ \\ (x, \gamma, v) &\mapsto \gamma |v_{i,j}| r_1(x + \gamma b(x), x + \gamma b(x) + \gamma \sum_{i,j,l=1}^d \partial_{x_l} \sigma_i(x) \sigma_{l,j}(x) v_{i,j}). \end{aligned}$$

We are going to prove that $\mathcal{E}_{ergo}(\sum_{i,j=1}^d \tilde{\Lambda}_{f,2}^{i,j}, \sum_{i,j,l=1}^d |\partial_{x_l} \sigma_i \sigma_{l,j}|)$ (see (12)) holds. First, we notice that $\mathfrak{B}(\phi)$ (61) and the fact that ϕ has sublinear growth, the functions b and $\partial_{x_l} \sigma_i \sigma_{l,j}$, $i, j, l \in \{1, \dots, d\}$, have sublinear growth: there exists $C_{b,\sigma} \geq 0$ such that $|b(x)| + \sum_{i,j,l=1}^d |\partial_{x_l} \sigma_i(x) \sigma_{l,j}(x)| \leq C_{b,\sigma}(1 + |x|)$ for every $x \in \mathbb{R}^d$. Therefore, in the same way as above, we obtain $\mathcal{E}_{ergo}(\sum_{i,j=1}^d \tilde{\Lambda}_{f,2}^{i,j}, \sum_{i,j,l=1}^d |\partial_{x_l} \sigma_i \sigma_{l,j}|)$ from $\sup_{n \in \mathbb{N}^*} \nu_n^\eta(\sum_{i,j,l=1}^d |\partial_{x_l} \sigma_i \sigma_{l,j}|) < \infty$. In order to treat the last term, we write

$$f(\bar{X}_{\Gamma_n}^3) - f(\bar{X}_{\Gamma_n}^2) = \langle \nabla f(\bar{X}_{\Gamma_n}^2), \Delta \bar{X}_n^3 \rangle + \frac{1}{2} \text{Tr}[D^2 f(\bar{X}_{\Gamma_n}^2)(\Delta \bar{X}_n^3)^{\otimes 2}] + R_2(\bar{X}_{\Gamma_{n-1}}^2, \bar{X}_{\Gamma_n}^3)$$

with $R_2(x, y) = f(y) - f(x) - \langle \nabla f(x), y - x \rangle - \frac{1}{2} \text{Tr}[D^2 f(x)(y - x)^{\otimes 2}]$. First we study

$$\mathbb{E}[\text{Tr}[(D^2 f(\bar{X}_{\Gamma_n}^2) - D^2 f(\bar{X}_{\Gamma_{n-1}}^2))(\Delta \bar{X}_n^3)^{\otimes 2}] | \bar{X}_{\Gamma_{n-1}}].$$

We define

$$\begin{aligned} \tilde{\Lambda}_{f,3}^1 : \mathbb{R}^d \times \mathbb{R}_+ \times \mathbb{R}^{d \times d} &\rightarrow \mathbb{R}_+ \\ (x, \gamma, v) &\mapsto \gamma (2 \text{Tr}[v] + d) |D^2 f(x + \gamma b(x) + \gamma \sum_{i,j,l=1}^d \partial_{x_l} \sigma_i(x) \sigma_{l,j}(x) v_{i,j}) - D^2 f(x)|, \end{aligned}$$

we have

$$\mathbb{E}[\text{Tr}[(D^2 f(\bar{X}_{\Gamma_n}^2) - D^2 f(\bar{X}_{\Gamma_{n-1}}^2))(\Delta \bar{X}_n^3)^{\otimes 2}] | \bar{X}_{\Gamma_{n-1}} = x] \leq C |\sigma|^2(x) \int \tilde{\Lambda}_{f,3}^1(x, \gamma_n, v) \mathfrak{W}(dv)$$

Using once again the fact that f has a compact support and the functions b and $\partial_{x_l} \sigma_i \sigma_{l,j}$, $i, j, l \in \{1, \dots, d\}$, have sublinear growth, in the same way as before from, it follows from $\sup_{n \in \mathbb{N}^*} \nu_n^\eta(|\sigma|^2) < \infty$ that $\mathcal{E}_{ergo}(\tilde{\Lambda}_{f,3}^1, |\sigma|^2)$ (with $\pi_3^1 = \mathfrak{W}$) holds.

Now, we consider the other term. Similarly as before, we define the function from $\mathbb{R}^d \times \mathbb{R}^d$ to \mathbb{R}_+ as follows

$$r_2(x, y) = \sup_{\theta \in [0,1]} |D^2 f(x + \theta(y - x)) - D^2 f(x)|. \quad (80)$$

Then r_2 is a bounded continuous function such that $r_2(x, x) = 0$. Moreover, we have

$$R_2(x, y) \leq r_2(x, y) |y - x|^2.$$

We define now

$$\begin{aligned} \tilde{\Lambda}_{f,3}^2 : \mathbb{R}^d \times \mathbb{R}_+ \times \mathbb{R}^{d \times d} &\rightarrow \mathbb{R}_+ \\ (x, \gamma, v) &\mapsto \gamma (2 \text{Tr}[v] + d) \sum_{\zeta \in \{-1,1\}^d} r_2(x + \gamma b(x) + \gamma \sum_{l=1}^d \partial_{x_l} \sigma_i(x) \sigma_{l,j}(x) v_{i,j}, \\ &\quad x + \gamma b(x) + \gamma \sum_{i,j,l=1}^d \partial_{x_l} \sigma_i(x) \sigma_{l,j}(x) v_{i,j} + \sqrt{\gamma} \sum_{i=1}^d \sigma_i(x) \zeta_i \sqrt{2v_{i,i} + 1}) \end{aligned}$$

It follows that

$$\mathbb{E}[|\Delta \bar{X}_n^3|^2 r_2(\bar{X}_{\Gamma_n}^2, \bar{X}_{\Gamma_{n-1}}^3) | \bar{X}_{\Gamma_{n-1}} = x] \leq C |\sigma|^2(x) \int \tilde{\Lambda}_{f,3}^2(x, \gamma_n, v) d\mathfrak{W}(dv)$$

Once again, since b , σ and $\partial_{x_l} \sigma_i \sigma_{l,j}$, $i, j, l \in \{1, \dots, d\}$, have sublinear growth, it follows from $\sup_{n \in \mathbb{N}^*} \nu_n^\eta(|\sigma|^2) < \infty$ that $\mathcal{E}_{ergo}(\tilde{\Lambda}_{f,3}^2, |\sigma|^2)$ (with $\pi_3^2 = \mathfrak{W}$) holds. We gather all the terms together and the proof is completed. \square

4.2.3 Proof of (14) and (16)

Polynomial case

Proposition 4.7. *Let $p \geq 0, a \in (0, 1], \rho, s \in (1, 2]$ and, $\psi(y) = y^p, \phi(y) = y^a$ and $\epsilon_{\mathcal{I}}(t) = t^{\rho/2}$.*

We suppose that $(U_n)_{n \in \mathbb{N}^}$ is a sequence of independent random variables such that U satisfies $M_{\mathcal{N}, 2}(U)$ (see (56)) and $M_{2p \vee 2pp/s \vee 2}(U)$ (see (57)). Moreover, we assume that $(\mathcal{W}_n)_{n \in \mathbb{N}^*}$ is a sequence of independent and centered random variables such that $M_{2p \vee 2pp/s \vee 2}(\mathcal{W})$ (see (58)) holds.*

We also assume that (59), $\mathfrak{B}(\phi)$ (see (61)) and \mathcal{R}_p (see (62)), with this p , hold. We also suppose that $\mathcal{SW}_{\mathcal{I}, \gamma, \eta}(\rho, \epsilon_{\mathcal{I}})$ (see (26)) hold.

Then $\mathcal{SW}_{\mathcal{I}, \gamma, \eta}(V^{p \vee 1 + a - 1}, \rho, \epsilon_{\mathcal{I}})$ (see (19)) holds and we have the following properties

A. *If $\mathcal{SW}_{\mathcal{II}, \gamma, \eta}(V^{p/s})$ (see (20)) and $\mathcal{SW}_{\text{pol}}(p, a, s, \rho)$ (see (85)) hold, then*

$$\mathbb{P}\text{-a.s.} \quad \sup_{n \in \mathbb{N}^*} -\frac{1}{H_n} \sum_{k=1}^n \eta_k \tilde{A}_k^\gamma (\psi \circ V)^s (\bar{X}_{\Gamma_{k-1}}) < \infty, \quad (81)$$

and we also have,

$$\mathbb{P}\text{-a.s.} \quad \sup_{n \in \mathbb{N}^*} \nu_n^\eta (V^{p/s + a - 1}) < \infty. \quad (82)$$

Moreover, when $p/s \leq p \vee 1 + a - 1$, the assumption $\mathcal{SW}_{\mathcal{II}, \gamma, \eta}(V^{p/s})$ (see (20)) can be replaced by $\mathcal{SW}_{\mathcal{I}, \gamma, \eta}$ (see (27)). Besides, if we also suppose that \mathcal{L}_V (see (7)) holds and that $p/s + a - 1 > 0$, then $(\nu_n^\eta)_{n \in \mathbb{N}^}$ is tight*

B. *If $f \in \mathcal{D}(A)$ and (22) is satisfied, then*

$$\mathbb{P}\text{-a.s.} \quad \lim_{n \rightarrow \infty} \frac{1}{H_n} \sum_{k=1}^n \eta_k \tilde{A}_k^\gamma f(\bar{X}_{\Gamma_{k-1}}) = 0 \quad (83)$$

Proof. The result is an immediate consequence of Lemma 3.1. It remains to check the assumption of this Lemma.

First, we show $\mathcal{SW}_{\mathcal{I}, \gamma, \eta}(V^{p \vee 1 + a - 1}, \rho, \epsilon_{\mathcal{I}})$ (see (19)). First we notice that for any $p \leq 1$ then \mathcal{R}_p (see (62)) implies \mathcal{R}_1 . Since (59), $\mathfrak{B}(\phi)$ (see (61)) and \mathcal{R}_p (see (62)) hold, it follows from Proposition 4.5 that $\mathcal{I}_{Q, V}(\tilde{\psi}, \phi)$ (see (8)) is satisfied with the function $\tilde{\psi} : [v_*, \infty) \rightarrow \mathbb{R}_+$ defined by $\tilde{\psi}(y) = y^{p \vee 1}$. Then, using $\mathcal{SW}_{\mathcal{I}, \gamma, \eta}(\rho, \epsilon_{\mathcal{I}})$ (see (26)) with Lemma 3.3, gives $\mathcal{SW}_{\mathcal{I}, \gamma, \eta}(V^{p \vee 1 + a - 1}, \rho, \epsilon_{\mathcal{I}})$ (see (19)). In the same way, for $p/s \leq a + p - 1$, we deduce from $\mathcal{SW}_{\mathcal{II}, \gamma, \eta}$ (see (27)) and Lemma 3.3 that $\mathcal{SW}_{\mathcal{II}, \gamma, \eta}(V^{p/s})$ (see (20)) holds.

Now, we are going to prove $\tilde{\mathcal{I}}_X(f, V^{a+p \vee 1 - 1}, \mathfrak{X}, \rho, \epsilon_{\mathcal{I}})$ (see (18)) for $f \in \mathcal{D}(A)$ and $f = V^{p/s}$ and the proof of (81) and (83) will be completed. Notice that (82) will follow from $\mathcal{I}_{Q, V}(\psi, \phi)$ (see (8)) and Theorem 3.1. The proof is a consequence of Lemma 4.4 which is given below. We notice indeed that $\mathfrak{B}(\phi)$ (see (61)) gives $|\sigma \sigma^*|^{\rho/2} + \sum_{i=1}^d \sum_{l=1}^d |\partial_{x_l} \sigma_i(x) \sigma_{l,i}|^\rho \leq C V^{\rho a/2}$. This observation combined with (86) implies that for every $f \in \mathcal{D}(A)$ and $f = V^{p/s}$, there exists a sequence \mathfrak{X} , such that $\tilde{\mathcal{I}}_X(f, V^{a+p \vee 1 - 1}, \mathfrak{X}, \rho, \epsilon_{\mathcal{I}})$ (see (18)) holds and the proof is completed. \square

Lemma 4.4. *Let $p \geq 0, a \in (0, 1], \rho \in (1, 2]$ and, $\psi(y) = y^p$ and $\phi(y) = y^a$. We suppose that the sequence $(U_n)_{n \in \mathbb{N}^*}$ satisfies $M_{\rho \vee 2pp/s}(U)$ (see (57)) and that the sequence $(\mathcal{W}_n)_{n \in \mathbb{N}^*}$ satisfies $M_{\rho \vee 2pp/s}(\mathcal{W})$ (see (58)). Then, for every $n \in \mathbb{N}$, we have: for every $f \in \mathcal{D}(A)$,*

$$\mathbb{E}[|f(\bar{X}_{\Gamma_{n+1}}) - f(\bar{X}_{\Gamma_n}^1)|^\rho | \bar{X}_{\Gamma_n}] \leq C \gamma_{n+1}^{\rho/2} |\sigma \sigma^*(\bar{X}_{\Gamma_n})|^{\rho/2} + C \gamma_{n+1}^\rho \sum_{i=1}^d \sum_{l=1}^d |\partial_{x_l} \sigma_i(x) \sigma_{l,i}(x)|^\rho. \quad (84)$$

with $\mathcal{D}(A) = \mathcal{C}_K^2(\mathbb{R}^d)$. In other words, for every $f \in \mathcal{D}(A)$, we have $\tilde{\mathcal{I}}_X(f, g_\sigma, \mathfrak{X}, \rho, \epsilon_{\mathcal{I}})$ (see (18)) with $g_\sigma = |\sigma \sigma^|^{\rho/2} + \sum_{i=1}^d \sum_{l=1}^d |\partial_{x_l} \sigma_i(x) \sigma_{l,i}|^\rho$, $\mathfrak{X}_n = f(\bar{X}_{\Gamma_n}^1)$ for every $n \in \mathbb{N}$ and $\epsilon_{\mathcal{I}}(t) = t^{\rho/2}$ for every $t \in \mathbb{R}_+$.*

Moreover, if (59) and $\mathfrak{B}(\phi)$ (see (61)) hold and

$$\mathcal{SW}_{\text{pol}}(p, a, s, \rho) \begin{cases} s(2/\rho - 1)(a + p - 1) + s - 2 \geq 0, & \text{if } 2p/s < 1, \\ (2 - s)/(2 - \rho) \leq a \leq s/\rho, & \text{if } 2p/s \geq 1 \text{ and } p < 1, \\ \rho \leq s(1 - (1 - a)/p), & \text{if } p \geq 1. \end{cases} \quad (85)$$

Then, for every $n \in \mathbb{N}$, we have

$$\mathbb{E}[|V^{p/s}(\bar{X}_{\Gamma_{n+1}}) - V^{p/s}(\bar{X}_{\Gamma_n})|^\rho | \bar{X}_{\Gamma_n}] \leq C \gamma_{n+1}^{\rho/2} V^{p+a-1}(\bar{X}_{\Gamma_n}). \quad (86)$$

In other words, we have $\tilde{\mathcal{I}}_X(V^{p/s}, V^{p+a-1}, \mathfrak{X}, \rho, \epsilon_{\mathcal{I}})$ (see (18)) with $\mathfrak{X}_n = V^{p/s}(\bar{X}_{\Gamma_n})$ for every $n \in \mathbb{N}$ and $\epsilon_{\mathcal{I}}(t) = t^{\rho/2}$ for every $t \in \mathbb{R}_+$.

Proof. We begin by noticing that

$$|\bar{X}_{\Gamma_{n+1}} - \bar{X}_{\Gamma_n}^1| \leq C \gamma_{n+1}^{1/2} |\sigma \sigma^*(\bar{X}_{\Gamma_n})|^{1/2} |U_{n+1}| + C \gamma_{n+1} \left| \sum_{i=1}^d \sum_{l=1}^d |\partial_{x_i} \sigma_i(\bar{X}_{\Gamma_n}) \sigma_{l,i}(\bar{X}_{\Gamma_n})|^2 \right|^{1/2} |\mathcal{W}_{n+1}|$$

Let $f \in \mathcal{D}(A)$. Then f is Lipschitz and the previous inequality gives (84).

We focus now on the case $f = V^{p/s}$. We notice that $\mathfrak{B}(\phi)$ (see (61)) implies that for any $n \in \mathbb{N}$,

$$|\bar{X}_{\Gamma_{n+1}} - \bar{X}_{\Gamma_n}| \leq C \gamma_{n+1}^{1/2} \sqrt{\phi \circ V(\bar{X}_{\Gamma_n})} (1 + |U_{n+1}| + |\mathcal{W}_{n+1}|)$$

First, we assume that $2p/s \leq 1$. Let $x, y \in \mathbb{R}^d$. Then, the function from \mathbb{R}^+ to \mathbb{R}^+ such that $y \mapsto y^{2p/s}$ is concave and since \sqrt{V} is Lipschitz (from (59)), we deduce that

$$\begin{aligned} V^{p/s}(y) - V^{p/s}(x) &\leq 2p/s \sqrt{V}^{2p/s-1}(x) (\sqrt{V}(y) - \sqrt{V}(x)) \\ &\leq 2p/s [\sqrt{V}]_1 V^{p/s-1/2}(x) |y - x|. \end{aligned}$$

Now, since $2(s/\rho - 1)(a + p - 1) + s - 2 \geq 0$ and V takes values in $[v_*, \infty)$, we deduce that there exists $C > 0$ such that for every $x \in \mathbb{R}^d$, we have $V^{p/s-\rho/2}(x) \leq C V^{a(1-\rho)+p-1}(x)$.

Now, we assume that $2p/s \geq 1$. Using (29) with $\alpha = 2p/s$ and since \sqrt{V} is Lipschitz, we have

$$\begin{aligned} V^{p/s}(\bar{X}_{\Gamma_{n+1}}) - V^{p/s}(\bar{X}_{\Gamma_n}) &\leq 2^{2p/s} p/s (V^{p/s-1/2}(\bar{X}_{\Gamma_n}) |\sqrt{V}(\bar{X}_{\Gamma_{n+1}}, z) - \sqrt{V}(\bar{X}_{\Gamma_n})| \\ &\quad + |\sqrt{V}(\bar{X}_{\Gamma_{n+1}}) - \sqrt{V}(\bar{X}_{\Gamma_n}, z)|^{2p/s}) \\ &\leq 2^{2p/s} p/s ([\sqrt{V}]_1 V^{p/s-1/2}(\bar{X}_{\Gamma_n}) |\bar{X}_{\Gamma_{n+1}} - \bar{X}_{\Gamma_n}| \\ &\quad + [\sqrt{V}]_1^{2p/s} |\bar{X}_{\Gamma_{n+1}} - \bar{X}_{\Gamma_n}|^{2p/s}). \end{aligned}$$

In order to obtain (86), it remains to use the assumptions $\mathfrak{B}(\phi)$ (see (61)) and then $\rho \leq s(1 - (1-a)/p)$ if $p \geq 1$ and $(2-s)/(2-\rho) \leq a \leq s/\rho$ together with $2p/s \geq 1$ if $p < 1$. \square

Exponential case

Proposition 4.8. *Let $p \in [0, 1/2]$, $\lambda \geq 0$, $\rho \in (1, 2]$ and $\psi, \phi : [v_*, \infty) \rightarrow \mathbb{R}_+$ with $\psi(y) = \exp(\lambda y^p)$ and ϕ a continuous function such that $\phi(y) \leq Cy$ with $C > 0$ and $\epsilon_{\mathcal{I}}(t) = t^{\rho/2}$. We assume that (59), $\mathfrak{B}(\phi)$ (see (61)) and $\mathcal{R}_{p,\lambda}$ (see (64)) hold and that $\rho < s$. We also suppose that $\mathcal{SW}_{\mathcal{I},\gamma,\eta}(\rho, \epsilon_{\mathcal{I}})$ (see (26)) and (75) hold. Then $\mathcal{SW}_{\mathcal{I},\gamma,\eta}(V^{-1}\phi \circ V \exp(\lambda V^p), \rho, \epsilon_{\mathcal{I}})$ (see (19)) hold and we have the following properties*

A. *If $\mathcal{SW}_{\mathcal{I},\gamma,\eta}$ (see (27)) holds, then we have $\mathcal{SW}_{\mathcal{I},\gamma,\eta}(\exp(\lambda/s V^p))$ (see (20)) and*

$$\mathbb{P}\text{-a.s.} \quad \sup_{n \in \mathbb{N}^*} -\frac{1}{H_n} \sum_{k=1}^n \eta_k \tilde{A}_k^\gamma (\psi \circ V)^s(\bar{X}_{\Gamma_{k-1}}) < \infty, \quad (87)$$

and we also have,

$$\mathbb{P}\text{-a.s.} \quad \sup_{n \in \mathbb{N}^*} \nu_n^\eta (V^{-1}\phi \circ V \exp(\lambda/s V^p)) < \infty. \quad (88)$$

Besides, when \mathcal{L}_V (see (7)) holds, then $(\nu_n^\eta)_{n \in \mathbb{N}^*}$ is tight.

B. If $f \in \mathcal{D}(A)$ and (22) is satisfied, then

$$\mathbb{P}\text{-a.s.} \quad \lim_{n \rightarrow \infty} \frac{1}{H_n} \sum_{k=1}^n \eta_k \tilde{A}_k^\gamma f(\bar{X}_{\Gamma_{k-1}}) = 0 \quad (89)$$

Proof. The result is an immediate consequence of Lemma 3.1. It remains to check the assumption of this Lemma.

First, we show $\mathcal{SW}_{\mathcal{I}, \gamma, \eta}(V^{-1}\phi \circ V \exp(\lambda/sV^p), \rho, \epsilon_{\mathcal{I}})$ (see (19)). We begin by noticing that $\mathcal{R}_{p, \lambda}$ (see (64)) implies $\mathcal{R}_{p, \tilde{\lambda}}$ for every $\tilde{\lambda} \leq \lambda$. Since (59), $\mathfrak{B}(\phi)$ (see (61)), $\mathcal{R}_{p, \lambda}$ (see (64)) and (75) hold, it follows from Proposition 4.5 that $\mathcal{I}_{Q, V}(\tilde{\psi}, \phi)$ (see (8)) is satisfied for every function $\tilde{\psi} : [v_*, \infty) \rightarrow \mathbb{R}_+$ such that $\tilde{\psi}(y) = \exp(\tilde{\lambda}V^p)$ with $\tilde{\lambda} \leq \lambda$. At this point, we notice that this property and the fact that ϕ has sub-linear growth imply (90). Then, using $\mathcal{SW}_{\mathcal{I}, \gamma, \eta}(\rho, \epsilon_{\mathcal{I}})$ (see (26)) with Lemma 3.3, gives $\mathcal{SW}_{\mathcal{I}, \gamma, \eta}(V^{-1}\phi \circ V \exp(\lambda V^p), \rho, \epsilon_{\mathcal{I}})$ (see (19)). In the same way, we deduce from $\mathcal{SW}_{\mathcal{II}, \gamma, \eta}$ (see (27)) and Lemma 3.3 that $\mathcal{SW}_{\mathcal{II}, \gamma, \eta}(V^{-1}\phi \circ V \exp(\lambda/sV^p))$ (see (20)) holds.

Now, we are going to prove $\tilde{\mathcal{I}}_X(f, V^{-1}\phi \circ V \exp(\lambda V^p), \mathfrak{X}, \rho, \epsilon_{\mathcal{I}})$ (see (18)) for $f \in \mathcal{D}(A)$ and $f = V^{p/s}$ and the proof of (87) and (89) will be completed. Notice that (82) will follow from $\mathcal{I}_{Q, V}(\psi, \phi)$ (see (8)) and Theorem 3.1. The proof is a consequence of Lemma 4.4 (see (83)) and Lemma 4.5 which is given below. We notice indeed that $\mathfrak{B}(\phi)$ (see (61)) gives $|\sigma \sigma^*|^{\rho/2} + \sum_{i=1}^d \sum_{l=1}^d |\partial_{x_l} \sigma_i(x) \sigma_{l,i}|^\rho \leq (\phi \circ V)^\rho$. Moreover, we have already shown that (90) is satisfied. These observations combined with (91) imply that for every $f \in \mathcal{D}(A)$ and $f = \exp(\lambda/sV^p)$, there exists a sequence \mathfrak{X} , such that $\tilde{\mathcal{I}}_X(f, V^{-1}\phi \circ V \exp(\lambda V^p), \mathfrak{X}, \rho, \epsilon_{\mathcal{I}})$ (see (18)) holds and the proof is completed. \square

Lemma 4.5. Let $p \in [0, 1/2]$, $\lambda \geq 0$, $\rho, s \in (1, 2]$ and $\psi, \phi : [v_*, \infty) \rightarrow \mathbb{R}_+$ with $\psi(x) = \exp(\lambda x^p)$ and ϕ a continuous function such that $\phi(x) \leq Cx$ with $C > 0$. We assume that (59) and $\mathfrak{B}(\phi)$ (see (61)) hold, that $\rho < s$, and there exists $n_0 \in \mathbb{N}^*$, such that

$$\forall \tilde{\lambda} \leq \lambda, \exists C \geq 0, \forall n \geq n_0, \quad \mathbb{E}[\exp(\tilde{\lambda}V^p(\bar{X}_{\Gamma_{n+1}})) | \bar{X}_{\Gamma_n}] \leq C \exp(\tilde{\lambda}V^p(\bar{X}_{\Gamma_n})). \quad (90)$$

Then, for every $n \geq n_0$, we have

$$\mathbb{E}[|\exp(\lambda/sV^p(\bar{X}_{\Gamma_{n+1}})) - \exp(\lambda/sV^p(\bar{X}_{\Gamma_n}))|^\rho | \bar{X}_{\Gamma_n}] \leq C \gamma_{n+1}^{\rho/2} \frac{\phi \circ V(\bar{X}_{\Gamma_n})}{V(\bar{X}_{\Gamma_n})} \exp(\lambda V^p(\bar{X}_{\Gamma_n})), \quad (91)$$

In other words, we have $\tilde{\mathcal{I}}_X(\exp(\lambda/sV^p), V^{-1}\phi \circ V \exp(\lambda V^p), \mathfrak{X}, \rho, \epsilon_{\mathcal{I}})$ (see (18)) with $\mathfrak{X}_n = \exp(\lambda/sV^p(\bar{X}_{\Gamma_n}))$ for every $n \in \mathbb{N}$ and $\epsilon_{\mathcal{I}}(t) = t^{\rho/2}$ for every $t \in \mathbb{R}_+$.

Proof. Before we prove the result, we notice that $\mathfrak{B}(\phi)$ (see (61)) implies that for any $n \in \mathbb{N}$,

$$|\bar{X}_{\Gamma_{n+1}} - \bar{X}_{\Gamma_n}| \leq C \gamma_n^{1/2} \sqrt{\phi \circ V(\bar{X}_{\Gamma_n})} (1 + |U_{n+1}|^2 + |W_{n+1}|^2).$$

First, we assume that $p \leq 1/2$. Let $x, y \in \mathbb{R}^d$. First, since the function $x \mapsto x^{2p}$ is concave, we have

$$\begin{aligned} V^p(y) - V^p(x) &\leq 2p\sqrt{V}^{2p-1}(x)(\sqrt{V}(y) - \sqrt{V}(x)) \\ &\leq 2p[\sqrt{V}]_1 V^{p-1/2}(x) |y - x|. \end{aligned}$$

Moreover,

$$\exp(\lambda/sV^p(y)) - \exp(\lambda/sV^p(x)) \leq \frac{\lambda}{s} (\exp(\lambda/sV^p(y)) + \exp(\lambda/sV^p(x))) |V^p(y) - V^p(x)|.$$

We combine those two inequalities and use Hölder inequality in order to obtain

$$\begin{aligned} &\mathbb{E}[|\exp(\lambda/sV^p(\bar{X}_{\Gamma_{n+1}})) - \exp(\lambda/sV^p(\bar{X}_{\Gamma_n}))|^\rho | \bar{X}_{\Gamma_n}] \\ &\leq C \exp(\lambda\rho/sV^p(\bar{X}_{\Gamma_n})) V^{p\rho-\rho/2}(\bar{X}_{\Gamma_n}) \mathbb{E}[|\bar{X}_{\Gamma_{n+1}} - \bar{X}_{\Gamma_n}|^\rho | \bar{X}_{\Gamma_n}] \\ &\quad + C V^{p\rho-\rho/2}(\bar{X}_{\Gamma_n}) \mathbb{E}[\exp(\lambda\rho/sV^p(\bar{X}_{\Gamma_{n+1}})) | \bar{X}_{\Gamma_{n+1}} - \bar{X}_{\Gamma_n}|^\rho | \bar{X}_{\Gamma_n}] \\ &\leq C \exp(\lambda\rho/sV^p(\bar{X}_{\Gamma_n})) V^{p\rho-\rho/2}(\bar{X}_{\Gamma_n}) \mathbb{E}[|\bar{X}_{\Gamma_{n+1}} - \bar{X}_{\Gamma_n}|^\rho | \bar{X}_{\Gamma_n}] \\ &\quad + C V^{p\rho-\rho/2}(\bar{X}_{\Gamma_n}) \mathbb{E}[\exp(\lambda\rho\theta/sV^p(\bar{X}_{\Gamma_{n+1}})) | \bar{X}_{\Gamma_n}]^{1/\theta} \mathbb{E}[|\bar{X}_{\Gamma_{n+1}} - \bar{X}_{\Gamma_n}|^{\rho\theta/(\theta-1)} | \bar{X}_{\Gamma_n}]^{(\theta-1)/\theta}, \end{aligned}$$

for every $\theta > 1$. Now, we use (90) and since $\rho < s$, for every $\theta \in (1, s/\rho]$, we obtain

$$\mathbb{E}[\exp(\lambda\rho\theta/sV^p(\bar{X}_{\Gamma_{n+1}}))|\bar{X}_{\Gamma_n}] \leq C \exp(\lambda\rho\theta/sV^p(\bar{X}_{\Gamma_n})).$$

Rearranging the terms and since $\rho < s$, we conclude from $\mathfrak{B}(\phi)$ (see (61)) that

$$\begin{aligned} \mathbb{E}[|\exp(\lambda/sV^p(\bar{X}_{\Gamma_{n+1}})) - \exp(\lambda/sV^p(\bar{X}_{\Gamma_n}))|^\rho|\bar{X}_{\Gamma_{n+1}}] &\leq C\gamma_n^{\rho/2}V^{p\rho-\rho/2}(\bar{X}_{\Gamma_n})|\phi \circ V(\bar{X}_{\Gamma_n})|^{\rho/2} \exp(\lambda\rho/sV^p(\bar{X}_{\Gamma_n})) \\ &\leq C\gamma_n^{\rho/2}V^{-1}(\bar{X}_{\Gamma_n})\phi \circ V(\bar{X}_{\Gamma_n}) \exp(\lambda V^p(\bar{X}_{\Gamma_n})). \end{aligned}$$

□

4.3 Application to processes with jump

The purpose of this section is to build an invariant measure using a decreasing step Euler scheme for a Feller diffusion process with jump which is not necessarily a Levy process. This study extends the one in [13] where the author treat the convergence of $(\nu_n^\eta)_{n \in \mathbb{N}^*}$ for miscellaneous decreasing step Euler scheme for Levy processes. The interest of our approach is that we consider process with some general jump components which involve Levy processes but also diffusion process with censored jump or piecewise deterministic Markov processes. We consider weak mean reverting assumption that is $\phi(y) = y^a$, $a \in (0, 1]$ for every $y \in [v_*, \infty)$. Similarly as in its study we consider polynomial test functions ψ such that $\psi(y) = y^p$, $p \geq 0$ for every $y \in [v_*, \infty)$.

We consider a Poisson point process p with state space $(F; \mathcal{B}(F))$ where $\bar{F} = F \times \mathbb{R}_+$. We refer to [6] for more details. We denote by N the counting measure associated to p . We have $N([0, t) \times A) = \#\{0 \leq s < t; p_s \in A\}$ for $t \geq 0$ and $A \in \mathcal{B}(\bar{F})$. We assume that the associated intensity measure is given by $\hat{N}(dt, dz, dv) = dt \times \lambda(dz) \times \mathbb{1}_{[0, \infty)}(v)dv$ where $(z, v) \in \bar{F} = F \times \mathbb{R}_+$. We will use the notation $\tilde{N} = N - \hat{N}$. We also consider a d -dimension Brownian motion $(W_t)_{t \geq 0}$ independent from N . We are interested in the solution of the d dimensional stochastic equation

$$\begin{aligned} X_t = x + \int_0^t b(X_{s-})ds + \int_0^t \sigma(X_{s-})dW_s + \int_0^t \int_{\bar{F}} c(z, X_{s-})\mathbb{1}_{v \leq \zeta(z, X_{s-})}\mathbb{1}_{[0, h]}(|z|)\tilde{N}(ds, dz, dv) \\ + \int_0^t \int_{\bar{F}} c(z, X_{s-})\mathbb{1}_{v \leq \zeta(z, X_{s-})}\mathbb{1}_{(h, \infty)}(|z|)N(ds, dz, dv). \end{aligned} \quad (92)$$

where $h \geq 0$, $b : \mathbb{R}^d \rightarrow \mathbb{R}^d$, $\sigma : \mathbb{R}^d \rightarrow \mathbb{R}^{d \times d}$ and $c(z, \cdot) : \mathbb{R}^d \rightarrow \mathbb{R}^d$, $z \in \mathbb{R}^l$ are locally bounded and continuous functions. In this paper, we do not discuss existence of such processes. This processes can be seen as extension of Levy process (put $c(z, x) = c(x)z$ and $\zeta = 1$). Especially, if we want decomposition (92) to make sense, we must at least assume that for every $x \in \mathbb{R}^d$, we have

$$\int_F |c(z, x)|^2 \zeta(z, x) \mathbb{1}_{[0, h]}(|z|) \lambda(dz) < \infty, \quad (93)$$

and

$$\int_F |c(z, x)| \zeta(z, x) \mathbb{1}_{(h, \infty)}(|z|) \lambda(dz) < \infty. \quad (94)$$

The main difference with Levy processes is that the intensity of jump $\xi(x, z)\lambda(dz)$ may depend on the position of the process. Actually, this type of process can also be seen as an extension of SDE with censored jump component. Indeed, if for every $x \in \mathbb{R}^d$, we have

$$\int_F |c(z, x)| \zeta(z, x) \lambda(dz) < \infty, \quad (95)$$

it comes down to study the solution of following SDE with censored jump part:

$$X_t = x + \int_0^t \tilde{b}(X_{s-}) ds + \int_0^t \sigma(X_{s-}) dW_s + \int_0^t \int_{\bar{F}} c(z, X_{s-}) \mathbf{1}_{v \leq \zeta(z, X_{s-})} \mathbf{1}_{[0, \infty)}(|z|) N(ds, dz, dv). \quad (96)$$

with, for every $x \in \mathbb{R}^d$, $\tilde{b}(x) = b(x) + \int_{\bar{F}} c(z, x) \zeta(z, x) \mathbf{1}_{[0, h]}(|z|) \lambda(dz)$. The study of this family of processes in the literature is expensing. In [3], the author focus on the case $\sigma = 0$ and prove the existence of an absolutely continuous (with respect to the Lebesgue measure) density. In the PhD thesis [14], the author extends existence and uniqueness results for SDE with non null Brownian part and censored jump part and also show that they can be considered as limit processes of some general piecewise deterministic Markov processes. Besides, he studies ergodicity of those processes using a regenerated procedure. This procedure provides a Doeblin (locally lower Lebesgue bounded) condition which enables to prove recurrence in Harris sense and then ergodicity. Finally, one may notice that the study for SDE with censored jump part with form (96), is equivalent to the study of (92) with $h = 0$. Consequently, we study the approximation of invariant measures for solutions of (92) and the results we provide in this part apply to SDE with censored jump as soon as we put $h = 0$.

The infinitesimal generator of this process is given by

$$Af(x) = \langle b(x), \nabla f(x) \rangle + \sum_{i,j=1}^d (\sigma \sigma^*)_{i,j}(x) \frac{\partial^2 f}{\partial x_i \partial x_j}(x) + \int_{\bar{F}} (f(x + c(z, x)) - f(x) - \langle c(z, x), \nabla f(x) \rangle \mathbf{1}_{[0, h]}(|z|)) \zeta(z, x) \lambda(dz). \quad (97)$$

The first step is to consider an ‘truncated’ approximation for the process $(X_t)_{t \geq 0}$ with finite big jump intensity. In this case we can introduce an Euler scheme for this ‘truncated’ process and then prove convergence of the measure defined in (5) toward an invariant measure of the process $(X_t)_{t \geq 0}$. In order to show that the limit of the measure defined in (5) and built with the Euler scheme of the truncated process is an invariant measure for the process $(X_t)_{t \geq 0}$, it is necessary to introduce a supplementary hypothesis. This hypothesis enables to control of the distance between the generator of $(X_t)_{t \geq 0}$ and the generator of the ‘truncated’ process with jump size at most M . When $\lambda(\{z, h < |z|\}) = \infty$, we assume that

$$\lim_{M \rightarrow \infty} \lim_{n \rightarrow \infty} \nu_n^n(\bar{\lambda}_M) = 0 \quad \text{with} \\ \forall M \geq h, \forall x \in \mathbb{R}^d, \quad \bar{\lambda}_M(x) := \int_{\bar{F}} |c(z, x)| \zeta(z, x) \mathbf{1}_{(M, \infty)}(|z|) \lambda(dz) = 0. \quad (98)$$

4.3.1 Approach and preliminary results

Let $M \in \mathbb{R}_+ \cup \{+\infty\}$ such that $\lambda(\{z, h < |z| < M\}) < \infty$. We define $N^M(ds, dz, dv) := \mathbf{1}_{|z| < M} \mathbf{1}_{v \leq \|\zeta\|_\infty} N(ds, dz, dv)$. Now, we introduce the process $(X_t^M)_{t \geq 0}$ which satisfies the following equation

$$X_t^M = x + \int_0^t b(X_{s-}^M) ds + \int_0^t \sigma(X_{s-}^M) dW_s + \int_0^t \int_{\bar{F}} c(z, X_{s-}^M) \mathbf{1}_{v \leq \zeta(z, X_{s-}^M)} \mathbf{1}_{[0, h]}(|z|) \tilde{N}_M(ds, dz, dv) \\ + \int_0^t \int_{\bar{F}} c(z, X_{s-}^M) \mathbf{1}_{v \leq \zeta(z, X_{s-}^M)} \mathbf{1}_{(h, \infty)}(|z|) N_M(ds, dz, dv) \quad (99)$$

Since \hat{N}_M is finite, we represent the random measure N_M using a compound Poisson process. We introduce the Poisson processes $(J_t^M)_{t \geq 0}$, independent from \tilde{N}_M , with intensity $\|\zeta\|_\infty \lambda(\{z, h < |z| < M\})$ and jump times $(T_k^M)_{k \in \mathbb{N}^*}$. We introduce the sequences of independent random variables (and independent from J^M and \tilde{N}_M)

$$Z_k^M \sim \lambda(\{z, h < |z| < M\})^{-1} \mathbf{1}_{h < |z| < M} dz, \quad \text{and} \quad V_k \sim \|\zeta\|_\infty^{-1}(v) \mathbf{1}_{v \leq \|\zeta\|_\infty} dv.$$

Therefore, (99) can be rewritten

$$\begin{aligned} X_t^M &= x + \int_0^t b(X_s^M) ds + \int_0^t \int_{\bar{F}} c(z, X_s^M) \mathbb{1}_{v \leq \zeta(z, X_s^M)} \mathbb{1}_{[0, h]}(|z|) \tilde{N}_M(ds, dz, dv) \\ &+ \int_0^t \sigma(X_s^M) dW_s + \sum_{k=1}^{J_t^M} c(Z_k^M, X_{T_k^-}^M) \mathbb{1}_{V_k \leq \zeta(Z_k^M, X_{T_k^-}^M)} \end{aligned} \quad (100)$$

The infinitesimal generator of this process is given by

$$\begin{aligned} A^M f(x) &= \langle b(x), \nabla f(x) \rangle + \sum_{i,j=1}^d (\sigma \sigma^*)_{i,j}(x) \frac{\partial^2 f}{\partial x_i \partial x_j}(x) \\ &+ \int_{\bar{F}} (f(x + c(z, x)) - f(x) - \langle c(z, x), \nabla f(x) \rangle \mathbb{1}_{[0, h]}(|z|)) \mathbb{1}_{(h, M)}(|z|) \zeta(z, x) \lambda(dz). \end{aligned} \quad (101)$$

Now, we introduce an approximation for X^M . We will use a Euler type scheme such that for every $n \in \mathbb{N}$ and $t \in [\Gamma_n, \Gamma_{n+1}]$, we have

$$\begin{aligned} \bar{X}_t^M &= \bar{X}_{\Gamma_n}^M + (t - \Gamma_n) b(\bar{X}_{\Gamma_n}^M) + \int_{\Gamma_n}^t c(z, X_{\Gamma_n}^M) \mathbb{1}_{v \leq \zeta(z, X_{\Gamma_n}^M)} \mathbb{1}_{[0, h]}(|z|) \tilde{N}_M(ds, dz, dv) \\ &+ \sigma(\bar{X}_{\Gamma_n}^M) (W_t - W_{\Gamma_n}) + \sum_{k=1+J_{\Gamma_n}^M}^{J_t^M} c(Z_k^M, \bar{X}_{\Gamma_n}^M) \mathbb{1}_{V_k \leq \zeta(Z_k^M, \bar{X}_{\Gamma_n}^M)}. \end{aligned} \quad (102)$$

which is well defined since $\lambda(\{z, h < |z| < M\}) < \infty$. Now, for $\tilde{h} > 0$, $x \in \mathbb{R}^d$ and $t \geq 0$, we define

$$\begin{aligned} M_{1,t}^{\tilde{h}}(x) &:= \int_0^t \int_{\bar{F}} c(z, x) \mathbb{1}_{v \leq \zeta(z, x)} \mathbb{1}_{[0, \tilde{h}]}(|z|) N_M(ds, dz, dv), \\ \tilde{M}_{1,t}^{\tilde{h}}(x) &:= \int_0^t \int_{\bar{F}} c(z, x) \mathbb{1}_{v \leq \zeta(z, x)} \mathbb{1}_{[0, \tilde{h}]}(|z|) \tilde{N}_M(ds, dz, dv), \\ M_{2,t}^{\tilde{h}}(x) &:= \int_0^t \int_{\bar{F}} c(z, x) \mathbb{1}_{v \leq \zeta(z, x)} \mathbb{1}_{(\tilde{h}, M)}(|z|) N_M(ds, dz, dv), \\ \tilde{M}_{2,t}^{\tilde{h}}(x) &:= \int_0^t \int_{\bar{F}} c(z, x) \mathbb{1}_{v \leq \zeta(z, x)} \mathbb{1}_{(\tilde{h}, M)}(|z|) \tilde{N}_M(ds, dz, dv). \end{aligned} \quad (103)$$

In order to simplify the writing, we will use the notations:

$$\begin{aligned} \Delta \bar{X}_{n+1}^{M,1} &= \gamma_{n+1} b(\bar{X}_{\Gamma_n}^M), & \Delta \bar{X}_{n+1}^{M,2} &= \sigma(\bar{X}_{\Gamma_n}^M) (W_{\Gamma_{n+1}} - W_{\Gamma_n}), \\ \Delta \bar{X}_{n+1}^{M,3} &= \tilde{M}_{1, \gamma_{n+1}}^{\tilde{h}}(\bar{X}_{\Gamma_n}^M), & \Delta \bar{X}_{n+1}^{M,4} &= M_{2, \gamma_{n+1}}^{\tilde{h}}(\bar{X}_{\Gamma_n}^M), \end{aligned} \quad (104)$$

and $\bar{X}_{\Gamma_{n+1}}^{M,i} = \bar{X}_{\Gamma_n}^M + \sum_{k=1}^i \Delta \bar{X}_{n+1}^{M,k}$. At this point, we precise that we implicitly suppose that $\Delta \bar{X}_{n+1}^{M,3}$ can be simulated at time Γ_n . This assumption prevails in this paper. When it is not possible a solution is given in [13]. It consists in localizing the small jumps of $\bar{X}_{\Gamma_n}^M - \bar{X}_{\Gamma_{n-1}}^M$ on a strict subset $[\underline{h}_n, h]$ ($h_n > 0$) of $[0, h]$ with $\lim_{n \rightarrow \infty} \underline{h}_n = 0$ and in assuming that the small jumps with size contained in $[\underline{h}_n, h]$ can be simulated. This specific study in our case is very similar to the one for the Levy process made in [13], and also to the one we do when we suppose that $\Delta \bar{X}_{n+1}^{M,3}$ can be simulated. Consequently, we propose a study in which we assume $\Delta \bar{X}_{n+1}^{M,3}$ we invite the reader to refer [13] in order to generalize it to the case

where it can not be simulated.

For every $n \in \mathbb{N}^*$ and $t \in [\Gamma_n, \Gamma_{n+1}]$, the infinitesimal generator of $(\bar{X}_t^M)_{T \geq 0}$ is given by

$$\begin{aligned} A_{x_0}^M f(x) &= \langle b(x_0), \nabla f(x) \rangle + \text{Tr}[\sigma \sigma^*(x_0) D^2 f(x)] \\ &\quad + \int_F (f(x + c(z, x_0)) - f(x) - \langle c(z, x_0), \nabla f(x) \rangle \mathbb{1}_{[0, h]}(|z|) \zeta(z, x_0) \mathbb{1}_{[0, M]}(|z|) \lambda(dz), \end{aligned} \quad (105)$$

on the set $\{\bar{X}_{\Gamma_n}^M = x_0\}$, with the notation $D^2 f(x) = \frac{\partial^2 f}{\partial x_i \partial x_j}(x)$. At this point we notice that $Af(x) = A_x f(x)$ which will be a key property in order to prove the infinitesimal estimation $\mathcal{E}(\bar{A}^\gamma, A)$ (see (9)) in the sequel (in particular for the jump part).

In the sequel we will use the notation $U_{n+1} = \gamma_{n+1}^{-1/2}(W_{\Gamma_{n+1}} - W_{\Gamma_n})$. Actually, we introduce a weaker assumption than Gaussian distribution for the sequence $(U_n)_{n \in \mathbb{N}^*}$. Let $q \in \mathbb{N}^*$, $p \geq 0$. We suppose that $(U_n)_{n \in \mathbb{N}^*}$ is a sequence of independent random variables such that

$$M_{\mathcal{N}, q}(U) \quad \forall n \in \mathbb{N}^*, \forall \tilde{q} \in \{1, \dots, q\}, \quad \mathbb{E}[(U_n)^{\otimes \tilde{q}}] = \mathbb{E}[(\mathcal{N}(0, I_d))^{\otimes \tilde{q}}] \quad (106)$$

$$M_p(U) \quad \sup_{n \in \mathbb{N}^*} \mathbb{E}[|U_n|^p] < \infty \quad (107)$$

Now we introduce some hypothesis concerning the parameters. First, we introduce the hypothesis concerning the jump components. In the sequel, we will denote

$$\underline{\mathcal{I}}_{q, h}(x) = \int_F c(z, x)^{2q} \zeta(z, x) \mathbb{1}_{[0, h]}(|z|) \lambda(dz), \quad \bar{\tau}_{p, h}(x) = \int_F c(z, x)^{2p} \zeta(z, x) \mathbb{1}_{(h, M)}(|z|) \lambda(dz), \quad (108)$$

$$\text{and} \quad \tau_p(x) = \int_F c(z, x)^{2p} \zeta(z, x) \mathbb{1}_{[0, M]}(|z|) \lambda(dz),$$

for $p, q \geq 0$. We assume the following finiteness hypothesis: Let $p, q \geq 0$. for every $x \in \mathbb{R}^d$, we have

$$\underline{\mathcal{H}}_h^q \quad \underline{\mathcal{I}}_{q, h}(x) < \infty, \quad \bar{\mathcal{H}}_h^p \quad \bar{\tau}_{p, h}(x) < \infty, \quad \mathcal{H}^p \quad \tau_p(x) < \infty. \quad (109)$$

It is immediate to notice that \mathcal{H}^p implies both $\underline{\mathcal{H}}_h^p$ and $\bar{\mathcal{H}}_h^p$ for any $h > 0$. Moreover, we introduce the following classical hypothesis: $\underline{\mathcal{H}}_h^q$ implies $\underline{\mathcal{H}}_h^{q'}$ for $q \leq q'$ and $\bar{\mathcal{H}}_h^p$ implies $\bar{\mathcal{H}}_h^{p'}$ for $p \geq p'$. Now, we assume that the Lyapunov function $V : \mathbb{R}^d \rightarrow [v_*, \infty)$, $v_* > 0$, satisfies \mathcal{L}_V (see (7)) and

$$|\nabla V|^2 \leq C_V V, \quad \sup_{x \in \mathbb{R}^d} |D^2 V(x)| < \infty \quad (110)$$

We also define

$$\forall x \in \mathbb{R}^d, \quad \lambda_\psi(x) := \frac{1}{2} \lambda_{D^2 V(x) + \nabla V(x) \otimes \psi'' \circ V(x) \psi' \circ V(x)^{-1}}. \quad (111)$$

When $\psi(x) = |x|^p$, we will also use the notation λ_p instead of λ_ψ . We will suppose that, for every $x \in \mathbb{R}^d$,

$$\mathfrak{B}_{p, q}(\phi) \quad |b(x) + \kappa_{p, q}(x)|^2 + |\sigma \sigma^*(x)| \leq C \phi \circ V(x) \quad (112)$$

where

$$\kappa_{p, q}(x) = \int_{\mathbb{R}^d} c(z, x) \zeta(z, x) (\mathbb{1}_{p > 1/2} \mathbb{1}_{(h, M)}(|z|) - \mathbb{1}_{p, q \leq 1/2} \mathbb{1}_{[0, h]}(|z|)) \lambda(dz). \quad (113)$$

The reader may notice that $\kappa_{p, q}$ is well defined when $\underline{\mathcal{H}}_h^q$ and $\bar{\mathcal{H}}_h^p$ hold. When $p > 1/2$ we will also use the notation κ_p instead of $\kappa_{p, q}$. For $p, q \geq 0$ and ϕ a positive function, we introduce the following hypothesis

$$\begin{aligned} \underline{\mathcal{H}}_h^q(\phi, V) &\quad \underline{\mathcal{I}}_{q, h}(x) \leq C \phi \circ V(x)^q, \\ \bar{\mathcal{H}}_h^p(\phi, V) &\quad \bar{\tau}_{p, h}(x) \leq C \phi \circ V(x)^p, \\ \mathcal{H}^p(\phi, V) &\quad \tau_p(x) \leq C \phi \circ V(x)^p, \end{aligned} \quad (114)$$

for every $x \in \mathbb{R}^d$.

Remark 4.1. We notice that $\mathcal{H}^p(\phi, V)$ implies both $\underline{\mathcal{H}}_h^p(\phi, V)$ and $\overline{\mathcal{H}}_h^p(\phi, V)$ for any $h \geq 0$. Moreover, $\underline{\mathcal{H}}_h^q(\phi, V)$ implies $\underline{\mathcal{H}}_h^q(\phi, V)$ for any $\tilde{h} \in (0, h]$.

We now introduce the key hypothesis in order to obtain recursive control that we will use for polynomial functions ψ . We assume that there exists $\beta \in \mathbb{R}_+$, $\alpha > 0$, such that for every $x \in \mathbb{R}^d$, we have

$$\mathcal{R}_{p,q} \quad \langle \nabla V(x), b(x) + \kappa_{p,q}(x) \rangle + \chi_{p,q}(x) \leq \beta - \alpha \phi \circ V(x), \quad (115)$$

with

$$\chi_{p,q}(x) = \begin{cases} \|\lambda_1\|_\infty (\text{Tr}[\sigma\sigma^*(x)] + \tau_1(x)) + p^{-1}V^{1-p}(x)\tilde{\chi}_{p,q,h}(x) & \text{if } p \leq 1 \\ \|\lambda_p\|_\infty 2^{(2p-3)+} (\text{Tr}[\sigma\sigma^*] + \tau_1(x) + [\sqrt{V}]_1^{2p-2}V^{1-p}(x)\tau_p(x)) & \text{if } p \geq 1. \end{cases} \quad (116)$$

with

$$\begin{aligned} \tilde{\chi}_{p,q,h}(x) = & (\mathbb{1}_{p,q \leq 1/2} [\sqrt{V}]_1^{2(p \vee q)} v_*^{p-p \vee q} + \mathbb{1}_{p \leq 1/2} \mathbb{1}_{q > 1/2} C_q [V^{p-1} \nabla V]_{2q-1} \\ & + \mathbb{1}_{p > 1/2} C_{p \vee q} [V^{p \vee q - 1} \nabla V]_{2(p \vee q) - 1} \underline{\tau}_{p \vee q, h}(x) \\ & + (\mathbb{1}_{p \leq 1/2} [\sqrt{V}]_1^{2p} + \mathbb{1}_{p > 1/2} C_p [V^{p-1} \nabla V]_{2p-1}) \bar{\tau}_{p, h}(x) \end{aligned}$$

and C_q and $C_{p \vee q}$ the constant from the BDG inequality defined in (31).

4.3.2 Proof of the recursive control

In order to obtain the recursive mean reverting control, we require a first result concerning the evolution of the jump components.

Lemma 4.6. We have the following properties

A. Let $p \geq 0$. Assume that $\overline{\mathcal{H}}_h^p$ (see (109)) hold. There exists a locally bounded function $\epsilon : \mathbb{R}_+ \rightarrow \mathbb{R}$ which satisfies $\epsilon(t)/t \leq C$, and such that $\forall n \in \mathbb{N}$,

$$\mathbb{E}[|M_{2,t}^h(x)|^{2p}] \leq t(1 + \epsilon(t))\bar{\tau}_{p,h}(x). \quad (117)$$

B. Let $q \in [0, 1/2]$ and assume that $\underline{\mathcal{H}}_h^q$ (see (109)) hold. Then, $\forall n \in \mathbb{N}$,

$$\mathbb{E}[|M_{1,t}^h(x)|^{2q}] \leq t\underline{\tau}_{q,h}(x). \quad (118)$$

C. Let $q \in [1/2, 1]$ and assume that $\underline{\mathcal{H}}_h^q$ (see (109)) hold. Then, $\forall n \in \mathbb{N}$,

$$\mathbb{E}[|\tilde{M}_{1,t}^h(x)|^{2q}] \leq C_q t \underline{\tau}_{q,h}(x). \quad (119)$$

with C_q the constant which appears in the BDG inequality (see (31)).

D. Let $p > 1$. We assume that \mathcal{H}^p (see (109)) hold. Then, there exists $\xi > 1$, which does not depend on h , such that for we have

$$\mathbb{E}[|\tilde{M}_{1,t}^h(x) + \tilde{M}_{2,t}^h(x)|^{2p}] \leq t(\tau_p(x) + c_p \underline{\tau}_{p,h}(x)) + C_{p,h} t^\xi \phi \circ V(x)^p. \quad (120)$$

where $c_p \geq 0$, and $C_{p,h} \geq 0$ is finite if $\underline{\mathcal{H}}_h^1(\phi, V)$, $\underline{\mathcal{H}}_h^p(\phi, V)$, $\overline{\mathcal{H}}_h^{1/2}(\phi, V)$ and $\overline{\mathcal{H}}_h^p(\phi, V)$ (see (114)) hold.

Now, let $p \in [1/2, 1)$. Assume that \mathcal{H}^p (see (109)) holds. Then

$$\mathbb{E}[|\tilde{M}_{1,t}^h(x) + \tilde{M}_{2,t}^h(x)|^{2p}] \leq C_p t \tau_p(x) \quad (121)$$

with C_q the constant which appears in the BDG inequality (see (31)).

If we assume instead that $\underline{\mathcal{H}}^1$ and $\overline{\mathcal{H}}^p$ (see (109)) hold, there exists $C \geq 0$ such that

$$\mathbb{E}[|\tilde{M}_{1,t}^h(x) + \tilde{M}_{2,t}^h(x)|^{2p}] \leq C(t^p \underline{\tau}_1^p(x) + t \bar{\tau}_p(x)) \quad (122)$$

Finally, if $p = 1$, and only \mathcal{H}^1 (see (109)) hold then

$$\mathbb{E}[|\tilde{M}_{1,t}^h(x) + \tilde{M}_{2,t}^h(x)|^2] = t \tau_1(x) \quad (123)$$

Proof. We prove point **A.**. Let $(J_t^M)_{t \geq 0}$, a Poisson process with intensity $\|\zeta\|_\infty \mu(\{z, h < |z| < M\})$ and jump times $(T_k^M)_{k \in \mathbb{N}^*}$. We introduce the sequences of independent random variables (and independent from J^M)

$$Z_k^M \sim \lambda(\{z, h < |z| < M\})^{-1} \mathbf{1}_{h < |z| < M} dz, \quad \text{and} \quad V_k \sim \|\zeta\|_\infty^{-1}(v) \mathbf{1}_{v \leq \|\zeta\|_\infty} dv.$$

We rewrite,

$$\tilde{M}_{2,t}^h(x) = \sum_{k=1}^{J_t^M} c(Z_k^M, x) \mathbf{1}_{V_k \leq \zeta(Z_k^M, x)}.$$

Now we denote $\tilde{\lambda} = \|\zeta\|_\infty \lambda(\{z, h < |z| < M\})$. Therefore,

$$\begin{aligned} \mathbb{E}\left[\left|\sum_{k=1}^{J_t^M} c(Z_k^M, x) \mathbf{1}_{V_k \leq \zeta(Z_k^M, x)}\right|^{2p}\right] &= \mathbb{E}\left[\sum_{k \geq 1} \mathbf{1}_{k=J_t^M} \left|\sum_{l=1}^k c(Z_l^M, x) \mathbf{1}_{V_l \leq \zeta(Z_l^M, x)}\right|^{2p}\right] \\ &= \sum_{k \geq 1} \mathbb{E}\left[\left|\sum_{l=1}^k c(Z_l^M, x) \mathbf{1}_{V_l \leq \zeta(Z_l^M, x)}\right|^{2p}\right] \frac{e^{-\tilde{\lambda}t} (\tilde{\lambda}t)^k}{k!}. \end{aligned}$$

Now, using the inequality (30), it follows that

$$\begin{aligned} \mathbb{E}\left[\left|\sum_{l=1}^k c(Z_l^M, x) \mathbf{1}_{V_l \leq \zeta(Z_l^M, x)}\right|^{2p}\right] &\leq k^{(2p-1)_+} \sum_{l=1}^k \mathbb{E}\left[|c(Z_l^M, x) \mathbf{1}_{V_l \leq \zeta(Z_l^M, x)}|^{2p}\right] \\ &= k^{1+(2p-1)_+} \tilde{\lambda}^{-1} \int_{\mathbb{R}^d} c(z, \bar{X}_{\Gamma_n}^M)^{2p} \zeta(z, \bar{X}_{\Gamma_n}^M) \mathbf{1}_{(h,M)}(|z|) \lambda(dz). \end{aligned}$$

Moreover,

$$e^{-\tilde{\lambda}t} \sum_{k \geq 1} k^{1+(2p-1)_+} \frac{(\tilde{\lambda}t)^k}{k!} = e^{-\tilde{\lambda}t} \tilde{\lambda}t \sum_{k \geq 0} (k+1)^{(2p-1)_+} \frac{(\tilde{\lambda}t)^k}{k!}$$

Now, we are going to use the inequalities (28) and (29). If $p < 1$ then

$$\begin{aligned} e^{-\tilde{\lambda}t} \tilde{\lambda}t \sum_{k \geq 0} (k+1)^{(2p-1)_+} \frac{(\tilde{\lambda}t)^k}{k!} &\leq \tilde{\lambda}t + \tilde{\lambda}t \sum_{k \geq 0} k^{(2p-1)_+} \frac{(\tilde{\lambda}t)^k}{k!} \\ &= \tilde{\lambda}t + \tilde{\lambda}t e^{-\tilde{\lambda}t} \sum_{k \geq 1} k^{(2p-1)_+} \frac{(\tilde{\lambda}t)^k}{k!}. \end{aligned}$$

Using this reasoning recursively, we obtain

$$e^{-\tilde{\lambda}t} \tilde{\lambda}t \sum_{k \geq 0} (k+1)^{(2p-1)_+} \frac{(\tilde{\lambda}t)^k}{k!} \leq \tilde{\lambda}t \sum_{i=0}^{\lceil (2p-1)_+ \rceil} (\tilde{\lambda}t)^i,$$

and the proof is completed. Now, we assume that $p \geq 1$. Then

$$e^{-\tilde{\lambda}t} \tilde{\lambda}t \sum_{k \geq 0} (k+1)^{(2p-1)_+} \frac{(\tilde{\lambda}t)^k}{k!} \leq \tilde{\lambda}t + \tilde{\lambda}t(2p-1)_+ 2^{(2p-1)_+-1} \sum_{k \geq 0} (k+k^{(2p-1)_+}) \frac{(\tilde{\lambda}t)^k}{k!},$$

and similarly as before, a recursive approach yields (117).

We focus on the proof of point **B.**. We apply inequality (30) and compensation formula, and (118) follows from

$$\mathbb{E}[|M_{1,t}^h(x)|^{2q}] \leq \sum_{s \leq t} |M_{1,s}^h(x) - M_{1,s^-}^h(x)|^{2q} = t \mathcal{T}_{q,h}(x).$$

Point **C.** (see (119)) is a direct consequence of the BDG inequality (see (31)).

Finally, we consider the proof of point **D.**. First we treat the case $p = 1$. In this case, the process $(M_t)_{t \geq 0}$ such that $M_t := (\tilde{M}_{1,t}^h + \tilde{M}_{2,t}^h)^2 - \tau_1(x)$, is a martingale and then

$$\mathbb{E}[|\tilde{M}_{1,t}^h + \tilde{M}_{2,t}^h|^2] = t\tau_1(x).$$

Now, let $p > 1$. Let $h \in [0, h]$. Using the BDG inequality (see (31)), we obtain

$$\mathbb{E}[|\tilde{M}_{1,t}^h|^{2p}] \leq C_p \mathbb{E}\left[\left|\sum_{s \leq t} |\tilde{M}_{1,s}^h - \tilde{M}_{1,s-}^h|^2\right|^p\right] = C_p \mathbb{E}\left[\left|\sum_{s \leq t} |\Delta \tilde{M}_{1,s}^h|^2\right|^p\right].$$

In order to obtain our result, we are going to use a recursive approach. For any $k \in \mathbb{N}^*$, $\tilde{M}_{1,t}^{h,k} := \sum_{s \leq t} |\Delta \tilde{M}_{1,s}^h|^{2^k} - t\mathcal{I}_{2^{k-1},h}(x)$ is a martingale. Using (30) for the martingale $(\tilde{M}_{1,t}^{h,k})_{t \geq 0}$

$$\begin{aligned} \mathbb{E}\left[\left|\sum_{s \leq t} |\Delta \tilde{M}_{1,s}^h|^{2^k}\right|^{p/2^{k-1}}\right] &= \mathbb{E}\left[|\tilde{M}_{1,t}^{h,k} + t\mathcal{I}_{2^{k-1},h}(x)|^{p/2^{k-1}}\right] \\ &\leq 2^{(p/2^{k-1}-1)_+} \mathbb{E}\left[\left|\sum_{s \leq t} \Delta \tilde{M}_{1,s}^{h,k}\right|^{p/2^{k-1}}\right] + 2^{(p/2^{k-1}-1)_+} |t\mathcal{I}_{2^{k-1},h}(x)|^{p/2^{k-1}} \\ &\leq 2^{(p/2^{k-1}-1)_+} C_p \mathbb{E}\left[\left|\sum_{s \leq t} |\Delta \tilde{M}_{1,s}^h|^{2^{k+1}}\right|^{p/2^k}\right] + 2^{(p/2^{k-1}-1)_+} |t\mathcal{I}_{2^{k-1},h}(x)|^{p/2^{k-1}} \end{aligned}$$

Now, let $k_0 = \inf\{k \in \mathbb{N}^*; 2^k \geq p\}$. Using (28), we have

$$\mathbb{E}\left[\left|\sum_{s \leq t} |\Delta \tilde{M}_{1,s}^h|^{2^{k_0+1}}\right|^{p/2^{k_0}}\right] \leq \mathbb{E}\left[\sum_{s \leq t} |\Delta \tilde{M}_{1,s}^h|^{2p}\right] = t\mathcal{I}_{p,h}(x)$$

Since $2^k < p$ for any $k < k_0$, it follows that

$$\begin{aligned} \mathbb{E}[|\tilde{M}_{1,t}^h|^{2p}] &\leq c_p t\mathcal{I}_{p,h}(x) + c_p \sum_{k=1}^{k_0} |t\mathcal{I}_{2^{k-1},h}(x)|^{p/2^{k-1}} \\ &\leq c_p t\mathcal{I}_{p,h}(x) + c_p |t\mathcal{I}_{1,h}(x)|^p \vee |t\mathcal{I}_{2^{k_0-1},h}(x)|^{p/2^{k_0-1}} \end{aligned}$$

with $c_p \geq 0$ a constant which can change from line to line. Since we have $\underline{\mathcal{H}}_h^1(\phi, V)$ and $\underline{\mathcal{H}}_h^p(\phi, V)$, it follows that there exists $\xi > 1$ such that

$$\mathbb{E}[|\tilde{M}_{1,t}^h|^{2p}] \leq c_p t\mathcal{I}_{p,h}(x) + c_p t^\xi \phi \circ V(x)^p$$

Now, using (117), we have

$$\mathbb{E}[|M_{2,t}^h|^{2p}] \leq t(1 + \epsilon(t))\bar{\tau}_{p,h}(x).$$

From (29), it follows that

$$\mathbb{E}[|\tilde{M}_{2,t}^h|^{2p}] \leq t(1 + \epsilon(t))\bar{\tau}_{p,h}(x) + p2^{2p} \left(|t(1 + \epsilon(t))\bar{\tau}_{p,h}(x)|^{1-1/(2p)} |t\bar{\tau}_{1/2,h}(x)| + t^{2p} |\bar{\tau}_{1/2,h}(x)|^{2p} \right)$$

Since we have $\bar{\mathcal{H}}_h^{1/2}(\phi, V)$ and $\bar{\mathcal{H}}_h^p(\phi, V)$, it follows that there exists $\xi > 1$ such that

$$\mathbb{E}[|\tilde{M}_{2,t}^h|^{2p}] \leq t\bar{\tau}_{p,h}(x) + c_p t^\xi \phi \circ V(x)^p$$

Now since M_1^h and M_2^h are independent, using (29), we obtain

$$\begin{aligned} \mathbb{E}[|\tilde{M}_{1,t}^h + \tilde{M}_{2,t}^h|^{2p}] &\leq \mathbb{E}[|\tilde{M}_{2,t}^h|^{2p}] + p2^{2p} (\mathbb{E}[|\tilde{M}_{1,t}^h|] \mathbb{E}[|\tilde{M}_{2,t}^h|^{2p-1}] + \mathbb{E}[|\tilde{M}_{1,t}^h|^{2p}]) \\ &\leq t(\bar{\tau}_{p,h}(x) + c_p \mathcal{I}_{p,h}(x)) + c_p t^\xi \phi \circ V(x)^p, \end{aligned}$$

Finally, let $p \in [1/2, 1)$. Using the BDG inequality (see (31)), (28) and the compensation formula, we have

$$\begin{aligned} \mathbb{E}[|\tilde{M}_{1,t}^h + \tilde{M}_{2,t}^h|^{2p}] &\leq C_p \mathbb{E}\left[\sum_{s \leq t} |\Delta M_{1,s}^h + \Delta M_{2,s}^h|^2\right]^p \\ &\leq C_p \mathbb{E}\left[\sum_{s \leq t} |\Delta M_{1,s}^h + \Delta M_{2,s}^h|^{2p}\right] \\ &= C_p t \tau_p(x) \end{aligned}$$

Moreover, (122) follows from Jensen's inequality and the proof is completed. \square

Lemma 4.7. *Let $x \in \mathbb{R}^d$, $p, q \in [0, 1]$. We assume that $\overline{\mathcal{H}}_h^p$ and $\underline{\mathcal{H}}_h^{p \vee q}$ (see (109)) hold. Then, there exists $\epsilon : \mathbb{R}_+ \rightarrow \mathbb{R}$ a locally bounded function which satisfies $\epsilon(t)/t \leq C$ such that for every $x_0 \in \mathbb{R}^d$, we have*

$$\mathbb{E}[V^p(x_0 + \tilde{M}_{1,t}^h(x) + M_{2,t}^h(x) - \kappa_{p,q}(x)) - V^p(x_0)] \leq t \tilde{\chi}_{p,q,h}(x) + \mathbb{1}_{p \leq 1/2} [\sqrt{V}]_1^{2p} t \epsilon(t) \bar{\tau}_{p,h}(x)$$

where $\kappa_{p,q}$ defined in (113) and $\tilde{\chi}_{p,q,h}$ is defined in (116) and is given by

$$\begin{aligned} \tilde{\chi}_{p,q,h}(x) &= (\mathbb{1}_{p,q \leq 1/2} [\sqrt{V}]_1^{2(p \vee q)} v_*^{p-p \vee q} + \mathbb{1}_{p \leq 1/2} \mathbb{1}_{q > 1/2} C_q [V^{p-1} \nabla V]_{2q-1} \\ &\quad + \mathbb{1}_{p > 1/2} C_{p \vee q} [V^{p \vee q - 1} \nabla V]_{2(p \vee q - 1)} \underline{\tau}_{p \vee q, h}(x) \\ &\quad + (\mathbb{1}_{p \leq 1/2} [\sqrt{V}]_1^{2p} + \mathbb{1}_{p > 1/2} C_p [V^{p-1} \nabla V]_{2p-1}) \bar{\tau}_{p,h}(x) \end{aligned}$$

Let $p, q \leq 1/2$. Assume that $\overline{\mathcal{H}}_h^p$ and $\underline{\mathcal{H}}_h^{p \vee q}$ (see (109)) hold. Then, there exist $\epsilon : \mathbb{R}_+ \rightarrow \mathbb{R}$ a locally bounded function, which satisfies $\epsilon(t)/t \leq C$ such that for every $x_0 \in \mathbb{R}^d$, we have

$$\begin{aligned} \mathbb{E}[V^p(x_0 + M_{1,t}^{\tilde{h}}(x) + M_{2,t}^{\tilde{h}}(x)) - V^p(x_0)] &\leq [\sqrt{V}]_1^{2(p \vee q)} v_*^{p-p \vee q} t \underline{\tau}_{p \vee q, \tilde{h}}(x) \\ &\quad + [\sqrt{V}]_1^{2p} t (1 + \epsilon(t)) \bar{\tau}_{p, \tilde{h}}(x) \end{aligned} \quad (124)$$

Let $p \leq 1/2$ and $q > 1/2$. Assume that $\overline{\mathcal{H}}_h^p$ and $\underline{\mathcal{H}}_h^q$ (see (109)) hold. Then, there exist $\epsilon : \mathbb{R}_+ \rightarrow \mathbb{R}$ a locally bounded function, which satisfies $\epsilon(t)/t \leq C$ such that for every $x_0 \in \mathbb{R}^d$, we have

$$\begin{aligned} \mathbb{E}[V^p(x_0 + \tilde{M}_{1,t}^{\tilde{h}}(x) + M_{2,t}^{\tilde{h}}(x)) - V^p(x_0)] &\leq C_q [V^{p-1} \nabla V]_{2q-1} t \underline{\tau}_{q, \tilde{h}}(x) \\ &\quad + [\sqrt{V}]_1^{2p} t (1 + \epsilon(t)) \bar{\tau}_{p, \tilde{h}}(x) \end{aligned} \quad (125)$$

Let $p > 1/2$ and assume that $\overline{\mathcal{H}}_h^p$ and $\underline{\mathcal{H}}_h^{p \vee q}$ (see (109)) hold. Then, for every $x_0 \in \mathbb{R}^d$, we have

$$\begin{aligned} \mathbb{E}[V^p(x_0 + \tilde{M}_{1,t}^{\tilde{h}}(x) + \tilde{M}_{2,t}^{\tilde{h}}(x)) - V^p(x_0)] &\leq C_{p \vee q} [V^{p \vee q - 1} \nabla V]_{2(p \vee q - 1)} t \underline{\tau}_{p \vee q, \tilde{h}}(x) \\ &\quad + C_p [V^{p-1} \nabla V]_{2p-1} t \bar{\tau}_{p, \tilde{h}}(x) \end{aligned} \quad (126)$$

Proof. Assume first that $p \leq 1/2$. Using (28) with $\alpha = p/2$, and since \sqrt{V} is Lipschitz, it follows from the same approach as in the proof of Lemma 4.6, point **A.**, that

$$\begin{aligned} \mathbb{E}[V^p(x_0 + M_{2,t}^h(x)) - V^p(x_0)] &\leq \mathbb{E}[|\sqrt{V}(x_0 + M_{2,t}^h(x)) - \sqrt{V}(x_0)|^{2p}] \\ &\leq [\sqrt{V}]_1^{2p} \mathbb{E}[|M_{2,t}^h(x)|^{2p}] \\ &\leq [\sqrt{V}]_1^{2p} t (1 + \epsilon(t)) \int_F c(z, x)^{2p} \zeta(z, x) \mathbb{1}_{(h, M]}(|z|) \lambda(dz). \end{aligned}$$

with $\epsilon : \mathbb{R}_+ \rightarrow \mathbb{R}$ a locally bounded function, which satisfies $\epsilon(t)/t \leq C$. Let $x_0 \in \mathbb{R}^d$. We study

$$\mathbb{E}[V^p(x_0 + M_{1,t}^h(x)) - V^p(x_0)], \text{ if } q \leq 1/2 \text{ and } \mathbb{E}[V^p(x_0 + \tilde{M}_{1,t}^h(x)) - V^p(x_0)] \text{ if } q > 1/2.$$

First, we consider the case $q \leq p$. In this case $\overline{\mathcal{H}}_h^q$ implies $\underline{\mathcal{H}}_h^p$. Using once again (28) with $\alpha = p/2$, and since \sqrt{V} is Lipschitz, it follows from the same approach as in the proof of Lemma 4.6, point **B.**, that

$$\begin{aligned}
\mathbb{E}[V^p(x_0 + M_{1,t}^h(x)) - V^p(x_0)] &\leq \mathbb{E}[|\sqrt{V}(x_0 + M_{1,t}^h(x)) - \sqrt{V}(x_0)|^{2p}] \\
&\leq [\sqrt{V}]_1^{2p} \mathbb{E}[|M_{1,t}^h(x)|^{2p}] \\
&\leq [\sqrt{V}]_1^{2p} t \int_F c(z, x)^{2p} \zeta(z, x) \mathbb{1}_{[0,h]}(|z|) \lambda(dz).
\end{aligned}$$

Now let $p < q$. First, let $q \leq 1/2$. Using (28) with $\alpha = 2q$, the concavity of the function $y \mapsto y^{p/q}$, and since \sqrt{V} is Lipschitz, we deduce that

$$\begin{aligned}
\mathbb{E}[V^p(x_0 + M_{1,t}^h(x)) - V^p(x_0)] &\leq \mathbb{E}[|\sqrt{V}^{p/q}(x_0 + M_{1,t}^h(x)) - \sqrt{V}^{p/q}(x_0)|^{2q}] \\
&\leq V^{p-q}(x_0) \mathbb{E}[|\sqrt{V}(x_0 + M_{1,t}^h(x)) - \sqrt{V}(x_0)|^{2q}] \\
&\leq [\sqrt{V}]_1^{2q} v_*^{p-q} \mathbb{E}[|M_{1,t}^h(x)|^{2q}] \\
&\leq [\sqrt{V}]_1^{2q} v_*^{p-q} t \int_F c(z, x)^{2q} \zeta(z, x) \mathbb{1}_{[0,h]}(|z|) \lambda(dz).
\end{aligned}$$

We assume that $q > 1/2$. Using Taylor expansion of order one, we obtain

$$\mathbb{E}[V^p(x_0 + \tilde{M}_{1,t}^h(x)) - V^p(x_0)] = \mathbb{E}[V^{p-1}(\xi) \langle \nabla V(\xi), \tilde{M}_{1,t}^h(x) \rangle],$$

with $\xi \in [x_0, x_0 + \tilde{M}_{1,t}^h(x)]$. Now since \sqrt{V} is Lipschitz, we can prove that $x \mapsto V^{p-1}(x) \nabla V(x)$ is $2q - 1$ holder in this case (see [13], Lemma 3) and since $\tilde{M}_{1,t}^h(x)$ is centered, it follows from the same approach as in the proof of Lemma 4.6, point **C**. that

$$\begin{aligned}
\mathbb{E}[V^p(x_0 + \tilde{M}_{1,t}^h(x)) - V^p(x_0)] &\leq \mathbb{E}[V^{p-1}(x_0) \nabla V(x_0) \tilde{M}_{1,t}^h(x) + [V^{p-1} \nabla V]_{2q-1} |\tilde{M}_{1,t}^h(x)|^{2q}] \\
&= [V^{p-1} \nabla V]_{2q-1} \mathbb{E}[|\tilde{M}_{1,t}^h(x)|^{2q}] \\
&\leq C_q [V^{p-1} \nabla V]_{2q-1} t \int_F c(z, x)^{2q} \zeta(z, x) \mathbb{1}_{[0,h]}(|z|) \lambda(dz)
\end{aligned}$$

Now, we assume that $p > 1/2$. Let $p \geq q$. Using once again the fact $x \mapsto V^{p-1}(x) \nabla V(x)$ is $2p - 1$ Hölder, similarly as in Lemma 4.6, point **D**., we deduce that

$$\begin{aligned}
\mathbb{E}[V^p(x_0 + \tilde{M}_{1,t}^h(x) + \tilde{M}_{2,t}^h(x)) - V^p(x_0)] &\leq \mathbb{E}[V^{p-1}(x_0) \nabla V(x_0) (\tilde{M}_{1,t}^h(x) + \tilde{M}_{2,t}^h(x)) \\
&\quad + [V^{p-1} \nabla V]_{2p-1} |\tilde{M}_{1,t}^h(x) + \tilde{M}_{2,t}^h(x)|^{2p}] \\
&\leq C_p [V^{p-1} \nabla V]_{2p-1} t \int_F c(z, x)^{2p} \zeta(z, x) \mathbb{1}_{[0,M]}(|z|) \lambda(dz)
\end{aligned}$$

Now, let $p < q$. In the same way

$$\mathbb{E}[V^p(x_0 + \tilde{M}_{2,t}^h(x)) - V^p(x_0)] \leq C_p [V^{p-1} \nabla V]_{2p-1} t \int_F c(z, x)^{2p} \zeta(z, x) \mathbb{1}_{[h,M]}(|z|) \lambda(dz)$$

Finally, as in the proof for $q > 1/2 \geq p$, we obtain

$$\mathbb{E}[V^p(x_0 + \tilde{M}_{1,t}^h(x)) - V^p(x_0)] \leq C_q [V^{p-1} \nabla V]_{2q-1} t \int_F c(z, x)^{2q} \zeta(z, x) \mathbb{1}_{[0,h]}(|z|) \lambda(dz)$$

□

Now, we are able to present the recursive control under weak mean reverting assumption for test functions with polynomial growth.

Proposition 4.9. *Let $v_* > 0, p \geq 0, q \in [0, 1]$, and $\phi : [v_*, \infty) \rightarrow \mathbb{R}_+^*$ a continuous function such that $\phi(y) \leq Cy$ with $C > 0$ and define also $\psi : [v_*, \infty) \rightarrow \mathbb{R}_+$ such that $\psi(y) = y^p$.*

We assume that the sequence $(U_n)_{n \in \mathbb{N}^}$ satisfies $M_{\mathcal{N},2}(U)$ (see (106)) and $M_{2pV^2}(U)$ (see (107)) and that (110), $\mathfrak{B}_{p,q}(\phi)$ (see (112)) and $\mathcal{R}_{p,q}$ (115) hold.*

We have the following properties:

A. *Assume that $p \geq 1$. We also assume that $\mathcal{H}^p, \overline{\mathcal{H}}_h^p(\phi, V)$ and $\overline{\mathcal{H}}_h^1(\phi, V)$ (see (114)) are satisfied and that, if $p > 1$, $\underline{\mathcal{H}}_h^1(\phi, V), \underline{\mathcal{H}}_h^p(\phi, V), \overline{\mathcal{H}}_h^{1/2}(\phi, V)$ and $\overline{\mathcal{H}}_h^p(\phi, V)$ hold for any $\tilde{h} \in (0, h]$. Then, there exists $\alpha > 0, \beta \in \mathbb{R}_+$ and $n_0 \in \mathbb{N}^*$, such that*

$$\forall n \geq n_0, x \in \mathbb{R}^d, \quad \tilde{A}_n^\gamma \psi \circ V(x) \leq V^{-1}(x) \psi \circ V(x) (\beta - \alpha \phi \circ V(x)). \quad (127)$$

Moreover, when $\phi = Id$ we have

$$\sup_{n \in \mathbb{N}} \mathbb{E}[\psi \circ V(\overline{X}_{\Gamma_n}^M)] < \infty. \quad (128)$$

B. *Assume that $p \in [0, 1)$ and let $q \in [0, 1]$. Moreover, we assume that $\overline{\mathcal{H}}_h^p$ and $\underline{\mathcal{H}}_h^{p \vee q}$ (see (109)) hold and that we have $\overline{\mathcal{H}}_h^p(\phi, V)$ (see (114)) if $p \leq 1/2$. Then, there exists $\alpha > 0, \beta \in \mathbb{R}_+$ and $n_0 \in \mathbb{N}^*$, such that*

$$\forall n \geq n_0, x \in \mathbb{R}^d, \quad \tilde{A}_n^\gamma \psi \circ V(x) \leq V^{-1}(x) \psi \circ V(x) (\beta - \alpha \phi \circ V(x)). \quad (129)$$

Moreover, when $\phi = Id$ we have

$$\sup_{n \in \mathbb{N}} \mathbb{E}[\psi \circ V(\overline{X}_{\Gamma_n}^M)] < \infty. \quad (130)$$

Proof. We focus on the proof of **A.**. From the Taylor's formula and the definition of λ_ψ (see (111)), we have

$$\begin{aligned} \psi \circ V(\overline{X}_{\Gamma_{n+1}}^M) &= \psi \circ V(\overline{X}_{\Gamma_n}^M) + \langle \overline{X}_{\Gamma_{n+1}}^M - \overline{X}_{\Gamma_n}^M, \nabla V(\overline{X}_{\Gamma_n}^M) \rangle \psi' \circ V(\overline{X}_{\Gamma_n}^M) \\ &\quad + \frac{1}{2} (D^2 V(\xi_{n+1}) \psi' \circ V(\xi_{n+1}) + \nabla V(\xi_{n+1})^{\otimes 2} \psi'' \circ V(\xi_{n+1})) (\overline{X}_{\Gamma_{n+1}}^M - \overline{X}_{\Gamma_n}^M)^{\otimes 2}. \\ &\leq \psi \circ V(\overline{X}_{\Gamma_n}^M) + \langle \overline{X}_{\Gamma_{n+1}}^M - \overline{X}_{\Gamma_n}^M, \nabla V(\overline{X}_{\Gamma_n}^M) \rangle \psi' \circ V(\overline{X}_{\Gamma_n}^M) \\ &\quad + \lambda_\psi(\xi_{n+1}) |\psi' \circ V(\xi_{n+1}) (\overline{X}_{\Gamma_{n+1}}^M - \overline{X}_{\Gamma_n}^M)|^2. \end{aligned}$$

with $\xi_{n+1} \in (\overline{X}_{\Gamma_n}^M, \overline{X}_{\Gamma_{n+1}}^M)$. First, from (110), we have $\sup_{x \in \mathbb{R}^d} \lambda_p(x) < \infty$.

Now, since $W, N_M(dt, B_M \setminus \overline{B}_h, dv)$ and $\tilde{N}_M(dt, \overline{B}_h, dv)^2$ are independent, we have

$$\begin{aligned} \mathbb{E}[\overline{X}_{\Gamma_{n+1}}^M - \overline{X}_{\Gamma_n}^M | \overline{X}_{\Gamma_n}^M] &= \gamma_{n+1} b(\overline{X}_{\Gamma_n}^M) + \mathbb{E}[\Delta \overline{X}_{n+1}^{M,4} | \overline{X}_{\Gamma_n}^M], \\ \mathbb{E}[|\overline{X}_{\Gamma_{n+1}}^M - \overline{X}_{\Gamma_n}^M|^2 | \overline{X}_{\Gamma_n}^M] &= \gamma_{n+1} \text{Tr}[\sigma \sigma^*(\overline{X}_{\Gamma_n}^M)] + \mathbb{E}[|\Delta \overline{X}_n^{M,3}|^2 | \overline{X}_{\Gamma_n}^M] + \mathbb{E}[|\Delta \overline{X}_n^{M,4}|^2 | \overline{X}_{\Gamma_n}^M] \\ &\quad + \gamma_{n+1}^2 |b(\overline{X}_{\Gamma_n}^M)|^2 + 2\gamma_{n+1} \langle b(\overline{X}_{\Gamma_n}^M), \mathbb{E}[\Delta \overline{X}_{n+1}^{M,4} | \overline{X}_{\Gamma_n}^M] \rangle \end{aligned}$$

with,

$$\mathbb{E}[\Delta \overline{X}_n^{M,4} | \overline{X}_{\Gamma_n}^M] = \gamma_{n+1} \int_F c(z, \overline{X}_{\Gamma_n}^M) \zeta(z, \overline{X}_{\Gamma_n}^M) \mathbb{1}_{(h,M)}(|z|) \lambda(dz) = \gamma_{n+1} \kappa_1(\overline{X}_{\Gamma_n}^M),$$

and

$$\mathbb{E}[|\Delta \overline{X}_{n+1}^{M,3}|^2 | \overline{X}_{\Gamma_n}^M] = \gamma_{n+1} \int_F c(z, \overline{X}_{\Gamma_n}^M)^2 \zeta(z, \overline{X}_{\Gamma_n}^M) \mathbb{1}_{[0,h]}(|z|) \lambda(dz) = \gamma_{n+1} \tau_{1,h}(\overline{X}_{\Gamma_n}^M).$$

²with the notation $B_r = \{z \in F, 0 \leq |z| < r\}$ and $\overline{B}_r = \{z \in F, 0 \leq |z| \leq r\}$ for $r \geq 0$.

using the notations (114) and (108). Moreover, using $\overline{\mathcal{H}}_h^1(\phi, V)$, Lemma 4.6 (see (117) with $p = 1$) implies that there exists a locally bounded function $\epsilon : \mathbb{R}_+ \rightarrow \mathbb{R}$ which satisfies $|\epsilon(t)|/t \leq C$, such that for every $n \in \mathbb{N}$,

$$\mathbb{E}[|\Delta \overline{X}_{n+1}^{M,4}|^2 | \overline{X}_{\Gamma_n}^M] \leq \gamma_{n+1}(1 + \epsilon(\gamma_{n+1})) \overline{\tau}_{1,h}(\overline{X}_{\Gamma_n}^M)$$

Using the Cauchy Schwarz inequality it follows that

$$\mathbb{E}[\Delta \overline{X}_n^{M,4} | \overline{X}_{\Gamma_n}^M] \leq \sqrt{\gamma_{n+1}}(1 + \sqrt{|\epsilon(\gamma_{n+1})|}) \overline{\tau}_{1,h}(\overline{X}_{\Gamma_n}^M)^{1/2}$$

Assume first that $p = 1$. Using $\mathcal{B}_{p,q}(\phi)$ (see (112)) and $\overline{\mathcal{H}}_h^1(\phi, V)$ (see (114)), for every $\tilde{\alpha} \in (0, \alpha)$, there exists $n_0(\tilde{\alpha})$ such that for every $n \geq n_0(\tilde{\alpha})$,

$$\begin{aligned} \|\lambda_1\|_\infty |\gamma_{n+1}^2 |b(\overline{X}_{\Gamma_n}^M)|^2 + 2\gamma_{n+1}^2 \langle b(\overline{X}_{\Gamma_n}^M), \kappa_1(\overline{X}_{\Gamma_n}^M) \rangle \\ + \gamma_{n+1} \epsilon(\gamma_{n+1}) \overline{\tau}_{1,h}(\overline{X}_{\Gamma_n}^M) \leq \gamma_{n+1}(\alpha - \tilde{\alpha}) \phi \circ V(\overline{X}_{\Gamma_n}^M). \end{aligned} \quad (131)$$

From assumption (116) and since $\underline{\tau}_{1,h} + \overline{\tau}_{1,h} = \tau_1$, we conclude that

$$\tilde{A}_n^\gamma \psi \circ V(x) \leq \beta - \tilde{\alpha} \phi \circ V(x)$$

Assume now that $p > 1$. Since $|\nabla V| \leq C_V V$, then \sqrt{V} is Lipschitz. Using (30), it follows that

$$\begin{aligned} V^{p-1}(\xi_{n+1}) &\leq (\sqrt{V}(\overline{X}_{\Gamma_n}^M) + [\sqrt{V}]_1 |\overline{X}_{\Gamma_{n+1}}^M - \overline{X}_{\Gamma_n}^M|)^{2p-2} \\ &\leq 2^{(2p-3)+} (V^{p-1}(\overline{X}_{\Gamma_n}^M) + [\sqrt{V}]_1^{2p-2} |\overline{X}_{\Gamma_{n+1}}^M - \overline{X}_{\Gamma_n}^M|^{2p-2}) \end{aligned}$$

We focus on the study of the second term of the remainder. First, using $\mathfrak{B}_{p,q}$ (see (112)) and $\overline{\mathcal{H}}_h^p(\phi, V)$ (see (114)), for any $p \geq 1$,

$$|\Delta \overline{X}_{n+1}^{M,1} + \Delta \overline{X}_{n+1}^{M,2} + \mathbb{E}[\Delta \overline{X}_{n+1}^{M,4} | \overline{X}_{\Gamma_n}^M]|^{2p} \leq c_p \gamma_{n+1}^p \phi \circ V(\overline{X}_{\Gamma_n}^M)^p (1 + |U_{n+1}|^{2p}).$$

Moreover, using Lemma 4.6 (see (120)), there exists $\xi > 1$, such that for every $\tilde{h} \in [0, h]$, then $\forall n \in \mathbb{N}$,

$$\mathbb{E}\left[|\Delta \overline{X}_{n+1}^{M,3} + \Delta \overline{X}_{n+1}^{M,4} - \mathbb{E}[\Delta \overline{X}_{n+1}^{M,4} | \overline{X}_{\Gamma_n}^M]|^{2p} | \overline{X}_{\Gamma_n}^M\right] \leq \gamma_{n+1}(\tau_p(\overline{X}_{\Gamma_n}^M) + c_p \underline{\tau}_{p,\tilde{h}}(\overline{X}_{\Gamma_n}^M)) + C_{p,\tilde{h}} \gamma_{n+1}^\xi \phi \circ V(\overline{X}_{\Gamma_n}^M)^p.$$

Applying (29), it follows that

$$\begin{aligned} \mathbb{E}[|\overline{X}_{\Gamma_{n+1}}^M - \overline{X}_{\Gamma_n}^M|^{2p} | \overline{X}_{\Gamma_n}^M] &\leq \gamma_{n+1}(\tau_p(\overline{X}_{\Gamma_n}^M) + c_p \underline{\tau}_{p,\tilde{h}}(\overline{X}_{\Gamma_n}^M)) + C_{p,\tilde{h}} \gamma_{n+1}^\xi \phi \circ V(\overline{X}_{\Gamma_n}^M)^p \\ &\quad + 2p2^{2p-1} (C_p \gamma_{n+1}^p \phi \circ V(\overline{X}_{\Gamma_n}^M)^p (1 + \mathbb{E}[|U_{n+1}|^{2p}]) \\ &\quad + C_1 \gamma_{n+1}^{1/2} \phi \circ V(\overline{X}_{\Gamma_n}^M)^{1/2} (1 + \mathbb{E}[|U_{n+1}|]) \mathbb{E}[|\Delta \overline{X}_{n+1}^{M,3} + \Delta \overline{X}_{n+1}^{M,4} - \mathbb{E}[\Delta \overline{X}_{n+1}^{M,4} | \overline{X}_{\Gamma_n}^M]|^{2p-1} | \overline{X}_{\Gamma_n}^M]) \end{aligned}$$

Let $p' := 1 - 1/(2p)$. Using the Jensen inequality and (28), we have

$$\begin{aligned} \mathbb{E}\left[|\Delta \overline{X}_{n+1}^{M,3} + \Delta \overline{X}_{n+1}^{M,4} - \mathbb{E}[\Delta \overline{X}_{n+1}^{M,4} | \overline{X}_{\Gamma_n}^M]|^{2p-1} | \overline{X}_{\Gamma_n}^M\right] \\ \leq \mathbb{E}\left[|\Delta \overline{X}_{n+1}^{M,3} + \Delta \overline{X}_{n+1}^{M,4} - \mathbb{E}[\Delta \overline{X}_{n+1}^{M,4} | \overline{X}_{\Gamma_n}^M]|^{2p} | \overline{X}_{\Gamma_n}^M\right]^{p'} \\ \leq \gamma_{n+1}^{p'} \tau_p(\overline{X}_{\Gamma_n}^M)^{p'} + c_p \gamma_{n+1}^{p'} \underline{\tau}_{p,\tilde{h}}(\overline{X}_{\Gamma_n}^M)^{p'} + C_{p,\tilde{h}}^{p'} \gamma_{n+1}^{\xi p'} \phi \circ V(\overline{X}_{\Gamma_n}^M)^{pp'} \\ \leq \gamma_{n+1}^{p'} \phi \circ V(\overline{X}_{\Gamma_n}^M)^{pp'} + \gamma_{n+1}^{p'} \underline{\tau}_{p,\tilde{h}}(\overline{X}_{\Gamma_n}^M)^{p'} + C_{p,\tilde{h}}^{p'} \gamma_{n+1}^{\xi p'} \phi \circ V(\overline{X}_{\Gamma_n}^M)^{pp'} \\ = \gamma_{n+1}^{p'} \phi \circ V(\overline{X}_{\Gamma_n}^M)^{p-1/2} + \gamma_{n+1}^{p'} \underline{\tau}_{p,\tilde{h}}(\overline{X}_{\Gamma_n}^M)^{p'} + C_{p,\tilde{h}}^{p'} \gamma_{n+1}^{\xi p'} \phi \circ V(\overline{X}_{\Gamma_n}^M)^{pp'} \end{aligned}$$

Now, since $\underline{\mathcal{H}}_h^p(\phi, V)$ holds, then $\lim_{\tilde{h} \rightarrow 0} \underline{\tau}_{p,\tilde{h}}/(\phi \circ V)^p = 0$ and then for any $\varepsilon > 0$ there exists $h_0 > 0$ such that $\underline{\tau}_{p,h_0} < \varepsilon(\phi \circ V)^p$. Moreover C_{p,h_0} is finite since $\underline{\mathcal{H}}_{\tilde{h}}^1(\phi, V), \underline{\mathcal{H}}_{\tilde{h}}^p(\phi, V), \overline{\mathcal{H}}_{\tilde{h}}^{1/2}(\phi, V)$ and $\overline{\mathcal{H}}_{\tilde{h}}^p(\phi, V)$

hold for every $\tilde{h} \in (0, h]$. Let $\hat{\alpha} \in (0, \alpha)$. Since $p' > 1/2$, there exists $n_0(\hat{\alpha}) \in \mathbb{N}$ such that for any $n \geq n_0(\hat{\alpha})$, we have

$$\mathbb{E}[|\overline{X}_{\Gamma_{n+1}}^M - \overline{X}_{\Gamma_n}^M|^{2p} | \overline{X}_{\Gamma_n}^M] \leq \gamma_{n+1}(\tau_p(\overline{X}_{\Gamma_n}^M) + \phi \circ V(\overline{X}_{\Gamma_n}^M)^p \frac{\alpha - \hat{\alpha}}{\|\phi/I_d\|_\infty^{p-1} \|\lambda_p\|_\infty 2^{(2p-3)+} [\sqrt{V}]_1^{2p-2}})$$

To treat the other term we proceed as in (131) with $\|\lambda_1\|_\infty$ replaced by $\|\lambda_p\|_\infty 2^{2p-3} [\sqrt{V}]_1^{2p-2}$, α replace by $\hat{\alpha}$ and $\tilde{\alpha} \in (0, \hat{\alpha})$. We gather all the terms together and using $\mathcal{R}_{p,q}$ (see (115) and (116)), for every $n \geq n_0(\tilde{\alpha}) \vee n_0(\hat{\alpha})$, we obtain

$$\begin{aligned} \mathbb{E}[V^p(\overline{X}_{\Gamma_{n+1}}^M) - V^p(\overline{X}_{\Gamma_n}^M) | \overline{X}_{\Gamma_n}^M] &\leq \gamma_{n+1} p V^{p-1}(\overline{X}_{\Gamma_n}^M) (\beta - \alpha \phi \circ V(\overline{X}_{\Gamma_n}^M)) \\ &\quad + \gamma_{n+1} p V^{p-1}(\overline{X}_{\Gamma_n}^M) (\phi \circ V(\overline{X}_{\Gamma_n}^M) (\hat{\alpha} - \tilde{\alpha}) + (\alpha - \hat{\alpha}) \frac{V^{1-p}(\overline{X}_{\Gamma_n}^M) \phi \circ V(\overline{X}_{\Gamma_n}^M)^p}{\|\phi/I_d\|_\infty^{p-1}}) \\ &\leq \gamma_{n+1} V^{p-1}(\overline{X}_{\Gamma_n}^M) (\beta p - \tilde{\alpha} p \phi \circ V(\overline{X}_{\Gamma_n}^M)). \end{aligned}$$

which is exactly the recursive control for $p > 1$, that is (127). The proof of (128) is an immediate application of Lemma 3.2 as soon as we notice that the increments of the Euler scheme (102) have finite polynomial moments (under the hypothesis from **A.**) which implies (24).

Now, we prove point **B.**. Since $p \leq 1$, the function defined on (v_*, ∞) by $y \mapsto y^p$ is concave. Using then the Taylor dexpansion of order 2 of the function V , for every $x, y \in \mathbb{R}^d$, there exists $\lambda \in [0, 1]$ such that

$$\begin{aligned} V^p(y) - V^p(x) &\leq p V^{p-1}(x) (V(y) - V(x)) \\ &= p V^{p-1}(x) (\langle \nabla V(x), y - x \rangle + \frac{1}{2} \text{Tr}[D^2 V(\lambda x + (1 - \lambda)y) (y - x)^{\otimes 2}]) \end{aligned}$$

and then,

$$V^p(y) - V^p(x) \leq p V^{p-1}(x) (\langle \nabla V(x), y - x \rangle + \frac{1}{2} \|D^2 V\|_\infty |y - x|^2).$$

We apply this inequality, and with the notation (113), it follows that

$$\begin{aligned} \mathbb{E}[V^p(\overline{X}_{\Gamma_{n+1}}^{M,2} + \gamma_{n+1} \kappa_{p,q}(\overline{X}_{\Gamma_n}^M)) - V^p(\overline{X}_{\Gamma_n}^M) | \overline{X}_{\Gamma_n}^M] &\leq p V^{p-1}(\overline{X}_{\Gamma_n}^M) (\gamma_{n+1} \langle \nabla V(\overline{X}_{\Gamma_n}^M), b(\overline{X}_{\Gamma_n}^M) + \kappa_{p,q}(\overline{X}_{\Gamma_n}^M) \rangle \\ &\quad + \frac{1}{2} \|D^2 V\|_\infty \mathbb{E}[|\overline{X}_{\Gamma_{n+1}}^{M,2} + \gamma_{n+1} \kappa_{p,q}(\overline{X}_{\Gamma_n}^M) - \overline{X}_{\Gamma_n}^M|^2 | \overline{X}_{\Gamma_n}^M]). \end{aligned}$$

As in the proof of the case $p \geq 1$, it follows from $\mathfrak{B}_{p,q}(\phi)$ (see (112)) that there exists $\hat{\alpha} \in (0, \alpha)$ and $n_0(\hat{\alpha}) \in \mathbb{N}^*$ such that for every $n \geq n_0(\hat{\alpha})$, we have

$$\mathbb{E}[|\overline{X}_{\Gamma_{n+1}}^{M,2} + \kappa_{p,q}(\overline{X}_{\Gamma_n}^M) - \overline{X}_{\Gamma_n}^M|^2 | \overline{X}_{\Gamma_n}^M] \leq \gamma_{n+1} |\sigma \sigma^*(\overline{X}_{\Gamma_n}^M)| + \gamma_{n+1} \tau_1(x \overline{X}_{\Gamma_n}^M) + \gamma_{n+1} (\alpha - \bar{\alpha}) \phi \circ V(\overline{X}_{\Gamma_n}^M)$$

Now, it remains to study

$$\mathbb{E}[V^p(\overline{X}_{\Gamma_{n+1}}^M) - V^p(\overline{X}_{\Gamma_{n+1}}^{M,2} + \gamma_{n+1} \kappa_{p,q}(\overline{X}_{\Gamma_n}^M)) | \overline{X}_{\Gamma_n}^M].$$

Since $\overline{X}_{\Gamma_{n+1}}^M = \overline{X}_{\Gamma_{n+1}}^{M,2} + \Delta \overline{X}_{n+1}^{M,3} + \Delta \overline{X}_{n+1}^{M,4}$, we use Lemma 4.7 together with $\overline{\mathcal{H}}_h^p$ and $\underline{\mathcal{H}}_h^{p \vee q}$ (see (109)) and we obtain

$$\begin{aligned} \mathbb{E}[V^p(\overline{X}_{\Gamma_{n+1}}^M) - V^p(\overline{X}_{\Gamma_{n+1}}^{M,2} + \gamma_{n+1} \kappa_{p,q}(\overline{X}_{\Gamma_n}^M)) | \overline{X}_{\Gamma_n}^M] &\leq \gamma_{n+1} \tilde{\chi}_{p,q,h}(\overline{X}_{\Gamma_n}^M) \\ &\quad + \mathbb{1}_{p \leq 1/2} [\sqrt{V}]_1^{2p} \gamma_{n+1} \epsilon(\gamma_{n+1}) \bar{\tau}_{p,h}(\overline{X}_{\Gamma_n}^M), \end{aligned}$$

with $\epsilon : \mathbb{R}_+ \rightarrow \mathbb{R}$ a locally bounded function which satisfies $\epsilon(t)/t \leq C$. It follows from $\overline{\mathcal{H}}_h^p(\phi, V)$ (see (114)) when $p \leq 1/2$, that there exists $\tilde{\alpha} \in (0, \hat{\alpha})$ and $n_0(\tilde{\alpha}) \in \mathbb{N}^*$ such that for every $n \geq n_0(\tilde{\alpha})$, we have

$$\begin{aligned} \mathbb{E}[V^p(\overline{X}_{\Gamma_{n+1}}^M)] - V^p(\overline{X}_{\Gamma_{n+1}}^{M,2} + \gamma_{n+1}\kappa_{p,q}(\overline{X}_{\Gamma_n}^M))[\overline{X}_{\Gamma_n}^M] &\leq \gamma_{n+1}\tilde{\chi}_{p,q,h}(\overline{X}_{\Gamma_n}^M) \\ &+ \gamma_{n+1}(\hat{\alpha} - \tilde{\alpha})pV^{p-1}(\overline{X}_{\Gamma_n}^M)\phi \circ V(\overline{X}_{\Gamma_n}^M). \end{aligned}$$

Gathering all the terms together and using $\mathcal{R}_{p,q}$ (see (115)) yields the recursive control (129). The proof of (128) is an immediate application of Lemma 3.2 as soon as we notice that the increments of the Euler scheme (102) have finite polynomial moments (under the hypothesis from **A.**) which implies (24). \square

4.3.3 Proof of the infinitesimal estimation

In order to obtain the result, it is necessary to introduce some structural assumption concerning the jump process. For $x \in \mathbb{R}^d$, let us define the process $(M_t(x))_{t \geq 0}$ such that $M_t(x) = M_{1,t}^h(x) + M_{2,t}^h(x)$ (see (103) for notations) for every $t \geq 0$. We assume that

$$\forall z \in F, \lim_{|x| \rightarrow \infty} |c(z, x)|/|x| = 0, \quad \text{and} \quad \exists t_0 \in \mathbb{R}_+, \forall t \leq t_0, \quad \lim_{|x| \rightarrow \infty} |M_t(x)|/|x| = 0 \text{ a.s.} \quad (132)$$

Now, we give the result that provides the infinitesimal estimation.

Lemma 4.8. *Let $p \geq 0$ and let $q \in [0, 1]$. We consider the sequence $(U_n)_{n \in \mathbb{N}^*}$ which satisfies $M_{\mathcal{N},2}(U)$ (see (106)) and $M_2(U)$ (see (107)).*

Moreover, we assume that $\overline{\mathcal{H}}_h^p$ and $\underline{\mathcal{H}}_h^q$ (see (109)) hold. We also suppose that b and σ are locally bounded functions with sublinear growth, that (132) holds and that we have $\sup_{n \in \mathbb{N}^} \nu_n^n(|\sigma|^2) < \infty$, and also $\sup_{n \in \mathbb{N}^*} \nu_n^n(\tau_{q,h}) < \infty$ when $q \in (1/2, 1]$. Then, we have $\mathcal{E}(\tilde{A}^\gamma, A^M)$ (see (9)), with A^M defined in (101).*

Proof. In this proof we will use the function $\omega_{b,\sigma,t} : \mathbb{R}^d \times \mathbb{R}^d \rightarrow \mathbb{R}^d$ such that $\omega_{b,\sigma,t}(x, v) = x + tb(x) + \sqrt{t}\sigma(x)v$. We also introduce the random process $(M_t^h)_{t \geq 0}$ independent from $(U_n)_{n \in \mathbb{N}}$, such that for every $t \geq 0$, we have $M_t^h(x) = \tilde{M}_{1,t}^h(x) + M_{2,t}^h(x)$ (see (103) for notations). First we write

$$t^{-1}(\mathbb{E}[f(\omega_{b,\sigma,t}(x, v) + M_t^h(x))] - f(\omega_{b,\sigma,t}(x, v))) = A_3^M f(x) + R_{A_3}(x, t, v)$$

with

$$\begin{aligned} A_3^M f(x) &= \int_F (f(x + c(z, x)) - f(x) - \langle c(z, x), \nabla f(x) \rangle) \mathbb{1}_{[0,h]}(|z|) \zeta(z, x) \lambda(dz) \\ &+ \int_F (f(x + c(z, x)) - f(x)) \mathbb{1}_{(h,M)}(|z|) \zeta(z, x) \lambda(dz) \end{aligned}$$

It follows that we can decompose $R_{A_3}(x, t, v)$ in the following way: $R_{A_3}(x, t, v) = \underline{R}_{A_3}(x, t, v) + \overline{R}_{A_3}(x, t, v)$ with

$$\underline{R}_{A_3}(x, t, v) = \mathbb{E} \left[\int_0^1 \int_F \underline{\mathfrak{R}}_{A_3}(x, t, v, M_{t\theta}(x), z, \theta) \mathbb{1}_{[0,h]}(|z|) \lambda(dz) d\theta \right]$$

with

$$\begin{aligned} \underline{\mathfrak{R}}_{A_3} : \mathbb{R}^d \times \mathbb{R}_+ \times \mathbb{R}^d \times \mathbb{R}^d \times F \times [0, 1] &\rightarrow \mathbb{R}_+ \\ (x, t, u, v, z, \theta) &\mapsto f(\omega_{b,\sigma,t}(x, v) + u + c(z, x)) - f(\omega_{b,\sigma,t}(x, v) + u) \\ &- \langle c(z, x), \nabla f(\omega_{b,\sigma,t}(x, v) + u) \rangle \\ &- (f(x + c(z, x)) - f(x) - \langle c(z, x), \nabla f(x) \rangle) \zeta(z, x), \end{aligned}$$

and

$$\bar{R}_{A_3}(x, t, v) = \mathbb{E} \left[\int_0^1 \int_F \bar{\mathfrak{R}}_{A_3}(x, t, v, M_{t\theta}(x), z, \theta) \mathbb{1}_{(h, M)}(|z|) \lambda(dz) d\theta \right]$$

with

$$\begin{aligned} \bar{\mathfrak{R}}_{A_3} : \mathbb{R}^d \times \mathbb{R}_+ \times \mathbb{R}^d \times \mathbb{R}^d \times F \times [0, 1] &\rightarrow \mathbb{R}_+ \\ (x, t, u, v, z, \theta) &\mapsto (f(\omega_{b, \sigma, t}(x, v) + u + c(z, x)) - f(\omega_{b, \sigma, t}(x, v) + u)) \zeta(z, x) \\ &\quad - (f(x + c(z, x)) - f(x)) \zeta(z, x), \end{aligned}$$

$$\begin{aligned} \bar{R}_{A_3}(x, t, v) &= \mathbb{E} \left[\int_0^1 \int_F (f(\omega_{b, \sigma, t}(x, v) + M_{t\theta}^h(x) + c(z, x)) - f(\omega_{b, \sigma, t}(x, v) + M_{t\theta}^h(x))) \zeta(z, x) \mathbb{1}_{(h, M)}(|z|) \lambda(dz) d\theta \right] \\ &\quad - \mathbb{E} \left[\int_0^1 \int_F (f(x + c(z, x)) - f(x)) \zeta(z, x) \mathbb{1}_{(h, M)}(|z|) \lambda(dz) d\theta \right]. \end{aligned}$$

Case $\mathbf{q} \in [1/2, 1]$. We focus on \underline{R}_{A_3} . At this point, we assume that $\tau_{q, h}$ takes strictly positive values (otherwise $\underline{R}_{A_3} = 0$). We denote $\pi(d\Theta) = p_U(dv) \mathbb{1}_{[0, h]}(|z|) \lambda(dz) d\theta d$ for $\Theta = (v, z, \theta) \in \underline{G} = \mathbb{R}^d \times F \times [0, 1]$. It follows that (for $U \sim U_1$),

$$\mathbb{E}[\underline{R}_{A_3}(x, t, U)] = \tau_{q, h}(x) \mathbb{E} \left[\int_{\underline{G}} \tau_{q, h}^{-1}(x) \underline{\mathfrak{R}}_{A_3}(x, t, M_{t\theta}^h(x), v, z, \theta) \pi(d\Theta) \right]$$

First we show (12). We recall that b and σ have sublinear growth. Therefore, as a direct consequence of (132) and since f has a compact support, there exists $t_0 \geq 0$ such that

$$\forall \Theta = (v, z, \theta) \in \underline{G}, \quad \limsup_{|x| \rightarrow \infty} \sup_{t \in [0, t_0]} \tau_{q, h}^{-1}(x) \underline{\mathfrak{R}}_{A_3}(x, t, M_{t\theta}^h(x), \Theta) = 0 \text{ a.s.}$$

Finally, since f is continuous with compact support, then it is uniformly continuous and since $(M_{t\theta}(x))_{t \geq 0}$ is a left limited right continuous process, we deduce that for any compact subset K of \mathbb{R}^d , we have

$$\forall \Theta = (v, z, \theta) \in \underline{G}, \quad \limsup_{t \rightarrow 0} \sup_{x \in K} \tau_{q, h}^{-1}(x) \underline{\mathfrak{R}}_{A_3}(x, t, M_{t\theta}^h(x), \Theta) = 0 \text{ a.s.}$$

Consequently (12) holds. Now, we show that $\mathcal{E}_{ergo}(\tau_{q, h}^{-1} \underline{\mathfrak{R}}_{A_3}, \tau_{q, h})$ holds.

Using Taylor expansion of order one, we obtain

$$\begin{aligned} \underline{\mathfrak{R}}_{A_3}(x, t, u, v, z, \theta) &\leq \int_0^1 |c(z, x)| |\nabla f(\omega_{b, \sigma, \gamma}(x, v) + u + \vartheta c(z, x)) - \nabla f(\omega_{b, \sigma, \gamma}(x, v) + u) \\ &\quad + \nabla f(x + \vartheta c(z, x)) - \nabla f(x)| \zeta(z, x) |d\vartheta|. \end{aligned}$$

From Taylor expansion of order two, it also follows that

$$\underline{\mathfrak{R}}_{A_3}(x, t, u, v, z, \theta) \leq \frac{1}{2} \int_0^1 |c(z, x)|^2 |D^2 f(\omega_{b, \sigma, \gamma}(x, v) + u + \vartheta c(z, x)) - D^2 f(x + \vartheta c(z, x))| \zeta(z, x) d\vartheta$$

Therefore, for any $r \in [1, 2]$,

$$\underline{\mathfrak{R}}_{A_3}(x, t, u, v, z, \theta) \leq C \|D^2 f\|_\infty \vee \|\nabla f\|_\infty |c(z, x)|^r \zeta(z, x)$$

Taking $r = q$, the hypothesis $\underline{\mathcal{H}}_h^q$ (see (109)) brings

$$\sup_{x \in \mathbb{R}^d} \sup_{t \in \mathbb{R}_+} \tau_{q, h}^{-1}(x) \int_{\underline{G}} \mathbb{E}[\underline{\mathfrak{R}}_{A_3}(x, t, M_{t\theta}^h(x), \Theta)] \pi(d\Theta) < \infty,$$

and $\mathcal{E}_{ergo}(\mathcal{T}_{q,h}^{-1}\mathfrak{R}_{A_3}, \mathcal{T}_{q,h})$ follows from $\sup_{n \in \mathbb{N}^*} \nu_n^\eta(\mathcal{T}_{q,h}) < \infty$.

Now, we focus on \overline{R}_{A_3} . We denote $\overline{\pi}(d\overline{\Theta}) = p_U(dv)\mathbb{1}_{[0,h]}(|z|)\lambda(dz)d\theta d$ for $\overline{\Theta} = (v, z, \theta) \in \overline{G} = \mathbb{R}^d \times F \times [0, 1]$. It follows that (for $U \sim U_1$),

$$\mathbb{E}[\overline{R}_{A_3}(x, t, U)] = \mathbb{E}\left[\int_{\overline{G}} \overline{\mathfrak{R}}_{A_3}(x, t, M_{t\theta}^h(x), v, z, \theta) \overline{\pi}(d\overline{\Theta})\right]$$

First we show (12). We recall that b and σ have sublinear growth. Therefore, as a direct consequence of (132) and since f has a compact support, there exists $t_0 \geq 0$ such that

$$\forall \overline{\Theta} = (v, z, \theta) \in \overline{G}, \quad \lim_{|x| \rightarrow \infty} \sup_{t \in [0, t_0]} \overline{\mathfrak{R}}_{A_3}(x, t, M_{t\theta}^h(x), \overline{\Theta}) = 0 \text{ a.s.}$$

Finally, since f is continuous with compact support, then it is uniformly continuous and since $(M_{t\theta}(x))_{t \geq 0}$ is a left limited right continuous process, we deduce that for any compact subset K of \mathbb{R}^d , we have

$$\forall \overline{\Theta} = (v, z, \theta) \in \overline{G}, \quad \lim_{t \rightarrow 0} \sup_{x \in K} \overline{\mathfrak{R}}_{A_3}(x, t, M_{t\theta}^h(x), \overline{\Theta}) = 0 \text{ a.s.}$$

Consequently (12) holds. Now, we show that $\mathcal{E}_{ergo}(\overline{\mathfrak{R}}_{A_3}, 1)$ holds. As a direct consequence of $\lambda(\{z, h < |z| < M\}) < \infty$, we obtain

$$\sup_{x \in \mathbb{R}^d} \sup_{t \in \mathbb{R}_+} \int_{\overline{G}} \mathbb{E}[|\overline{\mathfrak{R}}_{A_3}(x, t, M_{t\theta}^h(x), \overline{\Theta})|] \overline{\pi}(d\overline{\Theta}) \leq C \|f\|_\infty \|\zeta\|_\infty \lambda(\{z, h < |z| < M\}) < \infty,$$

and $\mathcal{E}_{ergo}(\overline{\mathfrak{R}}_{A_3}, 1)$ follows. To complete the proof, it remains to study (for $U \sim U_1$)

$$t^{-1} \mathbb{E}[f(\omega_{b, \sigma, t}(x, U)) - f(x)].$$

This is already done in the proof of Proposition 4.6, so we invite the reader to refer to this part of the paper for more details.

Case $q \in [0, 1/2]$. In this case the study is the same as for $q \in [1/2, 1]$. We notice that $\kappa_{p,q}$ (see (113) for notations) is well defined and that (132) implies that it has sublinear growth. Consequently, since $\lambda(\{0 < |z| < M\}) < \infty$, if we replace b by $b - \kappa_{p,q}$ and then take $h = 0$, we obtain the result with a similar proof. \square

The following result show how to obtain $\lim_{n \rightarrow \infty} \nu_n^\eta(Af) = 0$ from Lemma 4.8. This is a key result which allow us to work with truncated jumps and nevertheless obtain convergence towards the invariant measure of the process with unbounded jumps.

Proposition 4.10. *We assume that $\lambda(\{z, h < |z|\}) = \infty$. For $\overline{M} \geq h$, we define*

$$\overline{\lambda}_{\overline{M}}(x) = \int_F |c(z, x)| |\zeta(z, x)| \mathbb{1}_{[\overline{M}, \infty)}(|z|) \lambda(dz).$$

We assume that (16) holds for every process that belongs to the family of processes $((\overline{X}_t^{\overline{M}})_{t \geq 0})_{\overline{M} \geq M_0}$, for some $M_0 \geq h$, that is:

$$\forall \overline{M} \geq M_0, \forall f \in \mathcal{D}(A) \quad \mathbb{P}\text{-a.s.} \quad \lim_{n \rightarrow \infty} \frac{1}{H_n} \sum_{k=1}^n \frac{\eta_k}{\gamma_k} \mathbb{E}[f(X_{\Gamma_k}^{\overline{M}}) - f(X_{\Gamma_{k-1}}^{\overline{M}}) | X_{\Gamma_{k-1}}^{\overline{M}}] = 0, \quad (133)$$

with $(\overline{X}_t^{\overline{M}})_{t \geq 0}$ defined in (102). We also suppose that (98) is satisfied, that is $\lim_{\overline{M} \rightarrow \infty} \lim_{n \rightarrow \infty} \nu_n^\eta(\overline{\lambda}_{\overline{M}}) = 0$. Finally we suppose that the hypothesis from Lemma 4.8 are satisfied with M replaced by \overline{M} (and $(\overline{X}_t^M)_{t \geq 0}$ replaced by $(\overline{X}_t^{\overline{M}})_{t \geq 0}$) for every $\overline{M} \geq M_0$. Then, we have

$$\forall f \in \mathcal{D}(A) \quad \mathbb{P}\text{-a.s.} \quad \lim_{\overline{M} \rightarrow \infty} \lim_{n \rightarrow \infty} \frac{1}{H_n} \sum_{k=1}^n \eta_k Af(X_{\Gamma_{k-1}}^{\overline{M}}) = 0. \quad (134)$$

Proof. We notice that $Af = A^{\overline{M}}f + R_{A_3, A_3^{\overline{M}}}f$ with

$$\begin{aligned} R_{A_3, A_3^{\overline{M}}}f(x) &= \int_F (f(x + c(z, x)) - f(x)) \mathbf{1}_{[\overline{M}, \infty)}(|z|) \zeta(z, x) \lambda(dz) \\ &\leq \|\nabla f\|_\infty \int_F c(z, x) \zeta(z, x) \mathbf{1}_{[\overline{M}, \infty)}(|z|) \lambda(dz) = \|\nabla f\|_\infty \overline{\lambda}_{\overline{M}}(x) \end{aligned}$$

Now, we write $\nu_n^\eta(A^{\overline{M}}f) = \nu_n^\eta(Af) + \nu_n^\eta(R_{A_3, A_3^{\overline{M}}}f)$ and then we obtain: $|\nu_n^\eta(Af)| \leq |\nu_n^\eta(A^{\overline{M}}f)| + \|\nabla f\|_\infty |\nu_n^\eta(\overline{\lambda}_{\overline{M}})|$. Since (133) holds, then Lemma 4.8 with Theorem 3.2 give,

$$\forall f \in \mathcal{D}(A) \quad \mathbb{P}\text{-a.s.} \quad \lim_{n \rightarrow \infty} \frac{1}{H_n} \sum_{k=1}^n \eta_k A^{\overline{M}}f(X_{\Gamma_{k-1}}^{\overline{M}}) = 0.$$

Finally, we let \overline{M} tends to infinity and since $\lim_{\overline{M} \rightarrow \infty} \lim_{n \rightarrow \infty} \nu_n^\eta(\overline{\lambda}_{\overline{M}}) = 0$, the proof is completed. \square

4.3.4 Proof of (14) and (16)

Proposition 4.11. *Let $p \geq 0, q \in [0, 1], a \in (0, 1], \rho, s \in (1, 2]$ and, $\psi(y) = y^p, \phi(y) = y^a$ and $\epsilon_{\mathcal{I}}(t) = t^{\rho/2}$. We assume that the sequence $(U_n)_{n \in \mathbb{N}^*}$ satisfies $M_{\mathcal{N}, 2}(U)$ (see (106)) and $M_{2p \vee 2}(U)$ (see (107)) and that (110) holds. Then, we have the following properties,*

A. *We assume that $\mathfrak{B}_{p,q}(\phi)$ (see (112)) and $\mathcal{R}_{p,q}$ (115) hold. We also suppose that $\mathcal{SW}_{\mathcal{I}, \gamma, \eta}(\rho, \epsilon_{\mathcal{I}})$ (see (26)) hold and that:*

- i) *If $p \geq 1$, we assume that $\mathcal{H}^p, \overline{\mathcal{H}}_h^p(\phi, V)$ and $\overline{\mathcal{H}}_h^1(\phi, V)$ (see (114)) are satisfied and that, if $p > 1$, $\underline{\mathcal{H}}_h^1(\phi, V), \underline{\mathcal{H}}_h^p(\phi, V), \overline{\mathcal{H}}_h^{1/2}(\phi, V)$ and $\overline{\mathcal{H}}_h^p(\phi, V)$ hold for any $\tilde{h} \in (0, h]$.*
- ii) *If $p \in (0, 1)$ and let $q \in [0, 1]$, we assume that $\overline{\mathcal{H}}_h^p$ and $\underline{\mathcal{H}}_h^{p \vee q}$ (see (109)) hold and that we have $\overline{\mathcal{H}}_h^p(\phi, V)$ (see (114)) if $p \leq 1/2$.*

Then $\mathcal{SW}_{\mathcal{I}, \gamma, \eta}(V^{p+a-1}, \rho, \epsilon_{\mathcal{I}})$ (see (19)) holds and we have the following property:

If in addition $\mathcal{SW}_{\mathcal{I}, \gamma, \eta}(V^{p/s})$ (see (20)), $\mathcal{SW}_{\text{pol}}(p, a, s, \rho)$ (see (139)), $\underline{\mathcal{H}}_h^{p \vee q}(\phi, V)$ and $\overline{\mathcal{H}}_h^p(\phi, V)$ are satisfied, then

$$\mathbb{P}\text{-a.s.} \quad \sup_{n \in \mathbb{N}^*} -\frac{1}{H_n} \sum_{k=1}^n \eta_k \tilde{A}_k^\gamma (\psi \circ V)^s (\overline{X}_{\Gamma_{k-1}}^M) < \infty, \quad (135)$$

and we also have,

$$\mathbb{P}\text{-a.s.} \quad \sup_{n \in \mathbb{N}^*} \nu_n^\eta(V^{p/s+a-1}) < \infty. \quad (136)$$

Moreover, when $p/s \leq p + a - 1$, the assumption $\mathcal{SW}_{\mathcal{I}, \gamma, \eta}(V^{p/s})$ (see (20)) can be replaced by $\mathcal{SW}_{\mathcal{I}, \gamma, \eta}$ (see (27)). Besides, if we also suppose that \mathcal{L}_V (see (7)) holds and that $p/s + a - 1 > 0$, then $(\nu_n^\eta)_{n \in \mathbb{N}^}$ is tight*

B. *If $f \in \mathcal{D}(A)$, (22), $\underline{\mathcal{H}}_h^q$ and $\overline{\mathcal{H}}_h^p$ (see (109)) are satisfied and $\mathcal{SW}_{\mathcal{I}, \gamma, \eta}(|\sigma \sigma^*|^{\rho/2} + \mathbf{1}_{q \neq 0} \overline{\mathcal{I}}_q^{1 \wedge \rho/(2q)} + \mathbf{1}_{p \neq 0} \overline{\mathcal{T}}_p^{1 \wedge \rho/(2p)}, \rho, \epsilon_{\mathcal{I}})$ (see (19)) holds, then*

$$\mathbb{P}\text{-a.s.} \quad \lim_{n \rightarrow \infty} \frac{1}{H_n} \sum_{k=1}^n \eta_k \tilde{A}_k^\gamma f(\overline{X}_{\Gamma_{k-1}}^M) = 0 \quad (137)$$

Remark 4.2. *The reader may notice that (137) remains true if we replace $\underline{\mathcal{H}}_h^q$ and $\overline{\mathcal{H}}_h^p$ by respectively $\underline{\mathcal{H}}_h^q(\phi, V)$ and $\overline{\mathcal{H}}_h^p(\phi, V)$ and if we also replace $\mathcal{SW}_{\mathcal{I}, \gamma, \eta}(|\sigma \sigma^*|^{\rho/2} + \overline{\mathcal{I}}_q^{1 \wedge \rho/(2q)} + \overline{\mathcal{T}}_p^{1 \wedge \rho/(2p)}, \rho, \epsilon_{\mathcal{I}})$ by $\mathcal{SW}_{\mathcal{I}, \gamma, \eta}(V^{a\rho/2}, \rho, \epsilon_{\mathcal{I}})$. A solution to obtain $\mathcal{SW}_{\mathcal{I}, \gamma, \eta}(V^{a\rho/2}, \rho, \epsilon_{\mathcal{I}})$ when $a\rho/2 \leq a + p - 1$ is provided by point **A.** that is $\mathcal{SW}_{\mathcal{I}, \gamma, \eta}(V^{p+a-1}, \rho, \epsilon_{\mathcal{I}})$. When $a\rho/2 > a + p - 1$ a possible solution consists if replacing p by p_0 in **A.** with p_0 satisfying $a\rho/2 \leq a + p_0 - 1$*

Proof. The result is an immediate consequence of Lemma 3.1. It remains to check the assumption of this Lemma.

We focus on the proof of (135) and (136). First, we show $\mathcal{SW}_{\mathcal{I},\gamma,\eta}(V^{p+a-1}, \rho, \epsilon_{\mathcal{I}})$ (see (19)). Since (110), $\mathfrak{B}_{p,q}(\phi)$ (see (112)) and $\mathcal{R}_{p,q}$ (see (115)) hold, it follows (using the hypothesis from point *i*) and point *ii*) from Proposition 4.9 that $\mathcal{I}_{Q,V}(\psi, \phi)$ (see (8)) is satisfied. Then, using $\mathcal{SW}_{\mathcal{I},\gamma,\eta}(\rho, \epsilon_{\mathcal{I}})$ (see (26)) with Lemma 3.3 gives $\mathcal{SW}_{\mathcal{I},\gamma,\eta}(V^{p+a-1}, \rho, \epsilon_{\mathcal{I}})$ (see (19)). In the same way, for $p/s \leq a + p - 1$, we deduce from $\mathcal{SW}_{\mathcal{I},\gamma,\eta}$ (see (27)) and Lemma 3.3 that $\mathcal{SW}_{\mathcal{I},\gamma,\eta}(V^{p/s})$ (see (20)) holds.

Now, we are going to prove $\tilde{\mathcal{I}}_X(V^{p/s}, V^{a+p-1}, \mathfrak{X}, \rho, \epsilon_{\mathcal{I}})$ (see (18)) and the proof of (135) will be completed. Notice that (136) will follow from $\mathcal{I}_{Q,V}(\psi, \phi)$ (see (8)) and Theorem 3.1. The proof is a consequence of Lemma 4.9 which is given below. We notice indeed that Lemma 4.9 (see (140)) implies that, there exists a sequence \mathfrak{X} , such that $\tilde{\mathcal{I}}_X(V^{p/s}, V^{a+p-1}, \mathfrak{X}, \rho, \epsilon_{\mathcal{I}})$ (see (18)) holds and the proof of (135) and (136) is completed.

We complete the proof of the Proposition by noticing that (137) follows directly from Lemma 4.9 (see (138)). \square

Lemma 4.9. *Let $p \geq 0, q \in [0, 1], a \in (0, 1], \rho \in (1, 2]$ and, $\psi(y) = y^p$ and $\phi(y) = y^a$. We suppose that the sequence $(U_n)_{n \in \mathbb{N}^*}$ satisfies $M_{\rho \vee 2p/s}(U)$ (see (107)) and that (110) holds. We also assume that $\overline{\mathcal{H}}_h^p$ and $\underline{\mathcal{H}}_h^q$ (see (109)) are satisfied.*

Then, for every $n \in \mathbb{N}$ we have: for every $f \in \mathcal{D}(A)$,

$$\begin{aligned} \mathbb{E}[|f(\overline{X}_{\Gamma_{n+1}}^M) - f(\overline{X}_{\Gamma_n}^{M,1})|^\rho | \overline{X}_{\Gamma_n}^M] &\leq C \gamma_{n+1}^{\rho/2} |\sigma \sigma^*(\overline{X}_{\Gamma_n}^M)|^{\rho/2} \\ &+ \gamma_{n+1}^{1 \wedge \rho/(2q)} \mathbb{1}_{q \neq 0} \underline{\mathcal{I}}_q^{1 \wedge \rho/(2q)}(\overline{X}_{\Gamma_n}^M) + C \gamma_{n+1} \mathbb{1}_{p \neq 0} \overline{\tau}_p^{1 \wedge \rho/(2p)}(\overline{X}_{\Gamma_n}^M). \end{aligned} \quad (138)$$

with $\mathcal{D}(A) = C_K^2(\mathbb{R}^d)$. In other words, for every $f \in \mathcal{D}(A)$, we have $\tilde{\mathcal{I}}_X(f, |\sigma \sigma^|^{\rho/2} + \mathbb{1}_{q \neq 0} \underline{\mathcal{I}}_q^{1 \wedge \rho/(2q)} + \mathbb{1}_{p \neq 0} \overline{\tau}_p^{1 \wedge \rho/(2p)}, \mathfrak{X}, \rho, \epsilon_{\mathcal{I}})$ (see (18)) with $\mathfrak{X}_n = f(\overline{X}_{\Gamma_n}^{M,1})$ for every $n \in \mathbb{N}$ and $\epsilon_{\mathcal{I}}(t) = t^{\rho/2}$ for every $t \in \mathbb{R}_+$.*

Now, we assume that (110), $\mathfrak{B}_{p,q}(\phi)$ (see (112)), $\underline{\mathcal{H}}_h^{p \vee q}(\phi, V)$ and $\overline{\mathcal{H}}_h^p(\phi, V)$ hold. Finally, we suppose that the following holds:

$$\mathcal{SW}_{\text{pol}}(p, a, s, \rho) \begin{cases} s(2/\rho - 1)(a + p - 1) + s - 2 \geq 0, & \text{if } 2p/s < 1, \\ (2 - s)/(2 - \rho) \leq a \leq s/\rho, & \text{if } 2p/s \geq 1 \text{ and } p < 1, \\ \rho \leq s(1 - (1 - a)/p), & \text{if } p \geq 1. \end{cases} \quad (139)$$

Then, for every $n \in \mathbb{N}$, we have

$$\mathbb{E}[|V^{p/s}(\overline{X}_{\Gamma_{n+1}}^M) - V^{p/s}(\overline{X}_{\Gamma_n}^M)|^\rho | \overline{X}_{\Gamma_n}^M] \leq C \gamma_{n+1}^{\rho/2} V^{p+a-1}(\overline{X}_{\Gamma_n}^M), \quad (140)$$

In other words, we have $\tilde{\mathcal{I}}_X(V^{p/s}, V^{p+a-1}, \mathfrak{X}, \rho, \epsilon_{\mathcal{I}})$ (see (18)) with $\mathfrak{X}_n = V^{p/s}(\overline{X}_{\Gamma_n}^M)$ for every $n \in \mathbb{N}$ and $\epsilon_{\mathcal{I}}(t) = t^{\rho/2}$ for every $t \in \mathbb{R}_+$.

Proof. Before we start the proof, we assume that $q \neq 0$ and $p \neq 0$. Otherwise, the proof is simpler so we leave it out.

Let $f \in \mathcal{D}(A)$. We introduce decomposition (with notations (104))

$$f(\overline{X}_{\Gamma_{n+1}}^M) - f(\overline{X}_{\Gamma_{n+1}}^{M,1}) = f(\overline{X}_{\Gamma_{n+1}}^{M,2}) - f(\overline{X}_{\Gamma_{n+1}}^{M,1}) + f(\overline{X}_{\Gamma_{n+1}}^{M,3}) - f(\overline{X}_{\Gamma_{n+1}}^{M,2}) + f(\overline{X}_{\Gamma_{n+1}}^M) - f(\overline{X}_{\Gamma_{n+1}}^{M,3})$$

First, we notice that since f is Lipschitz, we have

$$\mathbb{E}[|f(\overline{X}_{\Gamma_{n+1}}^{M,2}) - f(\overline{X}_{\Gamma_{n+1}}^{M,1})|^\rho | \overline{X}_{\Gamma_n}^M] \leq C \gamma_{n+1}^{\rho/2} |\sigma \sigma^*(\overline{X}_{\Gamma_n}^M)|^{\rho/2} \mathbb{E}[|U_{n+1}|^\rho].$$

Now we study $\Delta \overline{X}_{n+1}^{M,3}$. We distinguish two cases: $\rho/2 \leq q$ and $q < \rho/2$. First, let $\rho/2 \leq q$. Then from Cauchy Schwartz inequality, we obtain

$$\mathbb{E}[|\Delta \overline{X}_{n+1}^{M,3}|^\rho | \overline{X}_{\Gamma_n}^M] \leq \mathbb{E}[|\Delta \overline{X}_{n+1}^{M,3}|^{2q} | \overline{X}_{\Gamma_n}^M]^{\rho/(2q)}$$

Now if $q < \rho/2$, then since f is Lipschitz and defined on a compact set, it is also $2q/\rho$ -Hölder, and then for every $x_0 \in \mathbb{R}^d$, we have

$$\mathbb{E}[|f(x_0 + \Delta\bar{X}_{n+1}^{M,3}) - f(x_0)|^\rho | \bar{X}_{\Gamma_n}^M] \leq [f]_{2p/\rho} \mathbb{E}[|\Delta\bar{X}_{n+1}^{M,3}|^{2q} | \bar{X}_{\Gamma_n}^M]$$

Moreover, from Lemma 4.6 point **B.** and **C.**, it follows that

$$\mathbb{E}[|\Delta\bar{X}_{n+1}^{M,3}|^{2q}] \leq C\gamma_{n+1}\mathcal{I}_{q,h}(\bar{X}_{\Gamma_n}^M)$$

and we conclude that

$$\mathbb{E}[|f(\bar{X}_{\Gamma_{n+1}}^{M,3}) - f(\bar{X}_{\Gamma_n}^{M,2})|^\rho | \bar{X}_{\Gamma_n}^M] \leq C\gamma_{n+1}^{1 \wedge \rho/(2q)} \mathcal{I}_{q,h}^{1 \wedge \rho/(2q)}(\bar{X}_{\Gamma_n}^M).$$

Now we study $\Delta\bar{X}_{n+1}^{M,4}$. Once again we distinguish two cases: $\rho/2 \leq p$ and $p < \rho/2$. First, let $\rho/2 \leq p$. Using Lemma 4.6 point **A.** since we have $\bar{\mathcal{H}}^p$ and the Cauchy Schwartz inequality, it follows that

$$\mathbb{E}[|\Delta\bar{X}_{n+1}^{M,4}|^\rho] \leq \mathbb{E}[|\Delta\bar{X}_{n+1}^{M,4}|^{2p}]^{\rho/2p} \leq C\gamma_{n+1}\bar{\tau}_{p,h}^{\rho/(2p)}(\bar{X}_{\Gamma_n}^M),$$

Now if $p < \rho/2$, then since f is Lipschitz and defined on a compact set, it is also $2p/\rho$ -Hölder, and then for every $x_0 \in \mathbb{R}^d$, we have

$$\mathbb{E}[|f(x_0 + \Delta\bar{X}_{n+1}^{M,4}) - f(x_0)|^\rho | \bar{X}_{\Gamma_n}^M] \leq [f]_{2p/\rho} \mathbb{E}[|\Delta\bar{X}_{n+1}^{M,4}|^{2p} | \bar{X}_{\Gamma_n}^M]$$

Now, using Lemma 4.6 point **A.** since we have $\bar{\mathcal{H}}^p$, it follows that

$$\mathbb{E}[|\Delta\bar{X}_{n+1}^{M,4}|^{2p}] \leq C\gamma_{n+1}\bar{\tau}_{p,h}(\bar{X}_{\Gamma_n}^M),$$

and we conclude that

$$\mathbb{E}[|f(\bar{X}_{\Gamma_{n+1}}^M) - f(\bar{X}_{\Gamma_n}^{M,3})|^\rho | \bar{X}_{\Gamma_n}^M] \leq C\gamma_{n+1}\bar{\tau}_p^{1 \wedge \rho/(2p)}(\bar{X}_{\Gamma_n}^M).$$

and gathering all the terms as in the the initial decomposition gives (138).

We focus now on the case $f = V^{p/s}$. First, we assume that $2p/s \leq 1$. Let $x, y \in \mathbb{R}^d$. Then, the function from $\mathbb{R}^+ \times \mathbb{R}^+ \rightarrow \mathbb{R}^+$ such that $y \mapsto y^{2p/s}$ is concave and since \sqrt{V} is Lipschitz, we deduce that

$$\begin{aligned} V^{p/s}(y) - V^{p/s}(x) &\leq 2p/s V^{p/s-1/2}(x)(\sqrt{V}(y) - \sqrt{V}(x)) \\ &\leq 2p/s [\sqrt{V}]_1 V^{p/s-1/2}(x) |y - x|. \end{aligned}$$

Now, since $s(2/\rho - 1)(a + p - 1) + s - 2 \geq 0$ from $\mathcal{SW}_{pol}(p, a, s, \rho)$ (see (139)), and V takes values in $[v_*, \infty)$, we deduce that there exists $C > 0$ such that for every $x \in \mathbb{R}^d$, we have $V^{p/s-1/2}(x) \leq CV^{a(1-\rho/2)+p-1}(x)$. Using $\mathfrak{B}_{p,q}(\phi)$ (see (112)), we obtain

$$\mathbb{E}[|\bar{X}_{\Gamma_{n+1}}^{M,2} - \bar{X}_{\Gamma_n}^M|^\rho | \bar{X}_{\Gamma_n}^M = x] \leq C\gamma_{n+1}^{\rho/2} V^{a\rho/2}(x)$$

Now, we study $\Delta\bar{X}_{n+1}^{M,3}$. First we consider the case $2q > s$. Since $\rho \leq s$ (from $\mathcal{SW}_{pol}(p, a, s, \rho)$ (see (139))), we have $2q > \rho$ and it follows from Cauchy Schwartz inequality and Lemma 4.6 point **C.**, that

$$\mathbb{E}[|\Delta\bar{X}_{n+1}^{M,3}|^\rho | \bar{X}_{\Gamma_n}^M] \leq \mathbb{E}[|\Delta\bar{X}_{n+1}^{M,3}|^{2q} | \bar{X}_{\Gamma_n}^M]^{\rho/(2q)} \leq C\gamma_{n+1}^{\rho/(2q)} \mathcal{I}_{q,h}^{\rho/(2q)}(x)$$

Now, we notice that since $p/s \leq 1/2$ then $V^{p/s}$ is α -Hölder for any $\alpha \in [2p/s, 1]$ (see Lemma 3. in [13]). It follows that for $2q \leq s$, $V^{p/s}$ is $2(p \vee q)/s$ -Hölder. Then, using Cauchy-Schwartz inequality and since $\rho \leq s$ from $\mathcal{SW}_{pol}(p, a, s, \rho)$, we have

$$\begin{aligned} \mathbb{E}[|V^{p/s}(\bar{X}_{\Gamma_{n+1}}^{M,3}) - V^{p/s}(\bar{X}_{\Gamma_{n+1}}^{M,2})|^\rho | \bar{X}_{\Gamma_n}^M] &\leq [V^{p/s}]_{2(p \vee q)/s}^\rho \mathbb{E}[|\Delta\bar{X}_{n+1}^{M,3}|^{2(p \vee q)\rho/s} | \bar{X}_{\Gamma_n}^M] \\ &\leq [V^{p/s}]_{2(p \vee q)/s}^\rho \mathbb{E}[|\Delta\bar{X}_{n+1}^{M,3}|^{2(p \vee q)} | \bar{X}_{\Gamma_n}^M]^{\rho/s} \\ &\leq \gamma_{n+1}^{\rho/s} C_{\mathcal{I}_{(p \vee q),h}}^{\rho/s}(\bar{X}_{\Gamma_n}^M). \end{aligned}$$

where the last inequality is a consequence of Lemma 4.6 point **B.** and **C.**.

Using the fact that $V^{p/s}$ is $2p/s$ -Hölder, we deduce from (from Cauchy-Schwartz inequality and since $\rho \leq s$ from $\mathcal{SW}_{pol}(p, a, s, \rho)$ (see (139)) 4.6 point **A.**, that

$$\begin{aligned} \mathbb{E}[|V^{p/s}(\bar{X}_{\Gamma_{n+1}}^M) - V^{p/s}(\bar{X}_{\Gamma_{n+1}}^{M,3})|^\rho | \bar{X}_{\Gamma_n}^M] &\leq [V^{p/s}]_{2p/s}^\rho \mathbb{E}[|\Delta \bar{X}_{n+1}^{M,4}|^{2p\rho/s} | \bar{X}_{\Gamma_n}^M] \leq C \mathbb{E}[|\Delta \bar{X}_{n+1}^{M,4}|^{2p} | \bar{X}_{\Gamma_n}^M]^{\rho/s} \\ &\leq C \gamma_{n+1}^{\rho/s} \bar{\tau}_{p,h}^{\rho/s}(\bar{X}_{\Gamma_n}^M). \end{aligned}$$

We gather all the terms together and the proof is completed for the case $2p \leq s$. Now, we consider the case $2p > s$. Using (29) with $\alpha = 2p/s$, it follows that

$$\begin{aligned} V^{p/s}(\bar{X}_{\Gamma_{n+1}}^M) - V^{p/s}(\bar{X}_{\Gamma_n}^M) &\leq 2^{2p/s} p/s (V^{p/s-1/2}(\bar{X}_{\Gamma_n}^M) |\sqrt{V}(\bar{X}_{\Gamma_{n+1}}^M) - \sqrt{V}(\bar{X}_{\Gamma_n}^{M,1})| \\ &\quad + |\sqrt{V}(\bar{X}_{\Gamma_{n+1}}^M) - \sqrt{V}(\bar{X}_{\Gamma_n}^M)|^{2p/s}) \\ &\leq 2^{2p/s} p/s ([\sqrt{V}]_1 V^{p/s-1/2}(\bar{X}_{\Gamma_n}^M) |\bar{X}_{\Gamma_{n+1}}^M - \bar{X}_{\Gamma_n}^M| + [\sqrt{V}]_1^{2p/s} |\bar{X}_{\Gamma_{n+1}}^M - \bar{X}_{\Gamma_n}^M|^{2p/s}) \end{aligned}$$

We study $\Delta \bar{X}_{n+1}^{M,3}$. We recall that $\rho \leq s$ from $\mathcal{SW}_{pol}(p, a, s, \rho)$ (see (139)) and then $2p \geq \rho$ in this case. We distinguish the case $q \leq p$ and $q > p$. Using once again the Cauchy Schwartz inequality and point **A.** of Lemma 4.6, we obtain

$$\mathbb{E}[|\Delta \bar{X}_{n+1}^{M,3}|^{2p\rho/s} | \bar{X}_{\Gamma_n}^M] \leq C \gamma_{n+1} \bar{\tau}_{p,h}^{\rho/s}(\bar{X}_{\Gamma_n}^M) \quad \text{and} \quad \mathbb{E}[|\Delta \bar{X}_{n+1}^{M,3}|^\rho | \bar{X}_{\Gamma_n}^M] \leq C \gamma_{n+1} \bar{\tau}_{p,h}^{\rho/(2p)}(\bar{X}_{\Gamma_n}^M).$$

Now, we study $\Delta \bar{X}_{n+1}^{M,4}$. We recall that $\rho \leq s$ from $\mathcal{SW}_{pol}(p, a, s, \rho)$ (see (139)) and then $2p \geq \rho$ in this case. Using once again the Cauchy Schwartz inequality and point **A.** of Lemma 4.6, we obtain

$$\mathbb{E}[|\Delta \bar{X}_{n+1}^{M,4}|^{2p\rho/s} | \bar{X}_{\Gamma_n}^M] \leq C \gamma_{n+1} \bar{\tau}_{(p \vee q),h}^{p\rho/(s(p \vee q))}(\bar{X}_{\Gamma_n}^M) \quad \text{and} \quad \mathbb{E}[|\Delta \bar{X}_{n+1}^{M,4}|^\rho | \bar{X}_{\Gamma_n}^M] \leq C \gamma_{n+1} \bar{\tau}_{(p \vee q),h}^{\rho/(2(p \vee q))}(\bar{X}_{\Gamma_n}^M).$$

Using $\mathfrak{B}_{p,q}(\phi)$ (see (112)), $\underline{H}_h^{p \vee q}(\phi, V)$ and $\bar{H}_h^p(\phi, V)$, we obtain

$$\mathbb{E}[|\bar{X}_{\Gamma_{n+1}}^M - \bar{X}_{\Gamma_n}^M|^\rho | \bar{X}_{\Gamma_n}^M = x] \leq C \gamma_{n+1}^{\rho/2} V^{a\rho/2}(x).$$

and

$$\mathbb{E}[|\bar{X}_{\Gamma_{n+1}}^M - \bar{X}_{\Gamma_n}^M|^{2p\rho/s} | \bar{X}_{\Gamma_n}^M = x] \leq C \gamma_{n+1}^{p\rho/s \wedge 1} V^{ap\rho/s}(x)$$

In order to obtain (140), it remains to use $\rho \leq s(1 - (1-a)/p)$ if $p \geq 1$ and $(2-s)/(2-\rho) \leq a \leq s/\rho$ together with $2p/s \geq 1$ if $p < 1$. □

References

- [1] G.K. Basak, I. Hu, and C-Z Wei. Weak convergence of recursions. *Stochastic Processes and their Applications*, 68(1):65 – 82, 1997.
- [2] S. N. Ethier and T. G. Kurtz. *Markov processes*. Wiley Series in Probability and Mathematical Statistics: Probability and Mathematical Statistics. John Wiley & Sons, Inc., New York, 1986. Characterization and convergence.
- [3] N. Fournier. Jumping sdes: absolute continuity using monotonicity. *Stochastic Processes and their Applications*, 98(2):317–330, 2002.
- [4] H. Ganidis, B. Roynette, and F. Simonot. Convergence rate of some semi-groups to their invariant probability. *Stochastic Processes and their Applications*, 79(2):243–263, 1999.
- [5] R.J. Has'minskii. *Stochastic stability of differential equations*, volume 7 of *Monographs and Textbooks on Mechanics of Solids and Fluids : Mechanics and Analysis*. Sijthoff & Noordhoff, Alphen aan den Rijn, 1980.
- [6] N. Ikeda and S. Watanabe. *Stochastic differential equations and diffusion processes*. Kodansha scientific books. North-Holland, 1989.

- [7] D. Lamberton and G. Pagès. Recursive computation of the invariant distribution of a diffusion. *Bernoulli*, 8(3):367–405, 04 2002.
- [8] D. Lamberton and G. Pagès. Recursive computation of the invariant distribution of a diffusion: The case of a weakly mean reverting drift. *Stochastics and Dynamics*, 03(04):435–451, 2003.
- [9] Vincent Lemaire. *Estimation récursive de la mesure invariante d'un processus de diffusion*. PhD thesis, 2005. Thèse de doctorat dirigée par Lamberton, Damien et Pagès, Gilles Mathématiques appliquées Université de Marne-la-Vallée 2005.
- [10] H. Mei and G. Yin. Convergence and convergence rates for approximating ergodic means of functions of solutions to stochastic differential equations with markov switching. *Stochastic Processes and their Applications*, 125(8):3104 – 3125, 2015.
- [11] G.N. Milstein. Weak approximation of solutions of systems of stochastic differential equations. In *Numerical Integration of Stochastic Differential Equations*, volume 313 of *Mathematics and Its Applications*, pages 101–134. Springer Netherlands, 1995.
- [12] G. Pagès. Sur quelques algorithmes récursifs pour les probabilités numériques. *ESAIM Probab. Statist.*, 5:141–170 (electronic), 2001.
- [13] F. Panloup. Recursive computation of the invariant measure of a stochastic differential equation driven by a lévy process. *Ann. Appl. Probab.*, 18(2):379–426, 04 2008.
- [14] V. Rabet. *A stochastic equation with censored jumps related to multi-scale Piecewise Deterministic Markov Processes*. PhD thesis, Université Paris Est, Marne-la-Vallée, 2015.
- [15] D. Revuz and M. Yor. *Continuous martingales and Brownian motion*, volume 293 of *Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]*. Springer-Verlag, Berlin, third edition, 1999.
- [16] C. Soize. *The Fokker-Planck Equation for Stochastic Dynamical Systems and Its Explicit Steady State Solutions*. Advanced Series on Fluid Mechanics. World Scientific, 1994.
- [17] D. Talay. Second-order discretization schemes of stochastic differential systems for the computation of the invariant law. *Stochastics and Stochastic Reports*, 29(1):13–36, 1990.
- [18] D. Talay. Stochastic hamiltonian dissipative systems with non globally lipschitz coefficients: exponential convergence to the invariant measure and discretization by the implicit euler scheme. *Markov Process. Related Fields.*, (8):163–198, 2002.