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A NEW HYPOTHESIS FOR THE ELABORATION OF HEPTATONIC SCALES AND THEIR ORIGINS

Amine Beyhom

ولم يستعمل الذي بالكل مرتين مفعولاً إلى أكثر من أربعة عشر بعداً، والذي بالكل مفعولاً إلى أكثر من سبعة أبعاد، والذي بالخمسة إلى أكثر من أربعة أبعاد تحيط بها خمس نغم، والذي بالأربعة إلى أكثر من ثلاثة أبعاد تحيط بها أربع نغم، والطنيني أكثر من بعدين. وإنما دعا إلى ذلك حسن اختيار لا ضرورة [...]

‘...The double octave will not comprise, in practice, more than fourteen intervals; the octave, more than seven; the fifth, more than four intervals and five degrees; the fourth, more than three intervals and four notes; the tone, more than two intervals. It is experience and not the theoretical need which dictates it [...]’

Ibn Sīnā (Avicenna) – Kitāb-a-sh-Shifā’ [11th century]¹

The reason for having eight notes in one octave is an arbitrary concept. There are diverging explanations but none is satisfactory. The first part of this paper offers another view based on the author’s theory of Modal Systematics,² where basic principles are explained. The second part is a statistical analysis on the combination of intervals within the span of the just fourth, fifth and of the octave.³ The conclusion proposes two hypotheses, the first on the elaboration of the heptatonic scale and the second on the origins of heptatonism.

Introduction

The reasons given as to why the modern scale is made up of eight notes are unconvincing. Some suggest numerical relationships and their properties and others acoustic resonance. There are also propositions stating the obvious: it is as it is because it cannot be different.

The first reason is based on the properties of numbers. It offers two alternatives, firstly the magical properties of numbers, and secondly the ratios between them. The first alternative is dismissed because it does not relate to musical perception.⁴ Since Greek Antiquity, the second alternative has been the source of an ongoing dispute between the Pythagorean and the Aristoxenian schools.

The tetrad which was used by the Pythagoreans and their European followers provides the ratios of the predominant notes of the scale, as the Greeks perceived them (fig. 1: 176).⁵ However, it does not give any clues, and no other theory does, as to why the cycle of fifths should end after its seventh recurrence.

Later developments led to scales with twelve intervals, as in the modern European model, and seventeen with the Arabian⁶ and Persian paradigms.

There are no reasons either for the fourth⁷ to be made up of three, or for the fifth to be made up of four intervals.

Then the Aristoxenian school raised a point of particular importance when it pointed out that the practice of performance and the perception of intervals are the keys to theory.⁸

The Pythagorean construction of intervals, which in part is based on superparticular intervals,⁹ misled many theoreticians¹⁰ into believing that acoustic resonance might explain the construction of the scale, on the basis of its similarities with it. However, this is inconsistent with the predominance of the fourth in Greek theory and, for example, in Arabian theory and practice today. Acoustic resonance shows that the fourth is not the consequence of a direct process.¹¹

To put it simply, Pythagorean intervals are based on a relationship of numbers based on the tetrad, hence the following: 1:2:3:4. Therefore, any number can have a relationship with any other in the series. Figure 2, page 176, gives an example of an extension to 5 consecutive numbers: 1:2:3:4:5.

With time, new ratios appear. They come from combinations of the number 5 with the original four numbers in the tetrad. However, whenever acoustic resonance is assimilated with a generative theory, the only new ratios to appear are exclusively the consequence of number 5 in relation to the fundamental, (fig. 3: 177), and even if multiples equal to the powers of two of the frequency of the fundamental (upper octaves) can be equated with the fundamental, new ratios can appear but always excluding the fourth¹² (fig. 4: 177).

There are strong arguments in favour of the consonance with the just fourth.¹³ However, acoustic resonance fails in that neither can it generate modal scales, nor can it give satisfactory answers as to the number of eight pitches in the octave, or three in a fourth.¹⁴

Part I. Differentiation, combination, selection and classification of intervals in scale systems: basic modal systematics

The study of interval combination within a fourth or a fifth would have entertained scholars since music and mathematics were found to suit each other. Aristoxenus had limited combination techniques for his understanding of *genera*,¹⁵ but *Fārābī* saw them as systematic combinations.¹⁶ The combination of intervals must obey rules. Thus heptatonism is made up of a small number of consecutive intervals which we shall call conceptual. They are placed in larger containing intervals, such as the fourth, the fifth or the octave. Aristoxenus used the quarter-tone to define the size of conceptual intervals as well as for common denominator. With Cleonides it was the twelfth of the tone which was his common denominator¹⁷. *Fārābī* divided the octave in 144 equal parts¹⁸. This is twice the amount as in Cleonides. This shows how greatly *Fārābī* was influenced by the Harmonists, as Aristoxenus had them labelled. These scholars were focused on tonometry and generally used a small common denominator for a maximum of accuracy in their quantification¹⁹ of intervals. However, the Aristoxenian school favoured the largest possible common denominator, *i.e.*, an interval which can also be used as a conceptual interval (a second among intervals building up to larger containing intervals such as the fourth, the fifth or the octave).

Let us take a *genus* with a semi-tone or a quarter-tone as largest common denominator, within a fourth. To find out how many semi-tones make up a fourth, add semi-tones, one after the other until the fourth is filled up (tab. 1: 174). These intervals make a form of alphabet the letters of which being multiples of semi-tones.

In table 1, page 174, the intervals labelled 1, 2, etc., are integers. They are multiples of the largest common denominator which is the semi-tone. If we place three intervals in a fourth, other intervals may not fit in any longer. For example, if we place two of the smallest semi-tone intervals, the largest interval to fill up the fourth is one-tone-and-a-half, that is three semi-tones. When a fourth is made up of three intervals, the alphabet is reduced and has only intervals equating to one, two or three semi-tones.

The *genera* made from the systematic combination of the intervals in the alphabet constitute the well known six *genera* of semi-tone scales (tab. 2: 174), among which the first²⁰ and the fourth²¹, are mentioned by Aristoxenus. The first three *genera*²² have two classes of intervals: the semi-tone class, 1, and the one-and-a-half-tones class, 3. This also applies to the three other diatonic *genera*, but in this case with intervals of one semi-tone, 1 and one-tone, 2. Interval classes can be expressed as capacity vectors, according to the number

of intervals of each size they have (tab. 3: 174).

Another approach to the problem would devise a literal expression for the size of intervals expressed as multiples of the semi-tone, and then, arbitrarily, assigning the system amounting to the least integer number, as indicator of capacity. A good example is the *genera* with two one-semi-tone and one one-and-a-half-tones additional intervals (tab. 4: 174). The digits of the intervals are concatenated in a single integer. The lowest number in the series of three is 113. If we assign the smallest number in the series as a capacity vector, we need only count the number of occurrences of each interval. We start with the smallest one to find out what is the capacity of the corresponding scale systems. This is known as a hyper-system.

Taking, for example, vectors 2,0,1 and 1,2,0, with corresponding hyper-systems 113 and 122 as basis for generating remaining combinations. The intervals in each hyper-system can be combined differently in three sub-systems, or unique arrangements of intervals contained in the hyper-system. The reason for this is that each model contains a semi-tone which is repeated, in the first hyper-system and two one-tone intervals for the second. The outcome of the combination of intervals in a hyper-system containing three different intervals would be different. However, this configuration does not exist for semi-tone integer multiples.

Conceptual, quantification, and elementary intervals: Understanding theory and practice

In the Western equal temperament scale,²³ also known as the 12-ET system (equal-temperament with 12 intervals in the octave), both conceptual and quantification intervals may have the same value. The semi-tone is half of a tone. It is the smallest interval and therefore divides the fourth into five semi-tones. The fifth, is made of seven semi-tones: three-tones and one-half-tone. The octave has twelve semi-tones, that is six tones. The cent being equal to one hundredth of a semi-tone, appears to be more accurate. However, it has little purpose with the 12-ET since the semi-tone is the exact divider for all larger intervals.

With other systems²⁴, the smallest interval used, in theory, may neither be a divider of other intervals, nor a conceptual interval, or an interval which is used in the scales and melodies of a particular type of music. An example of it is the systematic scale defined in the first half of the 13th century by *Şafiy-y-a-d-Dīn al-Urmawī*, in his *Book of cycles*.²⁵ There, the smallest conceptual interval is the *limma*. The tone, is made up of two *limmata* and one *comma*, both Pythagorean.²⁶ The *limma* is equated²⁷ to the semi-tone. Therefore, a typical tone may take the form L + L + C, where 'L' stands for the *limma*, and 'C' for the *comma*. Therefore a pitch can be placed in a scale within the limits of these intervals.²⁸ In this case, the *limma*, and

the *comma* play the role of elementary intervals (they are used to make up other intervals in the scale). However, the *comma* is not a conceptual interval because it is never used as such between neighbouring pitches of a scale but only as part of another and larger conceptual interval. The *comma* and the *limma*, make up conceptual intervals used in the composition of other intervals such as the neutral second, called *mujannab* which, according to *Urmawī*, can be made up of two *limmata* (i.e., L + L) or with one *limma* plus one *comma* (i.e., L + C or C + L).

The difference between the two neutral seconds, i.e., the difference between two *limmata* and one *limma* plus one *comma* (fig. 5: 177), or $[(L+L) - (L+C) = (L-C)]$, is about 67 cents, almost three Pythagorean *commata*. Conceptually, however, the two possible forms of neutral seconds, with *Urmawī*, are equal. Both are called *mujannab* and considered as intermediate intervals placed between the *limma* and the tone.

Arabian theory has hardly changed since *Urmawī*.²⁹ Modern scholars give two principal representations of a scale with all possible locations of pitches.

The first is an approximation of the general scale with Holderian *comma*, HC, henceforth, and the second uses the quarter-tone for quantification.

A HC equates to $1/53^{\text{rd}}$ of an octave, about 23 cents (22.6415)³⁰. Therefore one *limma* equals to four HC, about 91 cents. This is close enough to the Pythagorean *limma*. The tone is 9 HC, or 204 cents, matching the Pythagorean tone. Typically, a tense diatonic *genus*³¹ is modelled as a succession of two Pythagorean tones of 9 HC each, plus a *limma* with 4 HC tones. The *mujannab* of *Urmawī*, which amounts to a neutral second, has two possible values, 6 HC or 7 HC, but they are considered as identical conceptual intervals.³²

The first division of the octave, the 53-ET giving the Holderian *comma* as a common divisor of all conceptual intervals, follows, in Arabian theory, complex rules that are given elsewhere.³³ The second division of the octave, in 24 theoretically equal quarter-tones, will demonstrate a privileged example of interval relationship.

At this point, it may be useful to explain how two intervals, which are different in size, can, according to *Urmawī*, be considered as identical conceptual intervals.³⁴ The best example is with the *maqām Bayāt*. It is based on the same scale as the *maqām Rāst*. The *Rāst* scale is composed of approximate three 'one-tone' and of four 'three-quarter-tones' intervals.

It could be noted as c d e- f g a b- c', with e- and b- being approximately one quarter-tone lower than their western equivalents. The scale of the *Bayāt* is close to the general structure of *maqām Rāst*, but begins with d and has a b flat (fig. 6: 178).

This gives d e- f g a b^b c' d'.³⁵ The note e-, which has the same name in all theories of the *maqām*,³⁶ is placed

differently according to the context of the performance, or depending on the local repertoire (fig. 7: 179).³⁷

In this *maqām*, the position of the degree *sīkā* has a lower pitch in Lebanese folk music than it has in Classical Arabian music in the Near-East. Should we decide to use a quarter-tone approximation for the intervals in Arabian music, as most modern theoreticians do, then the two neutral intervals between d and e- and between e- and f are conceptualised as two three-quarter-tones intervals (fig. 6: 178). However, with the *Dal'ūna*, in *maqām Bayāt*, Near-Eastern folk music has a lower e-, which, regardless, is considered as a *sīkā*, but the lower interval between d and e-, the lower *mujannab*, is smaller than an exact three-quarter-tone (fig. 7: 179), and the higher interval between e- and f is larger.³⁸

Furthermore, the positioning of the *sīkā* depends on which *maqām* is played as well as region and repertoire. A good example is in the difference between the position of the *sīkā* in the *maqām Bayāt* and the position of the same note in the *maqām Rāst* which in this case is higher in pitch, but lies approximately around the three-quarter-tone boundary. In the *maqām Sīkā*,³⁹ or one of its frequent variants, the *maqām Sīkā-Huzām*,⁴⁰ the position of *sīkā* is still higher and could sometimes reach the upper value of *Urmawī's* greater *mujannab*. This is the position assigned to this note in modern Turkish theory.⁴¹

The boundaries for these different positions for *sīkā* are not established in practice, and the study of its variations would require another paper.⁴²

This pitch is perceived as a *sīkā* anywhere the player may perform. The difference is quantitative. However, the relative positioning of the note which is placed between e^b and e, will always be perceived as a *sīkā*. Therefore, the conceptual understanding of the neutral second is not simply quantitative, but also relative and qualitative. Importantly, the *mujannab* is perceived as an intermediate interval between the one 'half-tone' and the 'one-tone' intervals. This applies for all other intervals such as the semi-tone which is an interval smaller than the *mujannab*, and to the 'one-tone' interval which is larger than the latter. The tonometric value of *mujannab* may vary,⁴³ but it is the relative position of the interval in the scale and its qualitative and relative size, compared to other conceptual intervals, which gives it its full value in the repertoire. To conclude on the nature of intervals in a scale, they are of three types:

1. An interval of measurement is an exact or approximate divider of other intervals. As a general rule, any musical system based on the equal division of the octave, as in an equal temperament, gives an interval of measurement, such as the semi-tone in the 12-ET, and with the quarter-tone in the 24-ET or the HC in the 53-ET divisions of the octave.

2. Conceptual interval. This is one of the consecutive

intervals of the second forming a musical system. For example, three seconds in a just fourth, four seconds in a just fifth, or seven seconds in an octave. Conceptual intervals can be measured either exactly or approximately with smaller intervals, usually of measurement, as in approximations using the quarter-tone or the HC.

3. Elementary intervals are used in combination to build up to consecutive conceptual intervals of seconds within a system. They can combine either with a similar elementary interval, such as with the two *limmata* in *Urmawī*'s general scale, which combine into a *mujannab* interval, or with another elementary interval, such as the *limma* + *comma*, for the second form of *mujannab*, with *Urmawī*.

These three types of intervals are not mutually exclusive. When the smallest conceptual interval is also the smallest common denominator of all conceptual intervals, as with the semi-tone in the 12-ET, then it becomes an interval of measurement, but it is also an elementary interval, although it remains conceptual when used as an interval of second within a musical system. The need to differentiate these three types of intervals arises within unequal temperaments, for example with *Urmawī*.⁴⁴

This distinction will provide with a better understanding of the combination processes applied to music intervals.

Applying the concept of qualitative differentiation of intervals on *Urmawī*'s scale

Urmawī's explanations about his scale show that the tone is composed of three elementary intervals and that no interval within the fourth may contain either three successive *limmata* or any two successive *commata* (fig. 8: 180).

The *comma* is neither a quantifying interval as it does not divide exactly other intervals such as the *mujannab* or the tone,⁴⁵ nor is it a conceptual interval, as it is never used as a melodic interval between two pitches in a modal scale.⁴⁶ Furthermore, a *comma* is never used as the first interval of a combination, with a notable exception for the *mujannab* which can hold the form 'C+L'.

A conceptual interval generally starts with itself or with another conceptual interval. The *limma*, for example, is used both as a conceptual interval, the smallest interval used in any of *Urmawī*'s modal scales and as an elementary interval used in the composition of other, relatively larger, conceptual intervals. With *Urmawī*, both the *comma* and the *limma*, are elementary intervals. However and additionally, the *limma* is also a conceptual interval.

In modal construction, and with an appropriate choice of pitches within the scale, with *Urmawī*, there are other conditions to be met. These include, for example, the inclusion of the fourth and of the fifth. They must be complementary in the octave. With such limitations,

we can conceptualise the intervals of adjacent seconds in *Urmawī*'s modes in the following way (fig. 8: 180):

1. A conceptual interval of one semi-tone is composed of a single interval, part of the scale. Since the smallest conceptual interval is the *limma*, we may conclude that the semi-tone is equivalent to a *limma*.

2. The *mujannab*, or neutral second conceptual interval is composed of two elementary intervals of the scale: the *mujannab* can be either composed of one *limma* + one *comma*, L+C, or of two consecutive *limmata*, L+L. It is the only interval with *Urmawī*, listed among intervals smaller than the fourth which may have two different sizes. As a corollary to this, two *mujannab* may follow each other, but only if they have a different composition such as when one is L+C and the other is L+L (or L+L then C+L).⁴⁷

3. The tone is composed of three elementary intervals. However, a) three *limmata* must not follow each other.⁴⁸ b) The *comma* must always be preceded or followed by a *limma*. In this case, the tone can only include two *limmata* and one *comma*, with two possible arrangements: L+L+C, or L+C+L.

4. The greater, or augmented conceptual interval of the tone is composed of four elementary intervals. It can only be made up of three *limmata* and one *comma*. They can only be arranged in this manner: L+L+C+L or L+C+L+L. This interval is not mentioned in the *Book of Cycles*. It is only assumed as part of *Urmawī*'s seconds.

5. The greatest conceptual interval of the second is made up of 5 elementary intervals because the fourth can only be composed of a maximum of seven elementary intervals, within the systematic general scale. However, two other intervals of second (conceptual interval) are needed for its completion. Since the smallest second is the semi-tone, the *limma*, the greatest conceptual interval is equal to the remainder coming from the subtraction of two *limmata* from the fourth. The fourth is composed of two tones and one semi-tone, i.e., $2 \times (2L+C) + L$, or $5L+2C$. Taking away two *limmata*, the resulting capacity of the greatest conceptual interval in a fourth is $3L+2C$. Applying the rules of construction of the intervals, such as no more than two *limmata* in a row, etc., the possible forms of the greatest second, or tone, in *Urmawī*-type scales are L+L+C+L+C, or L+C+L+L+C. This interval is not mentioned as such in the *Book of Cycles* but is also assumed.

The fourth needs a combination of smaller intervals so that their sum can add up to its capacity in terms of elementary intervals. In order to simplify the process, we shall use a simple handling of numbers equating to the conceptual intervals of the second with *Urmawī*.⁴⁹

1. The semi-tone equals number 1, as one elementary interval is needed to compose this conceptual interval.

2. *Mujannab* is given the value of 2 since two elementary intervals are needed to build it up to a conceptual interval.

3. The tone interval is given the value of 3 since it needs three elementary intervals.

4. The augmented tone has the value of 4 since it requires four elementary intervals.

5. The greatest interval of the second within a fourth has the value of 5 because it needs five elementary intervals.

Although having a quantitative function in terms of numbers of elementary intervals which make up a conceptual interval, numbers 1 to 5 express the intrinsic quality of the interval: its identification as a different conceptual interval from those represented with another number. As a common rule, the fourth is made up of three conceptual intervals. In order to comply with *Urmawī*, they must add up to seven elementary intervals.

Reduced to their hyper-systems, we have the following:

- a. 115, with $1+1+5 = 7$
(not in *Urmawī's Book of cycles*)
- b. 124, with $1+2+4 = 7$
(not in *Urmawī's Book of cycles*)
- c. 133, with $1+3+3 = 7$
- d. 223, with $2+2+3 = 7$

Therefore, in this case, a fourth may contain, either a) two semi-tones, 1, and one greatest interval of second, 5, or b) one semi-tone, one *mujannab*, or neutral tone, 2, and one augmented, or greater tone, 4, or c) one semi-tone and two intervals of one tone, 3, or d) two *mujannab*, or neutral tones and one one-tone interval. The algorithm for these hyper-systems is straight forward (fig. 9: 180).

To find the first hyper-system, (fig. 9: 180, first step) take the smallest conceptual interval, 1 twice in this case, and then deduce the value of the third interval by subtracting the quantitative value of the first two, which adds up to 2 elementary intervals, from the value of a fourth, or 7 elementary intervals, which gives 5.

The second hyper-system, the 124 hyper-system above, is obtained by decrementing the value of the last digit interval in the preceding first hyper-system, (fig. 9: 180, second step) and by incrementing accordingly the value of the interval standing just before in the series: the last digit in the first hypersystem is 5, which is decremented to 4, and the interval which precedes it, which is the central 1 in the 115 hyper-system, is incremented, accordingly, to 2.

The simultaneous decreasing of one interval value by one unit, or its decrementation, with the increasing of one other interval value by the same unit of one, or accordingly incrementing it, insures that the sum of the numbers in the series remains unchanged. Here it is equal to 7.

Applying the same process to the resulting hyper-system 124, (fig. 9: 180, second step — repeated) the third hyper-system is now 133. Applying the same process to this last hyper-system would result in 142.

The capacity of this series is, however, the same as for 124. The reason is that in the preceding 133, the last two intervals were equal but with the continuation of the process in the same way, interval values for the central three are the same as the preceding values for the last three, *i.e.*, 4 and 5, and reciprocally, which would result in the same composition of intervals, in terms of quantity, within the fourth. At this point, we need to improve the algorithm in order to find the remaining hyper-systems. This is done by decreasing the rank of the intervals to be modified by applying the same process to the interval the rank of which is immediately below the rank of the interval to which the decrementing process was last applied, *i.e.*, 133. The latter is the third interval in the series and now we must decrement the second interval in the series, and increment, accordingly, the preceding one, the first interval in the same series. Applying this process to 133 which we found in the preceding step, the second interval, central 3 (fig. 9: 180, 3rd step) is decremented to 2, and the first interval, 1, is also incremented to 2, whilst the third interval, which is the last 3, remains unchanged. This gives the new figure of 223. This is where the generation process ends since the two first intervals have now similar values. Any further step would generate a redundant hyper-system.⁵⁰

Now that we have determined the hyper-systems agreeing with *Urmawī*, we need extract all possible *genera* and shades to give the full range of intervals in the fourth. The next section will review combination processes of intervals, for any hyper-system.

Various forms of interval combination

There are different methods for combining intervals. One is the rotation and permutation process. It is the most common. Rotation was used, notably, by Aristoxenus in his *Elements of Harmonics*,⁵¹ and permutations were often used throughout history, and most probably by *Fārābī* in his *genera*, adding to Aristoxenus' range of tetrachords.⁵² Both processes are deficient since they do not give, in their simplest expression, a full account of all the possible combinations. The tree process given below has the whole range of results. However this is more related to statistical and probabilistic analyses.

There are other procedures, such as de-ranking, which can be considered as a general case of the Byzantine-wheel method. Modal systematics uses them all for the purpose of arranging and classification, with special recourse of the de-ranking process.

Rotation of intervals

Rotation (fig. 10: 181) is a straight forward process by which intervals may be combined, placing the first after the last one, or inversely, the last before the first, leaving the other intervals in their position.

The first method is a clockwise process which continues as long as the first interval does not come back to its initial position, obviously. Figure 10, page 181, shows that this process generates intervals in three different ways (the first does not rotate since it places the interval system in its original and basic position). However, the rotation process is defective, as it always gives three possible combinations of three intervals, whenever the combination possibilities for these three intervals allows for six different combinations.⁵³ For the purpose of his explanation, Aristoxenus used intervals of the enharmonic *genus* which are made up of two quarter-tones and one di-tone, that is two equal intervals out of three. Figure 11, page 181, shows intervals with subscript numbers so that they retain their initial rank in the basic configuration, that is a1 as the first interval of the basic configuration, a2 as the second and b3, as the third. Even then, the rotation process gives three distinct combinations. If the three intervals are equal to *Fārābī's* equal-tone distribution where each is 5/6 of a tone, a combination process, whatever it may be, will always give the same result as combining the three intervals a a a.

Other processes are more effective but Aristoxenus' use of this limited process might have been a consequence that he considered interval combination as a de-ranking process.

Permutation of intervals

Permutation exchanges one interval for another whilst others remain fixed. The same process is applied to another pair until all intervals have changed places.

With direct permutation (fig. 12: 181), interval a1 of the basic configuration a1 a2 b3 is first changed with interval a2. This results in combination a2 a1 b3. Then, coming back to the original configuration, with b3, which is the combination of b3 a2 a1. As a1 has already changed places with the two other intervals, we proceed with the second interval of the basic configuration, a2, with the others. This interval has already changed places, in the previous process, with a1: it should further change places with b3 only with the combination a1 b3 a2. The last interval has already changed places with both other intervals a1 and a2, and this is where the process ends.

If a1 is different from a2, and also from b3, then the second combination: a2 a1 b3 is different from the first combination, because it is a stand-alone interval system. The total number of distinct interval systems which result from the direct permutation process is 4, that is one more than with the rotational process. If both intervals are the same, however, if a1=a2, the two first combinations are equal. The process only gives three different combinations, similarly to those in the rotation process. In order to obtain the full range of possible combinations for these three intervals, we could apply the process of direct permutations, not only to the original configuration of a1 a2

b3, but also to each of the combinations which result from the direct combinations of a2 a1 b3, b3 a2 a1 and a1 b3 a2. If we apply this process to the second in the direct permutation process, combination b3 a2 a1, we obtain the following combinations:

1. New base: b3 a2 a1. This is the second combination in the direct permutation process. (fig. 12: 181)
2. Combination N°2: a2 b3 a1, consisting in exchanging the first interval with the second. This is a new combination, different from all the previous ones.
3. Combination N°3: a1 a2 b3, consisting in exchanging the first interval in the new basic configuration with the third one. This gives the same combination as the first one in the direct permutation process.
4. Combination N°4: b3, a1, a2, by exchanging the second interval in the new basic configuration with the third interval of the same. This is also a new combination.

Therefore we have two new interval combinations which added to the four distinct combinations of the direct permutation, amount to six different combinations of the three intervals a1, a2 and b3. These amount to the possible combinations with three distinct intervals. There is no need to apply the permutation process for the other combinations stemming from the first direct process. It is also possible to obtain a similar result with processes other than the successive permutations method, for example by applying rotation followed by a direct permutation process (fig. 13: 181). In this combination process, a direct permutation process is applied to each of the combinations coming from an initial rotation process (fig. 11: 181). This gives six independent and distinct combinations out of twelve. The six remaining combinations are redundant.

Let us be reminded of two characteristics of the reviewed combination processes:

1. The successive, or consecutive permutations and the alternate rotation/permutation processes generate a certain number of redundant combinations which have to be excluded from the outcome.
2. Out of six distinct resulting combinations obtained, three will be redundant if a1 equals a2. In this case, the outcome remains the same as for a simple rotation process (compare with fig. 11: 181).

Tree processing⁵⁴

Here, in the tree processing the combinations are based on an initial choice of intervals, rank by rank (fig. 14: 183). With the first rank, we may chose between the three intervals a1, a2, or b3 (the subscript plays here more the role of identifier for each interval, than the role of an initial rank number).

Having completed this first step, we still have two intervals of which one must be assigned to the second position in the series. The third step leaves us with one possibility since two out of three intervals have already been used.

The process is straightforward as it gives directly the six distinct combinations seen above. There are no redundancies although intervals a1 and a2 could be taken as equal. In this case, again, we only have three distinct combinations.

The tree processing method is rarely used for combination of intervals and this is one of the reasons why we have to explore further the de-ranking process which is of crucial importance in modal systematics⁵⁵ as it is a practical way for arranging and classifying large numbers of interval combinations, such as in the heptatonic scales.

The de-ranking process, or picking intervals ‘N’ in a row out of repeated series of ‘M’ conjunct intervals - Hyper-systems, systems, and sub-systems.

De-ranking is closely related to rotation. It is very useful and in the study of musical systems applies mostly to the double octave. In a reduced form, the de-ranking process takes it that a series of conjunct intervals is repeated a certain number of times, for example for in the series a1 a2 b3 a1 a2 b3 a1 a2 b3...⁵⁶ By de-ranking the first interval, we start the series of intervals by the first interval a2 instead of the first interval a1. We may consider this process as a rotation of intervals where the first a1 goes to the end of the extended series. If we choose N intervals out of a repeated pattern of N intervals, this process is a repeated rotation where N = M = 3. (fig. 15: 183)

In a more general application of this process, N intervals in a row are taken out of a series of M, repeated at least once, with both N and M being integer numbers. In the case of five intervals a b c d e repeated once in a row, for example (fig. 16: 184), we can pick up any series of three conjunct intervals to form a combination. The first ranking combination is a b c, the second b c d, the third c d e, etc.

If we apply this process to a double heptatonic tense diatonic scale, and in turn select seven conjunct intervals among the fourteen of the series (fig. 17: 184), beginning with the first interval, the second, the third, etc., and until the seventh, we obtain the seven different species of the scale.⁵⁷ In figure 17, page 184, the basic scale is 1 2 2 1 2 2, in which intervals are expressed as multiples of the semi-tone. This corresponds to the diatonic, and here also, the equal temperament western scale beginning with B or its equivalents (b, b', etc.), or B 1 (semi-tone) c 2 d 2 e 1 f 2 g 2 a 2 (b). Of all possible species of the double diatonic octave, this scale corresponds to the lowest value when expressing the concatenated intervals as an integer number.

With modal systematics, the first in a series of de-ranked combinations is considered as the basic system.⁵⁸

The others, in this example, are sub-systems of system 1 2 2 1 2 2 2 (fig. 18: 185). The hypersystem is the interval capacity indicator that we find in arranging all

intervals in a combination from the smallest to the largest. For example, in the hyper-system 1 1 2 2 2 2 2, from which we get that the capacity of all corresponding systems and sub-systems is equivalent to two one-semi-tone intervals and five one-tone intervals, there are other systems which are distinct from 1 2 2 1 2 2 2. They have the same capacity, within the same hyper-system. In order to find all systems and sub-systems originating from a hypersystem, one needs apply, for example, a combined process of rotations/permutations to its intervals.⁵⁹ This has been explained above⁶⁰. If we eliminate the redundant systems or sub-systems, we find two other systems for hyper-system 1 1 2 2 2 2 2. The first of these two distinct systems is the hyper-system itself, as it expresses an arrangement of intervals 1 1 2 2 2 2, which is different from 1 2 2 1 2 2 2, where the two semi-tones in the first combination are placed in a row. This system has in turn seven sub-systems. In this case, they are species. The remaining system which has the same interval capacity as the precedent ones but whose intervals are arranged following a different pattern where two semi-tones are separated alternately by one, then four, one-tone intervals, is 1 2 1 2 2 2 2, and has, accordingly, seven distinct sub-systems. Figure 19, page 185, shows how hyper-system 1 1 2 2 2 2 2 has intervals that can be combined in three distinct systems which in turn, give seven different combinations or sub-systems obtained from de-ranking.

This hyper-system is peculiar in that it is the only one composed exclusively of one ‘semi-tone’ or one ‘tone’ intervals. If to our alphabet of intervals, we add the ‘one-and-a-half-tones’ interval class in our model, we find two other hyper-systems, 1 1 1 1 2 3 3 and 1 1 1 2 2 2 3. These generate 15 and 20 distinct systems, respectively, or 105 and 140 distinct sub-systems. They are too numerous to be listed here, but an example of sub-system from the first hyper-system is the scale of the well known *Hijāz-Kār* Arabian mode, with two *Hijāz* tetrachords (1 3 1),⁶¹ separated by a one-tone interval: 1 3 1 [2] 1 3 1.⁶² Another example, related to the second hyper-system, is the scale of the contemporary Arabian *maqām Hijāz*, which commonly follows the scale 1 3 1 2 1 2 2 when reduced to a semi-tone scale without neutral intervals.

Now if we wanted to express the intervals of these hyper-systems in the equal-quarter-tones distribution of modern Arabian theory, then this would give:

- 2 2 4 4 4 4 4⁶³
- 2 2 2 4 4 4 6
- 2 2 2 2 4 6 6

If arranged in agreement with modal systematics classification, with the lesser values of hyper-systems holding the lower rank, their places would be reversed as:

1. 2 2 2 2 4 6 6
2. 2 2 2 4 4 4 6
3. 2 2 4 4 4 4 4

Let us now take in consideration the two neutral intervals used in modern Arabian theory. These are the three-quarter-tone ‘3’ and the five-quarter-tone ‘5’ intervals, which are conceptually differentiated from the one-semi-tone, one-tone, and one-tone-and-a-half. Combining the five intervals 2, 3, 4, 5, and 6 in seven possible positions, with the condition that the sum of the intervals must be equal to 24 quarter-tones, we end up having 19 hyper-systems (tab. 5: 175) with a possible number of 4795 sub-systems or scales. Among them, there are very few in usage. Scales used in semitone hyper-systems such as hyper-systems no. 1, 6 and 12 in the table, are limited to the diatonic and to the *Hijāz-Kār* or *Hijāz* type scales. For the remaining hyper-systems, scales used in the performance, practice and theory of Arabian, Persian and Turkish⁶⁴ music are remarkably few, no more than 150 to 200 when compared to the possible number of 4975, or out of more than eight thousand possible sub-systems with the extended alphabet (*i.e.*, with intervals greater than the one-and-a-half-tones), as we shall see in Part II.⁶⁵

Some preliminary remarks on the systems and sub-systems of the quarter-tone generative model can already here be expressed:

a) Homogeneity of interval composition within a hyper-system results in a lesser number of systems because of the redundancy factor. The less the interval contains different classes of intervals, for example hyper-system no.12 contains only two classes of intervals, the 2 and the 4, the less it generates systems and, consequently, sub-systems.⁶⁶

b) Two relatively homogeneous hyper-systems, no. 16 = 2334444 and 19 = 3333444, generate scales which are mostly used in Arabian music. Hyper-system no. 17, although very homogeneous, 3333336, is not in use because its intervals can add up neither to a fourth (sum=10) nor to a just fifth (sum=14).

c) Hyper-systems nos 1, 6 and 12, share with hyper-system no. 19 an important feature: more than half of their sub-systems have a fourth or a fifth beginning with the first interval.

d) System 2 4 4 2 4 4 4 in hyper-system no. 12 (this is the diatonic system that we have noted before) maximizes the number of fourths or fifths since six out of seven of its sub-systems contain a direct fourth and a direct fifth in relation to the tonic. Seven out of seven have either of them. This is the only system, among those generated with this model, with such qualities.

e) Hyper-systems which have the augmented seconds of Western music in a *Hijāz* tetrachordal combination (*i.e.*, containing at least one interval of one-and-a-half-tones – or 6 – and two intervals of one-semi-tone – or 2 – in the

form 2 6 2) generate large numbers of systems and sub-systems; these are hyper-systems no. 1, 6 and 9. This is an indication that these scales are a reservoir for modulation from and to diatonic scales.

f) Along with hyper-systems no. 12, 16 and 19, these generate about one hundred sub-systems that are the most frequently used or mentioned in specialized literature.

These scales, although stemming from hyper-systems with a reduced generative capacity, with about 22% of the total of sub-systems, form from two thirds to three quarters of the reservoir of scales used, or attested in Arabian music.⁶⁷ Their ratio of sub-systems with a double fourth and fifth from the tonic is close to 39% with most of the other sub-systems in usage (see rows with variants or ‘close to’ in the column of remarks of table 5, *i.e.*, hyper-systems nos 4, 10, 11 and 15 – the number of sub-systems marked FF for these represents a ratio of more than 46% of the total) contained in hyper-systems related to them.

g) In the ‘Remarks’ column with tab. 5, variants are mainly scales containing an alternative *Hijāz* tetrachord made up of intervals of 2, 3 and 5 quarter-tones. This is a possible indication that this tetrachord evolved from earlier forms such as 2 5 3 or 3 5 2, to our standardized form of 2 6 2, because of the pressure induced by the existence of the semi-tone equal temperament.⁶⁸

These remarks, made on the basis of the quarter-tone generative model of modal systematics, suggest already some criteria which may be applied in statistical studies of systems and sub-systems as we shall apply in Part II of the present paper.

These criteria will help answer the question why out of 12 possible intervals in a semi-tone distribution, or out of almost 24 intervals in a quarter-tone distribution, only seven are combined, in most music, to form an octave? And why are there three intervals in a fourth and four in a fifth, generally.

Before answering these questions, we must return to *Urmawī’s genera*, in order to have a better understanding of how, by applying the qualitative interval differentiation concept, uneven divisions of the octave can amount to even ones.

Applying modal systematics to *Urmawī’s genera*

In *Urmawī’s* model, we have distinguished intervals of the second by means of the capacity of integers, from 1 to 5 (fig. 8: 180). If we combine these intervals in the frame of a fourth, the sum of which must be equal to seven elementary intervals, we obtained the following hyper-systems:

- 1 1 5
- 1 3 3
- 1 2 4
- 2 2 3

Hyper-systems within the fourth as a containing interval, with two identical intervals generate one single system equivalent to the generative hypersystem. They amount to three: 1 1 5, 1 3 3 and 2 2 3. Among them, the last two agree with *Urmawī* in the *Book of cycles*, with intervals not greater than the tone. By de-ranking, possible combinations of the intervals contained in the three aforementioned hyper-systems are, for the first, combinations 1 1 5, 1 5 1 and 5 1 1. For the second, combinations 1 3 3, 3 3 1 and 3 1 3. For the third, combinations 2 2 3, 2 3 2 and 3 2 2. The remaining hyper-system, 1 2 4, generates two systems resulting in six distinct combinations which stem from 1 2 4: 1 2 4, 2 4 1 and 4 1 2, and stemming from system 1 4 2: 1 4 2, 4 2 1 and 2 1 4 (fig. 20: 186, left).

All *genera* in hyper-systems 2 2 3 and 1 3 3 are known both to *Urmawī's Book of cycles* and to modern *maqām* theory of the quarter-tone division of the octave. (fig. 20: 186, right).

The possible and missing *genera* in the treatises have in common peculiar features: each of them contains two small intervals in a row, either two consecutive conceptual semi-tones or *limmata*, or a *limma* and a *mujannab* in a row, similar to the 1 and 2 intervals in *Urmawī's* qualitative model (fig. 20: 186, left), and the 2 and 3 quarter-tones intervals in the quarter-tone model (fig. 20: 186, right). This is another criterion which will be applied in the statistical study which follows.

At this point, we may also note that the connection between the quarter-tone model and the model in *Urmawī's* qualitative interval equivalents is straight forward: in order to shift from *Urmawī's* model to the quarter-tone model, add one unit to each interval in the first. (tab. 6: 176) All the scales of the quarter-tone model connect directly with *Urmawī's* qualitative representation, through a unitary vector subtracted from the interval values in the former. For example, the *maqām Hijāz* scale, 2 6 2 [4] 2 4 4 (sum=24) in modern *maqām* theory (the square brackets identify the disjunctive tone between two tetrachords), becomes 1 5 1 [3] 1 3 3 (sum=17) in *Urmawī's* model, and the *maqām Rāst* 4 3 3 [4] 4 3 3, in quarter-tones becomes 3 2 2 [3] 3 2 2, or two similar tetrachords composed of, successively, one 'one-tone' and two *mujannab* intervals, with a disjunctive one-tone [3] interval.

In the model applied to *Urmawī's* intervals which consist in a division of the octave in 17 equal parts, the total sum of the intervals must amount to 17 elementary intervals in one octave. The transition to the quarter-tone interval is straightforward, as by subtracting one unit in each conceptual interval of a heptatonic scale in the quarter-tone model, we end up subtracting seven units from the total of 24 quarter-tones, which gives the sum of 17.

All the scales of the quarter-tone model, arising from the hyper-systems in table 5, page 175 have equivalent counterparts in *Urmawī's* model,⁶⁹ which proves that the two models are, in essence, conceptually equivalent.⁷⁰ As

a further consequence, all the results from the statistical analysis, resulting from generations with the limited alphabet, from 2 to 6 quarter-tones, may be applied to Pythagorean equivalents in *Urmawī's* model.⁷¹ Another conclusion may be drawn at this stage. *Urmawī's* concept of the scale, regardless of Pythagorean procedures used to explain, or legitimize his ideas about music, is profoundly Aristoxenian and based on a combination model. This applies as a rule to the composition of conceptual intervals using elementary intervals. The intervals within a fourth are derived from a combinatory process where the fourth and the fifth add up to an octave.⁷²

Conclusion for part I

A quantitative model based on the equal division of the octave can be a qualitative model, taking in account the size of the intervals of which the scale is composed. They express the number of elementary intervals which build up each of the conceptual intervals. In the case of the quarter-tone model, the smallest elementary interval is the approximate quarter-tone (the measuring interval), the smallest conceptual interval is composed of two elementary intervals, or two approximate quarter-tones, etc. Combining the resulting conceptual intervals, we combine qualities of intervals that are differentiated by their capacity to contain elementary small intervals⁷³ (fig. 21: 186). This means that the scales which result from that type of generative model have intervals of seconds which, if measured exactly, would differ from one another even when having the same interval capacity (a one-tone interval in one scale may be slightly different from a one-tone interval in another scale).⁷⁴ However, these intervals, when taken in relation to other intervals in the scale carry a unique quality which differentiates them from the latter, which is typical of modal systematics.

Part II. Combining intervals in a system: statistical analysis

With modal systematics the basic process consists in combining intervals expressed as integers and then analysing the results in relation to both music practice and theory. The elements of the scale consist in a sequence of consecutive conceptual intervals.

Conceptual intervals are stand-alone units in the scale. They are distinct in theory and in practice. They are placed between the notes of the scale. Their function is qualitative.⁷⁵ For an immediate identification of any interval in a scale series, modal systematics determines the optimal (or the smallest, with the largest elementary interval) division of the scale, in such a way that the quantifying interval is the smallest conceptual interval and the elementary interval. In the semi-tone scale, the semi-tone is such that it fulfills the functions of quantifying, elementary and conceptual intervals.

With Arabian music⁷⁶, the semi-tone model is ineffective because conceptual intervals, such as the neutral tone or neutral augmented second - the *mujannab* and the greater tone in *Urmawī*'s model in fig. 8, or the three-quarter-tones and the five-quarter-tones intervals in the quarter-tone model (fig. 21: 186), cannot be distinguished and identified as conceptual intervals. Therefore, another division of the octave is necessary to provide qualification for all types of intervals. In this case it is the 17-ET, or the division of the octave in 17 equal intervals⁷⁷ which is needed, since this division allows for the distinction of all conceptual intervals. These small intervals have values (fig. 8: 180) of 1 to 5.

Integers segregate the semi-tone 1, the *mujannab* or neutral second 2, the tone 3, the neutral augmented tone, or greater tone above 4 and the fully augmented tone, greatest tone above 5.⁷⁸ However, the 17-ET model has a flaw which makes it difficult to see it as a representative division of the octave. If taken strictly as a measuring interval, the 17th of an octave is 71 cents (fig. 22: 187). Adding these intervals, we have 494 cents for a fourth and 706 cents for a fifth. These figures are close enough to the corrected values of the fourth and the fifth in the Pythagorean system, *i.e.*, 498 cents and 702 cents. The problem lies with the representation of the semi-tone. If the 17th of an octave is conceptualised as a semi-tone interval, the discrepancy with an equal temperament semi-tone, in approximation is 29 cents, or $100-71=29$, which is unacceptable to most musicologists. As a result, and although the measuring 17th of an octave interval which divides the octave in 17 equal parts is also an elementary⁷⁹ and a conceptual interval,⁸⁰ we shall take the quarter-tone model for Arabian-Persian-Turkish music bearing in mind the equivalence between the two models.⁸¹

The principle of economy: optimal balance between method and expression

In his first paragraph of his *Tonality of homophonic music*, Helmholtz said of the musician's liberty that:

'Music was forced first to select artistically, and then to shape for itself, the material on which it works. [...] Music alone finds an infinitely rich but totally shapeless plastic material in the tones of the human voice and artificial musical instruments, which must be shaped on purely artistic principles, unfettered by any reference to utility, as in architecture, or to the imitation of nature as in the fine arts, or to the existing symbolical meaning of sounds as in poetry. There is a greater and more absolute freedom in the use of the material for music than for any other of the arts. But certainly it is more difficult to make a proper use of absolute freedom, than to advance where external irremovable landmarks limit the width of the path which the artist has to traverse. Hence also the cultivation of the tonal material of music has [...] proceeded much more slowly than the development of the other arts. It is now our business to investigate this cultivation.⁸²

For thousands of years, freedom in music has been restricted by the necessity to produce recognisable pitch patterns making up melodies.⁸³ To this end, most cultures use heptatonic scales. They are a paradigm for composition. In order that a melody can be recognised, the degrees of the scale must be identifiable by pitches in relation to the other degrees of the scale. When these are expressed as intervals, they become conceptual intervals where each has its own quality so that they can be identified. Conceptual intervals must neither be too small as they would be too difficult to perceive, nor too big, as in both cases melodies may not be easily perceived.

Variations of intonation or subtle differences of intervals, especially with music which does not answer to any known temperament, are the consequence of impromptu performance, great mastery, regional variations, organology, particular tuning and so forth, all combined with the ability of the performer.⁸⁴ In a traditional process of knowledge transmission, however, these subtle variations, particularly in the domain of performance mastery and instant creativity, take place at a later stage of music understanding and perception. In order to transmit and receive, a basis must be found allowing for a firm structure of the musical discourse, whilst allowing the performer the possibility to further develop his freedom of interpretation. This basis, which is the essence of the melodic repertoire, is commonly named the scale.

When confronted with an audience, a traditional musician of average talent would try to perform with utmost expression and invention, keeping in mind the need for a melodic pattern that his listeners will recognise. This process should request the least possible energy whilst taking the least possible steps within the continuum of pitch, in the search for balance between technique (or complexity

of the means used) and expression (or the effect of the musical discourse on the audience). In order to achieve this goal, this musician would ideally need to have previously tested all possibilities, within an octave or other important containing intervals, and determine which would result in the maximum number of expressive possibilities.⁸⁵ The process for interval combination and the search for the optimal number of intervals within a 'fourth', a 'fifth' or an 'octave'⁸⁶, will be defined as stemming from the principle of economy.

A semi-tone and quarter-tone model for *genera* and scales

The two models in this study are the western and the Arabian. Whilst western music uses the semi-tone, the quarter-tone is the basis of conceptual divisions with the *maqām* where subtle refinements reveal modal complexity.⁸⁷ In both cases, the smallest conceptual interval is an approximate semi-tone. A recurrent objection to the use of the semi-tone interval as a smallest conceptual interval is that the Arabian quarter-tone is half its size. Some theoretical modern descriptions of *maqām Awj Āra*, for example, show indeed one quarter-tone intervals in the scale. This *maqām* is reminiscent of Turkish music as its scale is similar to the *maqām Hījāz-Kār*, but with *awj* (= b-)⁸⁸ as its starting note. This causes some cultural and technical problems with the organology of the *ūd*, because of the usage of the theoretical equal quartertone in the 1920s and 1930s.⁸⁹ These problems are easily resolved with the difference between conceptual and measuring intervals we have explained.⁹⁰

Another objection to Arabian performance is mainly with *maqām Sīkā*. It begins, as its name suggests, with *sīkā*, = e-. In the quarter-tone model, this scale equates to 3 4 4 3 3 4 3, beginning with e- and with a b- between the conjunct two-three-quarter-tones intervals. In the common Arabian tuning of the *ūd*,⁹¹ the open strings of lower pitch are often used as drones repeating the fundamental note of the *maqām*.⁹² In order to reinforce the fundamental, some contemporary lutenists tune the lowest auxiliary string to E-. Many, however, prefer to keep the original tuning⁹³ and use a technique of fast alternation between e- and a note about a third of a tone lower,⁹⁴ and quickly coming back and insisting on e- so that the fundamental e- is reinforced. The small interval used between e- and the slightly lower pitch is only a variation and is used intonationally. Its main function is to underline the importance of e-, the next, the lower degree in the scale being d, which is three quarter-tones away from e-. This is why the performer must use a smaller interval of intonation leading to the fundamental.

Now that these two main problems have been addressed and that the limitation of small intervals is taken in both models as equal to the conceptual semi-tone, two

possibilities have been considered regarding the largest interval in the scale. It must be firstly limited only by the size of the octave, and by the minimum of two intervals amounting to a scale element, or secondly by the largest conceptual interval in both models, *i.e.*, the one-and-a-half-tones interval.⁹⁵ As a result, each generative process uses alternatively two alphabets. In the semi-tone generation, the first alphabet is without limitations except for the semi-tone division. The largest interval in the alphabet is the largest possible allowed in a particular generation. The second alphabet is reduced to the three conceptual intervals of one semi-tone 1, one tone 2, and one and a half-tones, 3, or augmented second.

This also applies to the generation process for the quarter-tone, except that in this case, the interval increments are quarter-tones, with a limited alphabet of 2, 3, 4, 5 and 6 of them (fig. 21: 186). The generative process is simple. A computer programme detects all the combinations of a certain number of intervals given in an initial alphabet of conceptual intervals, and arranges the results as hyper-systems, systems and sub-systems. This process starts with the minimum possible number of intervals in the scale elements⁹⁶ and ends with the maximum possible number of elements in the containing interval. The minimum number of intervals in combination is two, and the maximum depends on the containing capacity of the intervals in the model. With both models this corresponds to the number of half-tones in a row which can be arranged in a containing interval, *i.e.*, five for a fourth, seven for a fifth and twelve for an octave.

Preliminary definitions and remarks

Specialised terms for scale systems will be used throughout this study, their definition follows:

1. A scale system is a sequence of numbers for different classes of conjunct (conceptual) intervals within the frame of a containing element. This is defined as an interval composed of conceptual intervals with the sum of the containing element equating to the number of elementary intervals building up to it set to a certain value. Containing intervals are equal to the fourth, with an ascending frequency ratio of 4/3, and the fifth, with a frequency ratio of 3/2, and the octave.

2. A hyper-system is a capacity indicator of conceptual intervals. It is a scale element in which these intervals and the numbers composing the sequence, are re-arranged to form the least integer when numbers are concatenated. Hyper-systems are arranged, in the frame of a generative process, from the smallest (when expressed in integer concatenated form) to the largest.

3. A system is a particular arrangement of intervals in a hyper-system. Systems are also scale elements. They are arranged from the lowest corresponding integer to the highest within the hyper-system. A hyper-system is identical to

the first ranking system it generates.

4. A sub-system is a particular arrangement of intervals inside a scale element which corresponds to a de-ranked system. The original system is the first sub-system, and each de-ranking produces the next ranking sub-system. The number of conceptual intervals, NI, henceforth, limits the number of sub-systems in a system, as some of the combinations resulting from the de-ranking process may be identical and therefore redundant. The number of non redundant sub-systems may be lesser than the corresponding NI. The first ranking sub-system in a system is identical to the head system.

5. NI is the number of conceptual intervals of conjunct seconds which constitute a scale element. In the statistical study below, NI is variable and extends from two conceptual intervals in a scale element, to the maximum possible number of smallest conceptual intervals in a row within the containing interval. In both models, the maximum number of conceptual intervals in a scale element is equal to the number of conjunct semi-tones - the smallest conceptual interval - required to build it up. The maximum number of conceptual intervals in a containing interval (NI_{max}) of a fourth is equal to the number of semi-tones needed, *i.e.*, five consecutive semi-tones (NI_{max}=5). A typical example of the relationship between hyper-systems, systems and sub-systems is shown in figure 19, page 185, and table 5 where the 19 hyper-systems of the quarter-tone model generation with the limited alphabet 2, 3, 4, 5, and 6, and with seven intervals (NI = 7) to the octave, are arranged in ascending integer values. A typical hyper-system generates diatonic scales, *i.e.*, hyper-system no. 12 in the generation with the reduced alphabet (tab. 5: 175). This hyper-system generates three systems (fig. 19: 185) for the corresponding semi-tone model, when each in turn generates 7 distinct sub-systems by de-ranking intervals in each system.⁹⁷ Table 5 is specific to the general combination process used in modal systematics. Since the containing interval is equivalent to the octave, the sum of the integers (in unconcatenated form) in each scale is 12 half-tones in the semi-tone, and 24 in the quarter-tone model.

With the fourth, the respective sums in the two models are 5 semi-tones or 10 quarter-tones, and in a just fifth 7 and 14 respectively. The equality of the intervals of the semi-tone and the quarter-tone models is straightforward. For the transition from a semi-tone interval system to its equivalent in the quarter-tone model, simply multiply the intervals of the integers by two. To reverse the process, divide all the integers in the quarter-tone model by two. However, intervals represented by odd integers in the quarter-tone model have no equivalents in the semi-tone model. This is the reason why the ranks of the hyper-systems in the semi-tone model are corrected to their rank in the quarter-tone model, as explained in the next section.

The main question is why the generally assessed

number of conceptual intervals in a modal scale is seven in an octave, or what is the optimal number of conceptual intervals in containing intervals with ratios 4/3, the fourth, 3/2, the fifth, and 2, the octave.

Combining intervals within a fourth: filters and criteria

In a combination process of conceptual intervals using the semi-tone as the smallest conceptual interval, the sum of the containing interval of the fourth⁹⁸ must be 5 in the semi-tone, and 10 in the quarter-tone models. Our first goal is to find all combinations of intervals of the alphabet that sum up to these values.

In the semi-tone generation (fig. 23: 188, top), the alphabet is unlimited, except by the semi-tone structure of the intervals. The smallest interval is the semitone, and the largest, for NI=2 (two intervals in combination) can therefore only be a 4 semi-tones interval, 4 in the concatenated sequence of intervals, '[14]' in the first hyper-system of the semi-tone scale generation with NI=2.

The sum of the two intervals in the first hypersystem is equal to 1+4=5. The other hyper-system for NI=2 is 23, with two intervals 2 and 3 (the semi-tone value is represented by the two digits).⁹⁹ The rank of the hyper-systems (first column to the left) is given both in the semi-tone (plain numbers) and the quartertone models (between brackets) if the two differ. If the hyper-system does not exist in the semi-tone model, only the rank of the corresponding quarter-tone hypersystem is given (one number between brackets). For NI=2, the two hyper-systems 14 and 23 both generate one single system, with two sub-systems for each system. For NI=3, we still have two (but different) hyper-systems (or capacity indicators) which generate each one single system, but with three sub-systems each (due to the three conjunct intervals in the system).

This generation corresponds to the commonly accepted number of three intervals in a fourth, and contains the tetrachord equivalent of the tense diatonic genus, hyper-system 122, and of the tone, or tense chromatic: 113. For each of NI=4 and NI=5, we obtain one single hyper-system, with four sub-systems for NI=4, and five identical, (with four which are redundant) sub-systems for NI=5. The total numbers of hyper-systems, systems and sub-systems in each case figure in the row below the last sub-system.

A first, and evident remark can be made. A small number of intervals, NI, implies that larger intervals have more chances to find a place in the system, whenever a larger NI results in an increased use of smaller intervals, notably here the semi-tone. Additional rows below the grand total give the numbers of remaining sub-systems for each NI whenever some eliminating conditions are met (the number of excluded sub-systems is shown in brackets, with a minus sign):

1. The total number of non-redundant sub-systems is equal to the initial total number of sub-systems minus the number of redundant sub-systems in each case. Redundancy occurs once in the semi-tone model, for NI=5. Here the hyper-system, system and sub-system(s) are identical, as one single interval class, the semi-tone, is used in the scale element. These redundant sub-systems, generated through the de-ranking process, are struck out in figure 23, page 188, and must be excluded from the generative process.

2. The 'double semi-tone criterion' (an asterisk is added at the end of each sub-system which responds to this criterion) excludes (separately from other filters) sub-systems containing two semi-tones in a row (conjunct semitones).¹⁰⁰ This filter, which has been inspired from Arabian music, is most effective when applied to sub-systems with a large number of intervals of greater values. If two consecutive semi-tones are present in a heptatonic scale, they are commonly found at the sides of the junction between two tetrachords, or at the junction between a scale and its equivalent to the octave, lower or higher.¹⁰¹ For larger containing intervals such as the fifth and the octave, this criterion is applied for three conjunct semi-tones.¹⁰²

3. The 'conjunct large intervals' filter (sub-systems marked with §) excludes scale elements containing at least two conjunct intervals larger than, or equal to, the one-tone interval, and among which one, at least, is larger than a tone. This is a general rule which is present in the heptatonic Arabian traditional scales. Examples of sub-systems with such characteristics are 46 and 55¹⁰³ for NI=2 in the quarter-tone model (fig. 23: 188, bottom). The criterion is most effective with smaller values of NI.¹⁰⁴

4. All these filters operate independently. If we combine them in one complex criterion, filtered subsystems will add up or merge ('neither nor', or a Boolean inversed 'OR' operator in the theory of ensembles) with a resulting number of filtered sub-systems in the row entitled 'Intersecting criteria 1'.

The aim is to compare, excluding all filtered sub-systems, the results of the generative process for different values of NI and to determine the optimal number¹⁰⁵ of conceptual intervals in the containing interval. The results of the semi-tone generation, with or without filters applied to them, are shown in the two graphics of figure 24, page 189. The generation with NI=3 (or three conceptual intervals in a containing fourth interval) gives the largest number of independent, non-redundant, sub-systems, *i.e.*, 6. The filters or criteria, accentuate this optimal value. If we exclude scale elements comprising large intervals (greater than the one-tone-and-a-half)¹⁰⁶ in addition to those excluded through the 'intersecting criteria 1' composed filter, the remaining two sub-systems in the case NI=2 would be equally eliminated, leaving thus the case NI=3 as the unique possibility concerning the ability to generate a just fourth (see 'intersecting criteria 2', fig. 23:188).¹⁰⁷ The same applies to the quarter-tone distribution, (fig. 25: 190) with however some

quantitative and qualitative differences in the contents of the two generations.

A first difference is that the quarter-tone model (fig. 23: 188, lower half and fig. 25: 190) generates, as expected, intermediate and additional hyper-systems containing neutral interval equivalents (or odd multiples of the quarter-tone) which are for example hyper-systems nos 2 and 4 in the case of NI=2. Whenever the smallest and largest intervals are the same, in both semi-tone and quarter-tone generations, for the same NI (due to the limitation of the smallest conceptual interval to the semi-tone in both cases), then the intermediate hyper-systems generate additional sub-systems in the quarter-tone model. The optimal number of intervals (the most economic choice) is still three. All the filters accentuate the optimal value by giving the two neighbouring sections of the line a smaller angle (in fig. 25: 190, top, 'intersecting criteria 1' give the most acute angle around value 13 for NI=3). Figure 23, page 188 shows, however, that the new possibilities in the quarter-tone model are not fully integrated, for NI=3, in Arabian music, although this case comprises no redundant sub-systems.¹⁰⁸ The new sub-systems 235, 532, 523 and 325 are seldom or never used in this music, as the only configuration for hypersystem 235, with its two systems 235 and 253, seems to be the one which places the largest interval in the middle (*i.e.*, sub-systems 253 and 352). If we were to add this criterion (*i.e.*, if we dismiss sub-systems containing suites of very small intervals such as 23 or 32, in the quarter-tone model) to the filters already used for the semi-tone model of the fourth, we would end up having NI=3 as the unique possibility for this containing interval, because the remaining sub-systems for NI=4 are excluded by this criterion (see the row 'intersecting criteria 2' for the quarter-tone model in figure 23, page 188).

The reverse pycnon rule

All the filters and criteria used with the fourth correspond to common practice and theory and their application provides with complementary information on the aesthetics of modal music, especially with the *maqām* and modal diatonic music. It would be interesting however to try to find one single criterion which would have the same effect as the four criteria explained above.¹⁰⁹ Taking a closer look at the composition of the sub-systems commonly in use in the diatonic and Arabian music (fig. 23: 188: quarter-tone model, in bold), and comparing the sums for any two conjunct intervals within them, we come up with a very interesting conclusion. All these sums are comprised between 6 and 8 quarter-tones. Fig. 26: 191 shows pairs of conjunct intervals in ascending values from the top and the left, beginning with a first interval of 2, and conjunct intervals (from top to bottom), beginning also with a 2, incremented until the maximum which is 8 quarter-tones, in order to complete the sum for the fourth.

The next column shows the same process, starting with interval 3 and a conjunct interval 2, with the conjunct interval incremented by one unit downwards. The largest interval for this column is 7, since the sum of the two intervals may not exceed 10, which is the value of the fourth in multiples of the quarter-tone.

The process continues for the other columns until all possibilities are given. Common bi-interval combinations are written in bold on grey background for combinations commonly used in Arabian music, or on black background for diatonic *genera*. Sums are given on the top right or bottom left corners of each bi-interval element. Equality of the sums follows oblique parallel lines, from bottom left to top right (or reciprocally).

All series with two conjunct intervals found in the commonly used tetrachords are concentrated in the three oblique rows with sums of 6, 7 or 8. Other combinations have sum values below or above. This is a very strong indicator for homogeneous interval distribution of the intervals within the scale. If we add to all these bi-interval combinations other intervals, to the left or to the right and check those which follow the rules of homogeneity (fig. 27: 191), we end up having only common tetrachords listed in figure 23, page 188. With a single criterion applied to the intervals of the sub-systems within the fourth, there is a model which is the closest possible to common practice and theory. Furthermore, this rule of homogeneity is the reciprocal of Aristoxenus' pycnon rule which says that a pycnon (a bi-interval scale element composed of two small intervals within a fourth) must be smaller or equal to the one tone interval.¹¹⁰ The rule of homogeneity observed with common *genera*, which we could also qualify as reverse pycnon, says the contrary (fig. 28: 192). The complement (here of any bi-interval combination inside the fourth) must have the same limitations as those for Aristoxenus' pycnon, and the bi-interval combination, although equal to, or greater than the one-tone interval (not a pycnon in Aristoxenus), has the same limitation as for the complement of the pycnon with Aristoxenus.

This important difference may have one of three causes.

1) With our modern music as with traditional forms, such as with the *maqām*, there has been important evolution diverging from their initial form, which initially, might have been close to Aristoxenus' descriptions.

2) Arabian music and Ancient Greek music were never connected, and the former evolved independently from the latter, or

3) Aristoxenus' description of the music of his time was not accurate or had no relation with practice.¹¹¹

Applying the reverse pycnon rule to the fifth and the octave The last filter has shown the most commonly used *genera*. Consequently, it would be interesting to apply this principle to the fifth, or to the octave. This is simple enough,

with the fifth, and consists in adding an interval at one end and at the other of the *genus*, within the rules of homogeneity (fig. 29: 192), and then verifying the sums of the resulting combinations.¹¹²

As expected, this shows that most of the pentachords resulting from this operation have their equivalence in both literature and practice, although some of possible pentachords do not appear in the series.¹¹³ Figure 30, page 193, shows an example of scale building beginning with the *Hijāz* tetrachord.

This is very similar to the generation of fifths, although less than half of the combinations (with a black background on the figure) exist in the literature or in the practice, with the remaining scale elements not found in the literature.¹¹⁴ Possibilities for some limited hexatonic elements (for example 626262 and 262626 in the figure) also exist. As a consequence, whenever the rule of homogeneity applies to commonly used *genera*, its extension to the fifth and octave intervals is either inadequate or too restrictive, although it shows that the full potential of Arabian music, even with such a restrictive criterion, is still not fulfilled.

There is a noticeable exception with the *maqām mukhālīf* which in Arabic means 'infringer', which has a limited scale of b- 3 c' 2 d^b 4 e^b 2 f^b where the two first intervals breach the rule of homogeneity. There are other *maqām* where conjunct tetrachords may form neighbouring semi-tones as for example in the *maqām Nawā-Athar* where the interval/tetrachord distribution is [4] {262} {262} (or a disjunctive one tone interval followed by two *Hijāz* tetrachords, where the two neighbouring semi-tones (in italics) also breach this rule (when applied to the octave). This is the main reason why, although the homogeneity rule is a perfect match-maker for tetrachords, we shall keep, for the following statistical studies in the frame of a fifth or an octave, the initial criteria given in figure 23, page 188.¹¹⁵

A little incursion in the eighth-of-a-tone model

The reader may be wondering why this study does not give more refined models, such as the eighth of a tone, for example. A first answer was given above and said that the purpose of interval generation was to use the least possible divisions in a containing interval with the utmost number of combinations, according to the principle of economy.

A second answer comes from the definition of the conceptual interval. Any interval in use in a scale should be relatively easily identified, both by performers and listeners alike (this procedure becomes difficult whenever the elementary intervals are smaller than the one-quarter-tone). However, and as a confirmation of the principle of relative size of intervals within a containing interval, we shall have a quick look at this possibility, in fourth.

When dealing with a new interval model, it must be first determined which are conceptual, elementary, or measurement intervals. When the measurement interval is one-eighth-of-a-tone, what would be the smallest conceptual interval? Two-eighths-of-a-tone would be too small because it equals a quarter-tone which is too small for being conceptual. A three-eighths-of-a-tone interval, as used by Aristoxenus in his hemiolic chromatic *genus*,¹¹⁶ with two conjunct intervals of three-eighths of a tone and one interval of fourteen-eighths – or seven-quarters-of-a tone, would restrict us to the 17-ET inspired by *Urmawi's* theory (fig. 31: 194, central one tone interval), with a three-eighths interval equivalent to a *limma*, an elementary quarter-tone used as an auxiliary interval, and with two possibilities for the *mujannab* interval (see the three one-tone intervals to the right of figure 31, page 194).

On the other hand, four-eighths-of-a-tone is equivalent to one-half-tone, and choosing such a small conceptual interval, we would lose the benefit of having a smaller division of the tone.

As long as we do not want to differentiate conceptual intervals using too small elementary intervals, dividing the tone further than the quarter-tone (with the smallest conceptual interval set to the semi-tone value) would be pointless for the model, but could bring a better approximation of intervals used in practice. Figure 32, page 194 shows the graphic results of a generation in eighths-of-a-tone with the smallest conceptual interval being a semi-tone (4/8 of a tone), and elementary intervals being one-eighth-of-a-tone. This leaves space between the semi-tone and the tone for three intermediate intervals of five, six and seven-eighths-of-a-tone.¹¹⁷

The optimal number of intervals remains three (NI=3) with changes to the general curve of the graph. With four intervals and a very small increment such as one-eighth-of-a-tone, we have more possibilities than we had with the quarter-tone generation (for example for the case NI=4), but NI=3 remains the optimal value. If we add the principle of memory to the principle of economy, or the need for performers of traditional music to memorise the elementary scale divisions of the fourth (or archetypal *genera*) in order to reproduce them effortlessly while performing, we end up concluding that the eighth-of-a-tone model simply gives too many possibilities, which would also be difficult to distinguish from one another. One would associate the difficulty of perceiving intermediate intervals for the audience and the performer (the eighth-of-a-tone is 25 cents in size, very close to the Pythagorean comma which is approximately 23 cents), with a major difficulty (a huge number of *genera* to memorise) which introduces a *quasi* impossibility for the existence of a traditional repertoire based, as already stated, on the memorisation and identification of melodic patterns. To conclude, let us note that within a fourth, the case NI=3 (intervals) is the only one (still) that does not generate re-

dundant sub-systems, a characteristic we have already underlined for the other two models (with semi-tones and quarter-tones). This discussion is continued at the end of next section.

Combining intervals within the fifth

Modelling the fifth in semi-tones or with quarter-tones (with the restriction to the semi-tone as the smallest conceptual interval) gives additional information on the internal structure of containing intervals (fig. 33: 195).¹¹⁸

NI = 4, in both models, is the optimal value although noticeable differences exist between the two. The optimal value for the semi-tone model is clearly shaped, and accentuated with the application of filters to the sub-systems.¹¹⁹ With the quarter-tone model, this optimal value has NI=5 as a competitor, and the filters give the latter a more important role, although less than for NI=4. Another difference is that the semi-tone model generates no redundancies (except for NI=5 which is a trivial case with 5 semi-tones in a row) whenever redundant sub-systems may be found in the quarter-tone model, including for N=4.¹²⁰ As a consequence, the semi-tone model is, within the fifth, more appropriate than the quarter-tone model.

For example, when reducing the results to the limited alphabet of 1, 2, 3 for the semi-tone model, the results (fig. 34: 196, compare with fig. 33: 195 above) show that the most effective filter is the disjunct large interval criterion which eliminates sub-systems containing intervals equal to, or greater than 3 semi-tones.

Discussing the preliminary results

Interval distribution within the fourth or the fifth provides with a preliminary answer to our greater question concerning heptatonism. Combination processes applied to conceptual intervals show that three intervals in a fourth and four intervals in a fifth correspond to an optimal value (a maximum of different *genera* for the least possible number of conceptual intervals) which reflects a balance between complexity (smaller interval identifiers such as the eighth of the tone or others, more intervals in a containing interval) and productivity in terms of independent (and fit for their role in music performance) interval combinations. This applies with or without the filters in resulting sub-systems. These filters reduce possibilities and give a hold on the internal mechanisms of modal music. Interval combinations chosen throughout history can be described and recognised – their positioning and qualitative sizes within the fourth or the fifth is not a coincidence. Furthermore, as we try to reduce the steps between intermediary intervals (as in the eighth-of-a-tone model of the fourth), the tendency towards a balance of the generations around the (same) optimal value remains, with however

quantitative differences between models. The semi-tone model seems to be best suited to the fifth, rather than to the fourth: the optimal value in the semi-tonal modelling of the fifth is very stable and the angle formed by the two bordering segments of the line is acute and (fig. 33: 195) accented in the case of a limited alphabet (fig. 34: 196); this optimal value still exists for the fourth, in the semi-tone model, but with a very limited number of combinations: in this case, only four major (diatonic) combinations may be used by the performer, which is somewhat limited compared to the twelve combinations in the quarter-tone model of the fourth (fig. 23: 188: nine *genera* are left if we filter the sub-systems with the second set of intersecting criteria). Twelve (or nine) combinations within a fourth seems a suitable reservoir for modal possibilities, alone or in combination, in performance or as paradigms for a repertoire as it gives the performer good possibilities for modulations, with the fourth as a starting containing interval that he can elaborate further and further (by modifying its internal structure – or interval composition), and then perhaps expand the span of the melody to the fifth or more. An eighth-of-the-tone model gives too many intermediate possibilities while adding perception difficulties (for example, an eighth-of-a-tone is much more difficult to recognize than a interval of one-quarter-tone, and the difference between a three-quarter-tone interval and a one-tone interval is much easier to distinguish from the difference between a six-eighths-of-a-tone interval and a seven-eighths-of-a-tone interval). With the fifth, however, the quarter-tone model becomes too rich,¹²¹ and too complicated. Almost seventy possible combinations are available to the performer, which would be difficult to memorise.¹²² It is easier to add a one-tone interval below or above the bordering intervals of the fourth. (fig. 29: 191). This is a process that would give some fifteen interval patterns available within the fifth. This is a practical means for enriching the repertoire with the least possible number of conceptual intervals. Even then, the semi-tone possibilities of the fifth compete with the potential of this last model. This is mainly due to the fact that the addition of one tone to the fourth reinforces the diatonic nature of interval combinations, as well as the possibilities for bi-fourth configurations (two intersecting fourths with successive ranks – fig. 35: 196).

Should we start our scale element with a one-tone interval (fig. 35: 196, left, the one-tone interval equates to the 4 quarter-tones), possible combinations complying with both rules of sum (for the adjacent fourth – in order to obtain a fifth) and the rule of homogeneity are more or less balanced between elements with neutral intervals (five) and elements with (exclusively) diatonic intervals (four). If we begin our element (ascending from left to right) with a neutral interval such as the three-quarter-tone interval

(fig. 35: 196, right), the remaining three intervals cannot make a fourth (their sum is always equal to 11 quarter-tones). In order to make a fifth, we are in some cases, for example 3345, 3434 and 3524, compelled to complete first the just fourth, then to add to it the one-tone interval, at the end. This process leaves us with only three possible combinations having both fourth and fifth, which is very little when compared to the nine possible combinations in the preceding case in which we have set the first interval to 4, and in which all combination have both fourth and fifth.

In an open process, however, not taking into account the fifth as a necessary step on the way to the octave, the reduced potential of the starting neutral three-quarter-tone interval widens up very quickly (before being restricted once again by the octave).

As a preliminary conclusion, we may say that the quarter-tone model is particularly suited to the just fourth, whilst the semi-tone model is better suited to the fifth as a containing interval. Both models, however, show that the number of four or three intervals within a fifth or a fourth, is not coincidental, but it is the result of an optimisation process between complexity and expressivity.

A further remark can be made concerning octave systems of scales. What is applicable to the fourth also applies to a combination of two fourths with a one tone interval, or to combinations of fourths and fifths within the octave. Adding up the numbers of optimised interval repartitions for two fourths (twice three optimal conceptual intervals) + a one-tone interval, the optimal number of intervals for the octave is seven – the same applies to the total optimised number of intervals from the combinations of fourths (three optimal intervals) and fifths (four optimal intervals).¹²³ However, not all scales do follow the fourth-plus-fifth, or the two-fourths plus a one-tone arrangement of interval combination. In the following section we shall repeat the process used for the fourth, and apply it to octave scale elements.

Generating scales in the semi-tone and quarter-tone approximation models: preliminary exposé

With modal systematics, octave scales are represented as suites of conjunct intervals the sums of which are equal to the number of elementary intervals within the octave. This means that they must be equal to 12 semi-tones in the semi-tone model, or to 24 quarter-tones in the quarter-tone model.

In both models, the smallest conceptual interval is the semi-tone. In this study, as for the statistical studies of the fourth and the fifth, we extend the definition of the conceptual intervals beyond the restricted alphabet¹²⁴ (2, 3, 4, 5 and 6 quarter-tones in the quarter-tone model). Thus,

the smallest number of conceptual intervals to an octave – or NI – is one, and the largest NI is equal to 12 (fig. 36: 197, for an example of results in the semi-tone model), or 12 semi-tones in a row (or the smallest conceptual interval twelve times in a row).¹²⁵

Intermediate cases (*i.e.*, NI = 2, 3, 4, 5, 6, 7, 8, 9, 10 and 11 conceptual intervals to the octave) have an intermediate behaviour, with a tendency to concentrate larger intervals for smaller values of NI, and semi-tone suites of intervals for larger values of NI. This is self evident from the cases for NI=1 and NI=12, but two further examples will help the reader better understand the phenomenon:

- NI=2 in figure 36, page 197, generates six different hyper-systems in the semi-tone model which are 1 11; 2 10; 3 9; 4 8; 5 7; 6 6. In turn they generate unique systems (identical to the hyper-systems) with two sub-systems for each configuration (there are only two possibilities for combining two numbers, here taken as a and b: a b and b a). System 6 6 is fully redundant (this means that whatever de-ranking – or other combinatory – process is applied to its intervals, we end up having the same configuration because all intervals are of the same class). The total number of generated independent sub-systems for the entirety of hyper-systems for this case (NI=2) is consequently equal to 11 (fig. 36, p. 197 var. NSS_NR).

- When NI increases, the largest possible intervals become smaller in size: for NI=3 with 19 systems, (same figure) for example, the largest possible interval is the ten-semi-tones interval which appears in the first hyper-system 1 1 10 (or two intervals of one-semi-tone in a row and one ten-semi-tones interval). The size of the largest interval decreases regularly with the increase of the NI, and reciprocally that is if we increment NI by one unit, the largest interval generated is one-semi-tone smaller than for the preceding generation (with a smaller NI). When we get closer to the upper limit, the largest generated conceptual interval has decreased in such a way that only small intervals are generated: for NI=11, for example, we obtain one single hyper-system of 1 1 1 1 1 1 1 1 1 2 (10 semi-tones in a row and one one-tone interval), which still generates one unique (identical) system, with however eleven different subsystems generated by the de-ranking process which are:

- 1 1 1 1 1 1 1 1 1 2
- 1 1 1 1 1 1 1 1 1 2 1
- 1 1 1 1 1 1 1 1 2 1 1
- 1 1 1 1 1 1 1 2 1 1 1
- 1 1 1 1 1 1 2 1 1 1 1
- 1 1 1 1 1 2 1 1 1 1 1
- 1 1 1 2 1 1 1 1 1 1 1
- 1 1 2 1 1 1 1 1 1 1 1
- 1 2 1 1 1 1 1 1 1 1 1
- 2 1 1 1 1 1 1 1 1 1 1

Consequently, the last case (NI=11, fig. 36: 197) generates, with one and only hyper-system which is identical to the one and only system it generates, the same number of independent (non-redundant) sub-systems as with NI=2 above (fig. 36: 197, var. NSS_NR).

This is a first indicator of symmetry for generations with different NI. This symmetry is obvious in figure 36, page 197, (var. NSS_NR) which shows the statistical results of a full scale generation in the semi-tone model of the octave. Values around NI=6 are symmetrically placed for the numbers of systems, (var. NS) however, this symmetry does not apply to hyper-systems (fig. 38: 198, var. NH: var. NSSU_NR is explained below).

- As a next step after determining the numbers of sub-systems, we exclude redundant sub-systems from the whole set (fig. 38 and 39: 198 – this also shows the numbers of hyper-systems). Redundant sub-systems occur whenever an interval configuration is repeated twice or more in order to cover the complete range of intervals within a system. System 4 4 4 in the semi-tone model (three – or NI=3 – successive di-tones which form an octave) is completely redundant, as any de-ranking process gives the same combination as the original one (this is a mono-interval element repeated 3 times in order to form an octave).

Another example is the one-tone scale used by Debussy, 2 2 2 2 2 2 (NI=6), which is also completely redundant. More elaborated semi-redundant systems (which generate a limited number of sub-systems – such as Messiaens’ scales with limited transpositions) exist, such as 1 1 2 1 1 2 1 1 2 containing three successive three-interval identical combinations of two conjunct semi-tones and one one-tone intervals (121 121 121 and 211 211 211 are independent sub-systems of the latter). There can only be three distinct sub-systems (scales).¹²⁶

- Results for sub-systems are then expressed, for both generations (*i.e.*, with the complete or with restricted alphabets), through the Unitary number of non-redundant sub-systems, or the total number of non-redundant sub-systems divided by the corresponding NI (see for example figs. 38 and 39: 198). This process is explained in details in the following section.

- To the results of the previous process, we apply then the two following filters and keep track of the results for both, as well as for the unitary numbers of sub-systems (figs 40, and 41: 199,¹²⁷ with these filters, successful combinations are kept, not excluded):¹²⁸

- Firstly find all sub-systems with a fourth starting with the first interval that we shall call a direct fourth. This limitation is due to the fact that a fourth, in second position, for example, in a sub-system is the first fourth of the lower ranking sub-system (by a deranking process).¹²⁹ The values on the graphics (var. NSS5U_NR, beginning with figure 40, page 199) indicate that the filtered remaining sub-systems have each a direct fifth (which starts with the first interval of the sub-system - these are labelled NSS5U_NR on the graphs,

for ‘Numbers of Sub-Systems in 5th Unitary, Non redundant’). As long as we are searching for statistical results, this is the same as searching for direct fourths, as a complement of the fifth (the fourth) can be obtained by de-ranking the sub-system four times.¹³⁰ This filter keeps the filtered sub-systems, and excludes the others (as if we excluded all sub-systems that do not have a direct fourth); original results with unitary sub-systems are however kept for further comparisons.

- The next step consists in verifying for systems with a direct fourth enclosed in a direct fifth (labelled FFU_NR, or ‘Fourth in a Fifth, Unitary and Non-Redundant’), for example in $\{(442) [4]\}$ (352). With the latter, the direct fourth is 442, and the direct fifth is $\{442[4]\}$ with the complement of the fourth within the fifth being the one-tone interval, or [4] – in such cases, the configuration of the sub-system is equivalent to a combination of two fourths and a one-tone interval ($4^{\text{th}} + T + 4^{\text{th}}$ – see example above). This filter is named the direct Fourth in a Fifth, or FF, process (same figures as above).

- Now that we have representative graphics for the overall statistical distribution of sub-systems, including the ones containing direct fourths and/or fifths, we may apply, separately, as a first approach, two additional filters which are very close to the ones used for the fourth and fifth containing intervals explored in the previous sections:

- The conjunct semi-tones criterion (which operates here for three or more semi-tones in a row – (figs 42 and 43: 200).¹³¹

- The conjunct large intervals criterion, which operates for intervals greater or equal to the one-tone-and-a-half interval (3 in semi-tones, 6 in quarter-tones (figs 44 and 45: 201).¹³²

- The final stage is reached by applying the last two filters simultaneously (figs 46 and 47: 202). All these graphics and filtering procedures are discussed in the next sections.

From hyper-systems to unitary sub-systems: an example based on the semi-tone model

We shall begin our investigation of the octave with a full scale generation in the semi-tone model using the complete alphabet, from the one-semi-tone interval to the twelve-semi-tone interval. A complete generation includes statistical results for numbers of conceptual intervals NI distributed between NI=1 to NI=12: the case for NI=1 (one single octave interval in the system) is shown on the first four graphs only.

A. Generation of octave systems with the full alphabet of conceptual intervals

This first example of octave generation in semi-tonal conceptual intervals shows that the results in numbers of systems for NI=1 have a symmetrical correspondent which is NI=11 (fig. 36: 197). The optimal

value for systems with this process is reached for NI=6, for which the number of systems is at its highest value (80 systems are produced for this number of conceptual intervals to the octave and 480 subsystems). Furthermore, results (for systems - NS - still) for intermediate values of NI (from NI=2 to NI=10) are symmetrically distributed around the optimal value (for NI=6).¹³³

However, the non-redundant sub-systems are also distributed symmetrically around the bi-optimal at NI=6,7. If we look at the numbers of hyper-systems (NH) generated by this process (fig. 38: 198)¹³⁴, they have a distribution which is different from the distribution of the number of systems (NS). This is because and, although for example NI=4 generates the largest number of hyper-systems (in this case 15), each hyper-system in this configuration can generate a limited number of systems since there are only a small number of positions (four in this case) in which conceptual intervals may be combined in order to obtain systems,¹³⁵ whenever the corresponding (symmetrical in terms of numbers of generated systems) case is NI=8, which generates a lesser number of hyper-systems and have the same number of systems because of its eight (twice more as for NI=4) possible positions for conceptual intervals. In the latter case, there are fewer possibilities for different classes of intervals within the hyper-system,¹³⁶ but more positions (eight) that conceptual intervals can fill. This explains why the results are balanced although we still have no explanation for the perfect symmetry of the resulting numbers of systems around NI=6.¹³⁷ The symmetry equally applies for the Unitary Number of Sub-Systems (NSSU), from which Non-Redundant sub-systems have been excluded (NSSU_NR on the graph in fig. 38: 198). The latter is a weighted variable which reproduces the effect of the principle of economy explained in the first section of the second part of the article. If we transpose this principle to the statistical generative models explored here, an increase of complexity (*i.e.*, of the number of conceptual intervals, or NI, needed in order to compose the octave), even if it produces more sub-systems must bring a relative increase of the latter. In other words, each supplementary interval needed to compose the octave must be justified by a proportional (qualitative) increase of the number of generated sub-systems, not only by augmentation of the total (quantitative) number of sub-systems.

This proportional increase criterion can be included from the results by dividing the total number of sub-systems, for each NI, by the number of conceptual intervals needed in order to complete the octave, that is by NI itself, which gives us $NSSU = NSS/NI$. This variable (NSSU) is equivalent to NS (or the total number of systems for each different NI),¹³⁸ and gives us, as such, no additional information. However, if we exclude the redundant (R) sub-systems¹³⁹ from the total number of sub-systems, we obtain the final (weighted) variable NSSU_NR which is equal to

the total number of sub-systems minus the number of redundant sub-systems for each NI, the whole being divided by NI itself, (or $NSSU_NR = (NSS - R)/NI$. $NSSU_NR$ is, as a result, a compound variable that integrates the principle of economy. It shows the need for each supplementary conceptual interval used in a scale to be justified by a proportional increase in the number or resulting sub-systems. In the graph of figure 38, page 198, the values for $NSSU_NR$ are close to the ones expressing the total number of systems (as there are few, comparatively, redundant sub-systems in each case, except for $NI=12$ which is a trivial case), and the corresponding broken line is also symmetrical around $NI=6$.¹⁴⁰

B. Generation of octave systems with reduced alphabet of conceptual intervals 1, 2, 3, compared to full alphabet generation

When we limit the largest interval in the semi-tone model to the one-and-a-half-tones interval (limited alphabet), the symmetry observed for systems and subsystems in the previous generation disappears (figs 37: 197 and 39: 198 - symmetry shifts around the bioptimal values $NI=7,8$, in fig. 37: 197, for non-redundant sub-systems) and the optimal value NI for systems (NS) shifts to the value $NI=7$ instead of $NI=6$, while smaller values for NI (up to 3) simply do not generate any scale element. The smallest productive value of NI in this generation is $NI=4$, with the unique hypersystem/system/sub-system 3 3 3 3.¹⁴¹ Furthermore, the number of pentatonic (with $NI=5$) systems diminishes considerably,¹⁴² and values of NS and $NSSU_NR$, and $NSSU_NR$ (fig. 39: 198) for slightly larger values of NI ($NI=6$ to $NI=8$) are also considerably reduced when compared to those of the full generation in figure 38, whenever results for still larger values of NI are less affected. Octatonic systems ($NI=8$) compete with the heptatonic models ($NI=7$) for the optimal value (especially in figure 37, where the numbers of sub-systems are not weighted) and, beginning with $NI=9$, generations are non-economical in all reviewed cases and figures, which means that increasing the number of conceptual intervals to more than eight in the octave gives a rapidly decreasing number of new systems.¹⁴³

Comparing figures 38 and 39, page 198 with figures 36 and 37, page 197 also shows that there is no direct correlation between sub-systems and systems. The optimal values are still, however, restricted to a limited number of possibilities, from $NI=6$ to $NI=8$. As a next step, we shall include the direct fourth (or direct fifth) and the fourth in a fifth filters in our models, and compare the results with those of the quarter-tone model.

Comparing generations in the semi-tone and quarter-tone models: looking for direct fourths and fifths

The direct fourth (*i.e.*, a fourth starting with the first interval of a sub-system) and the fourth in a fifth (see above) criteria may serve as supplementary filters for comparison with the remaining sub-systems. These filters are given at figures 40, page 199 (semi-tone model) and figure 41, page 199 (quarter-tone model), applied to the results of the realistic generation in the preceding stage, *i.e.*, to the unitary non-redundant¹⁴⁴ sub-systems with the limited intervallic alphabet of figure 37, page 197 and to its equivalent in the quarter-tone model.¹⁴⁵ A few remarks may be made about these results:

- The optimal value for the Unitary sub-systems occur, in both models, for $NI=7$,¹⁴⁶ although in the quarter-tone model this optimal value has a serious competitor for $NI=8$, with the latter being also the optimum for numbers of sub-systems including a direct fourth (or fifth), or a fourth in a fifth¹⁴⁷.

- The ratio of unitary sub-systems in the quarter-tone model (fig. 41, var. $NSSU_NR$) to the corresponding sub-systems in the semi-tone model (fig. 40: 199, var. $NSSU_NR$) is about 20 to 1, whenever this proportion diminishes, for sub-systems with the fourth or the fifth ($NSS5U_NR$ – around 10 to 1), it is even less for the fourth in a fifth criterion (6 to 1 for the latter). This means that, although the semi-tone model generates considerably fewer sub-systems, the proportion of sub-systems in this model with direct just fourths, or combined fourths and fifths (fourth in a fifth criterion), is larger than in the quarter-tone model.

However, the optimal value for these sub-systems occurs for $NI=8$. This is because larger numbers of NI work in favour of increased numbers of semi-tones in a scale. In turn, this applies in favour of the presence of fifths or fourths in a scale.¹⁴⁸

- All results for values of NI around the optimal value decrease in an almost exponential manner (for values of NI less than, or equal to, six, or greater than, or equal to, nine). This means that optimal generations are concentrated for values of NI between (and including) six and nine, which gives us a preliminary answer to our initial question in the introduction to this article. These results are, however, still not completely satisfactory, as they do not clearly show the expected optimal value for $N=7$. Let us remember that sub-systems in these generations may include tri-interval suites of semi-tones, or large conjunct intervals of the second, both of which do not fit with the aesthetics of modal traditional music. The results of the application for these complementary criteria are dealt with in the following section.

Using conjunct interval filters

As a complementary step towards a better understanding of heptatonic scales, excluding conjunctions of small or large intervals from the previous resulting sub-systems, seems to be a suitable filtering criterion. Since we have already shown in the sections dedicated to models of the fourth and the fifth, conjunctions of semi-tones are rare in heptatonic scales, and occur mainly between two conjunct tetrachords. The extension of the filter to three semi-tones in a row (which seems to be a non-existent combination in the scales of traditional music as we know it today)¹⁴⁹ makes a good aesthetic criterion when searching for generative optimal figures 42 and 43, page 200.¹⁵⁰ This filter is called ‘\umin’, or exclude ultra minimal combinations – here of three – semitones in a row. In figures 44, page 200 and 43, page 201, the results for the ‘\umin’ filter are shown separately.¹⁵¹ They are independent from the ‘\max(6)’ – or ‘\max(3)’ – filter, with inversed influences on the curves (figs 44 and 45: 201, for the latter): the conjunct semi-tones filter (figs 42 and 43: 200) affects only sub-systems for NI greater than or equal to 5 (compare with (figs 40 and 41: 199), with an increased effect for larger values of NI (the last three generations with NI=10, 11, 12 – in the semi-tone model – have zero values as a result)¹⁵². At this stage of the study, these results are obvious: semi-tones are predominant for larger values of NI, making of it a particularly effective filter.

On the far side of the alphabet, conjunct large intervals restrict combination possibilities, and make it impossible to get fourths, for example, as two conjunct one-and-a-half-tones intervals (which form a tri-tone) are already larger than the fourth (see results of the filter in figs 44 and 45: 201).^{153, 154} To exclude such sub-systems with two conjunct intervals equal to, or bigger than, the one-and-a-half-tones interval we must apply the second filter, ‘\max(3)’ or ‘\max(6)’ (or exclude sub-systems with two or more conjunct intervals equal to – or greater than – 3 semi-tones or 6 quarter-tones).

The evolution of the curves in figures 44 and 45, page 201, if compared with those of figures 40 and 41, page 199 is remarkable. Although it excludes conjunctions of large intervals, this filter has no effect for values of NI equal to, or greater than, 9, smaller values of this variable are the most affected with a tendency to favour the NI=8 generation as an optimal value. All systems for NI less than 5 are excluded. This may be explained by the fact that smaller values of NI facilitate the existence of larger intervals, whenever larger values of NI tend to exclude the latter from sub-systems.¹⁵⁵ The two filters for conjunct small (semi-tones) or big (larger than the one-and-a-half-tones interval) intervals have, when applied separately, complementary effects:

if applied simultaneously, they give most interesting results (see figs 46 and 47: 201)¹⁵⁶ all optimal values, for both the semi-tone¹⁵⁷ and quarter-tone¹⁵⁸ models, occur for NI=7, with neatly shaped acute angles around the latter, *i.e.*, with rapidly decreasing values as we move away from the optimal NI.

Whenever unfiltered generations of scale element show optimal values for a reduced ambitus of possible numbers of conceptual intervals to the octave (between 6 and 8 conceptual intervals to the octave), and although it is possible that, to start with, scales other than heptatonic may have been used in traditional modal music, further aesthetic (sizes of intervals and their patterns in conjunct forms) and economical (optimal productivity) considerations have stabilised this optimal value at NI=7, confirming thus the predominant role of heptatonism with this music.

Conclusion of the statistical study of the scale

Although other models and filters can be applied to the process of interval combination¹⁵⁹ or to particular subdivisions of modal music¹⁶⁰ we can draw a simple conclusion from this second part of the article. Heptatonism is, at least partly, the result of an optimisation process within interval structure of the containing intervals of the fourth, the fifth and the octave.

Synthesis

The results of the research shown in Parts I and II tend to prove that traditional choices for containing intervals such as the fourth, the fifth and the octave are not arbitrary decisions but the result of a real need for optimal melodic expression. Within the potentially infinite vertical space of pitches, melodic music seems to have followed a very rational, although intuitive and pragmatic, search for a limitation of combinations for conceptual intervals in order to arrange them as useful paradigms, notwithstanding the unlimited variations of pitches on the boundaries defined by the components of these interval combination paradigms. These variations have been the subject of endless speculations and mathematical expressions in terms, notably, of string and frequency ratios, which contributed in creating confusion between the two processes of (1) discrimination and (2) identification of intervals. The first process is mainly quantitative, whereas the second is purely qualitative. The first process is related to interval tonometry, while the second relates to the comparison of interval qualities within the frame of a scale or a melodic pattern (or formulae). These considerations led us to the formulation of new concepts including the differentiation between conceptual (qualitative) and measuring (quantitative) intervals.

Some small intervals within a combination have qualities that distinguish them, in the concept of melodic music, from others. These become stand-alone entities¹⁶¹ within larger containing intervals which, in turn, have other intrinsic qualities,¹⁶² making them a perfect receptacle for smaller conceptual intervals. With time, these larger intervals became the fourth, the fifth and the octave, because of the particular relevance of these terms in relation to their interval capacity. For these numbers of identified classes of smaller conceptual intervals, within the containing larger intervals, the number of useful paradigms is optimal, which means that the number of paradigms ready for immediate, or delayed, composition is at its maximal potential, although the number of identified smaller conceptual intervals is at the minimal which allows for their identification.

In parallel to the relative wealth of expression, the optimal numbers of conceptual intervals (within the larger containing intervals) carry other qualities, especially their ability to produce, when combining smaller conceptual intervals, unique patterns (combinations)¹⁶³ within the containing intervals. This non-redundancy among the potential musical paradigms increases the efficiency of the means available for melodic music. These characteristics make it possible today to formulate two hypotheses on (1) the process of formation of the heptatonic scale, and (2) on the new conceptual tools that may be used in the search for the possible origin of this scale.

A hypothesis for the formation of the heptatonic scale

The consonance of the fourth, the fifth and the octave seem to be the common denominator for a large variety and types of music in the world, whilst other intervals have historically been considered as dissonant.¹⁶⁴ This position has been supported by arithmetics (Pythagorean tetrad) or acoustics (theory of resonance) considerations. However, acoustic agreement between partial harmonics of different pitches is not the only criterion on which music is based. Although the Pythagorean tetrad is an ingenious means for ratios of the larger consonant¹⁶⁵ intervals, it remains, regardless that small conjunct and fluctuating intervals (dissonant) compose the immense majority of the traditional melodic repertoire related to *maqām* music.

Whereas consonance is not a real issue for these small intervals, the most important criteria, when composing melodies with a reduced span such as with most traditional music of heptatonic expression, are aesthetical adequacy and musical expressionism. Now I simply cannot imagine someone starting a musical repertoire, which would at the end of the way lead to the heptatonic meta-system of scales, with the help of interval leaps of combined fourths, fifths and/or octaves in order to arrange a couple of musical sounds together inside a melody. This

could be compared to travelling from one's own village to the large and far away city, then following another section of the highway in order to go to the village immediately across the valley. It just does not make sense, if there is a road between the two villages. If not, it is much easier to build the road between the two villages to start with, and then try to go to the large city (the octave)¹⁶⁶ or, before we reach it have a break at a pleasant inn in an average sized town on the way.¹⁶⁷ One can also wander off the road, or take shortcuts to the next break. This is the heart of interval fluctuations within a scale. Small discrepancies in comparison to the theoretical path assigned between two pitches, due to the morphology of each performer, the organological particularities of the instrument¹⁶⁸, it can be the voice or any other instrument, regional or cultural differences, etc. The way in which we walk to the medium sized town may be different,¹⁶⁹ and the particular place within the village, our destination, may be a little bit off limits (one might take a break at a different place within the village),¹⁷⁰ but the destination remains the same.

Combining a few conjunct intervals and going up or down the smaller scale, we may want to change direction and decide to play other pitches corresponding to different intervals, but that would still get us to the limits of the first established path between two pitches. The more possibilities we have in order to switch from one path to another, the more pleasant is our trip whenever we need to travel around a specific region, especially whenever we may find another intermediate middle sized town in which to set base for further explorations.¹⁷¹ This is the essence of modulation, or varying the paths by moving from one established pattern of pitches to another.

While improvising new ways, one must avoid perpetual change of intermediate stops; in order not to burden our fellow travellers we guide after all the explorations already undertaken. The guide may achieve the balance between complexity and expressionism, where the pleasure of reminiscence is mixed with the pleasure of perpetual discovery, and thus avoid excessive strain for the listeners.

This is the essence of *maqām* music as I came to understand it it.¹⁷²

On this basis, the formation process of a scale seems to become evident. Starting with a single pitch, neighbouring pitches may have been explored in succession until attaining the fourth or the fifth which, because of acoustic qualities and the need to mark a pause, or start a further stage, became the new turning point of the melody. From there on, our original performer may have chosen to go back to the starting pitch, and even beyond for a few notes and then back to it, then explore the same path, or change it for the sake of varying the original melodic pattern. Therefore, in a reduced span of one containing interval and with occasional overtaking of its boundaries, the performer can have obtained an ensemble of key-patterns of interval sizes, clearly distinguishable for the ear of

his listeners which became, in time, identified qualities of intervals within this first containing interval, the fourth (or the fifth – we explore this possibility in the next section).

Further differentiation of the small identified intervals begins to seem too esoteric for the listeners, as the discrepancies will seem too small (when they are smaller than a third of a tone, or a quarter-tone) to be clearly distinguished, the performer has reached a state of balance between his musical expressionism and the listeners ability to follow his more or less subtle modifications of the melody.¹⁷³

This is the point when spatial extension, in either of both directions must have become indispensable in order to pursue melodic composition (it may be spontaneous or delayed, as stated above). From there we have many possibilities, all of them, considering the original process of composing the fourth, leading to the same result: the heptatonic scale.

From the fourth to the octave

Possible ways of reaching the octave (or avoiding it as some *maqām do*)¹⁷⁴ are:

1) The exploration of the large containing octave interval in a linear manner, that is by testing conjunct intervals in succession or in alternation (in the latter case with intermediate pitches being part of the resulting scale),

2) The addition of smaller containing intervals to one another (for example two fourths and a one-tone interval, or a fourth and a fifth) and use each as an almost independent entity,

3) The expansion of a relatively small containing interval (a fourth or a fifth) by searching for successive or alternate notes inside the a) upper (or lower) fifth or fourth, the boundaries of either of the latter being the new starting point for this exploration, b) choosing any intermediate pitch in the original fourth or fifth (or any other initial configuration of conjunct intervals) and applying any of the three processes explained above, c) combining any of the above. With all these processes, the containing intervals must not be considered as imposing strictly delimited boundaries for the scale, but as indicating sizes of intervals justified by their acoustic characteristics. In other words, the three consonant containing intervals within the octave do not bind the performer (and the music), but guide him in the creative process of music composition.¹⁷⁵

Whatever process the performer chooses he will reach the same conclusion. The optimal repartition for intervals within the boundaries of the three containing intervals is three to a fourth, four to a fifth, and seven to an octave. The performer may decide to avoid the aesthetics implied in the process,¹⁷⁶ but optimal expressivity is reached with these numbers and remains an unavoidable conclusion.

On both sides of the octave

Now that we have reached the big city and that the intermediate stops are already explored, now that one has even determined alternative routes avoiding the heart of the city or the passage into smaller, intermediate towns, the performer may decide to conclude his composition or he may wish to undertake further explorations of the space beyond the boundaries. He could decide for example to jump from one pitch to another one a fourth or a fifth apart, and then come back, or go further, in order to explore the intermediate, or upper or lower, pitches until he and his listeners are satisfied with the new voyage where he guided them.

Eventually, with the increasing number of musicians in one location, performers came together to play alternative forms, each of them exploring parallel or separate ways of getting from one point to another of the containing interval, each of them with his own morphology, instrument(s), artistic taste, and origins. Each of them would listen to other musicians' performance and support or be inspired by it, or would be supported by those and inspire them himself. This process may have induced what we call today heterophony, in the large sense of the word, it may well be that, whenever this liberty of exploration vanished and became bounded by more or less strict patterns of progression of simultaneous musical parts, or whenever the octave (or largest) containing interval became prominent in a particular musical culture,¹⁷⁷ another form of music came to light, the one which is today called polyphony.

Clues about the possible origin of the heptatonic scale

If culture differs from one civilisation to another, some characteristics are common almost to all. Heptatonic scales, in the historical realm of modal music, are one of these common denominators. It seems that the number of seven conceptual intervals to the octave is the result of musically shared aesthetical criteria over a large region and for a long historical period. These criteria, which may probably be further enriched, are: 1) the consistency of bi-interval combinations (the use of middle-sized conceptual intervals) within a scale, *i.e.*: avoiding successions a) of very small (like the semi-tones) or large (like the one-tone-and-a-half) elements (intervals), b) of conjunctions of very small elements (like the semi-tone and the three-quarter-tone intervals), and c) of large elements within the fourth (like adjacent 'neutral' augmented seconds or more, or alternating tones and bigger intervals in conjunction, etc., 2) the use of an optimal step, also a smallest scale interval, for interval differentiation and identification, 3) the use of a limited alphabet of intervals of the second 4) the acoustic guidance of the main three large containing intervals (the fourth, the fifth, and the octave).

Other numbers of conceptual intervals may have been used for the octave, for example when these criteria did not apply very strictly, or when the need for particular combinations arose (for example on aesthetic or social grounds).

Whenever a specific culture decided to choose a lesser number of intervals in a scale, aesthetic criteria may have varied. In pentatonic music, for example, such a limitation as the three-semitone interval being the greatest conjunct interval in the scale, may have been set at a higher value. This makes it more difficult to create smaller containing intervals, especially the fourth, but leaves the larger containing intervals (like the fifth and the octave) play as acoustic guides for the performer (about) unchanged.

Choosing a number of intervals larger than seven, further possibilities appear. However, they are simple extensions of the optimised octave scales (containing seven conjunct conceptual intervals), or possible loop lines around some of the aesthetic criteria listed above (for example the inclusion of conjunct semitones).¹⁷⁸

If a culture decides that the acoustic characteristics of containing intervals are the leading criterion, the choice of the fourth may have led to the use of the intermediate neutral intervals composing it, in order to maximise its possibilities, whenever the choice of the fifth maximised the use of semitones, which favoured in its turn the appearance of tense diatonism (based on successions of tones and semitones).

The choice of the octave as the main acoustic criterion may, on the other side, have precipitated a process of equivalence between intervals with a difference of an octave (for example between a fourth and an octave-and-a-fourth), and the use of parallel lines in polyphonic music.

All of these criteria have different powers according to the culture in which they appear. The balance between them has led to different subdivisions of one main form of music, called heptatonic modality.

Later on, and in order to arrange musical systems of intervals within a coherent music theory, different civilisations have sometimes chosen different formulations, some to keep a firm connection with music performance, and some others based on a mathematical, seemingly more elegant basis, having some connections with musical practice or acoustic characteristics of musical intervals. With time theory became an entity of its own and was developed by scholars for the sake of the beauty of mathematical constructions which were confused by their promoters, and later by their followers, being a generative theory, and whenever any musical theory should first rely on practice. The mathematical expression of intervals through string ratios or through other, very small, quantifying intervals gave theoreticians the illusion that intervals do have exact sizes in performance, even if modal practice refutes this assertion. The map became the territory, whenever it should have

been, at most, a conventional sketch of the territory, or a more or less precise guide within the infinite possibilities of pitches within a containing interval. In order to remain a guide, and not become a rigid yoke to musical expressivity, theoretical expressions of scales should, first of all, differentiate between quantitative and qualitative intervals, and between conceptual, quantifying and elementary intervals, in order to stay, where possible, close to music performance and far from interval quantization.

As an overall conclusion to this study, this research gives a new, plausible explanation for heptatonism as a privileged receptacle for modal scales.

Some criteria underlined in the article, like the homogeneity rule, the insistence on the fourth or fifth, or any other indication of a calibration process of the scale, may give complementary information in the search for its origin.

TABLES

- 1 = 1 semi-tone
- 1 + 1 = 2 semi-tones, or one tone
- 1 + 1 + 1 = 3 semi-tones, or one-and-a-half-tones
- 1 + 1 + 1 + 1 = 4 semi-tones or a ditone
- 1 + 1 + 1 + 1 + 1 = 5 semi-tones or the approximative fourth

Table 1. Interval alphabet in an approximative fourth (in semi-tones).

- 1 1 3 (semi-tone, semi-tone, one-and-a-half-tones) - tonic chromatic of Aristoxenus
- 1 3 1 (semi-tone, one-and-a-half-tones, semi-tone)
- 3 1 1 (one-and-a-half-tones, semi-tone, semi-tone)
- 1 2 2 (semi-tone, tone, tone) - 'sharp diatonic' of Aristoxenus
- 2 1 2 (tone, semi-tone, tone)
- 2 2 1 (tone, tone, semi-tone)

Table 2. Species of *genera* made from multiples of the semi-tone.

Interval size: → Genera: ↓	1	2	3	Vectors
	Number of intervals of this size contained in the <i>genus</i>			
1 1 3	2	0	1	(2,0,1)
1 3 1	2	0	1	
3 1 1	2	0	1	
1 2 2	1	2	0	(1,2,0)
2 1 2	1	2	0	
2 2 1	1	2	0	

Table 3. Capacity vectors for *genera* with semi-tones.

From scalar systems to integer numbers			Capacity vector	Hyper-system
Sub-systems	concatenated number	Read		
1 1 3	1 1 3	one hundred and thirteen	(2,0,1)	113
1 3 1	1 3 1	one hundred and thirty one		
3 1 1	3 1 1	three hundred and eleven		

Table 4. Scale systems 1 1 3, 1 3 1 and 3 1 1 as integer numbers. Deriving the hyper-system.

Hyper-systems with internal composition in quartertone multiples – alphabet is composed of intervals 2, 3, 4, 5 and 6						
H.	Value	NS	NSS	NSS4	FF	Remarks
1	2 2 2 2 4 6 6	15	105	58	21	<i>Hijāz-Kār</i> , “gipsy” or “chromatic” scales
2	2 2 2 2 5 5 6	15	105	23	6	<i>Very rare: existence not confirmed as (5 2 2 6 2 5 2)</i>
3	2 2 2 3 3 6 6	30	210	54	0	<i>Rare: existence confirmed as (3 3 2 6 2 6 2)</i>
4	2 2 2 3 4 5 6	120	840	208	60	Variants of N°1
5	2 2 2 3 5 5 5	20	140	56	12	<i>Existence not confirmed to this day</i>
6	2 2 2 4 4 4 6	20	140	80	46	Frequent: generates <i>Hijāz</i> (2 6 2 4 2 4 4)
7	2 2 2 4 4 5 5	30	210	40	14	<i>Existence not confirmed to this day</i>
8	2 2 3 3 3 5 6	60	420	120	0	<i>Existence not confirmed to this day</i>
9	2 2 3 3 4 4 6	90	630	168	54	“Regular” <i>Şabā</i> (3 3 2 6 2 4 4) and other modes
10	2 2 3 3 4 5 5	90	630	210	54	Variants of N°1
11	2 2 3 4 4 4 5	60	420	96	36	Variants of N°6
12	2 2 4 4 4 4 4	3	21	12	9	<u>Semi-tonal “diatonism” and derivatives</u>
13	2 3 3 3 3 4 6	30	210	46	0	<i>Existence not confirmed to this day</i>
14	2 3 3 3 3 5 5	15	105	29	0	<i>Existence not confirmed to this day</i>
15	2 3 3 3 4 4 5	60	420	120	36	Scales are close to those of N°6
16	2 3 3 4 4 4 4	15	105	30	18	Frequent: Arabian <i>Bayāt</i> and other modes
17	3 3 3 3 3 3 6	1	7	0	0	<i>Existence not confirmed to this day</i>
18	3 3 3 3 3 4 5	6	42	12	0	<i>Existence not confirmed to this day</i>
19	3 3 3 3 4 4 4	5	35	18	9	Frequent: Arabian <i>Rāst</i> and other modes
<u>Total</u>	<u>19</u>	<u>685</u>	<u>4795</u>	<u>1380</u>	<u>385</u>	<ul style="list-style-type: none"> H: Hyper-system (numbers in this column correspond to the rank of the hyper-system) NS: Number of Systems in current H NSS: Number of Sub-Systems in current H NSS4: NSS in just fourth from tonic FF: NSS with a just Fourth in a just Fifth from tonic

Table 5. 19 hyper-systems generated within the quarter-tone model with the limited alphabet of intervals with 2, 3, 4, 5, and 6 quarter-tones. Columns to the right of the Value column express numbers of systems or sub-systems with or without conditions (fourth or fifth), rows with colour (or grey) background underline hyper-systems which generate most of the scales described in specialised literature.

<i>Urmawī</i>	Transition	Quartertones	Transition	<i>Urmawī</i>
1	+ 1	2	- 1	1
2	+ 1	3	- 1	2
3	+ 1	4	- 1	3
4	+ 1	5	- 1	4
5	+ 1	6	- 1	5

Table 6. Transition from *Urmawī*'s conceptual intervallic representation to the quarter-tone model, and reciprocally.

FIGURES

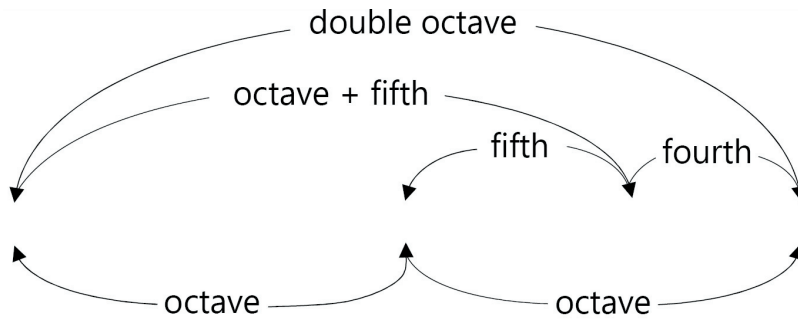


Fig. 1. Intervals resulting from the ratios between the four first integers, or tetrad. The ratios 1:2 and 2:4 give the octave; the ratio 1:4, the double octave; 1:3 the octave + the fifth; 2:3, the fifth and 3:4, the fourth. These intervals were the principle consonant intervals in Pythagorean and Aristoxenian theories. In order of their consonant quality, first comes the octave, then the fifth and lastly, the fourth.

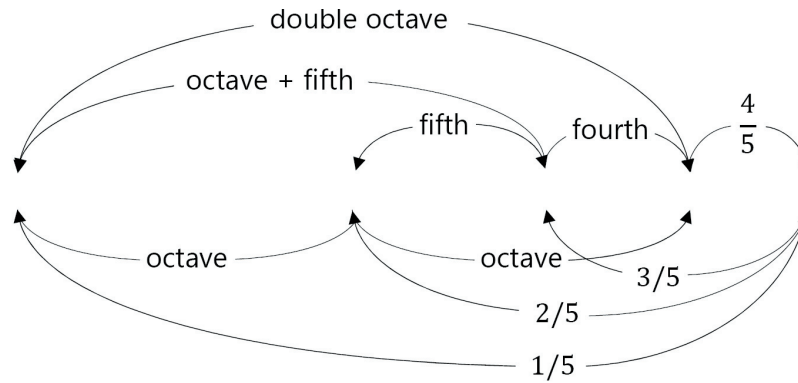


Fig. 2. The original tetrad extended to five numbers. Additional ratios include all relationships of the original four numbers with 5 added.

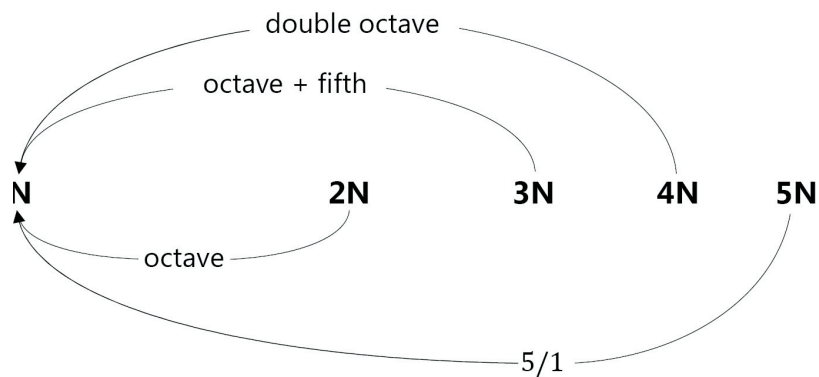


Fig. 3. In the procedure of generating notes with acoustic resonance, the pitches which result have frequencies corresponding to multiples of the fundamental to which they are uniquely related. The ratios are 2/1, 3/1, 4/1, 5/1, etc.

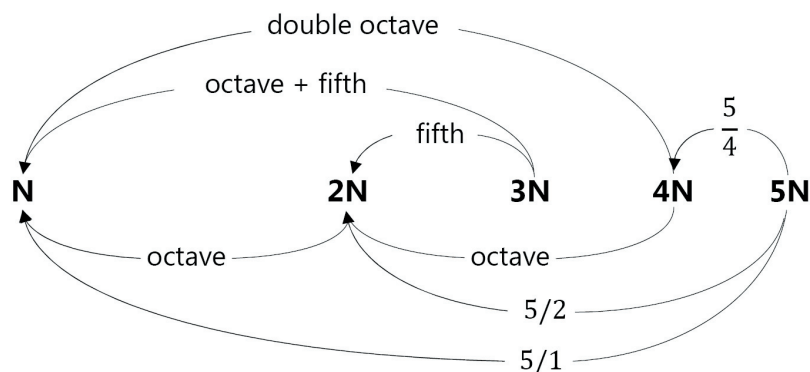


Fig. 4. Ratios for the first five harmonics with integration of successive octaves taken as equivalent pitches: the new resulting intervals include those with ratios 3/2, and 5/4 (plus 5/1 and 5/2), i.e. the fifth, and the harmonic third of the just intonation theory (and others).

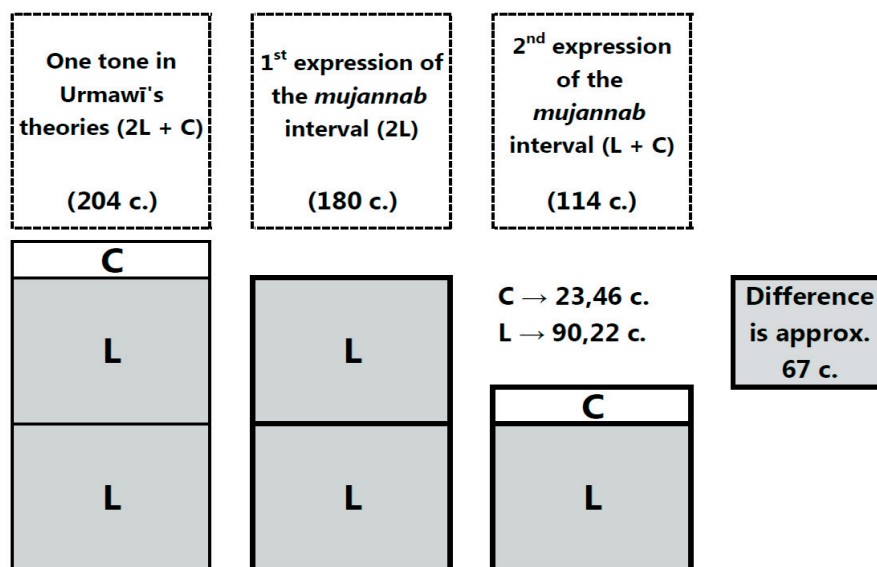


Fig. 5. Urmawī's tone and mujannab.

↓ <i>Rāst</i> 's ascending scale (one octave) ↓					Cumulative quarters	Elementary quarters	d'	Conceptual intervals' capacity	Conceptual intervals	
Conceptual intervals	Conceptual intervals' capacity	c'	Elementary quarters	Cumulative quarters	24	4	d'	4 quarter-tones	Tone	
				23	3					
				22	2					
				21	1					
<i>Mujannab</i>	3 quarter-tones	b'	3	24	20	4	b'	4 quarter-tones	Tone	
			2	23	19	3				
			1	22	18	2				
<i>Mujannab</i>	3 quarter-tones	a	3	21	17	1				a
			2	20	16	2				
			1	19	15	1				
Tone	4 quarter-tones	g	4	18	14	4	g	4 quarter-tones	Tone	
			3	17	13	3				
			2	16	12	2				
			1	15	11	1				
Tone	4 quarter-tones	f	4	14	10	4	f	4 quarter-tones	Tone	
			3	13	9	3				
			2	12	8	2				
			1	11	7	1				
<i>Mujannab</i>	3 quarter-tones	e'	3	10	6	3	e'	3 quarter-tones	<i>Mujannab</i>	
			2	9	5	2				
			1	8	4	1				
<i>Mujannab</i>	3 quarter-tones	d	3	7	3	3				d
			2	6	2	2				
			1	5	1	1				
Tone	4 quarter-tones	c	4	4	↑ <i>Bāyāt</i> 's ascending scale (one octave) ↑					
			3	3						
			2	2						
			1	1						

Fig.6. *Maqām Rāst* and *Bāyāt* scales in the modern quarter-tone theory.

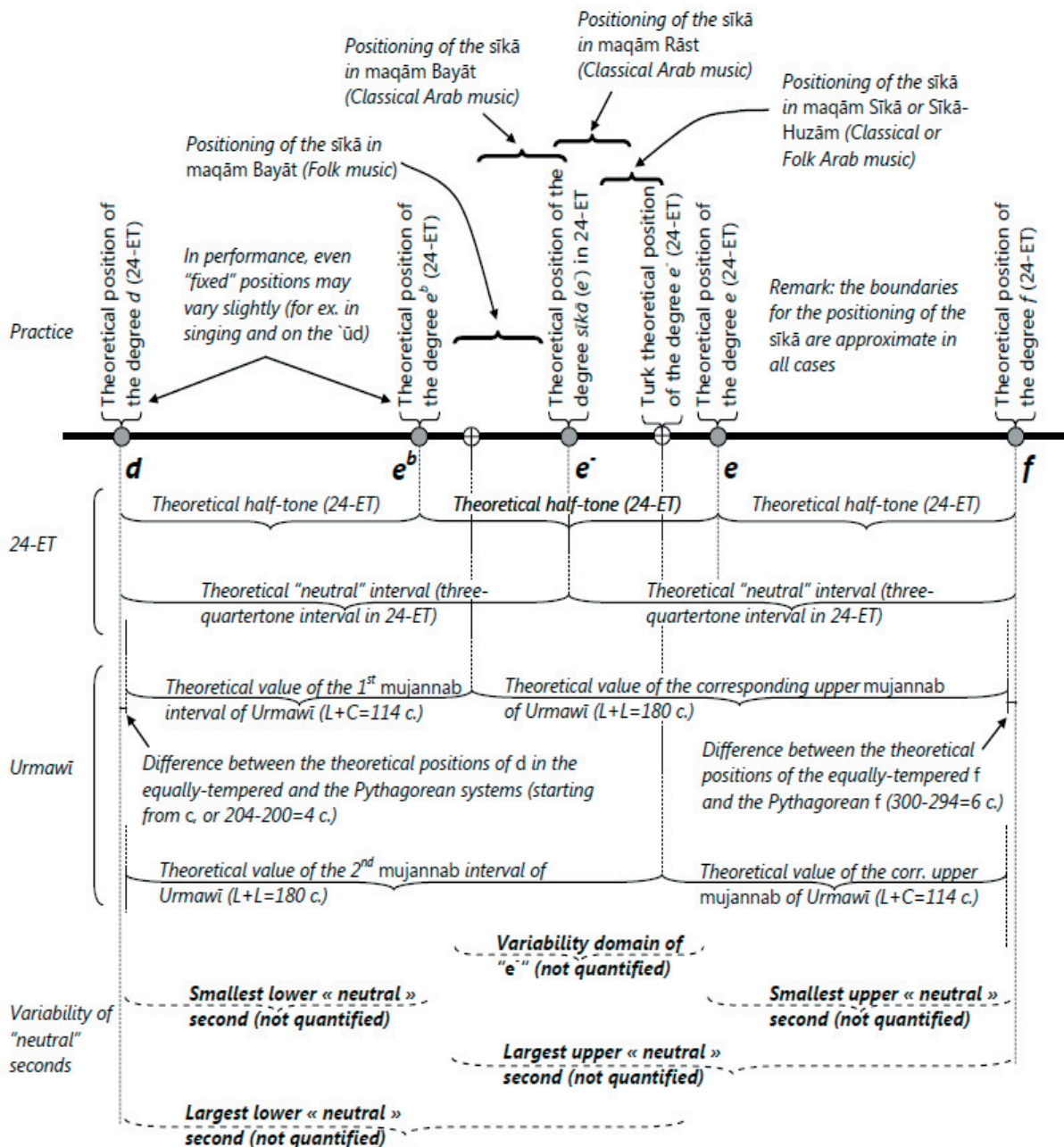


Fig. 7. Repertoire or regional variations of *sikā* and of the 'neutral' seconds.

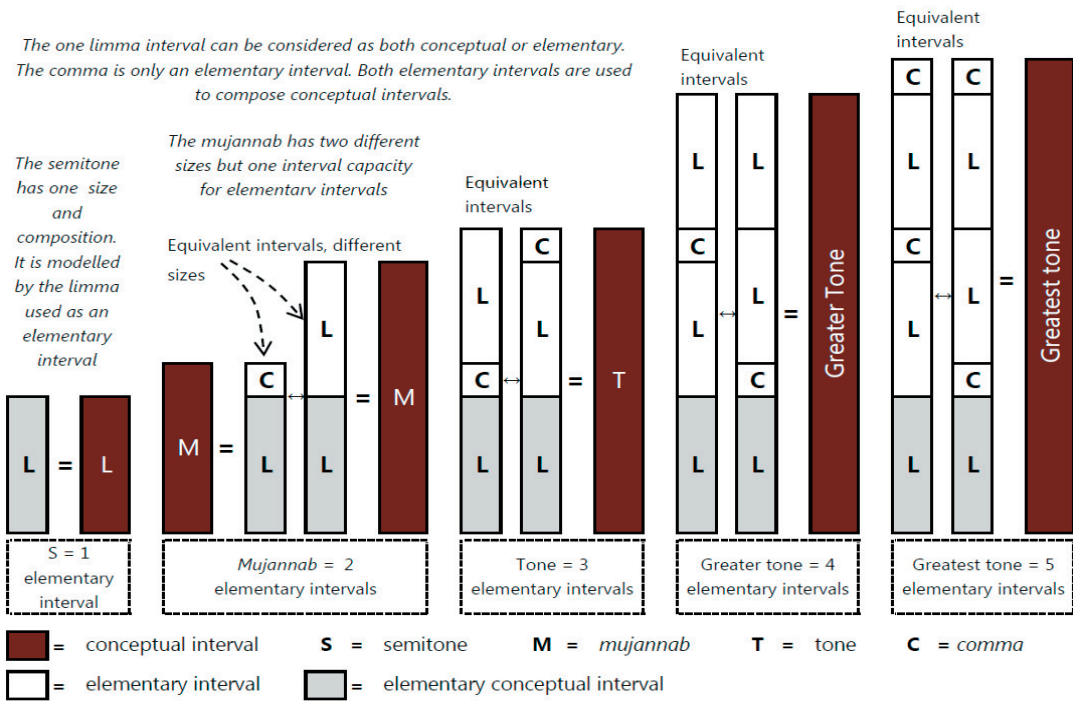


Fig. 8. Obtaining the 5 qualities of seconds in *Urmawī*'s theory: the semi-tone is the smallest conceptual interval, and is modelled with a *limma*. Other intervals within the fourth are modelled from a first *limma*, augmented with a combination of *commata* and *limmata*, bearing in mind that no more than two *limmata* in a row, and no successive *commata*, may be used. The *mujannab* has two possible sizes, but contains in both cases two elementary intervals. All intervals larger than the semi-tone have two different possibilities for combinations of elementary intervals.

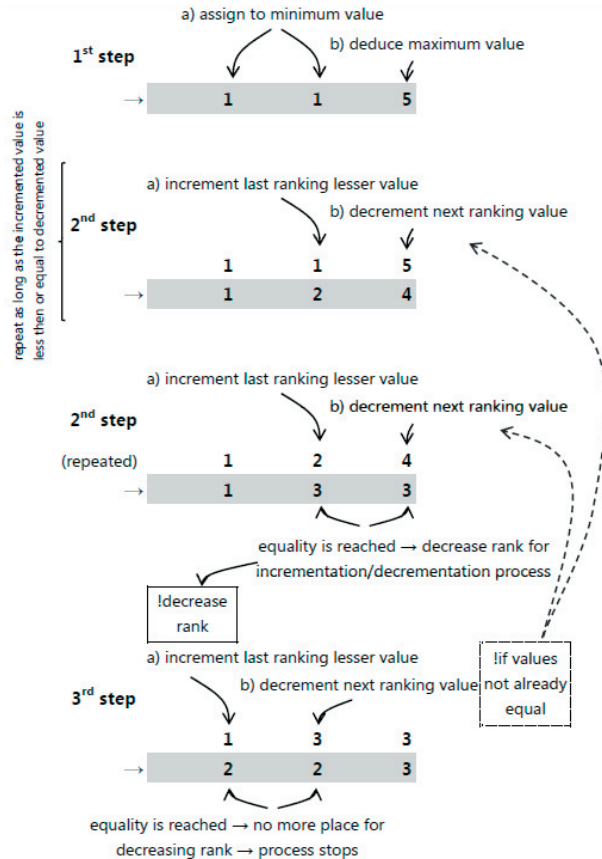


Fig. 9. Algorithm for hyper-systems.

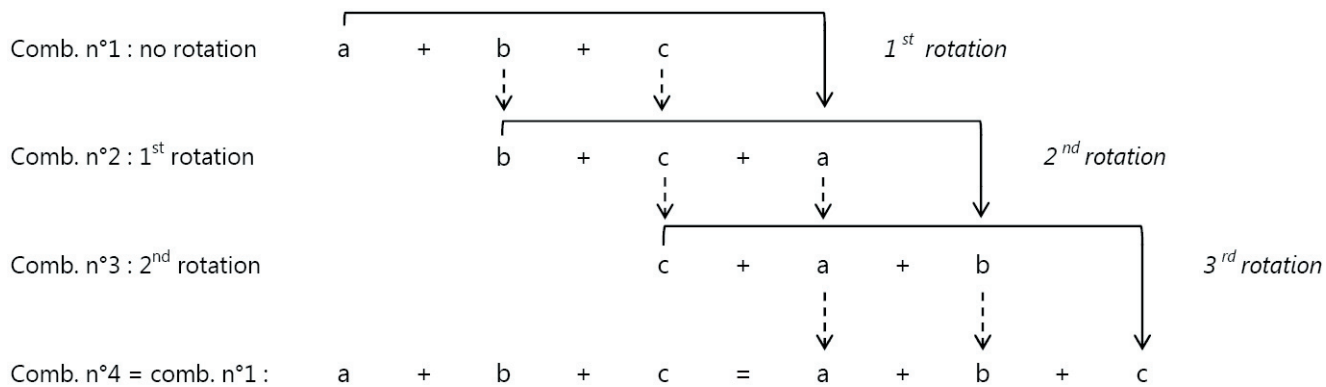


Fig. 10. Rotation of three distinct intervals a, b and c with the three resulting combinations.

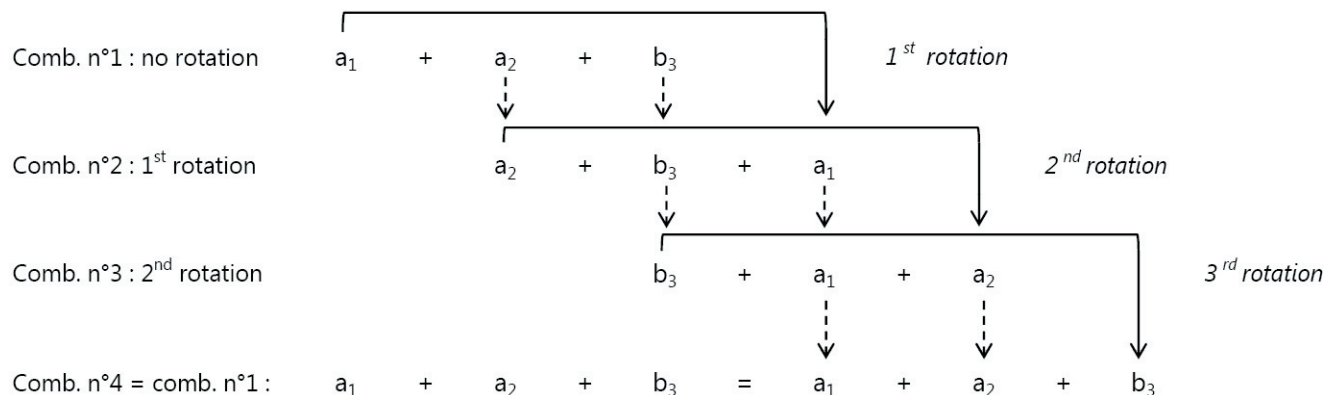


Fig. 11. Rotation of three intervals out of which two (the 'a' intervals) can be considered as equivalent (the subscript numbers identify the initial rank of each interval in the original – basic – combination): the outcome is still three distinct combinations.

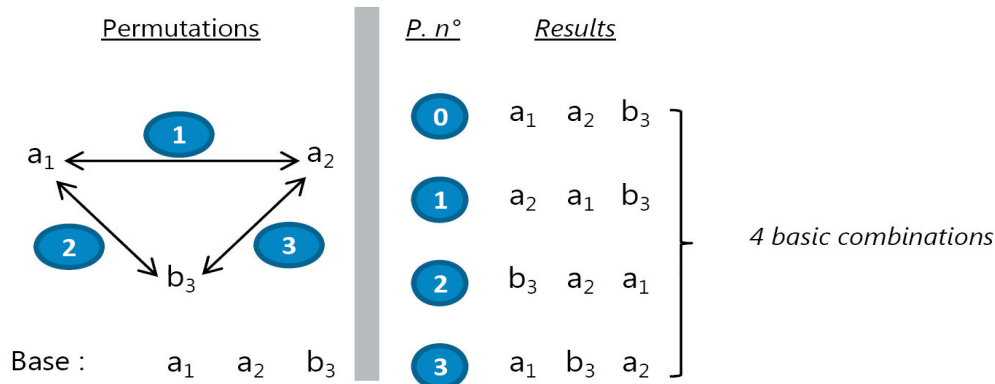


Fig. 12. Permutation of three intervals (the two “a” intervals may be considered as equal – if not, the subscript numbers, which identify the initial ranks of each interval in the original basic configuration, differentiate them): this procedure results in 4 different combinations, out of which one is redundant if the two intervals “a” be considered as equivalent.

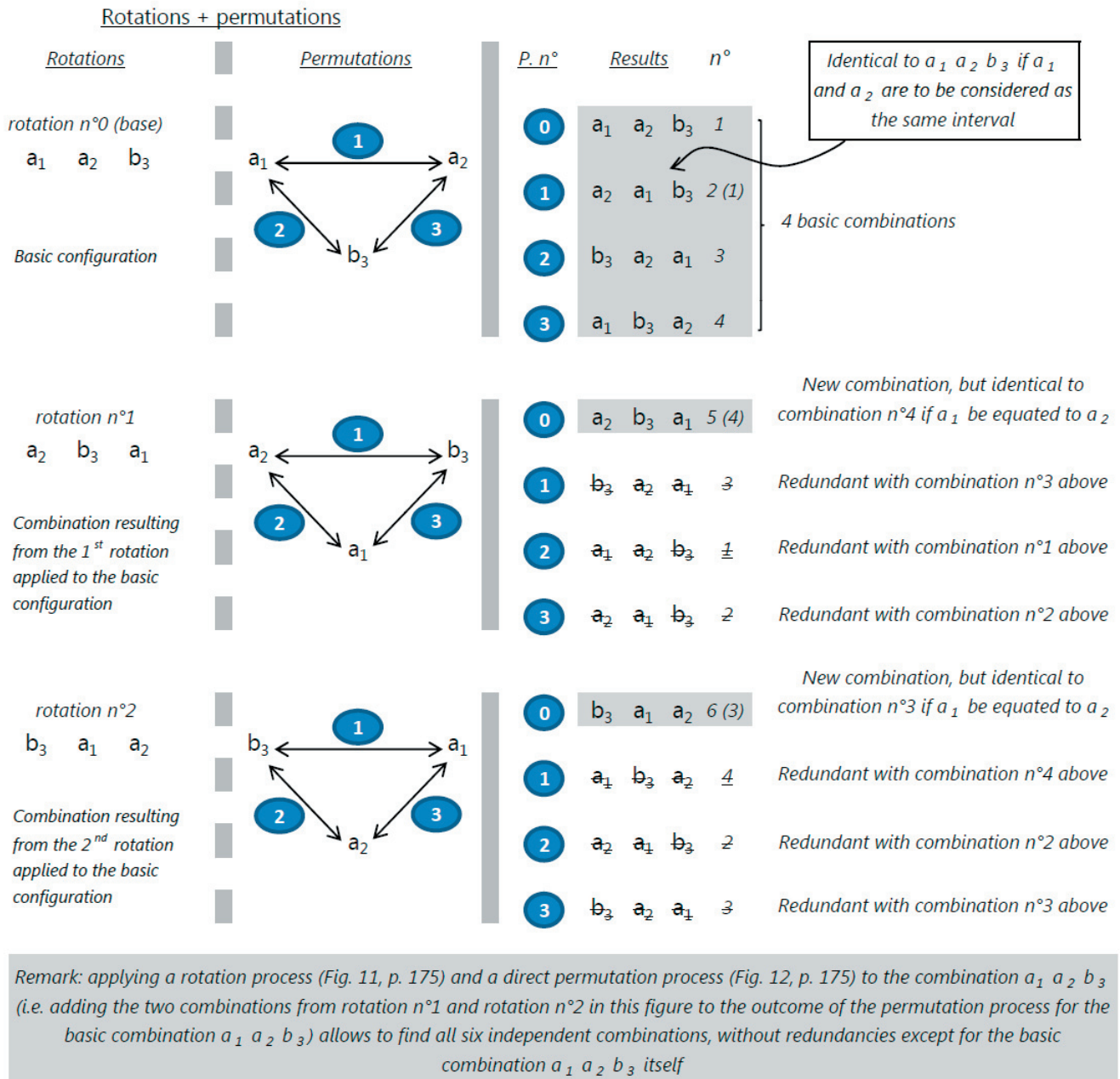


Fig. 13. Combining rotation and permutation for three intervals (the two 'a' are equal – if not, the numbers in subscript, which identify the initial ranks of each interval in the original basic configuration will differentiate them). The outcome here is 6 distinct combinations, but only 3 if 'a1' and 'a2' are identical.

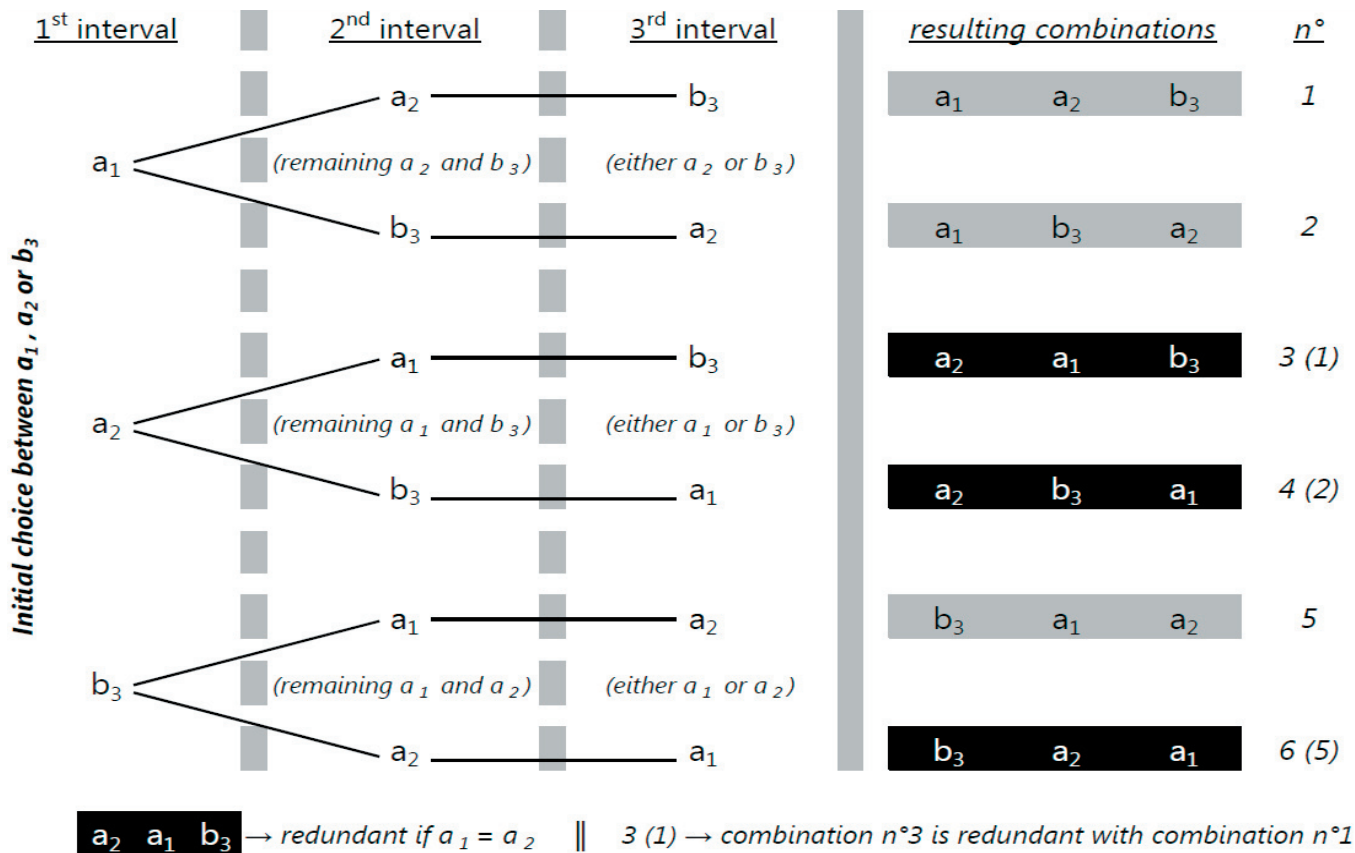


Fig. 14. Tree processing for three intervals (see previous figure). The outcome is 6 distinct combinations as in the rotation/permutation procedure, but the result is straight forward; however, if 'a1' and 'a2' be considered as identical, there would remain only three distinct combinations out of six possibilities.

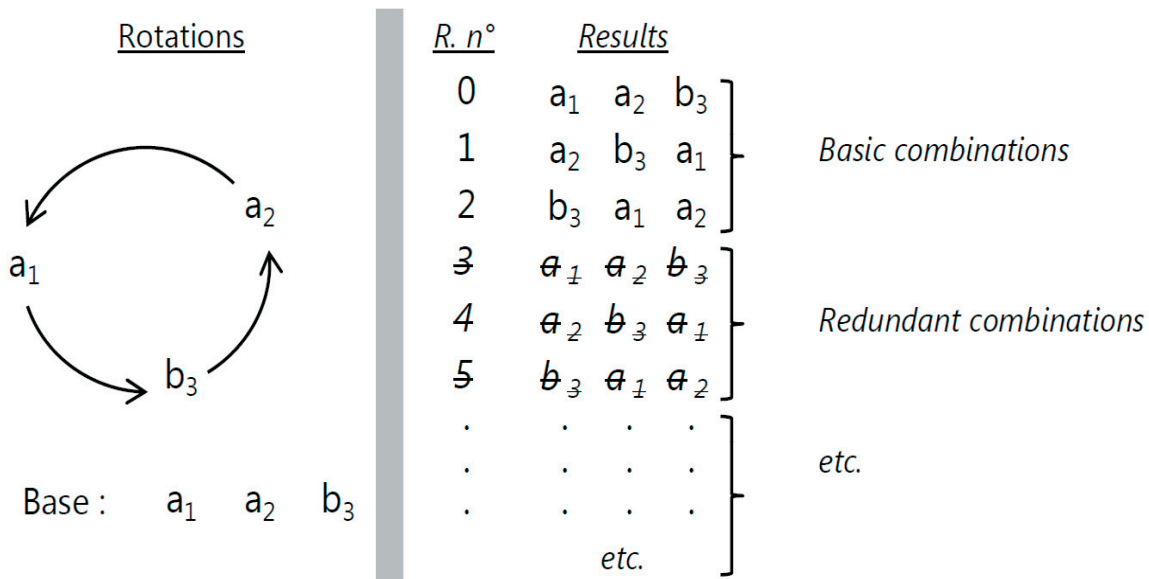


Fig. 15. Endless rotations of intervals as a particular case of the de-ranking procedure: by picking three (N) conjunct intervals, out of three ('M=N') endlessly repeated intervals, beginning with the first, then the second, etc., we end up applying a rotational procedure with, as a result, an endless series of redundant combinations.

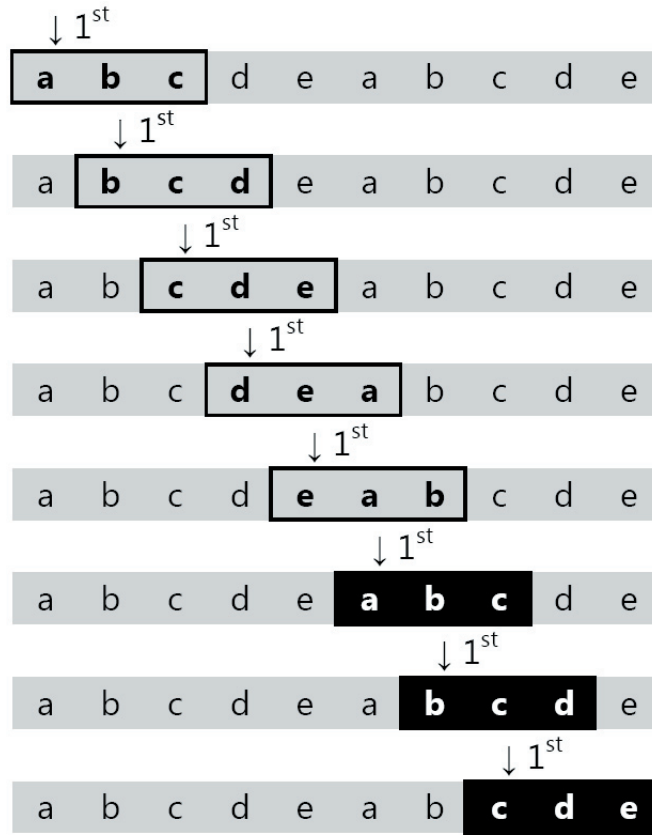


Fig. 16. De-ranking procedure applied to three successive intervals picked out from a double row of five intervals. There are five distinct combinations out of eight, the last three being redundant with the first three.

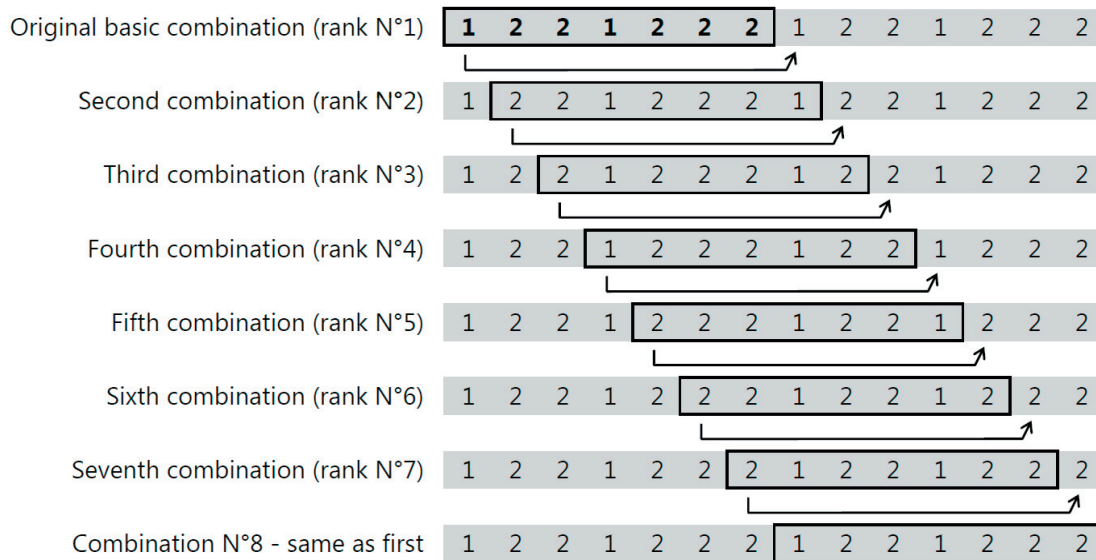


Fig. 17. De-ranking procedure applied to two identically composed octaves in a row. Seven species (or sub-systems in the theory of modal systematics) may be extracted through the procedure.

<i>(Hyper-system is</i> 1 1 2 2 2 2 2)	Rank of the sub-system	Beginning note (West.)	Intervals in multiples of the semi-tone
(Basic system has rank N°1)	1.	<i>B</i>	1 2 2 1 2 2 2
The seven sub-systems (including the basic system) extracted (by a de-ranking process) from basic system 1221222	2.	<i>c</i>	2 2 1 2 2 2 1
	3.	<i>d</i>	2 1 2 2 2 1 2
	4.	<i>e</i>	1 2 2 2 1 2 2
	5.	<i>f</i>	2 2 2 1 2 2 1
	6.	<i>g</i>	2 2 1 2 2 1 2
	7.	<i>a</i>	2 1 2 2 1 2 2

Fig. 18. With the de-ranking procedure applied to the western diatonic scale, the sub-system having the smallest figure as a whole number (as an integer concatenated form), is sub-system 1 2 2 1 2 2 2 (in concatenated form 1221222, or ‘one million two hundred and twenty one thousands and two hundred twenty-two’). All other sub-systems have a corresponding integer value that, if their intervals be concatenated to form an integer number, is larger than the former. Consequently, in modal systematics, the combination 1 2 2 1 2 2 2 holds the head rank among these 7 sub-systems and is considered as being the basic system from which the six others are deduced by the de-ranking procedure (the basic system is, besides being the head or base system, the first sub-system in the group of seven). The capacity indicator of these sub-systems is hypersystem 1 1 2 2 2 2 2 (two one-semi-tone and five one-tone intervals).

Hyper-system: 1 1 2 2 2 2 2		Rank of the sub-system	Beginning note (West.)	Intervals in multiples of the semi-tone
System N°1: 1 1 2 2 2 2 2	1.	<i>B</i>	1 1 2 2 2 2 2	
	2.		1 2 2 2 2 2 1	
	3.		2 2 2 2 2 1 1	
	4.		2 2 2 2 1 1 2	
	5.		2 2 2 1 1 2 2	
	6.		2 2 1 1 2 2 2	
	7.		2 1 1 2 2 2 2	
System N°2: 1 2 1 2 2 2 2	1.		1 2 1 2 2 2 2	
	2.		2 1 2 2 2 2 1	
	3.		1 2 2 2 2 1 2	
	4.		2 2 2 2 1 2 1	
	5.		2 2 2 1 2 1 2	
	6.		2 2 1 2 1 2 2	
	7.		2 1 2 1 2 2 2	
System N°3: 1 2 2 1 2 2 2	1.	<i>B</i>	1 2 2 1 2 2 2	
	2.	<i>c</i>	2 2 1 2 2 2 1	
	3.	<i>d</i>	2 1 2 2 2 1 2	
	4.	<i>e</i>	1 2 2 2 1 2 2	
	5.	<i>f</i>	2 2 2 1 2 2 1	
	6.	<i>g</i>	2 2 1 2 2 1 2	
	7.	<i>a</i>	2 1 2 2 1 2 2	

Fig. 19. Complete listing of the systems and sub-systems related to hyper-system 1 1 2 2 2 2 2 (in multiples of the half-tone): three systems are generated, one of which applies to the western regular scale (semi-tonal diatonic). The other two scale systems (or seven sub-systems for each system) are rarely used but are found in the specialised literature and used in contemporary music (see Beyhom, A., *systématique modale*, Vol. 3: 48-50 for more details).

		Rank of the sub-system	Intervals in qualitative numbers	Equivalents in quartertones
Hyper-system N°1: 1 1 5	System N°1: 1 1 5	1.	1 1 5 ?	Extended genera
		2.	1 5 1 exists	
		3.	5 1 1 ?	
	System N°1: 1 2 4	1.	1 2 4 ?	
		2.	2 4 1 exists	
		3.	4 1 2	
Hyper-system N°2: 1 2 4	System N°2: 1 4 2	1.	1 4 2 exists	
		2.	4 2 1 ? (rare)	
		3.	2 1 4 ?	
Hyper-system N°3: 1 3 3	System N°1: 1 3 3	1.	1 3 3 exists	Urmawī's genera
		2.	3 3 1 exists	
		3.	3 1 3 exists	
Hyper-system N°4: 2 2 3	System N°1: 2 2 3	1.	2 2 3 exists	
		2.	2 3 2 exists	
		3.	3 2 2 exists	

Fig. 20. *Urmawī's genera* within the fourth (the two hyper-systems below) and additional potential *genera*, in both conceptual (qualitative) interval modelling (left) and quarter-tone approximation model (right). The *genera* of *Urmawī* represent the full potential of the related hyper-systems; additional *genera* (and hyper-systems) exist only partly in literature (and practice) of traditional Arabian music.

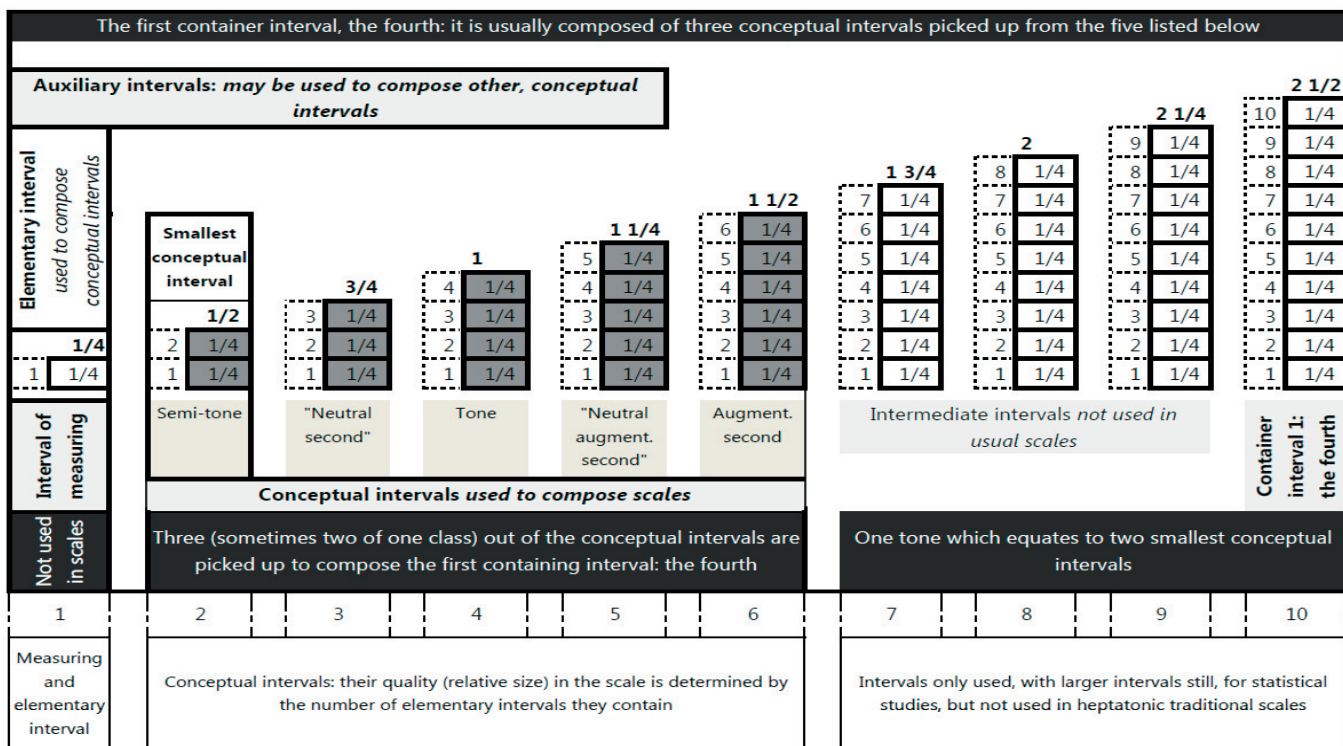


Fig. 21. Measurement, elementary, conceptual and containing intervals in the quarter-tone model. This figure introduces the concept of auxiliary intervals, i.e., smaller conceptual intervals which if combined with elementary intervals may be thought of as composing larger conceptual intervals, such as the neutral augmented second, which has five quarter-tones and which can be conceived as made up of a one-tone interval plus one elementary interval, that is a quarter-tone.

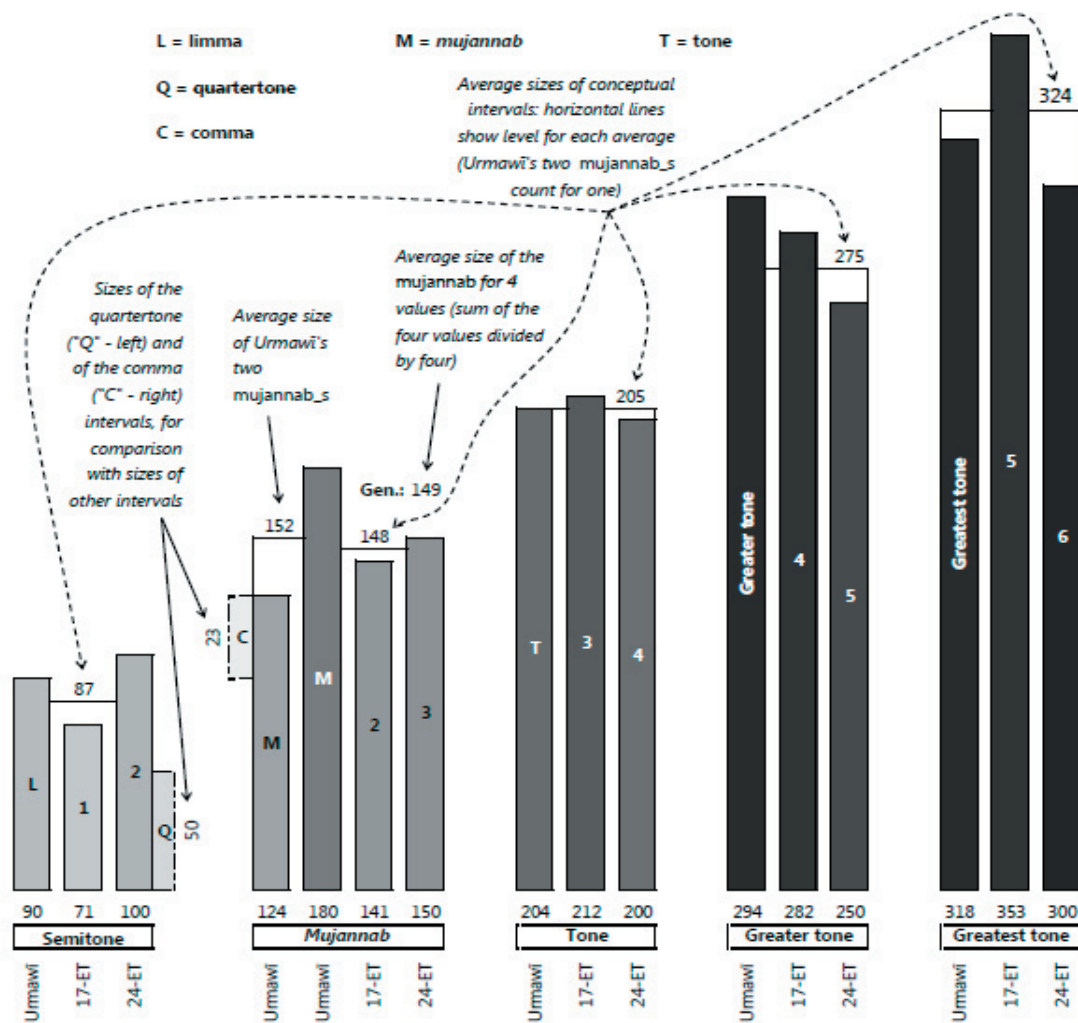


Fig. 22. Conceptual intervals from Pythagoras and *Urmawi's* 17-ET and 24-ET models. Averages show that the transition from one conceptual interval to another, respectively 61, 57, 70, 49, with the average value of 59, can be modelled either by the one-seventeenth of an octave, 71 cents, or the quarter-tone interval of 50 cents. The usage of *comma* and *limma* in *Urmawi's* model accentuate the unevenness with neutral intervals, the *mujannab* and the greater tone – the neutral augmented second.

Semitone model												
Rank of HS	NI = 2			NI = 3			NI = 4			NI = 5		
	HS	S	SS	HS	S	SS	HS	S	SS	HS	S	SS
1	14	14	14 ^{>} , 41 ^{>}	113	113	113*, 131, 311*	1112	1112	1112*, 1121*, 1211*, 2111*	11111	11111	11111*
(2)												11111*
2 (3)	23	23	23 [§] , 32 [§]									11111*
2 (4)				122	122	122, 221, 212						11111*
(5)												11111*
Total	2	2	4	2	2	6	1	1	4	1	1	5
Total non redundant (-)			(-0) 4			(-0) 6			(-0) 4			(-4) 1
Double semitone criterion (*)			(-0) 4			(-2) 4			(-4) 0			(-5) 0
Conjunct big intervals (§)			(-2) 2			(0) 6			(-0) 4			(-0) 5
Intersecting criteria 1 (OR)			(-2) 2			(-2) 4			(-4) 0			(-5) 0
Large intervals			(-2)			(-0)			(-0)			(-0)
Small conjunct different intervals			(-0)			(-0)			(-0)			(-0)
Intersecting criteria 2			(-4) 0			(-0) 4			(-4) 0			(-5) 0

Quartertone model												
Rank of HS	NI = 2			NI = 3			NI = 4			NI = 5		
	HS	S	SS	HS	S	SS	HS	S	SS	HS	S	SS
1	28	28	28 ^{>} , 82 ^{>}	226	226	226*, 262, 622*	2224	2224	2224*, 2242*, 2422*, 4222*	22222	22222	22222*
2	37	37	37 ^{>} , 73 ^{>}	235	235	235, 352, 523	2233	2233	2233*, 2332, 3322*, 3223*			22222*
(2) 3	46	46	46 [§] , 64 [§]		253	253, 532, 325		2323	2323, 3232, 2323 , 3232			22222*
4	55	55	55 [§] , 55 [§]	244	244	244, 442, 424						22222*
(4) 5				334	334	334, 343, 433						22222*
Total	4	4	8	4	5	15	2	3	12	1	1	5
Total non redundant (-)			(-1) 7			(-0) 15			(-2) 10			(-4) 1
Double semitone criterion (*)			(-0) 8			(-2) 13			(-7) 5			(-5) 0
Conjunct big intervals (§)			(-4) 4			(0) 15			(-0) 12			(-0) 5
Intersecting criteria 1 (OR)			(-4) 4			(-2) 13			(-9) 3			(-5) 0
Large intervals			(-4)			(-0)			(-0)			(-0)
Small conjunct different intervals			(-0)			(-4)			(-8)			(-0)
Intersecting criteria 2			(-8) 0			(-6) 9			(-12) 0			(-5) 0

Fig. 23. Combinations and filters in the frame of a fourth containing interval: HS = Hyper-system, S = system(s), SS = sub-systems.

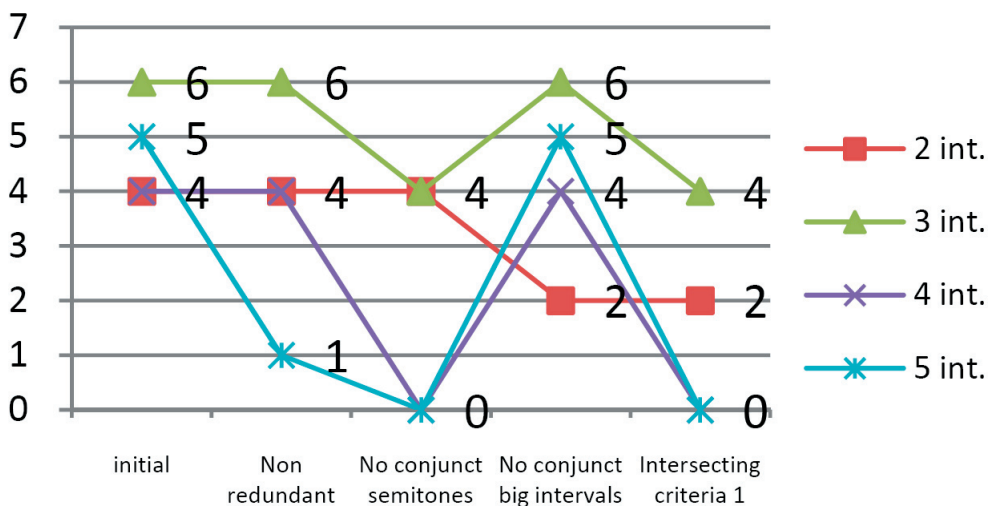
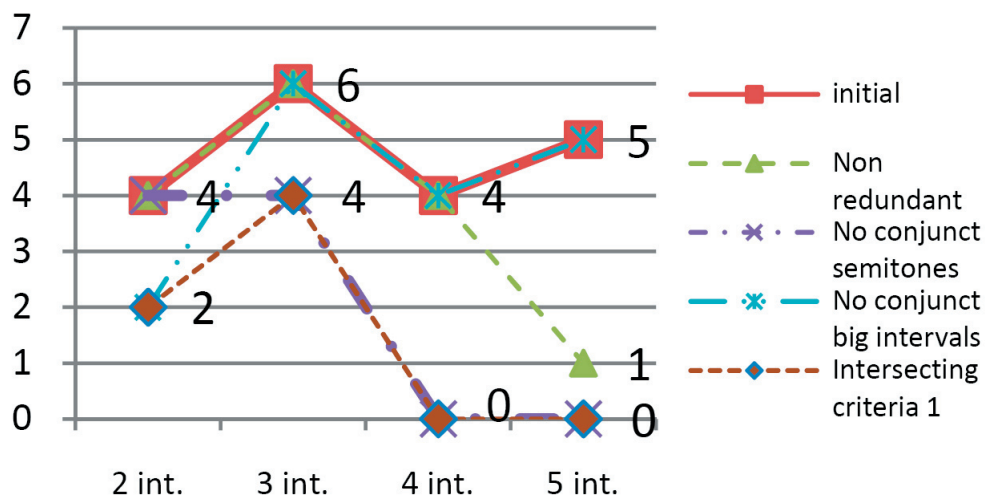


Fig. 24. Graphs of the distribution of sub-systems in a fourth, in relation with the number of intervals (conceptual conjunct intervals of second) in the scale element (above), and in relation with applied filters (below – cross-reference) for each case (NI = 2, 3, 4 and 5) – semi-tone model (results for filtered sub-systems should be compared to the values of the non-redundant line on the top-most graph, and to the corresponding values on the bottom one).

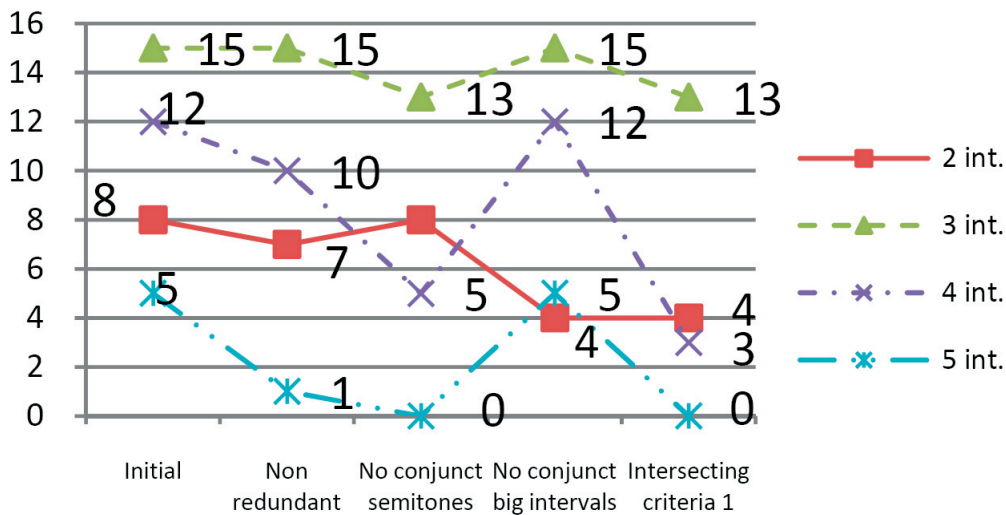
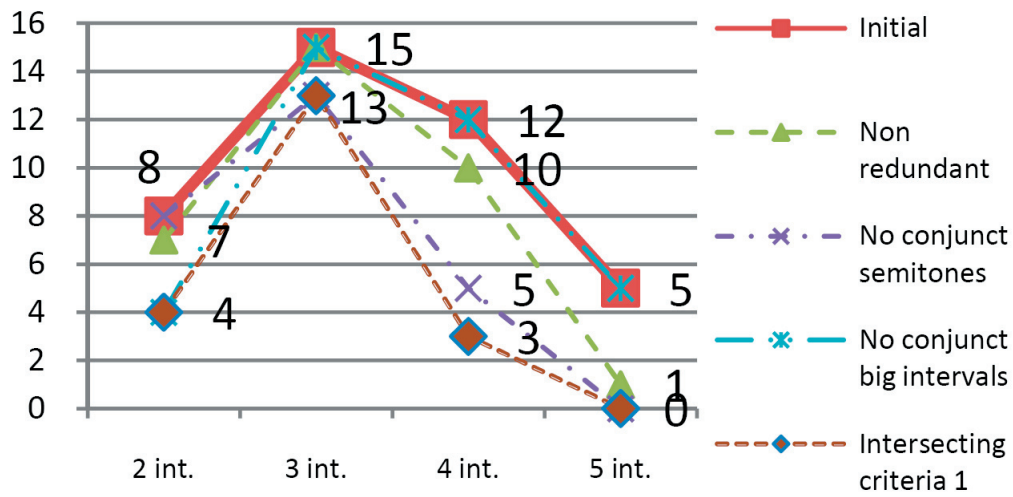


Fig. 25. Graphs of the distribution of sub-systems in a fourth, in relation with the number of conjunct conceptual intervals of second (NI) in the scale element (above), and (below) in relation with filters (cross-reference) applied in each case (NI=2, 3, 4 and 5). Quarter-tone model (compare with fig. 24 above).

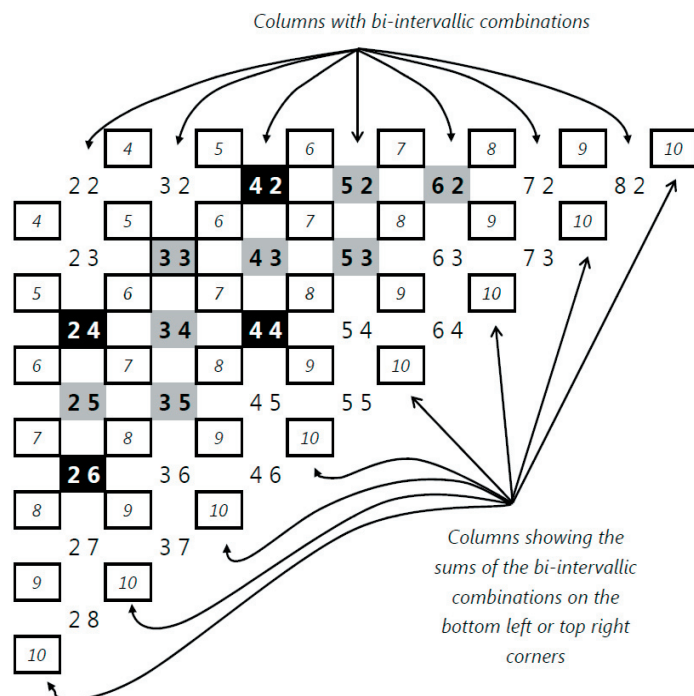


Fig. 26. Bi-interval elements of the generation for a containing interval of fourth in the quarter-tone model. Commonly used combinations are concentrated in (and occupy completely) the sector where sum values are comprised between 6 and 8 (both values included) – on black background: diatonic combinations; on black or grey background: Arabian combinations.

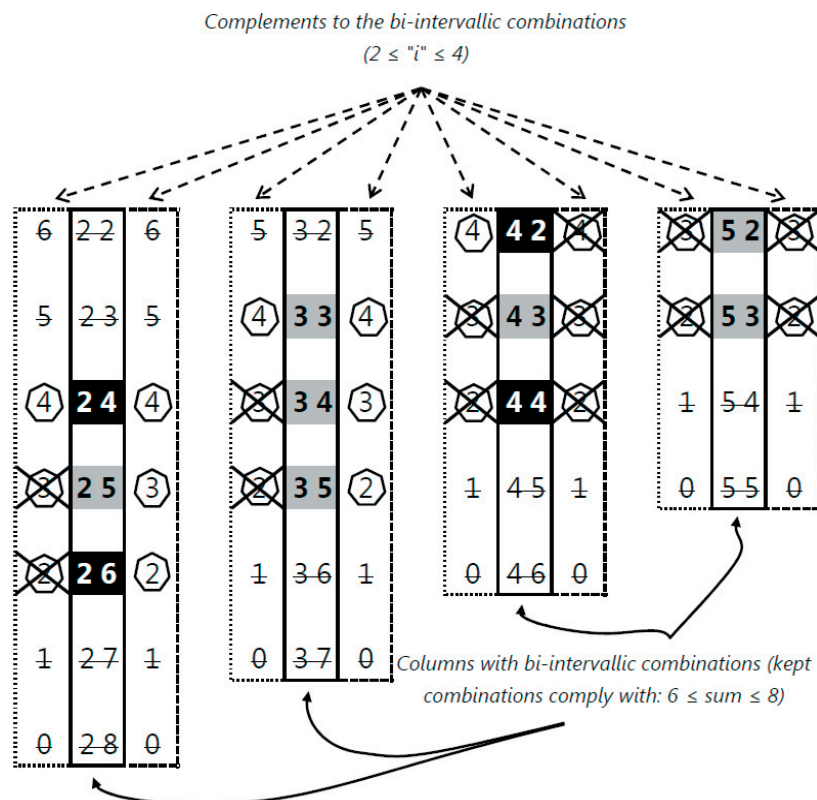


Fig. 27. Complements to the bi-interval elements of common use (black or grey background) on both sides of the elements, in order to obtain one tetrachord on each side. After redundant combinations (crossed intervals) are excluded, and by eliminating all combinations that do not comply with the homogeneity rule (which states that the sum of any two conjunct intervals must be such as $6 \leq \text{sum} \leq 8$) and its corollary (complement value to any two conjunct intervals is such as $2 \leq i \leq 4$ – where i is the complement value), the only remaining tetrachords in fourths are the commonly used tetrachords in both diatonic (semi-tone based – on black background in the figure) and Arabian music (both grey and black background).

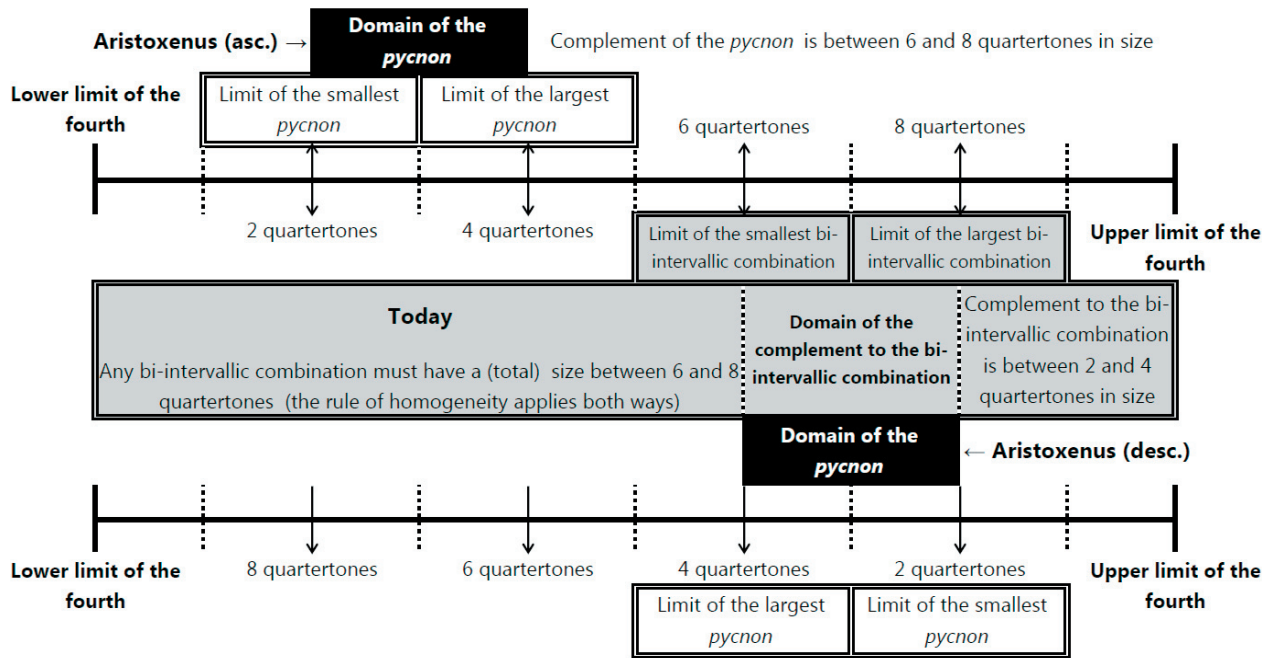


Fig. 28. The homogeneity rule, or reverse pycnon rule. If Aristoxenus' *genus* is falling, the domain of the *pycnon* is the domain of the complement of the bi-interval combination (within a fourth) in today's traditional heptatonic modal music. This applies to all *genera* in fourth of common use in Arabian music, including the chromatic *genus hijāz* (the symmetrical 262 in multiples of the quarter-tone) and its (most probably) original forms in 352 and 253 (the latter is more related to the *hijāz-kār maqām*).

<p>4262 <i>Nakrīz</i></p> <table border="1" style="border-collapse: collapse; text-align: center;"> <tr><td style="background-color: black; color: white;">4</td><td style="background-color: black; color: white;">262</td><td style="background-color: black; color: white;">4</td></tr> <tr><td>5</td><td style="background-color: grey;">Hijāz</td><td>5</td></tr> <tr><td>6</td><td style="background-color: grey;">Hijāz</td><td>6</td></tr> </table>	4	262	4	5	Hijāz	5	6	Hijāz	6	<p>4352 <i>Zāwīl</i></p> <table border="1" style="border-collapse: collapse; text-align: center;"> <tr><td>3</td><td style="background-color: grey;">352</td><td>3</td></tr> <tr><td>4</td><td style="background-color: grey;">"Awj-</td><td>4</td></tr> <tr><td>5</td><td style="background-color: grey;">Ārā_ṣaghīr"</td><td>5</td></tr> </table> <p>"A*-Ā*_ṣ*" 3524</p>	3	352	3	4	"Awj-	4	5	Ārā_ṣaghīr"	5
4	262	4																	
5	Hijāz	5																	
6	Hijāz	6																	
3	352	3																	
4	"Awj-	4																	
5	Ārā_ṣaghīr"	5																	
<p>4244 <i>Būsālīk</i> or <i>Nahawand</i></p> <table border="1" style="border-collapse: collapse; text-align: center;"> <tr><td style="background-color: black; color: white;">4</td><td style="background-color: black; color: white;">244</td><td style="background-color: black; color: white;">4</td></tr> <tr><td>5</td><td style="background-color: grey;">Kurd</td><td>5</td></tr> <tr><td>6</td><td style="background-color: grey;">Kurd</td><td>6</td></tr> </table>	4	244	4	5	Kurd	5	6	Kurd	6	<p>? 4344 <i>Rāst</i></p> <table border="1" style="border-collapse: collapse; text-align: center;"> <tr><td>3</td><td style="background-color: grey;">334</td><td>3</td></tr> <tr><td>4</td><td style="background-color: grey;">Bayāt</td><td>4</td></tr> <tr><td>5</td><td style="background-color: grey;">Bayāt</td><td>5</td></tr> </table> <p>Bayāt 3344</p>	3	334	3	4	Bayāt	4	5	Bayāt	5
4	244	4																	
5	Kurd	5																	
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<p>4442 ? ("Līdī")</p> <table border="1" style="border-collapse: collapse; text-align: center;"> <tr><td style="background-color: black; color: white;">2</td><td style="background-color: black; color: white;">442</td><td style="background-color: black; color: white;">2</td></tr> <tr><td>3</td><td style="background-color: grey;">'Ajām</td><td>3</td></tr> <tr><td style="background-color: black; color: white;">4</td><td style="background-color: black; color: white;">4</td><td style="background-color: black; color: white;">4</td></tr> </table> <p>'Ajām 4424</p>	2	442	2	3	'Ajām	3	4	4	4	<p>4343 (<i>Musta`ār</i>)</p> <table border="1" style="border-collapse: collapse; text-align: center;"> <tr><td>3</td><td style="background-color: grey;">343</td><td>3</td></tr> <tr><td>4</td><td style="background-color: grey;">`Irāq</td><td>4</td></tr> <tr><td>5</td><td style="background-color: grey;">`Irāq</td><td>5</td></tr> </table> <p>`Irāq 3434</p>	3	343	3	4	`Irāq	4	5	`Irāq	5
2	442	2																	
3	'Ajām	3																	
4	4	4																	
3	343	3																	
4	`Irāq	4																	
5	`Irāq	5																	
<p>Redundant ('Ajām 4424)</p> <table border="1" style="border-collapse: collapse; text-align: center;"> <tr><td style="background-color: black; color: white;">2</td><td style="background-color: black; color: white;">424</td><td style="background-color: black; color: white;">2</td></tr> <tr><td>3</td><td style="background-color: grey;">Būsālīk or Nahawand</td><td>3</td></tr> <tr><td style="background-color: black; color: white;">4</td><td style="background-color: black; color: white;">4</td><td style="background-color: black; color: white;">4</td></tr> </table> <p>check if sum=14</p> <p>check if sum=14</p>	2	424	2	3	Būsālīk or Nahawand	3	4	4	4	<p>4433 <i>Najd</i></p> <table border="1" style="border-collapse: collapse; text-align: center;"> <tr><td>2</td><td style="background-color: grey;">433</td><td>2</td></tr> <tr><td>3</td><td style="background-color: grey;">Rāst</td><td>3</td></tr> <tr><td>4</td><td style="background-color: grey;">Rāst</td><td>4</td></tr> </table> <p>Redundant ↓ (Rāst 4334)</p>	2	433	2	3	Rāst	3	4	Rāst	4
2	424	2																	
3	Būsālīk or Nahawand	3																	
4	4	4																	
2	433	2																	
3	Rāst	3																	
4	Rāst	4																	

Fig. 29. Extending the homogeneity rule to the fifth: names of *genera* in Arabian music stand below the tri-interval combinations. Names of the resulting pentachords in fifth (sum=14) figure at the sides of the successful combinations (diatonic combinations have a black background – left. Arabian configurations have a grey background - right). Different names for 4244 result from different positions of the tetrachord in the general scale of Arabian music theoretical literature; 3524 exists in one single and doubtful reference – the conclusion is that common pentachords in Arabian music are based on the fourth + one tone configuration, with one of the successful combinations ([4]253) not found in the reviewed literature.

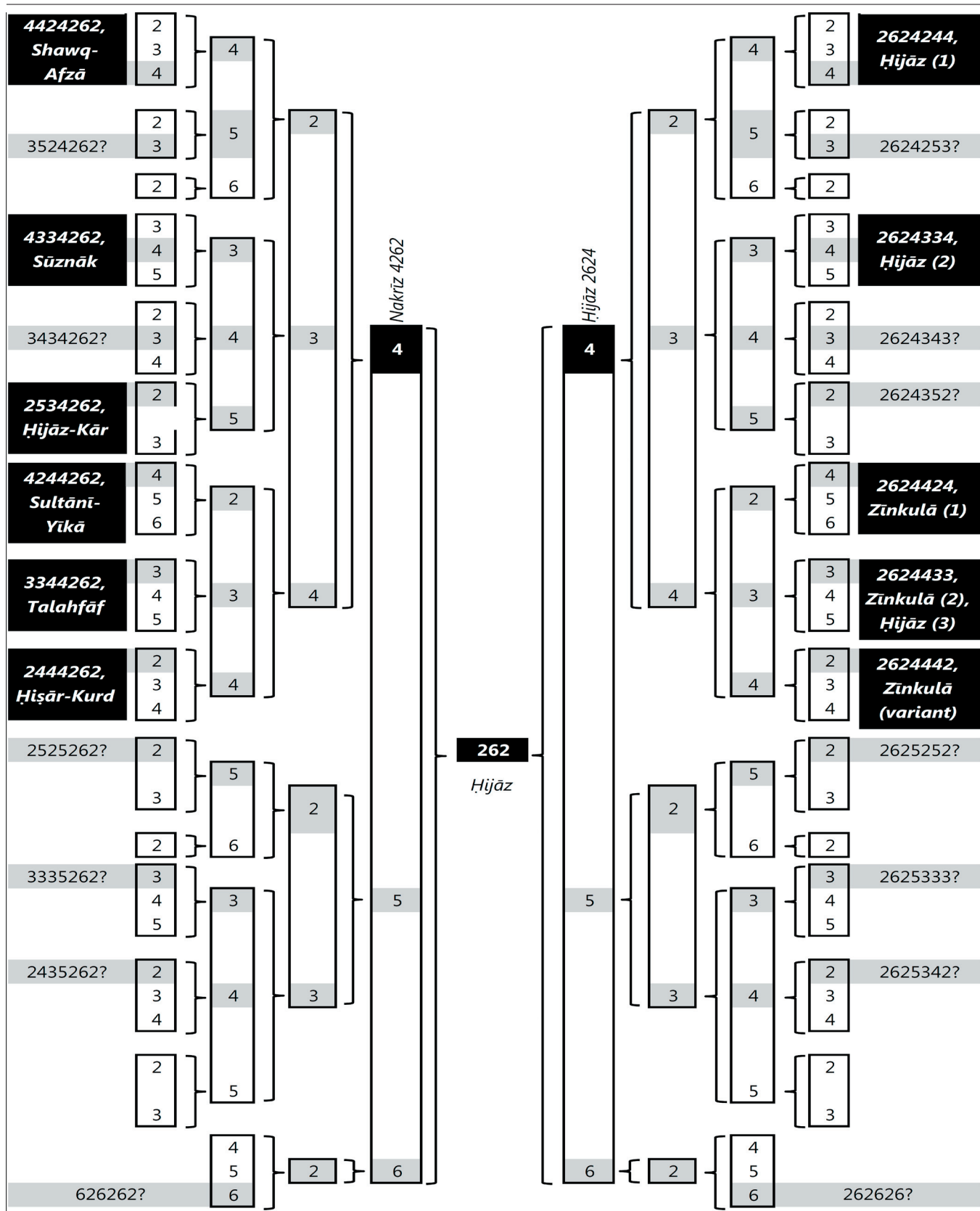


Fig. 30. Extending the homogeneity rule to the octave using tree processing of intervals, on the example of an initial *hijāz* genus. In case of success (the homogeneity rule is respected), intervals figure on a grey background, and names of resulting scales of Arabian *maqām* stand at the side of each attested combination (black background).

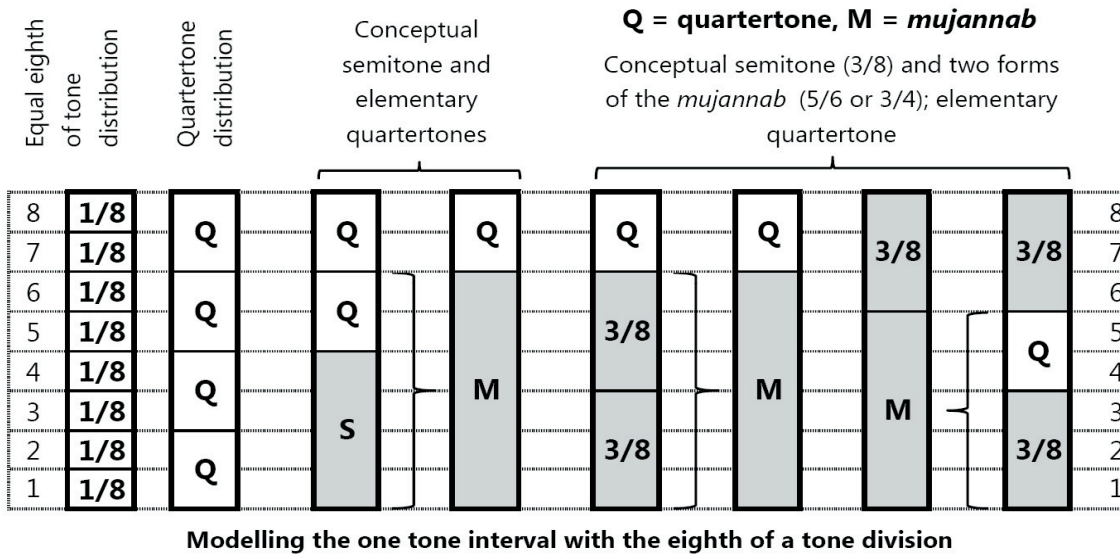


Fig. 31. Modelling the one-tone interval with eighths of a tone.

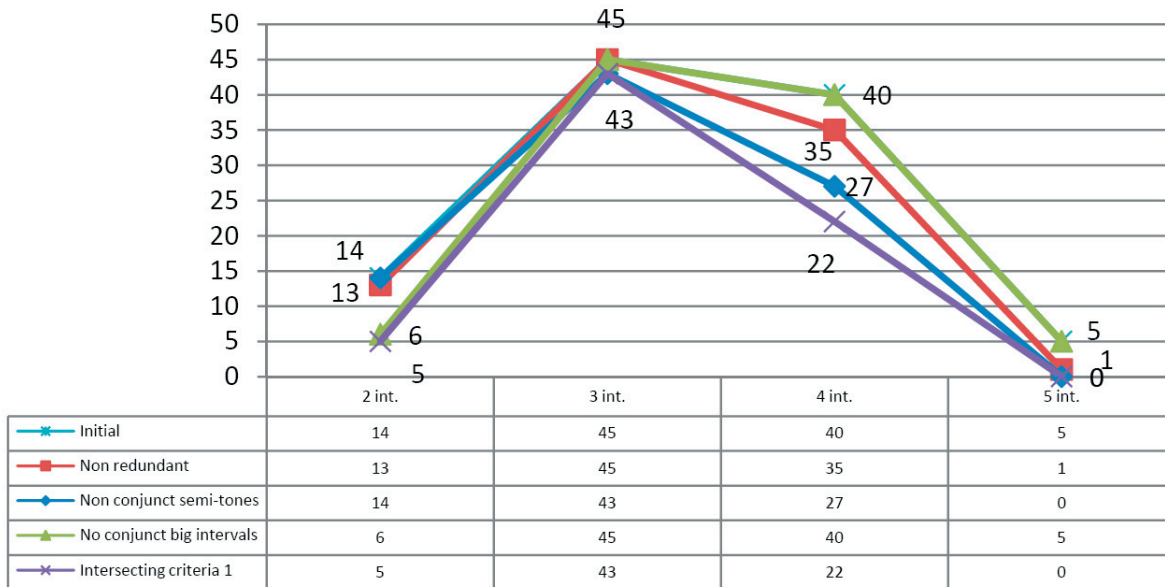


Fig. 32. Results of the generation for a just fourth in eighths of the tone, with the smallest conceptual interval chosen as the semi-tone (= one half-tone).

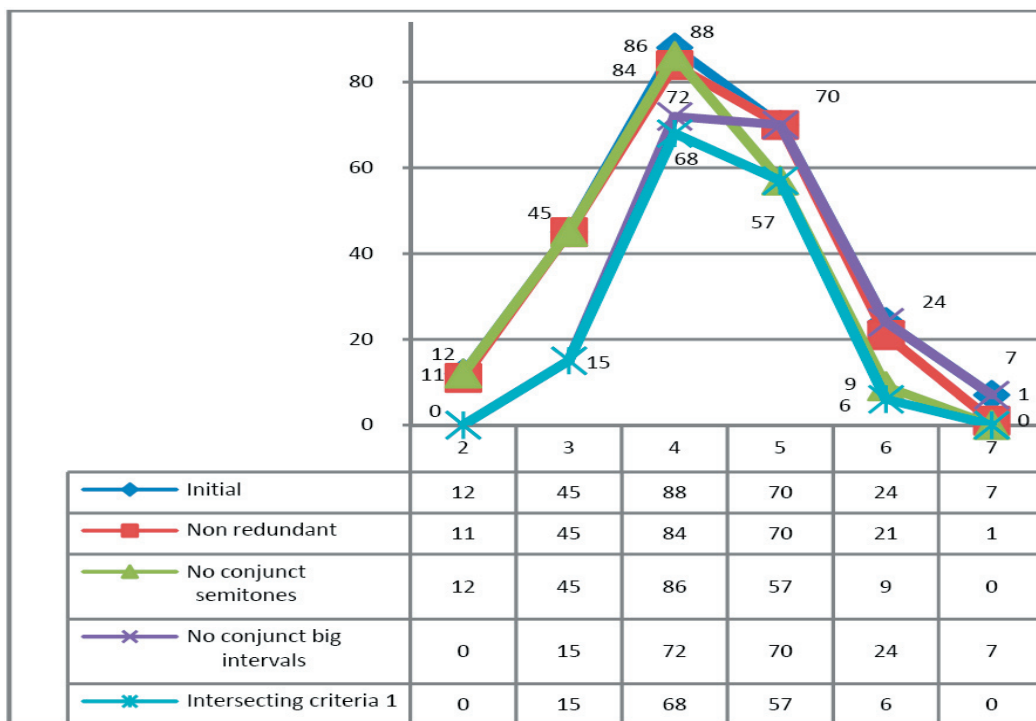
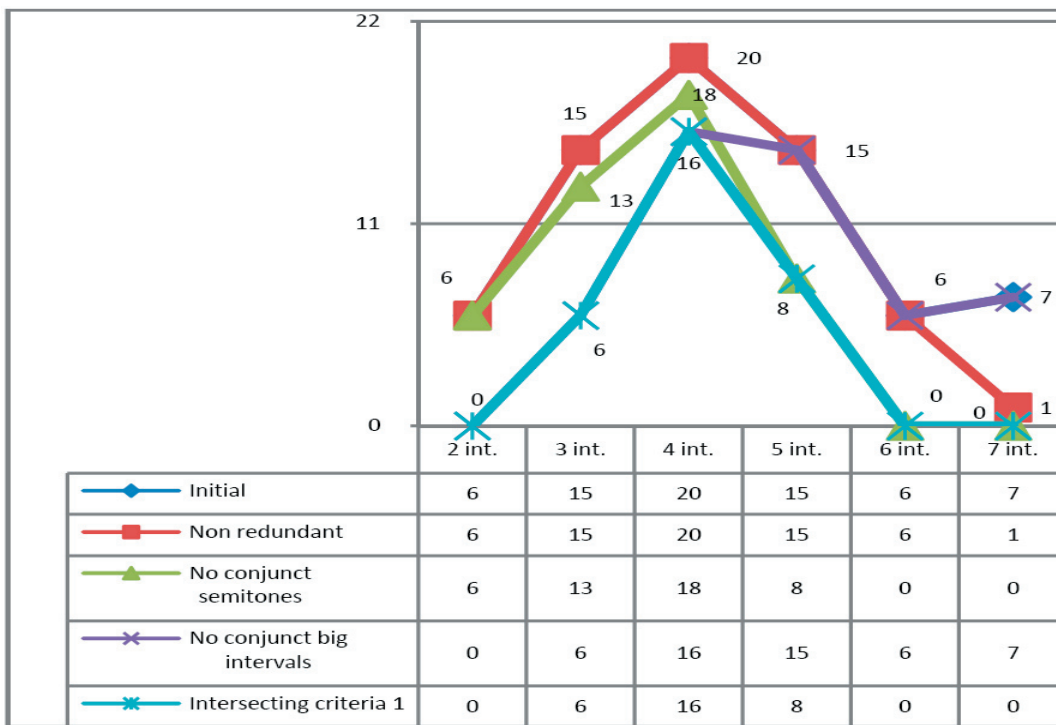


Fig. 33. Graphs for the semi-tone (above) and quarter-tone (below) of the fifth, with the unlimited alphabet: the optimal generations are at NI=4 in both cases, but in a clearly shaped form for the first (with intersecting criteria), whenever the quarter-tone model's optimal value at NI=4 has a competition at NI=5. The no conjunct semi-tones criterion applies to suites of three or more semi-tones in a row, and the no conjunct big intervals criterion to intervals equal to or greater than 3 semi-tones.

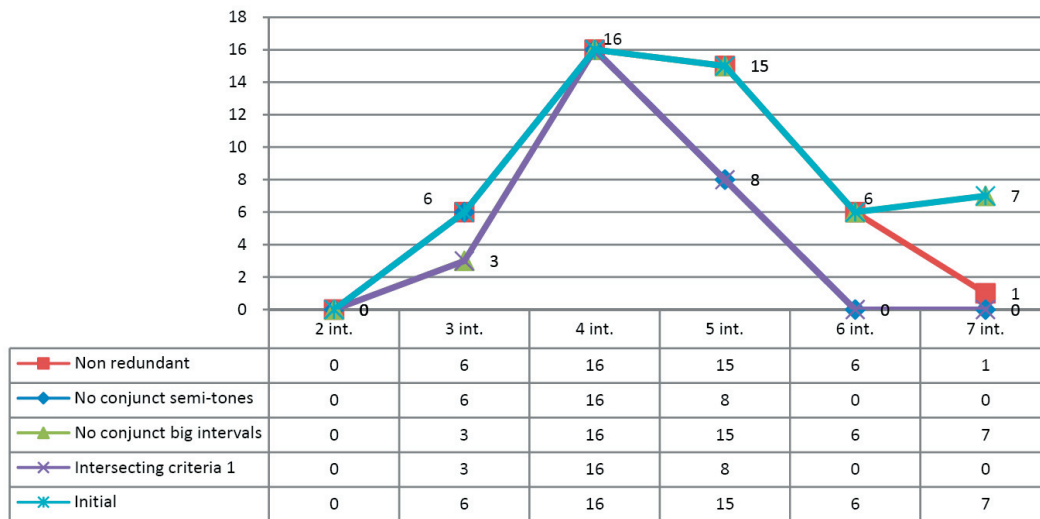


Fig. 34. Graph for the semi-tone model of the fifth with the limited alphabet 1, 2, 3: the shape of the intersecting criteria 1 line is narrower (values for NI=3 are relatively smaller than for an unlimited alphabet) and confirms the optimum for NI=4.

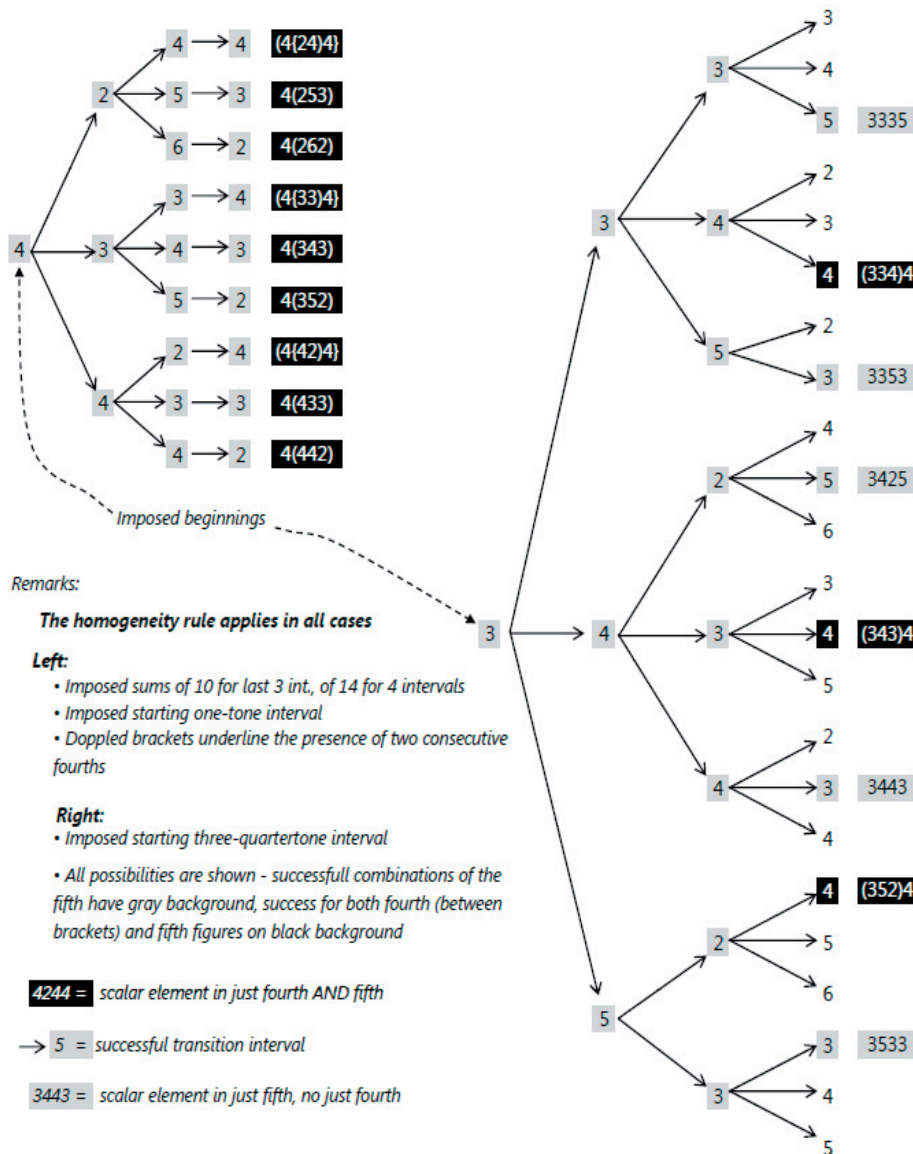


Fig. 35. Modelling the fifth with the one-tone interval initial condition and the homogeneity rule (left). Beginning with a one-tone interval increases the number of regular (and diatonic) fourths and fifths, as well as bi-fourth combinations within the fifth. Starting with a neutral interval such as the three-quarter-tones (right) lessens the possibilities for a fourth/fifth combination, as well as for bi-fourth configurations.

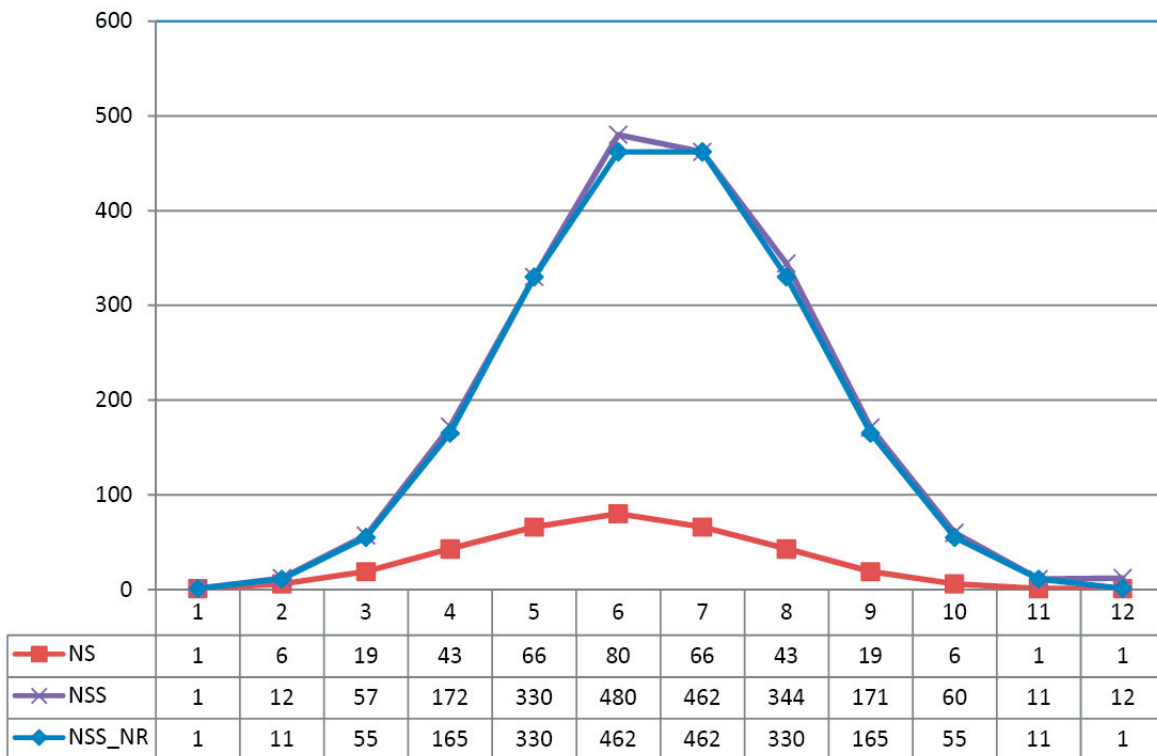


Fig. 36. Systems and sub-systems in an octave, from the initial generation and filtered for redundancies (NS = number of systems; NSS = number of sub-systems; NSS_NR = NSS with redundant sub-systems excluded) - semi-tone model. Full alphabet.

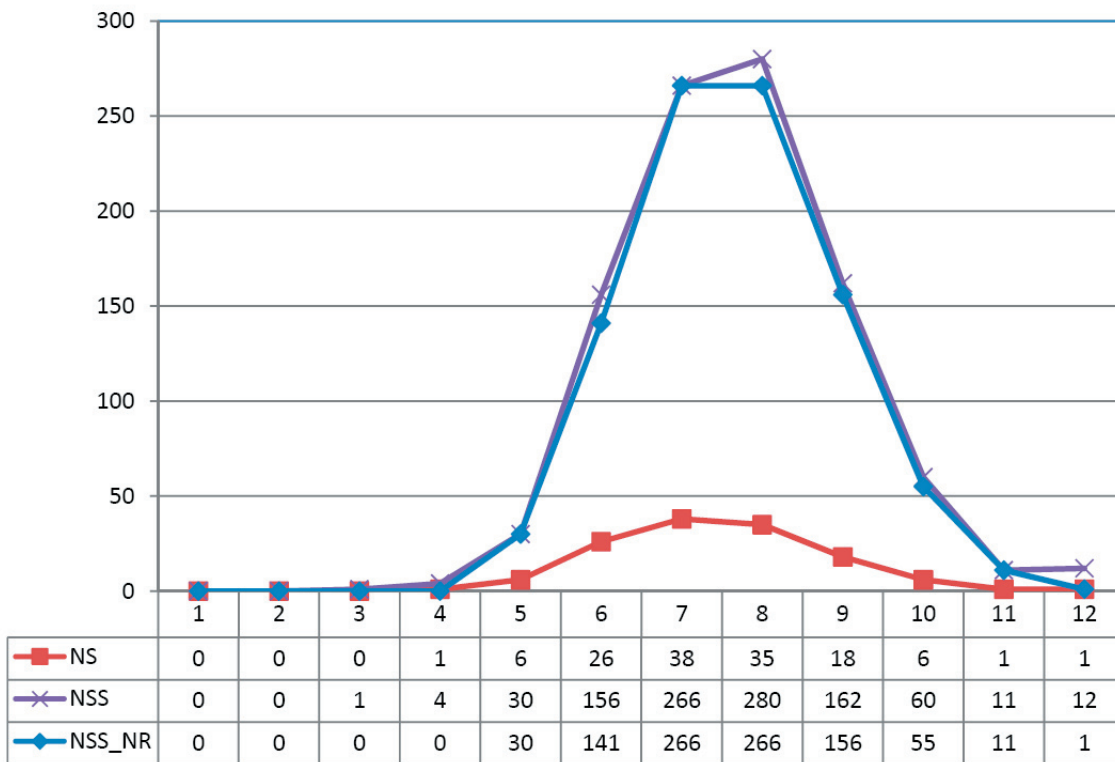


Fig. 37. Systems and sub-systems in an octave: restricted alphabet (1, 2 and 3 semi-tones only).

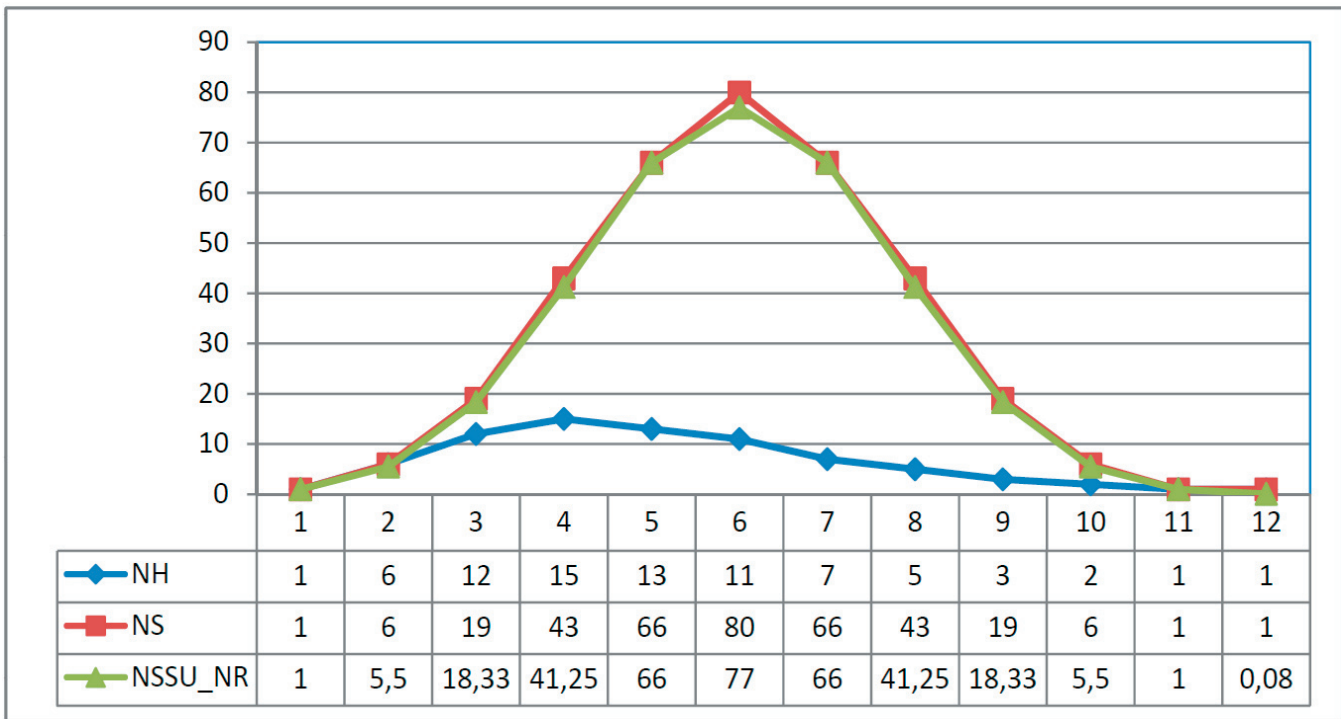


Fig.38. Hyper-systems and systems in an octave: unrestricted alphabet in multiples of the semi-tone (alphabet = 1 to 12); NH = Number of Hyper-systems (for each NI), NS=Number of Systems, NSSU_NR is the Number of Sub-Systems in Unitary weighing with the Redundant sub-systems excluded (NR=non-redundant).

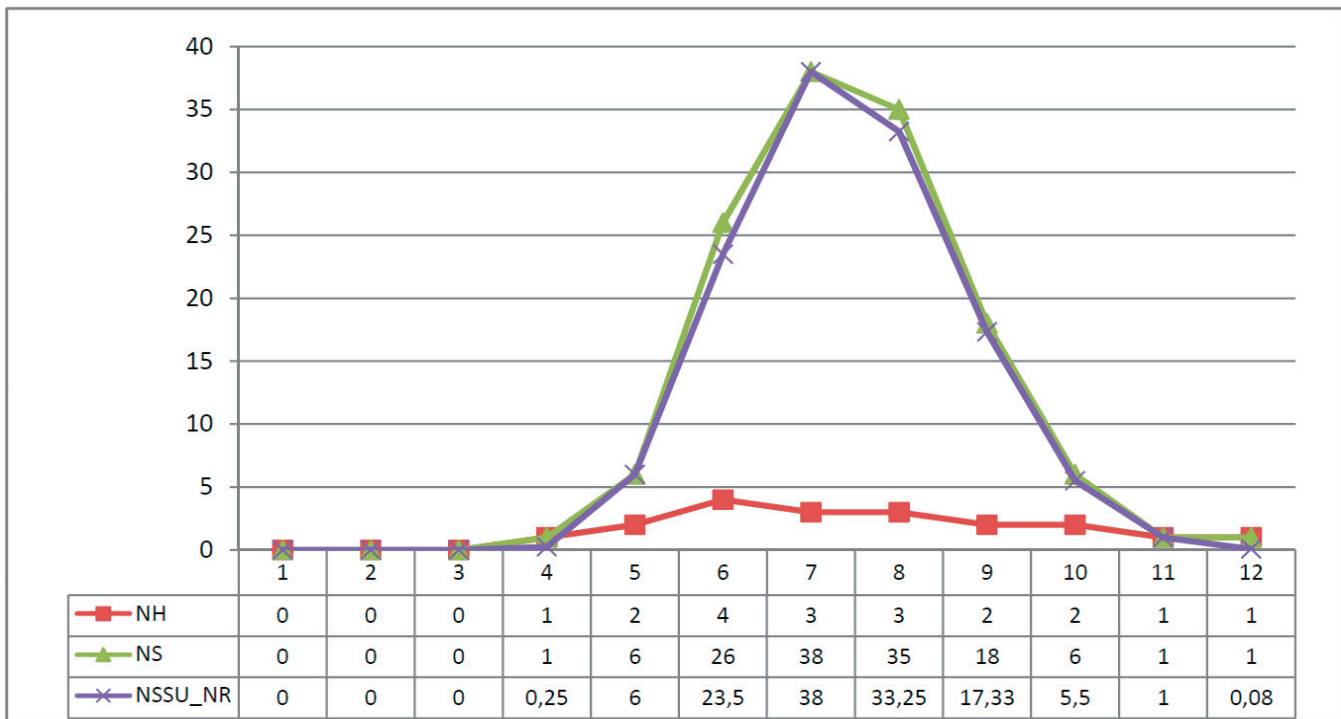


Fig. 39. Hyper-systems and systems in an octave: restricted alphabet in semi-tones (alphabet = '1, 2, 3').

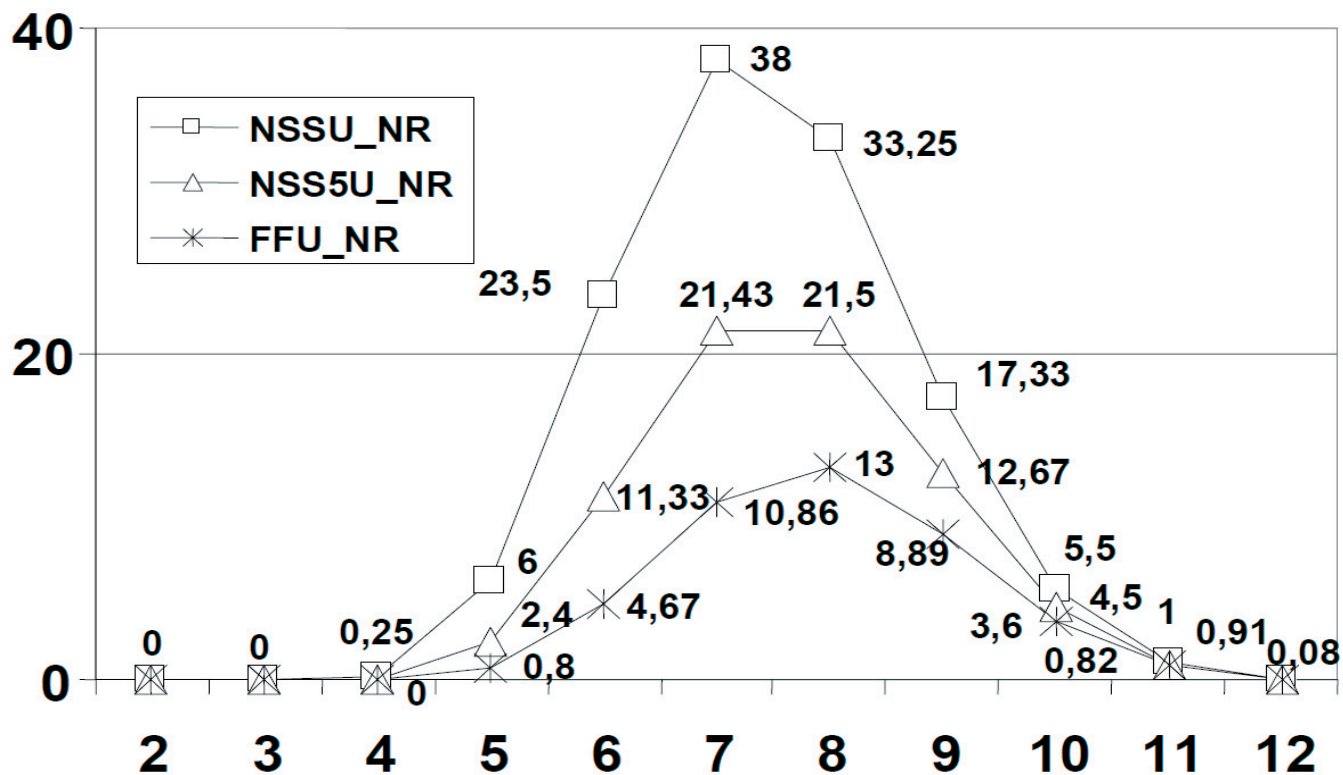


Fig. 40. Unitary non-redundant sub-systems in an octave, restricted alphabet (alphabet = 1, 2, 3) – semi-tone model: NSSU_NR is the Number of Sub-Systems in Unitary weighting with the redundant sub-systems excluded (NR=non-redundan). NSS5U_NR is the Unitary number of Non-Redundant Sub-Systems with a direct fifth (or fourth). FFU_NR sub-systems include a direct fourth in a fifth. See Appendix E, figure A1 for the full alphabet generation.

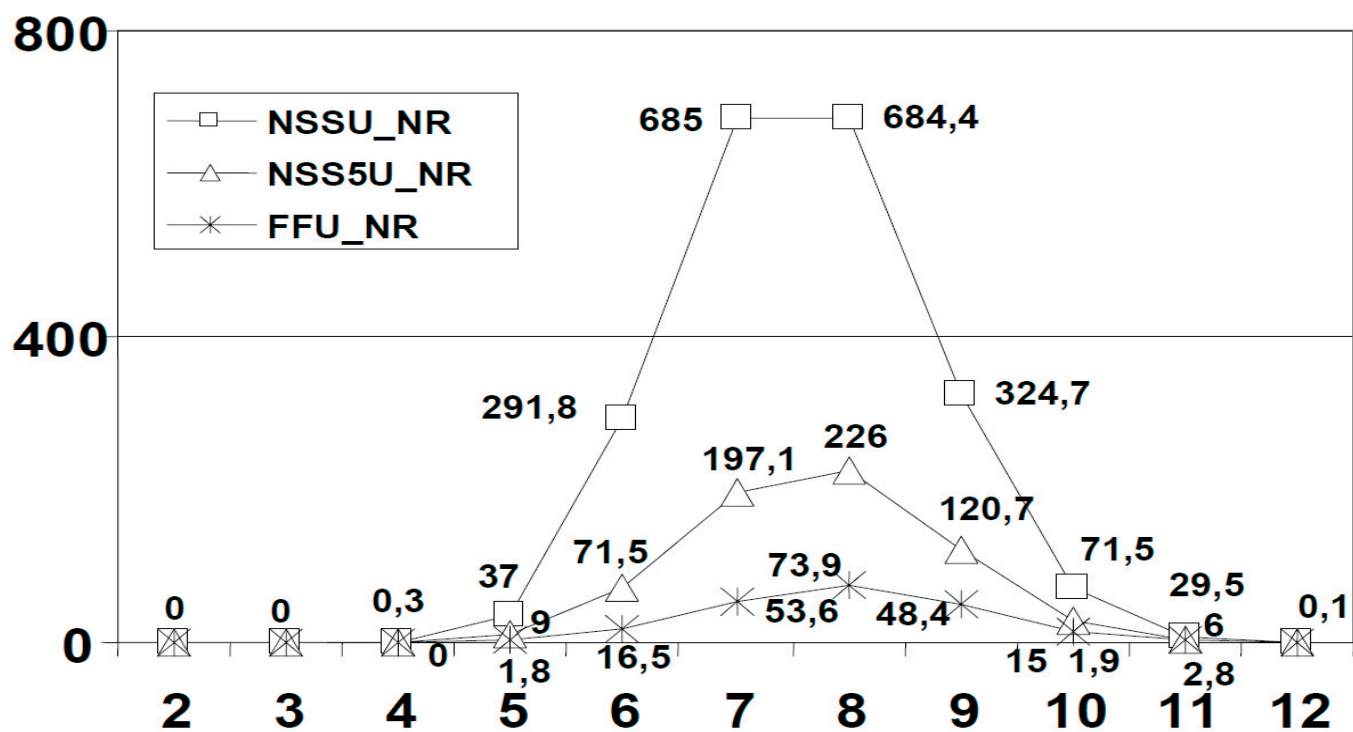


Figure 41. Same as above, but for the quarter-tone model (alphabet = 2, 3, 4, 5, 6 quarter-tones) – see Appendix E, figure A5 for the (nearly) full alphabet generation.

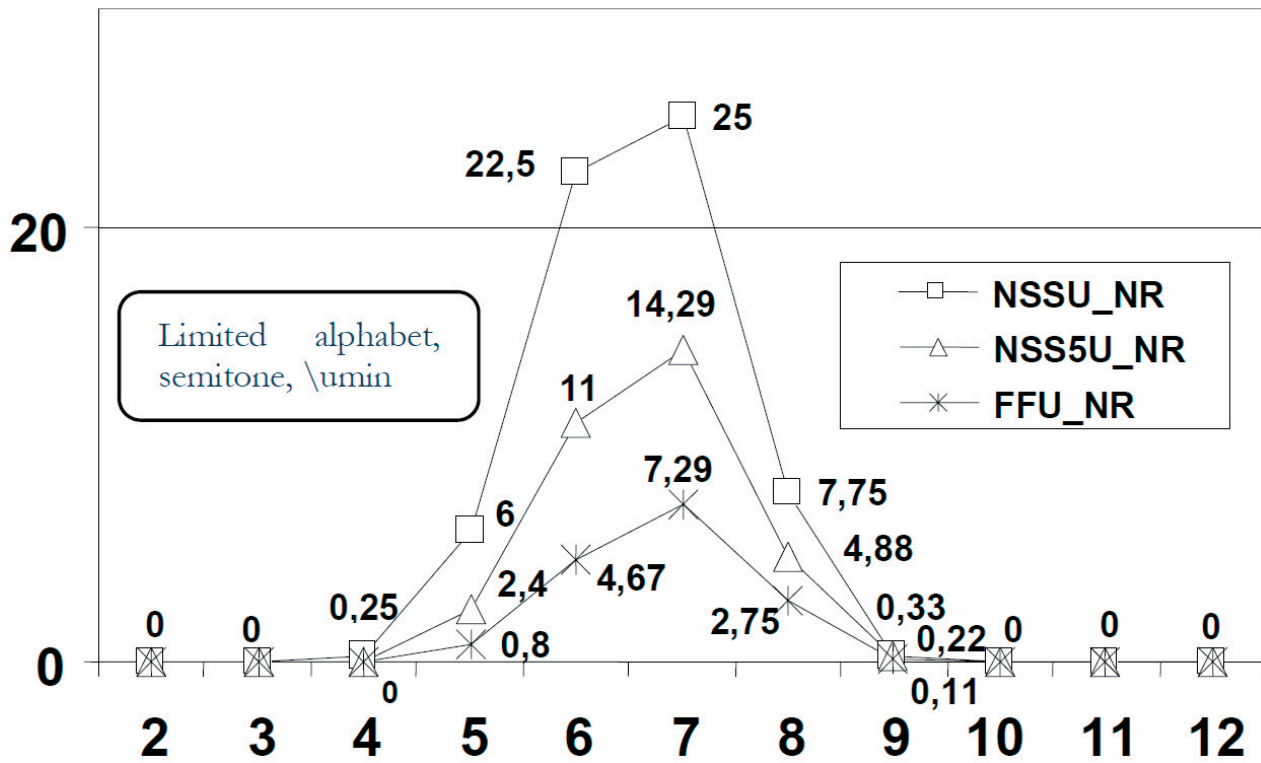


Fig. 42. Unitary non-redundant sub-systems in an octave, restricted alphabet – semi-tonal model with sub-systems containing tri-interval (or more) suites of semi-tones excluded. NSSU_NR = Number of Sub-Systems in Unitary weighting with the Non-redundant subsystems excluded, NSS5U_NR = unitary number of redundant sub-systems with a direct fifth, and FFU_NR with a direct fourth in a fifth – see Appendix E, figure A3 for the full alphabet generation

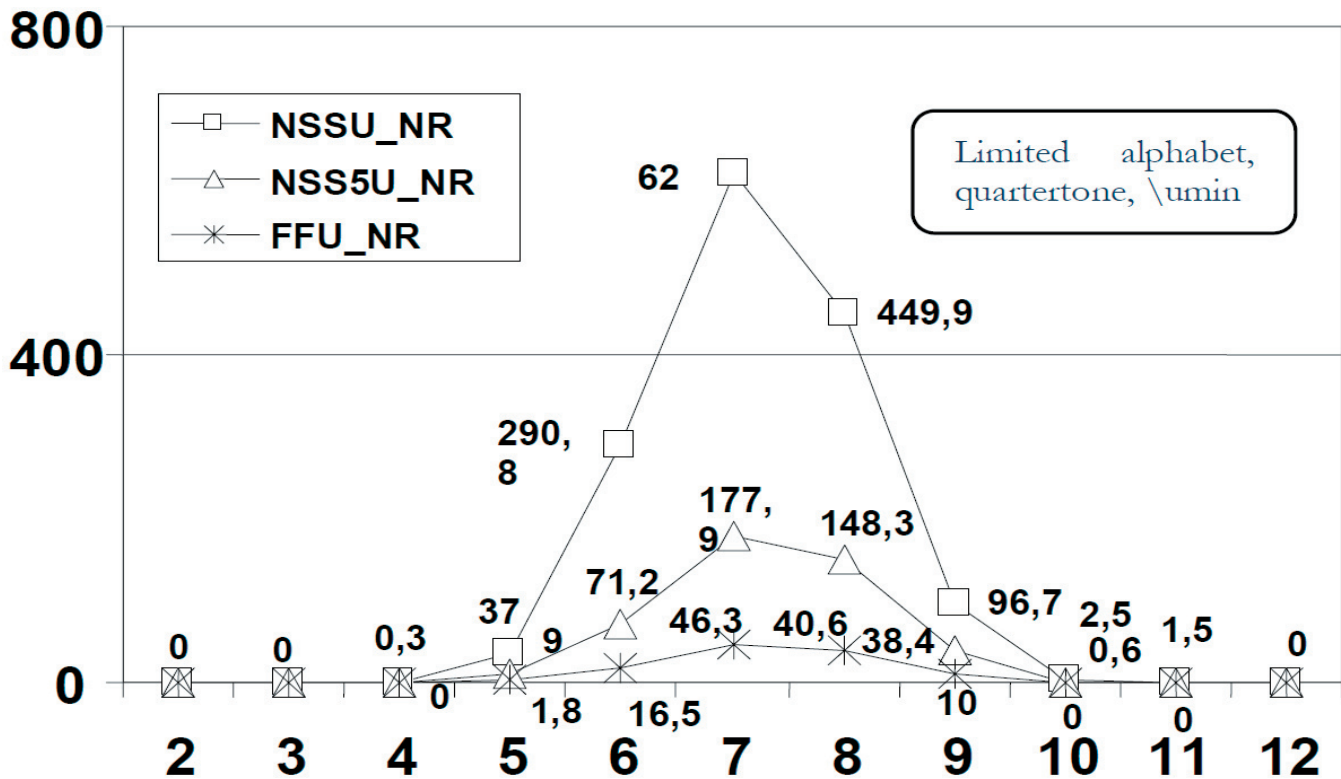


Fig. 43. Same as above, but for the quarter-tone model (alphabet = 2, 3, 4, 5, 6 quarter-tones) – see Appendix E, figure A7 for the full alphabet generation.

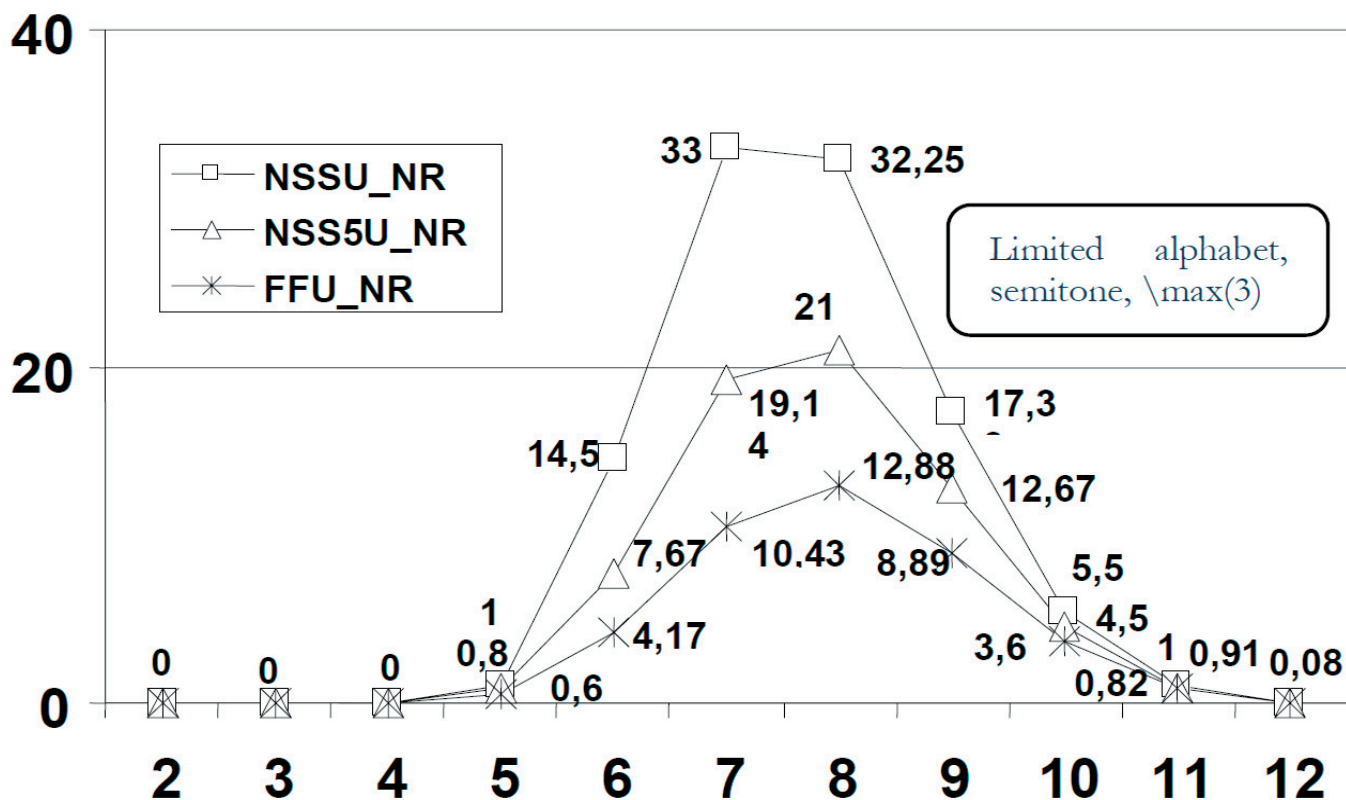


Fig. 44. Unitary non-redundant sub-systems in an octave, restricted alphabet (alphabet = 1, 2, 3) and filtering of sub-systems with bi-interval entries containing two conjunct intervals greater or equal to 3 semi-tones – semi-tone model. NSSU_NR is the Number of Sub-Systems in Unitary weighting with the redundant sub-systems excluded, NSS5U_NR is the unitary number of non-redundant sub-systems with a direct fifth, and FFU_NR with a direct fourth in a fifth – see Appendix E, figure A2 for the generation with the full alphabet.

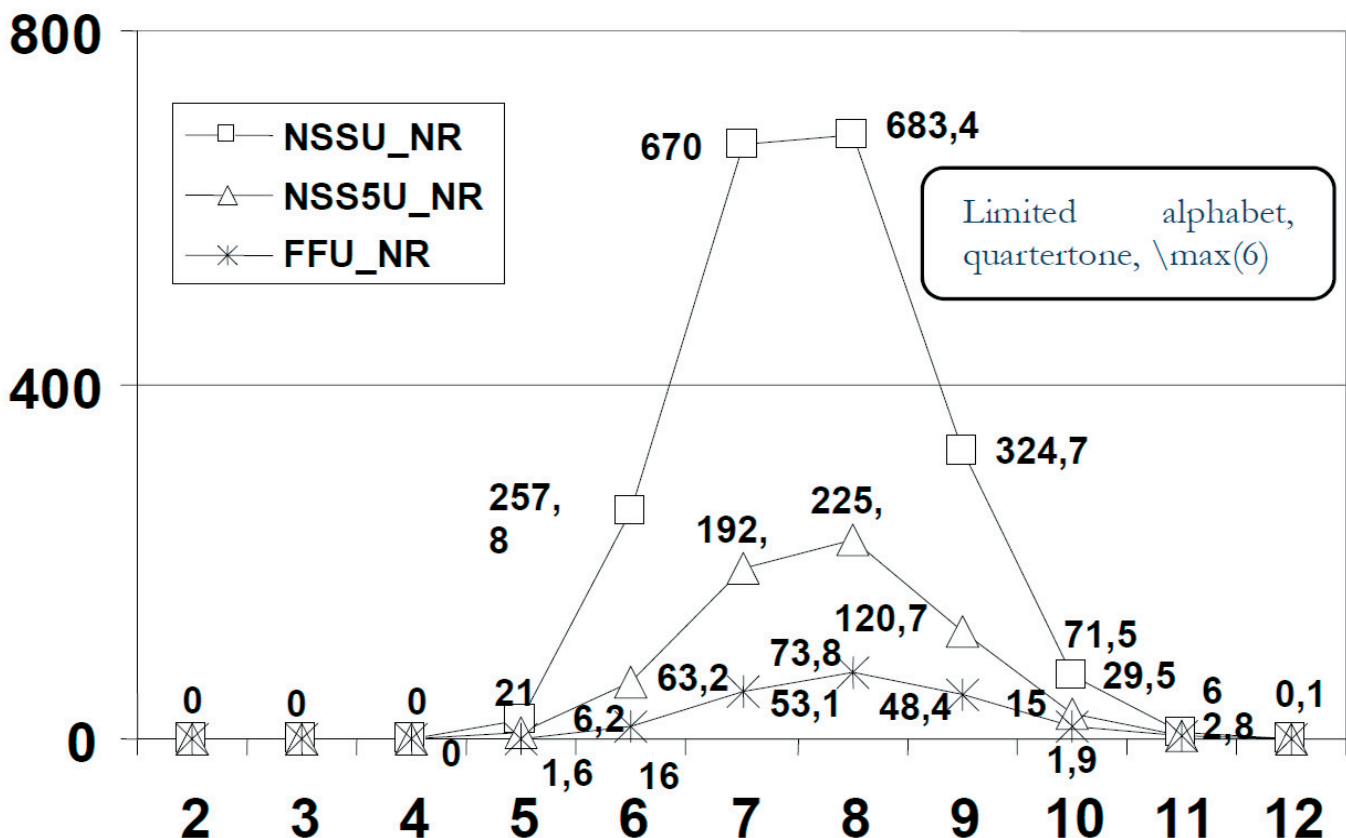


Fig. 45. Unitary non-redundant sub-systems in an octave, restricted alphabet (alphabet = 1, 2, 3) – semi-tone model with intersecting criteria (\umin OR \max(3) are excluded)! for variables, see previous captions.¹⁵⁶

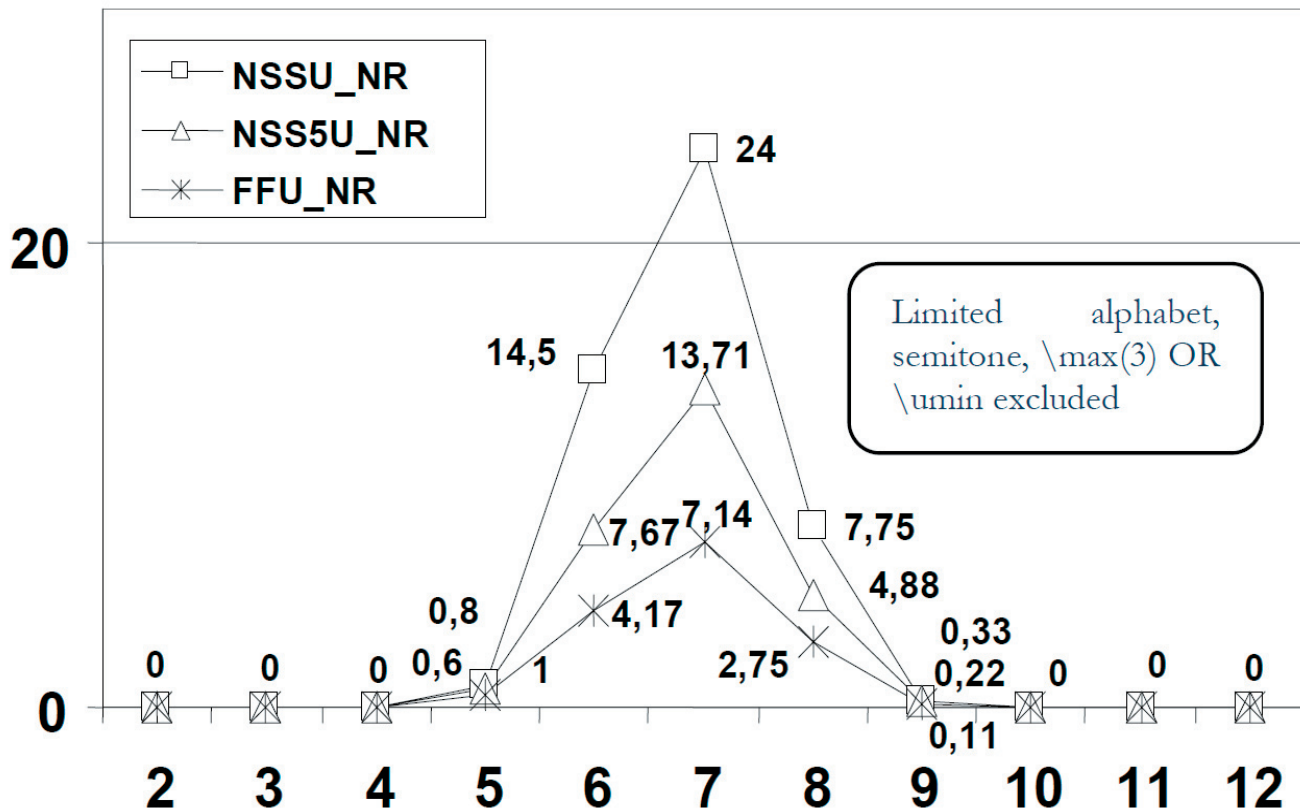


Fig. 46. Unitary non-redundant sub-systems in an octave, restricted alphabet (alphabet = 1, 2, 3) – semi-tone model with intersecting criteria ($\backslash\text{umin}$ OR $\backslash\text{max}(3)$ are excluded) for variables: see previous captions (see Appendix E, figure A4 for the full alphabet generation).

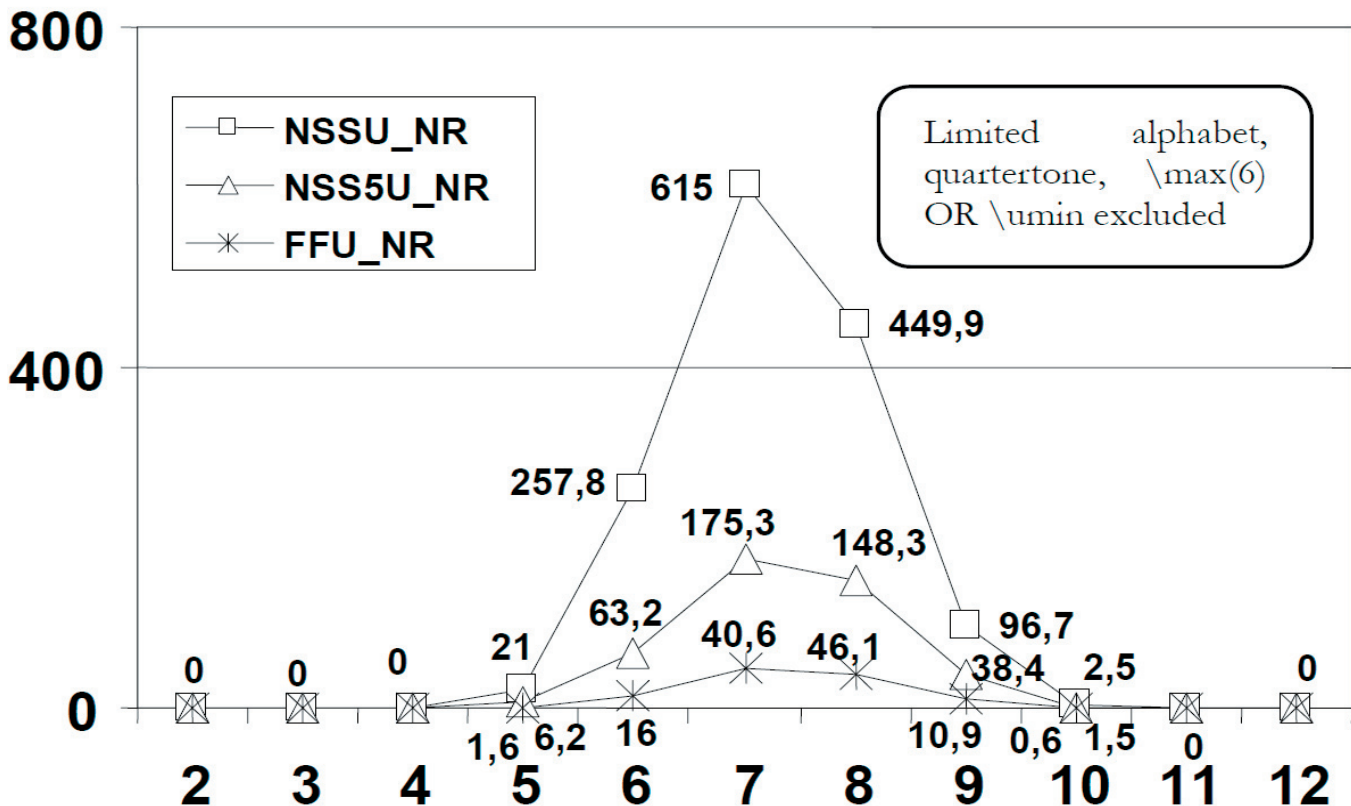


Fig. 47. Same as above, but for the quarter-tone model (alphabet = 2, 3, 4, 5, 6 quarter-tones) – see Appendix E, figure A8 for the full alphabet generation.

Notes

1 In Arabic: *Sīnā (Ibn)*, [*Kitāb al-sh-Shifā' - Riyādiyyāt 3 - Jawāmi' 'Ilm al-Mūsīqā*, text collated by *Zakariyyā Yūsūf* and revised by *Aḥmad Fu'ād al-Ihwānī* and *Maḥmūd Aḥmad al-Ḥifnī* from some ten manuscripts, *Wizārat a-t-Tarbiya wa-t-Ta'lim*, Cairo, *Al-Maṭba'a al-Amīriyya* [end of 10th, early 11th century AD], (1956), pp. 40-4; and in French: *Sīnā (Ibn)*, *Kitāb al-sh-Shifā'* (Mathematics, Chapter XII, Music), translated by Rodolphe d'Erlanger in *La musique arabe*, Tome II, Paris, (1936), Librairie orientaliste Paul Geuthner, p. 138. Erlanger worked from a copy of MS. 476 of the India Office, London, fols.152-74, dating from the end of 10th, early 11th century AD.

2 Beyhom, A., *Systématique modale*, 2003, unpublished doctoral thesis defended at the Université Sorbonne - Paris IV. The thesis and other publications from the present writer can be downloaded from http://www.beyhom.com/systematique/pages/excerpts_en.htm.

3 Appendices in the second part of the paper can be downloaded from <http://www.iconea.org>.

4 Numbers 3, 4, 5, and 7, may play a role in the outcome of interval combinations, as shown in Part II of this paper.

5 Crocker, R.L., 1963, Pythagorean Mathematics and Music, *The Journal of Aesthetics and Art Criticism*, Vol. 22, No. 2 (Winter, 1963), pp. 189-98, and No. 3 (Spring 1964), pp. 325-35.

6 We use the term 'Arabian music' as a generic concept applying to *maqām* practice, although Farmer, writes that the use of 'Arab' is well attested, in his 'Greek theorists of Music in Arabic Translation', *Isis*, Vol. 13, N°2 (1930), pp. 325-33. In note N°1, p. 325, he writes: 'I use the term "Arab" advisedly, just as I would use the word "English", at the same time implying the Scots, Irish, & Welsh. "Islamic" or "Muslim" will not serve, because Magians, Jews & Christians, contributed to this "Arabian culture." We shall include in this wide definition Turk and Persian music, as well as other *maqām* music with, mainly, heptatonic scales - and 'neutral' (this term is defined below) intervals used in the latter.

7 Additionally, the Pythagorean cycle of fifths does not generate a fourth. The scale is the consequence of an ascending cycle of fifths, bringing notes placed above the first octave back into it, hence F G A B c d e. The fourth, ascending from starting F, is F-B which is a Pythagorean tritone.

8 Aristoxenus, *Elements of Harmonics*, translated and commented by Macran, H.S., *Aristoxenou Harmonika Stoicheia: The Harmonics of Aristoxenus*, (Oxford, 1902), pp. 193-8, notably (pp. 193-4) 'For the apprehension of music depends on these two faculties, sense-perception and memory;' or p. 197: 'That no instrument is self-tuned, and that the harmonizing of it is the prerogative of the sense perception is obvious.'

9 Superparticular intervals with string ratios of the type (n+1)/n when n is a positive integer.

10 See Chailley, J., *Éléments de philologie musicale*, (Paris, 1985), pp. 64-5.

11 Acoustic resonance is not a generative process as such, but it is the consequence of the physical (and dimensional) properties of matter set to vibration. The integration of acoustic resonance within a generative theory is subjective as it admits that vertical relationships cannot be unidirectional, i.e., ascending.

12 A computer program has been used to test this hypothesis, up to the 1500th harmonic, and found no exact matches for the just fourth. A first approximation is found at the 341st harmonic, with about 496 cents, then 499 cents with the 683rd harmonic. The closest is the 1365th harmonic with 498 cents. The calculations were based on the formula: $i = 1200 \times \ln(R) / \ln(2)$, where 'i' is the interval in cents, 'R' the ratio of frequencies (the ratios of frequencies are 341, 683 and 1365, respectively), and then extracting modulo of (i/1200). In analytical terms, the problem consists in finding an integer J, which multiplies N, the frequency of the fundamental tone, and the ratio of which, to the nearest and lower octave (octaves of the sound with frequency N have the form 2^kN, where k is an integer number) is equal to 4:3, or $[(J \times N) / (2^k \times N) = (4/3)]$ (k is the power indicator of 2, with 2^k x N being simply an even multiple of N), which is not possible

because in this case $J = (2^k \times 4) / 3$, and neither 4 nor a power of 2 (2^k) can divide 3.

13 Helmholtz, H., *On the sensations of tone as a physiological basis for the theory of music*, (ed. Ellis, A.), (London 1895), pp. 192-4 (figs 60 A and 60 B, p. 193). The consonance of the fourth is explained in that two simultaneous notes at a fourth apart have some theoretical harmonics in common, as for example for two notes at (1) 300Hz and (2) at 400Hz, which have common harmonics with frequencies equal to 1200, 2400, 3600Hz (etc.), i.e., for every common multiple of 300 and 400.

14 In order to assemble a very approximative octave made up of the degrees of the diatonic scale, various resonance theories generally end up at the fifteenth harmonic, which is a 'b' if the fundamental is 'c' or, 'e' if the fundamental is an 'f'. This is an arbitrary proposition since no reason is given for having chosen the fifteenth harmonic as a last pitch and also this would require extraordinary hearing powers, since this fifteenth harmonic placed right below the fourth octave has generally little intensity. Therefore preceding pitches from the 7th, 11th and 14th harmonics, theoretically, should be heard much louder than the 15th harmonic.

15 This process is plainly explained in Barbera, A., 'Octave species', *The Journal of Musicology*, Vol. 3, N°3 (Summer, 1984), pp. 229-41, especially pp. 231-2: 'Aristoxenus has described the enharmonic genus in such a way that there can exist only three species of fourth. This is so because he has allowed only two different intervals, the enharmonic diesis or quarter-tone and the ditone, to enter his discussion. Thus we can arrange two quarter-tones and one ditone in at most three different ways. Had Aristoxenus considered a chromatic genus containing three different intervals, for example, 1/3 tone, 2/3 tone, and 1 1/2 tones, what would have been the result? Later writers make clear that the six possible arrangements of these three intervals were not all possible musically. In fact, only the first, second, and third species were musical possibilities, i.e., those species that are arrived at by making the highest interval the lowest or vice versa, leaving the rest of the sequence unchanged. The three arrangements that are not considered are neglected, I believe, because they are not species of a musical genus. A genus is, after all, a tuning, or more precisely, infinitely many tunings within firmly established boundaries. Such tunings presume a musical scale or system as background - a first note or string, a second note, third, and so forth. One can focus attention on any four consecutive notes of the scale and, depending upon the segment of the scale that is chosen, one can discern a variety of species. At no point, however, can one alter the sequence of notes of the scale. For instance, the third note of the system never becomes the second note. Therefore, because a system - the Greek musical scale - is assumed, and because species must be species of a genus, there can exist only three, not six, species of any specific tuning of a musical fourth.'

16 *Fārābī* [9th-10th century] in Erlanger, Baron R., *Kitāb al-Mūsīqī al-Kabīr*, La musique arabe, Vol.1, (Paris, 1930), p. 127: 'Should a consonant interval be repeated within a group, the small intervals could be situated at difference places in that group. Thus the fifth having been placed within a group with a certain arrangement of its small intervals, one can, within the same group have other fifths having their small intervals arranged in another way. For instance, the first interval in the first arrangement might be the last in another. In the case an interval is seen often in a group with its small intervals differently arranged, each of these arrangements of small intervals form a kind, a species, of a group. Within an interval, the arrangement of small intervals it contains can be classified as first, second, etc., until the various arrangements in this group are exhausted.'

17 Cleonides, *L'introduction harmonique*, (ed. and tr. Ruelle, Ch.), (Paris, 1884), notably §71 which says: 'Differences are produced numerically in the following manner. Having agreed that the tone is divided in twelve small parts each of which called a twelfth of a tone, all the other intervals have a proportional part in relation to the tone.'

18 *Fārābī*, *Kitāb al-Mūsīqī...*, pp. 59-.

19 Metrologic accuracy is essential to mathematical precision. However, *Fārābī* himself acknowledges that music performance dismisses very small intervals in the scale – see *Fārābī, Kitāb al-Mūsīqā* ..., pp. 174-6.

20 '1 1 3' (semi-tone, semi-tone, one-and-a-half-tones): *Aristoxenus, Elements...*, pp. 202-3.

21 '1 2 2' (semi-tone, tone, tone): *Aristoxenus, Elements...*, p. 204.

22 These could be called species and are defined as sub-systems in Modal systematics.

23 More than two thousand years ago, Ancient Greek theory included the semi-tone equal temperament which is in use in most Western music today (classical, to some extent, and pop music in general), together with modern Arabian quarter-tone divisions of the octave. (Aristoxenus' theory is based on a quarter-tone division. He defines the fourth as composed of five semi-tones. See Macran, H., *The harmonics of Aristoxenus...*, p. 208).

24 Such as many types of unequal, temperaments.

25 *Urmawī's Book of cycles* is extensively analysed by Wright, O., in *The Modal System of Arab and Persian Music A.D.1250-1300*, (Oxford, 1978). There appears to be no translation in English. There is a translation in French by Erlanger (1938) but there he refers to a commentary of the *Sharḥ Mawlānā Mubārak Shāh bar Adwār*, which he attributed to *Ṣafīy-y-a-d-Dīn al-Urmawī*, under the title of *Kitāb al-Adwār* [Livre des cycles musicaux], in *La Musique Arabe*, Vol.3, (ed. Erlanger, R.), (Paris, 1938). In the same volume, Farmer (p. XIII of Erlanger's translation) ascribes it to 'Alī ibn Muḥammad a-s-Sayyid a-sh-Sharīf al-Jurjānī.

26 The Pythagorean *comma* is the amount of six Pythagorean tones (8:9) from the sum of which one octave is taken away. The *comma* has the ratio of 524288:531441, which is about 23 cents. This discrepancy is the consequence of the Pythagorean tone, about 204 cents being slightly larger than the equal temperament tone at 200 cents. Therefore the octave is made up of five tones and two *limmata*. The fifth is made up of three tones and one *limma* (about 702 cents), and the fourth, of two tones and one *limma* (498 cents). The *limma* is the 'left over' quantity between two Pythagorean tones away from a fourth. This amounts to a ratio of 243:256, about 90 cents.

27 The *limma* is the complement of the Pythagorean ditone within the just fourth of ratio 3:4.

28 One of *Urmawī's* octave representations runs as: L L C, L L C, L, L L C, L L C, L L C, L. Placing notes at Pythagorean boundaries, we have c (L L C) d (L L C) e (L) f (L L C) g (L L C) a' (L L C) b' (L) c'. In the *maqām Rāst* of Arabian music, as defined by *Urmawī*, the boundaries stand differently: c (L L C) d (L L) e- (C L) f (L L C) g (L L C) a' (L L) b'- (C L) c'. The intervals between d and e- (or for the latter a pitch which stands between e flat and e sharp) and between e- and f are the *mujannab*, or neutral, seconds of *Urmawī*. The same applies to the intervals between a' and b'- and c'. Their value is (L+L) or (L+C), but both hold the same name of *mujannab*, whilst intervals such as the *limma* 'L' or the tone, have one single interval capacity, that is one *limma* for the semi-tone (with *Urmawī*), and two *limmata* and one *comma* for the tone.

29 The concept remains the same throughout history, and is based on the division of the tone into three small intervals and on the division of the 'neutral' second in two other, even smaller ones. See Beyhom, A., 'Arabité et modernité en musique, ou de quel modèle se démarquer', Congrès des Musiques dans le monde de l'Islam, Assilah, August 8-13 2007: <http://www.mcm.asso.fr/site02/music-wislam/articles/Beyhom-2007.pdf>.

30 Accuracy to the 4th decimal is needed only for computational purposes as in practice anything under two cents is hardly noticeable.

31 The tense diatonic *genus* is the Western paradigm. There are many other diatonic *genera* known from the Greeks (and the Arabs) which we shall not discuss here.

32 For example, *Ṣabbāgh, T, a-d-Dalīl al-Mūsīqā al-Ām fi Aṭrab al-Anḡām*, (Aleppo, 1950).

33 *Ṣabbāgh*, in his *Dalīl...* (p. 29 for example), uses the terms 'flat plus one quarter' for the note e- in the scale of the mode *Rāst*, although

the intervals that surround it are different in size (6 HC and 7 HC).

34 Conceptual intervals represent qualities of intervals when used in a melody or a scale. Compared one to another, each has a unique and identifying quality which relies on its relative size. These compose the fourth, the fifth or the octave, and play a distinct role in performance, bearing in mind fluctuations and regional preferences which will be stressed for the degree *sīkā* in Arabian music for example, (fig. 7, p. 173) and identified by the performer as a semi-tone, a *mujannab*, or a one-tone interval, and so forth. The Arabian usage of the HC agrees with the adepts of Pythagoras who insisted in the Pythagorean approximation of the Arabian scale, instead of an Equal Temperament. The reason is that the odd number of HC in one tone (nine) and its distribution among the Pythagorean *limma* (4 HC - sometimes called 'minor' semi-tone) and the Pythagorean *apotome* (5 HC - sometimes called 'major' semi-tone) are good enough approximations and represent two different intervals whenever the *mujannab* in Arabian music, conceptually equivalent to one and single interval, may also be approximated to two intervals of slightly different sizes, i.e., 6 HC and 7 HC, which, when added, equate to the augmented second of the Western scales.

35 Elsewhere, b- may be used for b^b.

36 Depending on the transliteration and, or, on local pronunciations: *sīkā, segah, seh-gāh*, etc.

37 The positions of the notes in the *maqām*, including the fundamental, may vary slightly during performance. See Beyhom, A., [Une étude comparée sur les intervalles des musiques orientales], *Actes du colloque Maqām et création (Fondation Royaumont, October 2005)*, p. 18-24.: http://www.royaumont.com/fondation_abbaye/fileadmin/user_upload/dossier_PDF/programmes_musicaux/ COLLOQUE_MQAM_ET_CREATION_OCTOBRE_2005.pdf, and Beyhom, A., 'Dossier: Mesures d'intervalles - méthodologie et pratique', *Revue des Traditions Musicales des Mondes Arabe et Méditerranéen (RTMMAM N°1)*, (Baabda-Lebanon, 2007), pp. 181-235.

38 This and the following explanations are based on the author's own experience while practicing Lebanese folk tunes, as well as on interval measurements of performance examples in various modes including the degree *sīkā*; on thorough discussions with teachers of Arabian music (mainly on the 'ūd), and also on an extensive and systematic study of contemporary *maqām* theory in the Near and Middle-East. For the latter see Beyhom, A., *Systématique...*

39 The mode *Sīkā* traditionally begins with the note *sīkā*.

40 The two are commonly used both with Classical and Folk Arabian music in the Near-East.

41 Signell, Makam, K., *Modal Practice in Turkish Art Music*, (Nokomis-Florida, 2004). Turkish modern theory uses the HC approximation for its intervals. In practice, however, as Signell stresses (pp. 37-47) and the way in which many contemporary Turkish musicians perform (as underlined for Kudsi Erguner on *Nāy* or for Fikret Karakaya on *Kemençe* in Beyhom, A., *Étude comparée...*), the note *sīkā* tends to be played lower than its assigned value (that is e minus one *comma* in Turkish theory), notably in *maqām Rāst, Ṣabā* and *Bayāt*.

42 The difference between the mobile notes of Ancient Greek theory and the variable position of the single note *sīkā* lies in the fact that mobile notes may move from one position to another in the general scale, whilst the variability of the degree *sīkā*, for example, involves only one position in the general scale, which varies. An example of mobility is a change from pitch e to pitch e^b for example, when a minor tetrachord d e f g modulates into a Kurdish tetrachord (or also as the introductory tetrachord in the flamenco scale, starting with 'd': d e^b f g), while the position of *sīkā* may vary depending on a certain number of factors, but its relative positioning in the scale remains the same (it is still considered as the same intermediate pitch between e^b and e, or e-).

43 For example the *sīkā* in Lebanese Folk music is lower than the *sīkā* in...

44 The urge for such a concept is even more evident with music not responding, partially or completely, to temperament, such as we have with traditional *a capella* singing worldwide.

45 At least in *Urmawī's* concept of the scale: it is much later in the history of music theory that some theoreticians began using the Holderian comma as a measuring interval for approximating Pythagorean intervals, but this can not apply to theoreticians of the Middle Ages who dealt mainly, if not exclusively, with Pythagorean interval (or string) ratios for interval handling.

46 This means that a melody would not, in the modal or *maqām* music described in *Urmawī's* theories, move directly from one pitch to another, one *comma* apart, unless this process is used in performance as an intonation variation within the original melody (in which case the size of the *comma* is very approximate). This is still the case with Arabian music, but where the quarter-tone is the elementary interval of the 24-ET - see the example of *maqām Awj Āra* in Part II and note 90.

47 The explanation of the role of two consecutive *mujannab* lies possibly in the perception of this interval as being the result of the division of the one-and-a-half-tones interval in two smaller intervals. More information about this process can be found in Beyhom, A., *Approche systématique de la musique arabe: genres et degrés système, De la théorie à l'art de l'improvisation* (ed. Ayari, M.), (Paris, 2005), pp 65-114, in which case, any two *mujannab* in a row must add up, at least in theory, to the greater tone shown in figure 8, i.e., composed of 3 *limmata* and one *comma*: the only possibility for this is that the two *mujannab* be of different sizes.

48 Because three small intervals are necessarily bigger than a *mujannab*, which means that their sum must necessarily be equal to the one-tone Pythagorean interval, which stands next in the row of conceptual intervals.

49 One could also use the corresponding letters, for example S, M, T, etc., for the combination process: numbers have the same discriminating power, but have the advantage of allowing a quick check of the sum of the elementary intervals in the series.

50 This simple algorithm is used for computer combination process and is very efficient for larger interval series as, for example, a heptatonic scale: it is applied in a more elaborate formulation in the generative procedures used by the Modal systematics theory, which allow a complete survey of hyper-systems, systems and sub-systems as they shall be defined below.

51 Aristoxenus, *Elements...*, Barbera, A., Octave species..., aforementioned.

52 The additional *genera* of *Fārābī* are what I call the neutral tones genus (which is equivalent to the Arab *Bayāt*), and the original equal tones tetrachord: expressed in multiples of quarter-tones, the first *genus* can be represented by 3 3 4, or three-quarter-tones, three-quarter-tones, and one one-tone, intervals. In its essence, it is equivalent to the equal diatonic *genus* of Ptolemy with successive string ratios of 11/12, 10/11 and 9/10. For a general survey of Greek genera, see Barbera, A., Arithmetic and Geometric Divisions of the Tetrachord, *Journal of Music Theory*, Vol.21, N°2, (Yale, 1977), p. 294-323, notably pp. 296, 298, 302, 303, 307, and Mathiesen, Th., *Appolo's Lyre*, University of Nebraska, 1999, pp. 468-75. The second addition of *Fārābī*, the equal-division *genus* (or equal-tone division of the *genus*), is composed of three identical intervals each of which has a size of 5/6 tone (see *Fārābī, Kitāb al-Mūsīqī...*, pp. 58-9).

53 The total number of combinations is obtained through the formula $N!$, in which N is the number of intervals to combine. Here, we have $3!$ (or three factorial) which is equal to $3 \times 2 \times 1 = 6$. On the other hand, any rotation of three identical intervals would give the same redundant combination, like in a a, a, a , for example.

54 This process is used in statistical and probability algorithmic, which is historically a recent domain in science.

55 And for music theory as a whole.

56 This process is called the Wheel by Byzantine chant theoreticians. It is applied to intervals composing a fifth repeated in a row. See Giannelos, D., *La musique byzantine*, (Paris, 1996), p. 89, 'Le système de la roue'.

57 Which are named sub-systems in the Modal systematics theory.

58 Together as the first sub-system of the series.

59 There are other more sophisticated algorithms for interval combinations in computer mathematics but our main purpose is to remain as close as possible to an intuitive handling of intervals.

60 The hyper-systems, systems and sub-systems are, in the general case of statistical research on scales (in Modal systematics), generated with the help of a computer program based on an extended version of the algorithm shown in figure 9, p. 174.

61 Or one semi-tone, one tone and a half, one semi-tone: this tetrachord is equivalent to the tonic chromatic tetrachord of Aristoxenus, with the semi-tones placed on both sides of the one-and-a-half-tones interval.

62 This mode is also frequently used in gypsy music, and also with film music, notably the score by Maurice Jarre for Lawrence of Arabia (dir. David Lean, 1962).

63 This is the most homogeneous system among the three, with only two different classes of intervals used.

64 In the case of the latter music, scales are notated differently but are conceived as being the same as Arabian corresponding scales. This is too lengthy a subject to be treated here, but the reader can have more information in Beyhom, A., *Étude comparée...*

65 The question arising here is why, out of this great number of potential scales, traditional music around the world would use only a few? A first answer to this question was given in Beyhom, A., *Systématique modale...*, in which some of the criteria suitable to scale system in order to verify if they correspond to musical practice as we know it are identified, such as the presence of a fourth or fifth from the tonic, and/or the absence of particular scale combinations (such as combining two large intervals in a row, or more than two semi-tones in a row, etc.). Applying these conditions, as well as others, to the scales of the quarter-tone generation which can be made up, we can get close enough to the configuration of scales used today, particularly in Arabian music. Exceptions can be dealt with separately, and will give valuable information about this particular music, and, of others, and the additional criteria applying to it.

66 There is a relatively simple empirical formula for the calculation of the number of systems which can be generated by a hyper-system provided that the total Number of Intervals in one hyper-system is NI intervals, and that different classes of intervals contained in the hyper-system have a capacity O_i (each interval i is reproduced O_i times in the hyper-system), the number of distinct permutations of intervals within the hyper-system is equal to $(NI!)/(O_1! \times O_2! \times O_3! \text{ etc.})$. In the case of hyper-system $N^\circ 12$, interval 2 occurs five times, and interval 4 twice, by replacing in the formula we obtain the number of distinct sub-systems or $[(7!)/(5! \times 2!)] = [5040/(120 \times 2)] = 21$. The structure of the formula explains why homogeneity of the conceptual intervals composing a hyper-system, is a factor that lessens the number of resulting (non-redundant) sub-systems.

67 Some of the scales found in the literature are questionable: a review of Arabian scales is given in Beyhom, A., *Systématique modale...* (see for example Vol. III, pp. 15-50).

68 This is discussed in Beyhom, A., Des critères d'authenticité dans les musiques métissées et de leur validation: exemple de la musique arabe, *filigrane*, 5, (Paris, 2007), pp. 63-91.

69 And reciprocally.

70 Recent research was commented on at the *Congrès des Musiques dans le monde de l'islam in Assilah – Morocco*, in 2007. This showed a continuity of the 17 unequal intervals per octave model (or seven intervals in a just fourth and three in a one-tone interval), throughout the history of Arabian theory, beginning with *Kindī* (9th century). The 17-ET model is a simplification of the 17 unequal intervals scheme(s) and is conceptually equivalent to the latter. This applies equally to the 24-ET model used in the statistical study in Part II of this article with a limitation of the smallest conceptual interval to the semi-tone.

71 The internal structure of the fourth or of the fifth may differ within the 17 intervals to an octave model and the quarter-tone model, when considering possibilities other than the three intervals to the

fourth and four intervals to the fifth. Furthermore the 17th of octave model allows a differentiation between the chromatic *genera*, based on hyper-system 1 2 4 in the 17th of octave model, and the enharmonic genus which may be represented by the system 1 1 5.

72 In his *book of cycles*, *Urmawī* takes the fifth (as was the case in Ancient Greek theory which inspired him) as a fourth to which a one-tone interval is added. With this concept of the scale, a fourth plus a fifth amounts to the same as combining two tetrachords (in fourth) and a one-tone interval in the frame of one octave, which, in Modal systematics, is equivalent to the combination of three intervals (among which two are equal) with a fixed sum.

73 *Urmawī*'s concept is that there are two such intervals: the *comma* and the *limma*. In modern Arabian quarter-tone theory, these would be the quarter-tone and the semi-tone, respectively.

74 Differences of intonation may occur, but the interval remains conceptually the same.

75 Although some theoreticians may consider them as an exact expression of the size of the intervals.

76 As well as for an imposing other types of music.

77 Which may be combined in order to compose conceptual intervals.

78 The sizes of the greater and greatest tones in the 17-ET model suggest that the augmented second could be less, or greater than, the equal temperament tone and a half. The *ḥijāz* tetrachord (which today is usually made up of, in this order: one-semi-tone, one-tone-and-a-half, and one-semi-tone) is not mentioned in *Urmawī*'s list of tetrachords. This is very strange since this tetrachord is a combinatory variant of the old tonic chromatic Greek *genus* and commonly used in contemporary traditional music. Comparing sizes of the greater and greatest tones in the extended model, the difference between them would be one *comma*, which is the same difference existing between the *limma* and the smaller *mujannab* (or the equivalent of an *apotome*). However, the relative size of one *comma*, compared to one *limma* or one *apotome*, is very different from its relative size when compared to the greater and greatest tones. The difference, which is already difficult to hear between, for example, a double-*limma* and a Pythagorean tone (add one *comma* to the former to obtain the latter), would be even less distinguishable between the two larger intervals. On the other hand, *Urmawī* could not have used the *limma* between the greater and the greatest tones in order to differentiate them, as this would not have allowed for space, in the frame of a fourth, for two additional semi-tones (or *limmata*) in a tri-intervallic configuration (fig. 8, p174 – if we add one *limma* to the greater tone, the capacity of the greatest tone would have to be one *comma* plus four *limmata*. The capacity of the fourth in a Pythagorean 17 intervals model, is two *commata* plus five *limmata* – i.e., a difference of one *comma* plus one *limma*. This leaves no space for the two additional *limmata*). This is possibly the reason why *Urmawī* gave up the *ḥijāz* tetrachord in its two (three) potential Pythagorean expressions, which would have been (a) M+Ts+S or a succession of one small *mujannab* (*limma* + *comma*, or *apotome*) plus one greater tone (tone + *limma*) plus one semi-tone (*limma*), (a') S+Ts+M' or a succession of one semi-tone plus one 'greater tone' plus one *mujannab*, and (b) the regular succession of one-semitone (*limma*), greatest tone (tone + small *mujannab* – or *apotome*) and one-semi-tone (*limma*) intervals (or L + greatest tone + L).

79 Used in the composition of other intervals.

80 Furthermore, that the numbers in the scale series express, before all, the quality of the intervals.

81 As a general remark on *Urmawī*'s Pythagorean model, the sizes of neutral intervals, particularly in the *Book of cycles*, seem a bit far from their counterparts in music practice (and in *Fārābī* and *Sinā*'s theories). Owen Wright has explored this at length in his *The Modal system...* We have shown, however, that *Urmawī*'s concept of the scale is not tonometric. It is qualitative. This is why the quantitative values of the intervals should not be taken into consideration for practice. Only their qualitative values should, of which the most important being the *mujannab* which lies somewhere (in size) in-between the one-*limma*

and the one-tone intervals.

82 Helmholtz, H., *On the Sensations of tone...*, p. 250.

83 Free Jazz or contemporary Western music break away from this principle and try to explore all the possibilities of sound. These attempts, although sometimes memorable, were never popular. It could be that music has an emotional power which may not exist with other forms of art, and that this emotion is induced by a process of reminiscence, predominant in music because of the long-term impossibility of recording it.

84 See Beyhom, A., *Étude comparée...* and *Des critères d'authenticité...*, notably, for the last reference (pp. 76-82), in connection with modal heterophony.

85 If intervals are not too small for reasons of clarity, or too large, there are no longer any scale, or any pattern for the melody.

86 The terms fourth, fifth and octave are in quotes because, in the statistical study, all possible compositions of these containing intervals are considered, i.e., with more than, or less than, three (or four, or seven) conceptual intervals to a just fourth (or to a just fifth, or to an octave).

87 Turkish music and Byzantine chant follow roughly the same rules as Arabian music. They used the *maqām* as a *lingua franca*. The Turkish model is an extension of *Urmawī*'s scale which might be better adapted to transpositions for the long necked *ṭunbūr*, and in the Chrysanthos of Maditos' version of 1818, Byzantine chant follows a 17-ET paradigm (extended as a measuring scale to the 68th of an octave, i.e., by dividing the 17th of an octave in four equal parts – called minutes). In the 1881 Byzantine version of the *Commission of Constantinople*, a 24-ET model, had each quarter-tone being further divided in three equal measuring intervals (or a semi-tone equal temperament, with each of the semi-tones divided in six equal minutes, resulting in a 72-ET model). In both types, conceptual intervals remain equivalent to those in Arabian music, with the greatest conjunct tone, in the enharmonic *genus*, of Byzantine chant equivalent to the greatest tone in *Urmawī*'s model. On the other hand, Ancient Indian music follows the same concept of interval quality because with the principle of 22 unequal *śrutis* the conceptual intervals are the result of a theoretical concatenation of smaller intervals, which are themselves elementary and auxiliary intervals. For an overview of the tonal systems of Indian music, see for example Powers, H. & Widdess, R., India, §3, 1, *The New Grove - Dictionary of Music and Musicians*, 12 (ed. Sadie S.), (Oxford, 2001), pp. 170-178. All these subdivisions of the scale, along with those of Javanese and other music, should be explored in detail in future publications.

88 *Maqām Ḥijāz-Kār* traditional beginning (and reference) note is *rāst*, commonly considered as equivalent to the Western note c.

89 This was mainly spread through the collective *Recueil des Travaux du Congrès de Musique Arabe qui s'est tenu au Caire en 1932* (Hég.1350), (Cairo, 1934), and Erlanger, R., *La musique arabe – Tome cinquième: Essai de codification des règles usuelles de la musique arabe moderne – Échelle générale des sons – Système modal*, (Paris, 1949).

90 For a detailed study of this problem, see Beyhom, A., *Systématique modale...* www.beyhom.com/download/thesis/pdf/analyse_ajw_ara.pdf. Furthermore, in a live performance, the author has heard only once in his lifetime an Arabian version of *maqām Awj-Āra*. This was played by a Moroccan lutenist named Saïd Chraïbi, in 2005. In a private conversation, the musician explained that he used the scale of *Awj-Āra* as given in Erlanger because he could not get a hold on a recorded Arabian version of this *maqām*. Chraïbi had already made at least two recordings including this *maqām*, which I later acquired under the titles *Souleimane* and *Taquassim Aoud*, with no references or commercial identification.

91 This instrument is the main reference in both theory and practice for Arabian music and musicians. It is commonly tuned in ascending fourths with an additional (lowest) variably tuned string. This string is sometimes tuned to e- whilst performing *maqām Sīkā*.

92 The drones are sometimes used to accentuate the role of a structural note of a particular scale.

93 The tuning of the *`ūd* is difficult and time consuming. One musician has confided to the author and other participants, during a work

shop at Royaumont, and probably with some exaggeration, that he had probably spent half of his twenty years of professional career tuning the *ūd*. See Beyhom, A., *Étude comparée...*

94 Mostly when coming back to the fundamental as a resting note.

95 Bearing in mind that the size of this interval may be, in performance, greater or lesser than the exact one-and-a-half-tones.

96 A scale element, here, is equivalent to a succession of conjunct intervals forming a containing interval. The minimal possible succession is made up of two intervals. The statistical study of the octave containing element (infra) shows sometimes the results for one single interval (NI = 1), to show symmetry with (NI = 12).

97 The full database of the hyper-systems, systems and sub-systems of the heptatonic scales in the quarter-tone model, with the limited alphabet of intervals, can be found in Appendix G, p. A35).

98 We shall use the terms fourth, fifth and octave henceforth, bearing in mind that the number of intervals in these containing intervals is variable, and represented by NI. The term just for each of these intervals is to be considered as an implicit quality.

99 These two intervals are taken as a successive one-tone and a one-and-a-half-tones.

100 This filter is one of the aesthetic criteria deduced from contemporary Arabian music and from *Urmawī's* model (which forbids two consecutive conceptual semi-tones). However, they do not necessarily apply, in the case of the fourth, to all modal music.

101 There must be no exception for the tetrachord 622 Erlanger recognises as *Sipahr* (see Erlanger, R., *La musique Arabe – Tome 5...*, p. 91). In his note to his first volume, p. 30, Erlanger says, that he felt this *genus* (which is the old tonic chromatic *genus* of Aristoxenus), should be included among other Arabian *genera*. In Volume 2 (Tome 2) of the same book (published in 1935), at p. 276, Erlanger (or Snoussi, his secretary, see Poché's introduction to the second edition of 2001) explains, nevertheless, that 'In *genera* theory, the most sensitive matter is the order in which the intervals decomposing the fourth in melodic sounds are placed, in relation to one another. With Arabian music, or at least in its urban form, that may be called classical, there is no occurrence of two consecutive semi-tones in the same tetrachord...'

102 See Powers, H. (e.a.), 2001, Mode, *The New Grove -Dictionary of Music and Musicians*, vol.16, (ed. Sadie, S.), (London,2001), pp. 775-860, sub Mode, §V, 3: Middle East and Asia: *Rāga* – (ii) Modal entities and the general scale, p. 838: 'There are a few evident parallels between South Asian and West Asian orderings of modal complex and general scale. For instance, in both cases a given modal entity will use only some of whatever pitch positions an octave span of the general scale makes available – in principle seven – and normally no more than two intervals of the semi-tone class will occur in a succession in a single modal complex.'

103 In multiples of the quarter-tone. These are hyper-systems three and 4 (for NI=2) in figure 23, bottom.

104 Sub-systems having intervals larger than the largest conceptual second (the greatest tone – in both models taken as equal to one tone and a half) are marked with a postpositioned '§' and kept 'as is', even when the conjunct large intervals filter is applied. However, their number is shown for each case (for each value of NI) in the Conjunct big intervals row.

105 The smallest NI giving the largest number of sub-systems, after eliminating sub-systems that do not comply with the aesthetic criteria listed in figure 23, 182.

106 This is equivalent to a generation with the limited alphabet of 1, 2, 3 in the semi-tone generation, and to 2, 3, 4, 5, 6 in the quarter-tone model.

107 The small conjunct different intervals criterion has no effect on the results of the semi-tonal generation.

108 This is because in order to generate redundant sub-systems, a system must contain a repetitive pattern, for example 112 (in the semi-tone multiples) in the 112112112 scale (an octave scale for which NI=9, and the sum of the conjunct intervals S=12) in the semi-tone model – there, of all the sub-systems generated by a de-ranking process,

only the first three are distinct. In other terms, and for a system to be able to generate redundant sub-systems, there must exist a suite of J integers repeated i times (this means that both i and J must necessarily be divisors of NI, as $i \times J = NI$) in such a way that $i \times (\sum N_j) = S$ (N_j is the size of an interval within a repetitive suite), in which i, j, N_j and S are integer numbers, with $1 \leq i \leq S$, $1 \leq j \leq S$, $1 \leq N_j \leq J$ and $1 \leq \sum N_j \leq S$, i and $\sum N_j$ being divisors of S, and i and J being divisors of NI. This formula is easily verified for the 112112112 scale above, as we repeat three (or i) times the suite of three (or J) intervals 112 whose sum $\sum N_j$ is equal to four, the whole of which verifies $3 \times 4 = 12$. However, if NI is equal to seven (if the scale be heptatonic), there simply exists no possibility of finding two integer numbers i and J which verify this formula, for both S=12 (semi-tone model) or S=24 (quarter-tone model), because NI is in this case a prime number that can be divided uniquely by itself or by one, and seven is not a divisor of twelve (of the sum S). In the case of the fourth in the quarter-tone model, and for NI=3, NI is once again a prime number that does not divide the sum S=10 (quarter-tones), neither does it divide S=5 (semi-tones). For the fifth, as is shown in figure 33, p. 189, the usual four intervals in the fifth generate independent (distinct) sub-systems only in the case of the semi-tone model, as NI=4 and S=7, and neither of the divisors of NI (i.e., the numbers 1, 2 and 4), except the trivial case 1, divides seven. In the quarter-tone model, however, S=14, and 2 divides fourteen so we may be able to find a suite of two (J) intervals repeated twice (i) systems provided that the sum of the two repeated intervals be equal to $14/2=7$ (or S/i); this is verified for the suites 43 (or a one-tone interval followed by a three-quarter-tone interval) or 34 repeated twice, as is shown in Appendix B (see hyper-system N°7, system N°2, p. A12) to the present article.

109 We do not count here the non redundancy criterion, as this filter is self-evident.

110 See Mathiesen, Th., *Appolo's Lyre...*, p. 49: 'If the interval between the *lichanos* and the *hypate* is smaller than the interval between the *lichanos* and the *mese*, the smaller interval is called a *pycnon*...'; and Mathiesen's figure 51, p. 313. The author gives the *pycnon* a range of 5 quarter-tones, although this would apply to the low diatonic *genus* of Aristoxenus, and the *pycnon* would then be equal to its complement in the just fourth. The *genus* with the greatest *pycnon* with Aristoxenus is the whole-tone color, the tense chromatic *genus* in Mathiesen with a *pycnon* (composed of the smallest two intervals) equal to the one-tone interval, i.e., smaller than its complement within a fourth. The smallest *pycnon* occurs, according to Aristoxenus, in the enharmonic *genus*, with a sum value of 2 quarter-tones.

111 Another possibility is that this was not accurately translated and explained until now.

112 This corresponds to a tree-like generative process with additional intervals chosen among the alphabet in order to comply with the homogeneity rule. Sums are checked afterwards to verify if the fifth is reached.

113 See Beyhom, A., *Systématique...*, Vol.3, pp. 7-13, and Appendix B – these pentachords are either rarely used, or are doubtful.

114 Most of these do not leave way for a possible combination of two tetrachords and a one-tone interval. The remaining set, i.e., 3524262, 3434262, 2624253, 2624343 and 2624352, are probably in tune with the aesthetic criteria of Arabian music, but may be difficult to perform on the *ūd* (for non-virtuoso performers) in its usual tuning (mainly in ascending fourths). See Beyhom, A., *Approche systématique...*

115 Also to clarify the effect of each criterion on the outcome of the generative process.

116 See Barbera, A., *Arithmetic and Geometric Divisions ...* p. 311.

117 The complete alphabet is, in multiples of the eighth of a tone, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, with the last value (16) representing the di-tone, which is the largest possible interval (in the frame of a fourth and with the semitone as smallest conceptual interval) in this model: for a complete listing of the results, see Appendix A.

118 The full results for the semi-tone model can be found in Appendix C.

119 The no conjunct semi-tones criterion applies to suites of

three or more semi-tones in a row.

120 Complete results in Appendices B and C.

121 Including redundant sub-systems in the optimal case for NI=4, which differs from all other above seen optimal cases.

122 Performers find it difficult to memorise more than a few dozens heptatonic scales, even when they are classified with the beginning genera and further combinations in Arabian theory. Modes may be taken as belonging to a family whose main characteristic is determined by the lowest tetrachord – this is a method which makes it easier to remember *maqāmāt*. However, this consists only of some 30 basic scale combinations. If such an arsenal is needed in order to memorise 30 scales, it seems clear that memorising 70 pentachords, with a subsequent and much greater number of octave scales, is simply an impossible task for the common musician.

123 This is a well known process in Ancient Greek and in Arabian theory. An example is given in details in Beyhom, A., *Systématique modale...*, pp. 301-312.

124 These intervals are too large, since they are greater than the one-and-a-half-tones interval and as such cannot be considered, ultimately, as conceptual intervals. However, the aim of the statistical study consists partly in determining the boundaries of the alphabet of these conceptual intervals.

125 The self-evident case for NI=1 appears only in this preliminary generative process.

126 As a general rule, scales with a NI as a prime number may not generate redundant sub-systems unless NI divides the sum of elementary elements within the scale (12 for the semi-tone model and 24 for the quarter-tone model. This is due to the characteristics of these numbers as explained in note N°108. For NI=2, with 2 being the second prime number (which divides itself and 1 only after 1 (NI=1 is a trivial case), two divides twelve and twenty four. As a consequence, there is a fully redundant system for NI=2, composed of two tri-tones (6 6 in semi-tones, or 12 12 in quarter-tones – for the latter, read twelve and twelve). The same applies for NI=3, 4 or 6, with hyper-systems 4 4 4, 3 3 3 3 and 2 2 2 2 2 2 in the semi-tone model.

127 Starting with these graphs, systematic comparison is undertaken between the two models (semi-tone and quarter-tone).

128 From this point on, only generations with a restricted alphabet are shown in the body of the article. For generations with the complete alphabets, with the exclusion of the one quarter-tone interval for the quarter-tone model, see Appendix E.

129 In the quarter-tone sub-system 3(244)362, for example, the fourth in second position (the 244 in brackets) is the first fourth of (244)3623, which is the next sub-system resulting from the deranking process.

130 This means that for each sub-system having a direct fifth, there is always a corresponding sub-system (which is obtained by de-ranking four times the initial sub-system with the direct fifth) with a direct fourth. In the previous sub-system (see note 129), the direct fourth is 244 with for complement 3623 fifth. De-ranking three times (*i.e.*, beginning with 2443623, 4436232, 436232, 3623244) we get a sub-system with a direct fifth 3623, but not necessarily a direct fourth. If we de-rank four times the last sub-system beginning with a direct fifth, we get the initial sub-system 2443623 with a direct fourth, but no direct fifth. Consequently, the number of sub-systems containing a direct fifth (in any of the generations shown) is equal to the number of sub-systems containing a direct fourth.

131 The extension from two to three semi-tones (in a row) in this filter allows for the existence of bi-fourth configurations (within a scale) with bordering semitones, for example two *hijāz* conjunct tetrachords – or (2 6 2) (2 6 2).

132 This filter is more permissive than the one used for the fourth and the fifth, due to the fact that some (very few, and mostly questionable) scales found in literature include conjunct one-tone and one-and-a-half-tones intervals – see Beyhom, A., *Systématique modale...*, Vol. III, notably pp. 33, 38 and 42.

133 I do not have yet a convincing explanation for this phenomenon,

and this question remains open for further discussions.

134 The shape of the broken lines representing distributions of NS and NSSU_NR in figure 38 can be compared to a regular normal (law of) distribution in statistical studies, or bell-type distribution – the results do not correspond, however, to the analytical expression of that law.

135 See Appendix D for a full list of hyper-systems for this generation, together with the numbers of systems and sub-systems in each hyper-system.

136 For NI=8 (where as the first hyper-system is 1 1 1 1 1 1 1 5) we may not use intervals larger than the five semi-tones, whenever we may use intervals as large as the nine-semi-tones interval for NI=4 (where the first hyper-system is 1 1 1 9 – or three one-semi-tone and one nine-semi-tones adding up to 4 conceptual intervals the sum of which equals to twelve half-tones, or the octave).

137 This *lacuna* is due to my own limitations in statistical and mathematical sciences. Any explanation of the phenomena by a specialist in this field would be greatly appreciated.

138 Because NSS is, by definition, equal to NS x NI.

139 These redundant sub-systems are useless in the traditional concept of modal music.

140 The use of the Unitary Number of Non Redundant Sub-Systems in the previous generations for the fourth and the fifth would have emphasized the optimum at NI=3 for the fourth, and at NI=4 for the fifth. The lesser numbers of results for the previous generations have allowed us, however, to try to go deep inside the structure of the fourth and the fifth, without having recourse to the weighted variables used for the octave generations. In the latter case, it would be too long a task because of the very important numbers of sub-systems involved. See Beyhom, A., *Systématique modale...*

141 In this system, three out of four sub-systems obtained by de-ranking are redundant. Consequently, this makes of it the unique sub-system.

142 With a limited alphabet, the two pentatonic hyper-systems come to 12333 and 22233 (see Appendix D). Only the last one allows for a simultaneous direct fourth and fifth configuration, or fourth in a fifth. In this case, the chosen alphabet can for example be extended to the di-tone (4), a step permitting the usage of four additional hyper-systems (namely 11244, 11334, 12234 and 22224), and multiplies by five the reservoir of systems (22224 is a poor candidate in this case as it generates one single system), and by four the reservoir of sub-systems which include a fourth in a fifth.

143 These results encourage questions about properties of numbers and their relations with the models in use.

144 Redundant sub-systems have a limited role in the quarter-tone model. Their weight in proportion to the total number of sub-systems is around 0,5%, whenever it is around 3% for the semi-tone model (with the exception for NI=12, with all sub-systems except that the original sub-system which is redundant). The qualitative results (optimal placements) are consequently not strongly affected by this criterion, in particular for the quarter-tone model, particularly for NI=7 (no redundancies).

145 The results in the following figures relate only to the restricted alphabet in order to give the most pertinent information. Graphic results for generations with the full alphabet are shown for both models, in Appendix E. Synoptical results for the quarter-tone model (full alphabet) are listed in Appendix F.

146 The full alphabet generation shows a maximal NSSU_NR value for N=6. All other optimal (NSS5U_NR and FFU_NR) occur for NI=7 – see Appendix E, figure A1, p. A28. In all the graphs, some of the results are corrected to the first decimal place.

147 The (almost, as the smallest interval is the semi-tone) full alphabet generation shows a steady optimal value for NI=7, shared in the case of FFU_NR with NI=8 – see Appendix E, figure A5, p. A30.

148 Semi-tones combine easily in the one-tone interval, as well as in the fourth or the fifth – see also figure 35 and section 'Discussing the preliminary results' above. 149 Ancient Arabian theory and practice seem to exclude these as well – other Ancient forms

of music must still be thoroughly checked for conformity with this criterion.

150 The full alphabet generation in semi-tones shows that optimal values occur for NI=6, except for NSSU_NR for which the optimal case is NI=5 (pentatonic scales) – see Appendix E, figure A3, p. A29.

151 The (almost) full alphabet generation in quarter-tones shows that the optimal values occur for NI=7 for NSSU_NR, and NI=8 for the other variables – see Appendix E, figure A7, p. A31

152 With the quarter-tone model, sub-systems for NI greater than nine subsist principally because of the possibility to use the three quarter-tone interval in conjunction with the semi-tone (for example combinations such as 223, 232, and 223): these combinations were excluded for the generations in just fourth, notably with the Conjunct small intervals criterion or through the homogeneity rule.

153 All optimal values of the full alphabet generation in semi-tones occur for NI=7 – see Appendix E, figure A2, p. A28.

154 The almost full alphabet generation in quarter-tones shows optimal values for NI=7, except for FFU_NR (or DQQU_NR) with NI=8 – see Appendix E, figure A6, p. A30.

155 Beginning with NI=10, the largest interval is one single one-and-a-half-tones interval – see Appendix D for details about the internal structure of hyper-systems for these generations for the semi-tone model.

156 This filter excludes sub-systems containing sequences of three or more conjunct semi-tones as well as sub-systems with two conjunct intervals equal to or greater than 3 (semi-tones) or 6 (quarter-tones).

157 The full alphabet generation shows that optimal values occur for NI=6 – see Appendix E, figure A4, p. A29.

158 The almost full alphabet generation shows that all optimal values occur for NI=7 – see Appendix E, figure A8, p. A31.

159 Refining filters for the quarter-tone model, for example, in order to verify better adequacy to the heptatonic model, setting the value of the largest interval of the alphabet to the 5 quarter-tones while testing for large conjunct intervals (this would tighten the results around NI=7), or by applying the conjunct small intervals criterion already used for generations within the ‘one fourth’ containing interval, or still by verifying the conformity of heptatonic sub-systems to the criteria of transitional two-interval semi-tones. This last one keeps only two-interval, and excludes three-interval conjunctions of semi-tones which occur on the transition from a fourth to a fifth, or from a fourth to another fourth, or from one octave to the other – see also the next note.

160 Other models include the ‘lo-go’ generations, with Lower than the Octave, or Greater than the Octave, sums for the sub-systems and models, etc. This can be equivalent to models of the octave in, for example, 23, 22, 21, or 25, 26, 27, etc. equal-temperament divisions of the octave). See Beyhom, A., *Systématique modale...* This generation confirms the adequacy of heptatonism in relation with the interval characteristics of modal music, notably in the domain of *maqām*.

161 Uncomposed within the containing interval, although measurable with the help of elementary, and measuring, intervals.

162 Notably acoustic.

163 Except for the quarter-tone model for the fifth containing interval, in which redundancies, although very limited, occur: we have seen that this model fits better the fourth containing interval, with the homogeneity rule leading to unique (non-redundant) *genera*, which represent all the common *genera* in Arabian music (the last one including all *genera* based on semi-tone classes of intervals).

164 Because of the possible disagreement between the harmonics which compose, in different proportions, their spectrum, or because of extra-musical reasons, sometimes linked to their numeric properties.

165 In the acoustic meaning.

166 If octave intervals were explored at that time.

167 The fourth or the fifth, for example.

168 In our geographical example: the particularities of the landscape.

169 One might decide to walk (or ride) through different villages.

170 It is not an exact temperament that is used by the performer.

171 Whenever we stop at a pitch other than the original beginning one, making the former, permanently or momentarily, the basis for new developments of the melody.

172 Mainly in its improvisation form in the 20th century.

173 This state of balance is reached by the performer depending on his ability to (1) identify these slightly different intervals, and/or (2) reproduce them with his voice or his instrument.

174 For example *maqām Ṣabā* in Arabian music, the scale of which may be expressed as d e- f gb a’ b^b c[#]; the upper octave is generally different from the lower one, and occurrences for d’ are exceptional (commonly, the transition from the first to the second octave uses d[#] e’).

175 This explains why, as an aesthetic choice, performers who are well aware of the importance of the three consonant containing intervals may deliberately ignore them in order to obtain a different combination (such as avoiding the octave in *maqām Ṣabā* – see previous note).

176 For example: (1) use relatively large intervals within a containing interval, (2) avoid the consonances of fourth, fifth or the octave, (3) use a certain number of conjunct semi-tones in a row, etc.

177 Or whenever this simultaneous emission of more or less parallel melodic patterns was part of the local culture – the hypothesis developed in this paragraph does not necessarily apply to this type of music as for example the ‘Are’Are music of the Solomon Islands, but may apply to improvised polyphonic music, in which the freedom of expression with the single performer is replaced by the freedom of vertical improvisation within a party of musicians. The hypothesis is that, even in the latter case, a preliminary process of scale calibration, as the one explained in the text above, is at the origin of heptatonic scales (if used in that particular music).

178 Octatonic or enneatonic scales found in some literature may also be the result of the inclusion of modulation variants for a scale, or for part of it, at least in musical theory.

APPENDICES to the article ‘A NEW HYPOTHESIS FOR THE ELABORATION OF HEPTATONIC SCALES AND THEIR ORIGINS’[♦]

BY: AMINE BEYHOM*

[♦] The original article is entitled ‘A new hypothesis for the elaboration of heptatonic scales and their origins’, *ICONEA 2008* – (2009 – London, ICONEA Publications) Dumbrill, R., and Finkel, I., (eds) 2010: *Proceedings of the International Conference of Near Eastern Archaeomusicology held at the British Museum – December 4, 5 and 6 2008*: pp. 151-205.

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APPENDIX A: SCALE ELEMENTS IN EIGHTHS OF THE
TONE, WITHIN THE CONTAINING INTERVAL OF THE
FOURTH (= 20 EIGHTHS)

Appendix A: Scale elements in eighths of the tone, within the containing interval of the fourth (= 20 eighths)

Legend:

>: large intervals

*: double semi-tone criterion

Bold: commonly used genera in Arab music

Italics and ~~stricken~~: redundant combinations

Underlined: semi-tone equivalent combinations

ni = 2

4 16>, 16 4>

5 15>, 15 5>

6 14>, 14 6>

7 13>, 13 7>

8 12, 12 8

9 11, 11 9

10 10, ~~10 10~~

ni = 3

4 4 12*, **4 12 4**, 12 4 4*

4 5 11, 5 11 4, 11 4 5

4 11 5, 11 5 4, 5 4 11

4 6 10, **6 10 4**, 10 4 6

4 10 6, 10 6 4, 6 4 10

4 7 9, 7 9 4, 9 4 7

4 9 7, 9 7 4, 7 4 9

4 8 8, **8 8 4**, **8 4 8**

5 5 10, 5 10 5, 10 5 5

5 6 9, 6 9 5, 9 5 6

5 9 6, 9 6 5, 6 5 9

5 7 8, 7 8 5, 8 5 7

5 8 7, 8 7 5, 7 5 8

6 6 8, **6 8 6**, **8 6 6**

6 7 7, 7 7 6, 7 6 7

ni = 4

4 4 4 8*, 4 4 8 4*, 4 8 4 4*, 8 4 4 4*

4 4 5 7*, 4 5 7 4, 5 7 4 4*, 7 4 4 5*

4 4 7 5*, 4 7 5 4, 7 5 4 4*, 5 4 4 7*

4 5 4 7, 5 4 7 4, 4 7 4 5, 7 4 5 4

4 4 6 6*, 4 6 6 4, 6 6 4 4*, 6 4 4 6*

4 6 4 6, 6 4 6 4, ~~4 6 4 6~~, ~~6 4 6 4~~

4 5 5 6, 5 5 6 4, 5 6 4 5, 6 4 5 5

4 5 6 5, 5 6 5 4, 6 5 4 5, 5 4 5 6

4 6 5 5, 6 5 5 4, 5 5 4 6, 5 4 6 5

5 5 5 5, ~~5 5 5 5~~, ~~5 5 5 5~~, ~~5 5 5 5~~

ni = 5

4 4 4 4 4*, ~~4 4 4 4 4~~*, ~~4 4 4 4 4~~*, ~~4 4 4 4 4~~*, ~~4 4 4 4 4~~*

APPENDIX B: FIGURES OF THE COMBINATION PROCESS
FOR A JUST FIFTH, WITH $N_I=4$ (QUARTER-TONE MODEL)
AND A REDUCED ALPHABET OF INTERVALS (FROM THE
SEMI-TONE, OR '2' QUARTER-TONES, TO THE ONE-AND-
A-HALF-TONES INTERVAL, OR '6' QUARTER-TONES)

Appendix B: Figures of the combination process for a just fifth, with ni=4 (quarter-tone model) and a reduced alphabet of intervals (from the semi-tone, or ‘2’ quarter-tones, to the one-and-a-half-tones interval – or ‘6’ quarter-tones)

Remarks:

ns: number of systems

**: conjunct semi-tones*

§: scale elements that contain ‘conjunct big intervals’, i.e., at least two conjunct intervals that are bigger or equal to the one-tone interval, and among which one is at least bigger than the tone

Bold font: main pentachords in use in Arab music

C: scale elements to which a semi-tone must be added in order to complete the Hijāz tetrachord (‘262’) – logique de construction par genres accolés

Italic font: pentachords that, to pur knowledge, are not in use in Arab music

~~*Italic font and stricken: redundant pentachords*~~

underlined with gray background: semi-tone-compatible pentachords

1st column: number of the hyper-system, intervallic composition of the hyper-system, and number of systems (‘ns’) related to it

2nd column: number of the hyper-system (‘N° hyp.’)

3^d column: number of the system (‘N° sys.’)

4th column: number of the sub-system or ‘pentachord’ (‘N° pent.’)

5th column: intervals of the sub-system in integer multiples of the quarter-tone (‘value’)

Hyper-system	N° hyp	N° sys	N° pent	value
N° 1 2 2 4 6 ns: 3	1	1	1	<u>2 2 4 6[§]</u>
	1	1	2	<u>2 4 6 2†</u>
	1	1	3	<u>4 6 2 2[§]</u>
	1	1	4	<u>6 2 2 4[*]</u>
	1	2	1	2 4 2 6^C
	1	2	2	4 2 6 2
	1	2	3	<u>2 6 2 4</u>
	1	2	4	<u>6 2 4 2</u>
	1	3	1	<u>2 6 4 2</u>
	1	3	2	<u>6 4 2 2[§]</u>
	1	3	3	<u>4 2 2 6[*]</u>
	1	3	4	<u>2 2 6 4[§]</u>

Hyper-system	N° hyp	N° sys	N° pent	value
N° 2 2 2 5 5 ns: 2	2	1	1	2 5 2 5
	2	1	2	5 2 5 2
	2	1	3	2 5 2 5
	2	1	4	5 2 5 2
	2	2	1	2 5 5 2 †
	2	2	2	5 5 2 2 [§]
	2	2	3	5 2 2 5 [§]
	2	2	4	2 2 5 5 [§]

Hyper-system	N° hyp	N° sys	N° pent	value
N° 3 2 3 3 6 ns: 3	3	1	1	2 3 3 6
	3	1	2	3 3 6 2
	3	1	3	3 6 2 3
	3	1	4	6 2 3 3
	3	2	1	2 3 6 3
	3	2	2	3 6 3 2
	3	2	3	6 3 2 3
	3	2	4	3 2 3 6
	3	3	1	2 6 3 3
	3	3	2	6 3 3 2
	3	3	3	3 3 2 6 ^c
3	3	4	3 2 6 3	

Hyper-system	N° hyp	N° sys	N° pent	value
N° 5 2 4 4 4 ns: 1	5	1	1	2 4 4 4
	5	1	2	4 4 4 2
	5	1	3	4 4 2 4
	5	1	4	4 2 4 4

Hyper-system	N° hyp	N° sys	N° pent	value
N° 4 2 3 4 5 ns: 6	4	1	1	2 3 4 5 ^f
	4	1	2	3 4 5 2 ^f
	4	1	3	4 5 2 3 ^f
	4	1	4	5 2 3 4
	4	2	1	2 3 5 4 ^f
	4	2	2	3 5 4 2 ^f
	4	2	3	5 4 2 3 ^f
	4	2	4	4 2 3 5
	4	3	1	2 4 3 5
	4	3	2	4 3 5 2
	4	3	3	3 5 2 4
	4	3	4	5 2 4 3
	4	4	1	2 4 5 3 ^f
	4	4	2	4 5 3 2 ^f
	4	4	3	5 3 2 4
	4	4	4	3 2 4 5 ^f
	4	5	1	2 5 3 4
	4	5	2	5 3 4 2
	4	5	3	3 4 2 5
	4	5	4	4 2 5 3
4	6	1	2 5 4 3 ^f	
4	6	2	5 4 3 2 ^f	
4	6	3	4 3 2 5	
4	6	4	3 2 5 4 ^f	

Hyper-system	N° hyp	N° sys	N° pent	value
N° 6 3 3 3 5 ns: 1	6	1	1	3 3 3 5
	6	1	2	3 3 5 3
	6	1	3	3 5 3 3
	6	1	4	5 3 3 3

Hyper-system	N° hyp	N° sys	N° pent	value
N° 7 3 3 4 4 ns: 2	7	1	1	3 3 4 4
	7	1	2	3 4 4 3
	7	1	3	4 4 3 3
	7	1	4	4 3 3 4
	7	2	1	3 4 3 4
	7	2	2	4 3 4 3
	7	2	3	3 4 3 4
	7	2	4	4 3 4 3

____ Interval '2' occurs 80 times (included 4 times in redundant sub-systems)

____ Interval '3' occurs 72 times (included 4 times in redundant sub-systems)

____ Interval '4' occurs 60 times (included 4 times in redundant sub-systems)

____ Interval '5' occurs 44 times (included 4 times in redundant sub-systems)

____ Interval '6' occurs 24 times (not in redundant sub-systems)

Nota: the total number of pentachords is 72, of which 4 are redundant

APPENDIX C: COMPLETE RESULTS OF THE SEMI-TONE
GENERATION WITHIN A CONTAINING INTERVAL OF
THE FIFTH

Appendix C: Complete results of the semi-tone generation within a containing interval of fifth

(The results were obtained through the computer program modes V. 5.2)

Remarks:

- ni == number of conceptual intervals per system
- $5tes$ == test on the presence of a just fifth beginning with the first interval
- $4tes$ == test on the presence of a just fourth beginning with the first interval
- D_QQ == test on the presence of a just fourth AND a just fifth beginning with the first interval
- UM == 'ultra-min' criterion for the detection of suites of three (or more) semi-tones ('1') in a row
- min == 'min' criterion for the detection of suites of two semi-tones ('1') in a row
- max == 'max' criterion for the detection of suites of two (or more) intervals equal or superior to 'it_maxc' (the latter's value is set for this generation to '3' semi-tones)
 - Additional remark for the last three criteria: these are equally effective for the detection of intervals in a double-fifth (the checked system is the double fifth composed of two identical sub-systems); for the results shown in the article (Fig. 33), the results were subsequently adapted for a single fifth
- $n^\circ hyp.$ == rank of the hyper-system for the current ni
- $n^\circ sys.$ == rank of the system for the current hyper-system
- $n^\circ s-sys.$ == rank of the sub-system for the current system
- $imin$ == smallest interval used for the generation
- $imax$ == largest interval used for the generation
- redundant sub-systems for $ni=7$ are in **bold**
- occurrences de l'intervalle '*': number of times interval '*' is detected in the sub-systems for the current ni

Base de données sous-systèmes: CALCUL N° 1 ni = 2 imin = 1 imax = 7 it_maxc = 3

hyper n° 1 ; val.: 1 6
 sys.: 1 ; 5tes: 2 ; 4tes 0 ; D_QQ 0

n° hyp.	n° sys.	n° s_sys.	valeur	5tes	4tes	D_QQ	UM	min	max
1	1	1	1 6	oui	non	non	non	non	non
1	1	2	6 1	oui	non	non	non	non	non

hyper n° 2 ; val.: 2 5
 sys.: 1 ; 5tes: 2 ; 4tes 1 ; D_QQ 1

n° hyp.	n° sys.	n° s_sys.	valeur	5tes	4tes	D_QQ	UM	min	max
2	1	1	2 5	oui	non	non	non	non	non
2	1	2	5 2	oui	oui	oui	non	non	non

hyper n° 3 ; val.: 3 4
 sys.: 1 ; 5tes: 2 ; 4tes 0 ; D_QQ 0

n° hyp.	n° sys.	n° s_sys.	valeur	5tes	4tes	D_QQ	UM	min	max
3	1	1	3 4	oui	non	non	non	non	oui
3	1	2	4 3	oui	non	non	non	non	oui

_____ occurrences de l'intervalle: 1 = 2
 _____ occurrences de l'intervalle: 2 = 2
 _____ occurrences de l'intervalle: 3 = 2
 _____ occurrences de l'intervalle: 4 = 2
 _____ occurrences de l'intervalle: 5 = 2
 _____ occurrences de l'intervalle: 6 = 2

Base de données sous-systèmes: CALCUL N° 2 ni = 3 imin = 1 imax = 7 it_maxc = 3

hyper n° 1 ; val.: 1 1 5
 sys.: 1 ; 5tes: 3 ; 4tes 1 ; D_QQ 1

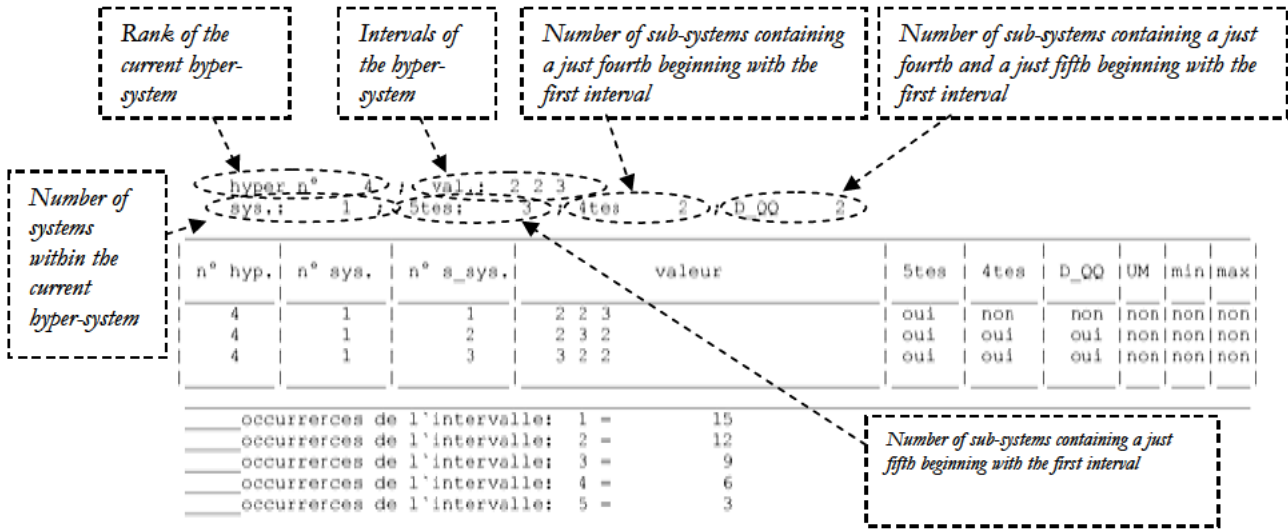
n° hyp.	n° sys.	n° s_sys.	valeur	5tes	4tes	D_QQ	UM	min	max
1	1	1	1 1 5	oui	non	non	non	oui	non
1	1	2	1 5 1	oui	non	non	non	oui	non
1	1	3	5 1 1	oui	oui	oui	non	oui	non

hyper n° 2 ; val.: 1 2 4
 sys.: 2 ; 5tes: 6 ; 4tes 2 ; D_QQ 2

n° hyp.	n° sys.	n° s_sys.	valeur	5tes	4tes	D_QQ	UM	min	max
2	1	1	1 2 4	oui	non	non	non	non	non
2	1	2	2 4 1	oui	non	non	non	non	non
2	1	3	4 1 2	oui	oui	oui	non	non	non
2	2	1	1 4 2	oui	oui	oui	non	non	non
2	2	2	4 2 1	oui	non	non	non	non	non
2	2	3	2 1 4	oui	non	non	non	non	non

hyper n° 3 ; val.: 1 3 3
 sys.: 1 ; 5tes: 3 ; 4tes 0 ; D_QQ 0

n° hyp.	n° sys.	n° s_sys.	valeur	5tes	4tes	D_QQ	UM	min	max
3	1	1	1 3 3	oui	non	non	non	non	oui
3	1	2	3 3 1	oui	non	non	non	non	oui
3	1	3	3 1 3	oui	non	non	non	non	oui



New ni

Base de données sous-systèmes: CALCUL N° 3 ni = 4 imin = 1 imax = 7 it_maxc = 3

hyper n° 1 ; val.: 1 1 1 4
sys.: 1 ; 5tes: 4 ; 4tes 2 ; D_QQ 2

n° hyp.	n° sys.	n° s_sys.	valeur	5tes	4tes	D_QQ	UM	min	max
1	1	1	1 1 1 4	oui	non	non	oui	oui	non
1	1	2	1 1 4 1	oui	non	non	non	oui	non
1	1	3	1 4 1 1	oui	oui	oui	oui	oui	non
1	1	4	4 1 1 1	oui	oui	oui	oui	oui	non

hyper n° 2 ; val.: 1 1 2 3
sys.: 3 ; 5tes: 12 ; 4tes 5 ; D_QQ 5

n° hyp.	n° sys.	n° s_sys.	valeur	5tes	4tes	D_QQ	UM	min	max
2	1	1	1 1 2 3	oui	non	non	non	oui	non
2	1	2	1 2 3 1	oui	non	non	non	non	non
2	1	3	2 3 1 1	oui	oui	oui	non	oui	non
2	1	4	3 1 1 2	oui	oui	oui	non	oui	non
2	2	1	1 1 3 2	oui	oui	oui	non	oui	non
2	2	2	1 3 2 1	oui	non	non	non	non	non
2	2	3	3 2 1 1	oui	oui	oui	non	oui	non
2	2	4	2 1 1 3	oui	non	non	non	oui	non
2	3	1	1 2 1 3	oui	non	non	non	non	non
2	3	2	2 1 3 1	oui	non	non	non	non	non
2	3	3	1 3 1 2	oui	oui	oui	non	non	non
2	3	4	3 1 2 1	oui	non	non	non	non	non

hyper n° 3 ; val.: 1 2 2 2
sys.: 1 ; 5tes: 4 ; 4tes 3 ; D_QQ 3

n° hyp.	n° sys.	n° s_sys.	valeur	5tes	4tes	D_QQ	UM	min	max
3	1	1	1 2 2 2	oui	oui	oui	non	non	non
3	1	2	2 2 2 1	oui	non	non	non	non	non
3	1	3	2 2 1 2	oui	oui	oui	non	non	non
3	1	4	2 1 2 2	oui	oui	oui	non	non	non

occurrences de l'intervalle: 1 =	40
occurrences de l'intervalle: 2 =	24
occurrences de l'intervalle: 3 =	12
occurrences de l'intervalle: 4 =	4

APPENDIX D: HYPER-SYSTEMS OF THE SEMI-TONE
OCTAVE COMPLETE ALPHABET GENERATION

Appendix D: Hyper-systems of the semi-tone octave complete alphabet generation

Remarks:

- *ni* == number of conceptual intervals per system
- *hyper* == hyper-system
- *Sys.* == number of systems for the current hyper-system
- *NSS5, NSS4* == number of systems with a just fifth, or a just fourth, beginning with the first interval
- *value* == explicit intervallic suite representing the hyper-system
- *ND_QQ* == number of systems that pass the test on the presence of a just fourth and a just fifth beginning with the first interval
- ‘*Italic*’ == systems or hyper-systems containing redundant sub-systems
- ‘**Italic**’ == completely redundant hyper-systems
- **On grey background** == hyper-systems that use the limited alphabet ‘1, 2, 3’

Hyper-systems that generate redundant sub-systems have a structure in which there is room for repeated patterns of intervals within the complete scale, as for example in hyper-system ‘1 1 1 1 4 4’ (for $ni = 6$); in this configuration of interval capacity, we may obtain the scheme ‘1 1’ ‘1 1’ ‘4 4’ (three pairs of identical intervals in all): this means that there exists at least one possibility of combining these intervals, in patterns of three conjunct intervals, in a system configuration which generates redundant sub-systems. One such system for the latter case is “1 1 4 1 1 4” in which the tri-intervallic combination “1 1 4” is repeated twice, and generates three redundant sub-systems (beginning with the fourth de-ranking process – the first one corresponding to the initial configuration) as shown below:

- 1st sub-system: 1 1 4 1 1 4
- 2nd sub-system: 1 4 1 1 4 1
- 3rd sub-system: 4 1 1 4 1 1
- 4th sub-system: 1 1 4 1 1 4 (identical to N°1)
- 5th sub-system: 1 4 1 1 4 1 (id. to N°2)
- 6th sub-system: 4 1 1 4 1 1 (id. to N°3)

ni = 2

hyper N° 1	value: 1 11	NS: 1	NSS5: 0	NSS4: 0	NDQQ: 0
hyper N° 2	value: 2 10	NS: 1	NSS5: 0	NSS4: 0	NDQQ: 0
hyper N° 3	value: 3 9	NS: 1	NSS5: 0	NSS4: 0	NDQQ: 0
hyper N° 4	value: 4 8	NS: 1	NSS5: 0	NSS4: 0	NDQQ: 0
hyper N° 5	value: 5 7	NS: 1	NSS5: 1	NSS4: 1	NDQQ: 0
hyper N° 6	value: 6 6	NS: 1	NSS5: 0	NSS4: 0	NDQQ: 0

ni = 3

hyper N° 1	value: 1 1 10	NS: 1	NSS5: 0	NSS4: 0	NDQQ: 0
hyper N° 2	value: 1 2 9	NS: 2	NSS5: 0	NSS4: 0	NDQQ: 0
hyper N° 3	value: 1 3 8	NS: 2	NSS5: 0	NSS4: 0	NDQQ: 0
hyper N° 4	value: 1 4 7	NS: 2	NSS5: 2	NSS4: 2	NDQQ: 0
hyper N° 5	value: 1 5 6	NS: 2	NSS5: 2	NSS4: 2	NDQQ: 0
hyper N° 6	value: 2 2 8	NS: 1	NSS5: 0	NSS4: 0	NDQQ: 0
hyper N° 7	value: 2 3 7	NS: 2	NSS5: 2	NSS4: 2	NDQQ: 0
hyper N° 8	value: 2 4 6	NS: 2	NSS5: 0	NSS4: 0	NDQQ: 0
hyper N° 9	value: 2 5 5	NS: 1	NSS5: 2	NSS4: 2	NDQQ: 1
hyper N° 10	value: 3 3 6	NS: 1	NSS5: 0	NSS4: 0	NDQQ: 0
hyper N° 11	value: 3 4 5	NS: 2	NSS5: 2	NSS4: 2	NDQQ: 0
hyper N° 12	value: 4 4 4	NS: 1	NSS5: 0	NSS4: 0	NDQQ: 0

<u>ni = 4</u>						
hyper N°	1	value: 1 1 1 9	NS: 1	NSS5: 0	NSS4: 0	NDQQ: 0
hyper N°	2	value: 1 1 2 8	NS: 3	NSS5: 0	NSS4: 0	NDQQ: 0
hyper N°	3	value: 1 1 3 7	NS: 3	NSS5: 3	NSS4: 3	NDQQ: 0
hyper N°	4	value: 1 1 4 6	NS: 3	NSS5: 4	NSS4: 4	NDQQ: 0
<i>hyper N°</i>	<i>5</i>	<i>value: 1 1 5 5</i>	<i>NS: 2</i>	<i>NSS5: 4</i>	<i>NSS4: 4</i>	<i>NDQQ: 1</i>
hyper N°	6	value: 1 2 2 7	NS: 3	NSS5: 3	NSS4: 3	NDQQ: 0
hyper N°	7	value: 1 2 3 6	NS: 6	NSS5: 4	NSS4: 4	NDQQ: 0
hyper N°	8	value: 1 2 4 5	NS: 6	NSS5: 10	NSS4: 10	NDQQ: 4
hyper N°	9	value: 1 3 3 5	NS: 3	NSS5: 3	NSS4: 3	NDQQ: 0
hyper N°	10	value: 1 3 4 4	NS: 3	NSS5: 4	NSS4: 4	NDQQ: 0
hyper N°	11	value: 2 2 2 6	NS: 1	NSS5: 0	NSS4: 0	NDQQ: 0
hyper N°	12	value: 2 2 3 5	NS: 3	NSS5: 7	NSS4: 7	NDQQ: 4
<i>hyper N°</i>	<i>13</i>	<i>value: 2 2 4 4</i>	<i>NS: 2</i>	<i>NSS5: 0</i>	<i>NSS4: 0</i>	<i>NDQQ: 0</i>
hyper N°	14	value: 2 3 3 4	NS: 3	NSS5: 4	NSS4: 4	NDQQ: 0
hyper N°	15	value: 3 3 3 3	NS: 1	NSS5: 0	NSS4: 0	NDQQ: 0

<u>ni = 5</u>						
hyper N°	1	value: 1 1 1 1 8	NS: 1	NSS5: 0	NSS4: 0	NDQQ: 0
hyper N°	2	value: 1 1 1 2 7	NS: 4	NSS5: 4	NSS4: 4	NDQQ: 0
hyper N°	3	value: 1 1 1 3 6	NS: 4	NSS5: 6	NSS4: 6	NDQQ: 0
hyper N°	4	value: 1 1 1 4 5	NS: 4	NSS5: 10	NSS4: 10	NDQQ: 4
hyper N°	5	value: 1 1 2 2 6	NS: 6	NSS5: 6	NSS4: 6	NDQQ: 0
hyper N°	6	value: 1 1 2 3 5	NS: 12	NSS5: 24	NSS4: 24	NDQQ: 10
hyper N°	7	value: 1 1 2 4 4	NS: 6	NSS5: 12	NSS4: 12	NDQQ: 4
hyper N°	8	value: 1 1 3 3 4	NS: 6	NSS5: 12	NSS4: 12	NDQQ: 0
hyper N°	9	value: 1 2 2 2 5	NS: 4	NSS5: 10	NSS4: 10	NDQQ: 6
hyper N°	10	value: 1 2 2 3 4	NS: 12	NSS5: 24	NSS4: 24	NDQQ: 8
hyper N°	11	value: 1 2 3 3 3	NS: 4	NSS5: 6	NSS4: 6	NDQQ: 0
hyper N°	12	value: 2 2 2 2 4	NS: 1	NSS5: 0	NSS4: 0	NDQQ: 0
hyper N°	13	value: 2 2 2 3 3	NS: 2	NSS5: 6	NSS4: 6	NDQQ: 4

<u>ni = 6</u>						
hyper N°	1	value: 1 1 1 1 1 7	NS: 1	NSS5: 1	NSS4: 1	NDQQ: 0
hyper N°	2	value: 1 1 1 1 2 6	NS: 5	NSS5: 8	NSS4: 8	NDQQ: 0
hyper N°	3	value: 1 1 1 1 3 5	NS: 5	NSS5: 14	NSS4: 14	NDQQ: 6
<i>hyper N°</i>	<i>4</i>	<i>value: 1 1 1 1 4 4</i>	<i>NS: 3</i>	<i>NSS5: 10</i>	<i>NSS4: 10</i>	<i>NDQQ: 5</i>
hyper N°	5	value: 1 1 1 2 2 5	NS: 10	NSS5: 27	NSS4: 27	NDQQ: 14
hyper N°	6	value: 1 1 1 2 3 4	NS: 20	NSS5: 58	NSS4: 58	NDQQ: 20
<i>hyper N°</i>	<i>7</i>	<i>value: 1 1 1 3 3 3</i>	<i>NS: 4</i>	<i>NSS5: 11</i>	<i>NSS4: 11</i>	<i>NDQQ: 0</i>
hyper N°	8	value: 1 1 2 2 2 4	NS: 10	NSS5: 26	NSS4: 26	NDQQ: 12
hyper N°	9	value: 1 1 2 2 3 3	NS: 16	NSS5: 44	NSS4: 44	NDQQ: 16
hyper N°	10	value: 1 2 2 2 2 3	NS: 5	NSS5: 17	NSS4: 17	NDQQ: 12
hyper N°	11	value: 2 2 2 2 2 2	NS: 1	NSS5: 0	NSS4: 0	NDQQ: 0

<u>ni = 7</u>						
hyper N°	1	value: 1 1 1 1 1 1 6	NS: 1	NSS5: 2	NSS4: 2	NDQQ: 0
hyper N°	2	value: 1 1 1 1 1 2 5	NS: 6	NSS5: 20	NSS4: 20	NDQQ: 10
hyper N°	3	value: 1 1 1 1 1 3 4	NS: 6	NSS5: 24	NSS4: 24	NDQQ: 12
hyper N°	4	value: 1 1 1 1 2 2 4	NS: 15	NSS5: 56	NSS4: 56	NDQQ: 28
hyper N°	5	value: 1 1 1 1 2 3 3	NS: 15	NSS5: 58	NSS4: 58	NDQQ: 21
hyper N°	6	value: 1 1 1 2 2 2 3	NS: 20	NSS5: 80	NSS4: 80	NDQQ: 46
hyper N°	7	value: 1 1 2 2 2 2 2	NS: 3	NSS5: 12	NSS4: 12	NDQQ: 9

<u>ni = 8</u>						
hyper N°	1	value: 1 1 1 1 1 1 1 5	NS: 1	NSS5: 4	NSS4: 4	NDQQ: 2
hyper N°	2	value: 1 1 1 1 1 1 2 4	NS: 7	NSS5: 34	NSS4: 34	NDQQ: 20
<i>hyper N°</i>	<i>3</i>	<i>value: 1 1 1 1 1 1 3 3</i>	<i>NS: 4</i>	<i>NSS5: 21</i>	<i>NSS4: 21</i>	<i>NDQQ: 11</i>
hyper N°	4	value: 1 1 1 1 1 2 2 3	NS: 21	NSS5: 108	NSS4: 108	NDQQ: 62
<i>hyper N°</i>	<i>5</i>	<i>value: 1 1 1 1 2 2 2 2</i>	<i>NS: 10</i>	<i>NSS5: 51</i>	<i>NSS4: 51</i>	<i>NDQQ: 34</i>

ni = 9

hyper N° 1	value: 1 1 1 1 1 1 1 1 4	NS: 1	NSS5: 6	NSS4: 6	NDQQ: 4
hyper N° 2	value: 1 1 1 1 1 1 1 2 3	NS: 8	NSS5: 52	NSS4: 52	NDQQ: 34
hyper N° 3	value: 1 1 1 1 1 1 2 2 2	NS: 10	NSS5: 66	NSS4: 66	NDQQ: 48

ni = 10

hyper N° 1	value: 1 1 1 1 1 1 1 1 1 3	NS: 1	NSS5: 8	NSS4: 8	NDQQ: 6
hyper N° 2	value: 1 1 1 1 1 1 1 1 2 2	NS: 5	NSS5: 41	NSS4: 41	NDQQ: 33

ni = 11

hyper N° 1	value: 1 1 1 1 1 1 1 1 1 1 2	NS: 1	NSS5: 10	NSS4: 10	NDQQ: 9
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ni = 12

hyper N° 1	value: 1 1 1 1 1 1 1 1 1 1 1 1	NS: 1	NSS5: 12	NSS4: 12	NDQQ: 12
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APPENDIX E: ADDITIONAL GRAPHS FOR OCTAVE
GENERATIONS WITH THE COMPLETE ALPHABET

Appendix E: Additional graphs for octave generations with the complete alphabet

Remarks:

- *ENESU (Évolution du Nombre d'Éléments Scalaires - Unitaire) == results in Unitary variables (weighted)*
- *NSSU_NR == **Number of Sub-Systems in Unitary weighting with the Non-redundant sub-systems excluded (Unitary)***
- *NSS5U_NR == **Unitary number of non-redundant sub-systems with a direct fifth (or fourth)***
- *DQQU_NR (nombre de sub-systems Non Redondants en Double Quarte ET Quinte justes – Unitaire) == **Unitary number of non-redundant sub-systems with a direct fourth in a fifth***
- *5tes == test on the presence of a just fifth beginning with the first interval*
- *4tes == test on the presence of a just fourth beginning with the first interval*
- *D_QQ == test on the presence of a just fourth AND a just fifth beginning with the first interval*
- *umin == 'ultra-min' criterion for the detection of suites of three (or more) semi-tones ('1') in a row*
- *exhaustive == generation with the complete alphabet of intervals (semi-tones)*
- *quasi-exhaustive == generation with the nearly complete alphabet of intervals (quarter-tones – the one-quarter-tone interval is excluded)*
- *max == 'max' criterion for the detection of suites of two (or more) intervals equal or superior to 'it_maxc' (the latter's value is set for this generation to '3' semi-tones)*
- *\umin == 'non-ultra-min' – without suites of three (or more) semi-tones ('1') in a row*
- *\max == 'non-max' – without suites of two (or more) intervals equal or superior to 'it_maxc' (the latter's value is set for this generation to '3' semi-tones)*

I. Semi-tone generations

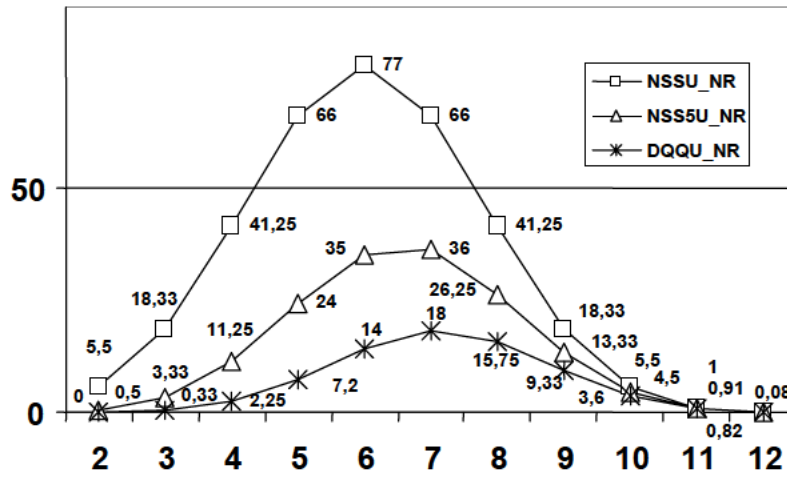


Figure A1: ENESU en 4te ou 5te (ou DQQ), f(NI): ½ ton, NR, exhaustive (compare with figure 40)

See 'Remarks' on p. A27

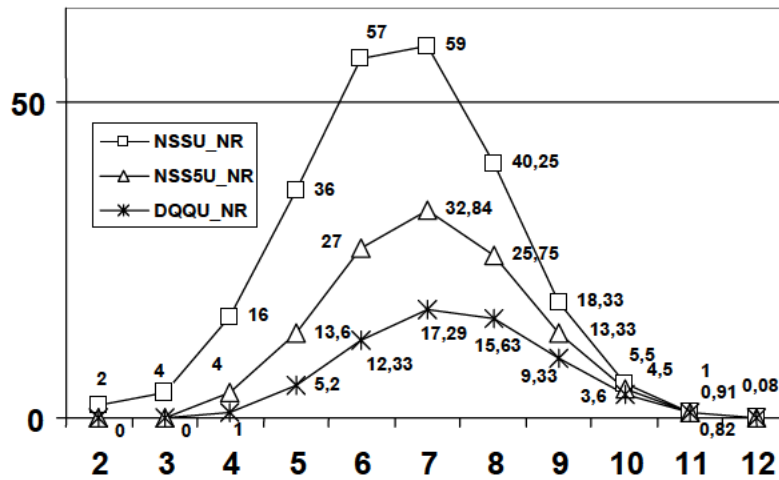


Figure A2: ENESU en 4te ou 5te (ou DQQ), f(NI): NR, \max, exhaustive ½ ton (compare with figure 44)

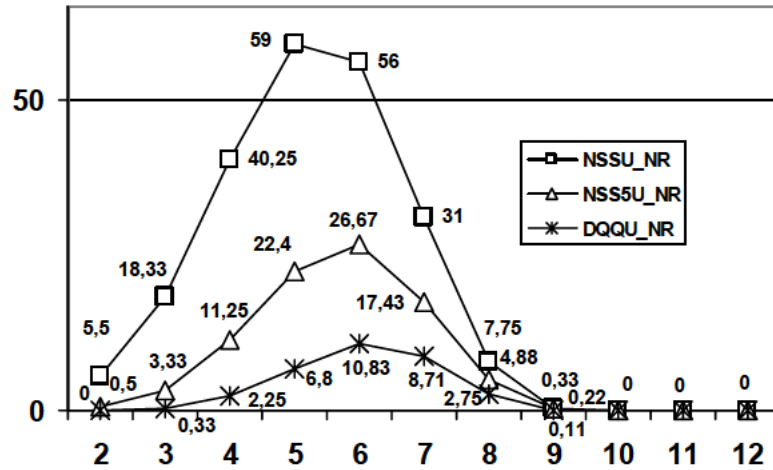


Figure A3: ENESU en 4te ou 5te (ou DQQ), f(NI): NR, \umin, exhaustive 1/2 ton (compare with figure 42)

See 'Remarks' on p. A27

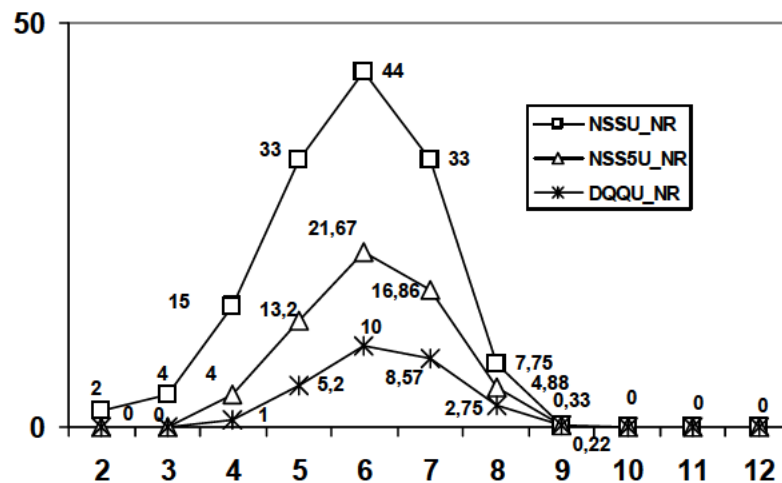


Figure A4: ENESU en 4te ou 5te (ou DQQ), f(NI): 1/2 ton, NR, composé \umin et \max, exhaustive (compare with figure 46)

II. Quarter-tone generations

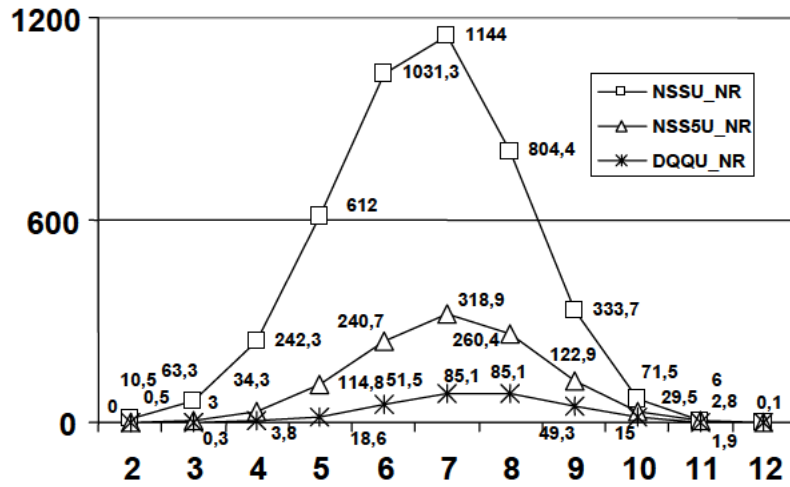


Figure A5: ENESU en 4te ou 5te (ou DQQ), $f(NI)$, $\frac{1}{4}$ ton, NR, quasi-exhaustive (compare with figure 41)

See 'Remarks' on p. A27

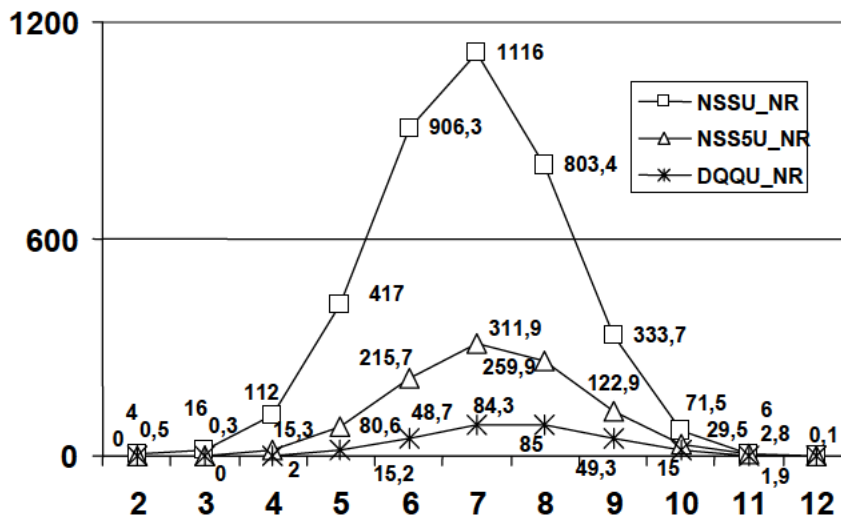


Figure A6: ENESU en 4te ou 5te (ou DQQ), $f(NI)$, $\frac{1}{4}$ ton, NR, $\backslash \max(6)$, quasi-exhaustive (compare with figure 45)

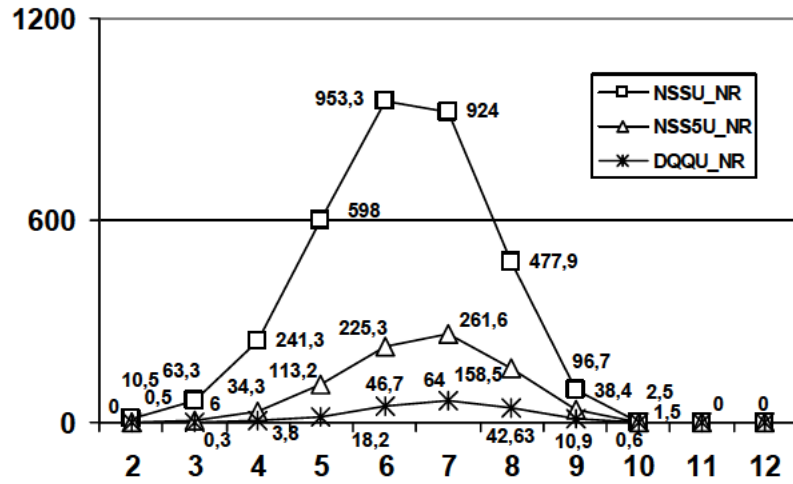


Figure A7: ENESU en 4te ou 5te (ou DQQ), $f(NI)$, $\frac{1}{4}$ ton, NR, \backslash_{umin} , quasi-exhaustive (compare with figure 43)

See 'Remarks' on p. A27

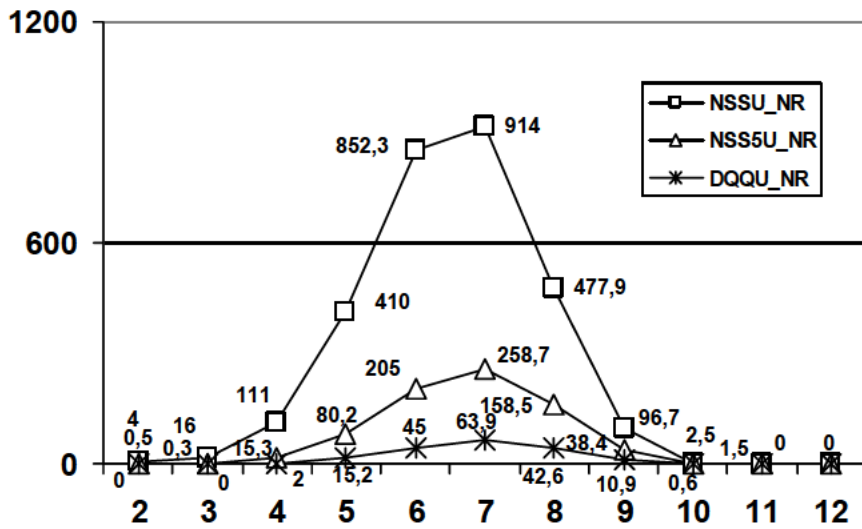


Figure A8: ENESU en 4te ou 5te (ou DQQ), $f(NI)$, $\frac{1}{4}$ ton, NR, $\backslash_{max(6)}$ et \backslash_{umin} (compare with figure 47)

APPENDIX F: SYNOPTICAL RESULTS FOR THE QUARTER-TONE GENERATIONS IN THE OCTAVE, WITH THE SMALLEST CONCEPTUAL INTERVAL (IMIN) SET TO '2' QUARTER-TONES AND NO LIMITATIONS FOR THE LARGEST INTERVAL

Appendix F: Synoptical results for the quarter-tone generations, with the smallest conceptual interval (imin) set to “2” quarter-tones and no limitations for the largest interval

- *Smallest interval in use: 2 quarter-tones*
- *Largest interval in use: 24 quarter-tones*
- *Range of NI: 2 to 12*
- *Presence of a direct fifth → ‘D. fifth’*
- *Presence of a direct fourth → ‘D. fourth’*
- *Conjunct two semi-tones → ‘min’*
- *Conjunct three semi-tones → ‘umin’*
- *Conjunct big intervals (6 quarter-tones or greater) → ‘max’*
- *‘R+’ → including redundancies*
- *‘R-’ → excluding redundancies*
- *Filters excluding sub-systems (‘NON’) are preceded by ‘\’*

NI	Subdivisions	Without filters		\max (6)		\min		\min AND \max		\umin		\umin AND \max	
		R+	R-	R+	R-	R+	R-	R+	R-	R+	R-	R+	R-
2	systems	11	10.5	4	4	11	10.5	4	4	11	10.5	4	4
	sub-systems: - D. fifth	.5	.5	0	0	.5	.5	0	0	.5	.5	0	0
	- D. fourth	.5	.5	0	0	.5	.5	0	0	.5	.5	0	0
	- D. fifth AND D. fourth	0	0	0	0	0	0	0	0	0	0	0	0
3	systems	64	63.33	16	16	63	62.33	15	15	64	63.33	16	16
	sub-systems: - D. fifth	6	6	.33	.33	6	6	.33	.33	6	6	.33	.33
	- D. fourth	6	6	.33	.33	6	6	.33	.33	6	6	.33	.33
	- D. fifth AND D. fourth	.33	.33	0	0	.33	.33	0	0	.33	.33	0	0
4	systems	245	242.25	114	112	229	226.25	107	105	244	241.25	113	111
	sub-systems: - D. fifth	34.5	34.25	15.5	15.25	33	32.75	15.5	15.25	34.5	34.25	15.5	15.25
	- D. fourth	34.5	34.25	15.5	15.25	33	32.75	15.5	15.25	34.5	34.25	15.5	15.25
	- D. fifth AND D. fourth	3.75	3.75	2	2	3.5	3.5	2	2	3.75	3.75	2	2
5	systems	612	612	417	417	507	507	355	355	598	598	410	410
	sub-systems: - D. fifth	114.8	114.81	80.6	80.6	97.6	97.6	71	71	113.2	113.2	80.2	80.2
	- D. fourth	114.8	114.81	80.6	80.6	97.6	97.6	71	71	113.2	113.2	80.2	80.2
	- D. fifth AND D. fourth	18.6	18.6	15.2	15.2	14.8	14.8	13	13	18.2	18.2	15.2	15.2
6	systems	1038	1031.33	913	906.33	679	672.83	628	621.83	960	953.33	859	852.33
	sub-systems: - D. fifth	242.17	240.67	217.17	215.67	162.17	161	153.67	152.5	226.83	225.33	26.5	205
	- D. fourth	242.17	240.67	217.17	215.67	162.17	161	153.67	152.5	226.83	225.33	26.5	205
	- D. fifth AND D. fourth	51.67	51.5	48.83	48.67	30	30	30	30	46.83	46.67	45.17	45

NI	Subdivisions	Without filters		\max (6)		\min		\min AND \max		\umin		\umin AND \ max	
		R+	R-	R+	R-	R+	R-	R+	R-	R+	R-	R+	R-
7	systems	1144	1144	1116	1116	474	474	473	473	924	924	914	914
	sub-systems: - D. fifth	318.86	318.86	311.86	311.86	134.86	134.86	134.57	134.57	261.57	261.57	258.71	258.71
	- D. fourth	318.86	318.86	311.86	311.86	134.86	134.86	134.57	134.57	261.57	261.57	258.71	258.71
	- D. fifth AND D. fourth	85.29	85.29	84.29	84.29	27.29	27.29	27.29	27.29	64	64	63.86	63.86
8	systems	810	804.58	809	803.38	142	138.38	142	138.38	483	477.88	483	477.88
	sub-systems: - D. fifth	262	260.38	262	259.88	46	45	46	45	160.25	158.5	160.25	158.5
	- D. fourth	262	260.38	262	259.88	46	45	46	45	160.25	158.5	160.25	158.5
	- D. fifth AND D. fourth	85.75	85.13	85.63	85	7.75	7.75	7.75	7.75	43	42.63	43	42.63
9	systems	335	333.67	335	333.67	9	8.33	9	8.33	98	96.67	98	96.67
	sub-systems: - D. fifth	123	122.89	123.56	122.89	3.78	3.56	3.78	3.56	39.11	38.44	39.11	38.44
	- D. fourth	123	122.89	123.56	122.89	3.78	3.56	3.78	3.56	39.11	38.44	39.11	38.44
	- D. fifth AND D. fourth	49.56	49.33	49.56	49.33	.56	.56	.56	.56	11.11	10.89	11	10.89
10	systems	73	71.5	73	71.5	0	0	0	0	3	2.5	3	2.5
	sub-systems: - D. fifth	30	29.5	30.5	29.5	0	0	0	0	1.8	1.5	1.8	1.5
	- D. fourth	30	29.5	30.5	29.5	0	0	0	0	1.8	1.5	1.8	1.5
	- D. fifth AND D. fourth	15.6	15	15.6	15	0	0	0	0	.7	.6	.7	.6
11	systems	6	6	6	6	0	0	0	0	0	0	0	0
	sub-systems: - D. fifth	2.82	2.82	2.82	2.82	0	0	0	0	0	0	0	0
	- D. fourth	2.82	2.82	2.82	2.82	0	0	0	0	0	0	0	0
	- D. fifth AND D. fourth	1.91	1.91	1.91	1.91	0	0	0	0	0	0	0	0
12	systems	1	.08	1	.8	0	0	0	0	0	0	0	0
	sub-systems: - D. fifth	1	.08	1.0	.08	0	0	0	0	0	0	0	0
	- D. fourth	1	.08	1.0	.08	0	0	0	0	0	0	0	0
	- D. fifth AND D. fourth	1	.08	1	.08	0	0	0	0	0	0	0	0

APPENDIX G: COMPLETE DATABASE OF THE QUARTER-TONE GENERATIONS FOR $N_I=7$ (HEPTATONIC SCALES) WITH THE LIMITED ALPHABET '2, 3, 4, 5, 6'

Appendix G: Complete database of the quarter-tone generations for NI=7 (heptatonic scales) with the limited alphabet '2, 3, 4, 5, 6'

I. Hyper-systems

H.	Value	NS	NSS	5ths
Hyper-system N° 1	2 2 2 2 4 6 6	15	105	58
Hyper-system N° 2	2 2 2 2 5 5 6	15	105	23
Hyper-system N° 3	2 2 2 3 3 6 6	30	210	54
Hyper-system N° 4	2 2 2 3 4 5 6	120	840	208
Hyper-system N° 5	2 2 2 3 5 5 5	20	140	56
Hyper-system N° 6	2 2 2 4 4 4 6	20	140	80
Hyper-system N° 7	2 2 2 4 4 5 5	30	210	40
Hyper-system N° 8	2 2 3 3 3 5 6	60	420	120
Hyper-system N° 9	2 2 3 3 4 4 6	90	630	168
Hyper-system N°10	2 2 3 3 4 5 5	90	630	210
Hyper-system N°11	2 2 3 4 4 4 5	60	420	96
Hyper-system N°12	2 2 4 4 4 4 4	3	21	12
Hyper-system N°13	2 3 3 3 3 4 6	30	210	46
Hyper-system N°14	2 3 3 3 3 5 5	15	105	29
Hyper-system N°15	2 3 3 3 4 4 5	60	420	120
Hyper-system N°16	2 3 3 4 4 4 4	15	105	30
Hyper-system N°17	3 3 3 3 3 3 6	1	7	0
Hyper-system N°18	3 3 3 3 3 4 5	6	42	12
Hyper-system N°19	3 3 3 3 4 4 4	5	35	18

II. Systems

H	S	value	5ths	4ths	FF
1	1	2222466	3	3	1
1	2	2222646	4	4	3
1	3	2222664	3	3	1
1	4	2224266	3	3	0
1	5	2224626	4	4	1
1	6	2226246	4	4	2
1	7	2226264	4	4	1
1	8	2226426	4	4	2
1	9	2226624	3	3	0
1	10	2242266	4	4	1
1	11	2242626	4	4	1
1	12	2246226	5	5	3
1	13	2262264	5	5	3
1	14	2262426	4	4	1
1	15	2262624	4	4	1

H	S	value	5ths	4ths	FF
2	1	2222556	2	2	1
2	2	2222565	0	0	0
2	3	2222655	2	2	1
2	4	2225256	1	1	0
2	5	2225265	0	0	0
2	6	2225526	3	3	1
2	7	2225625	0	0	0
2	8	2226255	3	3	1
2	9	2226525	1	1	0
2	10	2252256	1	1	0
2	11	2252265	1	1	0
2	12	2252526	2	2	0
2	13	2252625	1	1	0
2	14	2255226	4	4	2
2	15	2262525	2	2	0

H	S	value	5ths	4ths	FF
3	1	2223366	2	2	0
3	2	2223636	1	1	0
3	3	2223663	0	0	0
3	4	2226336	2	2	0
3	5	2226363	1	1	0
3	6	2226633	2	2	0
3	7	2232366	2	2	0
3	8	2232636	1	1	0
3	9	2232663	1	1	0
3	10	2233266	3	3	0
3	11	2233626	3	3	0
3	12	2236236	1	1	0
3	13	2236263	1	1	0
3	14	2236326	2	2	0
3	15	2236623	1	1	0
3	16	2262336	3	3	0
3	17	2262363	2	2	0
3	18	2262633	3	3	0
3	19	2263236	2	2	0
3	20	2263263	1	1	0
3	21	2263326	3	3	0
3	22	2263623	1	1	0
3	23	2266233	3	3	0
3	24	2266323	2	2	0
3	25	2323266	2	2	0
3	26	2323626	2	2	0
3	27	2326236	1	1	0
3	28	2326263	2	2	0
3	29	2326326	1	1	0
3	30	2332626	3	3	0

H	S	value	5ths	4ths	FF
4	1	2223456	1	1	0
4	2	2223465	1	1	0
4	3	2223546	3	3	2
4	4	2223564	3	3	1
4	5	2223645	1	1	0
4	6	2223654	1	1	0
4	7	2224356	2	2	0
4	8	2224365	1	1	0
4	9	2224536	2	2	0
4	10	2224563	1	1	0
4	11	2224635	3	3	1
4	12	2224653	3	3	1
4	13	2225346	3	3	2
4	14	2225364	3	3	1
4	15	2225436	1	1	0
4	16	2225463	1	1	0
4	17	2225634	1	1	0
4	18	2225643	1	1	0
4	19	2226345	1	1	0
4	20	2226354	2	2	0
4	21	2226435	3	3	2
4	22	2226453	3	3	2
4	23	2226534	2	2	0
4	24	2226543	1	1	0
4	25	2232456	1	1	0
4	26	2232465	1	1	0
4	27	2232546	3	3	2
4	28	2232564	2	2	1
4	29	2232645	1	1	0
4	30	2232654	0	0	0
4	31	2234256	1	1	0
4	32	2234265	0	0	0
4	33	2234526	2	2	0
4	34	2234625	1	1	0
4	35	2235246	4	4	2
4	36	2235264	3	3	1
4	37	2235426	3	3	1
4	38	2235624	2	2	0
4	39	2236245	0	0	0
4	40	2236254	0	0	0
4	41	2236425	1	1	0
4	42	2236524	1	1	0
4	43	2242356	3	3	1
4	44	2242365	1	1	0
4	45	2242536	3	3	1
4	46	2242563	1	1	0
4	47	2242635	2	2	0
4	48	2242653	2	2	0
4	49	2243256	2	2	1
4	50	2243265	0	0	0
4	51	2243526	3	3	1
4	52	2243625	0	0	0
4	53	2245236	2	2	1
4	54	2245263	0	0	0
4	55	2245326	3	3	1
4	56	2245623	0	0	0
4	57	2246235	3	3	1
4	58	2246253	3	3	1
4	59	2246325	2	2	1
4	60	2246523	2	2	1
4	61	2252346	3	3	2
4	62	2252364	2	2	1
4	63	2252436	1	1	0
4	64	2252463	1	1	0
4	65	2252634	0	0	0
4	66	2252643	1	1	0

4	67	2253246	4	4	2
4	68	2253264	3	3	1
4	69	2253426	3	3	1
4	70	2253624	2	2	0
4	71	2254236	1	1	0
4	72	2254263	0	0	0
4	73	2254326	2	2	0
4	74	2254623	1	1	0
4	75	2256234	0	0	0
4	76	2256243	0	0	0
4	77	2256324	1	1	0
4	78	2256423	1	1	0
4	79	2262345	2	2	0
4	80	2262354	3	3	1
4	81	2262435	3	3	1
4	82	2262453	3	3	1
4	83	2262534	3	3	1
4	84	2262543	2	2	0
4	85	2263245	1	1	0
4	86	2263254	2	2	1
4	87	2263425	1	1	0
4	88	2263524	3	3	1
4	89	2264235	4	4	2
4	90	2264253	4	4	2
4	91	2264325	3	3	2
4	92	2264523	3	3	2
4	93	2265234	2	2	1
4	94	2265243	1	1	0
4	95	2265324	3	3	1
4	96	2265423	1	1	0
4	97	2324256	0	0	0
4	98	2324265	1	1	0
4	99	2324526	1	1	0
4	100	2324625	2	2	0
4	101	2325246	2	2	0
4	102	2325264	2	2	0
4	103	2325426	2	2	1
4	104	2325624	1	1	0
4	105	2326245	2	2	1
4	106	2326254	1	1	0
4	107	2326425	2	2	0
4	108	2326524	0	0	0
4	109	2342526	1	1	0
4	110	2342625	2	2	1
4	111	2352426	3	3	1
4	112	2352624	3	3	1
4	113	2362425	1	1	0
4	114	2362524	0	0	0
4	115	2425263	0	0	0
4	116	2425326	3	3	1
4	117	2426253	3	3	1
4	118	2426325	1	1	0
4	119	2432526	2	2	1
4	120	2432625	1	1	0

H	S	value	5ths	4ths	FF
5	1	2223555	3	3	1
5	2	2225355	2	2	1
5	3	2225535	2	2	1
5	4	2225553	3	3	1
5	5	2232555	3	3	1
5	6	2235255	3	3	1
5	7	2235525	2	2	0
5	8	2252355	3	3	1
5	9	2252535	2	2	0
5	10	2252553	2	2	0
5	11	2253255	4	4	1

5	12	2253525	2	2	0
5	13	2255235	4	4	1
5	14	2255253	3	3	1
5	15	2255325	3	3	1
5	16	2255523	3	3	1
5	17	2325255	3	3	0
5	18	2325525	3	3	0
5	19	2352525	3	3	0
5	20	2525253	3	3	0

H	S	value	5ths	4ths	FF
6	1	2224446	4	4	3
6	2	2224464	5	5	4
6	3	2224644	5	5	4
6	4	2226444	4	4	3
6	5	2242446	5	5	4
6	6	2242464	4	4	3
6	7	2242644	3	3	1
6	8	2244246	5	5	3
6	9	2244264	3	3	1
6	10	2244426	4	4	2
6	11	2244624	3	3	1
6	12	2246244	3	3	1
6	13	2246424	4	4	3
6	14	2262444	4	4	2
6	15	2264244	5	5	3
6	16	2264424	5	5	4
6	17	2424246	3	3	0
6	18	2424264	3	3	0
6	19	2424426	4	4	2
6	20	2426244	4	4	2

H	S	value	5ths	4ths	FF
7	1	2224455	3	3	2
7	2	2224545	1	1	0
7	3	2224554	3	3	2
7	4	2225445	0	0	0
7	5	2225454	1	1	0
7	6	2225544	3	3	2
7	7	2242455	3	3	2
7	8	2242545	1	1	0
7	9	2242554	2	2	1
7	10	2244255	3	3	1
7	11	2244525	1	1	0
7	12	2245245	0	0	0
7	13	2245254	0	0	0
7	14	2245425	0	0	0
7	15	2245524	2	2	1
7	16	2252445	1	1	0
7	17	2252454	0	0	0
7	18	2252544	1	1	0
7	19	2254245	1	1	0
7	20	2254254	0	0	0
7	21	2254425	1	1	0
7	22	2254524	1	1	0
7	23	2255244	3	3	1
7	24	2255424	3	3	2
7	25	2424255	2	2	0
7	26	2424525	1	1	0
7	27	2425245	0	0	0
7	28	2425254	1	1	0
7	29	2425425	0	0	0
7	30	2442525	2	2	0

H	S	value	5ths	4ths	FF
8	1	2233356	2	2	0
8	2	2233365	1	1	0
8	3	2233536	2	2	0
8	4	2233563	2	2	0
8	5	2233635	2	2	0
8	6	2233653	3	3	0
8	7	2235336	2	2	0
8	8	2235363	2	2	0
8	9	2235633	3	3	0
8	10	2236335	1	1	0
8	11	2236353	2	2	0
8	12	2236533	2	2	0
8	13	2253336	2	2	0

8	14	2253363	1	1	0
8	15	2253633	2	2	0
8	16	2256333	1	1	0
8	17	2263335	2	2	0
8	18	2263353	2	2	0
8	19	2263533	2	2	0
8	20	2265333	2	2	0
8	21	2323356	1	1	0
8	22	2323365	2	2	0
8	23	2323536	2	2	0
8	24	2323563	3	3	0
8	25	2323635	3	3	0
8	26	2323653	3	3	0
8	27	2325336	2	2	0
8	28	2325363	3	3	0
8	29	2325633	2	2	0
8	30	2326335	2	2	0
8	31	2326353	2	2	0
8	32	2326533	1	1	0
8	33	2332356	2	2	0
8	34	2332365	2	2	0
8	35	2332536	3	3	0
8	36	2332563	2	2	0
8	37	2332635	3	3	0
8	38	2332653	2	2	0
8	39	2333256	1	1	0
8	40	2333265	1	1	0
8	41	2333526	2	2	0
8	42	2333625	1	1	0
8	43	2335236	2	2	0
8	44	2335263	1	1	0
8	45	2335326	2	2	0
8	46	2336235	3	3	0
8	47	2336253	2	2	0
8	48	2336325	2	2	0
8	49	2352363	3	3	0
8	50	2352633	2	2	0
8	51	2353236	2	2	0
8	52	2353263	2	2	0
8	53	2353326	2	2	0
8	54	2362533	1	1	0
8	55	2363253	3	3	0
8	56	2363325	2	2	0
8	57	2526333	1	1	0
8	58	2532633	3	3	0
8	59	2533263	2	2	0
8	60	2533326	2	2	0

H	S	value	5ths	4ths	FF
9	1	2233446	4	4	3
9	2	2233464	4	4	3
9	3	2233644	3	3	1
9	4	2234346	3	3	2
9	5	2234364	2	2	1
9	6	2234436	1	1	0
9	7	2234463	2	2	1
9	8	2234634	1	1	0
9	9	2234643	3	3	2
9	10	2236344	1	1	0
9	11	2236434	1	1	0
9	12	2236443	2	2	1
9	13	2243346	4	4	3
9	14	2243364	2	2	1
9	15	2243436	2	2	1
9	16	2243463	1	1	0
9	17	2243634	0	0	0
9	18	2243643	1	1	0
9	19	2244336	3	3	1
9	20	2244363	1	1	0
9	21	2244633	3	3	1
9	22	2246334	2	2	1
9	23	2246343	2	2	1
9	24	2246433	4	4	3
9	25	2263344	3	3	1
9	26	2263434	2	2	1
9	27	2263443	1	1	0
9	28	2264334	4	4	3
9	29	2264343	3	3	2

9	30	2264433	4	4	3
9	31	2323446	2	2	1
9	32	2323464	3	3	2
9	33	2323644	3	3	1
9	34	2324346	1	1	0
9	35	2324364	1	1	0
9	36	2324436	1	1	0
9	37	2324463	3	3	1
9	38	2324634	1	1	0
9	39	2324643	3	3	2
9	40	2326344	1	1	0
9	41	2326434	1	1	0
9	42	2326443	2	2	1
9	43	2332446	3	3	1
9	44	2332464	3	3	2
9	45	2332644	3	3	1
9	46	2334246	3	3	0
9	47	2334264	2	2	0
9	48	2334426	3	3	1
9	49	2334624	2	2	0
9	50	2336244	2	2	0
9	51	2336424	2	2	0
9	52	2342346	1	1	0
9	53	2342364	1	1	0
9	54	2342436	1	1	0
9	55	2342463	2	2	0
9	56	2342634	0	0	0
9	57	2342643	1	1	0
9	58	2343246	2	2	0
9	59	2343264	2	2	0
9	60	2343426	2	2	1
9	61	2343624	1	1	0
9	62	2344236	1	1	0
9	63	2344263	1	1	0
9	64	2344326	1	1	0
9	65	2346243	1	1	0
9	66	2346324	1	1	0
9	67	2362434	0	0	0
9	68	2362443	1	1	0
9	69	2363244	2	2	0
9	70	2363424	1	1	0
9	71	2364243	2	2	0
9	72	2364324	1	1	0
9	73	2424336	2	2	0
9	74	2424363	1	1	0
9	75	2424633	2	2	0
9	76	2426334	2	2	0
9	77	2426343	1	1	0
9	78	2426433	2	2	0
9	79	2432436	0	0	0
9	80	2432463	1	1	0
9	81	2432634	1	1	0
9	82	2432643	1	1	0
9	83	2433246	2	2	0
9	84	2433264	3	3	0
9	85	2433426	3	3	2
9	86	2434263	0	0	0
9	87	2434326	2	2	1
9	88	2442633	2	2	0
9	89	2443263	1	1	0
9	90	2443326	3	3	1

H	S	value	5ths	4ths	FF
10	1	2233455	3	3	2
10	2	2233545	1	1	0
10	3	2233554	2	2	1
10	4	2234355	2	2	1
10	5	2234535	1	1	0
10	6	2234553	3	3	1
10	7	2235345	1	1	0
10	8	2235354	1	1	0
10	9	2235435	2	2	1
10	10	2235453	3	3	1
10	11	2235534	2	2	0
10	12	2235543	3	3	1
10	13	2243355	2	2	1
10	14	2243535	1	1	0
10	15	2243553	2	2	0

15	42	2 4 3 5 4 3 3	1	1	0
15	43	2 4 4 3 3 3 5	3	3	1
15	44	2 4 4 3 3 5 3	3	3	1
15	45	2 4 4 3 5 3 3	1	1	0
15	46	2 4 4 5 3 3 3	1	1	0
15	47	2 4 5 3 3 3 4	2	2	0
15	48	2 4 5 3 3 4 3	2	2	0
15	49	2 4 5 3 4 3 3	2	2	0
15	50	2 4 5 4 3 3 3	1	1	0
15	51	2 5 3 3 3 4 4	3	3	1
15	52	2 5 3 3 4 3 4	3	3	1
15	53	2 5 3 3 4 4 3	3	3	1
15	54	2 5 3 4 3 3 4	4	4	2
15	55	2 5 3 4 3 4 3	4	4	2
15	56	2 5 3 4 4 3 3	3	3	1
15	57	2 5 4 3 3 3 4	2	2	0
15	58	2 5 4 3 3 4 3	4	4	2
15	59	2 5 4 3 4 3 3	3	3	1

15	60	2 5 4 4 3 3 3	2	2	1
H	S	value	5ths	4ths	FF
16	1	2 3 3 4 4 4 4	2	2	1
16	2	2 3 4 3 4 4 4	2	2	1
16	3	2 3 4 4 3 4 4	1	1	0
16	4	2 3 4 4 4 3 4	0	0	0
16	5	2 3 4 4 4 4 3	0	0	0
16	6	2 4 3 3 4 4 4	4	4	3
16	7	2 4 3 4 3 4 4	3	3	2
16	8	2 4 3 4 4 3 4	1	1	0
16	9	2 4 3 4 4 4 3	0	0	0
16	10	2 4 4 3 3 4 4	5	5	4
16	11	2 4 4 3 4 3 4	3	3	2
16	12	2 4 4 3 4 4 3	1	1	0
16	13	2 4 4 4 3 3 4	4	4	3
16	14	2 4 4 4 3 4 3	2	2	1
16	15	2 4 4 4 4 3 3	2	2	1

H	S	value	5ths	4ths	FF
17	1	3 3 3 3 3 3 6	0	0	0
H	S	value	5ths	4ths	FF
18	1	3 3 3 3 3 4 5	1	1	0
18	2	3 3 3 3 3 5 4	1	1	0
18	3	3 3 3 3 4 3 5	2	2	0
18	4	3 3 3 3 5 3 4	2	2	0
18	5	3 3 3 4 3 3 5	3	3	0
18	6	3 3 3 5 3 3 4	3	3	0
H	S	value	5ths	4ths	FF
19	1	3 3 3 3 4 4 4	2	2	1
19	2	3 3 3 4 3 4 4	3	3	1
19	3	3 3 3 4 4 3 4	3	3	1
19	4	3 3 4 3 3 4 4	5	5	3
19	5	3 3 4 3 4 3 4	5	5	3

III. Sub-systems

Remark: filter ‘Min’ works for suits of two conjunct semi-tones, whenever filter ‘Umin’ starts at three; M347 tests for two conjunct semi-tones in positions ‘3’, ‘4’, and ‘7’..

hyper N° 1; val.: 2222466

sys: 15; 5ths: 58; 4ths 58; FF 21

(0; 1; 1; 2 2 2 2 4 6 6); 5th = 3; 4th = 3; FF = 1; Umin = yes; Min = yes; Max = yes
00 sub-system N° 1; value: 2 2 2 2 4 6 6; 5th = no; 4th = no; FF = no; Umin = yes; min = yes; Max = yes; M347 = no
00 sub-system N° 2; value: 2 2 2 4 6 6 2; 5th = no; 4th = yes; FF = no; Umin = yes; min = yes; Max = yes; M347 = no
00 sub-system N° 3; value: 2 2 4 6 6 2 2; 5th = yes; 4th = no; FF = no; Umin = yes; min = yes; Max = yes; M347 = no
00 sub-system N° 4; value: 2 4 6 6 2 2 2; 5th = no; 4th = no; FF = no; Umin = yes; min = yes; Max = yes; M347 = no
00 sub-system N° 5; value: 4 6 6 2 2 2 2; 5th = no; 4th = yes; FF = no; Umin = yes; min = yes; Max = yes; M347 = no
00 sub-system N° 6; value: 6 6 2 2 2 2 4; 5th = yes; 4th = no; FF = no; Umin = yes; min = yes; Max = yes; M347 = no
00 sub-system N° 7; value: 6 2 2 2 2 4 6; 5th = yes; 4th = yes; FF = yes; Umin = yes; min = yes; Max = yes; M347 = no

(0; 1; 2; 2 2 2 2 6 4 6); 5th = 4; 4th = 4; FF = 3; Umin = yes; Min = yes; Max = no
00 sub-system N° 1; value: 2 2 2 2 6 4 6; 5th = yes; 4th = no; FF = no; Umin = yes; min = yes; Max = no; M347 = no
00 sub-system N° 2; value: 2 2 2 6 4 6 2; 5th = no; 4th = no; FF = no; Umin = yes; min = yes; Max = no; M347 = no
00 sub-system N° 3; value: 2 2 6 4 6 2 2; 5th = yes; 4th = yes; FF = yes; Umin = yes; min = yes; Max = no; M347 = no
00 sub-system N° 4; value: 2 6 4 6 2 2 2; 5th = no; 4th = no; FF = no; Umin = yes; min = yes; Max = no; M347 = no
00 sub-system N° 5; value: 6 4 6 2 2 2 2; 5th = no; 4th = yes; FF = no; Umin = yes; min = yes; Max = no; M347 = no
00 sub-system N° 6; value: 4 6 2 2 2 2 6; 5th = yes; 4th = yes; FF = yes; Umin = yes; min = yes; Max = no; M347 = no
00 sub-system N° 7; value: 6 2 2 2 2 6 4; 5th = yes; 4th = yes; FF = yes; Umin = yes; min = yes; Max = no; M347 = no

(0; 1; 3; 2 2 2 2 6 6 4); 5th = 3; 4th = 3; FF = 1; Umin = yes; Min = yes; Max = yes
00 sub-system N° 1; value: 2 2 2 2 6 6 4; 5th = yes; 4th = no; FF = no; Umin = yes; min = yes; Max = yes; M347 = no
00 sub-system N° 2; value: 2 2 2 6 6 4 2; 5th = no; 4th = no; FF = no; Umin = yes; min = yes; Max = yes; M347 = no
00 sub-system N° 3; value: 2 2 6 6 4 2 2; 5th = no; 4th = yes; FF = no; Umin = yes; min = yes; Max = yes; M347 = no
00 sub-system N° 4; value: 2 6 6 4 2 2 2; 5th = yes; 4th = no; FF = no; Umin = yes; min = yes; Max = yes; M347 = no
00 sub-system N° 5; value: 6 6 4 2 2 2 2; 5th = no; 4th = no; FF = no; Umin = yes; min = yes; Max = yes; M347 = no
00 sub-system N° 6; value: 6 4 2 2 2 2 6; 5th = yes; 4th = yes; FF = yes; Umin = yes; min = yes; Max = yes; M347 = no
00 sub-system N° 7; value: 4 2 2 2 2 6 6; 5th = no; 4th = yes; FF = no; Umin = yes; min = yes; Max = yes; M347 = no

(0; 1; 4; 2 2 2 4 2 6 6); 5th = 3; 4th = 3; FF = 0; Umin = yes; Min = yes; Max = yes
00 sub-system N° 1; value: 2 2 2 4 2 6 6; 5th = no; 4th = yes; FF = no; Umin = yes; min = yes; Max = yes; M347 = no
00 sub-system N° 2; value: 2 2 4 2 6 6 2; 5th = no; 4th = yes; FF = no; Umin = yes; min = yes; Max = yes; M347 = no
00 sub-system N° 3; value: 2 4 2 6 6 2 2; 5th = yes; 4th = no; FF = no; Umin = yes; min = yes; Max = yes; M347 = no
00 sub-system N° 4; value: 4 2 6 6 2 2 2; 5th = no; 4th = no; FF = no; Umin = yes; min = yes; Max = yes; M347 = no
00 sub-system N° 5; value: 2 6 6 2 2 2 4; 5th = yes; 4th = no; FF = no; Umin = yes; min = yes; Max = yes; M347 = no
00 sub-system N° 6; value: 6 6 2 2 2 4 2; 5th = yes; 4th = no; FF = no; Umin = yes; min = yes; Max = yes; M347 = no
00 sub-system N° 7; value: 6 2 2 2 4 2 6; 5th = no; 4th = yes; FF = no; Umin = yes; min = yes; Max = yes; M347 = no

(0; 1; 5; 2 2 2 4 6 2 6); 5th = 4; 4th = 4; FF = 1; Umin = yes; Min = yes; Max = no
00 sub-system N° 1; value: 2 2 2 4 6 2 6; 5th = no; 4th = yes; FF = no; Umin = yes; min = yes; Max = no; M347 = no
00 sub-system N° 2; value: 2 2 4 6 2 6 2; 5th = yes; 4th = no; FF = no; Umin = yes; min = yes; Max = no; M347 = no
00 sub-system N° 3; value: 2 4 6 2 6 2 2; 5th = yes; 4th = no; FF = no; Umin = yes; min = yes; Max = no; M347 = no
00 sub-system N° 4; value: 4 6 2 6 2 2 2; 5th = no; 4th = yes; FF = no; Umin = yes; min = yes; Max = no; M347 = no
00 sub-system N° 5; value: 6 2 6 2 2 2 4; 5th = yes; 4th = no; FF = no; Umin = yes; min = yes; Max = no; M347 = no
00 sub-system N° 6; value: 2 6 2 2 2 4 6; 5th = yes; 4th = yes; FF = yes; Umin = yes; min = yes; Max = no; M347 = no
00 sub-system N° 7; value: 6 2 2 2 4 6 2; 5th = no; 4th = yes; FF = no; Umin = yes; min = yes; Max = no; M347 = no

(0; 1; 6; 2 2 2 6 2 4 6); 5th = 4; 4th = 4; FF = 2; Umin = yes; Min = yes; Max = no
00 sub-system N° 1; value: 2 2 2 6 2 4 6; 5th = yes; 4th = no; FF = no; Umin = yes; min = yes; Max = no; M347 = no
00 sub-system N° 2; value: 2 2 6 2 4 6 2; 5th = no; 4th = yes; FF = no; Umin = yes; min = yes; Max = no; M347 = no
00 sub-system N° 3; value: 2 6 2 4 6 2 2; 5th = yes; 4th = yes; FF = yes; Umin = yes; min = yes; Max = no; M347 = no
00 sub-system N° 4; value: 6 2 4 6 2 2 2; 5th = no; 4th = no; FF = no; Umin = yes; min = yes; Max = no; M347 = no
00 sub-system N° 5; value: 2 4 6 2 2 2 6; 5th = yes; 4th = no; FF = no; Umin = yes; min = yes; Max = no; M347 = no
00 sub-system N° 6; value: 4 6 2 2 2 6 2; 5th = yes; 4th = yes; FF = yes; Umin = yes; min = yes; Max = no; M347 = no
00 sub-system N° 7; value: 6 2 2 2 6 2 4; 5th = no; 4th = yes; FF = no; Umin = yes; min = yes; Max = no; M347 = no

10 sub-system N° 7; value: 6 2 2 6 2 4 2; 5th = no; 4th = yes; FF = no; Umin = no; min = yes; Max = no; M347 = no

(0; 1; 15; 2 2 6 2 6 2 4); 5th = 4; 4th = 4; FF = 1; Umin = no; Min = yes; Max = no

00 sub-system N° 1; value: 2 2 6 2 6 2 4; 5th = no; 4th = yes; FF = no; Umin = no; min = yes; Max = no; M347 = no

00 sub-system N° 2; value: 2 6 2 6 2 4 2; 5th = no; 4th = yes; FF = no; Umin = no; min = yes; Max = no; M347 = yes

00 sub-system N° 3; value: 6 2 6 2 4 2 2; 5th = yes; 4th = no; FF = no; Umin = no; min = yes; Max = no; M347 = no

00 sub-system N° 4; value: 2 6 2 4 2 2 6; 5th = yes; 4th = yes; FF = yes; Umin = no; min = yes; Max = no; M347 = no

00 sub-system N° 5; value: 6 2 4 2 2 6 2; 5th = yes; 4th = no; FF = no; Umin = no; min = yes; Max = no; M347 = yes

00 sub-system N° 6; value: 2 4 2 2 6 2 6; 5th = no; 4th = yes; FF = no; Umin = no; min = yes; Max = no; M347 = yes

00 sub-system N° 7; value: 4 2 2 6 2 6 2; 5th = yes; 4th = no; FF = no; Umin = no; min = yes; Max = no; M347 = no

00 sub-system N° 3; value: 2 6 3 2 5 2 4; 5th = no; 4th = no; FF = no; Umin = no; min = no; Max = no; M347 = no
00 sub-system N° 4; value: 6 3 2 5 2 4 2; 5th = no; 4th = no; FF = no; Umin = no; min = no; Max = no; M347 = no
00 sub-system N° 5; value: 3 2 5 2 4 2 6; 5th = no; 4th = yes; FF = no; Umin = no; min = no; Max = no; M347 = no
00 sub-system N° 6; value: 2 5 2 4 2 6 3; 5th = no; 4th = no; FF = no; Umin = no; min = no; Max = no; M347 = no
00 sub-system N° 7; value: 5 2 4 2 6 3 2; 5th = no; 4th = no; FF = no; Umin = no; min = no; Max = no; M347 = no

(0; 4;119; 2 4 3 2 5 2 6); 5th = 2; 4th = 2; FF = 1; Umin = no; Min = no; Max = no

00 sub-system N° 1; value: 2 4 3 2 5 2 6; 5th = no; 4th = no; FF = no; Umin = no; min = no; Max = no; M347 = no
00 sub-system N° 2; value: 4 3 2 5 2 6 2; 5th = yes; 4th = no; FF = no; Umin = no; min = no; Max = no; M347 = no
00 sub-system N° 3; value: 3 2 5 2 6 2 4; 5th = no; 4th = yes; FF = no; Umin = no; min = no; Max = no; M347 = no
00 sub-system N° 4; value: 2 5 2 6 2 4 3; 5th = no; 4th = no; FF = no; Umin = no; min = no; Max = no; M347 = no
00 sub-system N° 5; value: 5 2 6 2 4 3 2; 5th = no; 4th = no; FF = no; Umin = no; min = no; Max = no; M347 = no
00 sub-system N° 6; value: 2 6 2 4 3 2 5; 5th = yes; 4th = yes; FF = yes; Umin = no; min = no; Max = no; M347 = no
00 sub-system N° 7; value: 6 2 4 3 2 5 2; 5th = no; 4th = no; FF = no; Umin = no; min = no; Max = no; M347 = no

(0; 4;120; 2 4 3 2 6 2 5); 5th = 1; 4th = 1; FF = 0; Umin = no; Min = no; Max = no

00 sub-system N° 1; value: 2 4 3 2 6 2 5; 5th = no; 4th = no; FF = no; Umin = no; min = no; Max = no; M347 = no
00 sub-system N° 2; value: 4 3 2 6 2 5 2; 5th = no; 4th = no; FF = no; Umin = no; min = no; Max = no; M347 = no
00 sub-system N° 3; value: 3 2 6 2 5 2 4; 5th = no; 4th = no; FF = no; Umin = no; min = no; Max = no; M347 = no
00 sub-system N° 4; value: 2 6 2 5 2 4 3; 5th = no; 4th = yes; FF = no; Umin = no; min = no; Max = no; M347 = no
00 sub-system N° 5; value: 6 2 5 2 4 3 2; 5th = no; 4th = no; FF = no; Umin = no; min = no; Max = no; M347 = no
00 sub-system N° 6; value: 2 5 2 4 3 2 6; 5th = no; 4th = no; FF = no; Umin = no; min = no; Max = no; M347 = no
00 sub-system N° 7; value: 5 2 4 3 2 6 2; 5th = yes; 4th = no; FF = no; Umin = no; min = no; Max = no; M347 = no

00 sub-system N° 7; value: 5 2 2 5 5 3 2; 5th = yes; 4th = no; FF = no; Umin = no; min = yes; Max = yes; M347 = no

(0; 5; 16; 2 2 5 5 5 2 3); 5th = 3; 4th = 3; FF = 1; Umin = no; Min = yes; Max = yes

00 sub-system N° 1; value: 2 2 5 5 5 2 3; 5th = yes; 4th = no; FF = no; Umin = no; min = yes; Max = yes; M347 = no
00 sub-system N° 2; value: 2 5 5 5 2 3 2; 5th = no; 4th = no; FF = no; Umin = no; min = yes; Max = yes; M347 = yes
00 sub-system N° 3; value: 5 5 5 2 3 2 2; 5th = no; 4th = yes; FF = no; Umin = no; min = yes; Max = yes; M347 = no
00 sub-system N° 4; value: 5 5 2 3 2 2 5; 5th = no; 4th = yes; FF = no; Umin = no; min = yes; Max = yes; M347 = no
00 sub-system N° 5; value: 5 2 3 2 2 5 5; 5th = yes; 4th = yes; FF = yes; Umin = no; min = yes; Max = yes; M347 = yes
00 sub-system N° 6; value: 2 3 2 2 5 5 5; 5th = yes; 4th = no; FF = no; Umin = no; min = yes; Max = yes; M347 = yes
00 sub-system N° 7; value: 3 2 2 5 5 5 2; 5th = no; 4th = no; FF = no; Umin = no; min = yes; Max = yes; M347 = no

(0; 5; 17; 2 3 2 5 2 5 5); 5th = 3; 4th = 3; FF = 0; Umin = no; Min = no; Max = yes

00 sub-system N° 1; value: 2 3 2 5 2 5 5; 5th = yes; 4th = no; FF = no; Umin = no; min = no; Max = yes; M347 = no
00 sub-system N° 2; value: 3 2 5 2 5 5 2; 5th = no; 4th = yes; FF = no; Umin = no; min = no; Max = yes; M347 = no
00 sub-system N° 3; value: 2 5 2 5 5 2 3; 5th = yes; 4th = no; FF = no; Umin = no; min = no; Max = yes; M347 = no
00 sub-system N° 4; value: 5 2 5 5 2 3 2; 5th = no; 4th = no; FF = no; Umin = no; min = no; Max = yes; M347 = no
00 sub-system N° 5; value: 2 5 5 2 3 2 5; 5th = yes; 4th = no; FF = no; Umin = no; min = no; Max = yes; M347 = no
00 sub-system N° 6; value: 5 5 2 3 2 5 2; 5th = no; 4th = yes; FF = no; Umin = no; min = no; Max = yes; M347 = no
00 sub-system N° 7; value: 5 2 3 2 5 2 5; 5th = no; 4th = yes; FF = no; Umin = no; min = no; Max = yes; M347 = no

(0; 5; 18; 2 3 2 5 5 2 5); 5th = 3; 4th = 3; FF = 0; Umin = no; Min = no; Max = yes

00 sub-system N° 1; value: 2 3 2 5 5 2 5; 5th = no; 4th = no; FF = no; Umin = no; min = no; Max = yes; M347 = no
00 sub-system N° 2; value: 3 2 5 5 2 5 2; 5th = no; 4th = yes; FF = no; Umin = no; min = no; Max = yes; M347 = no
00 sub-system N° 3; value: 2 5 5 2 5 2 3; 5th = yes; 4th = no; FF = no; Umin = no; min = no; Max = yes; M347 = no
00 sub-system N° 4; value: 5 5 2 5 2 3 2; 5th = no; 4th = yes; FF = no; Umin = no; min = no; Max = yes; M347 = no
00 sub-system N° 5; value: 5 2 5 2 3 2 5; 5th = yes; 4th = no; FF = no; Umin = no; min = no; Max = yes; M347 = no
00 sub-system N° 6; value: 2 5 2 3 2 5 5; 5th = yes; 4th = no; FF = no; Umin = no; min = no; Max = yes; M347 = no
00 sub-system N° 7; value: 5 2 3 2 5 5 2; 5th = no; 4th = yes; FF = no; Umin = no; min = no; Max = yes; M347 = no

(0; 5; 19; 2 3 5 2 5 2 5); 5th = 3; 4th = 3; FF = 0; Umin = no; Min = no; Max = no

00 sub-system N° 1; value: 2 3 5 2 5 2 5; 5th = no; 4th = yes; FF = no; Umin = no; min = no; Max = no; M347 = no
00 sub-system N° 2; value: 3 5 2 5 2 5 2; 5th = no; 4th = yes; FF = no; Umin = no; min = no; Max = no; M347 = no
00 sub-system N° 3; value: 5 2 5 2 5 2 3; 5th = yes; 4th = no; FF = no; Umin = no; min = no; Max = no; M347 = no
00 sub-system N° 4; value: 2 5 2 5 2 3 5; 5th = yes; 4th = no; FF = no; Umin = no; min = no; Max = no; M347 = no
00 sub-system N° 5; value: 5 2 5 2 3 5 2; 5th = yes; 4th = no; FF = no; Umin = no; min = no; Max = no; M347 = no
00 sub-system N° 6; value: 2 5 2 3 5 2 5; 5th = no; 4th = no; FF = no; Umin = no; min = no; Max = no; M347 = no
00 sub-system N° 7; value: 5 2 3 5 2 5 2; 5th = no; 4th = yes; FF = no; Umin = no; min = no; Max = no; M347 = no

(0; 5; 20; 2 5 2 5 2 5 3); 5th = 3; 4th = 3; FF = 0; Umin = no; Min = no; Max = no

00 sub-system N° 1; value: 2 5 2 5 2 5 3; 5th = yes; 4th = no; FF = no; Umin = no; min = no; Max = no; M347 = no
00 sub-system N° 2; value: 5 2 5 2 5 3 2; 5th = yes; 4th = no; FF = no; Umin = no; min = no; Max = no; M347 = no
00 sub-system N° 3; value: 2 5 2 5 3 2 5; 5th = yes; 4th = no; FF = no; Umin = no; min = no; Max = no; M347 = no
00 sub-system N° 4; value: 5 2 5 3 2 5 2; 5th = no; 4th = no; FF = no; Umin = no; min = no; Max = no; M347 = no
00 sub-system N° 5; value: 2 5 3 2 5 2 5; 5th = no; 4th = yes; FF = no; Umin = no; min = no; Max = no; M347 = no
00 sub-system N° 6; value: 5 3 2 5 2 5 2; 5th = no; 4th = yes; FF = no; Umin = no; min = no; Max = no; M347 = no
00 sub-system N° 7; value: 3 2 5 2 5 2 5; 5th = no; 4th = yes; FF = no; Umin = no; min = no; Max = no; M347 = no

00 sub-system N° 7; value: 4 2 2 6 4 2 4; 5th = yes; 4th = no; FF = no; Umin = no; min = yes; Max = no; M347 = no

(0; 6; 16; 2 2 6 4 4 2 4); 5th = 5; 4th = 5; FF = 4; Umin = no; Min = yes; Max = no
00 sub-system N° 1; value: 2 2 6 4 4 2 4; 5th = yes; 4th = yes; FF = yes; Umin = no; min = yes; Max = no; M347 = no
00 sub-system N° 2; value: 2 6 4 4 2 4 2; 5th = no; 4th = no; FF = no; Umin = no; min = yes; Max = no; M347 = yes
00 sub-system N° 3; value: 6 4 4 2 4 2 2; 5th = yes; 4th = yes; FF = yes; Umin = no; min = yes; Max = no; M347 = no
00 sub-system N° 4; value: 4 4 2 4 2 2 6; 5th = yes; 4th = yes; FF = yes; Umin = no; min = yes; Max = no; M347 = no
00 sub-system N° 5; value: 4 2 4 2 2 6 4; 5th = yes; 4th = yes; FF = yes; Umin = no; min = yes; Max = no; M347 = yes
00 sub-system N° 6; value: 2 4 2 2 6 4 4; 5th = no; 4th = yes; FF = no; Umin = no; min = yes; Max = no; M347 = yes
00 sub-system N° 7; value: 4 2 2 6 4 4 2; 5th = yes; 4th = no; FF = no; Umin = no; min = yes; Max = no; M347 = no

(0; 6; 17; 2 4 2 4 2 4 6); 5th = 3; 4th = 3; FF = 0; Umin = no; Min = no; Max = no
00 sub-system N° 1; value: 2 4 2 4 2 4 6; 5th = yes; 4th = no; FF = no; Umin = no; min = no; Max = no; M347 = no
10 sub-system N° 2; value: 4 2 4 2 4 6 2; 5th = no; 4th = yes; FF = no; Umin = no; min = no; Max = no; M347 = no
00 sub-system N° 3; value: 2 4 2 4 6 2 4; 5th = no; 4th = no; FF = no; Umin = no; min = no; Max = no; M347 = no
00 sub-system N° 4; value: 4 2 4 6 2 4 2; 5th = no; 4th = yes; FF = no; Umin = no; min = no; Max = no; M347 = no
00 sub-system N° 5; value: 2 4 6 2 4 2 4; 5th = yes; 4th = no; FF = no; Umin = no; min = no; Max = no; M347 = no
00 sub-system N° 6; value: 4 6 2 4 2 4 2; 5th = no; 4th = yes; FF = no; Umin = no; min = no; Max = no; M347 = no
00 sub-system N° 7; value: 6 2 4 2 4 2 4; 5th = yes; 4th = no; FF = no; Umin = no; min = no; Max = no; M347 = no

(0; 6; 18; 2 4 2 4 2 6 4); 5th = 3; 4th = 3; FF = 0; Umin = no; Min = no; Max = no
00 sub-system N° 1; value: 2 4 2 4 2 6 4; 5th = yes; 4th = no; FF = no; Umin = no; min = no; Max = no; M347 = no
00 sub-system N° 2; value: 4 2 4 2 6 4 2; 5th = no; 4th = yes; FF = no; Umin = no; min = no; Max = no; M347 = no
00 sub-system N° 3; value: 2 4 2 6 4 2 4; 5th = yes; 4th = no; FF = no; Umin = no; min = no; Max = no; M347 = no
00 sub-system N° 4; value: 4 2 6 4 2 4 2; 5th = no; 4th = no; FF = no; Umin = no; min = no; Max = no; M347 = no
00 sub-system N° 5; value: 2 6 4 2 4 2 4; 5th = yes; 4th = no; FF = no; Umin = no; min = no; Max = no; M347 = no
00 sub-system N° 6; value: 6 4 2 4 2 4 2; 5th = no; 4th = yes; FF = no; Umin = no; min = no; Max = no; M347 = no
00 sub-system N° 7; value: 4 2 4 2 4 2 6; 5th = no; 4th = yes; FF = no; Umin = no; min = no; Max = no; M347 = no

(0; 6; 19; 2 4 2 4 4 2 6); 5th = 4; 4th = 4; FF = 2; Umin = no; Min = no; Max = no
10 sub-system N° 1; value: 2 4 2 4 4 2 6; 5th = no; 4th = no; FF = no; Umin = no; min = no; Max = no; M347 = no
11 sub-system N° 2; value: 4 2 4 4 2 6 2; 5th = yes; 4th = yes; FF = yes; Umin = no; min = no; Max = no; M347 = no
11 sub-system N° 3; value: 2 4 4 2 6 2 4; 5th = no; 4th = yes; FF = no; Umin = no; min = no; Max = no; M347 = no
11 sub-system N° 4; value: 4 4 2 6 2 4 2; 5th = no; 4th = yes; FF = no; Umin = no; min = no; Max = no; M347 = no
11 sub-system N° 5; value: 4 2 6 2 4 2 4; 5th = yes; 4th = no; FF = no; Umin = no; min = no; Max = no; M347 = no
11 sub-system N° 6; value: 2 6 2 4 2 4 4; 5th = yes; 4th = yes; FF = yes; Umin = no; min = no; Max = no; M347 = no
10 sub-system N° 7; value: 6 2 4 2 4 4 2; 5th = yes; 4th = no; FF = no; Umin = no; min = no; Max = no; M347 = no

(0; 6; 20; 2 4 2 6 2 4 4); 5th = 4; 4th = 4; FF = 2; Umin = no; Min = no; Max = no
11 sub-system N° 1; value: 2 4 2 6 2 4 4; 5th = yes; 4th = no; FF = no; Umin = no; min = no; Max = no; M347 = no
11 sub-system N° 2; value: 4 2 6 2 4 4 2; 5th = yes; 4th = no; FF = no; Umin = no; min = no; Max = no; M347 = no
11 sub-system N° 3; value: 2 6 2 4 4 2 4; 5th = yes; 4th = yes; FF = yes; Umin = no; min = no; Max = no; M347 = no
11 sub-system N° 4; value: 6 2 4 4 2 4 2; 5th = no; 4th = no; FF = no; Umin = no; min = no; Max = no; M347 = no
00 sub-system N° 5; value: 2 4 4 2 4 2 6; 5th = no; 4th = yes; FF = no; Umin = no; min = no; Max = no; M347 = no
11 sub-system N° 6; value: 4 4 2 4 2 6 2; 5th = yes; 4th = yes; FF = yes; Umin = no; min = no; Max = no; M347 = no
11 sub-system N° 7; value: 4 2 4 2 6 2 4; 5th = no; 4th = yes; FF = no; Umin = no; min = no; Max = no; M347 = no

00 sub-system N° 7; value: 3 2 4 3 4 2 6; 5th = no; 4th = no; FF = no; Umin = no; min = no; Max = no; M347 = no

(0; 9; 87; 2 4 3 4 3 2 6); 5th = 2; 4th = 2; FF = 1; Umin = no; Min = no; Max = no

00 sub-system N° 1; value: 2 4 3 4 3 2 6; 5th = no; 4th = no; FF = no; Umin = no; min = no; Max = no; M347 = no

00 sub-system N° 2; value: 4 3 4 3 2 6 2; 5th = yes; 4th = no; FF = no; Umin = no; min = no; Max = no; M347 = no

00 sub-system N° 3; value: 3 4 3 2 6 2 4; 5th = no; 4th = yes; FF = no; Umin = no; min = no; Max = no; M347 = no

00 sub-system N° 4; value: 4 3 2 6 2 4 3; 5th = no; 4th = no; FF = no; Umin = no; min = no; Max = no; M347 = no

00 sub-system N° 5; value: 3 2 6 2 4 3 4; 5th = no; 4th = no; FF = no; Umin = no; min = no; Max = no; M347 = no

00 sub-system N° 6; value: 2 6 2 4 3 4 3; 5th = yes; 4th = yes; FF = yes; Umin = no; min = no; Max = no; M347 = no

00 sub-system N° 7; value: 6 2 4 3 4 3 2; 5th = no; 4th = no; FF = no; Umin = no; min = no; Max = no; M347 = no

(0; 9; 88; 2 4 4 2 6 3 3); 5th = 2; 4th = 2; FF = 0; Umin = no; Min = no; Max = no

00 sub-system N° 1; value: 2 4 4 2 6 3 3; 5th = no; 4th = yes; FF = no; Umin = no; min = no; Max = no; M347 = no

00 sub-system N° 2; value: 4 4 2 6 3 3 2; 5th = no; 4th = yes; FF = no; Umin = no; min = no; Max = no; M347 = no

00 sub-system N° 3; value: 4 2 6 3 3 2 4; 5th = no; 4th = no; FF = no; Umin = no; min = no; Max = no; M347 = no

00 sub-system N° 4; value: 2 6 3 3 2 4 4; 5th = yes; 4th = no; FF = no; Umin = no; min = no; Max = no; M347 = no

00 sub-system N° 5; value: 6 3 3 2 4 4 2; 5th = yes; 4th = no; FF = no; Umin = no; min = no; Max = no; M347 = no

00 sub-system N° 6; value: 3 3 2 4 4 2 6; 5th = no; 4th = no; FF = no; Umin = no; min = no; Max = no; M347 = no

00 sub-system N° 7; value: 3 2 4 4 2 6 3; 5th = no; 4th = no; FF = no; Umin = no; min = no; Max = no; M347 = no

(0; 9; 89; 2 4 4 3 2 6 3); 5th = 1; 4th = 1; FF = 0; Umin = no; Min = no; Max = no

00 sub-system N° 1; value: 2 4 4 3 2 6 3; 5th = no; 4th = yes; FF = no; Umin = no; min = no; Max = no; M347 = no

00 sub-system N° 2; value: 4 4 3 2 6 3 2; 5th = no; 4th = no; FF = no; Umin = no; min = no; Max = no; M347 = no

00 sub-system N° 3; value: 4 3 2 6 3 2 4; 5th = no; 4th = no; FF = no; Umin = no; min = no; Max = no; M347 = no

00 sub-system N° 4; value: 3 2 6 3 2 4 4; 5th = yes; 4th = no; FF = no; Umin = no; min = no; Max = no; M347 = no

00 sub-system N° 5; value: 2 6 3 2 4 4 3; 5th = no; 4th = no; FF = no; Umin = no; min = no; Max = no; M347 = no

00 sub-system N° 6; value: 6 3 2 4 4 3 2; 5th = no; 4th = no; FF = no; Umin = no; min = no; Max = no; M347 = no

00 sub-system N° 7; value: 3 2 4 4 3 2 6; 5th = no; 4th = no; FF = no; Umin = no; min = no; Max = no; M347 = no

(0; 9; 90; 2 4 4 3 3 2 6); 5th = 3; 4th = 3; FF = 1; Umin = no; Min = no; Max = no

10 sub-system N° 1; value: 2 4 4 3 3 2 6; 5th = no; 4th = yes; FF = no; Umin = no; min = no; Max = no; M347 = no

11 sub-system N° 2; value: 4 4 3 3 2 6 2; 5th = yes; 4th = no; FF = no; Umin = no; min = no; Max = no; M347 = no

11 sub-system N° 3; value: 4 3 3 2 6 2 4; 5th = no; 4th = yes; FF = no; Umin = no; min = no; Max = no; M347 = no

11 sub-system N° 4; value: 3 3 2 6 2 4 4; 5th = yes; 4th = no; FF = no; Umin = no; min = no; Max = no; M347 = no

11 sub-system N° 5; value: 3 2 6 2 4 4 3; 5th = no; 4th = no; FF = no; Umin = no; min = no; Max = no; M347 = no

11 sub-system N° 6; value: 2 6 2 4 4 3 3; 5th = yes; 4th = yes; FF = yes; Umin = no; min = no; Max = no; M347 = no

00 sub-system N° 7; value: 6 2 4 4 3 3 2; 5th = no; 4th = no; FF = no; Umin = no; min = no; Max = no; M347 = no

00 sub-system N° 7; value: 3 2 5 2 5 3 4; 5th = no; 4th = yes; FF = no; Umin = no; min = no; Max = no; M347 = no

(0; 10; 87; 2 5 2 5 4 3 3); 5th = 2; 4th = 2; FF = 0; Umin = no; Min = no; Max = no

00 sub-system N° 1; value: 2 5 2 5 4 3 3; 5th = yes; 4th = no; FF = no; Umin = no; min = no; Max = no; M347 = no

00 sub-system N° 2; value: 5 2 5 4 3 3 2; 5th = no; 4th = no; FF = no; Umin = no; min = no; Max = no; M347 = no

00 sub-system N° 3; value: 2 5 4 3 3 2 5; 5th = yes; 4th = no; FF = no; Umin = no; min = no; Max = no; M347 = no

00 sub-system N° 4; value: 5 4 3 3 2 5 2; 5th = no; 4th = no; FF = no; Umin = no; min = no; Max = no; M347 = no

00 sub-system N° 5; value: 4 3 3 2 5 2 5; 5th = no; 4th = yes; FF = no; Umin = no; min = no; Max = no; M347 = no

00 sub-system N° 6; value: 3 3 2 5 2 5 4; 5th = no; 4th = no; FF = no; Umin = no; min = no; Max = no; M347 = no

00 sub-system N° 7; value: 3 2 5 2 5 4 3; 5th = no; 4th = yes; FF = no; Umin = no; min = no; Max = no; M347 = no

(0; 10; 88; 2 5 3 2 5 3 4); 5th = 4; 4th = 4; FF = 1; Umin = no; Min = no; Max = no

00 sub-system N° 1; value: 2 5 3 2 5 3 4; 5th = no; 4th = yes; FF = no; Umin = no; min = no; Max = no; M347 = no

00 sub-system N° 2; value: 5 3 2 5 3 4 2; 5th = no; 4th = yes; FF = no; Umin = no; min = no; Max = no; M347 = no

00 sub-system N° 3; value: 3 2 5 3 4 2 5; 5th = no; 4th = yes; FF = no; Umin = no; min = no; Max = no; M347 = no

10 sub-system N° 4; value: 2 5 3 4 2 5 3; 5th = yes; 4th = yes; FF = yes; Umin = no; min = no; Max = no; M347 = no

00 sub-system N° 5; value: 5 3 4 2 5 3 2; 5th = yes; 4th = no; FF = no; Umin = no; min = no; Max = no; M347 = no

00 sub-system N° 6; value: 3 4 2 5 3 2 5; 5th = yes; 4th = no; FF = no; Umin = no; min = no; Max = no; M347 = no

00 sub-system N° 7; value: 4 2 5 3 2 5 3; 5th = yes; 4th = no; FF = no; Umin = no; min = no; Max = no; M347 = no

(0; 10; 89; 2 5 3 2 5 4 3); 5th = 4; 4th = 4; FF = 1; Umin = no; Min = no; Max = no

00 sub-system N° 1; value: 2 5 3 2 5 4 3; 5th = no; 4th = yes; FF = no; Umin = no; min = no; Max = no; M347 = no

00 sub-system N° 2; value: 5 3 2 5 4 3 2; 5th = no; 4th = yes; FF = no; Umin = no; min = no; Max = no; M347 = no

00 sub-system N° 3; value: 3 2 5 4 3 2 5; 5th = yes; 4th = yes; FF = yes; Umin = no; min = no; Max = no; M347 = no

00 sub-system N° 4; value: 2 5 4 3 2 5 3; 5th = yes; 4th = no; FF = no; Umin = no; min = no; Max = no; M347 = no

00 sub-system N° 5; value: 5 4 3 2 5 3 2; 5th = yes; 4th = no; FF = no; Umin = no; min = no; Max = no; M347 = no

00 sub-system N° 6; value: 4 3 2 5 3 2 5; 5th = yes; 4th = no; FF = no; Umin = no; min = no; Max = no; M347 = no

00 sub-system N° 7; value: 3 2 5 3 2 5 4; 5th = no; 4th = yes; FF = no; Umin = no; min = no; Max = no; M347 = no

(0; 10; 90; 2 5 3 3 2 5 4); 5th = 2; 4th = 2; FF = 1; Umin = no; Min = no; Max = no

00 sub-system N° 1; value: 2 5 3 3 2 5 4; 5th = no; 4th = yes; FF = no; Umin = no; min = no; Max = no; M347 = no

00 sub-system N° 2; value: 5 3 3 2 5 4 2; 5th = no; 4th = no; FF = no; Umin = no; min = no; Max = no; M347 = no

00 sub-system N° 3; value: 3 3 2 5 4 2 5; 5th = no; 4th = no; FF = no; Umin = no; min = no; Max = no; M347 = no

00 sub-system N° 4; value: 3 2 5 4 2 5 3; 5th = yes; 4th = yes; FF = yes; Umin = no; min = no; Max = no; M347 = no

00 sub-system N° 5; value: 2 5 4 2 5 3 3; 5th = no; 4th = no; FF = no; Umin = no; min = no; Max = no; M347 = no

00 sub-system N° 6; value: 5 4 2 5 3 3 2; 5th = no; 4th = no; FF = no; Umin = no; min = no; Max = no; M347 = no

00 sub-system N° 7; value: 4 2 5 3 3 2 5; 5th = yes; 4th = no; FF = no; Umin = no; min = no; Max = no; M347 = no

hyper N° 12; val.: 2 2 4 4 4 4 4

sys : 3 ; 5ths: 12 ; 4ths: 12 ; FF 9

(0; 12; 1; 2 2 4 4 4 4 4); 5th = 2; 4th = 2; FF = 1; Umin = no; Min = yes; Max = no
00 sub-system N° 1; value: 2 2 4 4 4 4 4; 5th = no; 4th = no; FF = no; Umin = no; min = yes; Max = no; M347 = no
00 sub-system N° 2; value: 2 4 4 4 4 4 2; 5th = yes; 4th = yes; FF = yes; Umin = no; min = yes; Max = no; M347 = yes
00 sub-system N° 3; value: 4 4 4 4 4 2 2; 5th = no; 4th = no; FF = no; Umin = no; min = yes; Max = no; M347 = no
00 sub-system N° 4; value: 4 4 4 4 2 2 4; 5th = no; 4th = no; FF = no; Umin = no; min = yes; Max = no; M347 = no
00 sub-system N° 5; value: 4 4 4 2 2 4 4; 5th = yes; 4th = no; FF = no; Umin = no; min = yes; Max = no; M347 = yes
00 sub-system N° 6; value: 4 4 2 2 4 4 4; 5th = no; 4th = yes; FF = no; Umin = no; min = yes; Max = no; M347 = yes
00 sub-system N° 7; value: 4 2 2 4 4 4 4; 5th = no; 4th = no; FF = no; Umin = no; min = yes; Max = no; M347 = no

(0; 12; 2; 2 4 2 4 4 4 4); 5th = 4; 4th = 4; FF = 3; Umin = no; Min = no; Max = no
00 sub-system N° 1; value: 2 4 2 4 4 4 4; 5th = no; 4th = no; FF = no; Umin = no; min = no; Max = no; M347 = no
11 sub-system N° 2; value: 4 2 4 4 4 4 2; 5th = yes; 4th = yes; FF = yes; Umin = no; min = no; Max = no; M347 = no
00 sub-system N° 3; value: 2 4 4 4 4 2 4; 5th = yes; 4th = yes; FF = yes; Umin = no; min = no; Max = no; M347 = no
00 sub-system N° 4; value: 4 4 4 4 2 4 2; 5th = no; 4th = no; FF = no; Umin = no; min = no; Max = no; M347 = no
11 sub-system N° 5; value: 4 4 4 2 4 2 4; 5th = yes; 4th = no; FF = no; Umin = no; min = no; Max = no; M347 = no
11 sub-system N° 6; value: 4 4 2 4 2 4 4; 5th = yes; 4th = yes; FF = yes; Umin = no; min = no; Max = no; M347 = no
11 sub-system N° 7; value: 4 2 4 2 4 4 4; 5th = no; 4th = yes; FF = no; Umin = no; min = no; Max = no; M347 = no

(0; 12; 3; 2 4 4 2 4 4 4); 5th = 6; 4th = 6; FF = 5; Umin = no; Min = no; Max = no
11 sub-system N° 1; value: 2 4 4 2 4 4 4; 5th = no; 4th = yes; FF = no; Umin = no; min = no; Max = no; M347 = no
11 sub-system N° 2; value: 4 4 2 4 4 4 2; 5th = yes; 4th = yes; FF = yes; Umin = no; min = no; Max = no; M347 = no
11 sub-system N° 3; value: 4 2 4 4 4 2 4; 5th = yes; 4th = yes; FF = yes; Umin = no; min = no; Max = no; M347 = no
11 sub-system N° 4; value: 2 4 4 4 2 4 4; 5th = yes; 4th = yes; FF = yes; Umin = no; min = no; Max = no; M347 = no
11 sub-system N° 5; value: 4 4 4 2 4 4 2; 5th = yes; 4th = no; FF = no; Umin = no; min = no; Max = no; M347 = no
11 sub-system N° 6; value: 4 4 2 4 4 2 4; 5th = yes; 4th = yes; FF = yes; Umin = no; min = no; Max = no; M347 = no
11 sub-system N° 7; value: 4 2 4 4 2 4 4; 5th = yes; 4th = yes; FF = yes; Umin = no; min = no; Max = no; M347 = no

hyper N° 17; val.: 3333336

sys: 1; 5ths: 0; 4ths 0; FF 0

(0; 17; 1; 3 3 3 3 3 3 6); 5th = 0; 4th = 0; FF = 0; Umin = no; Min = no; Max = no
00 sub-system N° 1; value: 3 3 3 3 3 6; 5th = no; 4th = no; FF = no; Umin = no; min = no; Max = no; M347 = no
00 sub-system N° 2; value: 3 3 3 3 3 6 3; 5th = no; 4th = no; FF = no; Umin = no; min = no; Max = no; M347 = no
00 sub-system N° 3; value: 3 3 3 3 6 3 3; 5th = no; 4th = no; FF = no; Umin = no; min = no; Max = no; M347 = no
00 sub-system N° 4; value: 3 3 3 6 3 3 3; 5th = no; 4th = no; FF = no; Umin = no; min = no; Max = no; M347 = no
00 sub-system N° 5; value: 3 3 6 3 3 3 3; 5th = no; 4th = no; FF = no; Umin = no; min = no; Max = no; M347 = no
00 sub-system N° 6; value: 3 6 3 3 3 3 3; 5th = no; 4th = no; FF = no; Umin = no; min = no; Max = no; M347 = no
00 sub-system N° 7; value: 6 3 3 3 3 3 3; 5th = no; 4th = no; FF = no; Umin = no; min = no; Max = no; M347 = no
