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Exact and asymptotic reductions of surface radiation integrals with complex exponential arguments to efficient contour integrals

J.M.L. Bernard

1) Introduction

We present here an efficient method for the reduction of surface radiation integrals to non-singular contour integrals, when we suppose known, as in Physical Optics (PO) [1]-[5], the analytic expression of surface currents whose electromagnetic radiation is calculated. This method applies to a large class of oscillatory surface integrals with complex exponential arguments, allowing to save computer time as the surface dimensions increase. In what follows, we look for near-field solutions, which signifies here that we seek for expressions which give a correct result from infinity to a short distance to the surface, when the illumination is not necessarily plane but only locally plane on the surface. Although many authors have studied this type of problems [6]-[22], we present here a novel practical solution, leading to simple non-singular expressions [23], in the case of large planar or moderately curved surfaces :

- (a) in monostatic near-field case, for the radiation of PO currents carried by a perfectly or imperfectly reflective surface, excited by a point source whose radiation can be of arbitrary pattern, without any approximation in the complex exponential argument of the integrand,
- (b) in bistatic near-field case, when we consider integrands that have complex exponential arguments which are quadratic functions of the coordinates, as for gaussian beam propagation and scattering problems, or when the exponential argument of the Green function for the radiation of plane or curved plates is approached by second order approximation.
- (c) in monostatic near-field case, for the radiation of PO currents supported by curved plates whose principal curvatures have arbitrary signs, for convex, concave or saddle geometries.

It is worth noticing that the functions involved in case (a) and case (b) are clearly different, even if case (b) can also be considered for a reduction in monostatic case. The case (c) profits of complex scaling transformations and exact results developed in case (b) to extend the results of case (a) to curved surfaces.

The already known methods are not general enough to solve our problems, either because they depend too closely on Helmholtz or Maxwell equations [6]-[12], or they assume specific geometries [9]-[11],[13]-[15] or specific illuminations [16]-[20], or they don't allow to

consider complex exponential arguments [17] or complex surfaces due to a complex change of variables [6]-[22]; besides, they often don't propose non-singular general expressions [6],[8],[12],[19],[22], which leads to severe drawbacks, either for an exact calculus (see sect. 1 of [12] on the use of Asvestas works [9]-[11]) or for asymptotic derivations (see sect. 4 of [6]). In contrast, our global reduction approach does not assume that integrands depend on Helmholtz or Maxwell equations, it gives explicit non singular reduction and avoids the use of special function, and it allows to consider changes of variables with complex coordinates.

This chapter is organized as follows. After presenting the problem in section 2, we develop our method for an analytic reduction of surface integrals in section 3, then detail its application to case (a) (sections 4 and 5) and to case (b) with complex scaling (section 6). The use of complex scaling is then generalized in sections 7 and 8, and the results, developed initially in case (a) for plane plates, are extended to surfaces with low curvatures (case (c)) in section 9. The comparisons with classic surface integration results show excellent agreements for both cases (a), (b) and (c) cases, in section 11 on numerical results. Some developments are given in appendices, concerning in particular the reductions for imperfectly reflective surfaces.

2) Formulation

Let S be a regular open surface of low curvature, defined, in the (x', y', z') cartesian coordinate system of origin O' , as a smooth perturbation of a surface S' of the plane $z' = 0$ with unit normal $\hat{\mathbf{z}}$. We search to develop a method that allows the reduction to a contour integral along $\partial S'$ of the electric \mathbf{E} and magnetic \mathbf{H} fields, in free space of impedance Z_0 and wavenumber $k = \frac{2\pi}{\lambda}$, for a time dependence $e^{i\omega t}$, which are radiated by surface currents \mathbf{J} and \mathbf{M} on S [1, sect. 1.26] following,

$$\begin{aligned} \mathbf{E}(\mathbf{r}) &= \frac{Z_0}{4\pi i k} \int_S (\text{grad}_{\mathbf{r}} \text{div}_{\mathbf{r}} + k^2)(\psi \mathbf{J}) dS - \frac{1}{4\pi} \int_S \text{curl}_{\mathbf{r}}(\psi \mathbf{M}) dS, \\ Z_0 \mathbf{H}(\mathbf{r}) &= \frac{Z_0}{4\pi} \int_S \text{curl}_{\mathbf{r}}(\psi \mathbf{J}) dS + \frac{1}{4\pi i k} \int_S (\text{grad}_{\mathbf{r}} \text{div}_{\mathbf{r}} + k^2)(\psi \mathbf{M}) dS, \\ \psi(\tilde{R}) &= \frac{e^{-ik\tilde{R}}}{\tilde{R}}, \tilde{R} = \sqrt{\mathbf{R} \cdot \mathbf{R}}, \mathbf{R} = \mathbf{a} + z\hat{\mathbf{z}} - (\mathbf{r}' + z'\hat{\mathbf{z}}) = \tilde{R} \mathbf{u}, \end{aligned} \quad (1)$$

where $\mathbf{r} = (\mathbf{a} + z\hat{\mathbf{z}}) \notin S$ is the observation point, and $(\mathbf{r}' + z'\hat{\mathbf{z}})$ is the current point of integration on S , with $z' = \zeta_s(\mathbf{r}')$, $\mathbf{r}' = x'\hat{\mathbf{x}} + y'\hat{\mathbf{y}} \in S'$. We then assume that \mathbf{J} and \mathbf{M} are analytically defined. For example, we can consider that the currents, when excited by a point source at \mathbf{r}_0 , are of the form,

$$\sum_n \mathbf{f}_n(\mathbf{u}_0) \frac{e^{-ik\tilde{R}_0}}{(k\tilde{R}_0)^n}, \quad (2)$$

where $\mathbf{R}_0 = (\mathbf{r}' + z'\hat{\mathbf{z}}) - \mathbf{r}_0$, $\tilde{R}_0 = \sqrt{\mathbf{R}_0 \cdot \mathbf{R}_0}$, $\mathbf{u}_0 = \frac{\mathbf{R}_0}{\tilde{R}_0}$, and the \mathbf{f}_n are regular functions.

In what follows, we study the reduction of different classes of integrals derived from (1),
 - in monostatic case, when $\mathbf{r} = \mathbf{r}_0$, and thus $\mathbf{R} = -\mathbf{R}_0$ in (1), for a planar (case (a)) or moderately curved (case (c)) surface without any approximation of the exponential arguments in the integrand,

- in bistatic case, when \mathbf{r} and \mathbf{r}_0 can be distinct, for S with low curvature and complex illuminations of gaussian type (case (b)), when the integrals can be approximated in the form,

$$\frac{e^{-ik(\tilde{r}+\tilde{r}_0)}}{\tilde{r} \tilde{r}_0} \int_{S'} e^{(\alpha_1 x' + \alpha_2 y') + (\beta_1 x'^2 + \beta_2 y'^2 + \beta_3 x' y')} h(\mathbf{r}') dS', \quad (3)$$

with some smooth function $h(\mathbf{r}')$, $\tilde{r} = \sqrt{\mathbf{r} \cdot \mathbf{r}}$, $\tilde{r}_0 = \sqrt{\mathbf{r}_0 \cdot \mathbf{r}_0}$, and some complex constants α_i, β_j ; In this situation, exact results, in particular when $h(\mathbf{r}')$ is a polynomial, are obtained.

It is worth noticing that we don't need to suppose here that \mathbf{J} and \mathbf{M} derive from fields satisfying Helmholtz or Maxwell equations as in [6]-[12], so that we can consider a large class of functions or asymptotic approximations of integrands.

We develop, two fundamental ways of investigation for the reduction of a surface integral, then some basic examples illustrating their complementarity, which subsequently leads us to efficient solutions for our problems. For that, complex changes of variables, and thus domains of integration with complex coordinates will be considered for the reduction.

3) A combination of two fundamental ways for an efficient reduction to a contour integral

The reduction that we propose is carried out by a correct combination of two fundamental ways of investigation and associated general basic results, that are perfectly adapted to solve in a simple manner the class of problems that we study.

Definition 1

Let the plane $z = 0$ be defined with unit normal $\hat{\mathbf{z}}$, and a unit standard basis composed of $\hat{\mathbf{x}}$ and $\hat{\mathbf{y}} = \hat{\mathbf{z}} \times \hat{\mathbf{x}}$. Each point is denoted $\mathbf{r}' = x'\hat{\mathbf{x}} + y'\hat{\mathbf{y}}$ in (x', y') coordinates. In the applications, analytical changes of variables can transform the initial domain of integration with real coordinates into a domain with complex coordinates, The dot and cross products of vectors with complex coordinates are here considered as in real space (without complex

conjugation), following,

$$\begin{aligned} \mathbf{u} \cdot \mathbf{v} &= u_x v_x + u_y v_y, \quad (\mathbf{u})^2 = \mathbf{u} \cdot \mathbf{u} = u_x u_x + u_y u_y, \\ \mathbf{u} \times \mathbf{v} &= (u_x v_y - u_y v_x) \hat{\mathbf{z}}, \quad \hat{\mathbf{z}} \times \mathbf{u} = -u_y \hat{\mathbf{x}} + u_x \hat{\mathbf{y}}, \end{aligned} \quad (4)$$

when $\mathbf{u} = u_x \hat{\mathbf{x}} + u_y \hat{\mathbf{y}}$ and $\mathbf{v} = v_x \hat{\mathbf{x}} + v_y \hat{\mathbf{y}}$ express with complex numbers $u_{x(y)}$ and $v_{x(y)}$. In this situation, we assume that S' is a subset of a complex manifold S'_0 of the plane $z = 0$, both simply connected and oriented. The domains S'_0 and S' both result from the transformation of simply connected (2-dimensional) real manifolds of the plane $z = 0$ with contours positively oriented, by an invertible function P , that we suppose thereafter linear to simplify, with oriented contours both resulting from the transform by P of the real contours. Letting $P(\mathbf{u}) = P_x(\mathbf{u}) \hat{\mathbf{x}} + P_y(\mathbf{u}) \hat{\mathbf{y}}$, we notice that $P(\mathbf{u}) \cdot P(\mathbf{u}) = \prod_{\pm} (P_x(\mathbf{u}) \pm iP_y(\mathbf{u}))$.

On this occasion, we have used that a linear invertible transform of a (closed) simply connected domain is a (closed) simply connected domain (note : for an injective continuous transform, see the Browner's domain invariance theorem).

Remark : if we remove a simply connected subset from a simply connected domain, we create a hole, but this new domain remains the union of two simply connected domains. Consequently, it can be assumed that our results on reduction of integrals on simply connected surfaces, described in what follows, can be extended to multiply connected domains of integration.

3.1) First way

3.1.1) Some elements of analysis in real space

Let S' be first a real domain in cylindrical coordinates (ρ, φ) , and $F(\rho^2, \varphi)$ be a function which is regular on any straight line from the origin to any point of real S' , we can write,

$$\int_{S'} \frac{\partial F(\rho^2, \varphi)}{\partial(\rho^2)} dS' = \int_{\mathcal{L}} \int_{0^+}^{\rho(\varphi)} \frac{\partial F(\rho^2, \varphi)}{\partial(\rho^2)} \rho d\rho d\varphi = \frac{1}{2} \int_{\mathcal{L}} (F(\rho^2, \varphi) - F(0, \varphi)) d\varphi. \quad (5)$$

where \mathcal{L} is positively oriented. In this expression, $d\varphi = (\hat{\mathbf{z}} \times \frac{\bar{\rho}}{\rho^2}) \cdot d\bar{\rho}$ with $\bar{\rho}$ the current point of integration. By another way, we can recover it from the Stokes theorem [1, equ. (34)

chap.2.9], using,

$$\begin{aligned} \text{curl}(\widehat{z} \times \frac{\bar{\rho}}{\rho^2}(F(\rho^2, \varphi) - F(0, \varphi))) &= \widehat{z} \text{div}(\frac{\bar{\rho}}{\rho^2}(F(\rho^2, \varphi) - F(0, \varphi))) \\ &= \widehat{z} \frac{2\partial((\bar{\rho} \cdot \frac{\bar{\rho}}{\rho^2})(F(\rho^2, \varphi) - F(0, \varphi)))}{\partial(\rho^2)} = \widehat{z} \frac{2\partial(F(\rho^2, \varphi))}{\partial(\rho^2)}, \end{aligned} \quad (6)$$

when $F(\rho^2, \varphi) - F(0, \varphi)$ is analytic on S' , so that we have,

$$\int_{S'} \frac{\partial(F(\rho^2, \varphi))}{\partial(\rho^2)} dS' = \frac{1}{2} \int_{\mathcal{L}} (F(\rho^2, \varphi) - F(0, \varphi)) (\widehat{z} \times \frac{\bar{\rho}}{\rho^2}) \cdot d\bar{\rho}. \quad (7)$$

We note that, shifting the origin to a real point \mathbf{a} (fig. 1), with $\bar{\rho} = \mathbf{r}' - \mathbf{a}$, $\frac{\mathbf{r}' - \mathbf{a}}{\sqrt{(\mathbf{r}' - \mathbf{a})^2}} = \widehat{x} \cos \varphi + \widehat{y} \sin \varphi$,

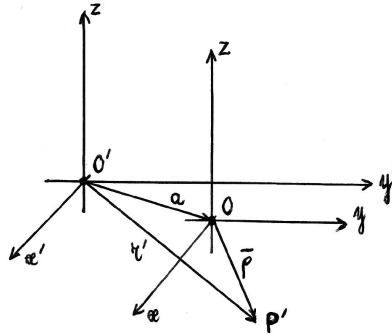


figure 1 : geometry and shift of the origin

we can write, when $F((\mathbf{r}' - \mathbf{a})^2, \varphi) - F(0, \varphi)$ remains analytic on S' ,

$$\int_{S'} \frac{\partial(F((\mathbf{r}' - \mathbf{a})^2, \varphi))}{\partial((\mathbf{r}' - \mathbf{a})^2)} dS' = \frac{1}{2} \int_{\mathcal{L}} (F((\mathbf{r}' - \mathbf{a})^2, \varphi) - F(0, \varphi)) \frac{\widehat{z} \times (\mathbf{r}' - \mathbf{a})}{(\mathbf{r}' - \mathbf{a})^2} \cdot d\mathbf{r}', \quad (8)$$

with $\int_{\mathcal{L}} d\varphi = 2\pi$ if $\mathbf{a} \in S'$ and $\int_{\mathcal{L}} d\varphi = 0$ if $\mathbf{a} \notin S'$, for real S' and \mathbf{a} .

3.1.2) Reduction in complex coordinates

The Stokes theorem, usually used in a domain with real coordinates, can be extended to domains with complex coordinates [24, p.213] by the formalism of exterior differential forms on complex manifolds, letting $dS' = (dx' \wedge dy')$ [25],[26, chap.7],[26, chap.3]. Considering the definition 1, S' is now a subset of a complex manifold S'_0 of the plane $z = 0$, both simply connected, resulting from a linear invertible transform of real connected domains with positively oriented contours.

Definition 2

Let $\mathbf{r}' = x'\hat{\mathbf{x}} + y'\hat{\mathbf{y}}$ denote a point of the 4-dimensional plane with complex coordinates x' and y' , and $\mathbf{a} = a_x\hat{\mathbf{x}} + a_y\hat{\mathbf{y}}$ denote a fixed complex point, with $\hat{\mathbf{z}} \times \hat{\mathbf{x}} = \hat{\mathbf{y}}$ and $\hat{\mathbf{y}} \times \hat{\mathbf{z}} = \hat{\mathbf{x}}$, we can write,

$$\begin{aligned} z'_\pm &= x' \pm iy' - (a_x \pm ia_y) = (x' - a_x) \pm i(y' - a_y) = (\mathbf{r}' - \mathbf{a}) \cdot (\hat{\mathbf{x}} \pm i\hat{\mathbf{y}}), \\ e^{\pm i\varphi} &= \frac{z'_\pm}{(z'_+ z'_-)^{\frac{1}{2}}}, \varphi = \mp \ln\left(\frac{z_\pm}{(z'_+ z'_-)^{\frac{1}{2}}}\right), (\mathbf{r}' - \mathbf{a})^2 = z'_+ z'_- = (z'_\pm e^{\mp i\varphi})^2, \\ \cos\varphi &= \frac{1}{2}\left(\frac{z_\pm}{(z'_+ z'_-)^{\frac{1}{2}}} + \left(\frac{z_\pm}{(z'_+ z'_-)^{\frac{1}{2}}}\right)^{-1}\right), \sin\varphi = \frac{1}{2i}\left(\frac{z_\pm}{(z'_+ z'_-)^{\frac{1}{2}}} - \left(\frac{z_\pm}{(z'_+ z'_-)^{\frac{1}{2}}}\right)^{-1}\right), \\ \mathbf{r}' - \mathbf{a} &= \frac{(z'_+ + z'_-)}{2}\hat{\mathbf{x}} + \frac{(z'_+ - z'_-)}{2i}\hat{\mathbf{y}} = z'_\pm e^{\mp i\varphi}(\cos\varphi \hat{\mathbf{x}} + \sin\varphi \hat{\mathbf{y}}), \end{aligned} \quad (9)$$

where the functions $(\cdot)^{1/2}$ and $\ln(\cdot)$ are analytical, apart from branch cuts, and the numbers $z'_+ z'_-$ and φ can be complex. We notice that $z_+ = z_- = 0$ only if $\mathbf{r}' = \mathbf{a}$, but that, in complex coordinates, the set of points satisfying $(\mathbf{r}' - \mathbf{a})^2 = 0$ are the two 2-dimensional straight lines verifying,

$$\begin{aligned} \mathbf{r}' &= \mathbf{a} + t(\hat{\mathbf{x}} \pm i\hat{\mathbf{y}}) \text{ where } z_\pm = 0, \text{ with arbitrary complex number } t, \\ \text{while } \hat{\mathbf{x}} \pm i\hat{\mathbf{y}} &= \frac{\mathbf{c} \pm i\hat{\mathbf{z}} \times \mathbf{c}}{c_x \mp ic_y} \text{ for any complex vector } \mathbf{c} \text{ with } \mathbf{c}^2 \neq 0. \end{aligned} \quad (10)$$

(note : if $(\mathbf{r}' - \mathbf{a})^2 \rightarrow 0$ while \mathbf{r}' does not tend to \mathbf{a} , $\cos\varphi$ and $\sin\varphi$ tend to infinity). We also note that we have,

$$d\varphi = \frac{-i}{2} d\ln\left(\frac{z'_+}{z'_-}\right) = \frac{-i}{2}\left(\frac{dz'_+}{z'_+} - \frac{dz'_-}{z'_-}\right) = (\hat{\mathbf{z}} \times \frac{\mathbf{r}' - \mathbf{a}}{(\mathbf{r}' - \mathbf{a})^2}) \cdot d\mathbf{r}'. \quad (11)$$

where the logarithm function has to be considered apart from its branch cut, or, more generally, by analytic continuation across it. The term $d\varphi$ is independent of the choice of $\hat{\mathbf{x}}$ and $\hat{\mathbf{y}} = \hat{\mathbf{z}} \times \hat{\mathbf{x}}$, and it vanishes along any line $\mathbf{r}' = \mathbf{a} + t\mathbf{v}$, with variable t and constant \mathbf{v} , so that φ remains identical for $\mathbf{r}' - \mathbf{a} = t\mathbf{v}$ as t varies. On this occasion, we also remark that $\frac{z'_+}{z'_-}$ describes a closed contour along any closed path in the (x', y') -plane.

Definition 3

Considering $\mathbf{a} = a_x\hat{\mathbf{x}} + a_y\hat{\mathbf{y}}$ a complex point of the plane, a function $F((\mathbf{r}' - \mathbf{a})^2, \varphi)$ is said to be c -regular relatively to $\mathbf{r}' = x'\hat{\mathbf{x}} + y'\hat{\mathbf{y}}$ on S'_0 (c for 'centered'), if it exists a 4-dimensional complex vicinity of S'_0 , where $F((\mathbf{r}' - \mathbf{a})^2, \varphi) - F(0, \varphi)$ is analytic relatively to $((\mathbf{r}' - \mathbf{a})^2)^{\frac{1}{2}}\cos\varphi = (x' - a_x)$ and $((\mathbf{r}' - \mathbf{a})^2)^{\frac{1}{2}}\sin\varphi = (y' - a_y)$, and vanishes when $(\mathbf{r}' - \mathbf{a})^2 \rightarrow 0$, while $\frac{\partial(F((\mathbf{r}' - \mathbf{a})^2, \varphi))}{\partial((\mathbf{r}' - \mathbf{a})^2)}$ is bounded. Noticing that the set of points verifying

$(\mathbf{r}' - \mathbf{a})^2 = 0$ is only a 2-dimensional manifold (see definition 2), $\frac{\partial(F((\mathbf{r}' - \mathbf{a})^2, \varphi))}{\partial((\mathbf{r}' - \mathbf{a})^2)}$ is also analytic relatively to $(x' - a_x)$ and $(y' - a_y)$ on S'_0 [28, sect. 23.2].

Let us assume that $F((\mathbf{r}' - \mathbf{a})^2, \varphi)$ is c-regular on S'_0 for any complex point \mathbf{a} belonging to it. We define the condition of continuation for $F((\mathbf{r}' - \mathbf{a})^2, \varphi)$ relatively to \mathbf{a} , from $\mathbf{a} = \mathbf{a}'$ to a complex point \mathbf{a} on S'_0 , as follows : it exists an open connected finite domain \mathcal{H} of complex points \mathbf{a} (4-dimensional set) including \mathbf{a}' and a point of S'_0 , where F remains c-regular on S'_0 .

Remark : considering c-regular functions $(\mathbf{r}' - \mathbf{a})^2(x' - a)^n$ and $(\mathbf{r}' - \mathbf{a})^2(y' - a)^n$, $n \geq 0$, we note that their partial derivatives with respects to $(\mathbf{r}' - \mathbf{a})^2$ are $(1 + n/2)(x' - a)^n$ and $(1 + n/2)(y' - a)^n$.

Remark : if $F(t, s)$ is an entire function of t independent of s , then the c-regularity on S' is verified for any \mathbf{a} . If $F((\mathbf{r}' - \mathbf{a})^2, \varphi)$ is c-regular on S'_0 , then it is also the case for $(\mathbf{r}' - \mathbf{a})^2(F((\mathbf{r}' - \mathbf{a})^2, \varphi) - F(0, \varphi))$, and for any primitive $\int^{(\mathbf{r}' - \mathbf{a})^2} (F(t, \varphi) - F(0, \varphi)) dt$ definite on S'_0 and at points with $(\mathbf{r}' - \mathbf{a})^2 = 0$.

Theorem 1

Let S' be a subdomain of a complex manifold S'_0 , both simply connected oriented complex manifolds, where S'_0 and thus S' result from a linear invertible transform of a real simply connected domain with positively oriented contours, as indicated in definition 1, with \mathcal{L} the oriented contour of S' resulting from this transformation. We consider $F((\mathbf{r}' - \mathbf{a})^2, \varphi)$ as a complex function satisfying the c-regularity on S'_0 , described in definition 3, for any point \mathbf{a} belonging to it. The equality

$$\int_{S'} \frac{\partial(F((\mathbf{r}' - \mathbf{a})^2, \varphi))}{\partial((\mathbf{r}' - \mathbf{a})^2)} dS' = \frac{1}{2} \int_{\mathcal{L}} (F((\mathbf{r}' - \mathbf{a})^2, \varphi) - F(0, \varphi)) \frac{\widehat{\mathbf{z}} \times (\mathbf{r}' - \mathbf{a})}{(\mathbf{r}' - \mathbf{a})^2} . d\mathbf{r}', \quad (12)$$

applies for $\mathbf{a} \in S'_0$, and for any \mathbf{a} verifying the condition of continuation in \mathcal{H} to S'_0 given in definition 3 (note : \mathbf{r}' denotes $x'\widehat{\mathbf{x}} + y'\widehat{\mathbf{y}}$ with $\widehat{\mathbf{y}} = \widehat{\mathbf{z}} \times \widehat{\mathbf{x}}$ and $\widehat{\mathbf{x}} = \widehat{\mathbf{y}} \times \widehat{\mathbf{z}}$). This equality is based on analytical continuation from real to complex coordinates, where it is worth noticing that $(\mathbf{r}' - \mathbf{a})^2 = 0$ does no more imply $\mathbf{r}' = \mathbf{a}$.

Proof. The equality (12) corresponds to a continuation of (8) for points with complex coordinates, in the domain of regularity of F . We first state (12) for $\mathbf{a} \in S'_0$. From the

properties of the exterior product and the definition of z'_+ and z'_- in (9), we can write,

$$\begin{aligned} dS' &= (dx' \wedge dy') = \frac{i}{2}(dz'_+ \wedge dz'_-) = \frac{-i}{4}(z'_- dz'_+ + z'_+ dz'_-) \wedge \left(\frac{dz'_+}{z'_+} - \frac{dz'_-}{z'_-} \right) \\ &= \frac{1}{2}d(z'_+ z'_-) \wedge \frac{-i}{2}d\ln\left(\frac{z'_+}{z'_-}\right) = \frac{1}{2}d(\mathbf{r}' - \mathbf{a})^2 \wedge d\varphi, \end{aligned} \quad (13)$$

From the definition 1, S'_0 results from a linear transformation of a simply connected real manifold relatively to (x', y') coordinates, and when $\mathbf{a} \in S'_0$, it can be mapped from \mathbf{a} to ∞ at constant φ with straight semi-lines, except on some cut(s) where φ is constant on both sides. In this case, we can integrate in the integral on S' relatively to $(\mathbf{r}' - \mathbf{a})^2$ variable from (13), and derive the contour integral along \mathcal{L} given in (12). Let us notice, that, when $\mathbf{a} \in S'$, we integrate from $\mathbf{r}' = \mathbf{a}$ to a point of \mathcal{L} at constant φ , while, when $\mathbf{a} \in S'_0$ but $\mathbf{a} \notin S'$, the limits of integration are two points of \mathcal{L} at constant φ .

From the condition of continuation given in definition 3, we can extend the validity of this equality to any point $\mathbf{a} = \mathbf{a}'$ belonging to the domain \mathcal{H} which comprises a subset S'_a of S'_0 . By definition of the c-regularity, the cuts in φ continue to have no influence; the left-hand side of (12) is analytic relatively to complex coordinates a_x and a_y ; the right-hand side, continuous at any point of \mathcal{H} , is analytic when $(\mathbf{r}' - \mathbf{a})^2 \neq 0$ on \mathcal{L} . That restriction disappears from the generalized principle of continuation [28, sect. 23.2], and the analyticity extends at any point \mathbf{a} of \mathcal{H} by noticing that $(\mathbf{r}' - \mathbf{a})^2|_{\mathbf{r}' \in \mathcal{L}} = 0$ describes a piecewise analytic p -dimensional manifold with $p \leq 3$. Therefore, considering that we have the analyticity in \mathcal{H} and that (12) is stated on S'_a , we can use the identity theorem [28, sect. 6.1], and conclude the validity of (12) for any point of \mathcal{H} .

Since the integration in (12) is on φ variable, the part of the contour integral, restricted to the factor $F(0, \varphi)$, could express in a simple form, and we now analyse it when $F(0, \varphi) \equiv 1$.

Proposition 1

let \mathcal{D}_1 be a smooth semi-line of the complex plane from the origin to ∞ , and \mathcal{L} a closed oriented contour of points $\mathbf{r}' = x'\hat{\mathbf{x}} + y'\hat{\mathbf{y}}$ with complex coordinates x' and y' . We have, when $(\mathbf{r}' - \mathbf{a})^2 \neq 0$ on \mathcal{L} ,

$$\int_{\mathcal{L}} (\hat{\mathbf{z}} \times \frac{\mathbf{r}' - \mathbf{a}}{(\mathbf{r}' - \mathbf{a})^2}) \cdot d\mathbf{r}' = \frac{-i}{2} \int_{\mathcal{L}} d\ln\left(\frac{z'_+}{z'_-}\right) = \pi(N_+ - N_-) \quad (14)$$

where N_+ (resp. N_-) is the number of times that the complex $\frac{z'_+}{z'_-}$ crosses \mathcal{D}_1 in the positive sense (resp. negative).

For a complex surface S' with complex contour \mathcal{L} verifying definition 1, i.e. resulting from a invertible linear transformation of a real simply connected surface with positively oriented contour, we can write, when $(\mathbf{r}' - \mathbf{a})^2 \neq 0$ on \mathcal{L} ,

$$\int_{\mathcal{L}} (\widehat{\mathbf{z}} \times \frac{\mathbf{r}' - \mathbf{a}}{(\mathbf{r}' - \mathbf{a})^2}) \cdot d\mathbf{r}' = \pi(\epsilon^+ 1_{S'_+}(a_x + ia_y) + \epsilon^- 1_{S'_-}(a_x - ia_y)). \quad (15)$$

Letting $x'_{\pm} = \text{Re}(x' \pm iy')$, $y'_{\pm} = \text{Im}(x' \pm iy')$, and noting that $x'_{\pm} + iy'_{\pm} = x' \pm iy'$, the functions $1_{S'_{\pm}}$ are the indicator functions of the domains S'_{\pm} of the complex plane, defined following,

$$\begin{aligned} 1_{S'_{\pm}}(z) &= \frac{1}{2\pi i(\pm \epsilon^{\pm})} \int_{\mathcal{L}_{\pm}} \frac{d(x'_{\pm} + iy'_{\pm})}{x'_{\pm} + iy'_{\pm} - z} = 1 \text{ if } z \in S'_{\pm}, = 0 \text{ if } z \notin S'_{\pm}, \\ S'_{\pm} &= \{z, 1_{S'_{\pm}}(z) = 1\} \equiv \{z = x' \pm iy', M(x', y') \in S'\}, \end{aligned} \quad (16)$$

where \mathcal{L}_{\pm} and ϵ^{\pm} verify,

$$\begin{aligned} \mathcal{L}_{\pm} &= \{z = x'_{\pm} + iy'_{\pm} (= x' \pm iy'), \text{ as } M(x', y') \text{ follows the contour } \mathcal{L}\}, \\ \pm \epsilon^{\pm} &= 1 \text{ (resp. } \pm \epsilon^{\pm} = -1) \text{ for } \mathcal{L}_{\pm} \text{ positively (resp. negatively) oriented.} \end{aligned} \quad (17)$$

Let us note that, for a real positively oriented contour \mathcal{L} and a point \mathbf{a} with real coordinates, we have $1_{S'_+}(a_x + ia_y) = 1_{S'_-}(a_x - ia_y)$, while the contours \mathcal{L}_+ and \mathcal{L}_- are respectively positively and negatively oriented, so that $\epsilon^{\pm} = 1$.

Proof. From (11), we have, when $(\mathbf{r}' - \mathbf{a})^2 \neq 0$ on \mathcal{L} ,

$$\int_{\mathcal{L}} (\widehat{\mathbf{z}} \times \frac{\mathbf{r}' - \mathbf{a}}{(\mathbf{r}' - \mathbf{a})^2}) \cdot d\mathbf{r}' = \frac{-i}{2} \int_{\mathcal{L}} d\ln\left(\frac{z'_+}{z'_-}\right), \quad (18)$$

where the logarithm function is considered by analytic continuation along the closed path covered by the uniform function $\frac{z'_+}{z'_-}$ on \mathcal{L} . We then obtain (14), since $(N_+ - N_-)$ is the number of turns done by $\frac{z'_+}{z'_-}$ around the origin, along the closed contour \mathcal{L} . For (15), (16) and (17), we write,

$$\begin{aligned} \int_{\mathcal{L}} (\widehat{\mathbf{z}} \times \frac{\mathbf{r}' - \mathbf{a}}{(\mathbf{r}' - \mathbf{a})^2}) \cdot d\mathbf{r}' &= \frac{-i}{2} \int_{\mathcal{L}} \left(\frac{dz'_+}{z'_+} - \frac{dz'_-}{z'_-} \right) \\ &= \frac{1}{2i} \int_{\mathcal{L}_+} \frac{d(x' + iy')}{x' + iy' - (a_x + ia_y)} - \frac{1}{2i} \int_{\mathcal{L}_-} \frac{d(x' - iy')}{x' - iy' - (a_x - ia_y)} \\ &= \epsilon^+ \pi 1_{S'_+}(a_x + ia_y) + \epsilon^- \pi 1_{S'_-}(a_x - ia_y), \end{aligned} \quad (19)$$

where the residue theorem is used, on the oriented complex contours $\mathcal{L}_\pm = \{x' \pm iy', \text{ as } M(x', y') \text{ follows the oriented contour } \mathcal{L}\}$ which limit the complex connected domains $S'_\pm = \{x' \pm iy', M(x', y') \in S'\}$, noticing that $x' \pm iy' = \text{Re}(x' \pm iy') + i\text{Im}(x' \pm iy')$.

Remark : A linear invertible transform of a (closed) simply connected domain is a (closed) simply connected domain (note : more generally, for an injective continuous transform, see the Browner's domain invariance theorem).

Proposition 2

Let \mathcal{W} be an open simply connected subset of the domain of analyticity of

$$\frac{\partial F((\mathbf{r}' - \mathbf{a})^2, \varphi)}{\partial (\mathbf{r}' - \mathbf{a})^2} = s(\mathbf{r}'), \quad (20)$$

relatively to the complex coordinates of $\mathbf{r}' = x'\hat{\mathbf{x}} + y'\hat{\mathbf{y}}$, that includes $\mathbf{r}' = \mathbf{a}$ and S' . Considering two points $\mathbf{r}'_{1,2}$ such that $(\mathbf{r}'_1 - \mathbf{a}) = t_1(\mathbf{r}'_2 - \mathbf{a})$ with $(\mathbf{r}'_2 - \mathbf{a})^2 \neq 0$, we suppose that it exists a complex path $\mathcal{T}_{\mathcal{L}}$ from $t = t_1$ to $t = 1$, defined such that $\mathbf{r}' = t(\mathbf{r}'_2 - \mathbf{a}) + \mathbf{a}$ remains in \mathcal{W} , for which, by definition, the complex angle φ of $(\mathbf{r}' - \mathbf{a})$ remains the same as t varies (see definition 2). Therefore, by analytic continuation in an open simply connected subset, we can integrate $s(\mathbf{r}')$ relatively to $(\mathbf{r}' - \mathbf{a})^2$ at constant φ along $\mathcal{T}_{\mathcal{L}}$ and write,

$$F((\mathbf{r}'_2 - \mathbf{a})^2, \varphi) - F((\mathbf{r}'_1 - \mathbf{a})^2, \varphi) = 2(\mathbf{r}'_2 - \mathbf{a})^2 \int_{\mathcal{T}_{\mathcal{L}}} s(t(\mathbf{r}'_2 - \mathbf{a}) + \mathbf{a}) t dt. \quad (21)$$

which could be used in (12) of theorem 1 when $\mathbf{r}'_1 = \mathbf{a}$.

Proof. For that we integrate s along $\mathcal{T}_{\mathcal{L}}$, relatively to $(\mathbf{r}' - \mathbf{a})^2$ variable at constant φ , by letting $\mathbf{r}' - \mathbf{a} = t(\mathbf{r}'_2 - \mathbf{a})$, and $d(\mathbf{r}' - \mathbf{a})^2 = 2(\mathbf{r}'_2 - \mathbf{a})^2 t dt$.

Remark : Let us notice that a complex anisotropic scaling can lead us to work on a complex manifold. If we dilate a real surface S' by a factor $p \neq 0$ along the direction \mathbf{c}_p with,

$$\mathcal{P} : \mathbf{r}' \rightarrow \mathbf{r}'_p = x'_p \hat{\mathbf{c}}_p + y'_p (\hat{\mathbf{z}} \times \hat{\mathbf{c}}_p), \quad x'_p = p(\mathbf{r}' \cdot \hat{\mathbf{c}}_p), \quad y'_p = \mathbf{r}' \cdot (\hat{\mathbf{z}} \times \hat{\mathbf{c}}_p), \quad (22)$$

where $\hat{\mathbf{c}}_p = \frac{\mathbf{c}_p}{\sqrt{\mathbf{c}_p \cdot \mathbf{c}_p}}$ (with $\mathbf{c}_p^2 \neq 0$ and $p \neq 0$), we obtain a complex surface S'_p , with

$$\begin{aligned} dS'_p &= dx'_p \wedge dy'_p = p dx' \wedge dy' = p dS', \\ \mathbf{v} \cdot \mathcal{P}(\mathbf{r}') &= \mathcal{P}(\mathbf{v}) \cdot \mathbf{r}', \quad (\hat{\mathbf{z}} \times \mathcal{P}(\mathbf{v})) \cdot \mathcal{P}(\mathbf{r}') = (\hat{\mathbf{z}} \times p\mathbf{v}) \cdot \mathbf{r}'. \end{aligned} \quad (23)$$

Remark : The choice to orient or not a domain stands out from the following equalities,

$$\int_{\chi([a,b])} f(x)dx = \int_{[a,b]} f(\chi(t))|\chi'(t)|dt, \quad \int_{\chi(a)}^{\chi(b)} f(x)dx = \int_a^b f(\chi(t))\chi'(t)dt, \quad (24)$$

by noticing that the first one, contrarily to the second, cannot be applied to complex a and b .

Remark : Let us notice that on each straight part of \mathcal{L} along $\hat{\mathbf{v}}$, with $d\mathbf{r}' = \hat{\mathbf{v}}ds$ and $\mathbf{r}' - \mathbf{a} = \mathbf{r}'_0 + \hat{\mathbf{v}}s$, we have $(\hat{\mathbf{z}} \times (\mathbf{r}' - \mathbf{a})) \cdot d\mathbf{r}' = \mathbf{g}_0 ds$, with constant \mathbf{r}'_0 and $\mathbf{g}_0 = \hat{\mathbf{z}} \times \mathbf{r}'_0$.

3.2) Second way

We consider here, as in previous section, the formalism of exterior differential forms, with $dS' = (dx' \wedge dy')$ [25]-[26, chap.7].

Theorem 2

Let S' be a simply connected oriented complex manifold of complex points $\mathbf{r}' = x'\hat{\mathbf{x}} + y'\hat{\mathbf{y}}$, satisfying the definition 1 and limited by the oriented contour \mathcal{L} , and a complex constant vector $\mathbf{a} = a_x\hat{\mathbf{x}} + a_y\hat{\mathbf{y}}$. We consider a function $F((\mathbf{r}' - \mathbf{a})^2, \varphi)$, analytic relatively to $((\mathbf{r}' - \mathbf{a})^2)^{\frac{1}{2}}\cos\varphi = (x' - a_x)$ and $((\mathbf{r}' - \mathbf{a})^2)^{\frac{1}{2}}\sin\varphi = (y' - a_y)$ on S' . We can write

$$\int_{S'} \mathbf{b} \cdot \text{grad}_{\mathbf{r}'}(F((\mathbf{r}' - \mathbf{a})^2, \varphi) - A_0)dS' = \int_{\mathcal{L}} (F((\mathbf{r}' - \mathbf{a})^2, \varphi) - A_0)(\hat{\mathbf{z}} \times \mathbf{b}) \cdot d\mathbf{r}', \quad (25)$$

where \mathbf{b} and A_0 are arbitrary constants, $\text{grad}_{\mathbf{r}'}$ is the gradient restricted to complex (x', y') coordinates of the plane $z = 0$, and the cut in φ can be ignored. From

$$\text{grad}_{\mathbf{r}'}(\cdot) = 2(\mathbf{r}' - \mathbf{a}) \frac{\partial(\cdot)}{\partial((\mathbf{r}' - \mathbf{a})^2)} + \frac{(\hat{\mathbf{z}} \times (\mathbf{r}' - \mathbf{a}))}{(\mathbf{r}' - \mathbf{a})^2} \frac{\partial(\cdot)}{\partial\varphi}, \quad (26)$$

we then have,

$$\begin{aligned} & \int_{S'} (\mathbf{r}' - \mathbf{a}) \cdot \left(\mathbf{b} \frac{\partial(F((\mathbf{r}' - \mathbf{a})^2, \varphi))}{\partial((\mathbf{r}' - \mathbf{a})^2)} - \frac{1}{2} \frac{(\hat{\mathbf{z}} \times \mathbf{b})}{(\mathbf{r}' - \mathbf{a})^2} \frac{\partial(F((\mathbf{r}' - \mathbf{a})^2, \varphi))}{\partial\varphi} \right) dS' \\ &= \frac{1}{2} \int_{\mathcal{L}} (F((\mathbf{r}' - \mathbf{a})^2, \varphi) - A_0)(\hat{\mathbf{z}} \times \mathbf{b}) \cdot d\mathbf{r}'. \end{aligned} \quad (27)$$

Proof. In real coordinates, we could use the Stokes theorem [1, chap.2.9], in combination with $\text{grad}(F((\mathbf{r}' - \mathbf{a})^2, \varphi)) \times \hat{\mathbf{d}} = \text{curl}(\hat{\mathbf{d}}F((\mathbf{r}' - \mathbf{a})^2, \varphi))$, where $\hat{\mathbf{d}}$ is a constant vector. More generally, in the formalism of exterior forms for complex manifolds, we prefer here to

use gradient in cartesian coordinates with $dS' = dx' \wedge dy' = -dy' \wedge dx'$, following

$$\int_{S'} b \cdot \text{grad}_{\mathbf{r}'}(F) dS' = \int_{S'} b \cdot \hat{\mathbf{x}} \frac{\partial F}{\partial x'} dx' \wedge dy' - \int_{S'} b \cdot \hat{\mathbf{y}} \frac{\partial F}{\partial y'} dy' \wedge dx'. \quad (28)$$

From the analyticity of F relatively to x' and y' , the cut in φ is omitted. We integrate $\frac{\partial F}{\partial x'} dx'$ and $\frac{\partial F}{\partial y'} dy'$ parts, while noticing $(\hat{\mathbf{z}} \times \mathbf{b}) \cdot d\mathbf{r}' = b \cdot \hat{\mathbf{x}} dy' - b \cdot \hat{\mathbf{y}} dx'$, and obtain (25). Then, from

$$\begin{aligned} dF &= \frac{\partial F}{\partial (\mathbf{r}' - \mathbf{a})^2} d(\mathbf{r}' - \mathbf{a})^2 + \frac{\partial F}{\partial \varphi} d\varphi = \text{grad}_{\mathbf{r}'}(F) \cdot d(\mathbf{r}' - \mathbf{a}), \\ d(\mathbf{r}' - \mathbf{a}) &= \frac{1}{2} \frac{\mathbf{r}' - \mathbf{a}}{(\mathbf{r}' - \mathbf{a})^2} d(\mathbf{r}' - \mathbf{a})^2 + (\hat{\mathbf{z}} \times (\mathbf{r}' - \mathbf{a})) d\varphi, \end{aligned} \quad (29)$$

we derive (26), which, combined with (25), gives (27).

Remark : We can simply recover the Cauchy integral theorem from (25). Letting $\epsilon = +1$ or -1 , we take $\mathbf{b} = \frac{1}{2}(\hat{\mathbf{x}} + i\epsilon\hat{\mathbf{y}})$ and $A_0 = 0$ in (25), with $F((\mathbf{r}' - \mathbf{a})^2, \varphi) = u(z_+, z_-)$ analytical on S' . We can use that $\mathbf{b} \cdot \text{grad}_{\mathbf{r}'} \equiv \frac{1}{2}(\frac{\partial}{\partial x'} + i\epsilon\frac{\partial}{\partial y'})$, $(\hat{\mathbf{z}} \times \mathbf{b}) = -\frac{i\epsilon}{2}(\hat{\mathbf{x}} + i\epsilon\hat{\mathbf{y}})$, $(\hat{\mathbf{x}} + i\epsilon\hat{\mathbf{y}}) \cdot d\mathbf{r}' = dz_\epsilon$, and write from (25),

$$\int_{S'} \left(\frac{\partial}{\partial x'} + i\epsilon \frac{\partial}{\partial y'} \right) u(z_+, z_-) dS' = -i\epsilon \int_{\mathcal{L}} u(z_+, z_-) dz_\epsilon. \quad (30)$$

In the particular case where $u(z_+, z_-) = v(z_\epsilon)$, considering the regularity on S' , we have $\frac{1}{2}(\frac{\partial}{\partial x'} \pm i\frac{\partial}{\partial y'})v(z_\pm) = \frac{\partial}{\partial z_\mp}v(z_\pm) = [\frac{\partial}{\partial z_\mp}(z_\pm)]v'(z_\pm) = 0$ on S' , and we recover the Cauchy integral theorem,

$$\int_{\mathcal{L}} v(z_\pm) dz_\pm = 0 \text{ when } v'(z_\pm) \text{ exists on whole } S'. \quad (31)$$

Remark : Using (25) with $F((\mathbf{r}' - \mathbf{a})^2, \varphi) = g(\mathbf{r}')F_1((\mathbf{r}' - \mathbf{a})^2)$, we have,

$$\int_{S'} \mathbf{b} \cdot \text{grad}_{\mathbf{r}'}(g(\mathbf{r}')F_1((\mathbf{r}' - \mathbf{a})^2)) dS' = \int_{\mathcal{L}} (g(\mathbf{r}')F_1((\mathbf{r}' - \mathbf{a})^2) - A_0)(\hat{\mathbf{z}} \times \mathbf{b}) \cdot d\mathbf{r}', \quad (32)$$

and it is worth noticing that, in asymptotics, when $F_1((\mathbf{r}' - \mathbf{a})^2)$ is highly oscillatory and $g(\mathbf{r}')$ remains smooth as $(\mathbf{r}' - \mathbf{a})^2$ varies, we have,

$$\begin{aligned} \text{grad}_{\mathbf{r}'}(g(\mathbf{r}')F((\mathbf{r}' - \mathbf{a})^2)) &= \text{grad}_{\mathbf{r}'}(g(\mathbf{r}'))F((\mathbf{r}' - \mathbf{a})^2) \\ &+ 2(\mathbf{r}' - \mathbf{a})g(\mathbf{r}') \frac{\partial F((\mathbf{r}' - \mathbf{a})^2)}{\partial ((\mathbf{r}' - \mathbf{a})^2)} \sim 2(\mathbf{r}' - \mathbf{a})g(\mathbf{r}') \frac{\partial F((\mathbf{r}' - \mathbf{a})^2)}{\partial ((\mathbf{r}' - \mathbf{a})^2)}. \end{aligned} \quad (33)$$

Proposition 3

Let $F_0((\mathbf{r}' - \mathbf{a})^2)$ be a scalar function of complex points $\mathbf{r}' = x'\hat{\mathbf{x}} + y'\hat{\mathbf{y}}$ analytic on S' , and \mathbf{F}_{0v} a vectorial function, whose each component is analytic on S' , we have,

$$\begin{aligned} \int_{S'} b \cdot (\mathbf{r}' - \mathbf{a}) \frac{\partial(F_0((\mathbf{r}' - \mathbf{a})^2))}{\partial((\mathbf{r}' - \mathbf{a})^2)} dS' &= \frac{1}{2} \int_{\mathcal{L}} F_0((\mathbf{r}' - \mathbf{a})^2) (\hat{\mathbf{z}} \times b) \cdot d\mathbf{r}' \\ \int_{S'} (\mathbf{r}' - \mathbf{a}) \cdot \frac{\partial(\mathbf{F}_{0v}((\mathbf{r}' - \mathbf{a})^2))}{\partial((\mathbf{r}' - \mathbf{a})^2)} dS' &= \frac{1}{2} \int_{\mathcal{L}} (\hat{\mathbf{z}} \times \mathbf{F}_{0v}((\mathbf{r}' - \mathbf{a})^2)) \cdot d\mathbf{r}'. \end{aligned} \quad (34)$$

Proof. Using the theorem 2, with $F((\mathbf{r}' - \mathbf{a})^2, \varphi) = F_0((\mathbf{r}' - \mathbf{a})^2)$ independent of φ , we obtain the first line, then the second line by considering each component of \mathbf{F}_{0v} as F_0 . On this occasion, let us remark that,

$$\begin{aligned} \text{if } \frac{\partial F_0((\mathbf{r}' - \mathbf{a})^2)}{\partial((\mathbf{r}' - \mathbf{a})^2)} &= \frac{e^{-v\tilde{R}}}{\tilde{R}} \text{ then } F_0((\mathbf{r}' - \mathbf{a})^2) = 2 \int^{\tilde{R}} e^{-vt} dt = \frac{-2e^{-v\tilde{R}}}{v} + C, \\ \text{if } \frac{\partial F_0((\mathbf{r}' - \mathbf{a})^2)}{\partial((\mathbf{r}' - \mathbf{a})^2)} &= e^{-v(\mathbf{r}' - \mathbf{a})^2} \text{ then } F_0((\mathbf{r}' - \mathbf{a})^2) = \frac{-e^{-v(\mathbf{r}' - \mathbf{a})^2}}{v} + C, \end{aligned} \quad (35)$$

where $\tilde{R} = \sqrt{\mathbf{R} \cdot \mathbf{R}} = \sqrt{(\mathbf{r}' - \mathbf{a})^2 + z^2}$, and C, v are some arbitrary constants.

Theorem 3

Let $B_0((\mathbf{r}' - \mathbf{a})^2, \varphi)$ be a function, analytic relatively to $((\mathbf{r}' - \mathbf{a})^2)^{\frac{1}{2}} \cos \varphi = (x' - a_x)$ and $((\mathbf{r}' - \mathbf{a})^2)^{\frac{1}{2}} \sin \varphi = (y' - a_y)$ on S' , we have, for $n \geq 0$,

$$\begin{aligned} \mathbf{d}_{n+1} \cdot \int_{S'} \prod_{j=1}^n (\mathbf{d}_j \cdot (\mathbf{r}' - \mathbf{a})) \text{grad}(B_0((\mathbf{r}' - \mathbf{a})^2, \varphi)) dS' \\ = \int_{\mathcal{L}} \prod_{j=1}^n (\mathbf{d}_j \cdot (\mathbf{r}' - \mathbf{a})) B_0((\mathbf{r}' - \mathbf{a})^2, \varphi) (\hat{\mathbf{z}} \times \mathbf{d}_{n+1}) \cdot d\mathbf{r}' \\ - \mathbf{d}_{n+1} \cdot \sum_{m=1}^n (\mathbf{d}_m \int_{S'} \prod_{j \neq m}^n (\mathbf{d}_j \cdot (\mathbf{r}' - \mathbf{a})) B_0((\mathbf{r}' - \mathbf{a})^2, \varphi) dS'), \end{aligned} \quad (36)$$

where the \mathbf{d}_j are constants, and we can let $\prod_j^0 (\dots) = 1, \prod_{j \neq m}^1 (\dots) = 1, \prod_{j \neq m}^0 (\dots) = 0$.

Proof. Since $f \text{grad}(h) = \text{grad}(fh) - h \text{grad}(f)$ and $\text{grad}(\mathbf{d}_m \cdot \mathbf{r}') = \mathbf{d}_m$, we have

$$\begin{aligned} \prod_{j=1}^n (\mathbf{d}_j \cdot (\mathbf{r}' - \mathbf{a})) \text{grad}(B_0((\mathbf{r}' - \mathbf{a})^2, \varphi)) &= \text{grad}(\prod_{j=1}^n (\mathbf{d}_j \cdot (\mathbf{r}' - \mathbf{a})) B_0((\mathbf{r}' - \mathbf{a})^2, \varphi)) \\ &- \sum_{m=1}^n \mathbf{d}_m \prod_{j \neq m}^n (\mathbf{d}_j \cdot (\mathbf{r}' - \mathbf{a})) B_0((\mathbf{r}' - \mathbf{a})^2, \varphi), \end{aligned} \quad (37)$$

Multiplying this equation by \mathbf{d}_{n+1} and integrating, then using (25), we obtain the proposition. This result will be useful, in particular when $B_0((\mathbf{r}' - \mathbf{a})^2, \varphi)$ is independent of the variable φ , as we will see in theorem 4.

3.3) Why these distinct ways are complementary: a practical example

The first and second theorems permit to consider respective reductions of surface integrals for functions of types $F((\mathbf{r}' - \mathbf{a})^2, \varphi)$ and $(\mathbf{r}' - \mathbf{a})F((\mathbf{r}' - \mathbf{a})^2, \varphi)$, which, correctly combined, avoid the use of fractional powers of $(\mathbf{r}' - \mathbf{a})^2$ and of special functions. Let us illustrate it by some basic results.

For that, let us show how to reduce, for arbitrary integer $n + 1 \geq 0$,

$$\begin{aligned} I_{n+1} &= \int_{S'} \prod_{j=1}^{n+1} (\mathbf{d}_j \cdot (\mathbf{r}' - \mathbf{a})) F_0((\mathbf{r}' - \mathbf{a})^2) dS', \\ \text{with } I_0 &= \int_{S'} F_0((\mathbf{r}' - \mathbf{a})^2) dS', \end{aligned} \quad (38)$$

to a contour integral of the plane $z = 0$, where the polynomial $\prod_{j=1}^{n+1} (\mathbf{d}_j \cdot (\mathbf{r}' - \mathbf{a}))$ is defined with arbitrary complex constant vectors \mathbf{d}_j of the plane $z = 0$, and $\prod_{j=1}^0 (\dots) = 1$.

We now state, from theorem 1, a specific proposition for the even case $n + 1 = 2q$, then, using the theorem 2, a general proposition for arbitrary integer n . In this example, the use of both theorem will be necessary to solve the problem in whole generality. We consider in what follows, that we are in an open connected domain of the variable \mathbf{a} including a point of S' , where $F_0((\mathbf{r}' - \mathbf{a})^2)$ remains analytic on S' , which allows the application of theorems 1 and 2.

3.3.1) Exact reduction for an even polynomial factors

Proposition 4

We have, from the theorem 1, for $n + 1 = 2q \geq 0$,

$$\begin{aligned}
 I_{2q} &= \int_{S'} \prod_{j=1}^{2q} (\mathbf{d}_j \cdot (\mathbf{r}' - \mathbf{a})) F_0((\mathbf{r}' - \mathbf{a})^2) dS' \\
 &= \frac{1}{2} \int_{\mathcal{L}} G_{2q}(\varphi) (B_{2q}((\mathbf{r}' - \mathbf{a})^2) - B_{2q}(0)) [(\hat{\mathbf{z}} \times \frac{\mathbf{r}' - \mathbf{a}}{(\mathbf{r}' - \mathbf{a})^2}) \cdot d\mathbf{r}'] \quad (39)
 \end{aligned}$$

where,

$$\begin{aligned}
 B_{2q}((\mathbf{r}' - \mathbf{a})^2) &= \int_0^{(\mathbf{r}' - \mathbf{a})^2} t^q F_0(t) dt, \quad G_{2q}(\varphi) ((\mathbf{r}' - \mathbf{a})^2)^q = \prod_{j=1}^{2q} \mathbf{d}_j \cdot (\mathbf{r}' - \mathbf{a}), \\
 B_0((\mathbf{r}' - \mathbf{a})^2) &= \int_0^{(\mathbf{r}' - \mathbf{a})^2} F_0(t) dt, \quad G_0(\varphi) = 1, \quad (40)
 \end{aligned}$$

with constant \mathbf{d}_j , and the case $q = 0$ is considered with $\prod_{j=1}^0 (\cdot) \equiv 1$.

Proof. Letting $\frac{\partial(F((\mathbf{r}' - \mathbf{a})^2), \varphi)}{\partial((\mathbf{r}' - \mathbf{a})^2)} = G_{2q}(\varphi) (\mathbf{r}' - \mathbf{a})^{2q} F_0((\mathbf{r}' - \mathbf{a})^2)$ in theorem 1, we obtain the equality (39), where we notice that B_{2q} depends on $q - 1$ successive primitives of F_0 . It is worth noticing that we have,

$$\begin{aligned}
 B_{2q}((\mathbf{r}' - \mathbf{a})^2) &= \int_0^{(\mathbf{r}' - \mathbf{a})^2} t^q e^{-vt} dt + C \text{ if } F_0((\mathbf{r}' - \mathbf{a})^2) = e^{-v(\mathbf{r}' - \mathbf{a})^2}, \\
 B_{2q}((\mathbf{r}' - \mathbf{a})^2) &= 2 \int_{\sqrt{z^2}}^{\tilde{R}} (t^2 - z^2)^q \frac{e^{-vt}}{t^{m-1}} dt + C \text{ if } F_0((\mathbf{r}' - \mathbf{a})^2) = \frac{e^{-v\tilde{R}}}{\tilde{R}^m}, \quad (41)
 \end{aligned}$$

where $\tilde{R} = \sqrt{\mathbf{R} \cdot \mathbf{R}} = \sqrt{(\mathbf{r}' - \mathbf{a})^2 + z^2}$, $\mathbf{r}' = x' \hat{\mathbf{x}} + y' \hat{\mathbf{y}}$, while \mathbf{a} , v , C and z are some constants. In these cases, the function $B_{2q}((\mathbf{r}' - \mathbf{a})^2) - B_{2q}(0)$ can be expressed in closed form or asymptotically.

3.3.2) Exact reduction for polynomial factors of arbitrary order

A direct use of the theorem 1 for I_{n+1} with odd $n + 1$, would lead to a contour integral with fractional powers of $(\mathbf{r}' - \mathbf{a})^2$. To avoid it, we now state a relation that lowers the degree of the polynomial in whole generality. The repeated use of it, in combination of results for $n = 0$ (proposition 3) and $n + 1 = 0$ (proposition 4), allows the reduction without fractional powers for polynomial factors of arbitrary order.

Theorem 4

We have,

$$\begin{aligned} I_0 &= \int_{S'} F_0((\mathbf{r}' - \mathbf{a})^2) dS' \\ &= \frac{1}{2} \int_{\mathcal{L}} (B_0((\mathbf{r}' - \mathbf{a})^2) - B_0(0)) [(\hat{\mathbf{z}} \times \frac{\mathbf{r}' - \mathbf{a}}{(\mathbf{r}' - \mathbf{a})^2}) \cdot d\mathbf{r}'], \end{aligned} \quad (42)$$

and for arbitrary integer $n \geq 0$,

$$\begin{aligned} I_{n+1} &= \int_{S'} \mathbf{d}_{n+1} \cdot (\mathbf{r}' - \mathbf{a}) \prod_{j=1}^n (\mathbf{d}_j \cdot (\mathbf{r}' - \mathbf{a})) F_0((\mathbf{r}' - \mathbf{a})^2) dS' \\ &= \frac{1}{2} \int_{\mathcal{L}} \prod_{j=1}^n (\mathbf{d}_j \cdot (\mathbf{r}' - \mathbf{a})) B_0((\mathbf{r}' - \mathbf{a})^2) (\hat{\mathbf{z}} \times \mathbf{d}_{n+1}) \cdot d\mathbf{r}' \\ &\quad - \frac{1}{2} \sum_{m=1}^n ((\mathbf{d}_{n+1} \cdot \mathbf{d}_m) \int_{S'_{j \neq m}} \prod_{j=1}^n (\mathbf{d}_j \cdot (\mathbf{r}' - \mathbf{a})) B_0((\mathbf{r}' - \mathbf{a})^2) dS'), \end{aligned} \quad (43)$$

where B_0 is a primitive of F_0 , the \mathbf{d}_j , numbered from 1 to $n + 1$, are arbitrary constant vectors of the plane $z = 0$, $\mathbf{a} = a_x \hat{\mathbf{x}} + a_y \hat{\mathbf{y}}$ is a constant vector, $\mathbf{r}' = x' \hat{\mathbf{x}} + y' \hat{\mathbf{y}}$ is the current point of integration on the plane, and $\prod_j^0 (\dots) = 1$, $\prod_{j \neq m}^1 (\dots) = 1$, and $\prod_{j \neq m}^0 (\dots) = 0$.

When $n + 1$ is odd or even, this equation allows, by repeated applications, to reduce I_{n+1} to a combination of contour integrals, depending on $M = \lfloor (n + 3)/2 \rfloor$ successive primitives of F_0 , denoted $F_0^{-(q+1)}(x) = \frac{1}{q!} \int^x (x - t)^q F_0(t) dt$, $q = 0, \dots, M - 1$.

Proof. Using the theorem 3 with the expression (26) of the gradient, we derive (43) of the proposition. The case $n = 0$ corresponds to proposition 3. The equation (43), after repeated applications, allows the reduction of the polynomial rank in surface integral term, until a complete reduction of I_{n+1} to a combination of contour integrals. The result then depend on a set of $\lfloor (n + 1)/2 \rfloor + 1$ (i.e. $\lfloor (n + 3)/2 \rfloor$) successive primitives F_0^{-q} of F_0 , which can be expressed by the use of integration by parts following $F_0^{-q}(x) = \frac{1}{q!} \int^x (x - t)^q F_0(t) dt$.

A generalization of the proposition will be derived in theorem 13, by considering a function of \mathcal{R}^2 instead of $(\mathbf{r}' - \mathbf{a})^2$, where \mathcal{R}^2 is an arbitrary polynomial of x' and y' of rank 2.

Remark : The reduction to a contour integral is not unique. When we use the theorem 4, the order of the \mathbf{d}_j could have an influence on the form of final reduction.

Remark : Letting $I_{n+1} = \mathbf{d}_{n+1} \cdot \mathcal{I}_n$, we note that,

$$\begin{aligned}
 \mathcal{I}_n &= \int_{S'} (\mathbf{r}' - \mathbf{a}) \prod_{j=1}^n (\mathbf{d}_j \cdot (\mathbf{r}' - \mathbf{a})) F_0((\mathbf{r}' - \mathbf{a})^2) dS' \\
 &= \frac{-1}{2} ((\hat{\mathbf{z}} \times \int_{\mathcal{L}} \prod_{j=1}^n (\mathbf{d}_j \cdot (\mathbf{r}' - \mathbf{a})) B_0((\mathbf{r}' - \mathbf{a})^2) d\mathbf{r}') \\
 &\quad + \sum_{m=1}^n (\mathbf{d}_m \int_{S'_{j \neq m}} \prod_{j=1}^n (\mathbf{d}_j \cdot (\mathbf{r}' - \mathbf{a})) B_0((\mathbf{r}' - \mathbf{a})^2) dS')). \tag{44}
 \end{aligned}$$

Remark : The reduction to a contour integral generally supposes the knowledge of a primitive relatively to $(\mathbf{r}' - \mathbf{a})^2$. When it cannot be expressed in closed form, its expression can often be derived asymptotically as in lemma 2 (see appendix).

4) Reduction to contour integrals in case (a), for a pure dipolar illumination of a large perfectly conducting flat plate

We study here the reduction of Physical Optics radiation integrals belonging to case (a) for pure electric and magnetic dipolar illumination, from results of previous sections. A rewriting of the PO integrals, and the expressions of primitives involved in the reduction, are given for flat surfaces. A generalization for non dipolar illuminations and for the case of S with low curvature will be respectively given in section 5 and 9.

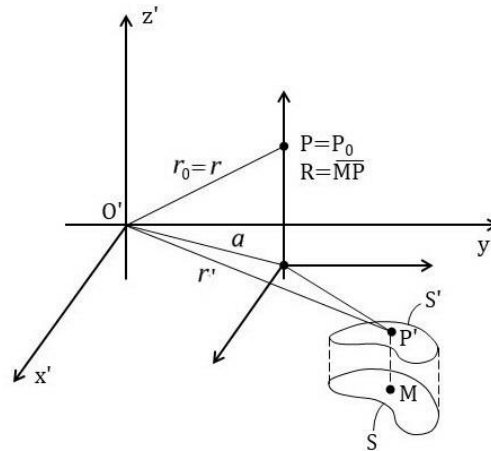


figure 2 : diffraction by S illuminated by source at $\mathbf{r}_0 = \mathbf{r}$ (monostatic case)

Our method allows to study the PO diffraction for perfectly and imperfectly conducting surfaces. To simplify the reading, we choose to describe the latter case (surface with impedance boundary conditions) in appendix.

4.1) Basic expansion of kernels in general radiation surface integral expressions

Let us consider the expressions of the electric and magnetic fields, \mathbf{E} and \mathbf{H} , radiated by electric and magnetic currents \mathbf{J} and \mathbf{M} impressed on a real arbitrary surface S , in free space with impedance Z_0 and wavenumber $k = \frac{2\pi}{\lambda}$, at the observation point $\mathbf{r} = (\mathbf{a} + z\hat{\mathbf{z}}) \notin S$ [1], given in (1) of section 2, where $\psi(\tilde{R}) = \frac{e^{-ik\tilde{R}}}{\tilde{R}}$, $\tilde{R} = \sqrt{\mathbf{R} \cdot \mathbf{R}}$, $\mathbf{R} = \mathbf{a} + z\hat{\mathbf{z}} - (\mathbf{r}' + z'\hat{\mathbf{z}})$, $\mathbf{R} = \tilde{R} \mathbf{u}$, and $(\mathbf{r}' + z'\hat{\mathbf{z}})$ is the point of integration on S . We can write in this expression,

$$\begin{aligned} (\text{grad}_r \text{div}_r + k^2)(\psi \mathbf{J}) &= \tilde{R} \left(\frac{\psi'}{\tilde{R}} \right)' (\mathbf{J} \cdot \mathbf{u}) \mathbf{u} + \frac{\psi'}{\tilde{R}} \mathbf{J} + k^2 \psi \mathbf{J}, \\ \text{curl}_r(\psi \mathbf{J}) &= -\mathbf{J} \times \text{grad}_r \psi = \frac{\psi'}{\tilde{R}} \mathbf{R} \times \mathbf{J}, \end{aligned} \quad (45)$$

where $\psi' = \partial_{\tilde{R}} \psi$ and $\left(\frac{\psi'}{\tilde{R}} \right)' = \partial_{\tilde{R}} \left(\frac{\psi'}{\tilde{R}} \right)$. To simplify, from $\Delta \psi = \frac{1}{\tilde{R}^2} \left((\tilde{R})^3 \frac{\psi'}{\tilde{R}} \right)' = -k^2 \psi$, we then note that $\tilde{R} \left(\frac{\psi'}{\tilde{R}} \right)' + k^2 \psi = -3 \frac{\psi'}{\tilde{R}}$, and with $(\mathbf{J} \cdot \mathbf{u}) \mathbf{u} = \mathbf{u} \times (\mathbf{u} \times \mathbf{J}) + \mathbf{J}$, we derive,

$$(\text{grad}_r \text{div}_r + k^2)(\psi \mathbf{J}) = \tilde{R} \left(\frac{\psi'}{\tilde{R}} \right)' \mathbf{u} \times (\mathbf{u} \times \mathbf{J}) - \frac{2\psi'}{\tilde{R}} \mathbf{J}. \quad (46)$$

This gives us, at $\mathbf{r} = (\mathbf{a} + z\hat{\mathbf{z}}) \notin S$,

$$\begin{aligned} \mathbf{E} &= \frac{Z_0}{4\pi i k} \int_S \left(\tilde{R} \left(\frac{\psi'}{\tilde{R}} \right)' \mathbf{u} \times (\mathbf{u} \times \mathbf{J}) - \frac{2\psi'}{\tilde{R}} \mathbf{J} \right) dS - \frac{1}{4\pi} \int_S \frac{\psi'}{\tilde{R}} \mathbf{R} \times \mathbf{M} dS, \\ Z_0 \mathbf{H} &= \frac{Z_0}{4\pi} \int_S \frac{\psi'}{\tilde{R}} \mathbf{R} \times \mathbf{J} dS + \frac{1}{4\pi i k} \int_S \left(\tilde{R} \left(\frac{\psi'}{\tilde{R}} \right)' \mathbf{u} \times (\mathbf{u} \times \mathbf{M}) - \frac{2\psi'}{\tilde{R}} \mathbf{M} \right) dS. \end{aligned} \quad (47)$$

We now study the case where S is equal to its projection S' on the plane $z = 0$. A generalization for the case of S with low curvature will be given in sections 8 and 9.

Remark : letting $\mathbf{E} = \int_S \mathcal{G}_e dS$ and $\mathbf{H} = \int_S \mathcal{G}_h dS$, we have $\mathcal{G}_e \sim Z_0 (\mathcal{G}_h \times \mathbf{u})$ for large $k\tilde{R}$.

4.2) Exact reduction to contour integrals when S is a flat plate S' perfectly conducting

In monostatic case, the primary point source and the observation point have the same position. The monostatic Physical Optics radiation integrals in perfectly conducting case ($\mathbf{M}|_S = 0$), for electric and magnetic dipolar illuminations, can be then reworked in such a form that all surface integrals can be exactly reduced to contour integrals, with all required primitives in closed forms.

Lemma 1

Let us consider a surface integral on a flat plate S' of the plane $z = 0$, which is a combination of integrals of the type,

$$\int_{S'} B(\mathbf{R}^2) dS', \int_{S'} \frac{B(\mathbf{R}^2)(\mathbf{R}^2 - z^2)}{\mathbf{R}^2} G(\varphi) dS' \text{ and } \int_{S'} (\mathbf{a} - \mathbf{r}') B(\mathbf{R}^2) dS', \quad (48)$$

where $\mathbf{R}^2 = (\mathbf{r}' - \mathbf{a})^2 + z^2$, $G(\varphi) = \frac{((\mathbf{r}' - \mathbf{a}) \cdot \hat{\mathbf{d}})(\mathbf{r}' - \mathbf{a})}{(\mathbf{r}' - \mathbf{a})^2}$, with $z \neq 0$ and $\hat{\mathbf{d}}$ some constants. If the point \mathbf{a} belongs to an open connected domain \mathcal{H} of analyticity which includes a point of S' (see definition 3), where $B(\mathbf{R}^2)$ and $B(\mathbf{R}^2)/\mathbf{R}^2$ are analytic on S' , we can, with expressions of primitives of $B(t)$ and $B(t)/t$, reduce the surface integrals to non-singular contour integrals, from the proposition 4 (derived from theorem 1) and the proposition 3 (derived from theorem 2).

Proof. To reduce the integrals to contour integrals, we let successively $\frac{\partial(F((\mathbf{r}' - \mathbf{a})^2, \varphi))}{\partial((\mathbf{r}' - \mathbf{a})^2)}$ as $B(\mathbf{R}^2)$ and $\frac{B(\mathbf{R}^2)(\mathbf{R}^2 - z^2)}{\mathbf{R}^2} G(\varphi)$ in proposition 4, involving the primitives of $B(t)$ and $B(t)/t$, and $\frac{\partial(F((\mathbf{r}' - \mathbf{a})^2)}{\partial((\mathbf{r}' - \mathbf{a})^2)}$ as $B(\mathbf{R}^2)$ in proposition 3, involving the primitive of $B(t)$.

4.2.1) Physical Optics diffracted fields and reduction, for an electric dipolar source

Theorem 5

The Physical Optics diffracted field, radiated by a perfectly conducting flat surface S' of the plane $z' = 0$, when illuminated by a electric dipole $\mathbf{J}_0(\mathbf{r}_1) = \hat{\mathbf{d}}\delta(\mathbf{r}_1 - \mathbf{r}_0)$, can be written, at $\mathbf{r} = z\hat{\mathbf{z}} + \mathbf{a} \notin S'$, when $\mathbf{r}_0 = \mathbf{r}$ (i.e. in monostatic case), following,

$$\begin{aligned} \mathbf{E} &= \frac{-Z_0 \hat{\mathbf{d}}}{16\pi^2 i k} \int_{S'} z \tilde{\mathbf{R}} \left(\left(\frac{\psi'}{\tilde{\mathbf{R}}} \right)^2 \right)' dS' + \frac{Z_0}{16\pi^2 i k} \int_{S'} z \tilde{\mathbf{R}} \left(\left(\frac{\psi'}{\tilde{\mathbf{R}}} \right)^2 \right)' (\mathbf{u} \cdot \hat{\mathbf{d}}) \mathbf{u} dS' \\ &+ \frac{Z_0 (\hat{\mathbf{z}} \cdot \hat{\mathbf{d}})}{4\pi^2 i k} \int_{S'} \left(\frac{\psi'}{\tilde{\mathbf{R}}} \right)^2 (\mathbf{a} - \mathbf{r}') dS' + \frac{Z_0 z ((\hat{\mathbf{z}} \cdot \hat{\mathbf{d}}) \hat{\mathbf{z}} - \hat{\mathbf{d}})}{4\pi^2 i k} \int_{S'} \left(\frac{\psi'}{\tilde{\mathbf{R}}} \right)^2 dS', \\ \mathbf{H} &= \frac{-\hat{\mathbf{d}}}{8\pi^2} \times \int_{S'} z \mathbf{R} \left(\frac{\psi'}{\tilde{\mathbf{R}}} \right)^2 dS', \end{aligned} \quad (49)$$

where $(\cdot)' \equiv \partial_{\tilde{R}}(\cdot)$, $\psi(\tilde{R}) = \frac{e^{-ik\tilde{R}}}{\tilde{R}}$, $\tilde{R} = \sqrt{\mathbf{R} \cdot \mathbf{R}}$, while noting that, on S' (i.e. at $z' = 0$), we have $\mathbf{R} = (\mathbf{a} - \mathbf{r}') + z\hat{\mathbf{z}}$, $\mathbf{R}^2 = (\mathbf{r}' - \mathbf{a})^2 + z^2$ and,

$$\begin{aligned} (\mathbf{u} \cdot \hat{\mathbf{d}})\mathbf{u} &= \frac{(\hat{\mathbf{d}} \cdot \hat{\mathbf{z}})\hat{\mathbf{z}}z^2}{\mathbf{R}^2} + \frac{(\hat{\mathbf{d}} \cdot \hat{\mathbf{z}} + \hat{\mathbf{z}}(\hat{\mathbf{d}} \cdot (\cdot)))z(\mathbf{a} - \mathbf{r}')}{\mathbf{R}^2} \\ &+ \frac{\mathbf{R}^2 - z^2}{\mathbf{R}^2} \frac{((\mathbf{r}' - \mathbf{a}) \cdot \hat{\mathbf{d}})(\mathbf{r}' - \mathbf{a})}{(\mathbf{r}' - \mathbf{a})^2}. \end{aligned} \quad (50)$$

In this case, using the following exact primitives,

$$\begin{aligned} \int \frac{((\frac{\psi'}{\tilde{R}})^2)'}{\tilde{R}} d\mathbf{R}^2 &= \frac{2(ik + \frac{1}{\tilde{R}})^2 e^{-i2k\tilde{R}}}{\mathbf{R}^4}, \quad \int (\frac{\psi'}{\tilde{R}})^2 d\mathbf{R}^2 = -\frac{(1 + 2ik\tilde{R})e^{-i2k\tilde{R}}}{2\mathbf{R}^4}, \\ \int \tilde{R} ((\frac{\psi'}{\tilde{R}})^2)' d\mathbf{R}^2 &= 2(-k^2 + \frac{3ik}{\tilde{R}} + \frac{3}{2\mathbf{R}^2}) \frac{e^{-i2k\tilde{R}}}{\mathbf{R}^2}, \end{aligned} \quad (51)$$

(see details in appendix), the lemma 1 applies, and we can exactly reduce every integral of (49) to a non-singular contour integral whose integrand involves no special function.

Proof. We can write the Physical Optics field with (47), considering that $\mathbf{J} = 2\hat{\mathbf{z}} \times \mathbf{H}_i$ and $\mathbf{M} = 0$ on S' , where \mathbf{H}_i is radiated by the source $\mathbf{J}_0(\mathbf{r}_1) = \hat{\mathbf{d}}\delta(\mathbf{r}_1 - \mathbf{r}_0)$, and thus

$$2\pi\mathbf{J} = \hat{\mathbf{z}} \times (\mathbf{R}_0 \times \frac{\psi'_0}{\tilde{R}_0} \hat{\mathbf{d}}) = (\hat{\mathbf{z}} \cdot \hat{\mathbf{d}}) \frac{\psi'_0}{\tilde{R}_0} \mathbf{R}_0 + z_0 \frac{\psi'_0}{\tilde{R}_0} \hat{\mathbf{d}}, \quad \mathbf{R}_0 = (\mathbf{r}' - \mathbf{r}_0), \quad (52)$$

In the monostatic case, when $\mathbf{r} = \mathbf{r}_0 = z\hat{\mathbf{z}} + \mathbf{a} \notin S'$ and $\mathbf{R} = -\mathbf{R}_0 = \tilde{R}\mathbf{u}$, we can write,

$$\begin{aligned} \mathbf{E} &= \frac{Z_0}{8\pi^2 ik} \int_{S'} z\tilde{R} (\frac{\psi'}{\tilde{R}})' \mathbf{u} \times (\mathbf{u} \times \frac{\psi'}{\tilde{R}} \hat{\mathbf{d}}) + 2(\frac{\psi'}{\tilde{R}})^2 ((\hat{\mathbf{z}} \cdot \hat{\mathbf{d}})\mathbf{R} - z\hat{\mathbf{d}}) dS' \\ &= \frac{-Z_0 \hat{\mathbf{d}}}{16\pi^2 ik} \int_{S'} z\tilde{R} ((\frac{\psi'}{\tilde{R}})^2)' dS' + \frac{Z_0}{16\pi^2 ik} \int_{S'} z\tilde{R} ((\frac{\psi'}{\tilde{R}})^2)' (\mathbf{u} \cdot \hat{\mathbf{d}})\mathbf{u} dS' + \\ &+ \frac{Z_0(\hat{\mathbf{z}} \cdot \hat{\mathbf{d}})}{4\pi^2 ik} \int_{S'} (\frac{\psi'}{\tilde{R}})^2 (\mathbf{a} - \mathbf{r}') dS' + \frac{Z_0((\hat{\mathbf{z}} \cdot \hat{\mathbf{d}})\hat{\mathbf{z}} - \hat{\mathbf{d}})}{4\pi^2 ik} \int_{S'} z(\frac{\psi'}{\tilde{R}})^2 dS', \end{aligned} \quad (53)$$

and

$$\mathbf{H} = \frac{-1}{8\pi^2} \int_{S'} (\frac{\psi'}{\tilde{R}})^2 \mathbf{R} \times (\hat{\mathbf{z}} \times (\mathbf{R} \times \hat{\mathbf{d}})) dS' = \frac{-\hat{\mathbf{d}}}{8\pi^2} \times \int_{S'} z\mathbf{R} (\frac{\psi'}{\tilde{R}})^2 dS', \quad (54)$$

and thus (49). The lemma 1 then applies, and we can reduce the expressions of \mathbf{E} and \mathbf{H} to contour integrals, by using the exact primitives given in (51) (derived in appendice A). Let us

notice that, for large $k\tilde{R}$ in the integrals,

$$\mathbf{E} \sim \frac{Z_0}{8\pi^2 ik} \int_{S'} \frac{z\tilde{R} \left(\left(\frac{\psi'}{\tilde{R}}\right)^2\right)'}{2} ((\mathbf{u} \cdot \hat{\mathbf{d}})\mathbf{u} - \hat{\mathbf{d}}) dS', \quad \mathbf{H} \sim \frac{-\hat{\mathbf{d}}}{8\pi^2} \times \int_{S'} z\mathbf{R} \left(\frac{\psi'}{\tilde{R}}\right)^2 dS'. \quad (55)$$

4.2.2) Physical Optics diffracted fields and reduction, for a magnetic dipolar source

Theorem 6

The Physical Optics diffracted field, radiated by a perfectly conducting surface S' of the plane $z' = 0$, when illuminated by a magnetic dipole $\mathbf{M}_0(\mathbf{r}_1) = \hat{\mathbf{d}}\delta(\mathbf{r}_1 - \mathbf{r}_0)$, can be written, at $\mathbf{r} = z\hat{\mathbf{z}} + \mathbf{a} \notin S'$, when $\mathbf{r}_0 = \mathbf{r}$ (monostatic case), following,

$$\begin{aligned} \mathbf{E} &= \frac{1}{8(\pi ik)^2} \int_{S'} z \left(\frac{\psi'}{\tilde{R}}\right)' \left(\frac{\psi'}{\tilde{R}}\right)' (\mathbf{R} \times \hat{\mathbf{d}}) dS' - \\ &\quad - \frac{1}{4(\pi ik)^2} \int_{S'} \hat{\mathbf{z}} \times \left(\tilde{R} \left(\left(\frac{\psi'}{\tilde{R}}\right)^2\right)' (\mathbf{u}(\mathbf{u} \cdot \hat{\mathbf{d}}) - \hat{\mathbf{d}}) - 2\left(\frac{\psi'}{\tilde{R}}\right)^2 \hat{\mathbf{d}}\right) dS', \\ \mathbf{H} &= \frac{1}{8\pi^2 ik Z_0} \int_{S'} \frac{z\tilde{R}}{2} \left(\left(\frac{\psi'}{\tilde{R}}\right)^2\right)' (\hat{\mathbf{d}} - \mathbf{u}(\mathbf{u} \cdot \hat{\mathbf{d}})) - 2\left(\frac{\psi'}{\tilde{R}}\right)^2 \mathbf{R} \times (\hat{\mathbf{z}} \times \hat{\mathbf{d}}) dS', \end{aligned} \quad (56)$$

with $(.)' \equiv \partial_{\tilde{R}}(.)$, $\psi(\tilde{R}) = \frac{e^{-ik\tilde{R}}}{\tilde{R}}$ and (50). Using the exact primitives given in (51) and,

$$\int \left(\frac{\psi'}{\tilde{R}}\right)' \left(\frac{\psi'}{\tilde{R}}\right)' d\mathbf{R}^2 = ik^3 \frac{\psi^2}{\tilde{R}} + 3\left(\frac{\psi}{\tilde{R}}\right)^2 \left(\frac{k^2}{2} - \left(ik + \frac{1}{\tilde{R}}\right)^2\right), \quad (57)$$

the lemma 1 applies, and we can exactly reduce every integral of (56) to non-singular contour integral whose integrand involves no special function.

Proof. We can write the Physical Optics field with (47), considering that the sources $\mathbf{J} = 2\hat{\mathbf{z}} \times \mathbf{H}_i$ and $\mathbf{M} = 0$ on S' , where \mathbf{H}_i is radiated by $\mathbf{M}_0(\mathbf{r}_1) = \hat{\mathbf{d}}\delta(\mathbf{r}_1 - \mathbf{r}_0)$, and thus,

$$\mathbf{J} = \frac{\hat{\mathbf{z}} \times \left(\tilde{R}_0 \left(\frac{\psi'_0}{\tilde{R}_0}\right)' \mathbf{u}_0 \times (\mathbf{u}_0 \times \hat{\mathbf{d}}) - \frac{2\psi'_0}{\tilde{R}_0} \hat{\mathbf{d}}\right)}{2\pi ik Z_0}, \quad \mathbf{R}_0 = (\mathbf{r}' - \mathbf{r}_0), \quad \mathbf{u}_0 = \frac{\mathbf{R}_0}{\tilde{R}_0}. \quad (58)$$

In the monostatic case $\mathbf{R} = -\mathbf{R}_0 = z\hat{\mathbf{z}} + \mathbf{a} - \mathbf{r}'$, we then obtain,

$$\begin{aligned}
 \mathbf{E} &= \frac{1}{8(\pi ik)^2} \int_{S'} z \tilde{R} \left(\frac{\psi'}{\tilde{R}} \right)' (\mathbf{u} \times \left(\frac{\psi'}{\tilde{R}} \right)' \hat{\mathbf{d}}) - 2\psi' \left(\frac{\psi'}{\tilde{R}} \right)' \hat{\mathbf{z}} \times (\mathbf{u} \times (\mathbf{u} \times \hat{\mathbf{d}})) dS' \\
 &\quad - \frac{1}{8(\pi ik)^2} \int_{S'} \frac{2\psi'}{\tilde{R}} \hat{\mathbf{z}} \times \left(\tilde{R} \left(\frac{\psi'}{\tilde{R}} \right)' \mathbf{u} \times (\mathbf{u} \times \hat{\mathbf{d}}) - \frac{2\psi'}{\tilde{R}} \hat{\mathbf{d}} \right) dS' \\
 &= \frac{1}{8(\pi ik)^2} \int_{S'} z \left(\frac{\psi'}{\tilde{R}} \right)' \left(\frac{\psi'}{\tilde{R}} \right)' (\mathbf{R} \times \hat{\mathbf{d}}) dS' - \\
 &\quad - \frac{1}{4(\pi ik)^2} \int_{S'} \hat{\mathbf{z}} \times \left(\tilde{R} \left(\left(\frac{\psi'}{\tilde{R}} \right)^2 \right)' (\mathbf{u}(\mathbf{u} \cdot \hat{\mathbf{d}}) - \hat{\mathbf{d}}) - 2 \left(\frac{\psi'}{\tilde{R}} \right)^2 \hat{\mathbf{d}} \right) dS', \tag{59}
 \end{aligned}$$

and

$$\begin{aligned}
 \mathbf{H} &= \frac{1}{8\pi^2 ik Z_0} \int_{S'} \frac{\psi'}{\tilde{R}} \mathbf{R} \times \left(\hat{\mathbf{z}} \times \left(\tilde{R} \left(\frac{\psi'}{\tilde{R}} \right)' \mathbf{u} \times (\mathbf{u} \times \hat{\mathbf{d}}) - \frac{2\psi'}{\tilde{R}} \hat{\mathbf{d}} \right) \right) dS' \\
 &= \frac{1}{8\pi^2 ik Z_0} \int_{S'} z \tilde{R} \left(\left(\frac{\psi'}{\tilde{R}} \right)^2 \right)' (\hat{\mathbf{d}} - \mathbf{u}(\mathbf{u} \cdot \hat{\mathbf{d}})) - 2 \left(\frac{\psi'}{\tilde{R}} \right)^2 \mathbf{R} \times (\hat{\mathbf{z}} \times \hat{\mathbf{d}}) dS'. \tag{60}
 \end{aligned}$$

and thus (56). Considering these expressions with (50), the lemma 1 applies, and we can completely reduce them to contour integrals, by using the primitives given in (51) and (57) (derived in appendice A). Let us notice that, for large $k\tilde{R}$,

$$\mathbf{E} \sim \frac{-\hat{\mathbf{d}} \times}{8\pi^2 (ik)^2} \int_{S'} z \mathbf{R} \left(\left(\frac{\psi'}{\tilde{R}} \right)' \right)^2 dS', \quad \mathbf{H} \sim \frac{1}{Z_0} \int_{S'} \frac{z \tilde{R} \left(\left(\frac{\psi'}{\tilde{R}} \right)^2 \right)' (\hat{\mathbf{d}} - \mathbf{u}(\mathbf{u} \cdot \hat{\mathbf{d}}))}{2ik8\pi^2} dS'. \tag{61}$$

5) Reduction to contour integrals in case (a), for a weighted dipolar illumination of a large perfectly conducting flat plate

5.1) Asymptotic surface integral expressions for a weighted dipolar illumination

We now consider the incident field, as the combination of the radiations of electric and magnetic dipoles $\mathbf{J}_0(\mathbf{r}_1) = \hat{\mathbf{d}}_e \delta(\mathbf{r}_1 - \mathbf{r}_0)$ and $\mathbf{M}_0(\mathbf{r}_1) = \hat{\mathbf{d}}_h \delta(\mathbf{r}_1 - \mathbf{r}_0)$ weighted by $f_{e,h}(\mathbf{u}_0)$, following,

$$\begin{aligned}
\mathbf{E}_i(\mathbf{r}) &\sim \frac{1}{4\pi ik} (f_e(\mathbf{u}_0) \tilde{R}_0 \left(\frac{\psi'_0}{\tilde{R}_0}\right)' (\mathbf{u}_0 \times (\mathbf{u}_0 \times Z_0 \hat{\mathbf{d}}_e)) - ik f_h(\mathbf{u}_0) \frac{\psi'_0}{\tilde{R}_0} \mathbf{R}_0 \times \hat{\mathbf{d}}_h) \\
&\sim \frac{ik}{4\pi} \frac{e^{-ik\tilde{R}_0}}{\tilde{R}_0} (f_e(\mathbf{u}_0) (\mathbf{u}_0 \times (\mathbf{u}_0 \times Z_0 \hat{\mathbf{d}}_e)) + f_h(\mathbf{u}_0) (\mathbf{u}_0 \times \hat{\mathbf{d}}_h)), \\
\mathbf{H}_i(\mathbf{r}) &\sim \frac{1}{4\pi ik Z_0} (ik f_e(\mathbf{u}_0) \frac{\psi'_0}{\tilde{R}_0} (\mathbf{R}_0 \times Z_0 \hat{\mathbf{d}}_e) + f_h(\mathbf{u}_0) \left(\frac{\psi'_0}{\tilde{R}_0}\right)' \mathbf{R}_0 \times (\mathbf{u}_0 \times \hat{\mathbf{d}}_h)) \\
&\sim \frac{ik}{4\pi Z_0} \frac{e^{-ik\tilde{R}_0}}{\tilde{R}_0} (-f_e(\mathbf{u}_0) (\mathbf{u}_0 \times Z_0 \hat{\mathbf{d}}_e) + f_h(\mathbf{u}_0) (\mathbf{u}_0 \times (\mathbf{u}_0 \times \hat{\mathbf{d}}_h))), \tag{62}
\end{aligned}$$

for large $k\tilde{R}_0$, where $\mathbf{R}_0 = \mathbf{r} - \mathbf{r}_0 = \tilde{R}_0 \mathbf{u}_0$, $\tilde{R}_0 = \sqrt{\mathbf{R}_0 \cdot \mathbf{R}_0}$, and the functions $f_{e,h}(\mathbf{u}_0) (\mathbf{u}_0 \times \hat{\mathbf{d}}_h)$ are analytic, with $\mathbf{E}_i(\mathbf{r}) \sim -Z_0 \mathbf{u}_0 \times \mathbf{H}_i(\mathbf{r})$.

In this case, the near-field monostatic diffracted field expressions at $\mathbf{r} = \mathbf{r}_0 \notin S'$, for a perfectly conducting plate S' illuminated by a source of arbitrary radiation pattern, can be derived in PO from (47) with $\mathbf{J} = 2\hat{\mathbf{z}} \times \mathbf{H}_i$ and $\mathbf{M} = 0$ on S' . These PO expressions then remains valid from infinity to a distance qualified to be large relatively to the wavelength λ , but that can be small relatively to the dimensions of S' .

Therefore, we obtain, in monostatic case, from a combination of (55) and (61) weighted by $f_{e,h}$, the expressions,

$$\begin{aligned}
\mathbf{E}(\mathbf{r}) &\sim \frac{1}{8\pi^2 (ik)^2} \left(\int_{S'} z f_e(-\mathbf{u}) \frac{ik}{2} \left(\left(\frac{\psi'}{\tilde{R}}\right)'\right)^2 (\mathbf{R} \times (\mathbf{u} \times Z_0 \hat{\mathbf{d}}_e)) dS' \right. \\
&\quad \left. + \int_{S'} z f_h(-\mathbf{u}) \left(\left(\frac{\psi'}{\tilde{R}}\right)'\right)^2 (\mathbf{R} \times \hat{\mathbf{d}}_h) dS' \right), \\
\mathbf{H}(\mathbf{r}) &\sim \frac{1}{8\pi^2 (ik)^2 Z_0} \left(\int_{S'} z f_e(-\mathbf{u}) (ik)^2 \left(\frac{\psi'}{\tilde{R}}\right)^2 (\mathbf{R} \times Z_0 \hat{\mathbf{d}}_e) dS' \right. \\
&\quad \left. - \int_{S'} z f_h(-\mathbf{u}) \frac{ik}{2} \left(\left(\frac{\psi'}{\tilde{R}}\right)'\right)^2 (\mathbf{R} \times (\mathbf{u} \times \hat{\mathbf{d}}_h)) dS' \right), \tag{63}
\end{aligned}$$

for large $k\tilde{R}$, with $\mathbf{r} = \mathbf{r}_0 = \mathbf{a} + z\hat{\mathbf{z}}$, $\tilde{R} = \sqrt{\mathbf{R} \cdot \mathbf{R}}$, $\mathbf{R} = \tilde{R} \mathbf{u} = \mathbf{a} + z\hat{\mathbf{z}} - \mathbf{r}'$, $\mathbf{u} = -\mathbf{u}_0$. In this frame, let us notice, that, if we want to consider a source with a spherical pattern, instead of a dipole with toric pattern, we can apply the weighting factors $f_{e,h}(\mathbf{u}_0) = \beta_{e,h} / \sqrt{(\mathbf{u}_0 \times \hat{\mathbf{d}}_{e,h})^2}$, with $(\mathbf{u}_0 \times \hat{\mathbf{d}}_{e,h})^2 = 1 - (\mathbf{u}_0 \cdot \hat{\mathbf{d}}_{e,h})^2$.

5.2) Reduction to contour integrals for even or odd weight functions $f_{e,h}$

Thereafter, we reduce \mathbf{E} in (63) to a contour integral, for $f_h \equiv 0$ (resp. $f_e \equiv 0$), when f_e (resp. f_h) is even or odd relatively to coordinates of $\frac{\mathbf{r}' - \mathbf{a}}{\tilde{R}}$ in the integral expressions, for the cases with $\hat{\mathbf{d}}_{e,h} \cdot \hat{\mathbf{z}} = 0$.

5.2.1) Electric case with $f_h \equiv 0$, $\hat{\mathbf{d}} = \hat{\mathbf{d}}_e$

Let us consider the case with $f_h \equiv 0$, $\hat{\mathbf{d}} = \hat{\mathbf{d}}_e$. The plate S' is then illuminated by,

$$\mathbf{E}_i \sim \frac{ikZ_0}{4\pi} \frac{e^{-ik\tilde{R}_0}}{\tilde{R}_0} f_e(\mathbf{u}_0)(\mathbf{u}_0(\mathbf{u}_0 \cdot \hat{\mathbf{d}}) - \hat{\mathbf{d}}), \quad (64)$$

and the PO diffracted field at $\mathbf{r} = \mathbf{a} + z\hat{\mathbf{z}} \notin S'$, when $\mathbf{r} = \mathbf{r}_0$ (monostatic case), is given by,

$$\mathbf{E} \sim -\frac{Z_0}{8\pi k^2} \frac{i}{\lambda} \int_{S'} z\tilde{R} \left(\left(\frac{\psi'}{\tilde{R}} \right)' \right)^2 f_e(-\mathbf{u}) ((\mathbf{u} \cdot \hat{\mathbf{d}})\mathbf{u} - \hat{\mathbf{d}}) dS'. \quad (65)$$

Theorem 7

When $f_e(-\mathbf{u}) = \frac{f(\frac{z}{\tilde{R}}, \frac{\mathbf{r}' - \mathbf{a}}{\tilde{R}}) + f(\frac{-z}{\tilde{R}}, \frac{\mathbf{a} - \mathbf{r}'}{\tilde{R}})}{2} = \frac{g(\frac{z}{\tilde{R}}, \frac{\mathbf{a} - \mathbf{r}'}{\tilde{R}})}{\sqrt{1 - (\mathbf{u} \cdot \hat{\mathbf{d}})^2}}$ and $\hat{\mathbf{d}} \cdot \hat{\mathbf{z}} = 0$, the integral reduces to,

$$\begin{aligned} \mathbf{E} &\sim \frac{-i}{\lambda} \left(\frac{1}{2} \int_{\mathcal{L}} (F_1((\mathbf{r}' - \mathbf{a})^2, \varphi) f_e(-\mathbf{u}) - F_1(0, \varphi) g(1, 0)) [(\hat{\mathbf{z}} \times \frac{\mathbf{r}' - \mathbf{a}}{(\mathbf{r}' - \mathbf{a})^2}) \cdot d\mathbf{r}'] \right) + \\ &+ \frac{i}{\lambda} \left(\frac{1}{2} \int_{\mathcal{L}} (F_2((\mathbf{r}' - \mathbf{a})^2) f_e(-\mathbf{u}) - F_2(0) g(1, 0)) [(\hat{\mathbf{z}} \times \hat{\mathbf{d}}) \cdot d\mathbf{r}'] \right), \end{aligned} \quad (66)$$

where $\mathcal{L} \equiv \partial S'$ is oriented counterclockwise, and the primitives F_1 and F_2 satisfy,

$$\begin{aligned} F_1 &= \frac{Z_0 z}{8\pi k^2} \int \tilde{R} \left(\left(\frac{\psi'}{\tilde{R}} \right)' \right)^2 \left(\frac{(\mathbf{r}' - \mathbf{a})^2 G(\varphi)}{\mathbf{R}^2} - \hat{\mathbf{d}} \right) d\mathbf{R}^2, \\ F_2 &= \frac{Z_0 z^2 \hat{\mathbf{z}}}{8\pi k^2} \int \frac{\tilde{R} \left(\left(\frac{\psi'}{\tilde{R}} \right)' \right)^2}{\mathbf{R}^2} d\mathbf{R}^2, \\ F_1 &\sim -\frac{Z_0 z e^{-2ik\tilde{R}}}{4\pi \mathbf{R}^2} \left(\frac{(\mathbf{r}' - \mathbf{a})^2 G(\varphi)}{\mathbf{R}^2} - \hat{\mathbf{d}} \right), \quad F_2 \sim -\frac{Z_0 z^2 \hat{\mathbf{z}} e^{-2ik\tilde{R}}}{4\pi \mathbf{R}^4}, \end{aligned} \quad (67)$$

for $z \neq 0$, with $G(\varphi) = \frac{((\mathbf{r}' - \mathbf{a}) \cdot \hat{\mathbf{d}})(\mathbf{r}' - \mathbf{a})}{(\mathbf{r}' - \mathbf{a})^2}$, $F_1(0, \varphi) = \frac{Z_0}{4\pi} \frac{\hat{\mathbf{d}} e^{-2ikz}}{z}$ and $F_2(0) = -\frac{Z_0}{4\pi} \frac{\hat{\mathbf{z}} e^{-2ikz}}{z^2}$, $f_e(-\mathbf{u})$ regular on S' . The VV polarization term is given by $\mathbf{E} \cdot \hat{\mathbf{d}}$ when $\mathbf{r}_0 \cdot \hat{\mathbf{d}} = \hat{\mathbf{z}} \cdot \hat{\mathbf{d}} = 0$, i.e. $\hat{\mathbf{d}} = \frac{\mathbf{r}_0 \times \hat{\mathbf{z}}}{\sqrt{(\mathbf{r}_0 \times \hat{\mathbf{z}})^2}}$ as $\mathbf{r}_0 \times \hat{\mathbf{z}} \neq 0$.

Proof. We can detail (65), noticing that, when $\hat{\mathbf{d}} \cdot \hat{\mathbf{z}} = 0$, we have

$$(\mathbf{u} \cdot \hat{\mathbf{d}})\mathbf{u} = \frac{(\mathbf{r}' - \mathbf{a})^2 G(\varphi)}{\mathbf{R}^2} + \frac{(\mathbf{a} - \mathbf{r}') \cdot \hat{\mathbf{d}}}{\mathbf{R}^2} z\hat{\mathbf{z}}, \quad G(\varphi) = \frac{((\mathbf{r}' - \mathbf{a}) \cdot \hat{\mathbf{d}})(\mathbf{r}' - \mathbf{a})}{(\mathbf{r}' - \mathbf{a})^2}, \quad (68)$$

Considering that $\frac{g(\frac{z}{\tilde{R}}, \frac{\mathbf{a} - \mathbf{r}'}{\tilde{R}})}{\sqrt{1 - (\mathbf{u} \cdot \hat{\mathbf{d}})^2}}$ is a regular even function of $\frac{\mathbf{a} - \mathbf{r}'}{\tilde{R}}$ for $z \neq 0$, we use the theorem 1

with the proposition 13 (see appendix) for the part of the integrand with $(\frac{(\mathbf{r}'-\mathbf{a})^2 G(\varphi)}{R^2} - \hat{\mathbf{d}})$ factor, then the proposition 14 (see appendix) for the part of the integrand with $(\mathbf{a} - \mathbf{r}') \cdot \hat{\mathbf{d}}$ factor, and thus derive (66).

For an illumination with an isotropic pattern, we can choose $g \equiv 1$, and write,

$$\mathbf{E} \cdot \hat{\mathbf{d}} \sim -\frac{Z_0}{4\pi} \frac{i}{\lambda} \left(\frac{1}{2} \int_{\mathcal{L}} \frac{(\frac{ze^{-2ik\tilde{R}}}{R^2} \sqrt{1 - (\frac{\mathbf{r}'-\mathbf{a}}{\tilde{R}} \cdot \hat{\mathbf{d}})^2} - \frac{e^{-2ikz}}{z})}{(\mathbf{r}' - \mathbf{a})^2} [(\hat{\mathbf{z}} \times (\mathbf{r}' - \mathbf{a})) \cdot d\mathbf{r}'] \right), \quad (69)$$

which is the VV polarization term when $\mathbf{r}_0 \cdot \hat{\mathbf{d}} = 0$, i.e. $\hat{\mathbf{d}} = \frac{\mathbf{r}_0 \times \hat{\mathbf{z}}}{\sqrt{(\mathbf{r}_0 \times \hat{\mathbf{z}})^2}}$ as $\mathbf{r}_0 \times \hat{\mathbf{z}} \neq 0$.

Let us notice that, $(\hat{\mathbf{z}} \times (\mathbf{r}' - \mathbf{a})) \cdot \hat{\mathbf{v}}$ is constant on each straight part of \mathcal{L} along $\hat{\mathbf{v}}$, and that, for real $\mathbf{a} \notin S'$, the parts of the integrals with $\frac{e^{-2ikz}}{z}$ factor vanish.

Theorem 8

When $f_e(-\mathbf{u}) = \frac{f(\frac{z}{\tilde{R}}, \frac{\mathbf{r}'-\mathbf{a}}{\tilde{R}}) - f(\frac{z}{\tilde{R}}, \frac{\mathbf{a}-\mathbf{r}'}{\tilde{R}})}{2} = \frac{(\mathbf{r}'-\mathbf{a})}{\tilde{R}} \cdot \frac{\mathbf{h}(\frac{z}{\tilde{R}}, \frac{\mathbf{a}-\mathbf{r}'}{\tilde{R}})}{\sqrt{1-(\mathbf{u} \cdot \hat{\mathbf{d}})^2}}$ and $\hat{\mathbf{d}} \cdot \hat{\mathbf{z}} = 0$, it reduces to,

$$\begin{aligned} \mathbf{E} \sim & -\frac{i}{\lambda} \left(\frac{1}{2} \int_{\mathcal{L}} F_1((\mathbf{r}' - \mathbf{a})^2, \varphi) [(\hat{\mathbf{z}} \times \mathbf{h}(\frac{z}{\tilde{R}}, \frac{\mathbf{a} - \mathbf{r}'}{\tilde{R}})) \cdot d\mathbf{r}'] + \right. \\ & \left. + \frac{1}{2} \int_{\mathcal{L}} F_2((\mathbf{r}' - \mathbf{a})^2)' \left(\frac{\hat{\mathbf{d}} \cdot (\mathbf{a} - \mathbf{r}') f_e(-\mathbf{u})}{\tilde{R}} \right) [(\hat{\mathbf{z}} \times \frac{\mathbf{r}' - \mathbf{a}}{(\mathbf{r}' - \mathbf{a})^2}) \cdot d\mathbf{r}'] \right), \end{aligned} \quad (70)$$

where, for $z \neq 0$, the primitives F_1 and F_2 are written following,

$$\begin{aligned} F_1 &= \frac{Z_0 z}{8\pi k^2} \int \left(\left(\frac{\psi'}{\tilde{R}} \right)^2 \right)' \left(\frac{(\mathbf{r}' - \mathbf{a})^2 G(\varphi)}{R^2} - \hat{\mathbf{d}} \right) dR^2, \quad F_2 = \frac{Z_0 z^2 \hat{\mathbf{z}}}{8\pi k^2} \int \left(\left(\frac{\psi'}{\tilde{R}} \right)^2 \right)' dR^2, \\ F_1 &\sim -\frac{Z_0}{4\pi} \frac{ze^{-2ik\tilde{R}}}{\tilde{R} R^2} \left(\frac{(\mathbf{r}' - \mathbf{a})^2 G(\varphi)}{R^2} - \hat{\mathbf{d}} \right), \quad F_2 \sim -\frac{Z_0}{4\pi} \frac{z^2 \hat{\mathbf{z}} e^{-2ik\tilde{R}}}{\tilde{R} R^2}, \end{aligned} \quad (71)$$

with $G(\varphi) = \frac{((\mathbf{r}'-\mathbf{a}) \cdot \hat{\mathbf{d}})(\mathbf{r}'-\mathbf{a})}{(\mathbf{r}'-\mathbf{a})^2}$, $f_e(-\mathbf{u})$ regular on S' , $\mathcal{L} \equiv \partial S'$ oriented counterclockwise. The VV polarization term is given by $\mathbf{E} \cdot \hat{\mathbf{d}}$, when $\hat{\mathbf{d}} = \frac{\mathbf{r}_0 \times \hat{\mathbf{z}}}{\sqrt{(\mathbf{r}_0 \times \hat{\mathbf{z}})^2}}$ as $\mathbf{r}_0 \times \hat{\mathbf{z}} \neq 0$.

Proof. Considering that $\frac{\mathbf{h}(\frac{z}{\tilde{R}}, \frac{\mathbf{a}-\mathbf{r}'}{\tilde{R}})}{\sqrt{1-(\mathbf{u} \cdot \hat{\mathbf{d}})^2}}$ is a regular even function of the coordinates of $\frac{\mathbf{r}'-\mathbf{a}}{\tilde{R}}$, $\hat{\mathbf{d}} \cdot \hat{\mathbf{z}} = 0$, we use the proposition 14 and the theorem 1 with proposition 13 (see appendix), and derive (70) with (71).

5.2.2) Magnetic case with $f_e \equiv 0$, $\hat{\mathbf{d}} = \hat{\mathbf{d}}_h$

Let us consider the case with $f_e \equiv 0$, $\hat{\mathbf{d}} = \hat{\mathbf{d}}_h$. The plate S' is then illuminated by,

$$\mathbf{E}_i \sim \frac{ik}{4\pi} (\mathbf{u}_0 \times \hat{\mathbf{d}}) f_h(\mathbf{u}_0) \frac{e^{-ik\tilde{R}_0}}{\tilde{R}_0}, \quad (72)$$

and the PO diffracted field at $\mathbf{r} = \mathbf{a} + z\hat{\mathbf{z}} \notin S'$, when $\mathbf{r} = \mathbf{r}_0$ (monostatic case), satisfies,

$$\mathbf{E} \cdot \mathbf{v}_0 \sim \frac{-z}{8(\pi ik)^2} \int_{S'} \left(\left(\frac{\psi'}{\tilde{R}} \right)' \right)^2 \mathbf{w}_0 \cdot \mathbf{R} f_h(-\mathbf{u}) dS' \quad \text{with } \mathbf{w}_0 = \mathbf{v}_0 \times \hat{\mathbf{d}}. \quad (73)$$

Theorem 9

When $f_h(-\mathbf{u}) = \frac{f(\frac{-z}{\tilde{R}}, \frac{\mathbf{r}'-\mathbf{a}}{\tilde{R}}) + f(\frac{-z}{\tilde{R}}, \frac{\mathbf{a}-\mathbf{r}'}{\tilde{R}})}{2} = \frac{g(\frac{z}{\tilde{R}}, \frac{\mathbf{a}-\mathbf{r}'}{\tilde{R}})}{\sqrt{1-(\mathbf{u} \cdot \hat{\mathbf{d}})^2}}$ and $\hat{\mathbf{d}} \cdot \hat{\mathbf{z}} = 0$, the integral reduces to,

$$\mathbf{E} \cdot \mathbf{v}_0 \sim \frac{ik}{4\pi} \int_{\mathcal{L}} \frac{\left(\frac{ze^{-2ik\tilde{R}} f_h(-\mathbf{u})}{\tilde{R}^2} - \frac{\tilde{R} e^{-2ikz} g(1,0)}{z} \right) (\hat{\mathbf{z}} \times \frac{-\mathbf{w}_0(\mathbf{r}'-\mathbf{a})^2 + (\mathbf{r}'-\mathbf{a})(zw_{0z})}{\tilde{R}})}{4\pi(\mathbf{r}'-\mathbf{a})^2} d\mathbf{r}' \quad (74)$$

where $\mathbf{w}_0 = \mathbf{v}_0 \times \hat{\mathbf{d}}$. For $\mathbf{v}_0 = (\hat{\mathbf{d}} \times \frac{\mathbf{r}_0}{r_0})$, with $\hat{\mathbf{d}} = \frac{\mathbf{r}_0 \times \hat{\mathbf{z}}}{\sqrt{(\mathbf{r}_0 \times \hat{\mathbf{z}})^2}}$ as $\mathbf{r}_0 \times \hat{\mathbf{z}} \neq 0$, we have,

$$\mathbf{E} \cdot \mathbf{v}_0 \sim \frac{ik}{4\pi} \int_{\mathcal{L}} \frac{\frac{ze^{-2ik\tilde{R}} f_h(-\mathbf{u})}{\tilde{R}^2} - \frac{\tilde{R} e^{-2ikz} g(1,0)}{z}}{4\pi(\mathbf{r}'-\mathbf{a})^2} (\hat{\mathbf{z}} \times \left(\frac{(\mathbf{r}'-\mathbf{a})\tilde{R}}{\tilde{r}_0} - \frac{\mathbf{r}'(\mathbf{r}'-\mathbf{a})^2}{\tilde{R}\tilde{r}_0} \right)) d\mathbf{r}', \quad (75)$$

denoted the HH polarization term. The contour $\mathcal{L} \equiv \partial S'$ is oriented counterclockwise.

Proof. Noticing that $w_0 \cdot \mathbf{R} = -\mathbf{w}_0 \cdot (\mathbf{r}' - \mathbf{a}) + zw_{0z}$, and $f_h(-\mathbf{u}) = \frac{f(\frac{-z}{\tilde{R}}, \frac{\mathbf{r}'-\mathbf{a}}{\tilde{R}}) + f(\frac{-z}{\tilde{R}}, \frac{\mathbf{a}-\mathbf{r}'}{\tilde{R}})}{2}$ is even relatively to the coordinates of $\frac{\mathbf{r}'-\mathbf{a}}{\tilde{R}}$, we derive, from the theorem 1 and proposition 14,

$$\begin{aligned} \mathbf{E} \cdot \mathbf{v}_0 &\sim \frac{i}{\lambda} \frac{1}{2} \int_{\mathcal{L}} (F((\mathbf{r}'-\mathbf{a})^2) - F(0)) (\hat{\mathbf{z}} \times (-\mathbf{w}_0 + \frac{\mathbf{r}'-\mathbf{a}}{(\mathbf{r}'-\mathbf{a})^2} (zw_{0z}))) d\mathbf{r}', \\ F &= \frac{z}{4\pi k^2 ik} \int \left(\left(\frac{\psi'}{\tilde{R}} \right)' \right)^2 f_h(-\mathbf{u}) d\mathbf{R}^2. \end{aligned} \quad (76)$$

From the proposition 13 (see appendix), we derive $F \sim \frac{1}{4\pi} \frac{ze^{-2ik\tilde{R}}}{\tilde{R}^3} f_h(-\mathbf{u})$, and thus (74).

For polarization HH, $\mathbf{v}_0 = (\hat{\mathbf{d}} \times \frac{\mathbf{r}_0}{r_0})$ with $\hat{\mathbf{d}} = \frac{\mathbf{r}_0 \times \hat{\mathbf{z}}}{\sqrt{(\mathbf{r}_0 \times \hat{\mathbf{z}})^2}}$ as $\mathbf{r}_0 \times \hat{\mathbf{z}} \neq 0$ and $\mathbf{w}_0 = \frac{\mathbf{r}_0}{r_0} = \frac{\mathbf{a} + z\hat{\mathbf{z}}}{\tilde{r}_0}$, we then derive (78) from,

$$-\mathbf{w}_0(\mathbf{r}'-\mathbf{a})^2 + (\mathbf{r}'-\mathbf{a})(zw_{0z}) = \frac{(-\mathbf{r}' + (\mathbf{r}'-\mathbf{a}))(\mathbf{r}'-\mathbf{a})^2 + (\mathbf{r}'-\mathbf{a})z^2}{\tilde{r}_0}, \quad (77)$$

For an illumination with isotropic pattern, i.e $g \equiv 1$ and $f_h(-\mathbf{u}) = \frac{1}{\sqrt{1-(\mathbf{u} \cdot \hat{\mathbf{d}})^2}}$, we have,

$$\mathbf{E} \cdot \mathbf{v}_0 \sim \frac{ik}{4\pi} \int_{\mathcal{L}} \frac{\left(\frac{ze^{-2ik\tilde{R}}}{R^2 \sqrt{1-(\mathbf{u} \cdot \hat{\mathbf{d}})^2}} - \frac{\tilde{R} e^{-2ikz}}{z} \right)}{4\pi(\mathbf{r}' - \mathbf{a})^2} (\hat{\mathbf{z}} \times \left(\frac{(\mathbf{r}' - \mathbf{a})\tilde{R}}{\tilde{r}_0} - \frac{\mathbf{r}'(\mathbf{r}' - \mathbf{a})^2}{\tilde{R}\tilde{r}_0} \right)) \cdot d\mathbf{r}'. \quad (78)$$

Let us notice that, $(\hat{\mathbf{z}} \times (\mathbf{r}' - \mathbf{a})) \cdot \hat{\mathbf{v}}$ is constant on each straight part of \mathcal{L} along $\hat{\mathbf{v}}$, and that, for real $\mathbf{a} \notin S'$, the parts of the integrals with $\frac{\tilde{R} e^{-2ikz}}{z}$ factor vanish.

Theorem 10

When $f_h(-\mathbf{u}) = \frac{f(\frac{-z}{\tilde{R}}, \frac{\mathbf{r}' - \mathbf{a}}{\tilde{R}}) - f(\frac{-z}{\tilde{R}}, \frac{\mathbf{a} - \mathbf{r}'}{\tilde{R}})}{2} = \frac{(\mathbf{r}' - \mathbf{a})}{\tilde{R}} \cdot \frac{h(\frac{z}{\tilde{R}}, \frac{\mathbf{a} - \mathbf{r}'}{\tilde{R}})}{\sqrt{1-(\mathbf{u} \cdot \hat{\mathbf{d}})^2}}$ and $\hat{\mathbf{d}} \cdot \hat{\mathbf{z}} = 0$, the integral reduces to,

$$\begin{aligned} \mathbf{E} \cdot \mathbf{v}_0 &\sim \frac{ik}{4\pi} \left(\int_{\mathcal{L}} \frac{ze^{-2ik\tilde{R}} \left(\frac{(\mathbf{w}_0 \cdot (\mathbf{a} - \mathbf{r}')) f_h(-\mathbf{u})}{\tilde{R}} \right)}{4\pi R^2} (\hat{\mathbf{z}} \times \frac{(\mathbf{r}' - \mathbf{a})}{(\mathbf{r}' - \mathbf{a})^2}) \cdot d\mathbf{r}' \right. \\ &\quad \left. + \int_{\mathcal{L}} \frac{ze^{-2ik\tilde{R}} (zw_{0z})}{4\pi R^4} (\hat{\mathbf{z}} \times \frac{h(\frac{z}{\tilde{R}}, \frac{\mathbf{a} - \mathbf{r}'}{\tilde{R}})}{\sqrt{1-(\mathbf{u} \cdot \hat{\mathbf{d}})^2}}) \cdot d\mathbf{r}' \right). \end{aligned} \quad (79)$$

For $\mathbf{v}_0 = (\hat{\mathbf{d}} \times \frac{\mathbf{r}_0}{r_0})$ with $\hat{\mathbf{d}} = \frac{\mathbf{r}_0 \times \hat{\mathbf{z}}}{\sqrt{(\mathbf{r}_0 \times \hat{\mathbf{z}})^2}}$ as $\mathbf{r}_0 \times \hat{\mathbf{z}} \neq 0$, and $\mathbf{w}_0 = \frac{\mathbf{r}_0}{r_0} = \frac{\mathbf{a} + z\hat{\mathbf{z}}}{\tilde{r}_0}$, $\mathbf{E} \cdot \mathbf{v}_0$ is the HH polarization term.

Proof. $\frac{h(\frac{z}{\tilde{R}}, \frac{\mathbf{a} - \mathbf{r}'}{\tilde{R}})}{\sqrt{1-(\mathbf{u} \cdot \hat{\mathbf{d}})^2}}$ is a regular even function of the coordinates of $\frac{\mathbf{r}' - \mathbf{a}}{\tilde{R}}$ with $\hat{\mathbf{d}} \cdot \hat{\mathbf{z}} = 0$. We can then use the proposition 14 and the theorem 1 with proposition 13 (see appendix), and derive (79).

6) Definition of complex scaling in case (b) and elementary exact surface integral reduction

As defined in section 2, we consider the case (b) and study the reduction of a plane surface radiation integral whose integrand has an exponential term with an argument composed of a complex polynomial of rank 2 relatively to plane coordinates. This form is usually encountered, if the surface of integration S' is a plane scatterer (a flat plate), when it is illuminated by a gaussian beam [13]-[15], or when its largest dimension D is such that $\sqrt{kD^3}$ is much smaller than the source and observation distances from S' [2]-[4] (see (208) in appendix). In the latter case, it is worth noticing that, if the scatterer is flat (a flat plate), the importance of the non linear part of the polynomial vanishes in the far field region (i.e. at large distance d relatively to λ , with $d > 2D^2/\lambda$ so that phase tolerance is within $\pi/8$ [1]), while it remains significant if the scatterer is curved (a curved plate).

We analyse in this section the exact reduction of the elementary surface integral,

$$\int_{S'} e^{\mathcal{R}^2} dS' \text{ with } \mathcal{R}^2 = (\alpha_1 x' + \alpha_2 y') - ik(h_1 x'^2 + h_2 y'^2 + h_3 x' y'), \quad (80)$$

for arbitrary complex α_i and h_j , to a contour integral without any use of special functions. For that, we define a complex scaling transformation to extend the domain of application of results given in previous sections. More complex integrands will be considered in sections 7 and 8.

6.1) Transformation of the polynomial argument \mathcal{R}^2

We begin by developing a particular rewriting of polynomial \mathcal{R}^2 . This will lead us to an efficient complex scaling transform which will permit the extension of our contour reduction method.

Proposition 5

We can write, for any h_j coefficients with $h_3^2 \neq 4h_1 h_2$,

$$h_1 x^2 + h_2 y^2 + h_3 xy = \xi((1 - c_{p,x}^2)x^2 + (1 - c_{p,y}^2)y^2 - 2c_{p,x}c_{p,y}xy), \quad (81)$$

where c_{px} and c_{py} are the (complex) coordinates of a vector \mathbf{c}_p . Subsequently, we have,

$$\begin{aligned} 2c_{p,x}c_{p,y} &= -h_3/\xi, \quad c_{p,x}^2 = 1 - h_1/\xi, \quad c_{p,y}^2 = 1 - h_2/\xi, \\ \xi &= \frac{(h_1 + h_2) + \epsilon \sqrt{(h_1 - h_2)^2 + h_3^2}}{2}, \end{aligned} \quad (82)$$

where the choice of $\epsilon = +1$ or -1 is free. We can then let,

$$\begin{aligned} c_{p,x} &= \epsilon'_x \sqrt{1 - h_1/\xi}, \quad c_{p,y} = \epsilon'_y \sqrt{1 - h_2/\xi}, \\ \epsilon_x'^2 &= \epsilon_y'^2 = 1, \quad \epsilon'_x \epsilon'_y = \frac{-h_3/\xi}{2\sqrt{1 - h_1/\xi}\sqrt{1 - h_2/\xi}} \text{ for } h_3 \neq 0, \end{aligned} \quad (83)$$

or

$$\begin{aligned} c_{p,x} &= \epsilon_x \sqrt{\xi - h_1}/\sqrt{\xi}, \quad c_{p,y} = \epsilon_y \sqrt{\xi - h_2}/\sqrt{\xi}, \\ \epsilon_x^2 &= \epsilon_y^2 = 1, \quad \epsilon_x \epsilon_y = \frac{-h_3}{2\sqrt{\xi - h_1}\sqrt{\xi - h_2}} \text{ for } h_3 \neq 0, \end{aligned} \quad (84)$$

where we can choose arbitrarily ϵ_x (or ϵ'_x) equal to $+1$ or -1 , and take indifferently $-\mathbf{c}_p$ in place of \mathbf{c}_p (note : in the case $h_3 = 0$, every $\epsilon_{x(y)}$ and $\epsilon'_{x(y)}$ can be chosen either $+1$ or -1 freely). Letting $(\mathbf{c}_p)_\epsilon$ for \mathbf{c}_p when $\epsilon = \pm 1$, it is worth noticing that,

$$\frac{(c_{px})_\epsilon}{(c_{py})_\epsilon} = -\frac{(c_{py})_{-\epsilon}}{(c_{px})_{-\epsilon}}, \quad (\mathbf{c}_p)_\epsilon^2 - 1 = \frac{1}{(\mathbf{c}_p)_{-\epsilon}^2 - 1}, \quad (85)$$

and thus,

$$\begin{aligned} (\mathbf{c}_p)_\epsilon &= \frac{(c_{py})_\epsilon}{(c_{px})_{-\epsilon}} (- (c_{py})_{-\epsilon} \hat{\mathbf{x}} + (c_{px})_{-\epsilon} \hat{\mathbf{y}}) = \frac{(c_{py})_\epsilon}{(c_{px})_{-\epsilon}} (\hat{\mathbf{z}} \times (\mathbf{c}_p)_{-\epsilon}), \\ \frac{(\mathbf{c}_p)_\epsilon^2}{(\mathbf{c}_p)_{-\epsilon}^2} &= \frac{(c_{py})_\epsilon^2}{(c_{px})_{-\epsilon}^2} = \frac{(c_{px})_\epsilon^2}{(c_{py})_{-\epsilon}^2} = (\mathbf{c}_p)_\epsilon^2 - 1. \end{aligned} \quad (86)$$

For the particular case where $h_3 = 0$, we obtain, after choosing $\epsilon = +1$ or $\epsilon = -1$ in the definition of ξ ,

$$\begin{aligned} \xi = h_1, c_{p,x} = 0, c_{p,y} &= \sqrt{h_1 - h_2}/\sqrt{h_1}, p^2 = 1 - (\mathbf{c}_p)^2 = h_2/h_1, \\ \xi = h_2, c_{p,x} &= \sqrt{h_2 - h_1}/\sqrt{h_2}, c_{p,y} = 0, p^2 = 1 - (\mathbf{c}_p)^2 = h_1/h_2. \end{aligned} \quad (87)$$

We also note that, when $h_3^2 \rightarrow 4h_1h_2$, we can take $\xi = h_1 + h_2$ and write,

$$\begin{aligned} (h_1x'^2 + h_2y'^2 + h_3x'y') &\rightarrow (\sqrt{h_1}x' - \epsilon' \sqrt{h_2}y')^2, \quad \epsilon' = -h_3/(2\sqrt{h_1}\sqrt{h_2}), \\ \xi = h_1 + h_2, c_{p,x} &= \frac{\sqrt{h_2}}{\sqrt{h_1 + h_2}}, c_{p,y} = \epsilon' \frac{\sqrt{h_1}}{\sqrt{h_1 + h_2}}, p^2 = 1 - (\mathbf{c}_p)^2 = 0. \end{aligned} \quad (88)$$

Proof. Letting,

$$(h_1x'^2 + h_2y'^2 + h_3x'y') = \xi((\mathbf{r}')^2 - (\mathbf{c}_p \cdot \mathbf{r}')^2), \quad (89)$$

we obtain (81) with (82), then $2c_{p,x}c_{p,y} = -h_3/\xi$, $c_{p,x}^2 = 1 - h_1/\xi$, $c_{p,y}^2 = 1 - h_2/\xi$. Thus, $4(1 - h_1/\xi)(1 - h_2/\xi) = (h_3/\xi)^2$ and we have,

$$\begin{aligned} \xi &= \frac{(h_1 + h_2) + \epsilon \sqrt{(h_1 - h_2)^2 + 4(h_3/2)^2}}{2}, \\ c_{p,x} &= \epsilon'_x \sqrt{1 - h_1/\xi} = \frac{\epsilon_x \sqrt{\xi - h_1}}{\sqrt{\xi}}, \quad c_{p,y} = \epsilon'_y \sqrt{1 - h_2/\xi} = \frac{\epsilon_y \sqrt{\xi - h_2}}{\sqrt{\xi}}, \\ \epsilon'_x \epsilon'_y &= \frac{-h_3/(2\xi)}{\sqrt{1 - \frac{h_1}{\xi}} \sqrt{1 - \frac{h_2}{\xi}}}, \quad \epsilon_x \epsilon_y = \frac{-h_3/2}{\sqrt{\xi - h_1} \sqrt{\xi - h_2}}, \quad \epsilon_{x(y)}^2 = \epsilon_{x(y)}'^2 = 1, \end{aligned} \quad (90)$$

where the choice of ϵ , ϵ_x and ϵ'_x , either $+1$ or -1 , is free (note : we can take $-\mathbf{c}_p$ in place of \mathbf{c}_p). Letting ξ_ϵ and $(\mathbf{c}_p)_\epsilon$, specifying the choice of ϵ , in place of ξ and \mathbf{c}_p , we can write,

$$\begin{aligned}
 \frac{(c_{px})_\epsilon}{(c_{py})_\epsilon} &= \frac{(c_{px})_\epsilon (c_{py})_\epsilon}{(c_{py})_\epsilon (c_{py})_\epsilon} = \frac{-h_3}{2(\xi_\epsilon - h_2)} = \frac{-h_3}{(h_1 - h_2) + \epsilon \sqrt{(h_1 - h_2)^2 + h_3^2}} \\
 &= \frac{(h_1 - h_2) - \epsilon \sqrt{(h_1 - h_2)^2 + h_3^2}}{h_3} = -\frac{(c_{py})_{-\epsilon}}{(c_{px})_{-\epsilon}}, \\
 (\mathbf{c}_p)_\epsilon^2 &= 2 - \frac{h_1 + h_2}{\xi_\epsilon} = 1 - 2 \frac{-\xi_\epsilon + (h_1 + h_2)}{(h_1 + h_2) + \epsilon \sqrt{(h_1 - h_2)^2 + h_3^2}} \\
 &= 1 - \frac{(h_1 + h_2) - \epsilon \sqrt{(h_1 - h_2)^2 + h_3^2}}{(h_1 + h_2) + \epsilon \sqrt{(h_1 - h_2)^2 + h_3^2}} = 1 - \frac{1}{1 - (\mathbf{c}_p)_{-\epsilon}^2} = \frac{(\mathbf{c}_p)_{-\epsilon}^2}{(\mathbf{c}_p)_{-\epsilon}^2 - 1}. \quad (91)
 \end{aligned}$$

Proposition 6

We can write,

$$\begin{aligned}
 (\alpha_1 x' + \alpha_2 y') - ik(h_1 x'^2 + h_2 y'^2 + h_3 x' y') &= \mathbf{w}' \cdot \mathbf{r}' - ik\xi((\mathbf{r}')^2 - (\mathbf{c}_p \cdot \mathbf{r}')^2) \\
 &= \mathbf{w}'_m \cdot \mathbf{r}'_p - ik\xi(\mathbf{r}'_p)^2 = v(\mathbf{r}'_p - \mathbf{a}_p)^2 + v_1, \quad (92)
 \end{aligned}$$

with $\mathbf{w}' \cdot \mathbf{r}' = \alpha_1 x' + \alpha_2 y'$, $\mathbf{w}' \cdot \mathbf{r}' = \mathbf{w}'_m \cdot \mathbf{r}'_p$, $(\mathbf{r}'_p)^2 = (\mathbf{r}')^2 - (\mathbf{c}_p \cdot \mathbf{r}')^2$, $v = -ik\xi$ and \mathbf{c}_p defined in proposition 5. Letting $\mathbf{c}_p = \sqrt{\mathbf{c}_p^2} \hat{\mathbf{c}}_p$ and $p = \sqrt{1 - \mathbf{c}_p^2}$, the complex points \mathbf{r}'_p , \mathbf{w}'_m , and \mathbf{a}_p verify,

$$\begin{aligned}
 \mathbf{r}'_p &= (p(\hat{\mathbf{c}}_p \cdot \mathbf{r}') \hat{\mathbf{c}}_p + (\mathbf{r}' - \hat{\mathbf{c}}_p(\hat{\mathbf{c}}_p \cdot \mathbf{r}')) = \mathbf{r}' - \mathbf{c}_p(\mathbf{c}_p \cdot \mathbf{r}') / (1 + p), \\
 \mathbf{w}'_m &= \frac{(\hat{\mathbf{c}}_p \mathbf{w}')}{p} \hat{\mathbf{c}}_p + (\mathbf{w}' - \hat{\mathbf{c}}_p(\hat{\mathbf{c}}_p \mathbf{w}')) = \mathbf{w}' + \mathbf{c}_p(\mathbf{c}_p \cdot \mathbf{w}') / (p(1 + p)), \\
 \mathbf{a}_p &= \frac{\mathbf{w}'_m}{2ik\xi}, \quad v = -ik\xi, \quad v_1 = ik\xi(\mathbf{a}_p)^2 = \frac{\mathbf{w}'_m^2}{4ik\xi}, \quad (93)
 \end{aligned}$$

where ξ and p satisfy,

$$\begin{aligned}
 \xi &= \frac{(h_1 + h_2) + \epsilon \sqrt{(h_1 - h_2)^2 + h_3^2}}{2} = \frac{2(h_1 h_2 - (h_3/2)^2)}{(h_1 + h_2) - \epsilon \sqrt{(h_1 - h_2)^2 + h_3^2}}, \\
 p &= \sqrt{1 - \mathbf{c}_p^2} = \sqrt{-1 + \frac{h_1 + h_2}{\xi}} = \sqrt{\frac{(h_1 + h_2) - \epsilon \sqrt{(h_1 - h_2)^2 + h_3^2}}{(h_1 + h_2) + \epsilon \sqrt{(h_1 - h_2)^2 + h_3^2}}}. \quad (94)
 \end{aligned}$$

with $\epsilon = +1$ or -1 , $(p\xi)^2 = h_1 h_2 - (\frac{h_3}{2})^2$.

This can be considered in general, and for some particular cases, in the sense of the limit. So, if $h_3^2/(4h_1h_2) \rightarrow 1$, following the choice of ϵ for ξ , we have $\xi \rightarrow 0$ with $p \rightarrow \infty$ or $\xi \rightarrow h_1 + h_2$ with $p \rightarrow 0$, and we can take,

$$\begin{aligned}\widehat{c}_{p,x} &= \frac{\sqrt{h_1}}{\sqrt{h_1+h_2}}, \widehat{c}_{p,y} = \frac{-\epsilon' \sqrt{h_2}}{\sqrt{h_1+h_2}}, \text{ with } p \sim \sqrt{\frac{4(h_1+h_2)^2}{4h_1h_2-h_3^2}} \rightarrow \infty, \\ \widehat{c}_{p,x} &= \frac{\sqrt{h_2}}{\sqrt{h_1+h_2}}, \widehat{c}_{p,y} = \frac{\epsilon' \sqrt{h_1}}{\sqrt{h_1+h_2}}, \text{ with } p \sim \sqrt{\frac{4h_1h_2-h_3^2}{4(h_1+h_2)^2}} \rightarrow 0,\end{aligned}\quad (95)$$

with $\epsilon' = \frac{-h_3/2}{\sqrt{h_1}\sqrt{h_2}}$. Besides, if $-h_3^2/(h_1-h_2)^2 \rightarrow 1$, we have $\xi \rightarrow \frac{h_1+h_2}{2}$ with $p \rightarrow 1$, and thus $\mathbf{c}_p^2 \rightarrow 0$ (note : $\mathbf{c}_p^2 = 0$ does not imply $\mathbf{c}_p = 0$ in complex coordinates), and we can take,

$$\begin{aligned}\mathbf{r}'_p &\rightarrow \mathbf{r}' - \frac{(\mathbf{c}_p \cdot \mathbf{r}') \mathbf{c}_p}{2} \text{ and } \mathbf{w}'_m \rightarrow \mathbf{w}' + \frac{(\mathbf{c}_p \cdot \mathbf{w}') \mathbf{c}_p}{2}, \\ \text{with } c_{p,x} &= \frac{\sqrt{h_2-h_1}}{\sqrt{h_1+h_2}}, c_{p,y} = \epsilon_y \frac{\sqrt{h_1-h_2}}{\sqrt{h_1+h_2}}, \epsilon_y = \frac{-h_3}{\sqrt{h_2-h_1}\sqrt{h_1-h_2}},\end{aligned}\quad (96)$$

which implies, for (h_1-h_2) and $h_3 \rightarrow 0$, that we have $\mathbf{c}_p \rightarrow 0$ and,

$$\mathbf{r}'_p \rightarrow \mathbf{r}', \mathbf{w}'_m \rightarrow \mathbf{w}', \mathbf{a}_p \rightarrow \frac{\mathbf{w}'}{2ik\xi} \text{ with } \xi \rightarrow h_{1(,2)}.\quad (97)$$

Proof. Considering (92), we choose \mathbf{w}'_m and \mathbf{r}'_p verifying,

$$\mathbf{w}'_m \cdot \mathbf{r}'_p = \mathbf{w}' \cdot \mathbf{r}' = \alpha_1 x' + \alpha_2 y', (\mathbf{r}'_p)^2 = ((\mathbf{r}')^2 - (\mathbf{c}_p \cdot \mathbf{r}')^2),\quad (98)$$

as in (93), with $\widehat{\mathbf{c}}_p = \frac{\mathbf{c}_p}{\sqrt{\mathbf{c}_p \cdot \mathbf{c}_p}}$ and \mathbf{c}_p defined in proposition 5. We then have,

$$\begin{aligned}(\alpha_1 x' + \alpha_2 y') - ik(h_1 x'^2 + h_2 y'^2 + h_3 x' y') &= \\ = \mathbf{w}'_m \cdot \mathbf{r}'_p - ik\xi (\mathbf{r}'_p)^2 &= -ik\xi \left((\mathbf{r}'_p - \frac{\mathbf{w}'_m}{2ik\xi})^2 - \frac{\mathbf{w}'_m \cdot \mathbf{w}'_m}{(2ik\xi)^2} \right) = v(\mathbf{r}'_p - \mathbf{a}_p)^2 + v_1,\end{aligned}\quad (99)$$

with $\mathbf{a}_p = \frac{\mathbf{w}'_m}{2ik\xi}$, $v = -ik\xi$, $v_1 = \frac{\mathbf{w}'_m \cdot \mathbf{w}'_m}{4ik\xi}$, which completes (93). Using $p \sim 1 - \mathbf{c}_p^2/2$ when $\mathbf{c}_p^2 \rightarrow 0$, we derive (96). On this occasion, let us notice that we could have chosen to take $p = -\sqrt{1 - \mathbf{c}_p^2}$ in place of $p = \sqrt{1 - \mathbf{c}_p^2}$ in the definitions.

6.2) Reduction of the surface integral to a contour integral

Considering the expression of complex points \mathbf{r}'_p given in proposition 6, we define a complex change of variables (complex scaling) and an efficient reduction to contours integrals.

6.2.1) Complex scaling transform \mathcal{P} and definition of S'_p

Definition 4

Using \mathbf{r}'_p given in (93) of proposition 6, we consider \mathcal{P} , the complex anisotropic dilation (or scaling) by a factor $p = \sqrt{1 - \mathbf{c}_p^2}$ along the vector $\widehat{\mathbf{c}}_p = \mathbf{c}_p / \sqrt{\mathbf{c}_p^2}$ (with $\mathbf{c}_p^2 \neq 0$), defined by,

$$\begin{aligned} \mathcal{P}: \mathbf{r}' &= x' \widehat{\mathbf{x}} + y' (\widehat{\mathbf{z}} \times \widehat{\mathbf{x}}) \rightarrow \mathbf{r}'_p = x'_p \widehat{\mathbf{c}}_p + y'_p (\widehat{\mathbf{z}} \times \widehat{\mathbf{c}}_p), \\ \text{with } x'_p &= p (\mathbf{r}' \cdot \widehat{\mathbf{c}}_p), y'_p = \mathbf{r}' \cdot (\widehat{\mathbf{z}} \times \widehat{\mathbf{c}}_p), \end{aligned} \quad (100)$$

when $h_3^2 \neq 4h_1h_2$, i.e. $p \neq 0$. This function is linear invertible, and satisfies,

$$\begin{aligned} \mathcal{P}(\mathbf{r}') &= (p(\widehat{\mathbf{c}}_p \cdot \mathbf{r}') \widehat{\mathbf{c}}_p + (\mathbf{r}' - \widehat{\mathbf{c}}_p(\widehat{\mathbf{c}}_p \cdot \mathbf{r}')) = \mathbf{r}' - \frac{\mathbf{c}_p(\mathbf{c}_p \cdot \mathbf{r}')}{1+p} = \mathbf{r}'_p, \\ \mathcal{P}^{-1}(\mathbf{r}'_p) &= \frac{(\widehat{\mathbf{c}}_p \cdot \mathbf{r}'_p)}{p} \widehat{\mathbf{c}}_p + (\mathbf{r}'_p - \widehat{\mathbf{c}}_p(\widehat{\mathbf{c}}_p \cdot \mathbf{r}'_p)) = \mathbf{r}'_p + \frac{\mathbf{c}_p(\mathbf{c}_p \cdot \mathbf{r}'_p)}{p(1+p)} = \mathbf{r}', \\ \mathcal{P}(\mathbf{c}_p) &= p\mathbf{c}_p, \mathcal{P}((\widehat{\mathbf{z}} \times \mathbf{c}_p)) = (\widehat{\mathbf{z}} \times \mathbf{c}_p), \end{aligned} \quad (101)$$

As in definition 1, we then define a complex simply connected manifold S'_p and its oriented contour \mathcal{L}_p as the transformation by \mathcal{P} of a real simply connected manifold of the plane $z = 0$, now named S' , and its real positively oriented contour \mathcal{L} . Letting $dS' = dx' \wedge dy'$ and $dS'_p = dx'_p \wedge dy'_p$, we have $p dS' = dS'_p$.

Besides, we remark that,

$$\begin{aligned} \text{grad}_{\mathbf{r}'}(g(\mathbf{r}')) &= \mathcal{P}(\text{grad}_{\mathbf{r}'_p}(g(\mathbf{r}'))), \\ \mathcal{P}^2(\mathbf{r}') &= \mathbf{r}' - (\mathbf{c}_p \cdot \mathbf{r}') \mathbf{c}_p, p^2 \mathcal{P}^{-2}(\mathbf{r}') = p^2 \mathbf{r}' + (\mathbf{c}_p \cdot \mathbf{r}') \mathbf{c}_p, \end{aligned} \quad (102)$$

and notice, in the limit case $-h_3^2/(h_1 - h_2)^2 \rightarrow 1$, i.e. when $p \rightarrow 1$ and thus $(\mathbf{c}_p)^2 \rightarrow 0$,

$$\mathbf{r}'_p = \mathcal{P}(\mathbf{r}') \rightarrow \mathbf{r}' - \frac{(\mathbf{c}_p \cdot \mathbf{r}') \mathbf{c}_p}{2} \text{ and } \mathbf{w}'_m = \mathcal{P}^{-1}(\mathbf{w}') \rightarrow \mathbf{w}' + \frac{(\mathbf{c}_p \cdot \mathbf{w}') \mathbf{c}_p}{2}, \quad (103)$$

(note : $(\mathbf{c}_p)^2 \rightarrow 0$ implies $\mathbf{c}_p \rightarrow A_1(\widehat{\mathbf{x}} \pm i\widehat{\mathbf{y}})$, where $\widehat{\mathbf{x}} \pm i\widehat{\mathbf{y}} = \frac{\mathbf{e} \pm i\widehat{\mathbf{z}} \times \mathbf{e}}{c_x \mp ic_y}$ for any complex vector \mathbf{c} with $c_x \mp ic_y \neq 0$). Therefore, for $(h_1 - h_2) \rightarrow 0$ and $h_3 \rightarrow 0$, we have $\mathbf{c}_p \rightarrow 0$ and,

$$\mathcal{P}(\mathbf{r}') \rightarrow \mathbf{r}' \text{ and } \mathbf{w}'_m = \mathcal{P}^{-1}(\mathbf{w}') \rightarrow \mathbf{w}' \text{ with } \xi \rightarrow h_{1(2)}. \quad (104)$$

Remark : For $h_3 = 0$, we note that we have the two following choices ,

$$\begin{aligned} \xi &= h_1, \widehat{\mathbf{c}}_p = \widehat{\mathbf{y}}, p^2 = h_2/h_1, \mathcal{P}(\mathbf{r}') = (\widehat{\mathbf{x}} \cdot \mathbf{r}') \widehat{\mathbf{x}} + p (\widehat{\mathbf{y}} \cdot \mathbf{r}') \widehat{\mathbf{y}}, \\ \xi &= h_2, \widehat{\mathbf{c}}_p = \widehat{\mathbf{x}}, p^2 = h_1/h_2, \mathcal{P}(\mathbf{r}') = p (\widehat{\mathbf{x}} \cdot \mathbf{r}') \widehat{\mathbf{x}} + (\widehat{\mathbf{y}} \cdot \mathbf{r}') \widehat{\mathbf{y}}. \end{aligned} \quad (105)$$

Remark : We note that, for $n \geq 1$,

$$\mathcal{P}^{2n}(\mathbf{r}') = \mathbf{r}' - \left(\sum_{k=0}^{n-1} p^{2k} \right) (\mathbf{c}_p \cdot \mathbf{r}') \mathbf{c}_p, \quad \mathcal{P}^{2n+1}(\mathbf{r}') = \mathcal{P}(\mathbf{r}') - \left(p \sum_{k=0}^{n-1} p^{2k} \right) (\mathbf{c}_p \cdot \mathbf{r}') \mathbf{c}_p, \quad (106)$$

with $\sum_{k=0}^{n-1} p^{2k} = \frac{1-p^{2n}}{1-p^2}$ for $n \geq 1$, from which we derive $\mathcal{P}^{-m}(\mathbf{r}')$ from $\mathcal{P}^{-1}(\mathbf{c}_p) = \mathbf{c}_p/p$.

Remark : From (26), we have $\partial_{(\mathbf{r}'_p - \mathbf{a}_p)^2} \equiv \frac{(\mathbf{r}'_p - \mathbf{a}_p)}{2(\mathbf{r}'_p - \mathbf{a}_p)^2} \cdot \text{grad}_{\mathbf{r}'_p}$ and using $\text{grad}_{\mathbf{r}'_p} \equiv \mathcal{P}^{-1}(\text{grad}_{\mathbf{r}'})$, we note that $\partial_{(\mathbf{r}'_p - \mathbf{a}_p)^2} \equiv \frac{(\mathbf{r}' - \mathcal{P}^{-1}(\mathbf{a}_p))}{2(\mathbf{r}'_p - \mathbf{a}_p)^2} \cdot \text{grad}_{\mathbf{r}'}$.

6.2.2) First reduction of integrals along the transformed contour \mathcal{L}_p

Proposition 7

Considering $\mathbf{r}'_p = \mathcal{P}(\mathbf{r}')$, with $\mathbf{a}_p = \frac{\mathbf{w}'_m}{2ik\xi}$, v and p given in (93), and an arbitrary constant A_0 , we can write,

$$\begin{aligned} & \int_{S'} e^{(\alpha_1 x' + \alpha_2 y') - ik(h_1 x'^2 + h_2 y'^2 + h_3 x' y')} \cdot dS' = \\ & = \frac{1}{2p} \int_{\mathcal{L}_p} \left(\frac{\widehat{\mathbf{z}} \times (\mathbf{r}'_p - \mathbf{a}_p) (e^{\mathbf{w}'_m \cdot \mathbf{r}'_p - ik\xi(\mathbf{r}'_p)^2} - A_0)}{\mathbf{w}'_m \cdot \mathbf{r}'_p - ik\xi(\mathbf{r}'_p)^2 - \frac{\mathbf{w}'_m{}^2}{4ik\xi}} \right) \cdot d\mathbf{r}'_p \\ & + \frac{(A_0 - e^{\frac{\mathbf{w}'_m{}^2}{4ik\xi}})}{-2ikp\xi} \int_{\mathcal{L}_p} \left(\widehat{\mathbf{z}} \times \frac{\mathbf{r}'_p - \mathbf{a}_p}{(\mathbf{r}'_p - \mathbf{a}_p)^2} \right) \cdot d\mathbf{r}'_p, \end{aligned} \quad (107)$$

where $\mathbf{w}'_m \cdot \mathbf{r}'_p - ik\xi(\mathbf{r}'_p)^2 - \frac{\mathbf{w}'_m{}^2}{4ik\xi} = v(\mathbf{r}'_p - \mathbf{a}_p)^2$, $v = -ik\xi$. Let \mathcal{D}_1 be a smooth semi-line of the complex plane from the origin to ∞ , we note that, when $(\mathbf{r}'_p - \mathbf{a}_p)^2 \neq 0$ on \mathcal{L}_p ,

$$\int_{\mathcal{L}_p} \left(\widehat{\mathbf{z}} \times \frac{\mathbf{r}'_p - \mathbf{a}_p}{(\mathbf{r}'_p - \mathbf{a}_p)^2} \right) \cdot d\mathbf{r}'_p = \frac{-i}{2} \int_{\mathcal{L}_p} d\ln(\chi(\mathbf{r}'_p)) = \pi(N_+ - N_-), \quad (108)$$

where N_+ (resp. N_-) is the number of times $\chi(\mathbf{r}'_p)$ crosses \mathcal{D}_1 in positive sense (resp. negative) along the closed contour \mathcal{L}_p , and,

$$\chi(\mathbf{r}'_p) = \frac{z'_{\mathbf{a}^+}}{z'_{\mathbf{a}^-}}, \quad z'_{\mathbf{a}^\pm} = x'_a \pm iy'_a \text{ as } \mathbf{r}'_p - \mathbf{a}_p = x'_a \widehat{\mathbf{c}}_p + y'_a (\widehat{\mathbf{z}} \times \widehat{\mathbf{c}}_p). \quad (109)$$

The first contour integral in (107) is in whole generality non singular as we take $A_0 = e^{\frac{\mathbf{w}'_m{}^2}{4ik\xi}}$, but, when $(\mathbf{r}'_p - \mathbf{a}_p)^2 \neq 0$ on \mathcal{L}_p , the choice $A_0 = 0$ could be suitable for asymptotic evaluations as v is large.

Proof. From the proposition 6 and $p dS' = dS'_p$, we have,

$$\int_{S'} e^{v(\alpha_1 x' + \alpha_2 y') - ik(h_1 x'^2 + h_2 y'^2 + h_3 x' y')} dS' = \frac{1}{p} \int_{S'_p} e^{v(\mathbf{r}'_p - \mathbf{a}_p)^2 + v_1} dS'_p. \quad (110)$$

We can then apply the theorem 1 to the surface integral, following,

$$\begin{aligned} \int_{S'_p} e^{v(\mathbf{r}'_p - \mathbf{a}_p)^2 + v_1} dS_p &= \frac{1}{2} \left(\int_{\mathcal{L}_p} \left(\frac{\widehat{\mathbf{z}} \times (\mathbf{r}'_p - \mathbf{a}_p) (e^{-ik\xi(\mathbf{r}'_p - \mathbf{a}_p)^2 + v_1} - A_0)}{v(\mathbf{r}'_p - \mathbf{a}_p)^2} \right) \cdot d\mathbf{r}'_p + \right. \\ &\left. + (A_0 - e^{v_1}) \int_{\mathcal{L}_p} \left(\widehat{\mathbf{z}} \times \frac{\mathbf{r}'_p - \mathbf{a}_p}{v(\mathbf{r}'_p - \mathbf{a}_p)^2} \right) \cdot d\mathbf{r}'_p \right), \end{aligned} \quad (111)$$

and derive (107), definite for $(\mathbf{r}'_p - \mathbf{a}_p)^2 \neq 0$ on \mathcal{L}_p , or, in whole generality, for $A_0 = e^{v_1}$. We obtain (108) from proposition 1.

Remark : When the terms $\frac{(\mathbf{r}'_p \pm i(\mathbf{r}'_p \times \widehat{\mathbf{z}})) \cdot \widehat{\mathbf{c}}_p}{(\mathbf{a}_p \pm i(\mathbf{a}_p \times \widehat{\mathbf{z}})) \cdot \widehat{\mathbf{c}}_p}$ are small as $\mathbf{r}'_p \in \mathcal{L}_p$, we note that,

$$\max |\operatorname{Im}(\ln(\frac{1 - \frac{(\mathbf{r}'_p + i(\mathbf{r}'_p \times \widehat{\mathbf{z}})) \cdot \widehat{\mathbf{c}}_p}{(\mathbf{a}_p + i(\mathbf{a}_p \times \widehat{\mathbf{z}})) \cdot \widehat{\mathbf{c}}_p}}{1 - \frac{(\mathbf{r}'_p - i(\mathbf{r}'_p \times \widehat{\mathbf{z}})) \cdot \widehat{\mathbf{c}}_p}{(\mathbf{a}_p - i(\mathbf{a}_p \times \widehat{\mathbf{z}})) \cdot \widehat{\mathbf{c}}_p}}))|_{\mathbf{r}'_p \in \mathcal{L}_p} \ll 1, \quad (112)$$

and thus, the term $\operatorname{Im}(\ln(\frac{z'_{a+}}{z'_{a-}}))$ little varies, $\frac{z'_{a+}}{z'_{a-}}$ does not run round the origin, $N_+ = N_-$ and the winding number $(N_+ - N_-)$ is null in (108).

6.2.3) Some properties of $\mathcal{P}(\mathbf{r}') = \mathbf{r}'_p$ to analyse and transform (107)

Proposition 8

From the definitions of \mathcal{P} and \mathcal{P}^{-1} (see definition 4), we have,

$$\mathcal{P}(\mathbf{u}) \cdot \mathbf{v} = \mathbf{u} \cdot \mathcal{P}(\mathbf{v}), \mathcal{P}(\widehat{\mathbf{z}} \times \mathbf{u}) = p \widehat{\mathbf{z}} \times \mathcal{P}^{-1}(\mathbf{u}), \mathcal{P}(\widehat{\mathbf{z}} \times \mathcal{P}(\mathbf{v})) = p(\widehat{\mathbf{z}} \times \mathbf{v}), \quad (113)$$

for any vectors \mathbf{u} and \mathbf{v} of the plane. Thus, we can write,

$$\mathbf{u} \cdot d\mathbf{r}'_p = \mathcal{P}(\mathbf{u}) \cdot d\mathbf{r}', (\widehat{\mathbf{z}} \times \mathbf{r}'_p) \cdot d\mathbf{r}'_p = p(\widehat{\mathbf{z}} \times \mathbf{r}') \cdot d\mathbf{r}', \quad (114)$$

and consider that, as $\mathbf{w}'_m \cdot \mathbf{r}'_p = \mathbf{w}' \cdot \mathbf{r}'$ with $\mathbf{r}'_p = \mathcal{P}(\mathbf{r}')$,

$$\mathcal{P}(\mathbf{w}'_m) = \mathbf{w}', \mathbf{w}'_m = \mathcal{P}^{-1}(\mathbf{w}'). \quad (115)$$

Concerning the properties relatively to the choice of $\epsilon = \pm 1$ (factor used in (90) for ξ), we can let $(\mathbf{c}_p)_\pm = (\widehat{\mathbf{c}}_p)_\pm \sqrt{(\mathbf{c}_p^2)_\pm}$, p_\pm , ξ_\pm , $(\mathcal{P})_\pm$ and $(\mathbf{r}'_p)_\pm$, be the respective terms $(\mathbf{c}_p) = (\widehat{\mathbf{c}}_p) \sqrt{(\mathbf{c}_p^2)}$, ξ , p , \mathcal{P} and (\mathbf{r}'_p) for $\epsilon = \pm 1$, and notice that,

$$\begin{aligned}
p_{\pm}^2 &= \frac{\xi_{\mp 1}}{\xi_{\pm 1}} = \frac{1}{p_{\mp}^2}, (\xi_{\pm 1} p_{\pm})^2 = (\xi_{\mp 1} p_{\mp})^2 = \xi_+ \xi_- = h_1 h_2 - (h_3/2)^2, \\
p_{\pm}^2 &= 1 - (\mathbf{c}_p^2)_{\pm} = -1 + \frac{h_1 + h_2}{\xi_{\pm}}, (\mathbf{r}'_p)_{\pm} = p_{\pm} (\mathbf{r}'_p)_{\mp}, (\mathcal{P})_{\pm} = p_{\pm} (\mathcal{P})_{\mp}, \\
(\widehat{\mathbf{c}}_p)_{\pm} &= \epsilon_{c\pm} (\widehat{\mathbf{z}} \times (\widehat{\mathbf{c}}_p)_{\mp}), \epsilon_{c\pm} = \frac{(c_{py})_{\pm} \sqrt{(\mathbf{c}_p^2)_{\mp}}}{(c_{px})_{\mp} \sqrt{(\mathbf{c}_p^2)_{\pm}}} = -\frac{(c_{px})_{\pm} \sqrt{(\mathbf{c}_p^2)_{\mp}}}{(c_{py})_{\mp} \sqrt{(\mathbf{c}_p^2)_{\pm}}}, \quad (116)
\end{aligned}$$

where $(\epsilon_c)^2 = 1$, and $-\epsilon \equiv \mp$ as $\epsilon \equiv \pm$. We remark that $\xi_{-\epsilon} = 0$ implies $h_3^2 = 4h_1 h_2$, $\xi_{\epsilon} = h_1 + h_2$, and $(\mathbf{c}_p^2)_{\epsilon} = 1$ if $h_1 + h_2 \neq 0$, while $(\mathbf{c}_p^2)_{\epsilon} = 1$ implies $\xi_{-\epsilon} = 0$ and $\xi_{\epsilon} = h_1 + h_2$, and notice that $(\mathbf{c}_p^2)_{\epsilon} = 0$ implies $(\mathbf{c}_p^2)_{-\epsilon} = 0$, $2\xi_{\pm} = h_1 + h_2$ and $p_{\pm} = 1$. (note : even if $(\xi p)^2 = h_1 h_2 - (h_3/2)^2$, we have not to calculate $p\xi$ as $\sqrt{(p\xi)^2}$, but with each term p and ξ determined independently, to avoid the cut of square root).

Proof. The properties in (113) derives from linearity. For (116), we consider (82) and notice that,

$$\begin{aligned}
1 - (\mathbf{c}_p^2)_{\pm} &= -1 + \frac{h_1 + h_2}{\xi_{\pm}} = \frac{\xi_{\mp}}{\xi_{\pm}} = \frac{1}{1 - (\mathbf{c}_p^2)_{\mp}}, \\
\frac{(\widehat{\mathbf{c}}_{px})_{\epsilon}}{(\widehat{\mathbf{c}}_{py})_{\epsilon}} &= \frac{(c_{px})_{\epsilon} (c_{py})_{\epsilon}}{(c_{py})_{\epsilon} (c_{py})_{\epsilon}} = \frac{-h_3}{2(\xi_{\epsilon} - h_2)} = \frac{-h_3}{(h_1 - h_2) + \epsilon \sqrt{(h_1 - h_2)^2 + h_3^2}} \\
&= \frac{(h_1 - h_2) - \epsilon \sqrt{(h_1 - h_2)^2 + h_3^2}}{h_3} = -\frac{(c_{py})_{-\epsilon}}{(c_{px})_{-\epsilon}} = -\frac{(\widehat{\mathbf{c}}_{py})_{-\epsilon}}{(\widehat{\mathbf{c}}_{px})_{-\epsilon}}. \quad (117)
\end{aligned}$$

Proposition 9

Let from now on,

$$\begin{aligned}
\mathcal{M}(\mathbf{w}') &= p^2 \mathcal{P}^{-1}(\mathbf{w}'_m), \text{ as } p\mathcal{P}(\widehat{\mathbf{z}} \times \mathbf{w}'_m) = \widehat{\mathbf{z}} \times \mathcal{M}(\mathbf{w}'), \\
\mathcal{N}(\mathbf{w}') &= p \mathbf{w}'_m, \text{ as } v_1 = ik\xi(\mathbf{a}_p)^2 = \frac{\mathbf{w}'_m{}^2}{4ik\xi} = \frac{(\mathcal{N}(\mathbf{w}'))^2}{4ik\xi p^2}, \quad (118)
\end{aligned}$$

with $\mathbf{a}_p = \frac{\mathbf{w}'_m}{2ik\xi}$, $\mathbf{w}'_m = \mathcal{P}^{-1}(\mathbf{w}')$, and $p = \sqrt{1 - \mathbf{c}_p^2}$. We can write, for $\mathbf{r}'_p = \mathcal{P}(\mathbf{r}')$,

$$\frac{(\widehat{\mathbf{z}} \times \frac{(\mathbf{r}'_p - \mathbf{a}_p)}{(\mathbf{r}'_p - \mathbf{a}_p)^2}) \cdot d\mathbf{r}'_p}{-2ikp\xi} = \frac{\frac{1}{2}(\widehat{\mathbf{z}} \times (\mathbf{r}' - \frac{\mathcal{M}(\mathbf{w}')}{2ik\xi p^2})) \cdot d\mathbf{r}'}{\mathbf{w}' \cdot \mathbf{r}' - ik(h_1 x'^2 + h_2 y'^2 + h_3 x' y') - \frac{(\mathcal{N}(\mathbf{w}'))^2}{4ik\xi p^2}}, \quad (119)$$

where, $\mathcal{P}^{-1}(\mathbf{r}'_p) = \frac{(\widehat{\mathbf{c}}_p \mathbf{r}'_p)}{p} \widehat{\mathbf{c}}_p + (\mathbf{r}'_p - \widehat{\mathbf{c}}_p (\widehat{\mathbf{c}}_p \mathbf{r}'_p)) = \mathbf{r}'_p + \frac{c_p (\mathbf{c}_p \cdot \mathbf{r}'_p)}{p(1+p)} = \mathbf{r}'$

$$\begin{aligned}\frac{\mathcal{M}(\mathbf{w}')}{2ik\xi p^2} &= \mathcal{P}^{-1}(\mathbf{a}_p) = \frac{(1 - \mathbf{c}_p^2)\mathbf{w}' + (\mathbf{c}_p \cdot \mathbf{w}')\mathbf{c}_p}{2ik\xi p^2} \\ &= \frac{1}{2ik(\xi p)^2} \left((h_2 w'_x - \frac{h_3 w'_y}{2}) \hat{\mathbf{x}} + (h_1 w'_y - \frac{h_3 w'_x}{2}) \hat{\mathbf{y}} \right),\end{aligned}\quad (120)$$

and,

$$\begin{aligned}\frac{(\mathcal{N}(\mathbf{w}'))^2}{4ik\xi p^2} &= \frac{\mathbf{w}'_m{}^2}{4ik\xi} = \frac{(1 - \mathbf{c}_p^2)\mathbf{w}'^2 + (\mathbf{c}_p \cdot \mathbf{w}')^2}{4ik\xi p^2} \\ &= \frac{1}{4ik(\xi p)^2} (h_2 w'_x{}^2 + h_1 w'_y{}^2 - h_3 w'_x w'_y),\end{aligned}\quad (121)$$

with $(\xi p)^2 = \xi_+ \xi_- = h_1 h_2 - (h_3/2)^2$. Let us notice that when $h_3 = 0$ and $h_1 = h_2 \rightarrow 0$,

$$\frac{(\hat{\mathbf{z}} \times \frac{(\mathbf{r}'_p - \mathbf{a}_p)}{(\mathbf{r}'_p - \mathbf{a}_p)^2}) \cdot d\mathbf{r}'_p}{-2ikp\xi} = \frac{(\hat{\mathbf{z}} \times \mathbf{w}') \cdot d\mathbf{r}'}{(\mathbf{w}')^2}.\quad (122)$$

Proof. Considering (92) and (114)-(118), we obtain (119), then we derive (120) from,

$$\begin{aligned}\mathcal{M}(\mathbf{w}') &= p^2 \mathcal{P}^{-2}(\mathbf{w}') \\ &= (\hat{\mathbf{c}}_p \cdot \mathbf{w}') \hat{\mathbf{c}}_p + (1 - \mathbf{c}_p^2)(\mathbf{w}' - \hat{\mathbf{c}}_p(\hat{\mathbf{c}}_p \cdot \mathbf{w}')) = (1 - \mathbf{c}_p^2)\mathbf{w}' + (\mathbf{c}_p \cdot \mathbf{w}')\mathbf{c}_p \\ &= \left(-1 + \frac{h_1 + h_2}{\xi}\right)\mathbf{w}' + \left(1 - \frac{h_1}{\xi}\right)w'_x \hat{\mathbf{x}} + \left(1 - \frac{h_2}{\xi}\right)w'_y \hat{\mathbf{y}} - \frac{h_3}{2\xi}(w'_y \hat{\mathbf{x}} + w'_x \hat{\mathbf{y}}) \\ &= \frac{1}{\xi} (h_2 w'_x \hat{\mathbf{x}} + h_1 w'_y \hat{\mathbf{y}} - \frac{h_3}{2}(w'_y \hat{\mathbf{x}} + w'_x \hat{\mathbf{y}})) \\ &= \frac{1}{\xi} \left((h_2 w'_x - \frac{h_3 w'_y}{2}) \hat{\mathbf{x}} + (h_1 w'_y - \frac{h_3 w'_x}{2}) \hat{\mathbf{y}} \right),\end{aligned}\quad (123)$$

and,

$$\begin{aligned}(\mathcal{N}(\mathbf{w}'))^2 &= \frac{\mathbf{w}'_m{}^2}{4ik\xi} = (p \mathcal{P}^{-1}(\mathbf{w}'))^2 = (\hat{\mathbf{c}}_p \cdot \mathbf{w}')^2 + (1 - \mathbf{c}_p^2)(\mathbf{w}' - \hat{\mathbf{c}}_p(\hat{\mathbf{c}}_p \cdot \mathbf{w}'))^2 \\ &= (1 - \mathbf{c}_p^2)\mathbf{w}'^2 + (\mathbf{c}_p \cdot \mathbf{w}')^2 \\ &= \left(-1 + \frac{h_1 + h_2}{\xi}\right)(w'_x{}^2 + w'_y{}^2) + \left(1 - \frac{h_1}{\xi}\right)w'_x{}^2 + \left(1 - \frac{h_2}{\xi}\right)w'_y{}^2 - \frac{h_3}{\xi} w'_x w'_y \\ &= \frac{1}{\xi} (h_2 w'_x{}^2 + h_1 w'_y{}^2 - h_3 w'_x w'_y).\end{aligned}\quad (124)$$

Proposition 10

Let $z'_{\mathbf{a}\pm} = (\mathbf{r}'_p - \mathbf{a}_p) \cdot (\widehat{\mathbf{x}} \pm i\widehat{\mathbf{y}}) = x'_a \pm iy'_a$, where $\mathbf{r}'_p - \mathbf{a}_p = x'_a \widehat{\mathbf{c}}_p + y'_a (\widehat{\mathbf{z}} \times \widehat{\mathbf{c}}_p)$ with $\mathbf{r}'_p = \mathcal{P}(\mathbf{r}')$, then $z'_{\mathbf{a}\pm} = 0$ on lines $\mathbf{r}' = \mathcal{P}^{-1}(\mathbf{a}_p) + t\mathbf{u}_c$, $\mathbf{u}_c = \widehat{\mathbf{c}}_p \pm ip(\widehat{\mathbf{z}} \times \widehat{\mathbf{c}}_p)$. For real \mathbf{r}' , each equation $z'_{\mathbf{a}\pm} = 0$ has no or an infinity of solutions if $\frac{\mp ip\xi + \frac{h_3}{2}}{h_2} (= \frac{h_1}{\pm ip\xi + \frac{h_3}{2}})$ is real, and a single one \mathbf{r}_{\pm} else. In the latter case, we notice that $\mathbf{r}_+ = \mathbf{r}_- = \mathcal{P}^{-1}(\mathbf{a}_p)$ if $\mathcal{P}^{-1}(\mathbf{a}_p)$ is a point with real coordinates, for example when \mathbf{a}_p and $\mathcal{P}(\mathbf{r}')$ are real.

Proof. From $z'_{\mathbf{a}\pm} = (\mathbf{r}'_p - \mathbf{a}_p) \cdot (\widehat{\mathbf{x}} \pm i\widehat{\mathbf{y}})$, $(\widehat{\mathbf{x}} \pm i\widehat{\mathbf{y}})^2 = 0$ and $\widehat{\mathbf{x}} \pm i\widehat{\mathbf{y}} = \frac{\mathbf{c} \pm i\widehat{\mathbf{z}} \times \mathbf{c}}{c_x \mp ic_y}$ as $\mathbf{c}^2 \neq 0$, we have $z'_{\mathbf{a}\pm} = 0$ as $\mathbf{r}' = \mathcal{P}^{-1}(\mathbf{a}_p) + t\mathbf{u}_c$, $\mathbf{u}_c = p\mathcal{P}^{-1}(\widehat{\mathbf{c}}_p \pm i(\widehat{\mathbf{z}} \times \widehat{\mathbf{c}}_p))$, and $z'_{\mathbf{a}\pm} = \frac{(\mathbf{r}' - \mathcal{P}^{-1}(\mathbf{a}_p)) \cdot \mathbf{v}_c}{c_x \mp ic_y}$ with $\mathbf{v}_c = \mathcal{P}(\mathbf{c}_p \pm i\widehat{\mathbf{z}} \times \mathbf{c}_p) = p\mathbf{c}_p \pm i\widehat{\mathbf{z}} \times \mathbf{c}_p$; we can consider solutions of $\mathbf{v}_c \cdot \mathbf{r}' = 0$ with

$$\begin{aligned} \frac{v_{cx}}{v_{cy}} &= \frac{pc_x \mp ic_y}{pc_y \pm ic_x} = \frac{\mp ip(c_x^2 + c_y^2) + c_x c_y (p^2 - 1)}{p^2 c_y^2 + c_x^2} \\ &= \frac{\mp ip - c_x c_y}{1 - c_y^2} = \frac{\mp ip\xi + \frac{h_3}{2}}{h_2}, \end{aligned} \quad (125)$$

and deduce last statement. Let us remark that $\mathbf{u}_c \cdot \mathbf{v}_c = 0$ and $(p\xi)^2 = h_1 h_2 - (\frac{h_3}{2})^2$.

6.2.4) Derivation of integrals along the initial contour \mathcal{L}

Theorem 11

We can reduce the integral of an exponential with quadratic argument on S' , to an integral along the contour \mathcal{L} of S' , following,

$$\begin{aligned} \int_{S'} e^{\mathbf{w}' \cdot \mathbf{r}' - ik(h_1 x'^2 + h_2 y'^2 + h_3 x' y')} dS' &= \\ &= \frac{1}{2} \int_{\mathcal{L}} \left[\frac{(e^{\mathbf{w}' \cdot \mathbf{r}' - ik(h_1 x'^2 + h_2 y'^2 + h_3 x' y')} - A_0)}{\mathbf{w}' \cdot \mathbf{r}' - ik(h_1 x'^2 + h_2 y'^2 + h_3 x' y') - \frac{\mathbf{w}'_m{}^2}{4ik\xi}} \right] \\ & [(\widehat{\mathbf{z}} \times (\mathbf{r}' - \frac{\mathcal{M}(\mathbf{w}')}{2ik\xi p^2})) \cdot d\mathbf{r}'] + \frac{(A_0 - e^{\frac{\mathbf{w}'_m{}^2}{4ik\xi}})}{-2ikp\xi} \int_{\mathcal{L}_p} (\widehat{\mathbf{z}} \times \frac{\mathbf{r}'_p - \mathbf{a}_p}{(\mathbf{r}'_p - \mathbf{a}_p)^2}) \cdot d\mathbf{r}'_p, \end{aligned} \quad (126)$$

with $\frac{\mathcal{M}(\mathbf{w}')}{2ik\xi p^2}$ and $\frac{\mathbf{w}'_m{}^2}{4ik\xi}$ detailed in proposition 9, arbitrary A_0 , and $\widehat{\mathbf{y}} = \widehat{\mathbf{z}} \times \widehat{\mathbf{x}}$. The integral on \mathcal{L} is non-singular when $A_0 = e^{\frac{\mathbf{w}'_m{}^2}{4ik\xi}}$. For $A_0 \neq e^{\frac{\mathbf{w}'_m{}^2}{4ik\xi}}$, we notice that, when $(\mathbf{r}'_p - \mathbf{a}_p)^2 \neq 0$ on \mathcal{L} ,

$$\int_{\mathcal{L}_p} (\widehat{\mathbf{z}} \times \frac{\mathbf{r}'_p - \mathbf{a}_p}{(\mathbf{r}'_p - \mathbf{a}_p)^2}) \cdot d\mathbf{r}'_p = \frac{-i}{2} \int_{\mathcal{L}} d\ln(\chi(\mathcal{P}(\mathbf{r}'))) = \pi(N_+ - N_-), \quad (127)$$

where N_+ (resp. N_-) is the number of times the complex $\chi(\mathcal{P}(\mathbf{r}'))$ crosses any smooth semi-line \mathcal{D}_1 from the origin to ∞ , in positive sense (resp. negative), with $\chi(\mathcal{P}(\mathbf{r}')) = \frac{x'_a + iy'_a}{x'_a - iy'_a}$, $\mathcal{P}(\mathbf{r}') - \mathbf{a}_p = x'_a \widehat{\mathbf{c}}_p + y'_a (\widehat{\mathbf{z}} \times \widehat{\mathbf{c}}_p)$, $\mathcal{P}(\mathbf{r}') = \mathbf{r}'_p$. Let us notice that, when \mathbf{a}_p is large and satisfies (112), in particular if $h_1 h_2 \rightarrow (h_3/2)^2$, we have $\int_{\mathcal{L}} d \ln(\chi(\mathcal{P}(\mathbf{r}'))) = 0$.

The first contour integral is in whole generality non singular as we take $A_0 = e^{\frac{w_p^2}{4ik\xi}}$. However, when the parameters give no divergence of integrands along \mathcal{L} , the choice $A_0 = 0$ could be suitable for an asymptotic evaluation as k is large.

Proof. For that, we rewrite the proposition 7 with the use of propositions 8 and 9.

Remark : From (9), we have,

$$\mathbf{r}' = \mathcal{P}^{-1}(\mathbf{a}_p) + h'_p \mathbf{u}_{\varphi_p}, \quad \mathbf{u}_{\varphi_p} = \mathcal{P}^{-1}(\cos \varphi_p \widehat{\mathbf{x}} + \sin \varphi_p \widehat{\mathbf{y}}), \quad h'_p{}^2 = (\mathbf{r}'_p - \mathbf{a}_p)^2, \quad (128)$$

except on lines $\mathbf{r}' = \mathcal{P}^{-1}(\mathbf{a}_p) + t \mathbf{u}_c$ with $\mathbf{u}_c = \widehat{\mathbf{c}}_p \pm ip(\widehat{\mathbf{z}} \times \widehat{\mathbf{c}}_p)$ where $(\mathbf{r}'_p - \mathbf{a}_p)^2 = 0$ and \mathbf{u}_{φ_p} is undefined.

7) Complex scaling and extended cases of applications for exact reductions

We now consider the surface integral of a function $g(\mathbf{r}')f(\mathcal{R}^2)$, where \mathcal{R}^2 is a polynomial of coordinates,

$$\begin{aligned} \mathcal{R}^2 &= d_0 + (\alpha_1 x' + \alpha_2 y') - ik(h_1 x'^2 + h_2 y'^2 + h_3 x' y') \\ &= d_0 + v(\mathbf{r}'_p - \mathbf{a}_p)^2 + v_1, \end{aligned} \quad (129)$$

with $\mathbf{r}'_p = \mathcal{P}(\mathbf{r}')$, and \mathcal{P} a complex scaling, following the proposition 6 and definition 4. For the reduction of the surface integral,

$$\int_{S'} g(\mathbf{r}') f(\mathcal{R}^2) dS', \quad (130)$$

we then define S'_p of contour \mathcal{L}_p as the transform of S' by \mathcal{P} , with $dS' = dx' \wedge dy'$, $dS'_p = dx'_p \wedge dy'_p$, and $dS' = dS'_p/p$, while the primitive of f , satisfying the conditions given in definition 3, is known, and $g(\mathbf{r}')$ is a polynomial of coordinates.

7.1) Exact reduction when $g \equiv 1$

Theorem 12

Let F_v be a primitive of f , we can write,

$$\begin{aligned} \int_{S'} f(\mathcal{R}^2) dS' &= \frac{1}{2} \int_{\mathcal{L}} \left[\frac{F_v(\mathcal{R}^2) - A_0}{\mathcal{R}^2 - \left(\frac{w_m'^2}{4ik\xi} + d_0\right)} \right] [(\widehat{\mathbf{z}} \times (\mathbf{r}' - \mathcal{P}^{-1}(\mathbf{a}_p)))] \cdot d\mathbf{r}' \\ &+ \frac{A_0 - F_v\left(\frac{w_m'^2}{4ik\xi} + d_0\right)}{-2ikp\xi} \int_{\mathcal{L}_p} \left(\widehat{\mathbf{z}} \times \frac{\mathbf{r}'_p - \mathbf{a}_p}{(\mathbf{r}'_p - \mathbf{a}_p)^2} \right) \cdot d\mathbf{r}'_p, \end{aligned} \quad (131)$$

where A_0 is an arbitrary constant, $p = \sqrt{1 - \mathbf{c}_p^2}$, and,

$$\begin{aligned} \mathcal{P}^{-1}(\mathbf{a}_p) &= \frac{1}{2ik(h_1h_2 - (h_3/2)^2)} \left((h_2w'_x - \frac{h_3w'_y}{2}) \widehat{\mathbf{x}} + (h_1w'_y - \frac{h_3w'_x}{2}) \widehat{\mathbf{y}} \right), \\ \frac{w_m'^2}{4ik\xi} &= \frac{1}{4ik(h_1h_2 - (h_3/2)^2)} (h_2w_x'^2 + h_1w_y'^2 - h_3w'_xw'_y), \end{aligned} \quad (132)$$

while $\int_{\mathcal{L}_p} \left(\widehat{\mathbf{z}} \times \frac{\mathbf{r}'_p - \mathbf{a}_p}{(\mathbf{r}'_p - \mathbf{a}_p)^2} \right) \cdot d\mathbf{r}'_p$ can be determined with (127). (note : for most generality, instead to calculate $p\xi$ on second line of (131) as $\sqrt{(p\xi)^2}$, it will be preferable to determine each term p and ξ separately, to avoid problems with the cut of square root.)

Proof. In proposition 7 and theorem 11, we change the exponential function for f , and let,

$$\frac{F_v(\mathcal{R}^2)}{v} = \int f(d_0 + v(\mathbf{r}'_p - \mathbf{a}_p)^2 + v_1) d(\mathbf{r}'_p - \mathbf{a}_p)^2 = \frac{1}{v} \int f(\mathcal{R}^2) d\mathcal{R}^2. \quad (133)$$

We can then write,

$$\begin{aligned} \int_{S'} f(\mathcal{R}^2) dS' &= \frac{1}{p} \int_{S'_p} f(d_0 + v(\mathbf{r}'_p - \mathbf{a}_p)^2 + v_1) dS'_p \\ &= \int_{\mathcal{L}_p} \frac{F_v(\mathcal{R}^2) - A_0}{2p} \frac{(\widehat{\mathbf{z}} \times (\mathbf{r}'_p - \mathbf{a}_p)) \cdot d\mathbf{r}'_p}{\mathcal{R}^2 - (v_1 + d_0)} \\ &+ \frac{(A_0 - F(v_1 + d_0))}{2vp} \int_{\mathcal{L}_p} \left(\widehat{\mathbf{z}} \times \frac{\mathbf{r}'_p - \mathbf{a}_p}{(\mathbf{r}'_p - \mathbf{a}_p)^2} \right) \cdot d\mathbf{r}'_p, \end{aligned} \quad (134)$$

and (131) from $(\widehat{\mathbf{z}} \times (\mathbf{r}'_p - \mathbf{a}_p)) \cdot d\mathbf{r}'_p = p(\widehat{\mathbf{z}} \times (\mathbf{r}' - \mathcal{P}^{-1}(\mathbf{a}_p))) \cdot d\mathbf{r}'$.

7.2) Exact reduction when $g(\mathbf{r}')$ is a polynomial of coordinates

Considering that $\mathbf{r}' = \mathcal{P}^{-1}(\mathbf{r}'_p)$, and

$$\begin{aligned} \widehat{\mathbf{x}} \cdot (\mathbf{r}' - \mathcal{P}^{-1}(\mathbf{a}_p)) &= \widehat{\mathbf{x}} \cdot \mathcal{P}^{-1}(\mathbf{r}'_p - \mathbf{a}_p) = \mathcal{P}^{-1}(\widehat{\mathbf{x}}) \cdot (\mathbf{r}'_p - \mathbf{a}_p), \\ \widehat{\mathbf{y}} \cdot (\mathbf{r}' - \mathcal{P}^{-1}(\mathbf{a}_p)) &= \widehat{\mathbf{y}} \cdot \mathcal{P}^{-1}(\mathbf{r}'_p - \mathbf{a}_p) = \mathcal{P}^{-1}(\widehat{\mathbf{y}}) \cdot (\mathbf{r}'_p - \mathbf{a}_p), \end{aligned} \quad (135)$$

we notice, from Taylor expansion, that we have, for any polynomial $g(\mathbf{r}')$ of order m ,

$$g(\mathbf{r}') = \sum_{j=0}^m \frac{1}{j!} \sum_{k=0}^j \binom{j}{k} \frac{\partial^j g|_{\mathbf{r}'=\mathcal{P}^{-1}(\mathbf{a}_p)}}{\partial x'^k \partial y'^{j-k}} [(\mathcal{P}^{-1}(\hat{\mathbf{x}}) \cdot (\mathbf{r}'_p - \mathbf{a}_p))^k (\mathcal{P}^{-1}(\hat{\mathbf{y}}) \cdot (\mathbf{r}'_p - \mathbf{a}_p))^{j-k}]. \quad (136)$$

Theorem 13

Considering a polynomial $g(\mathbf{r}')$ of order m , we can develop the exact reduction of the integral $\int_{S'} g(\mathbf{r}') f(\mathcal{R}^2) dS'$ on the surface S' of the plane. For that, we need the reduction of integral

$$I_0 = \int_{S'} f(\mathcal{R}^2) dS', \quad (137)$$

given by theorem 12 in (131), and of the elementary integrals,

$$I_{n+1} = \int_{S'} \mathbf{d}_{n+1} \cdot (\mathbf{r}'_p - \mathbf{a}_p) \prod_{j=1}^n (\mathbf{d}_j \cdot (\mathbf{r}'_p - \mathbf{a}_p)) f(\mathcal{R}^2) dS', \quad (138)$$

with $1 \leq n+1 \leq m$, constants \mathbf{d}_j , and complex scaling \mathcal{P} for (129). For that, we generalize the theorem 4, and write,

$$I_{n+1} = \frac{-1}{2ik\xi} \left(\frac{1}{p} \int_{\mathcal{L}} \prod_{j=1}^n (\mathbf{d}_j \cdot (\mathbf{r}'_p - \mathbf{a}_p)) F_v(\mathcal{R}^2) \mathcal{P}(\hat{\mathbf{z}} \times \mathbf{d}_{n+1}) \cdot d\mathbf{r}' - \sum_{m=1}^n ((\mathbf{d}_{n+1} \cdot \mathbf{d}_m) \int_{S'} \prod_{j \neq m}^n (\mathbf{d}_j \cdot (\mathbf{r}'_p - \mathbf{a}_p)) F_v(\mathcal{R}^2) dS') \right), \quad (139)$$

where F_v is a primitive of f , $\mathcal{P}(\hat{\mathbf{z}} \times \mathbf{d}_{n+1}) = p \hat{\mathbf{z}} \times \mathcal{P}^{-1}(\mathbf{d}_{n+1})$, and $\prod_{j=1}^0 (\dots) = 1$, $\prod_{j \neq i}^1 (\dots) = 1$, and $\prod_{j \neq i}^0 (\dots) = 0$. Subsequently, we have decreased the rank of polynomial in surface integral term, and it becomes possible, by a repeated application of (139), to reduce I_{n+1} to a combination of contour integrals depending on $M = \lfloor (n+3)/2 \rfloor$ successive primitives of f , denoted $\frac{1}{q!} \int^x (x-t)^q f(t) dt$, $q = 0, \dots, M-1$.

Proof. Taking up the theorem 4, $(\mathbf{r}'_p - \mathbf{a}_p)$ and dS'_p instead of $(\mathbf{r}' - \mathbf{a})$ and dS' , with $v = -ik\xi$ and $dS'_p \equiv p dS'$, we derive (139). By repeated applications, we can decrease the rank, and using theorem 12, reduce I_{n+1} to a combination of contour integrals for any $n \geq 0$,

using repeated integrals of f (see the results given in appendix for successive primitives and their asymptotics).

Remark : Letting $\mathcal{I}_{n+1} = \mathcal{P}^{-1}(\mathbf{d}_{n+1}) \cdot \mathcal{I}_n$ in (139), we use $\mathcal{P}(\widehat{\mathbf{z}} \times \mathbf{d}_{n+1}) = p\widehat{\mathbf{z}} \times \mathcal{P}^{-1}(\mathbf{d}_{n+1})$ and $\mathbf{d}_{n+1} \cdot \mathbf{d}_m = \mathcal{P}^{-1}(\mathbf{d}_{n+1}) \cdot \mathcal{P}(\mathbf{d}_m)$, and then note that,

$$\begin{aligned} \mathcal{I}_n &= \frac{1}{2ik\xi} (\widehat{\mathbf{z}} \times \int_{\mathcal{L}} \prod_{j=1}^n (\mathbf{d}_j \cdot (\mathbf{r}'_p - \mathbf{a}_p)) F_v(\mathcal{R}^2) d\mathbf{r}' \\ &+ \sum_{m=1}^n (\mathcal{P}(\mathbf{d}_m) \int_{S'} \prod_{j \neq m}^n (\mathbf{d}_j \cdot (\mathbf{r}'_p - \mathbf{a}_p)) F_v(\mathcal{R}^2) dS')). \end{aligned} \quad (140)$$

Remark : When $(h_1 - h_2) \rightarrow 0$ and $h_3 \rightarrow 0$, we have $\mathbf{r}'_p \rightarrow \mathbf{r}'$, $p \rightarrow 1$, $\mathbf{a}_p \rightarrow \frac{\mathbf{w}'}{2ik\xi}$, $\xi \rightarrow h_{1(,2)}$, and using $\frac{F_v(\mathcal{R}^2)}{v} = \int f(\mathcal{R}^2) d(\mathbf{r}'_p - \mathbf{a}_p)^2$ with $v = -ik\xi$, we recover the theorem 4.

Remark : From (139), we note that,

$$\begin{aligned} \int_{S'} \mathbf{d} \cdot (\mathbf{r}' - \mathcal{P}^{-1}(\mathbf{a}_p)) f(\mathcal{R}^2) dS' &= \frac{1}{p} \int_{S'_p} \mathcal{P}^{-1}(\mathbf{d}) \cdot (\mathbf{r}'_p - \mathbf{a}_p) f(\mathcal{R}^2) dS'_p \\ &= \frac{1}{2p} \int_{\mathcal{L}} \frac{[F_v(\mathcal{R}^2) - A_0]}{v} [\mathcal{P}(\widehat{\mathbf{z}} \times \mathcal{P}^{-1}(\mathbf{d})) \cdot d\mathbf{r}'], \end{aligned} \quad (141)$$

where $\frac{\mathcal{P}(\widehat{\mathbf{z}} \times \mathcal{P}^{-1}(\mathbf{d}))}{\xi p} = \frac{(\widehat{\mathbf{z}} \times \mathcal{P}^{-2}(\mathbf{d}))}{\xi}$ with $\frac{\mathcal{P}^{-2}(\mathbf{d})}{\xi} = \frac{\mathbf{d} \cdot \widehat{\mathbf{c}}_p}{p^2 \xi} \widehat{\mathbf{c}}_p + \frac{\mathbf{d} \cdot (\widehat{\mathbf{z}} \times \widehat{\mathbf{c}}_p)}{\xi} \widehat{\mathbf{z}} \times \widehat{\mathbf{c}}_p$. For $h_3 = 0$, we have, using (87), $\frac{\mathcal{P}^{-2}(\mathbf{d})}{\xi} = \frac{\mathbf{d} \cdot \widehat{\mathbf{x}}}{h_1} \widehat{\mathbf{x}} + \frac{\mathbf{d} \cdot \widehat{\mathbf{y}}}{h_2} \widehat{\mathbf{y}}$.

8) Complex scaling and applications to asymptotic reductions

We now consider asymptotic reduction of $\int_{S'} g(\mathbf{r}') f(\mathcal{R}^2) dS'$ when $f(\mathcal{R}^2) = e^{\mathcal{R}^2}$ or $f(\mathcal{R}^2) = \frac{e^{2\sqrt{\mathcal{R}^2}}}{\sqrt{\mathcal{R}^2}}$ are highly oscillating function, while $g(\mathbf{r}')$ is a smooth analytical function on S' and \mathcal{R}^2 is a polynomial.

8.1) The case with $f(\mathcal{R}^2) = e^{\mathcal{R}^2}$ and semi-asymptotic reduction

To simplify the reading, we choose to detail the case with $f(\mathcal{R}^2) = e^{\mathcal{R}^2}$, for \mathcal{R}^2 given by a polynomial of order 2 like (129), and a large class of smooth function $g(\mathbf{r}')$, in appendix, with proposition 15 and semi-asymptotic reduction given in theorem 19.

8.2) The case with $f(\mathcal{R}^2) = \frac{e^{2\sqrt{\mathcal{R}^2}}}{\sqrt{\mathcal{R}^2}}$ and asymptotic reduction for curved plate

If we want to generalize the results of section 5 and take account of a moderate curvature of the surface of integration in the radiation of surface currents, we have to consider $(z - z')$ instead of z in (63) with polynomial approximation of z' relatively to x' and y' coordinates, as $z' = (C_1x'^2 + C_2y'^2)$, and derive plane surface integral from the use of a projection and its Jacobian. For that, we let,

$$\begin{aligned}\sqrt{\mathcal{R}^2} &= -ik\sqrt{(x' - a_x)^2 + (y' - a_y)^2 + ((C_1x'^2 + C_2y'^2) - z)^2} \\ &= -ik\sqrt{h'_1(x' - a'_x)^2 + h'_2(y' - a'_y)^2 + z''^2 + \epsilon(x', y')} \\ &= \sqrt{\mathcal{R}_0^2 - k^2\epsilon(x', y')},\end{aligned}\tag{142}$$

with $\epsilon(x', y') = (C_1x'^2 + C_2y'^2)^2$ and,

$$\begin{aligned}h'_j &= 1 - 2zC_j, \quad a'_x = \frac{a_x}{h'_1}, \quad a'_y = \frac{a_y}{h'_2}, \quad z''^2 = z^2 - \left(\frac{1 - h'_1}{h'_1}a_x^2 + \frac{1 - h'_2}{h'_2}a_y^2\right), \\ \sqrt{\mathcal{R}_0^2} &= -ik\sqrt{h'_1(x' - a'_x)^2 + h'_2(y' - a'_y)^2 + z''^2},\end{aligned}\tag{143}$$

as k and z'' are large, when $\epsilon(x', y') \ll z^2$. Similarly to (105), we can use one of complex scaling \mathcal{P} verifying,

$$\begin{aligned}p^2 &= h'_2/h'_1, \quad \mathcal{P}(\mathbf{r}') = (\hat{\mathbf{x}}.\mathbf{r}')\hat{\mathbf{x}} + p(\hat{\mathbf{y}}.\mathbf{r}')\hat{\mathbf{y}}, \quad \xi = h'_1, \\ p^2 &= h'_1/h'_2, \quad \mathcal{P}(\mathbf{r}') = p(\hat{\mathbf{x}}.\mathbf{r}')\hat{\mathbf{x}} + (\hat{\mathbf{y}}.\mathbf{r}')\hat{\mathbf{y}}, \quad \xi = h'_2,\end{aligned}\tag{144}$$

such that,

$$\begin{aligned}v'(\mathbf{r}'_p - \mathbf{a}_p)^2 &= h'_1(x' - a'_x)^2 + h'_2(y' - a'_y)^2, \quad d_0 + v_1 = z''^2, \\ \text{with } \mathbf{r}'_p &= \mathcal{P}(\mathbf{r}'), \quad \mathcal{P}^{-1}(\mathbf{a}_p) = a'_x\hat{\mathbf{x}} + a'_y\hat{\mathbf{y}}, \quad v' = \xi,\end{aligned}\tag{145}$$

which then permits to reduce asymptotically the surface integral $\int_{S'} g(r') f(\mathcal{R}^2) dS'$ with $f(\mathcal{R}^2) = \frac{e^{2\sqrt{\mathcal{R}^2}}}{\sqrt{\mathcal{R}^2}}$ from theorems 1, 12 and lemma 2.

Theorem 14

Letting $f(\mathcal{R}^2) = \frac{e^{2\sqrt{\mathcal{R}^2}}}{\sqrt{\mathcal{R}^2}}$, we have for its primitive $F_v(\mathcal{R}^2) = e^{2\sqrt{\mathcal{R}^2}}$. We can use the theorems 1 and 12, with lemma 2 and complex scaling \mathcal{P} , and then write, for large k ,

$$\begin{aligned}
& \int_{S'} g(\mathbf{r}') f(\mathcal{R}_0^2) dS' \\
& \sim \int_{\mathcal{L}} \frac{\frac{1}{2}(g(\mathbf{r}')F_v(\mathcal{R}_0^2(\mathbf{r}')) - g(\mathbf{a}')F_v(\mathcal{R}_0^2(\mathbf{a}')))(\widehat{\mathbf{z}} \times (\mathbf{r}' - \mathcal{P}^{-1}(\mathbf{a}_p))) \cdot d\mathbf{r}'}{-k^2(h_1'(x' - a'_x)^2 + h_2'(y' - a'_y)^2)} \\
& \sim \int_{\mathcal{L}} \frac{\frac{1}{2}(g(\mathbf{r}')F_v(\mathcal{R}_0^2(\mathbf{r}')) - g(\mathbf{a}')F_v(\mathcal{R}_0^2(\mathbf{a}')))(\widehat{\mathbf{z}} \times (\mathbf{r}' - \mathbf{a}')) \cdot d\mathbf{r}'}{-k^2(h_1'(x' - a'_x)^2 + h_2'(y' - a'_y)^2)}. \tag{146}
\end{aligned}$$

For small $\epsilon(x', y')$, we can then choose to take $\frac{g(\mathbf{r}')f(\mathcal{R}^2)}{f(\mathcal{R}_0^2)} = \frac{g(\mathbf{r}')\sqrt{\mathcal{R}_0^2}}{\sqrt{\mathcal{R}^2}} e^{2(\sqrt{\mathcal{R}^2} - \sqrt{\mathcal{R}_0^2})}$ in place of $g(\mathbf{r}')$, and obtain,

$$\begin{aligned}
& \int_{S'} g(\mathbf{r}') f(\mathcal{R}^2) dS', \\
& \sim \int_{\mathcal{L}} \frac{\frac{1}{2}(g_1(\mathbf{r}')F_v(\mathcal{R}^2(\mathbf{r}')) - g_1(\mathbf{a}')F_v(\mathcal{R}^2(\mathbf{a}')))(\widehat{\mathbf{z}} \times (\mathbf{r}' - \mathbf{a}')) \cdot d\mathbf{r}'}{-k^2(h_1'(x' - a'_x)^2 + h_2'(y' - a'_y)^2)}, \tag{147}
\end{aligned}$$

with $g_1(\mathbf{r}') = \frac{g(\mathbf{r}')\sqrt{\mathcal{R}_0^2}}{\sqrt{\mathcal{R}^2}}$, when the h_j' are both real and of the same sign. Besides, in order to consider $\frac{e^{2\sqrt{\mathcal{R}^2}}}{(\sqrt{\mathcal{R}^2})^n}$ in place of $\frac{e^{2\sqrt{\mathcal{R}^2}}}{\sqrt{\mathcal{R}^2}}$, we can let $\frac{g(\mathbf{r}')}{(\sqrt{\mathcal{R}^2})^{n-1}}$ in place of $g(\mathbf{r}')$ in (147).

For more generality, in particular to cover the case of real h_j' of different signs, we take,

$$h_j' - i\delta \text{ instead of } h_j', \text{ then } \delta \text{sign}(\arg(ik)) \rightarrow 0^+, \tag{148}$$

as $0 < |\arg(ik)| \leq \pi/2$, to avoid any line of singularities of denominator in (147).

Proof. These expressions are derived from theorems 1 and 12, used for primitives defined asymptotically with the lemma 2 (see appendix). For that, we take $g(\mathbf{r}')f(\mathcal{R}^2)$ in place of $f(\mathcal{R}^2)$ in the theorem 12, considering that its primitive is asymptotically $\sim g(\mathbf{r}')F_v(\mathcal{R}^2)$ from lemma 2. The small real parameter δ permits to avoid any line of singularities. Concerning the sign of δ , we choose it so that $e^{2\sqrt{\mathcal{R}^2}} < \infty$ when $\mathbf{r}' \rightarrow \infty$ and $0 < |\arg(ik)| \leq \pi/2$, which implies $0 < \delta \text{sign}(\arg(ik)) \rightarrow 0^+$.

This theorem will be used in next section and validated in numerical results, letting $g_1(\mathbf{r}') \simeq g(\mathbf{r}')$.

Remark : Considering (92), we can let,

$$\begin{aligned}
v'(\mathbf{r}'_p - \mathbf{a}'_p)^2 &= (h_1' - i\delta)(x' - a'_x)^2 + (h_2' - i\delta)(y' - a'_y)^2 = \\
&= \prod_{\pm} (\sqrt{(h_1' - i\delta)(x' - a'_x)} \pm i\sqrt{(h_2' - i\delta)(y' - a'_y)}), \tag{149}
\end{aligned}$$

with small real $\delta \neq 0$ such that $(h'_1 - i\delta) \neq -(h'_2 - i\delta)$, and $v' = \xi$. The sign chosen for δ has a crucial influence. For example, we note, from proposition 9 and 1, that,

$$\begin{aligned} \int_{\mathcal{L}} \frac{\frac{1}{2}(\widehat{\mathbf{z}} \times (\mathbf{r}' - \mathbf{a}')) \cdot d\mathbf{r}'}{(h'_1 - i\delta)(x' - a'_x)^2 + (h'_2 - i\delta)(y' - a'_y)^2} &= \frac{1}{2p\xi} \int_{\mathcal{L}_p} (\widehat{\mathbf{z}} \times \frac{\mathbf{r}'_p - \mathbf{a}'_p}{(\mathbf{r}'_p - \mathbf{a}'_p)^2}) \cdot d\mathbf{r}'_p \\ &= \frac{\pi(\epsilon^+ 1_{S'_+}(a'_x + ia'_y) + \epsilon^- 1_{S'_-}(a'_x - ia'_y))}{2p\xi}, \end{aligned} \quad (150)$$

where $\xi = (h'_1 - i\delta)$ and $p = \sqrt{\frac{(h'_2 - i\delta)}{\xi}} \sim \sqrt{\frac{h'_2}{h'_1}(1 - i\delta(\frac{1}{h'_2} - \frac{1}{h'_1}))}$ (see (105)). For real h'_1 and h'_2 of opposite signs, we note the importance of the sign of δ , since we have,

$$\lim_{\delta \rightarrow 0^\pm} (2p\xi) = \pm ih'_1 \text{sgn}(h'_2) \sqrt{-(h'_2/h'_1)}. \quad (151)$$

Remark : It is worth noticing that we have not to take the term $p\xi$ as $\sqrt{(p\xi)^2}$, but as the product of the terms p and ξ evaluated separately. Indeed, for example, we note that, for real δ , the term $(1 - i\delta)\sqrt{\frac{(-1-i\delta)}{(1-i\delta)}}$ is defined, while the term $\sqrt{(1 - i\delta)(-1 - i\delta)}$ is not, since $(1 - i\delta)(-1 - i\delta) = -1 - \delta^2$ is on the cut of the square root function.

9) Detailed reduction in case (c) for surface radiation integrals on curved plates

Let us consider the expressions of the electric \mathbf{E} in (47), radiated by electric and magnetic currents, $\mathbf{J} = 2\widehat{\mathbf{z}} \times \mathbf{H}_i$ and $\mathbf{M} = 0$, excited on a real perfectly conducting curved surface S by weighted dipolar illuminations, as defined with $f_{e,h}$ in (62) (note : for flat plate, \mathbf{E} was given by (63) of section 5).

We take this time a curved plate S instead of a flat one, with current point \mathbf{R}' defined by,

$$\mathbf{R}' = \mathbf{r}' + \zeta_s(\mathbf{r}')\widehat{\mathbf{z}}, \quad \zeta_s(\mathbf{r}') = C_1 x'^2 + C_2 y'^2, \quad \zeta_s(\mathbf{0}) = 0, \quad (152)$$

with $C_j < 0$ (resp. $C_j > 0$) for convex (resp. concave) surface inflections. We note that, for purely convex (resp. purely concave) surface with $C_1 C_2 > 0$, S is below (resp. above) the plane $z = 0$, while, for saddle surfaces with $C_1 C_2 < 0$, S is partly below and partly above the plane.

In what follows, the surface S' is the projection of S on the plane $z = 0$, \mathbf{R}' is the current point of integration on S while \mathbf{r}' is the one on S' , \mathcal{L} is the positively oriented contour of S' , $\mathbf{R}_1 = (\mathbf{a} + z\widehat{\mathbf{z}})$ is the observation point, and \mathbf{R} is the vector between the observation point and the current point of S , following,

$$\begin{aligned}\mathbf{R} &= \mathbf{R}_1 - \mathbf{R}' = (\mathbf{a} + z\hat{\mathbf{z}}) - (\mathbf{r}' + \zeta_s(\mathbf{r}')\hat{\mathbf{z}}), \\ \mathbf{R}^2(r') &= (x' - a_x)^2 + (y' - a_y)^2 + ((C_1x'^2 + C_2y'^2) - z)^2,\end{aligned}\quad (153)$$

for which we can consider the equalities in (142)-(145) and the theorem 14, when we are in the region Ω_C defined with $z \gg (C_1x'^2 + C_2y'^2)$ and large $k\sqrt{\mathbf{R}^2}$.

In Physical Optics, the radii of curvature $\frac{1}{2|C_j|}$ are assumed to be large relatively to λ , and it is worth noticing that, with this assumption, the region Ω_C is generally much wider than the far field region, where $|\mathbf{R}_1| > d_F = 8\max(\frac{(\mathbf{R}'^2)_S}{\lambda})$, $k|\mathbf{R}_1|$ is large and $|\mathbf{R}'|/|\mathbf{R}_1|$ is small (note : $d_F = 2D^2/\lambda$ is Fraunhofer distance [1], $D = 2\max(|\mathbf{R}'|_S)$).

Remark : Considering $R_c = \min(\frac{1}{2|C_j|})$, we notice that $|C_1x^2 + C_2y^2|_S \leq z_\lambda \frac{\lambda}{2R_c}$ with $z_\lambda = \max(\frac{(x^2+y^2)_S}{\lambda}) = \frac{z_F}{8}$, and thus, that $|C_1x^2 + C_2y^2|_S \leq \frac{z}{\alpha}$ when $z \geq z_F(\frac{\alpha\lambda}{16R_c})$ is verified.

9.1) Reduction to a contour integral for radiation of curved plates when $f_h \equiv 0$.

9.1.1) A contour integral for radiation at finite distance, large relatively to wavelength

The Physical Optics (PO) diffracted field at $\mathbf{r} = \mathbf{a} + z\hat{\mathbf{z}} \notin S$, when $\mathbf{r} = \mathbf{r}_0$ (monostatic case) and $f_h \equiv 0$, satisfies,

$$\mathbf{E} \sim -\frac{Z_0}{8(\pi k)^2} \frac{ik}{2} \int_{S'} [(\hat{\mathbf{n}} \cdot \mathbf{R}) J^s](\mathbf{r}') \tilde{R} \left(\left(\frac{\psi'}{\tilde{R}} \right)^2 \right)' f_e(-\mathbf{u}) ((\mathbf{u} \cdot \hat{\mathbf{d}})\mathbf{u} - \hat{\mathbf{d}}) dS'. \quad (154)$$

where $\hat{\mathbf{n}} J^{as} = \partial_x \mathbf{R}'|_S \times \partial_y \mathbf{R}'|_S$, S' is the projection on the plane $z = 0$ of the curved surface S of points \mathbf{R}' and outward normal vector $\hat{\mathbf{n}}$, $\psi(\tilde{R}) = \frac{e^{-ik\tilde{R}}}{\tilde{R}}$, $(\cdot)' \equiv \partial_{\tilde{R}}(\cdot)$. We then use that,

$$\begin{aligned}\mathbf{R}^2(r') &= (1 - 2zC_1)(x' - a'_x)^2 + (1 - 2zC_2)(y' - a'_y)^2 + z''^2 + \epsilon(\mathbf{r}'), \\ a'_x &= \frac{a_x}{1 - 2zC_1}, \quad a'_y = \frac{a_y}{1 - 2zC_2}, \quad z''^2 = z^2 - \left(\frac{2zC_1}{1 - 2zC_1} a_x^2 + \frac{2zC_2}{1 - 2zC_2} a_y^2 \right), \\ J^{as}(\mathbf{r}') &= \left| \frac{\partial \mathbf{R}'}{\partial x'} \times \frac{\partial \mathbf{R}'}{\partial y'} \right|, \quad \hat{\mathbf{n}} = \left(\frac{\partial \mathbf{R}'}{\partial x'} \times \frac{\partial \mathbf{R}'}{\partial y'} \right) / \left| \frac{\partial \mathbf{R}'}{\partial x'} \times \frac{\partial \mathbf{R}'}{\partial y'} \right|, \\ [(\hat{\mathbf{n}} \cdot \mathbf{R}) J^{as}](\mathbf{r}') &= -2(C_1x'(a_x - x') + C_2y'(a_y - y')) + z - \zeta_s(\mathbf{r}'),\end{aligned}\quad (155)$$

where $\epsilon(\mathbf{r}') = (\zeta_s(\mathbf{r}'))^2$, $\zeta_s(\mathbf{r}') = C_1x'^2 + C_2y'^2$, and let $\mathbf{R}^2 = v(\mathbf{r}'_p - \mathbf{a}_p)^2 + z''^2 + \epsilon(\mathbf{r}')$ with $\mathbf{a}_p = \mathcal{P}(\mathbf{a}')$ as in (142)-(145). In what follows, we consider that the term $\epsilon(x, y)$ has moderate influence on the terms in front of exponential terms. For that, we take z sufficiently large so that $|(C_1x^2 + C_2y^2)|_S \ll z$.

Theorem 15

Considering the theorem 14, we can write the surface integral (154), when k is large and $|C_1x^2 + C_2y^2|_S \ll z$, as the contour integral,

$$\mathbf{E} \sim \frac{-Z_0 ik}{(4\pi)^2} \int_{\mathcal{L}} \frac{\left(\frac{e^{-2ik\tilde{R}(\mathbf{r}')}}{R^2(\mathbf{r}')} \mathcal{W}_e(\mathbf{r}') - \frac{e^{-2ik\tilde{R}(\mathbf{a}')}}{R^2(\mathbf{a}')} \mathcal{W}_e(\mathbf{a}') \right) [(\hat{\mathbf{z}} \times (\mathbf{r}' - \mathbf{a}')) \cdot d\mathbf{r}']}{(1 - 2zC_1 - i\delta)(x' - a'_x)^2 + (1 - 2zC_2 - i\delta)(y' - a'_y)^2}, \quad (156)$$

where,

$$\begin{aligned} \tilde{R}(\mathbf{r}') &= \sqrt{(\mathbf{r}' - \mathbf{a})^2 + (\zeta_s(\mathbf{r}') - z)^2}, \\ \mathcal{W}_e(\mathbf{r}') &= -f_e(-\mathbf{u})((\mathbf{u} \cdot \hat{\mathbf{d}})\hat{\mathbf{u}} - \hat{\mathbf{d}})[(\hat{\mathbf{n}} \cdot \mathbf{R})J^{as}](\mathbf{r}'), \end{aligned} \quad (157)$$

with \mathcal{L} the positively oriented contour of S' on the plane $z = 0$. The term δ is a small real constant tending to 0, with $\delta \text{sign}(\arg(ik)) \rightarrow 0^+$, necessary for the cases where $(1 - 2zC_1)(1 - 2zC_2) < 0$.

When $f_e(-u) = \frac{1}{\sqrt{1-(u \cdot \hat{\mathbf{d}})^2}}$, we then have the VV polarisation term $\mathbf{E} \cdot \hat{\mathbf{d}}$, with $\hat{\mathbf{d}} = \frac{\mathbf{r}_0 \times \hat{\mathbf{z}}}{\sqrt{(\mathbf{r}_0 \times \hat{\mathbf{z}})^2}}$ as $\mathbf{r}_0 \times \hat{\mathbf{z}} \neq 0$, (already considered for a flat plate in (69)) following,

$$\begin{aligned} \mathbf{E} \cdot \hat{\mathbf{d}} &= -\frac{Z_0 ik}{(4\pi)^2} \int_{\mathcal{L}} \mathcal{V}(\mathbf{r}') [(\hat{\mathbf{z}} \times (\mathbf{r}' - \mathbf{a}')) \cdot d\mathbf{r}'], \\ \mathcal{V}(\mathbf{r}') &= \frac{\frac{[(\hat{\mathbf{n}} \cdot \mathbf{R})J^s](\mathbf{r}') e^{-2ik\tilde{R}(\mathbf{r}')}}{R^2(\mathbf{r}')} \sqrt{1 - \left(\frac{\mathbf{r}' - \mathbf{a}}{\tilde{R}(\mathbf{r}')}\right) \cdot \hat{\mathbf{d}})^2} - \frac{[(\hat{\mathbf{n}} \cdot \mathbf{R})J^s](\mathbf{a}') e^{-2ik\tilde{R}(\mathbf{a}')}}{R^2(\mathbf{a}')} \sqrt{1 - \left(\frac{\mathbf{a}' - \mathbf{a}}{\tilde{R}(\mathbf{a}')}\right) \cdot \hat{\mathbf{d}})^2}}{(1 - 2zC_1 - i\delta)(x' - a'_x)^2 + (1 - 2zC_2 - i\delta)(y' - a'_y)^2}. \end{aligned} \quad (158)$$

9.1.2) Particular asymptotic cases : at very large distance or at focal points

For $z \rightarrow \infty$, we can take in surface and contour integral,

$$\sqrt{\mathbf{R}^2} \sim \tilde{r}_0 - \frac{\mathbf{r}_0}{\tilde{r}_0} \cdot (\mathbf{r}' + \hat{\mathbf{z}}\zeta_s(\mathbf{r}')), \quad z'' \sim \tilde{r}_0 - \frac{\mathbf{r}_0}{\tilde{r}_0} \cdot (\mathbf{a}' + \hat{\mathbf{z}}\zeta_s(\mathbf{a}')), \quad (159)$$

with $[(\hat{\mathbf{n}} \cdot \mathbf{R})J^{as}](\mathbf{r}') \rightarrow -2(C_1x'a_x + C_2y'a_y) + z$, and the expression (156) becomes,

$$\begin{aligned} \mathbf{E} &\sim -\frac{Z_0 ik}{(4\pi)^2} \frac{e^{-2ik\tilde{r}_0}}{r_0^2} f_e\left(-\frac{\mathbf{r}_0}{\tilde{r}_0}\right) \left(\hat{\mathbf{d}} - \left(\frac{\mathbf{r}_0}{\tilde{r}_0} \cdot \hat{\mathbf{d}}\right) \frac{\mathbf{r}_0}{\tilde{r}_0}\right) \\ &\int_{\mathcal{L}} \frac{\mathcal{H}(\mathbf{r}') e^{2ik\left(\frac{\mathbf{r}_0}{\tilde{r}_0} \cdot (\mathbf{r}' + \hat{\mathbf{z}}\zeta_s(\mathbf{r}'))\right)} - \mathcal{H}(\mathbf{a}') e^{2ik\left(\frac{\mathbf{r}_0}{\tilde{r}_0} \cdot (\mathbf{a}' + \hat{\mathbf{z}}\zeta_s(\mathbf{a}'))\right)}}{\left(\frac{2zC_1-1}{2z} + i\delta\right)(x' - a'_x)^2 + \left(\frac{2zC_2-1}{2z} + i\delta\right)(y' - a'_y)^2} [(\hat{\mathbf{z}} \times (\mathbf{r}' - \mathbf{a}')) \cdot d\mathbf{r}'], \end{aligned} \quad (160)$$

where $\mathcal{H}(\mathbf{r}') = (C_1x'(\frac{a_x}{z}) + C_2y'(\frac{a_y}{z}) - \frac{1}{2})$.

At focal points, as $\mathcal{K}_m = \min(1 - 2zC_j)|_{j=1,2} \rightarrow 0$, we have $\mathbf{a}' \rightarrow \infty$ and $z''^2 \rightarrow \infty$, but a'/z''^2 and $\mathcal{K}_m a'$ tend to finite constant vectors, and we can write for (156),

$$\mathbf{E} \sim \frac{Z_0 ik}{(4\pi)^2} \left(\int_{\mathcal{L}} \frac{(\frac{e^{-2ik\tilde{R}(\mathbf{r}')}}{R^2(\mathbf{r}')} \mathcal{W}_e(\mathbf{r}') - \frac{e^{-2ik\tilde{R}(\mathbf{a}')}}{R^2(\mathbf{a}')} \mathcal{W}_e(\mathbf{a}')) (\widehat{\mathbf{z}} \times \mathcal{K}_m \mathbf{a}') \cdot d\mathbf{r}'}{\mathcal{K}_m (\mathbf{R}_\delta^2 - z^2 - \epsilon(\mathbf{r}')) + (\frac{2zC_1 \mathcal{K}_m}{(1-2zC_1)} a_x^2 + \frac{2zC_2 \mathcal{K}_m}{(1-2zC_2)} a_y^2)} \right). \quad (161)$$

with $\mathbf{R}_\delta^2 = (1 - 2zC_1 - i\delta)(x' - a'_x)^2 + (1 - 2zC_2 - i\delta)(y' - a'_y)^2 + z''^2 + \epsilon(\mathbf{r}')$.

9.2) Reduction to a contour integral for diffraction by curved plates when $f_e \equiv 0$.

9.2.1) A contour integral for radiation at finite distance, large relatively to wavelength

The PO diffracted field at $\mathbf{r} = \mathbf{a} + z\widehat{\mathbf{z}} \notin S'$, when $\mathbf{r} = \mathbf{r}_0$ (monostatic case) and $f_e \equiv 0$, satisfies,

$$\mathbf{E} \cdot \mathbf{v}_0 \sim \frac{-1}{8(\pi ik)^2} \int_{S'} [(\widehat{\mathbf{n}} \cdot \mathbf{R}) J^{as}](\mathbf{r}') \left(\left(\frac{\psi'}{\tilde{R}} \right)' \right)^2 \mathbf{w}_0 \cdot \mathbf{R} f_h(-\mathbf{u}) dS', \quad (162)$$

where $\mathbf{v}_0 = (\widehat{\mathbf{d}} \times \frac{\mathbf{r}_0}{r_0})$ with $\widehat{\mathbf{d}} = \frac{\mathbf{r}_0 \times \widehat{\mathbf{z}}}{\sqrt{(\mathbf{r}_0 \times \widehat{\mathbf{z}})^2}}$ as $\mathbf{r}_0 \times \widehat{\mathbf{z}} \neq 0$, $\mathbf{w}_0 = \frac{\mathbf{r}_0}{r_0} = \frac{\mathbf{a} + z\widehat{\mathbf{z}}}{r_0}$, and $(\widehat{\mathbf{n}} \cdot \mathbf{R}) J^{as}$ is defined as previously.

Considering that $\mathbf{w}_0 \cdot \mathbf{R} = \mathbf{w}_0 \cdot (\mathbf{R}_1 + \mathbf{R}_2)$ with $\mathbf{R}_2(\mathbf{r}') = (\mathbf{a} - \mathbf{a}' + (z - \zeta_s(\mathbf{r}'))\widehat{\mathbf{z}})$ and $\mathbf{R}_1(\mathbf{r}') = (\mathbf{a}' - \mathbf{r}')$, we could use (141) for the part with \mathbf{R}_1 term, as already used in theorem 8, but we here prefer to directly apply the theorem 14.

Theorem 16

Considering the theorem 14, we can write the surface integral (162), when k is large and $|(C_1 x^2 + C_2 y^2)|_S \ll z$, as the contour integral,

$$\mathbf{E} \cdot \mathbf{v}_0 \sim \frac{ik}{(4\pi)^2} \int_{\mathcal{L}} \frac{(\frac{\mathcal{W}_h(\mathbf{r}') e^{-2ik\tilde{R}(\mathbf{r}')}}{(\tilde{R}(\mathbf{r}'))^3} - \frac{\mathcal{W}_h(\mathbf{a}') e^{-2ik\tilde{R}(\mathbf{a}')}}{(\tilde{R}(\mathbf{a}'))^3}) (\widehat{\mathbf{z}} \times (\mathbf{r}' - \mathbf{a}')) \cdot d\mathbf{r}'}{(1 - 2zC_1 - i\delta)(x' - a'_x)^2 + (1 - 2zC_2 - i\delta)(y' - a'_y)^2}, \quad (163)$$

with $\mathcal{W}_h(\mathbf{r}') = f_h(-\mathbf{u}) [(\widehat{\mathbf{n}} \cdot \mathbf{R}) J^{as}](\mathbf{r}') [\mathbf{w}_0 \cdot \mathbf{R}](\mathbf{r}')$. The term δ is a small real constant tending to 0, with $\delta \text{sign}(\arg(ik)) \rightarrow 0^+$, necessary for the cases where we have $(1 - 2zC_1)(1 - 2zC_2) < 0$. The HH polarization is obtained when we let $\mathbf{w}_0 = \frac{\mathbf{r}_0}{r_0} = \frac{\mathbf{a} + z\widehat{\mathbf{z}}}{r_0}$. We can then consider (as in (78) for a flat plate) an isotropic illumination with $f_h(\mathbf{u}) = 1/\sqrt{1 - (\frac{\mathbf{r}' - \mathbf{a}}{\tilde{R}} \cdot \widehat{\mathbf{d}})^2}$, $\widehat{\mathbf{d}} = \frac{\mathbf{r}_0 \times \widehat{\mathbf{z}}}{\sqrt{(\mathbf{r}_0 \times \widehat{\mathbf{z}})^2}}$ as $\mathbf{r}_0 \times \widehat{\mathbf{z}} \neq 0$, and $\mathbf{w}_0 = \frac{\mathbf{r}_0}{r_0} = \frac{\mathbf{a} + z\widehat{\mathbf{z}}}{r_0}$.

9.2.2) Particular asymptotic cases : at very large distance or at focal points

For $z \rightarrow \infty$, we use (159) in previous expression, and we can write,

$$\begin{aligned} \mathbf{E} \cdot \mathbf{v}_0 &\sim \frac{ik}{(4\pi)^2} \frac{e^{-2ik\tilde{R}}}{\tilde{r}_0^2} f_h\left(-\frac{\mathbf{r}_0}{\tilde{r}_0}\right) (\mathbf{w}_0 \cdot \mathbf{r}_0) \\ &\int_{\mathcal{L}} \frac{\mathcal{H}(\mathbf{r}') e^{2ik(\frac{r'_0}{r'_0} \cdot (\mathbf{r}' + \hat{\mathbf{z}}\zeta_s(\mathbf{r}'))} - \mathcal{H}(\mathbf{a}') e^{2ik(\frac{r'_0}{r'_0} \cdot (\mathbf{a}' + \hat{\mathbf{z}}\zeta_s(\mathbf{a}'))}}}{\left(\frac{2zC_1-1}{2z} + i\delta\right)(x' - a'_x)^2 + \left(\frac{2zC_2-1}{2z} + i\delta\right)(x' - a'_y)^2} (\hat{\mathbf{z}} \times (\mathbf{r}' - \mathbf{a}')) \cdot d\mathbf{r}', \end{aligned} \quad (164)$$

where $\mathcal{H}(\mathbf{r}') = (C_1 x'(\frac{a_x}{z}) + C_2 y'(\frac{a_y}{z}) - \frac{1}{2})$.

At focal points, as $\mathcal{K}_m = \min(|1 - 2zC_j|_{j=1,2}) \rightarrow 0$, we have $a' \rightarrow \infty$ and $z''^2 \rightarrow \infty$, but \mathbf{a}'/z''^2 tends to a finite constant vector, and we can then write

$$\mathbf{E} \cdot \mathbf{v}_0 \sim \frac{-ik}{(4\pi)^2} \int_{\mathcal{L}} \frac{\left(\frac{\mathcal{W}_h(\mathbf{r}') e^{-2ik\tilde{R}(\mathbf{r}')}}{\tilde{R}\mathbf{R}^2} - \frac{\mathcal{W}_h(\mathbf{a}') e^{-2ik\tilde{R}(\mathbf{a}')}}{(\tilde{R}(\mathbf{a}'))^3}\right) (\hat{\mathbf{z}} \times \mathcal{K}_m \mathbf{a}') \cdot d\mathbf{r}'}{\mathcal{K}_m (\mathbf{R}_\delta^2 - z^2 - \epsilon(\mathbf{r}')) + \left(\frac{2zC_1\mathcal{K}_m}{1-2zC_1} a_x^2 + \frac{2zC_2\mathcal{K}_m}{1-2zC_2} a_y^2\right)}. \quad (165)$$

with $\mathbf{R}_\delta^2 = (1 - 2zC_1 - i\delta)(x' - a'_x)^2 + (1 - 2zC_2 - i\delta)(y' - a'_y)^2 + z''^2 + \epsilon(\mathbf{r}')$.

Remark : Considering $x \in [x' - \Delta/2, x' + \Delta/2]$ with $0 < |(cx)^2 + y^2| \ll 1$, $0 < \Delta \ll 1$, and smooth function $v(x)$, we can generally write $\int_{x'-\Delta/2}^{x'+\Delta/2} \frac{v(x)dx}{(cx)^2 + y^2} \simeq \frac{iv(x')}{2cy} [\ln(\frac{iy+cx}{iy-cx})]_{x'-\Delta/2}^{x'+\Delta/2}$.

10) Miscellaneous results

Proposition 11

Considering

$$\begin{aligned} \mathcal{R}^2 &= (\alpha_1 x' + \alpha_2 y') - ik(h_1 x'^2 + h_2 y'^2 + h_3 x' y') \\ &= \mathbf{w}' \cdot \mathbf{r}' - ik(h_1 x'^2 + h_2 y'^2 + h_3 x' y'), \end{aligned} \quad (166)$$

we have, for arbitrary constants \mathbf{d} and A_0 ,

$$\int_{S'} (\mathbf{d} \cdot \mathbf{r}') f(\mathcal{R}^2) dS' = A_1 \cdot \left(\int_{\mathcal{L}} [F_v(\mathcal{R}^2) - A_0] d\mathbf{r}' - (\hat{\mathbf{z}} \times \mathbf{w}') \int_{S'} f(\mathcal{R}^2) dS' \right), \quad (167)$$

where F_v is a primitive of f and $A_1 = \frac{\mathcal{P}(\hat{\mathbf{z}} \times \mathcal{P}^{-1}(\mathbf{d}))}{-2ikp\xi}$. In the particular cases,

$$\begin{aligned} p^2 \xi &= \frac{(h_1 + h_2) - \epsilon \sqrt{(h_1 - h_2)^2 + 4(h_3/2)^2}}{2} \rightarrow 0, \\ \text{or } \xi &= \frac{(h_1 + h_2) + \epsilon \sqrt{(h_1 - h_2)^2 + 4(h_3/2)^2}}{2} \rightarrow 0, \end{aligned} \quad (168)$$

we have,

$$\begin{aligned} h_3^2/(4h_1h_2) &\rightarrow 1, \text{ i.e. } 2\epsilon' \sqrt{h_1h_2} \rightarrow h_3 \text{ with } (\epsilon')^2 = 1, \\ (h_1x'^2 + h_2y'^2 + h_3x'y') &\rightarrow (\sqrt{h_1}x' + \epsilon' \sqrt{h_2}y')^2, \end{aligned} \quad (169)$$

In these circumstances, the equality (167) has for consequence that,

$$\begin{aligned} \lim_{\xi \rightarrow 0} \int_{S'} f(\mathcal{R}^2) dS' &= \lim_{\xi \rightarrow 0} \frac{\widehat{\mathbf{c}}_p}{\widehat{\mathbf{c}}_p \cdot (\widehat{\mathbf{z}} \times \mathbf{w}')} \cdot \int_{\mathcal{L}} [F(\mathcal{R}^2) - A_0] d\mathbf{r}', \\ \lim_{p^2\xi \rightarrow 0} \int_{S'} f(\mathcal{R}^2) dS' &= \lim_{p^2\xi \rightarrow 0} \frac{(\widehat{\mathbf{z}} \times \widehat{\mathbf{c}}_p)}{(\widehat{\mathbf{z}} \times \widehat{\mathbf{c}}_p) \cdot (\widehat{\mathbf{z}} \times \mathbf{w}')} \cdot \int_{\mathcal{L}} [F(\mathcal{R}^2) - A_0] d\mathbf{r}'. \end{aligned} \quad (170)$$

Letting

$$\begin{aligned} \mathbf{v}' &= ((\xi - h_1)\widehat{x} + \frac{-h_3}{2}\widehat{y}) \text{ for } \xi \neq h_1, \\ \text{or } \mathbf{v}' &= (\frac{-h_3}{2}\widehat{x} + (\xi - h_2)\widehat{y}) \text{ for } \xi \neq h_2, \end{aligned} \quad (171)$$

we can then write, for $\xi \neq h_1$ or $\xi \neq h_2$,

$$\begin{aligned} \lim_{\xi \rightarrow 0} \int_{S'} f(\mathcal{R}^2) dS' &= \lim_{\xi \rightarrow 0} \frac{\mathbf{v}'}{\mathbf{v}' \cdot (\widehat{\mathbf{z}} \times \mathbf{w}')} \cdot \int_{\mathcal{L}} [F(\mathcal{R}^2) - A_0] d\mathbf{r}', \\ \lim_{p^2\xi \rightarrow 0} \int_{S'} f(\mathcal{R}^2) dS' &= \lim_{p^2\xi \rightarrow 0} \frac{(\widehat{\mathbf{z}} \times \mathbf{v}')}{(\widehat{\mathbf{z}} \times \mathbf{v}') \cdot (\widehat{\mathbf{z}} \times \mathbf{w}')} \cdot \int_{\mathcal{L}} [F(\mathcal{R}^2) - A_0] d\mathbf{r}', \end{aligned} \quad (172)$$

while, for $\xi = h_1 = h_2$, which implies $h_3 = 0$ and $p^2\xi = \xi$, we have,

$$(\widehat{\mathbf{z}} \times \mathbf{w}') \lim_{\xi \rightarrow 0} \int_{S'} f(\mathcal{R}^2) dS' = \lim_{\xi \rightarrow 0} \int_{\mathcal{L}} [F(\mathcal{R}^2) - A_0] d\mathbf{r}'. \quad (173)$$

Proof. Using that $\mathbf{d} \cdot \mathbf{r}' = \mathcal{P}^{-1}(\mathbf{d}) \cdot (\mathbf{r}'_p - \mathbf{a}_p) + \mathcal{P}^{-1}(\mathbf{d}) \cdot \mathbf{a}_p$, then the proposition 3, we derive,

$$\begin{aligned} \int_{S'} \mathbf{d} \cdot \mathbf{r}' f(\mathcal{R}^2) dS' &= \mathcal{P}^{-1}(\mathbf{d}) \cdot \mathbf{a}_p \int_{S'} f(\mathcal{R}^2) dS' + \frac{1}{p} \int_{S'_p} \mathcal{P}^{-1}(\mathbf{d}) \cdot (\mathbf{r}'_p - \mathbf{a}_p) f(\mathcal{R}^2) dS'_p \\ &= \mathcal{P}^{-1}(\mathbf{d}) \cdot \mathbf{a}_p \int_{S'} f(\mathcal{R}^2) dS' + \frac{1}{2p} \int_{\mathcal{L}} \frac{[F_v(\mathcal{R}^2) - A_0]}{v} [\mathcal{P}(\widehat{\mathbf{z}} \times \mathcal{P}^{-1}(\mathbf{d})) \cdot d\mathbf{r}'], \end{aligned} \quad (174)$$

and thus (167), if we notice that,

$$\begin{aligned} \mathcal{P}^{-1}(\mathbf{d}) \cdot \mathbf{a}_p &= (\widehat{\mathbf{z}} \times \mathcal{P}^{-1}(\mathbf{d})) \cdot (\widehat{\mathbf{z}} \times \mathbf{a}_p) = \frac{\widehat{\mathbf{z}} \times \mathcal{P}^{-1}(\mathbf{d}) \cdot \widehat{\mathbf{z}} \times \mathcal{P}^{-1}(\mathbf{w}')}{2ik\xi} \\ &= \frac{\widehat{\mathbf{z}} \times \mathcal{P}^{-1}(\mathbf{d}) \cdot \mathcal{P}(\widehat{\mathbf{z}} \times \mathbf{w}')}{2ikp\xi} = \frac{\mathcal{P}(\widehat{\mathbf{z}} \times \mathcal{P}^{-1}(\mathbf{d})) \cdot (\widehat{\mathbf{z}} \times \mathbf{w}')}{2ikp\xi}, \end{aligned} \quad (175)$$

where we have used (113)-(114) with $\mathbf{a}_p = \frac{\mathcal{P}^{-1}(\mathbf{w}')}{2ik\xi}$. Considering that,

$$\begin{aligned} \frac{\mathcal{P}(\widehat{\mathbf{z}} \times \mathcal{P}^{-1}(\mathbf{d}))}{p\xi} &= \frac{(\widehat{\mathbf{z}} \times \mathcal{P}^{-2}(\mathbf{d}))}{\xi}, \\ \frac{\mathcal{P}^{-2}(\mathbf{d})}{\xi} &= \frac{\mathbf{d} \cdot \widehat{\mathbf{c}}_p}{p^2\xi} \widehat{\mathbf{c}}_p + \frac{\mathbf{d} \cdot (\widehat{\mathbf{z}} \times \widehat{\mathbf{c}}_p)}{\xi} \widehat{\mathbf{z}} \times \widehat{\mathbf{c}}_p, \end{aligned} \quad (176)$$

we note that, in right side of (167), $\frac{\mathcal{P}(\widehat{\mathbf{z}} \times \mathcal{P}^{-1}(\mathbf{d}))}{2ikp\xi}$ tends to ∞ for $\xi \rightarrow 0$ or $p^2\xi \rightarrow 0$, while left side of (167) is finite for arbitrary bounded \mathbf{d} . We can then write,

$$\begin{aligned} \lim_{\xi \rightarrow 0} \widehat{\mathbf{c}}_p \cdot [(\widehat{\mathbf{z}} \times \mathbf{w}') \int_{S'} f(\mathcal{R}^2) dS' - \int_{\mathcal{L}} [F(\mathcal{R}^2) - A_0] d\mathbf{r}'] &= 0, \\ \lim_{p^2\xi \rightarrow 0} (\widehat{\mathbf{z}} \times \widehat{\mathbf{c}}_p) \cdot [(\widehat{\mathbf{z}} \times \mathbf{w}') \int_{S'} f(\mathcal{R}^2) dS' - \int_{\mathcal{L}} [F(\mathcal{R}^2) - A_0] d\mathbf{r}'] &= 0, \end{aligned} \quad (177)$$

and thus (170). We then obtain (172), if we use (82)-(84) and,

$$\begin{aligned} c_{p,x} &= \epsilon_x \sqrt{\xi - h_1} / \sqrt{\xi}, \quad c_{p,y} = \epsilon_y \sqrt{\xi - h_2} / \sqrt{\xi}, \\ \epsilon_x^2 &= \epsilon_y^2 = 1, \quad \epsilon_y \epsilon_x = \frac{\epsilon_y}{\epsilon_x} = \frac{\epsilon_x}{\epsilon_y} = \frac{-h_3}{2\sqrt{\xi - h_1} \sqrt{\xi - h_2}}, \\ \xi &= \frac{(h_1 + h_2) + \epsilon \sqrt{(h_1 - h_2)^2 + h_3^2}}{2}, \\ (\mathbf{c}_p)_\epsilon^2 &= 1 - \frac{(h_1 + h_2) - \epsilon \sqrt{(h_1 - h_2)^2 + h_3^2}}{(h_1 + h_2) + \epsilon \sqrt{(h_1 - h_2)^2 + h_3^2}}, \\ p^2 &= 1 - (\mathbf{c}_p)^2, \quad p^2\xi = \frac{(h_1 + h_2) - \epsilon \sqrt{(h_1 - h_2)^2 + h_3^2}}{2}. \end{aligned} \quad (178)$$

and (173) from (177), noticing that $p^2\xi \rightarrow \xi$ when $\xi \rightarrow h_1 = h_2$.

Remark : Using (116) of proposition 8, we have, as $\epsilon = \pm 1$,

$$\begin{aligned} (\widehat{\mathbf{c}}_p)_\pm &= \pm \epsilon_c (\widehat{\mathbf{z}} \times (\widehat{\mathbf{c}}_p)_\mp), \quad (p^2\xi)_\pm = (\xi)_\mp, \\ \left(\frac{\mathbf{d} \cdot (\widehat{\mathbf{c}}_p)}{p^2\xi} (\widehat{\mathbf{c}}_p) \right)_\pm &= \left(\frac{\mathbf{d} \cdot (\widehat{\mathbf{z}} \times \widehat{\mathbf{c}}_p)}{\xi} \widehat{\mathbf{z}} \times \widehat{\mathbf{c}}_p \right)_\mp, \end{aligned} \quad (179)$$

with $(\epsilon_c)^2 = 1$, and we then note that,

$$\frac{\mathcal{P}^{-2}(\mathbf{d})}{\xi} = \frac{\mathbf{d} \cdot \widehat{\mathbf{c}}_p}{p^2\xi} \widehat{\mathbf{c}}_p + \frac{\mathbf{d} \cdot (\widehat{\mathbf{z}} \times \widehat{\mathbf{c}}_p)}{\xi} \widehat{\mathbf{z}} \times \widehat{\mathbf{c}}_p \text{ is independent of } \epsilon. \quad (180)$$

Proposition 12

Let F_v be a primitive of f , and $\xi = h_1 = h_2$, we then have $h_3 = 0$, and $\mathcal{R}^2 \rightarrow \mathbf{w}' \cdot \mathbf{r}'$ for $\xi \rightarrow 0$. Therefore, we can write, from proposition 11,

$$\int_{S'} f(\mathbf{w}' \cdot \mathbf{r}') dS' = \frac{(\widehat{\mathbf{z}} \times \mathbf{w}')}{(\mathbf{w}')^2} \cdot \int_{\mathcal{L}} (F_v(\mathbf{w}' \cdot \mathbf{r}') - F_v(0)) d\mathbf{r}'. \quad (181)$$

Proof. We can obtain this equality from (173), or by using $\text{grad}(F_v(\mathbf{w}' \cdot \mathbf{r}')) = \mathbf{w}' f(\mathbf{w}' \cdot \mathbf{r}')$.

Remark : Letting $\mathbf{w}' = a\mathbf{w}'_1$ with $a \rightarrow 0$, we note from (181) that,

$$\int_{S'} dS' = \frac{(\widehat{\mathbf{z}} \times \mathbf{w}'_1)}{(\mathbf{w}'_1)^2} \cdot \int_{\mathcal{L}} (\mathbf{w}'_1 \cdot \mathbf{r}') d\mathbf{r}'. \quad (182)$$

11) Numerical results

11.1) Exact contour reduction of a surface integral whose integrand has an exponential term with a quadratic argument

11.1.1) For $e^{\mathbf{w}' \cdot \mathbf{r}' - ik\xi(\mathbf{r}'^2 - (\mathbf{c}_p \cdot \mathbf{r}')^2)}$ integrand in surface integral

We choose to validate our reduction given in theorem 11, by verifying,

$$\begin{aligned} & \frac{k}{\sqrt{\pi}} \int_{S'} e^{\mathbf{w}' \cdot \mathbf{r}' - ik\xi(\mathbf{r}'^2 - (\mathbf{c}_p \cdot \mathbf{r}')^2)} dS' = \\ & = \frac{k}{2\sqrt{\pi}} \int_{\mathcal{L}} \left[\frac{(e^{\mathbf{w}' \cdot \mathbf{r}' - ik\xi(\mathbf{r}'^2 - (\mathbf{c}_p \cdot \mathbf{r}')^2)} - e^{\frac{\mathbf{w}'_m \cdot 2}{4ik\xi}})}{(\mathbf{w}' \cdot \mathbf{r}' - ik\xi(|\mathbf{r}'|^2 - (\mathbf{c}_p \cdot \mathbf{r}')^2)) - \frac{\mathbf{w}'_m \cdot 2}{4ik\xi}} \right] \\ & \left[(\widehat{\mathbf{z}} \times \left(\mathbf{r}' - \frac{\mathcal{M}(\mathbf{w}')}{2ik\xi(1 - \mathbf{c}_p^2)} \right)) \cdot d\mathbf{r}' \right], \end{aligned} \quad (183)$$

when \mathbf{w}' and \mathbf{c}_p vary, in the figures 3a and 3b. We take S' , a square of 1m side length, whose center is at $\mathbf{r}_c = (0.5 \widehat{\mathbf{x}} + 0.5 \widehat{\mathbf{y}})$, $k = 2\pi/\lambda$, $\lambda = 2 \text{ cm}$, $\xi = e^{i\Phi}/|\mathbf{r}|$, $|\mathbf{r}| = 1 \text{ m}$. The term $\mathbf{c}_p = \sin\theta\cos\varphi\widehat{\mathbf{x}} + \sin\theta\sin\varphi\widehat{\mathbf{y}}$ is considered for $\varphi = \pi/4$ as θ varies, and $\mathbf{w}' = 2ik\mathbf{c}_p$. The modulus of each side in (183) are given in dB , for $\xi = e^{i\Phi}/|\mathbf{r}|$ with $\Phi = 0$, $\Phi = -\frac{\pi}{18}$ and $\Phi = -\frac{\pi}{2}$. The respective complexity of integrands in surface and contour integrals are equal, and the save in computation time grows linearly with the ratio of surface to its contour. We notice the perfect agreement of the respective results.

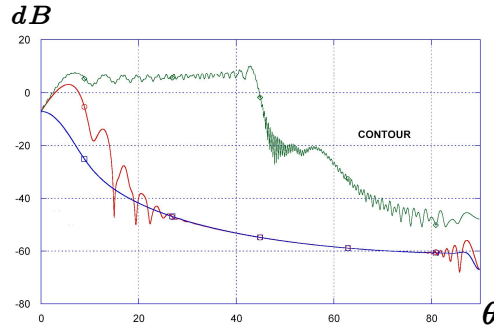


figure 3a : contour integral term in (183), for $\xi = e^{i\Phi}/|\mathbf{r}'|$ with $\Phi = 0$ (green), $\Phi = -\frac{\pi}{18}$ (red) and $\Phi = -\frac{\pi}{2}$ (blue curve)

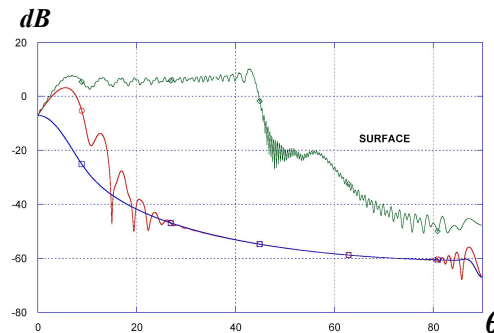


figure 3b : surface integral term in (183), for $\xi = e^{i\Phi}/|\mathbf{r}'|$ with $\Phi = 0$ (green), $\Phi = -\frac{\pi}{18}$ (red) and $\Phi = -\frac{\pi}{2}$ (blue)

11.1.2) For $\sqrt{a' + b'x' + cx'^2} e^{w'.r' - ik\xi(r'^2 - (c_p.r')^2)}$ integrand in surface integral

We here choose to study the example with $\sqrt{a' + b'x' + cx'^2} e^{w'.r' - ik\xi(|r'|^2 - (c_p.r')^2)}$ integrand. The exponential argument is similar to the ones considered in previous section with $\Phi = -\frac{\pi}{4}$. The constants a' , b' and c' considered in figure 4 are respectively, $a' = 1$ with $b' = c' = 0$, and $a' = 1$ with $b' = c' = 100$. In both cases, we then apply the theorem 19 of appendix with $b_s = 4$. The agreement of results given by contour and surface integrals are excellent.

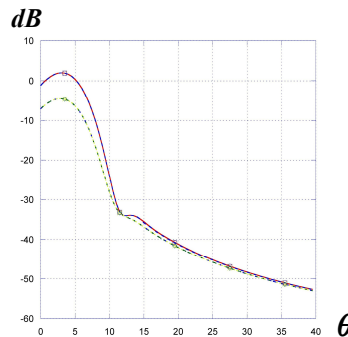


figure 4 : amplitude in dB of the 'surface' integrals with $\sqrt{a' + b'x' + cx'^2} e^{w'.r' - ik\xi(|r'|^2 - (c_p.r')^2)}$ integrand, and its 'contour' reduction given by theorem 19 in appendix : for $\xi = e^{i\Phi}/|\mathbf{r}'|$ with $\Phi = -\pi/4$, $a' = 1$ with $b' = c' = 0$ in yellow (contour) and green (surface) curves, and $a' = 1$ with $b' = c' = 100$ in blue (contour) and red (surface) curves.

11.2) Asymptotic contour reduction for near-field radiation surface integral in Physical Optics for a large plate illuminated by a point source with spherical pattern.

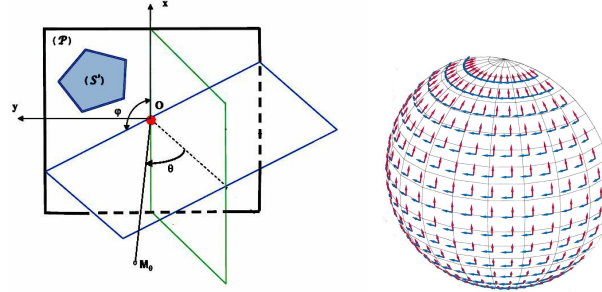


figure 5 : Geometry for the near-field RCS for a source illuminating scatterers with a spherical pattern

11.2.1) Asymptotic reduction for large flat plates

We consider the monostatic Physical Optics radiation term $U = \mathbf{E} \cdot \hat{\mathbf{d}}$ in VV polarisation (resp. $U = \mathbf{E} \cdot \mathbf{v}_0$ in HH) at constant φ when θ varies (figure 5), for a point source at \mathbf{r}_0 with isotropic pattern, where $\hat{\mathbf{d}} = \frac{\mathbf{r}_0 \times \hat{\mathbf{z}}}{\sqrt{(\mathbf{r}_0 \times \hat{\mathbf{z}})^2}}$ and $\mathbf{v}_0 = \hat{\mathbf{d}} \times \frac{\mathbf{r}_0}{r_0}$ as $\mathbf{r}_0 \times \hat{\mathbf{z}} \neq 0$, and its reduction (69) (resp. ((78))) from theorem 7 (resp. 9), for a flat plate S' of large dimension. We then compare the backscattering pattern $\frac{\sqrt{4\pi} \tilde{r} U}{|E_i|_{O'}}$ (RCS) given by contour and surface integral expressions, in figure 6, when we consider a square plate with 1 m side length, shifted from the origin O' and centered at $(0, 0.5 \text{ m})$, observed in the plane $\varphi = 20 \text{ deg}$, at the distance $\tilde{r} = \tilde{r}_0 = 1 \text{ m}$, for $\lambda = 2 \text{ cm}$.

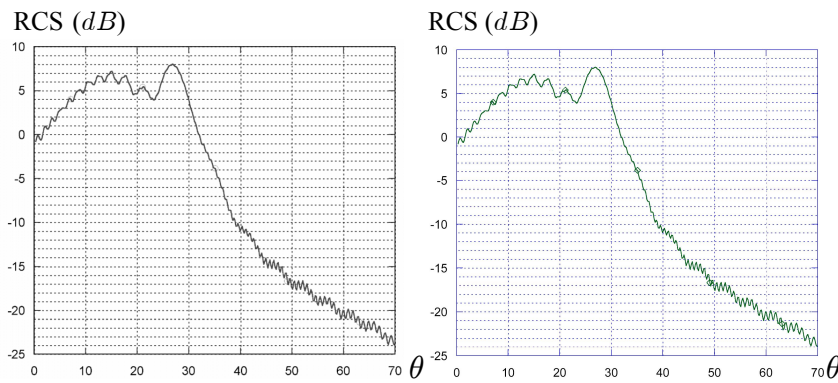


figure 6 : RCS for polarisation VV at 1m, contour (left-hand) and surface (right-hand) integrals at $\varphi = 20^\circ$

In figure 7, we consider a larger shift of the plate center, disposed at $(0, 3 \text{ m})$, and we observe the field at $\tilde{r} = \tilde{r}_0 = 3.5 \text{ m}$ in the plane $\varphi = 80 \text{ deg}$. We distinguish more clearly differences between the polarizations VV and HH (VV is clearly larger than HH), and notice

that the agreement between our contour integral and the initial surface integral continues to be excellent.

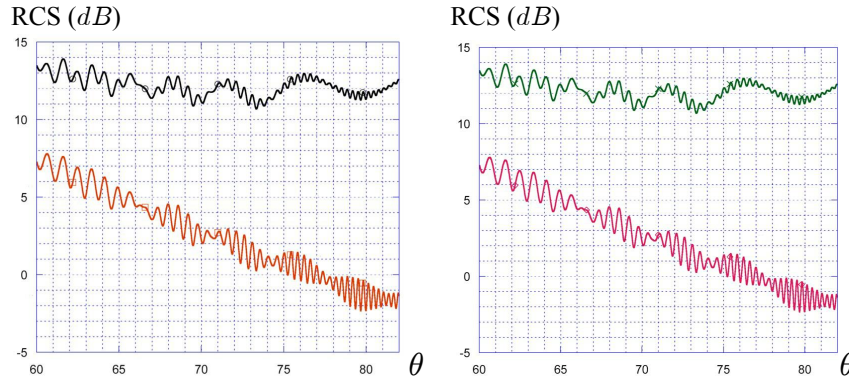


figure 7 : RCS for polarisation VV and HH at 3.5m, contour (left-hand) and surface (right-hand) integrals at $\varphi = 80^\circ$

11.2.2) Asymptotic reduction for large curved plates : convex, concave and saddle surfaces

We consider the monostatic Physical Optics radiation term $U = \mathbf{E} \cdot \hat{\mathbf{d}}$ in VV polarisation (resp. $U = \mathbf{E} \cdot \mathbf{v}_0$ in HH) at constant φ when θ varies (fig. 5), for a point source at \mathbf{r}_0 with isotropic pattern, where $\hat{\mathbf{d}} = \frac{\mathbf{r}_0 \times \hat{\mathbf{z}}}{\sqrt{(\mathbf{r}_0 \times \hat{\mathbf{z}})^2}}$ and $\mathbf{v}_0 = \hat{\mathbf{d}} \times \frac{\mathbf{r}_0}{r_0}$ as $\mathbf{r}_0 \times \hat{\mathbf{z}} \neq 0$. Its contour reduction is calculated by (158) (resp. (163)) from theorem 15 (resp. 16), for a curved plate S whose projection is a square of side length $D = 1\text{m}$, centered at the origin, and observed in planes $\varphi = 70^\circ$ and 20° , at different distances, for $k = 100\pi$ ($\lambda = 2\text{cm}$), with $|C_{1,2}|D^2 \ll 4z$. We then compare the backscattering pattern $\frac{\sqrt{4\pi} \hat{\mathbf{r}} U}{|E_i|_{\theta'}}$ given by contour and surface integral expressions, and analyse the modifications as distance, angles, and curvatures vary, for convex, concave and saddle surfaces.

In figure 8, at large distance $z = 100\text{m}$, we note the influence of curvature on the width of the plateau, which is similar for convex and concave case, and vanishes when the plate is flat.

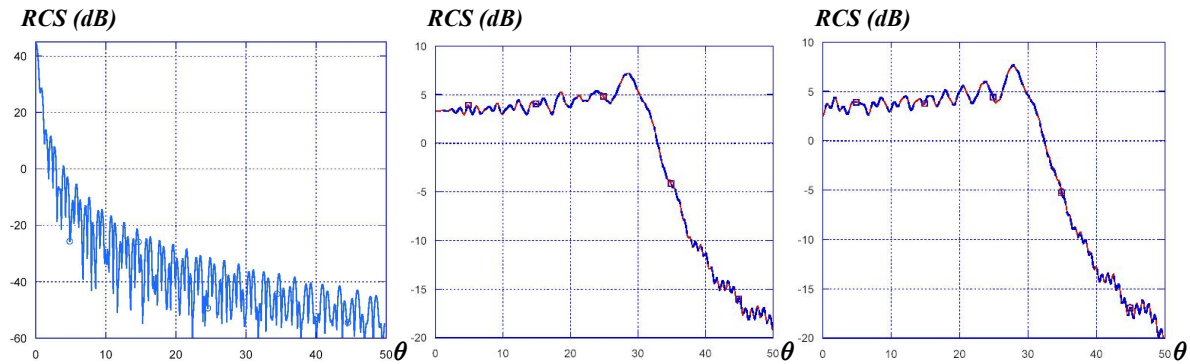


figure 8 : RCS VV at 100m , for flat plate $C_{1,2} = 0$ (left), for convex plate with $C_{1,2} = -0.6$ (middle), or concave with $C_{1,2} = +0.6$ (right), with surface integral (blue) and contour integral (red) (perfect agreement) at $\varphi = 70^\circ$

In figure 9 and 10, we let vary the curvature in the near field, at a distance of 6m, and give the results from surface and contour integrals, in convex and concave cases.

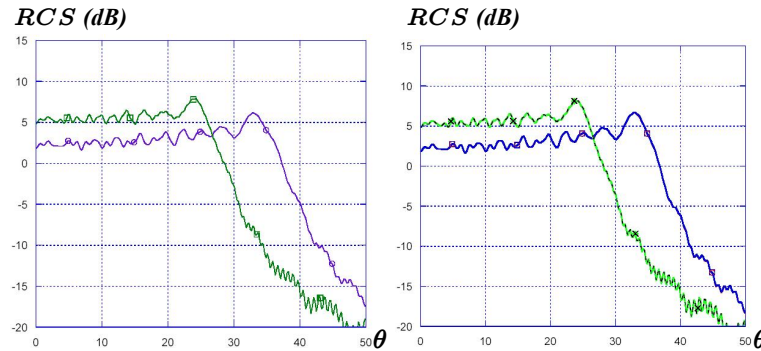


figure 9: RCS VV and HH at 6m for surface integral (on left) et contour integral (on right), for $C_1 = C_2 = -0.4$ (green) et $C_1 = C_2 = -0.6$ (blue) (purely convex), at $\varphi = 70^\circ$

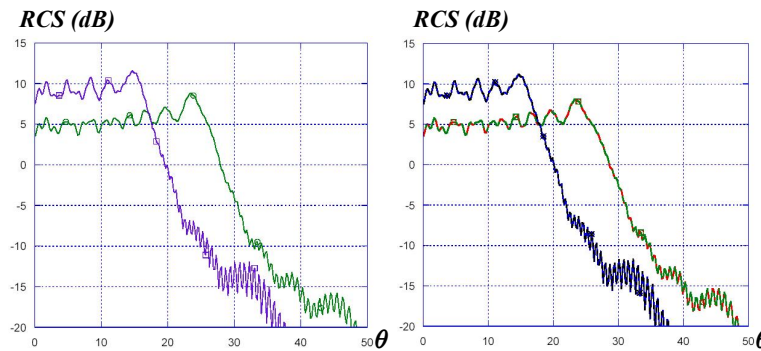


figure 10: RCS VV and HH at 6m for surface integral (on left) et contour integral (on right), for $C_1 = C_2 = 0.4$ (blue) and $C_1 = C_2 = 0.6$ (green) (purely concave), at $\varphi = 70^\circ$

At 3m, the figures 11, 12 and 13 show the variation of patterns as we let vary the curvature. In figure 13, we detail the concave case, and show a minimum of the width of the plateau near the focal point when $1 - 2zC = 0$, which varies monotonically else. In figure 14, we let vary the distance and observe that the plateau, very different in near field, becomes similar at large distance.

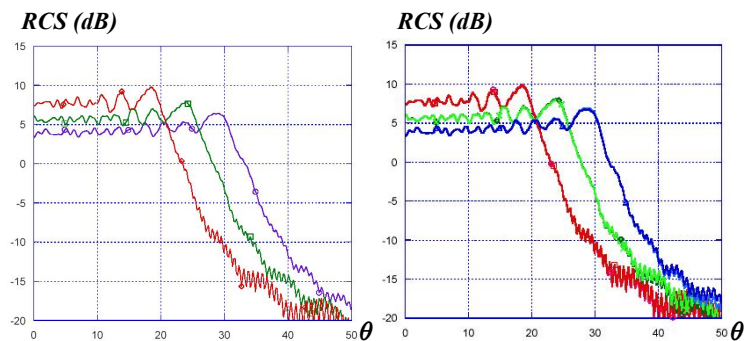


figure 11 : Comparison of RCS VV and HH at 3m for surface integral (on left) et contour integral (on right), for $C_1 = C_2 = -0.2$ (red), $= -0.3$ (green), $= -0.4$ (blue) at $\varphi = 70^\circ$

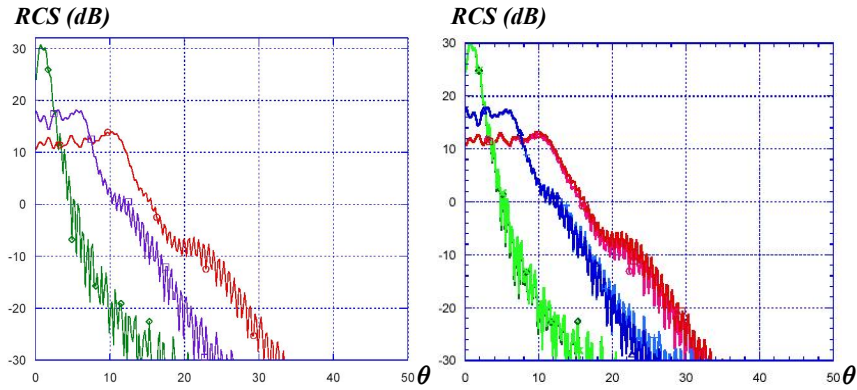


figure 12 : Comparison of RCS VV and HH at 3m for surface integral (on left) et contour integral (on right),
for $C_1 = C_2 = 0.2$ (green), $= 0.3$ (blue), $= 0.4$ (red) at $\varphi = 70^\circ$

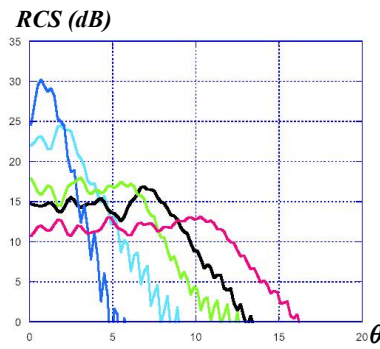


figure 13 : Variation in purely concave cases with $C_j = C$, in vicinity of $1 - 2zC = 0$ (focal case) at $3m$ in VV,
 $C = 0.4$ (red), $C = 0.3$ (vert), $C = 0.2$ (blue), $C = 0.1$ (clear blue), $C = 0.$ (black) at $\varphi = 70^\circ$

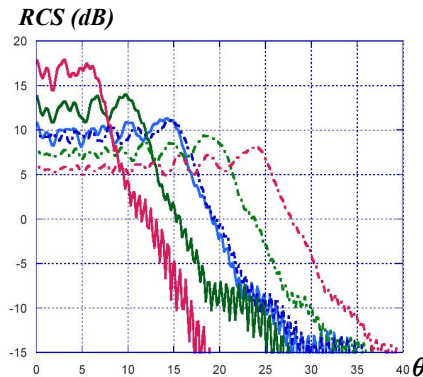


figure 14 : purely convex cases with negative C_j (dashed lines) ou concave cases with positive C_j (continuous lines) in VV,
for $|C_1| = |C_2| = 0.3$, at $3m$ (red), $6m$ (green), et $100m$ (blue) at $\varphi = 70^\circ$

In figure 15 and 16, we give results for a saddle surface. The asymetry of the plate is illustrated by the change of pattern in the cases $\varphi = 70^\circ$ and $\varphi = 20^\circ$ in figure 15, and we observe the changes as we let vary the distance in figure 16. In all these cases, we notice the very good agreement of the surface and our contour integrals.

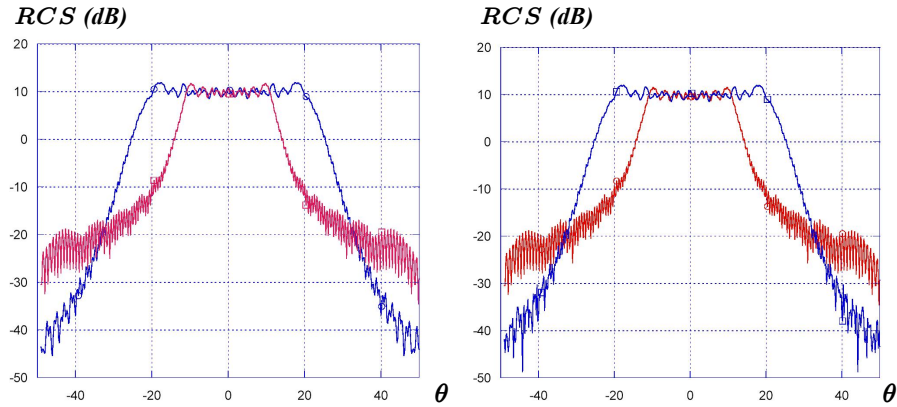


figure 15: Comparison of the RCS VV for surface integral (on left) et contour integral (on right), at 6m, for saddle surface with $C_1 = 0.3$ et $C_2 = -0.3$ at $\varphi = 70^\circ$ (in blue) and at $\varphi = 20^\circ$ (in red)

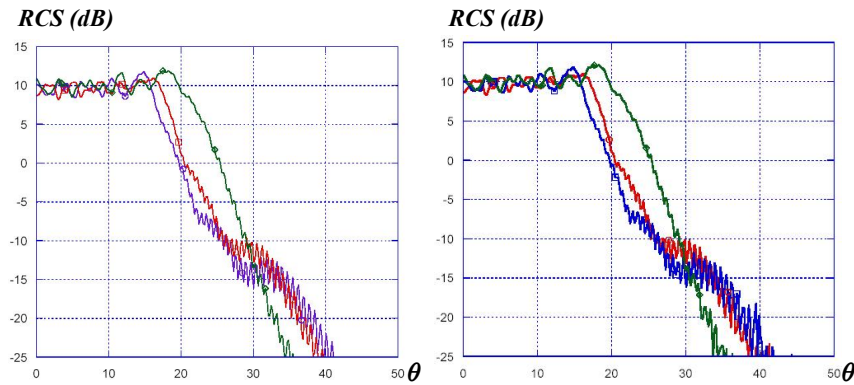


figure 16: Comparison of the RCS VV for surface integral (on left) et contour integral (on right), at 6m (green), 20m (red) and 100m (blue), for $C_1 = 0.3$ et $C_2 = -0.3$ (saddle surface, convex in the plane of incidence at $\varphi = 70^\circ$)

12) Conclusion

How to reduce fast oscillating surface integrals with exponentials of complex arguments, an important problem commonly encountered in wave physics, was studied in this chapter. For that, we have here derived a general method of reduction to non-singular contour integrals, for a large class of radiation surface integrals, when the currents are analytically defined on the surface of integration, as in Physical Optics.

We began first with simple near-field reductions in monostatic case, without special function, that are valid from small to large distance, which applies for arbitrary dimensions of objects, illumination patterns, impedance boundary conditions (perfectly or imperfectly reflective surfaces), and for large planar or moderately curved plates. Secondly, specific original expressions are presented in bistatic cases, in particular for the reduction of surface integrals whose integrands have exponentials with complex arguments, that are arbitrary complex

polynomial of the coordinates of rank 2, corresponding to quadratic approximations which are commonly used for Green's function, or in gaussian beam propagation problems. Because the respective complexity of integrands in surface and line integrals are equal, the save in computation time grows linearly with the ratio of surface to its contour. As made evident by the numerical examples, the presented formulae are highly accurate, and they ought to find wide domains of applications in accelerating the computation.

Appendix A : Primitives for theorems 5 and 6, and properties

A.1) Primitives for the theorem 5

Let us show that,

$$\begin{aligned} \int \frac{1}{\tilde{R}} \left(\left(\frac{\psi'}{\tilde{R}} \right)^2 \right)' d\mathbf{R}^2 &= 2 \left(\frac{\psi'}{\tilde{R}} \right)^2 = 2 \frac{(ik + \frac{1}{\tilde{R}})^2}{\mathbf{R}^4} e^{-i2k\tilde{R}}, \\ \int \left(\frac{\psi'}{\tilde{R}} \right)^2 d\mathbf{R}^2 &= -2 \frac{(1 + 2ik\tilde{R})}{4\mathbf{R}^4} e^{-i2k\tilde{R}} \\ \int \tilde{R} \left(\left(\frac{\psi'}{\tilde{R}} \right)^2 \right)' d\mathbf{R}^2 &= 2 \left(-k^2 + \frac{3ik}{\tilde{R}} + \frac{3}{2\mathbf{R}^2} \right) \frac{e^{-i2k\tilde{R}}}{\mathbf{R}^2} \end{aligned} \quad (184)$$

with $\tilde{R} = \sqrt{\mathbf{R} \cdot \mathbf{R}}$, $(\cdot)' \equiv \partial_{\tilde{R}}(\cdot)$, $\psi(\tilde{R}) = \frac{e^{-ik\tilde{R}}}{\tilde{R}}$, $\mathbf{R}^2 = (\mathbf{r}' - \mathbf{a})^2 + z^2 \neq 0$. The first line is trivial since $d\mathbf{R}^2 = 2\tilde{R} d\tilde{R}$, while the second and third line need some manipulations.

We remark first that ψ verifies $\Delta\psi + k^2\psi = 0$ as $\mathbf{R}^2 \neq 0$, and thus,

$$\frac{1}{\mathbf{R}^2} (\tilde{R}^3 \left(\frac{\psi'}{\tilde{R}} \right))' = \tilde{R} \left(\frac{\psi'}{\tilde{R}} \right)' + 3 \frac{\psi'}{\tilde{R}} = -k^2\psi, \quad (185)$$

which multiplied by $\frac{\psi'}{\tilde{R}}$, leads us to

$$\tilde{R} \left(\left(\frac{\psi'}{\tilde{R}} \right)^2 \right)' = -k^2 \frac{((\psi)^2)'}{\tilde{R}} - 6 \left(\frac{\psi'}{\tilde{R}} \right)^2. \quad (186)$$

Considering that we have, by integration by parts,

$$\int \tilde{R} \left(\left(\frac{\psi'}{\tilde{R}} \right)^2 \right)' \tilde{R} d\tilde{R} = (\psi')^2 - 2 \int \left(\frac{\psi'}{\tilde{R}} \right)^2 \tilde{R} d\tilde{R}, \quad (187)$$

we can use (186), and write,

$$\int -k^2 ((\psi)^2)' d\tilde{R} - \int 6 \left(\frac{\psi'}{\tilde{R}} \right)^2 \tilde{R} d\tilde{R} = (\psi')^2 - 2 \int \left(\frac{\psi'}{\tilde{R}} \right)^2 \tilde{R} d\tilde{R}, \quad (188)$$

so that we obtain the second line of (184) following,

$$\int \left(\frac{\psi'}{\widetilde{R}}\right)^2 \widetilde{R} d\widetilde{R} = \frac{1}{4}((ik\psi)^2 - (\psi')^2) = -\left(2ik + \frac{1}{\widetilde{R}}\right) \frac{\psi^2}{4\widetilde{R}} \left(= \frac{(\widetilde{R} \psi^2)'}{4\mathbf{R}^2} \right). \quad (189)$$

We can then rewrite (187), and derive the third line of (184),

$$\int \widetilde{R} \left(\left(\frac{\psi'}{\widetilde{R}}\right)'\right) \widetilde{R} d\widetilde{R} = \frac{3(\psi')^2 - (ik\psi)^2}{2} = \left(-k^2 + \frac{3ik}{\widetilde{R}} + \frac{3}{2\mathbf{R}^2}\right) \psi^2. \quad (190)$$

A.2) Primitives for the theorem 6

For the theorem 6, we have to complete (184). For this, we write,

$$\begin{aligned} \int \left(\frac{\psi'}{\widetilde{R}}\right)' \left(\frac{\psi'}{\widetilde{R}}\right)' d\mathbf{R}^2 &= -2 \int \left(\frac{\psi'}{\widetilde{R}}\right)' \left(k^2\psi + 3\frac{\psi'}{\widetilde{R}}\right) d\widetilde{R} = \\ &= \left(-2k^2 \frac{\psi'\psi}{\widetilde{R}} + k^2 \int \left(\frac{\psi'}{\widetilde{R}}\right)^2 d\mathbf{R}^2\right) - 3\left(\frac{\psi'}{\widetilde{R}}\right)^2 \\ &= 2k^2 \left(ik + \frac{1}{\widetilde{R}}\right) \frac{\psi^2}{\widetilde{R}} - k^2 \left(ik + \frac{1}{2\widetilde{R}}\right) \frac{\psi^2}{\widetilde{R}} - 3\left(\frac{\psi'}{\widetilde{R}}\right)^2 \\ &= k^2 \left(ik + \frac{3}{2\widetilde{R}}\right) \frac{\psi^2}{\widetilde{R}} - 3\left(ik + \frac{1}{\widetilde{R}}\right)^2 \left(\frac{\psi}{\widetilde{R}}\right)^2, \end{aligned} \quad (191)$$

where we have successively used (185), an integration by parts, and (189).

A3) Some expressions and properties of successive primitives of $\frac{e^{-v\widetilde{R}}}{\widetilde{R}^n}$

Let $\widetilde{R} = \sqrt{\mathbf{R} \cdot \mathbf{R}}$ with $\mathbf{R}^2 = (\mathbf{r}' - \mathbf{a})^2 + z^2 \neq 0$. We have for $n \geq 1$,

$$\begin{aligned} \int_{\infty}^{\mathbf{R}^2} \frac{e^{-v\widetilde{R}}}{\widetilde{R}^n} d\mathbf{R}^2 &= 2v^{n-2} V_{n-1}(v\widetilde{R}) \sim -\frac{2}{v} \frac{e^{-v\widetilde{R}}}{\widetilde{R}^{n-1}} \left(1 - \frac{n-1}{v\widetilde{R}} + \dots\right) \text{ if } v\widetilde{R} \rightarrow \infty \\ V_n(u) &= \int_{\infty}^u \frac{e^{-t}}{t^n} dt = -\frac{E_n(u)}{u^{n-1}}, V_n(u) \sim -\frac{e^{-u}}{u^n} \left(1 - \frac{n}{u} + \frac{n(n+1)}{u^2} - \dots\right), \\ E_{n+1}(u) &= \frac{1}{n} (e^{-u} - uE_n(u)), V_n(u) + nV_{n+1}(u) = -\frac{e^{-u}}{(u)^n}, \end{aligned} \quad (192)$$

while, for p successive primitives, we can write,

$$\begin{aligned}
& (p-1)! \int_{\infty}^{R'^2} d\mathbf{R}^2 \dots \int_{\infty}^{R'^2} d\mathbf{R}^2 \frac{e^{-v\tilde{R}}}{\tilde{R}^n} = \int_{\infty}^{R'^2} (\mathbf{R}'^2 - \mathbf{R}^2)^{p-1} \frac{e^{-v\tilde{R}}}{\tilde{R}^n} d\mathbf{R}^2 \\
& = \sum_{0 \leq m \leq p-1} (-1)^{p-1-m} C_{p-1}^m (\tilde{R}')^{2m} \int_{\infty}^{R'^2} \frac{e^{-v\tilde{R}}}{\tilde{R}^{n-2(p-1-m)}} d\mathbf{R}^2 \\
& = 2 \sum_{0 \leq m \leq p-1} (-1)^{p-1-m} C_{p-1}^m (\tilde{R}')^{2m} (v^{n-2(p-1-m)-2} V_{n-2(p-1-m)-1}(v\tilde{R}')) \\
& = -2 \sum_{0 \leq m \leq p-1} (-1)^{p-1-m} C_{p-1}^m (\tilde{R}')^{2m} \left(\frac{E_{n-2(p-1-m)-1}(v\tilde{R}')}{(\tilde{R}')^{n-2(p-1-m)-2}} \right) \\
& = \frac{-2}{(\tilde{R}')^{n-2p}} \sum_{0 \leq m \leq p-1} (-1)^{p-1-m} C_{p-1}^m E_{n-2(p-1-m)-1}(v\tilde{R}'), \tag{193}
\end{aligned}$$

where we remark that the first asymptotic orders of the sum, when $v\tilde{R} \gg 1$, vanish. For example, using $\sum_{0 \leq m \leq p-1} (-1)^{p-1-m} C_{p-1}^m = 0$ in this expression for $p = 2$, we have,

$$\begin{aligned}
& \int_{\infty}^{R'^2} d\mathbf{R}^2 \int_{\infty}^{R'^2} d\mathbf{R}^2 \frac{e^{-v\tilde{R}}}{\tilde{R}^n} \sim \\
& \sim \frac{-2}{v} \frac{e^{-v|R'|}}{\tilde{R}'^{n-(2p-1)}} \sum_{0 \leq m \leq 1} (-1)^{p-1-m} C_{p-1}^m \left(1 - \frac{n-2(p-m)+1}{v\tilde{R}'} \right) \Big|_{p=2} \\
& \sim \frac{4}{v^2} \frac{e^{-v|R'|}}{(\tilde{R}')^{n-(2p-2)}} \sum_{0 \leq m \leq 1} (-1)^{p-1-m} m C_{p-1}^m \Big|_{p=2} = \frac{4}{v^2} \frac{e^{-v|R'|}}{(\tilde{R}')^{n-2}}. \tag{194}
\end{aligned}$$

Remark : we note that, from,

$$\int_{\infty}^{R'^2} \frac{-ve^{-v\tilde{R}} \left(1 + \frac{n}{v\tilde{R}} \right)}{\tilde{R}^{n+1}} d\mathbf{R}^2 = \frac{2e^{-v\tilde{R}'}}{\tilde{R}'^n}, \tag{195}$$

we have,

$$(vR')^n e^{v\tilde{R}'} (V_n(v\tilde{R}') + nV_{n+1}(v\tilde{R}')) = -1. \tag{196}$$

Appendix B : Reduction of Physical Optics surface integrals to contour integrals in bistatic case, and quadratic approximation of exponential arguments

B.1) Physical Optics surface integrals and asymptotics

Let us consider (47) for the fields, diffracted by the surface S , when it is illuminated by incident field \mathbf{E}_i and \mathbf{H}_i , with tangential fields on scatterer approximated as in Physical Optics (PO). At the observation point $\mathbf{R}_1 = \mathbf{a} + z\hat{\mathbf{z}} \notin S$, we can then write,

$$\begin{aligned}\mathbf{E} &= \frac{Z_0}{4\pi ik} \int_S (\tilde{R} (\frac{\psi'}{\tilde{R}})' \mathbf{u} \times (\mathbf{u} \times \mathbf{J}) - \frac{2\psi'}{\tilde{R}} \mathbf{J}) dS - \frac{1}{4\pi} \int_S \frac{\psi'}{\tilde{R}} \mathbf{R} \times \mathbf{M} dS, \\ Z_0 \mathbf{H} &= \frac{Z_0}{4\pi} \int_S \frac{\psi'}{\tilde{R}} \mathbf{R} \times \mathbf{J} dS + \frac{1}{4\pi ik} \int_S (\tilde{R} (\frac{\psi'}{\tilde{R}})' \mathbf{u} \times (\mathbf{u} \times \mathbf{M}) - \frac{2\psi'}{\tilde{R}} \mathbf{M}) dS, \\ \mathbf{J} &= \hat{\mathbf{n}} \times \mathbf{H} \sim \hat{\mathbf{n}} \times \mathbf{H}_{GO}, \mathbf{M} = -\hat{\mathbf{n}} \times \mathbf{E} \sim -\hat{\mathbf{n}} \times \mathbf{E}_{GO},\end{aligned}\quad (197)$$

where,

$$\begin{aligned}\psi(\tilde{R}) &= \frac{e^{-ik\tilde{R}}}{\tilde{R}}, (\cdot)' \equiv \partial_{\tilde{R}}(\cdot), \tilde{R} = \sqrt{\mathbf{R} \cdot \mathbf{R}}, \\ \mathbf{R} &= \mathbf{R}_1 - \mathbf{R}' = \tilde{R} \mathbf{u}, \mathbf{R}_1 = \sqrt{\mathbf{R}_1 \cdot \mathbf{R}_1} \mathbf{u}_1,\end{aligned}\quad (198)$$

with $\mathbf{R}' = (\mathbf{r}' + z'\hat{\mathbf{z}})$ the point of integration on S . The fields \mathbf{E}_{GO} and \mathbf{H}_{GO} , derive from the Geometrical Optics fields on S , and we can write,

$$\begin{aligned}\mathbf{H}_{GO} &\sim \mathcal{D}_H(\mathbf{u}_0, \mathcal{F}_i(\mathbf{u}_0); \mathbf{R}') \psi(\tilde{R}_i), \mathbf{E}_{GO} \sim \mathcal{D}_E(\mathbf{u}_0, \mathcal{F}_i(\mathbf{u}_0); \mathbf{R}') \psi(\tilde{R}_i), \\ \mathbf{R}_0 &= \mathbf{R}' - \mathbf{R}_i = \tilde{R}_0 \mathbf{u}_0, \tilde{R}_0 = \sqrt{\mathbf{R}_0 \cdot \mathbf{R}_0}, \mathbf{R}_i = \sqrt{\mathbf{R}_i \cdot \mathbf{R}_i} \mathbf{u}_i, \\ \mathbf{E}_i &= \mathbf{u}_0 \times \mathcal{F}_i(\mathbf{u}_0) \psi(\tilde{R}_0), Z_0 \mathbf{H}_i = \mathbf{u}_0 \times \mathbf{E}_i,\end{aligned}\quad (199)$$

where $\mathbf{R}_i = (\mathbf{a}_i + z_i \hat{\mathbf{z}}) \notin S$ is the origin of phase for \mathbf{E}_i and \mathbf{H}_i (see (62) for an example of \mathbf{E}_i and \mathbf{H}_i). We take the example of a curved plate S defined by,

$$\mathbf{R}' = \mathbf{r}' + \zeta_s(\mathbf{r}') \hat{\mathbf{z}}, \zeta_s(\mathbf{r}') = C_1 x'^2 + C_2 y'^2, \zeta_s(\mathbf{0}) = 0, \quad (200)$$

with $C_j < 0$ (resp. $C_j > 0$) for convex (resp. concave) surface inflection, where \mathbf{r}' is the current point on S' , the projection of S on the plane $z = 0$. We then have,

$$\begin{aligned}\mathbf{R} &= (\mathbf{a} + z\hat{\mathbf{z}}) - (\mathbf{r}' + \zeta_s(\mathbf{r}') \hat{\mathbf{z}}), \\ \mathbf{R}^2(r') &= (x' - a_x)^2 + (y' - a_y)^2 + ((C_1 x'^2 + C_2 y'^2) - z)^2,\end{aligned}\quad (201)$$

and we can express the fields as integrals on S' , following,

$$\begin{aligned}
\mathbf{E} &= \frac{Z_0}{4\pi ik} \int_{S'} (\tilde{R} (\frac{\psi'}{\tilde{R}})') \mathbf{u} \times (\mathbf{u} \times \mathbf{J}_p) - \frac{2\psi'}{\tilde{R}} \mathbf{J}_p dS_{r'} \\
&\quad - \frac{1}{4\pi} \int_{S'} \frac{\psi'}{\tilde{R}} \mathbf{R} \times \mathbf{M}_p dS_{r'}, \\
Z_0 \mathbf{H} &= \frac{1}{4\pi ik} \int_{S'} (\tilde{R} (\frac{\psi'}{\tilde{R}})') \mathbf{u} \times (\mathbf{u} \times \mathbf{M}_p) - \frac{2\psi'}{\tilde{R}} \mathbf{M}_p dS_{r'} \\
&\quad + \frac{Z_0}{4\pi} \int_{S'} \frac{\psi'}{\tilde{R}} \mathbf{R} \times \mathbf{J}_p dS_{r'}, \\
\mathbf{J}_p &\sim \hat{\mathbf{n}} J^s(\mathbf{r}') \times \mathbf{H}_{GO}, \quad \mathbf{M}_p \sim -\hat{\mathbf{n}} J^s(\mathbf{r}') \times \mathbf{E}_{GO},
\end{aligned} \tag{202}$$

where

$$\begin{aligned}
J^s(\mathbf{r}') &= \left| \frac{\partial \mathbf{R}'}{\partial x'} \wedge \frac{\partial \mathbf{R}'}{\partial y'} \right|, \quad \hat{\mathbf{n}} = \left(\frac{\partial \mathbf{R}'}{\partial x'} \wedge \frac{\partial \mathbf{R}'}{\partial y'} \right) / \left| \frac{\partial \mathbf{R}'}{\partial x'} \wedge \frac{\partial \mathbf{R}'}{\partial y'} \right|, \\
\hat{\mathbf{n}} J^s(\mathbf{r}') &= -2C_1 x' \hat{\mathbf{x}} - 2C_2 y' \hat{\mathbf{y}} + \hat{\mathbf{z}}.
\end{aligned} \tag{203}$$

Considering large $k\mathbf{R}$, we can neglect high order of $1/\tilde{R}$ in the integral, following,

$$\begin{aligned}
\mathbf{E} &= \frac{ikZ_0}{4\pi(\tilde{R})^3} \int_{S'} e^{-ik\tilde{R}} \mathbf{R} \times (\mathbf{R} \times \mathbf{J}_p) dS + \frac{ik}{4\pi(\tilde{R})^2} \int_{S'} e^{-ik\tilde{R}} \mathbf{R} \times \mathbf{M}_p dS, \\
Z_0 \mathbf{H} &= \frac{-ikZ_0}{4\pi(\tilde{R})^2} \int_{S'} e^{-ik\tilde{R}} \mathbf{R} \times \mathbf{J}_p dS + \frac{ik}{4\pi(\tilde{R})^3} \int_{S'} e^{-ik\tilde{R}} \mathbf{R} \times (\mathbf{R} \times \mathbf{M}_p) dS,
\end{aligned} \tag{204}$$

where,

$$\begin{aligned}
\mathbf{J}_p &\sim \hat{\mathbf{n}} J^s \times \mathbf{H}_{GO} \sim \hat{\mathbf{n}} J^s \times \frac{e^{-ik\tilde{R}_0}}{\tilde{R}_0} \mathcal{D}_H(\mathbf{u}_0, \mathcal{F}_i(\mathbf{u}_0); \mathbf{R}'), \\
\mathbf{M}_p &\sim -\hat{\mathbf{n}} J^s(\mathbf{r}') \times \mathbf{E}_{GO} \sim -\hat{\mathbf{n}} J^s \times \frac{e^{-ik\tilde{R}_0}}{\tilde{R}_0} \mathcal{D}_E(\mathbf{u}_0, \mathcal{F}_i(\mathbf{u}_0); \mathbf{R}'),
\end{aligned} \tag{205}$$

with $\hat{\mathbf{n}} J^s(\mathbf{r}') = -2C_1 x' \hat{\mathbf{x}} - 2C_2 y' \hat{\mathbf{y}} + \hat{\mathbf{z}}$.

B.2) An asymptotic development of \tilde{R}

Let us consider the asymptotics of $\tilde{R} = \sqrt{\mathbf{R}^2}$ when $\mathbf{R} = \mathbf{R}_1 - \mathbf{R}_2$ as $|\mathbf{R}_1| \gg |\mathbf{R}_2|$. For that, we write,

$$\begin{aligned}
\mathbf{R}^2 &= (\mathbf{R}_1 - \mathbf{R}_2)^2 = (\sqrt{\mathbf{R}_1^2} \mathbf{u}_1 - \mathbf{R}_2)^2 \\
&= (\sqrt{\mathbf{R}_1^2} - \mathbf{R}_2 \cdot \mathbf{u}_1)^2 + \mathbf{R}_2^2 - (\mathbf{R}_2 \cdot \mathbf{u}_1)^2,
\end{aligned} \tag{206}$$

which leads us to,

$$\begin{aligned}\sqrt{\mathbf{R}^2} &= (\sqrt{\mathbf{R}_1^2} - \mathbf{R}_2 \cdot \mathbf{u}_1) + \frac{\mathbf{R}_2^2 - (\mathbf{R}_2 \cdot \mathbf{u}_1)^2}{\sqrt{\mathbf{R}^2} + (\sqrt{\mathbf{R}_1^2} - \mathbf{R}_2 \cdot \mathbf{u}_1)}, \\ \frac{\sqrt{\mathbf{R}^2}}{\sqrt{\mathbf{R}_1^2}} &= 1 - \frac{\mathbf{R}_2 \cdot \mathbf{u}_1}{\sqrt{\mathbf{R}_1^2}} + \frac{\mathbf{R}_2^2 - (\mathbf{R}_2 \cdot \mathbf{u}_1)^2}{2\mathbf{R}_1^2(1 + \frac{1}{2}((\frac{\sqrt{\mathbf{R}^2}}{\sqrt{\mathbf{R}_1^2}} - 1) - \frac{\mathbf{R}_2 \cdot \mathbf{u}_1}{\sqrt{\mathbf{R}_1^2}}))}.\end{aligned}\quad (207)$$

Letting $\delta_{as} = \frac{1}{2}((\frac{\sqrt{\mathbf{R}^2}}{\sqrt{\mathbf{R}_1^2}} - 1) - \frac{\mathbf{R}_2 \cdot \mathbf{u}_1}{\sqrt{\mathbf{R}_1^2}})$, we can then write,

$$\begin{aligned}\frac{\sqrt{\mathbf{R}^2}}{\sqrt{\mathbf{R}_1^2}} &= 1 - \frac{\mathbf{R}_2 \cdot \mathbf{u}_1}{\sqrt{\mathbf{R}_1^2}} + \frac{\mathbf{R}_2^2 - (\mathbf{R}_2 \cdot \mathbf{u}_1)^2}{2\mathbf{R}_1^2} \\ &+ \frac{\mathbf{R}_2^2 - (\mathbf{R}_2 \cdot \mathbf{u}_1)^2}{2\mathbf{R}_1^2} \left(\sum_{n=1}^N (-\delta_{as})^n + \frac{(-\delta_{as})^{N+1}}{1 + \delta_{as}} \right),\end{aligned}\quad (208)$$

which gives us a complete asymptotics of $\sqrt{\mathbf{R}^2}$ for $|\mathbf{R}_1| \gg |\mathbf{R}_2|$ at any order N with explicit remaining term.

B.3) Quadratic approximation of exponential arguments and reduction of surface integrals to contour ones

Let us consider quadratic exponential arguments approximations in (204) from (208), following,

$$\begin{aligned}\tilde{R}_0 &\sim \sqrt{\mathbf{R}_i^2} - \mathbf{R}' \cdot \mathbf{u}_i + \frac{\mathbf{R}'^2 - (\mathbf{R}' \cdot \mathbf{u}_i)^2}{2\sqrt{\mathbf{R}_i^2}}, \\ \tilde{R} &\sim \sqrt{\mathbf{R}_1^2} - \mathbf{R}' \cdot \mathbf{u}_1 + \frac{\mathbf{R}'^2 - (\mathbf{R}' \cdot \mathbf{u}_1)^2}{2\sqrt{\mathbf{R}_1^2}},\end{aligned}\quad (209)$$

as $\sqrt{\mathbf{R}_i^2}$ and $\sqrt{\mathbf{R}_1^2}$ are large relatively to $|\mathbf{R}'| \sqrt{k|\mathbf{R}'|}$, so that higher order terms can be neglected. We then obtain (204) as combinations of elementary integrals of the form,

$$\int_{S'} g(\mathbf{r}') f(\mathcal{R}^2) dS', \quad (210)$$

where f is the exponential function, while \mathcal{R}^2 and $g(\mathbf{r}')$ are the polynomial functions,

$$\begin{aligned}\mathcal{R}^2 &= (\alpha_1 x' + \alpha_2 y') - ik(h_1 x'^2 + h_2 y'^2 + h_3 x' y') \\ &= v(r'_p - a_p)^2 + v_1, \\ g(\mathbf{r}') &= \sum_{j=0}^p \frac{1}{j!} \sum_{k=0}^j \binom{j}{k} \frac{\partial^j g|_{r'=\mathcal{P}^{-1}(a_p)}}{\partial x'^k \partial y'^{j-k}} [(\mathcal{P}^{-1}(\hat{x}) \cdot (r'_p - a_p))^k \\ &\quad (\mathcal{P}^{-1}(\hat{y}) \cdot (r'_p - a_p))^{j-k}],\end{aligned}\quad (211)$$

with $r'_p = \mathcal{P}(r')$ the complex scaling considered in (135) and (136). We can then reduce the surface integrals to contour integrals, using the theorems 12 and 13.

Remark : we can also write,

$$\begin{aligned}\mathbf{R}^2(r') &= (x' - a_x)^2 + (y' - a_y)^2 + ((C_1 x'^2 + C_2 y'^2) - z)^2 \\ &= (1 - 2zC_1)(x' - a'_x)^2 + (1 - 2zC_2)(y' - a'_y)^2 + z'^2 + \epsilon(\mathbf{r}'), \\ a'_x &= \frac{a_x}{1 - 2zC_1}, \quad a'_y = \frac{a_y}{1 - 2zC_2}, \quad z'^2 = z^2 - \left(\frac{2zC_1}{1 - 2zC_1} a_x^2 + \frac{2zC_2}{1 - 2zC_2} a_y^2 \right),\end{aligned}\quad (212)$$

with $\epsilon(\mathbf{r}') = (\zeta_s(\mathbf{r}'))^2$, $\zeta_s(\mathbf{r}') = C_1 x'^2 + C_2 y'^2$ (note : $\mathbf{R}^2 = (r'_p - a_p)^2 + z'^2 + \epsilon(\mathbf{r}')$ with $a_p = \mathcal{P}(a')$).

Appendix C : Reduction of Physical Optics radiation integrals in monostatic case for imperfectly conducting surface

We consider a point source illuminating at large distance (relatively to the wavelength) an isotropic impedance flat surface with normal $\hat{\mathbf{z}}$. We consider the incident field as locally plane, combination of TE (\mathbf{E} //surface) and TM (\mathbf{H} // surface) waves, respectively denoted $(\mathbf{E}_e^i, \mathbf{H}_e^i)$ and $(\mathbf{E}_h^i, \mathbf{H}_h^i)$. The currents \mathbf{J} and \mathbf{M} , respectively denoted $\mathbf{J}_{e(h)}|_S = \hat{\mathbf{z}} \times \mathbf{H}_{e(h)}$ and $\mathbf{M}_{e(h)}|_S = -\hat{\mathbf{z}} \times \mathbf{E}_{e(h)}$, then satisfy,

$$\mathbf{J}_{e(h)} = 2b'_{e(h)}(-\mathbf{u}_0 \cdot \hat{\mathbf{z}})[\hat{\mathbf{z}} \times \mathbf{H}_{e(h)}^i], \quad \mathbf{M}_{e(h)} = -c'(-\mathbf{u}_0 \cdot \hat{\mathbf{z}})[\hat{\mathbf{z}} \times \mathbf{J}_{e(h)}], \quad (213)$$

where c' and $b'_{e,h}$ are regular functions, depending on the angle of the incidence vector \mathbf{u}_0 with $\hat{\mathbf{z}}$. In the case of constant impedance, the terms $b'_{e(h)}$ and c' are constants [29]-[30], which are $b'_{e(h)} \equiv 1$ and $c' \equiv 0$ in perfectly conducting case.

C.1) Physical Optics diffracted fields and reduction, for an electric dipolar source

Theorem 17

The Physical Optics field, diffracted by the surface S' with an impedance condition, illuminated by the electric dipole $\mathbf{J}_0 = \hat{\mathbf{d}}\delta(\mathbf{r}' - \mathbf{r}_0)$ is given by (47) with (213). It can be written, in monostatic case, when $\hat{\mathbf{d}} \cdot \hat{\mathbf{z}} = 0$, following,

$$\begin{aligned} \mathbf{E} \sim & \frac{Z_0}{8\pi^2 ik} \int_{S'} \frac{z}{2} \tilde{R} \left(\left(\frac{\psi'}{\tilde{R}} \right)^2 \right)' ((\mathbf{u} \cdot \mathbf{h}) \mathbf{u} - \mathbf{h}) dS' + \\ & + \frac{1}{8\pi^2} \int_{S'} c' \left(\frac{z}{\tilde{R}} \right) (\psi')^2 \left(\frac{\mathbf{R}^2 - z^2}{\mathbf{R}^2} \left(\frac{\mathbf{r}' - \mathbf{a}}{(\mathbf{r}' - \mathbf{a})^2} \times \hat{\mathbf{z}} \right) \left(((\mathbf{r}' - \mathbf{a}) \times \hat{\mathbf{z}}) \cdot \mathbf{h} \right) \right. \\ & \left. + ((\mathbf{u} \cdot \mathbf{h}) \mathbf{u} - \mathbf{h}) \right) dS', \end{aligned} \quad (214)$$

for large $k\tilde{R}$, where,

$$\begin{aligned} \mathbf{h} &= b'_h \left(\frac{z}{\tilde{R}} \right) (\hat{\mathbf{d}} \cdot \hat{\mathbf{d}}_1) \hat{\mathbf{d}}_1 + b'_e \left(\frac{z}{\tilde{R}} \right) (\hat{\mathbf{d}} \cdot \hat{\mathbf{d}}_2) \hat{\mathbf{d}}_2, \\ \hat{\mathbf{d}}_1 &= \frac{(\mathbf{a} - \mathbf{r}')}{\sqrt{(\mathbf{a} - \mathbf{r}')^2}}, \quad \hat{\mathbf{d}}_2 = \hat{\mathbf{z}} \times \hat{\mathbf{d}}_1. \\ (\mathbf{u} \cdot \mathbf{h}) \mathbf{u} &= \frac{z\hat{\mathbf{z}}(\mathbf{h} \cdot (\mathbf{a} - \mathbf{r}'))}{\mathbf{R}^2} + \frac{(\mathbf{R}^2 - z^2)(\mathbf{h} \cdot (\mathbf{a} - \mathbf{r}'))(\mathbf{a} - \mathbf{r}')}{\mathbf{R}^2 (\mathbf{a} - \mathbf{r}')^2}. \end{aligned}$$

Considering the propositions 13 and 14, we can use,

$$\int (\psi')^2 d\mathbf{R}^2 \sim -\frac{ike^{-i2k\tilde{R}}}{\tilde{R}}, \quad \int z\tilde{R} \left(\left(\frac{\psi'}{\tilde{R}} \right)^2 \right)' d\mathbf{R}^2 \sim -\frac{2k^2 ze^{-i2k\tilde{R}}}{\mathbf{R}^2}, \quad (215)$$

and reduce asymptotically each integral to a non-singular contour integral.

Proof. We consider, using (213), that S' is illuminated by the radiation of the electrical dipole $\mathbf{J}_0 = \hat{\mathbf{d}}\delta(\mathbf{r}' - \mathbf{r}_0)$ with $\hat{\mathbf{d}} \cdot \hat{\mathbf{z}} = 0$ ($\hat{\mathbf{d}} \parallel$ plane). We distinguish locally TE et TM waves in incident waves and apply $b_{e,h}$ coefficients. We can then write,

$$\begin{aligned} 2\pi\mathbf{J}|_S &\sim \left(\frac{z}{\tilde{R}} \right) \hat{\mathbf{z}} \times (\mathbf{R}_0 \times \frac{\psi'_0}{\tilde{R}_0} \mathbf{h}), \\ 2\pi\mathbf{M}|_S &\sim -c' \left(\frac{z}{\tilde{R}} \right) (\hat{\mathbf{z}} \times (\hat{\mathbf{z}} \times (\mathbf{R}_0 \times \frac{\psi'_0}{\tilde{R}_0} \mathbf{h}))) \\ &= -c' \left(\frac{z}{\tilde{R}} \right) ((\hat{\mathbf{z}} \cdot (\mathbf{R}_0 \times \frac{\psi'_0}{\tilde{R}_0} \mathbf{h})) \hat{\mathbf{z}} - (\mathbf{R}_0 \times \frac{\psi'_0}{\tilde{R}_0} \mathbf{h})), \\ \mathbf{h} &= b'_h \left(\frac{z}{\tilde{R}} \right) (\hat{\mathbf{d}} \cdot \hat{\mathbf{d}}_1) \hat{\mathbf{d}}_1 + b'_e \left(\frac{z}{\tilde{R}} \right) (\hat{\mathbf{d}} \cdot \hat{\mathbf{d}}_2) \hat{\mathbf{d}}_2, \quad \hat{\mathbf{d}}_1 = \frac{(\mathbf{a} - \mathbf{r}')}{\sqrt{(\mathbf{a} - \mathbf{r}')^2}}, \quad \hat{\mathbf{d}}_2 = \hat{\mathbf{z}} \times \hat{\mathbf{d}}_1, \end{aligned} \quad (216)$$

with $\mathbf{R}_0 = (\mathbf{r}' - \mathbf{r}_0)$ and $\mathbf{u}_0 = \frac{\mathbf{R}_0}{R_0}$. In the case where $\mathbf{R} = -\mathbf{R}_0 = z\hat{\mathbf{z}} + \mathbf{a} - \mathbf{r}'$ (monostatic

case), we then obtain,

$$\begin{aligned} \mathbf{E} &\sim \frac{Z_0}{8\pi^2 ik} \int_{S'} \frac{z}{2\tilde{R}} \left(\left(\frac{\psi'}{\tilde{R}} \right)^2 \right)' (\mathbf{u} \times (\mathbf{u} \times \mathbf{h})) dS' \\ &+ \frac{1}{8\pi^2} \int_{S'} a' \left(\frac{z}{\tilde{R}} \right) \left(\frac{\psi'}{\tilde{R}} \right)^2 (-\mathbf{R} \times \hat{\mathbf{z}} (\hat{\mathbf{z}} \cdot (\mathbf{R} \times \mathbf{h})) + \mathbf{R} \times (\mathbf{R} \times \mathbf{h})) dS', \end{aligned} \quad (217)$$

and thus (214). We can then, following the theorems 1 and 2, and the propositions 13 and 14, reduce the field \mathbf{E} to a contour integral. For this, we can use (184), and also,

$$\begin{aligned} \int (\psi')^2 d\mathbf{R}^2 &= \int \mathbf{R}^2 \left(\frac{\psi'}{\tilde{R}} \right)^2 d\mathbf{R}^2 = \frac{(\tilde{R} \psi^2)'}{2} - \int \frac{(\tilde{R} \psi^2)'}{2\mathbf{R}^2} d\mathbf{R}^2 \\ &= -\frac{(3 + 2ik\tilde{R})}{2\mathbf{R}^2} e^{-i2k\tilde{R}} - \int \frac{e^{-i2k\tilde{R}}}{2\mathbf{R}^4} d\mathbf{R}^2 = -\frac{ik}{\tilde{R}} e^{-i2k\tilde{R}} \left(1 + O\left(\frac{1}{k\tilde{R}}\right) \right). \end{aligned} \quad (218)$$

C.2) Physical Optics diffracted fields and reduction, for a magnetic dipolar source

Theorem 18

The Physical Optics field, diffracted by the surface S' with an impedance condition, illuminated by the magnetic dipole $\mathbf{M}_0 = \hat{\mathbf{d}}\delta(\mathbf{r}' - \mathbf{r}_0)$, is given by (47) with (213). It can be written, in monostatic case, when $\hat{\mathbf{d}} \cdot \hat{\mathbf{z}} = 0$, following

$$\begin{aligned} \mathbf{E} &\sim \frac{-z}{8(\pi ik)^2} \int_{S'} \mathbf{h} \times \mathbf{R} \left(\left(\frac{\psi'}{\tilde{R}} \right)' \right)^2 dS' \\ &- \frac{1}{8\pi^2 ik Z_0} \int_{S'} \hat{\mathbf{z}} \times (\mathbf{R} \times \hat{\mathbf{z}}) \frac{c'(\frac{z}{\tilde{R}})}{2} \tilde{R} \left(\left(\frac{\psi'}{\tilde{R}} \right)^2 \right)' \left(\frac{z}{\mathbf{R}^2} (\mathbf{R} \cdot \mathbf{h}) \right) dS' \\ &- \frac{1}{8\pi^2 ik Z_0} \int_{S'} \mathbf{h} \times \mathbf{R} \frac{c'(\frac{z}{\tilde{R}})}{2} \tilde{R} \left(\left(\frac{\psi'}{\tilde{R}} \right)^2 \right)' dS', \end{aligned} \quad (219)$$

for large $k\tilde{R}$, with

$$\begin{aligned} \mathbf{h} &= b'_e \left(\frac{z}{\tilde{R}} \right) (\hat{\mathbf{d}} \cdot \hat{\mathbf{d}}_1) \hat{\mathbf{d}}_1 + b'_h \left(\frac{z}{\tilde{R}} \right) (\hat{\mathbf{d}} \cdot \hat{\mathbf{d}}_2) \hat{\mathbf{d}}_2, \\ \hat{\mathbf{d}}_1 &= \frac{(\mathbf{a} - \mathbf{r}')}{\sqrt{(\mathbf{a} - \mathbf{r}')^2}}, \quad \hat{\mathbf{d}}_2 = \hat{\mathbf{z}} \times \hat{\mathbf{d}}_1, \\ (\mathbf{R} \times \hat{\mathbf{z}})(\mathbf{R} \cdot \mathbf{h}) &= (\mathbf{R}^2 - z^2) \frac{(\mathbf{h} \cdot (\mathbf{a} - \mathbf{r}'))(\mathbf{a} - \mathbf{r}') \times \hat{\mathbf{z}}}{(\mathbf{a} - \mathbf{r}')^2}, \end{aligned}$$

Considering the propositions 13 and 14, we can use,

$$\begin{aligned} \int \left(\frac{\psi'}{\tilde{R}}\right)' \left(\frac{\psi'}{\tilde{R}}\right)' d\mathbf{R}^2 &\sim k^2 ik \frac{e^{-i2k\tilde{R}}}{\tilde{R}^3}, \quad \int \tilde{R} \left(\left(\frac{\psi'}{\tilde{R}}\right)^2\right)' d\mathbf{R}^2 \sim -\frac{2k^2 e^{-i2k\tilde{R}}}{\mathbf{R}^2}, \\ \int \frac{1}{\tilde{R}} \left(\left(\frac{\psi'}{\tilde{R}}\right)^2\right)' d\mathbf{R}^2 &\sim -\frac{2k^2 e^{-i2k\tilde{R}}}{\mathbf{R}^4}, \end{aligned} \quad (220)$$

and reduce asymptotically each surface integral to a non-singular contour integral.

Proof. We consider, using (213), that S' is illuminated by the radiation of the magnetic dipole $\mathbf{M}_0 = \hat{\mathbf{d}}\delta(\mathbf{r}' - \mathbf{r}_0)$ with $\hat{\mathbf{d}} \cdot \hat{\mathbf{z}} = 0$ ($\hat{\mathbf{d}}//\text{plane}$), and thus

$$\begin{aligned} \mathbf{J}|_S &= \frac{b'(\frac{z}{\tilde{R}})}{2\pi ik Z_0} \hat{\mathbf{z}} \times (\tilde{R}_0 \left(\frac{\psi'}{\tilde{R}_0}\right)' \mathbf{u}_0 \times (\mathbf{u}_0 \times \mathbf{h}) - \frac{2\psi'}{\tilde{R}} \mathbf{h}), \quad \mathbf{R}_0 = (\mathbf{r}' - \mathbf{r}_0), \\ \mathbf{h} &= b'_e \left(\frac{z}{\tilde{R}}\right) (\hat{\mathbf{d}} \cdot \hat{\mathbf{d}}_1) \hat{\mathbf{d}}_1 + b'_h \left(\frac{z}{\tilde{R}}\right) (\hat{\mathbf{d}} \cdot \hat{\mathbf{d}}_2) \hat{\mathbf{d}}_2, \quad \hat{\mathbf{d}}_1 = \frac{(\mathbf{a} - \mathbf{r}')}{\sqrt{(\mathbf{a} - \mathbf{r}')^2}}, \quad \hat{\mathbf{d}}_2 = \hat{\mathbf{z}} \times \hat{\mathbf{d}}_1, \\ \mathbf{M}|_S &= \frac{-c'(\frac{z}{\tilde{R}})}{2\pi ik Z_0} \hat{\mathbf{z}} \times (\hat{\mathbf{z}} \times (\tilde{R}_0 \left(\frac{\psi'}{\tilde{R}_0}\right)' \mathbf{u}_0 \times (\mathbf{u}_0 \times \mathbf{h}) - \frac{2\psi'}{\tilde{R}} \mathbf{h})), \end{aligned} \quad (221)$$

where $\mathbf{R}_0 = (\mathbf{r}' - \mathbf{r}_0)$, $\mathbf{u}_0 = \frac{\mathbf{R}_0}{R_0}$. In the case where $\mathbf{R} = -\mathbf{R}_0 = z\hat{\mathbf{z}} + \mathbf{a} - \mathbf{r}'$ (monostatic case), we then obtain,

$$\begin{aligned} \mathbf{E} &= \frac{1}{8(\pi ik)^2} \int_{S'} z \left(\frac{\psi'}{\tilde{R}}\right)' \left(\frac{\psi'}{\tilde{R}}\right)' (\mathbf{R} \times \mathbf{h}) dS' \\ &\quad - \frac{1}{4(\pi ik)^2} \int_{S'} \hat{\mathbf{z}} \times (\tilde{R} \left(\left(\frac{\psi'}{\tilde{R}}\right)^2\right)' (\mathbf{u}(\mathbf{u} \cdot \mathbf{h}) - \mathbf{h}) - 2\left(\frac{\psi'}{\tilde{R}}\right)^2 \mathbf{h}) dS' \\ &\quad + \frac{1}{8\pi^2 ik Z_0} \int_{S'} c' \left(\frac{z}{\tilde{R}}\right) \frac{\psi'}{\tilde{R}} \mathbf{R} \times (\hat{\mathbf{z}} \times (\hat{\mathbf{z}} \times (\tilde{R} \left(\frac{\psi'}{\tilde{R}}\right)' \mathbf{u} \times (\mathbf{u} \times \mathbf{h}) \\ &\quad - \frac{2\psi'}{\tilde{R}} \mathbf{h}))) dS' \sim \frac{-z}{8(\pi ik)^2} \int_{S'} \left(\left(\frac{\psi'}{\tilde{R}}\right)'\right)^2 \mathbf{h} \times \mathbf{R} dS' \\ &\quad + \frac{1}{8\pi^2 ik Z_0} \int c' \left(\frac{z}{\tilde{R}}\right) \tilde{R} \frac{\psi'}{\tilde{R}} (\mathbf{R} \times \hat{\mathbf{z}}) (\hat{\mathbf{z}} \cdot (\mathbf{u} \times (\mathbf{u} \times \mathbf{h}))) \left(\frac{\psi'}{\tilde{R}}\right)' dS' \\ &\quad - \frac{1}{8\pi^2 ik Z_0} \int c' \left(\frac{z}{\tilde{R}}\right) \tilde{R} \frac{\psi'}{\tilde{R}} \mathbf{R} \times \left(\left(\frac{\psi'}{\tilde{R}}\right)' \mathbf{u} \times (\mathbf{u} \times \mathbf{h})\right) dS', \end{aligned} \quad (222)$$

and thus (219). We can then, following the theorems 1 and 2, and the propositions 13 and 14, reduce the field \mathbf{E} to a contour integral. For this, we use (184), (191), and

$$(\mathbf{R} \times \hat{\mathbf{z}}) \tilde{R} \frac{(\mathbf{R} \cdot \mathbf{h})}{\mathbf{R}^2} = \frac{\mathbf{R}^2 - z^2}{\tilde{R}} \frac{(\mathbf{h} \cdot (\mathbf{a} - \mathbf{r}')) ((\mathbf{a} - \mathbf{r}') \times \hat{\mathbf{z}})}{(\mathbf{a} - \mathbf{r}')^2}. \quad (223)$$

Appendix D : Particular uses of asymptotics with theorems 1 and 2

The reduction to a contour integral generally supposes the knowledge of a primitive, as in theorems 1 and 2. When it cannot be expressed in closed form, its expression can often be derived asymptotically.

Lemma 2

Let $F_1((\mathbf{r}' - \mathbf{a})^2, \varphi) = \frac{e^{-v\tilde{R}}}{\tilde{R}^n}$ be analytic on the surface S' of complex points $\mathbf{r}' = x'\hat{\mathbf{x}} + y'\hat{\mathbf{y}}$ of the plane $z = 0$, with $\tilde{R} = \sqrt{\mathbf{R} \cdot \mathbf{R}} = \sqrt{(\mathbf{r}' - \mathbf{a})^2 + z^2}$, constants n and v with $\text{Re}(v) > 0$, $n \geq 1$, and,

$$\frac{\partial(F((\mathbf{r}' - \mathbf{a})^2, \varphi))}{\partial((\mathbf{r}' - \mathbf{a})^2)} \equiv g_0((\mathbf{r}' - \mathbf{a})^2, \varphi) \frac{\partial(F_1((\mathbf{r}' - \mathbf{a})^2, \varphi))}{\partial((\mathbf{r}' - \mathbf{a})^2)}, \quad (224)$$

which is $O(e^{-v\tilde{R}})$ as we let, from any definite point on S' , the parameter $w = (\mathbf{r}' - \mathbf{a})^2$ varies from 0 to ∞ at φ constant. Considering $g_0(w, \varphi)$ and $\frac{\partial g_0(w, \varphi)}{\partial(w)}$ as analytic function, relatively to $x' - a_x$ and $y' - a_y$ variables, bounded and smooth as w and v vary, we can write, for large $v\tilde{R}$,

$$\begin{aligned} & \int_{\infty}^{(\mathbf{r}' - \mathbf{a})^2} \frac{\partial(F(w, \varphi))}{\partial(w)} dw = \\ & = g_0((\mathbf{r}' - \mathbf{a})^2, \varphi) F_1((\mathbf{r}' - \mathbf{a})^2, \varphi) - \int_{\infty}^{(\mathbf{r}' - \mathbf{a})^2} \frac{\partial g_0(w, \varphi)}{\partial(w)} F_1(w, \varphi) dw \\ & = F_1((\mathbf{r}' - \mathbf{a})^2, \varphi) (g_0((\mathbf{r}' - \mathbf{a})^2, \varphi) + \frac{2\tilde{R}}{v} \frac{\partial g_0((\mathbf{r}' - \mathbf{a})^2, \varphi)}{\partial((\mathbf{r}' - \mathbf{a})^2)} + o(1/v)). \end{aligned} \quad (225)$$

More generally, with the assumption that $\frac{\partial g_0(w, \varphi)}{\partial(\sqrt{w})}$ is bounded and smooth when w varies from 0 to ∞ , we can write for large vz ,

$$\int_{\infty}^{(\mathbf{r}' - \mathbf{a})^2} \frac{\partial(F(w, \varphi))}{\partial(w)} dw = F_1((\mathbf{r}' - \mathbf{a})^2, \varphi) (g_0((\mathbf{r}' - \mathbf{a})^2, \varphi) + O(\frac{1}{z^{n-1}\sqrt{vz}})). \quad (226)$$

which can be used in combination with the theorems 1 and 2.

Proof. The function $F_1((\mathbf{r}' - \mathbf{a})^2, \varphi) = \frac{e^{-v\tilde{R}}}{\tilde{R}^n}$ is analytic on S' . We note that

$$\int_{\infty}^{(\mathbf{r}' - \mathbf{a})^2} \frac{2e^{-v\tilde{R}}}{\tilde{R}^{n-1}} (\tilde{R})' d(\mathbf{r}' - \mathbf{a})^2 = \int_{\infty}^{(\mathbf{r}' - \mathbf{a})^2} \frac{2e^{-v\tilde{R}}}{\tilde{R}^{n-1}} d\tilde{R} \sim -\frac{2\tilde{R}}{v} F_1((\mathbf{r}' - \mathbf{a})^2), \quad (227)$$

with $(\tilde{R})' = \partial_{(\mathbf{r}' - \mathbf{a})^2}(\tilde{R}) = 1/2\tilde{R}$, and considering that the branch point of \tilde{R} is not on S' , we can derive (225) from the use of integration by parts. More generally, if $g_1(\sqrt{w}) = \frac{\partial g_0(w, \varphi)}{\partial(\sqrt{w})}$ remains smooth when w varies from 0 to ∞ , we note that, for large vz ,

$$\begin{aligned} \int_{\infty}^{(\mathbf{r}' - \mathbf{a})^2} \frac{\partial g_0(w, \varphi)}{\partial(w)} F_1(w, \varphi) dw &= \int_{\infty}^{(\mathbf{r}' - \mathbf{a})^2} \frac{2\sqrt{w} \partial g_0(w, \varphi)}{\partial(w)} \frac{e^{-v\tilde{R}}}{\tilde{R}^n} d\sqrt{w} \\ &= \int_{\infty}^{\sqrt{(\mathbf{r}' - \mathbf{a})^2}} \frac{g_1(u) e^{-v\sqrt{u^2 + z^2}}}{\sqrt{u^2 + z^2}^n} du = O\left(\int_{\infty}^0 \frac{g_1(u) e^{-v\sqrt{u^2 + z^2}}}{\sqrt{u^2 + z^2}^n} du\right) = O\left(\frac{1/z^{n-1}}{\sqrt{vz}}\right), \end{aligned} \quad (228)$$

which implies (226).

D.1) A particular use of lemma 2 with theorem 1

Proposition 13

Let $F_1((\mathbf{r}' - \mathbf{a})^2, \varphi) = \frac{e^{-v\tilde{R}}}{\tilde{R}^n}$ on a flat real plate S' , with $\tilde{R} = \sqrt{\mathbf{R} \cdot \mathbf{R}} = \sqrt{(\mathbf{r}' - \mathbf{a})^2 + z^2}$.

For large $v\tilde{R}$ on S' , and constants $n, v, z_1 \neq 0$, we can write,

$$\begin{aligned} &\int_{S'} h_0\left(\frac{z_1}{\tilde{R}}, \varphi\right) \frac{\partial(F_1((\mathbf{r}' - \mathbf{a})^2))}{\partial((\mathbf{r}' - \mathbf{a})^2)} dS' \\ &\sim \frac{1}{2} \int_{\mathcal{L}} \left(h_0\left(\frac{z_1}{\tilde{R}}, \varphi\right) F_1((\mathbf{r}' - \mathbf{a})^2) \left(1 - \frac{1}{v\tilde{R}} \left(\frac{z_1}{\tilde{R}} \frac{h'_0}{h_0}\left(\frac{z_1}{\tilde{R}}, \varphi\right)\right)\right) \right. \\ &\quad \left. - F_1(0) \left(h_0\left(\frac{z_1}{\tilde{R}}, \varphi\right) - \frac{z_1}{z} \frac{h'_0\left(\frac{z_1}{\tilde{R}}, \varphi\right)}{v\tilde{R}}\right) \Big|_{(\mathbf{r}' - \mathbf{a})^2=0} \right) \left(\frac{\hat{\mathbf{z}} \times (\mathbf{r}' - \mathbf{a})}{(\mathbf{r}' - \mathbf{a})^2}\right) \cdot d\mathbf{r}', \end{aligned} \quad (229)$$

when $\frac{\partial(F_1((\mathbf{r}' - \mathbf{a})^2, \varphi))}{\partial((\mathbf{r}' - \mathbf{a})^2)} \equiv h_0\left(\frac{z_1}{\tilde{R}}, \varphi\right) \frac{\partial(F_1((\mathbf{r}' - \mathbf{a})^2, \varphi))}{\partial((\mathbf{r}' - \mathbf{a})^2)}$ and $h'_0(\tau, \varphi) = \partial_{\tau} h_0(\tau, \varphi)$ is smooth, with $\tau = \frac{z_1}{\tilde{R}}$. Using $\partial_{(\mathbf{r}' - \mathbf{a})^2}(\mathbf{R}^2) = 2\tilde{R} (\tilde{R})' = 1$ and $\frac{n}{v\tilde{R}} = o(1)$, we then have,

$$\begin{aligned} &\int_{S'} h_0\left(\frac{z_1}{\tilde{R}}, \varphi\right) \frac{e^{-v\tilde{R}}}{\tilde{R}^{n+1}} dS' \\ &\sim \frac{-1}{v} \int_{\mathcal{L}} \left(h_0\left(\frac{z_1}{\tilde{R}}, \varphi\right) \frac{e^{-v\tilde{R}}}{\tilde{R}^n} - h_0\left(\frac{z_1}{z}, \varphi\right) \frac{e^{-vz}}{z^n} \right) \left(\frac{\hat{\mathbf{z}} \times (\mathbf{r}' - \mathbf{a})}{(\mathbf{r}' - \mathbf{a})^2}\right) \cdot d\mathbf{r}'. \end{aligned} \quad (230)$$

(note : this expression remains valid when $\sqrt{(\mathbf{r}' - \mathbf{a})^2} h'_0\left(\frac{z_1}{\tilde{R}}, \varphi\right)$ has a smooth behaviour).

Proof. Let us consider the theorem 1 with

$$\frac{\partial(F((\mathbf{r}' - \mathbf{a})^2, \varphi))}{\partial((\mathbf{r}' - \mathbf{a})^2)} = h_0\left(\frac{z_1}{\tilde{R}}, \varphi\right) \frac{\partial(F_1((\mathbf{r}' - \mathbf{a})^2, \varphi))}{\partial((\mathbf{r}' - \mathbf{a})^2)}, \quad (231)$$

when $\frac{z_1}{\tilde{R}}$ and $h'(\frac{z_1}{\tilde{R}}, \varphi)$ are regular and bounded. In this case, we can generally obtained an asymptotic expression of F if F_1 has steep variation for large \tilde{R} . For this, letting $(\mathbf{r}' - \mathbf{a})^2 = w$ in the expressions, we consider the lemma 2, writing,

$$\begin{aligned} & \int_{\infty}^{(\mathbf{r}' - \mathbf{a})^2} \frac{\partial(F(w, \varphi))}{\partial(\tau)} dw = \\ & = h_0\left(\frac{z_1}{\tilde{R}}, \varphi\right) F_1((\mathbf{r}' - \mathbf{a})^2, \varphi) + \int_{\infty}^{(\mathbf{r}' - \mathbf{a})^2} \frac{F_1(w, \varphi)(\tilde{R})'}{\tilde{R}} \frac{z_1 h'_0\left(\frac{z_1}{\tilde{R}}, \varphi\right)}{\tilde{R}} dw \\ & \sim h_0\left(\frac{z_1}{\tilde{R}}, \varphi\right) F_1((\mathbf{r}' - \mathbf{a})^2, \varphi) + \frac{z_1 h'_0\left(\frac{z_1}{\tilde{R}}, \varphi\right)}{\tilde{R}^2} \int_{\infty}^{(\mathbf{r}' - \mathbf{a})^2} F_1(w, \varphi)(\tilde{R})' dw, \end{aligned} \quad (232)$$

with $h'_0(\tau, \varphi) = \partial_{\tau} h_0(\tau, \varphi)$, $(\tilde{R})' = \partial_{(\mathbf{r}' - \mathbf{a})^2}(\tilde{R})$. We then obtain (229) from theorem 1 when the integrals remain convergent, then (230). More generally, from lemma 2, (230) can be also applied when $\sqrt{(\mathbf{r}' - \mathbf{a})^2} h'_0(\frac{z_1}{\tilde{R}}, \varphi)$ has a smooth behaviour.

D.2) A particular use of lemma 2 with theorem 2

Proposition 14

Let $w = (\mathbf{r}' - \mathbf{a})^2$, $\mathbf{R}^2 = w + z^2$, and $F(w, \varphi) = F_0(w) h_1\left(\frac{w^q}{\mathbf{R}^{2q}} G(\varphi)\right)$ be analytic relatively to $(w)^{\frac{1}{2}} \cos \varphi = (x' - a_x)$ and $(w)^{\frac{1}{2}} \sin \varphi = (y' - a_y)$ on a flat real plate S' . Considering $F_0(w) = \frac{e^{-v\tilde{R}}}{\tilde{R}^{n-1}}$ with $\tilde{R} = \sqrt{\mathbf{R}^2}$, we have,

$$\begin{aligned} & \int_{S'} \mathbf{b} \cdot (\mathbf{r}' - \mathbf{a}) h_1\left(\frac{(\mathbf{r}' - \mathbf{a})^{2q}}{\mathbf{R}^{2q}} G(\varphi)\right) \frac{\partial F_0((\mathbf{r}' - \mathbf{a})^2)}{\partial((\mathbf{r}' - \mathbf{a})^2)} dS' \\ & \sim \frac{1}{2} \int_{\mathcal{L}} F((\mathbf{r}' - \mathbf{a})^2, \varphi) (\hat{\mathbf{z}} \times \mathbf{b}) \cdot d\mathbf{r}', \end{aligned} \quad (233)$$

for large $v\tilde{R}$ on S' , with $v, \mathbf{b}, z \neq 0$ some constants, and $n, q \geq 1$. Generalizing this result to a vector function \mathbf{h}_v with $F_v(w, \varphi) = F_0(w) \mathbf{h}_v\left(\frac{w^q}{\mathbf{R}^{2q}} G(\varphi)\right)$, we have,

$$\begin{aligned} & \int_{S'} (\mathbf{r}' - \mathbf{a}) \cdot \mathbf{h}_v\left(\frac{(\mathbf{r}' - \mathbf{a})^{2q}}{\mathbf{R}^{2q}} G(\varphi)\right) \frac{\partial F_0((\mathbf{r}' - \mathbf{a})^2)}{\partial((\mathbf{r}' - \mathbf{a})^2)} dS' \\ & \sim \frac{1}{2} \int_{\mathcal{L}} (\hat{\mathbf{z}} \times F_v((\mathbf{r}' - \mathbf{a})^2, \varphi)) \cdot d\mathbf{r}'. \end{aligned} \quad (234)$$

where $\frac{\partial F_0((\mathbf{r}'-\mathbf{a})^2)}{\partial((\mathbf{r}'-\mathbf{a})^2)} \sim (\frac{-v\tilde{R}F_0((\mathbf{r}'-\mathbf{a})^2)}{2\mathbf{R}^2})$. Let us note that these results remain valid for not constant z as $\text{grad}_{\mathbf{r}'}(v\mathbf{R}^2) = 2v(\mathbf{r}' - \mathbf{a}) + o(1)$ for large $v\tilde{R}$ on S' .

Proof. Using $\frac{\partial}{\partial((\mathbf{r}'-\mathbf{a})^2)}(F_0((\mathbf{r}' - \mathbf{a})^2)) = (\frac{-v\tilde{R}F_0((\mathbf{r}'-\mathbf{a})^2)}{2\mathbf{R}^2})(1 + \frac{n}{v\tilde{R}})$, and

$$\begin{aligned} \frac{\partial(h_1(\frac{(\mathbf{r}'-\mathbf{a})^{2q}}{\mathbf{R}^{2q}}G(\varphi)))}{\partial((\mathbf{r}' - \mathbf{a})^2)} &= \frac{q}{\mathbf{R}^2} \left(\frac{z^2}{\mathbf{R}^2} \frac{(\mathbf{r}' - \mathbf{a})^{2q-2}}{\mathbf{R}^{2q-2}} G(\varphi) h_1'(\frac{(\mathbf{r}' - \mathbf{a})^{2q}}{\mathbf{R}^{2q}} G(\varphi)) \right), \\ \frac{\partial(h_1(\frac{(\mathbf{r}'-\mathbf{a})^{2q}}{\mathbf{R}^{2q}}G(\varphi)))}{(\mathbf{r}' - \mathbf{a})^2 \partial\varphi} &= \frac{1}{\mathbf{R}^2} \left(\frac{(\mathbf{r}' - \mathbf{a})^{2q-2}}{\mathbf{R}^{2q-2}} G'(\varphi) h_1'(\frac{(\mathbf{r}' - \mathbf{a})^{2q}}{\mathbf{R}^{2q}} G(\varphi)) \right), \end{aligned} \quad (235)$$

we have, for large $v\tilde{R}$,

$$h(\frac{(\mathbf{r}' - \mathbf{a})^{2q}}{\mathbf{R}^{2q}} G(\varphi)) \text{grad}_{\mathbf{r}'}(F_0((\mathbf{r}' - \mathbf{a})^2)) \sim \text{grad}_{\mathbf{r}'}(F((\mathbf{r}' - \mathbf{a})^2, \varphi)). \quad (236)$$

We can then derive (233) from theorem 2, and use it for each component of \mathbf{h}_v to derive (234).

Appendix E : Complements on the use of proposition 2 when $s(\mathbf{r}') = g(r')f(\mathcal{R}^2)$, as $f(\mathcal{R}^2) = e^{\mathcal{R}^2}$ and $g(\mathbf{r}')$ is a smooth analytical function on S' .

Let us note \mathcal{R}^2 a polynomial of order 2 as in (129), with $h'_p{}^2 = (\mathbf{r}'_p - \mathbf{a}_p)^2$, $\mathbf{r}'_p = \mathcal{P}(\mathbf{r}')$,

$$\mathcal{R}^2 = d_0 + (\alpha_1 x' + \alpha_2 y') - ik(h_1 x'^2 + h_2 y'^2 + h_3 x' y') = d_0 + v h'_p{}^2 + v_1, \quad (237)$$

We detail here the reduction of $\int_{S'} g(r') f(\mathcal{R}^2) dS'$, when $f(\mathcal{R}^2) = e^{\mathcal{R}^2}$ has large variations on S' at constant φ_p , from theorems 1 and 2 in combination with proposition 2. We give here some complements on evaluation of primitives using in parts asymptotics for this purpose.

Proposition 15

We consider the proposition 2 with

$$\frac{\partial F((\mathbf{r}' - \mathbf{a})^2, \varphi)}{\partial(\mathbf{r}' - \mathbf{a})^2} = s(\mathbf{r}') = g(\mathbf{r}') e^{\mathcal{R}^2}, \quad (238)$$

with $\mathbf{a} = \mathcal{P}^{-1}(\mathbf{a}_p)$ and $F_{vs} \equiv vF$, assuming that, at any point $\mathbf{r}' \in S'$, it exists a path $\mathcal{T}_{\mathcal{L}}$ in the t -complex plane from $t = 0$ to $t = 1$, defined so that $t(\mathbf{r}' - \mathbf{a}') + \mathbf{a}'$, with $\mathbf{a}' = \mathcal{P}^{-1}(\mathbf{a}_p)$, remains in an open simply connected subset \mathcal{W}_s of the domain of analyticity of s . A primitive $F_{vs} \equiv vF$ of s relatively to $vh'_p{}^2$, satisfying

$$\frac{\partial F_{vs}((\mathbf{r}' - \mathbf{a})^2, \varphi)}{v \partial (\mathbf{r}' - \mathbf{a})^2} = s(\mathbf{r}') = g(\mathbf{r}') e^{\mathcal{R}^2}, \quad (239)$$

verifies,

$$F_{vs}(\mathbf{r}') - F_{vs}(\mathbf{a}') = 2vh_p'^2 e^{d_0+v_1} \int_{\mathcal{I}_c} g(t(\mathbf{r}' - \mathbf{a}') + \mathbf{a}') e^{t^2vh_p'^2} t dt. \quad (240)$$

Proof. From its definition and proposition 2, the function F_{vs} verifies

$$\begin{aligned} & F_{vs}(\mathbf{r}' = \mathcal{P}^{-1}(\mathbf{a}_p) + h'_p \mathbf{u}_{\varphi_p}) - F_{vs}(\mathbf{r}' = \mathcal{P}^{-1}(\mathbf{a}_p)) \\ &= \int_0^{vh_p'^2} s(\mathcal{P}^{-1}(\mathbf{a}_p) + h' \mathbf{u}_{\varphi_p}) v dh'^2 \\ &= 2v(\mathbf{r}'_p - \mathbf{a}_p)^2 \int_{\mathcal{I}_c} g(t\mathcal{P}^{-1}(\mathbf{r}'_p - \mathbf{a}_p) + \mathcal{P}^{-1}(\mathbf{a}_p)) f(d_0 + v_1 + t^2vh_p'^2) t dt, \end{aligned} \quad (241)$$

For that, we have let $h' = th'_p$ as a new variable of integration, with $\mathbf{r}'_p - \mathbf{a}_p = h'_p \mathcal{P}(\mathbf{u}_{\varphi_p})$, $h_p'^2 = (\mathbf{r}'_p - \mathbf{a}_p)^2$, $\mathbf{u}_{\varphi_p} = \mathcal{P}^{-1}(\cos\varphi_p \hat{\mathbf{x}} + \sin\varphi_p \hat{\mathbf{y}})$, which finally implies (240).

Theorem 19

When the proposition 15 applies, the surface integral $\int_{S'} g(\mathbf{r}') e^{\mathcal{R}^2} dS'$ can be reduced semi-asymptotically for large v , by changing in theorem 12, $f(\mathcal{R}^2)$ for $g(\mathbf{r}') e^{\mathcal{R}^2}$, $F_v(\mathcal{R}^2)$ for $F_a(\mathbf{r}')$ and $F_v(\frac{\mathbf{w}_m'^2}{4ik\xi} + d_0)$ for 0, taking,

$$F_a(\mathbf{r}') = \mathcal{K}_s(\mathbf{r}') + U_s(|vh_p'^2| - b_s)(\mathcal{K}_l(\mathbf{r}')), \quad (242)$$

with $vh_p'^2 = v(\mathbf{r}'_p - \mathbf{a}_p)^2$, while, letting $u(t) = g(t\mathcal{P}^{-1}(\mathbf{r}'_p - \mathbf{a}_p) + \mathcal{P}^{-1}(\mathbf{a}_p))$, we write,

$$\begin{aligned} \mathcal{K}_l(\mathbf{r}') &= (u(1) - \frac{\partial_t u(1)}{2vh_p'^2}) e^{(vh_p'^2 + d_0 + v_1)} - (u(t_s) - \frac{\partial_t u(t_s)}{2t_svh_p'^2}) e^{(t_s^2vh_p'^2 + d_0 + v_1)}, \\ \mathcal{K}_s(\mathbf{r}') &= 2vh_p'^2 e^{d_0+v_1} \int_0^{t_s} u(t) e^{t^2vh_p'^2} t dt, \end{aligned} \quad (243)$$

with $t_s = 1 + U_s(|vh_p'^2| - b_s) (\sqrt{\frac{b_s}{|vh_p'^2|}} - 1) \leq 1$, $U_s(x) = \begin{cases} 1 & \text{if } x \geq 0 \\ 0 & \text{if } x < 0 \end{cases}$. The number b_s is some chosen arbitrary large constant, and \mathcal{K}_l is the asymptotic expression of the part of integral with $t^2vh_p'^2 \geq b_s$, used in (242) for $|vh_p'^2| \geq b_s$.

Proof. For large $t^2vh'_p{}^2 > b_s$, we can use an integration by parts in (240), and remark that,

$$\begin{aligned}
 & 2vh'_p{}^2 e^{d_0+v_1} \int_{t_s}^1 g(t(\mathbf{r}' - \mathcal{P}^{-1}(\mathbf{a}_p)) + \mathcal{P}^{-1}(\mathbf{a}_p)) (e^{t^2vh'_p{}^2} t) dt \\
 &= e^{d_0+v_1} (u(1)e^{vh'_p{}^2} - u(t_s)e^{t_s^2vh'_p{}^2} \\
 & - \int_{t_s}^1 \partial_t g(t(\mathbf{r}' - \mathcal{P}^{-1}(\mathbf{a}_p)) + \mathcal{P}^{-1}(\mathbf{a}_p)) e^{t^2vh'_p{}^2} dt) \\
 &\sim e^{d_0+v_1} \left((u(1) - \frac{\partial_t u(1)}{2vh'_p{}^2}) e^{vh'_p{}^2} - (u(t_s) - \frac{\partial_t u(t_s)}{2vh'_p{}^2 t_s}) e^{t_s^2vh'_p{}^2} \right), \tag{244}
 \end{aligned}$$

Dividing the path of integration in (240) into two parts, we then derive (242).

Remark : From proposition 10, we have, when $\frac{\mp i\sqrt{h_1 h_2 - (\frac{h_3}{2})^2} + \frac{h_3}{2}}{h_2}$ are not real, two roots \mathbf{r}'_{\pm} with $h'_p{}^2 = 0$ on the real plane, and if $g(r_{\pm}) = g(\mathcal{P}^{-1}(a_p))$, $\frac{(r'_p - a_p)(g(r') - g(\mathcal{P}^{-1}(a_p)))}{(r'_p - a_p)^2} < \infty$ from,

$$(r'_p - a_p) \frac{g(r') - g(r_{\pm})}{z'_{a+} z'_{a-}} \Big|_{r' \rightarrow r_{\pm}} = \frac{(r'_p - a_p)}{z'_{a\mp}} \left[\frac{(r' - r_{\pm})}{z'_{a\pm}} \cdot \mathbf{grad}_{r'}(g(\mathbf{r}')) \right] \Big|_{r' \rightarrow r_{\pm}} < \infty. \tag{245}$$

Appendix F : Complements on theorem 19 for some complicated case

Usual methods (as Filon's) can be used for \mathcal{K}_s in (242) with some precaution concerning analyticity of u . For example, we can let $\int_0^{t_s} g(s_1 t + s_0) e^{t^2vh'_p{}^2} t dt \simeq \sum_{0 \leq t_{s,n} \leq t_s} e^{t_{s,n}^2vh'_p{}^2} \int_{t_{s,n-1}}^{t_{s,n}} g(s_1 t + s_0) t dt$. Generally, we can assume that $g(s_1 t + s_0)$ is such that,

when the path from 0 to t_s is real, it does not cross any branch cut of g . When cuts are present on S' , we can split S' , which is equivalent to arbitrarily suppress the part of integration through the cut.

In many cases, a closed form primitive of $tg(s_1 t + s_0)$ can be used. When $g(\mathbf{r}') = \sqrt{a' + bx' + c'x'^2}$, i.e. $g(s_1 t + s_0) = \sqrt{a + bt + ct^2}$, the primitive $H(t) = \int^t t \sqrt{R} dt$ with $R = a + bt + ct^2 = \frac{(2ct+b)^2 + \Delta'}{4c} = c \prod_{\pm} (t - t_{\pm})$, $t_{\pm} = \frac{b(-1 \pm \sqrt{1 - \frac{4ac}{b^2}})}{2c}$,

$\Delta' = 4ac - b^2$, is given by (see [31] 2.262.2),

$$\begin{aligned}
 H|_{c \neq 0} &= \frac{1}{c^2 \sqrt{c}} \left(\sqrt{c} \sqrt{R} \left(\frac{cR}{3} - \frac{(2ct+b)b}{8} \right) - \frac{b\Delta'}{16} \ln \left(\frac{2\sqrt{c}\sqrt{R} + 2ct + b}{b\sqrt{1 - \frac{4ac}{b^2}}} \right) \right), \\
 H|_{c=0, b \neq 0} &= \frac{2t\sqrt{R^3}}{3b} - \frac{4\sqrt{R^5}}{15b^2}, \quad H|_{c=b=0} = \frac{t^2\sqrt{a}}{2}. \tag{246}
 \end{aligned}$$

Considering a Taylor expansion for $H|_{|c|\ll|a|,|c|\ll|b|}$ when $0 \leq t \leq 1$, we have,

$$\begin{aligned} H|_{|c|\ll|a|,|c|\ll|b| \neq 0} &\sim H|_{c=0,b \neq 0} + c \frac{\sqrt{R_{c=0}} \left(\left(\frac{R_{c=0}}{7} - \frac{3a}{5} \right) R_{c=0} + a^2 \right) R_{c=0} - a^3}{b^4}, \\ H|_{|c|\ll|b|\ll|a|} &\sim H|_{c=0,|b|\ll|a|} + c \int \frac{t^3}{2\sqrt{a+bt}} dt - \frac{c^2}{2} \int \frac{t^5}{4(\sqrt{a+bt})^3} dt \\ &\sim H|_{c=b=0} + b \frac{t^3}{2\sqrt{a}} \left(\frac{1}{3} - b \frac{t}{16a} \right) + c \frac{t^4}{4\sqrt{a}} \left(\frac{1}{2} - \frac{t}{a} \left(\frac{b}{5} + \frac{ct}{12} \right) \right). \end{aligned} \quad (247)$$

In practice, we can consider for (247) that $u \ll v$ if $u \leq v/20$.

Remark : Let $P_{2n}(z) = z^{2n} + \sum_{k=0}^{2n-1} a_k z^k = z^{2n} (1 + z^{-1} \sum_{k=0}^{2n-1} a_k z^{k+1-2n})$. For $g \equiv (P_{2n})^{1/2}$, it exists a polynomial $g_0(z) = z^n (1 + \sum_{k=1}^n b_k z^{-k})$ such that $g(z) = g_0(z) + O(z^{-1})$.

Appendix G : Some equalities in relation with proposition 11

The expression of $\frac{\mathcal{P}^{-2}(\mathbf{d})}{\xi}$, used in proof of proposition 11 and given by,

$$\frac{\mathcal{P}^{-2}(\mathbf{d})}{\xi} = \frac{\mathbf{d} \cdot \widehat{\mathbf{c}}_p}{p^2 \xi} \widehat{\mathbf{c}}_p + \frac{\mathbf{d} \cdot (\widehat{\mathbf{z}} \times \widehat{\mathbf{c}}_p)}{\xi} \widehat{\mathbf{z}} \times \widehat{\mathbf{c}}_p, \quad (248)$$

is independent of the choice of the term ϵ used in the expression of ξ . In this expression, we have,

$$\begin{aligned} (\mathbf{d} \cdot \widehat{\mathbf{c}}_p) \widehat{\mathbf{c}}_p &= \frac{\mathbf{d} \cdot (\sqrt{\xi - h_1} \widehat{x} + \frac{\epsilon_y}{\epsilon_x} \sqrt{\xi - h_2} \widehat{y})}{\epsilon \sqrt{(h_1 - h_2)^2 + h_3^2}} (\sqrt{\xi - h_1} \widehat{x} + \frac{\epsilon_y}{\epsilon_x} \sqrt{\xi - h_2} \widehat{y}) \\ &= \frac{\mathbf{d} \cdot ((\xi - h_1) \widehat{x} + \frac{-h_3}{2} \widehat{y})}{(\xi - h_1) \epsilon \sqrt{(h_1 - h_2)^2 + h_3^2}} ((\xi - h_1) \widehat{x} + \frac{-h_3}{2} \widehat{y}) \\ &= \frac{\mathbf{d} \cdot (\frac{-h_3}{2} \widehat{x} + (\xi - h_2) \widehat{y})}{(\xi - h_2) \epsilon \sqrt{(h_1 - h_2)^2 + h_3^2}} (\frac{-h_3}{2} \widehat{x} + (\xi - h_2) \widehat{y}), \end{aligned} \quad (249)$$

and,

$$\begin{aligned}
 (\mathbf{d} \cdot (\hat{\mathbf{z}} \times \hat{\mathbf{c}}_p))(\hat{\mathbf{z}} \times \hat{\mathbf{c}}_p) &= \frac{(\xi - h_1) \mathbf{d} \cdot (\hat{\mathbf{y}} - \frac{\epsilon_y \sqrt{\xi - h_2}}{\epsilon_x \sqrt{\xi - h_1}} \hat{\mathbf{x}})}{\epsilon \sqrt{(h_1 - h_2)^2 + h_3^2}} (\hat{\mathbf{y}} - \frac{\epsilon_y \sqrt{\xi - h_2}}{\epsilon_x \sqrt{\xi - h_1}} \hat{\mathbf{x}}) \\
 &= \frac{\mathbf{d} \cdot ((\xi - h_1) \hat{\mathbf{y}} - \frac{h_3}{2} \hat{\mathbf{x}})}{(\xi - h_1) \epsilon \sqrt{(h_1 - h_2)^2 + h_3^2}} ((\xi - h_1) \hat{\mathbf{y}} - \frac{h_3}{2} \hat{\mathbf{x}}) \\
 &= \frac{\mathbf{d} \cdot (\frac{-h_3}{2} \hat{\mathbf{y}} - (\xi - h_2) \hat{\mathbf{x}})}{(\xi - h_2) \epsilon \sqrt{(h_1 - h_2)^2 + h_3^2}} (\frac{-h_3}{2} \hat{\mathbf{y}} - (\xi - h_2) \hat{\mathbf{x}}), \tag{250}
 \end{aligned}$$

where $\epsilon \sqrt{(h_1 - h_2)^2 + h_3^2} = \xi(\mathbf{c}_p)_\epsilon^2$. We note, for example, that we have,

$$\begin{aligned}
 \frac{\hat{\mathbf{x}} \mathcal{P}^{-2}(\hat{\mathbf{x}})}{\xi} &= \frac{(\hat{\mathbf{x}} \cdot \hat{\mathbf{c}}_p)^2}{p^2 \xi} + \frac{(\hat{\mathbf{x}} \cdot (\hat{\mathbf{z}} \times \hat{\mathbf{c}}_p))^2}{\xi} = \frac{1}{\xi(\mathbf{c}_p)_\epsilon^2} \left(\frac{\xi - h_1}{p^2 \xi} + \frac{\xi - h_2}{\xi} \right) \\
 &= \frac{1}{\xi(\mathbf{c}_p)_\epsilon^2} \sum_{\pm} \frac{\pm (h_2 - h_1) + \epsilon \sqrt{(h_1 - h_2)^2 + h_3^2}}{(h_1 + h_2) \mp \epsilon \sqrt{(h_1 - h_2)^2 + h_3^2}} \\
 &= \frac{4h_2 \epsilon \sqrt{(h_1 - h_2)^2 + h_3^2}}{\xi(\mathbf{c}_p)_\epsilon^2} \prod_{\pm} \frac{1}{(h_1 + h_2) \pm \epsilon \sqrt{(h_1 - h_2)^2 + h_3^2}} \\
 &= \frac{h_2}{(h_1 h_2 - h_3^2/4)}. \tag{251}
 \end{aligned}$$

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