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# **Exact and asymptotic reductions in electromagnetics for surface radiation integrals with complex phases to efficient contour integrals**

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## ***1) Introduction***

We present here an efficient method for the reduction of surface radiation integrals to non-singular contour integrals, when we suppose known, as in Physical Optics (PO) [1]-[5], the analytic expression of surface currents whose electromagnetic radiation is calculated. This method applies to a large class of oscillatory surface integrals with complex phase, allowing to save computer time as the surface dimensions increase. In what follows, we look for near-field solutions, which signifies here that we seek for expressions which give a correct result from infinity to a short distance to the surface, when the illumination is not necessarily plane but only locally plane on the surface. Although many authors have studied this type of problems [6]-[22], we present here a novel practical solution, leading to simple non-singular expressions [23], in the case of large planar or moderately curved surfaces :

(a) in monostatic near-field case, for the radiation of PO currents carried by a perfectly or imperfectly reflective surface, excited by a point source whose radiation can be of arbitrary pattern, without any approximation in the phase of the integrand,

(b) in bistatic near-field case, when we consider integrands that have a complex phase which is a quadratic function of the coordinates, as for gaussian beam propagation or in second order phase approximation of the radiation Green function, for curved surfaces.

(c) in monostatic near-field case, for the radiation of PO currents supported by curved plates whose principal curvatures have arbitrary signs, for convex, concave or saddle curved surfaces .

It is worth noticing that the functions involved in case (a) and case (b) are clearly different, even if case (b) can also be considered for a reduction in monostatic case. The case (c) profits of complex scaling transformations and exact results developed in case (b) to extend the results of case (a) to curved surfaces.

The already known methods are not general enough to solve our problems, either because they depend too closely on Helmholtz or Maxwell equations [6]-[12], or they assume specific geometries [9]-[11],[13]-[15] or specific illuminations [16]-[20], or they don't allow to

consider complex phases [17] or complex surfaces due to a change of variables [6]-[22]; besides, they often don't propose non-singular general expressions [6],[8],[12],[19],[22], which leads to severe drawbacks, either for an exact calculus (see sect. 1 of [12] on the use of Asvestas works [9]-[11]) or for asymptotic derivations (see sect. 4 of [6]). In contrast, our global reduction approach does not assume that integrands depend on Helmholtz or Maxwell equations, it gives explicit non singular reduction and avoids the use of special function, and it allows to consider changes of variables with complex coordinates.

This chapter is organized as follows. After presenting the problem in section 2, we develop our method for an analytic reduction of surface integrals in section 3, then detail its application to case (a) (sections 4 and 5) and to case (b) with complex scaling (section 6). Developed first for plane surfaces of integration, the use of complex scaling is generalized in section 7, and the method is extended to surfaces with low curvatures (case (c)) in section 8. Comparisons with classic surface integration results show excellent agreements for both cases (a), (b) and (c) cases (section 10). Some developments are given in appendices, concerning in particular the reductions for imperfectly reflective surfaces.

## 2) Formulation

Let  $S$  be a regular open surface of low curvature, defined, in the  $(x', y', z')$  cartesian coordinate system of origin  $O'$ , as a smooth perturbation of a surface  $S'$  of the plane  $z' = 0$  with unit normal  $\hat{\mathbf{z}}$ . We develop a method that allows the reduction to a contour integral along  $\partial S'$  of the electric  $\mathbf{E}$  and magnetic  $\mathbf{H}$  fields in free space of impedance  $Z_0$  and wavenumber  $k = \frac{2\pi}{\lambda}$ , radiated by surface currents  $\mathbf{J}$  and  $\mathbf{M}$  on  $S$  [1, sect. 1.26], following,

$$\begin{aligned} \mathbf{E} &= \frac{Z_0}{4\pi ik} \int_S (\text{grad}_{\mathbf{r}} \text{div}_{\mathbf{r}} + k^2)(\psi \mathbf{J}) dS - \frac{1}{4\pi} \int_S \text{curl}_{\mathbf{r}}(\psi \mathbf{M}) dS, \\ Z_0 \mathbf{H} &= \frac{Z_0}{4\pi} \int_S \text{curl}_{\mathbf{r}}(\psi \mathbf{J}) dS + \frac{1}{4\pi ik} \int_S (\text{grad}_{\mathbf{r}} \text{div}_{\mathbf{r}} + k^2)(\psi \mathbf{M}) dS, \\ \psi(\tilde{R}) &= \frac{e^{-ik\tilde{R}}}{\tilde{R}}, \tilde{R} = \sqrt{\mathbf{R} \cdot \mathbf{R}}, \mathbf{R} = \mathbf{a} + z\hat{\mathbf{z}} - (\mathbf{r}' + z'\hat{\mathbf{z}}) = \tilde{R} \mathbf{u}, \end{aligned} \quad (1)$$

where  $\mathbf{r} = (\mathbf{a} + z\hat{\mathbf{z}}) \notin S$  is the observation point, and  $(\mathbf{r}' + z'\hat{\mathbf{z}})$  is the current point of integration on  $S$ , with  $z' = \zeta_s(\mathbf{r}')$ ,  $\mathbf{r}' = x'\hat{\mathbf{x}} + y'\hat{\mathbf{y}} \in S'$ . We then assume that  $\mathbf{J}$  and  $\mathbf{M}$  are analytically defined. For example, we can consider that the currents, when excited by a point source at  $\mathbf{r}_0$ , are of the form,

$$\sum_n \mathbf{f}_n(\mathbf{u}_0) \frac{e^{-ik\tilde{R}_0}}{(k\tilde{R}_0)^n}, \quad (2)$$

where  $\mathbf{R}_0 = (\mathbf{r}' + z'\hat{\mathbf{z}}) - \mathbf{r}_0$ ,  $\tilde{R}_0 = \sqrt{\mathbf{R}_0 \cdot \mathbf{R}_0}$ ,  $\mathbf{u}_0 = \frac{\mathbf{R}_0}{\tilde{R}_0}$ , and the  $\mathbf{f}_n$  are regular functions.

In what follows, we study the reduction of different classes of integrals derived from (1),  
 - in monostatic case, when  $\mathbf{r} = \mathbf{r}_0$ , and thus  $\mathbf{R} = -\mathbf{R}_0$  in (0), for a planar (case (a)) or moderately curved (case (c)) surface without any approximation in the phase of the integrand,  
 - in bistatic case, when  $\mathbf{r}$  and  $\mathbf{r}_0$  can be distinct, for  $S$  with low curvature and complex illuminations of gaussian type (case (b)), when the integrals can be approximated in the form,

$$\frac{e^{-ik(\tilde{r}+\tilde{r}_0)}}{\tilde{r} \tilde{r}_0} \int_{S'} e^{(\alpha_1 x' + \alpha_2 y') + (\beta_1 x'^2 + \beta_2 y'^2 + \beta_3 x' y')} h(\mathbf{r}') dS', \quad (3)$$

with some smooth function  $h(\mathbf{r}')$ ,  $\tilde{r} = \sqrt{\mathbf{r} \cdot \mathbf{r}}$ ,  $\tilde{r}_0 = \sqrt{\mathbf{r}_0 \cdot \mathbf{r}_0}$ , and some complex constants  $\alpha_i, \beta_j$ ; In this situation, exact results, in particular when  $h(\mathbf{r}')$  is a polynomial, are obtained.

It is worth noticing that we don't need to suppose here that  $\mathbf{J}$  and  $\mathbf{M}$  derive from fields satisfying Helmholtz or Maxwell equations as in [6]-[12], so that we can consider a large class of functions or asymptotic approximations of integrands.

We develop, two fundamental ways of investigation for the reduction of a surface integral, then some basic examples illustrating their complementarity, which subsequently leads us to efficient solutions for our problems. For that, complex changes of variables, and thus domains of integration with complex coordinates will be considered for the reduction.

### ***3) A combination of two fundamental ways for an efficient reduction to a contour integral***

The reduction that we propose is carried out by a correct combination of two fundamental ways of investigation and associated basic results, that are perfectly adapted to solve in a simple manner the class of problems that we study.

#### **Definition 1**

*Let the plane  $z = 0$  be defined with unit normal  $\hat{\mathbf{z}}$ , and a unit standard basis composed of  $\hat{\mathbf{x}}$  and  $\hat{\mathbf{y}} = \hat{\mathbf{z}} \times \hat{\mathbf{x}}$ . Each point is denoted  $\mathbf{r}' = x'\hat{\mathbf{x}} + y'\hat{\mathbf{y}}$  in  $(x', y')$  coordinates. In the applications, analytical changes of variables can transform the initial domain of integration with real coordinates into a domain with complex coordinates. The dot and cross products of vectors with complex coordinates are here considered as in real space (without complex conjugation), with  $\mathbf{u} \cdot \mathbf{v} = u_x v_x + u_y v_y$  when  $\mathbf{u} = u_x \hat{\mathbf{x}} + u_y \hat{\mathbf{y}}$  and  $\mathbf{v} = v_x \hat{\mathbf{x}} + v_y \hat{\mathbf{y}}$ .*

In this situation, we assume that  $S'$  is a subset of a complex manifold  $S'_0$  of the plane  $z = 0$ , both simply connected and oriented, with  $\mathcal{L} \equiv \partial S'$ . The domain  $S'_0$  results from the transformation of a simply connected (2-dimensional) real oriented manifold of the plane  $z = 0$ , by an invertible function  $P$ , that we suppose thereafter linear to simplify. Letting  $P(\mathbf{u}) = P_x(\mathbf{u})\hat{\mathbf{x}} + P_y(\mathbf{u})\hat{\mathbf{y}}$ , we notice that  $P(\mathbf{u}).P(\mathbf{u}) = \prod_{\pm} (P_x(\mathbf{u}) \pm iP_y(\mathbf{u}))$ .

### 3.1) First way

#### 3.1.1) Some elements of analysis in real space

Let  $S'$  be a real domain in cylindrical coordinates  $(\rho, \varphi)$  and  $F(\rho^2, \varphi)$  be a function which is regular on any straight line from the origin to any point of real  $S'$ , we can write,

$$\int_{S'} \frac{\partial F(\rho^2, \varphi)}{\partial(\rho^2)} dS' = \int_{\mathcal{L}} \int_{0+}^{\rho(\varphi)} \frac{\partial F(\rho^2, \varphi)}{\partial(\rho^2)} \rho d\rho d\varphi = \frac{1}{2} \int_{\mathcal{L}} (F(\rho^2, \varphi) - F(0, \varphi)) d\varphi. \quad (4)$$

where  $\mathcal{L}$  is positively oriented. In this expression,  $d\varphi = (\hat{\mathbf{z}} \times \frac{\bar{\rho}}{\rho^2}).d\bar{\rho}$  with  $\bar{\rho}$  the current point of integration. We can recover it with the Stokes theorem [1, equ. (34) chap.2.9], by using,

$$\begin{aligned} \text{curl}(\hat{\mathbf{z}} \times \frac{\bar{\rho}}{\rho^2} (F(\rho^2, \varphi) - F(0, \varphi))) &= \hat{\mathbf{z}} \text{div}(\frac{\bar{\rho}}{\rho^2} (F(\rho^2, \varphi) - F(0, \varphi))) \\ &= \hat{\mathbf{z}} \frac{2\partial((\bar{\rho} \cdot \frac{\bar{\rho}}{\rho^2})(F(\rho^2, \varphi) - F(0, \varphi)))}{\partial(\rho^2)} = \hat{\mathbf{z}} \frac{2\partial(F(\rho^2, \varphi))}{\partial(\rho^2)}, \end{aligned} \quad (5)$$

when  $F(\rho^2, \varphi) - F(0, \varphi)$  is analytic on  $S'$ , so that we have,

$$\int_{S'} \frac{\partial(F(\rho^2, \varphi))}{\partial(\rho^2)} dS' = \frac{1}{2} \int_{\mathcal{L}} (F(\rho^2, \varphi) - F(0, \varphi)) (\hat{\mathbf{z}} \times \frac{\bar{\rho}}{\rho^2}).d\bar{\rho}. \quad (6)$$

Shifting the origin to a real point  $\mathbf{a}$  (fig. 1) with,  $\bar{\rho} = \mathbf{r}' - \mathbf{a}$ ,  $\frac{\mathbf{r}' - \mathbf{a}}{\sqrt{(\mathbf{r}' - \mathbf{a})^2}} = \hat{\mathbf{x}}\cos\varphi + \hat{\mathbf{y}}\sin\varphi$ ,

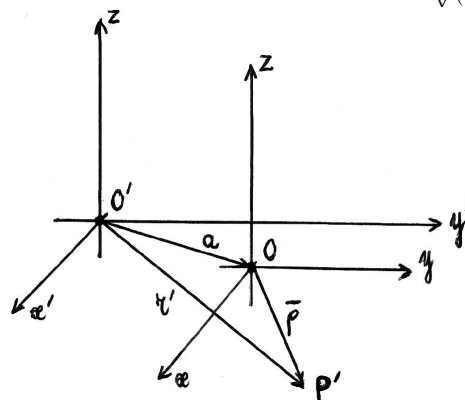


figure 1 : geometry and shift of the origin

we can write, when  $F((\mathbf{r}' - \mathbf{a})^2, \varphi) - F(0, \varphi)$  remains analytic on  $S'$ ,

$$\int_{S'} \frac{\partial(F((\mathbf{r}' - \mathbf{a})^2, \varphi))}{\partial((\mathbf{r}' - \mathbf{a})^2)} dS' = \frac{1}{2} \int_{\mathcal{L}} (F((\mathbf{r}' - \mathbf{a})^2, \varphi) - F(0, \varphi)) (\widehat{\mathbf{z}} \times \frac{\mathbf{r}' - \mathbf{a}}{(\mathbf{r}' - \mathbf{a})^2}) \cdot d\mathbf{r}', \quad (7)$$

with  $\int_{\mathcal{L}} d\varphi = 2\pi$  if  $\mathbf{a} \in S'$  and  $\int_{\mathcal{L}} d\varphi = 0$  if  $\mathbf{a} \notin S'$ , for real  $S'$  and  $\mathbf{a}$ .

### 3.1.2) Reduction in complex coordinates

The Stokes theorem, usually used in a domain with real coordinates, can be extended to domains with complex coordinates [24, p.213] by the formalism of exterior differential forms on complex manifolds, letting  $dS' = (dx' \wedge dy')$  [25],[26, chap.7],[26, chap.3].

#### Definition 2

Let  $\mathbf{r}' = x'\widehat{\mathbf{x}} + y'\widehat{\mathbf{y}}$  with  $\widehat{\mathbf{y}} = \widehat{\mathbf{z}} \times \widehat{\mathbf{x}}$  denote a point of the 4-dimensional plane with complex coordinates  $x'$  and  $y'$ , and  $\mathbf{a} = a_x\widehat{\mathbf{x}} + a_y\widehat{\mathbf{y}}$  denote a fixed complex point, we can write,

$$\begin{aligned} z'_\pm &= x' \pm iy' - (a_x \pm ia_y) = (x' - a_x) \pm i(y' - a_y) = (\mathbf{r}' - \mathbf{a}) \cdot (\widehat{\mathbf{x}} \pm i\widehat{\mathbf{y}}), \\ e^{\pm i\varphi} &= \frac{z'_\pm}{(z'_+ z'_-)^{\frac{1}{2}}}, \varphi = \mp \text{iln}\left(\frac{z_\pm}{(z'_+ z'_-)^{\frac{1}{2}}}\right), e^{\pm 2i\varphi} = \frac{z'_\pm}{z'_\mp}, (\mathbf{r}' - \mathbf{a})^2 = z'_+ z'_- = (z'_\pm e^{\mp i\varphi})^2, \\ \cos\varphi &= \frac{1}{2} \left( \left(\frac{z_\pm}{(z'_+ z'_-)^{\frac{1}{2}}}\right) + \left(\frac{z_\pm}{(z'_+ z'_-)^{\frac{1}{2}}}\right)^{-1} \right), \sin\varphi = \frac{1}{2i} \left( \left(\frac{z_\pm}{(z'_+ z'_-)^{\frac{1}{2}}}\right) - \left(\frac{z_\pm}{(z'_+ z'_-)^{\frac{1}{2}}}\right)^{-1} \right), \\ \mathbf{r}' - \mathbf{a} &= \frac{(z'_+ + z'_-)}{2} \widehat{\mathbf{x}} + \frac{(z'_+ - z'_-)}{2i} \widehat{\mathbf{y}} = z'_\pm e^{\mp i\varphi} (\cos\varphi \widehat{\mathbf{x}} + \sin\varphi \widehat{\mathbf{y}}), \end{aligned} \quad (8)$$

where the functions  $(\cdot)^{1/2}$  and  $\ln(\cdot)$  are analytical, apart from branch cuts. We notice that, in complex coordinates, the set of points verifying  $(\mathbf{r}' - \mathbf{a})^2 = 0$  are the 2-dimensional straight lines  $\mathbf{r}' = \mathbf{a} + t(\widehat{\mathbf{x}} \pm i\widehat{\mathbf{y}})$  where  $z_\pm = 0$ , with complex number  $t$ , while we remark that  $\widehat{\mathbf{x}} \pm i\widehat{\mathbf{y}} = \frac{e \pm i\widehat{\mathbf{z}} \times \mathbf{c}}{c_x \mp ic_y}$  for any complex vector  $\mathbf{c}$  with  $\mathbf{c}^2 \neq 0$ , and  $z_+ = z_- = 0$  only if  $\mathbf{r}' = \mathbf{a}$  (note : if  $(\mathbf{r}' - \mathbf{a})^2 \rightarrow 0$  and  $\mathbf{r}'$  does not tend to  $\mathbf{a}$ ,  $\cos\varphi$  and  $\sin\varphi$  tend to infinity). Apart from branch cut(s), we have,

$$d\varphi = \frac{-i}{2} d\ln\left(\frac{z'_+}{z'_-}\right) = (\widehat{\mathbf{z}} \times \frac{\mathbf{r}' - \mathbf{a}}{(\mathbf{r}' - \mathbf{a})^2}) \cdot d\mathbf{r}'. \quad (9)$$

It is worth noticing that  $d\varphi$  is independent of the choice of  $\widehat{\mathbf{x}}$  and  $\widehat{\mathbf{y}} = \widehat{\mathbf{z}} \times \widehat{\mathbf{x}}$ , and it vanishes along any line  $\mathbf{r}' = \mathbf{a} + t\mathbf{v}$ , with variable  $t$  and constant  $\mathbf{v}$ . On this occasion, we remark that  $\frac{z'_+}{z'_-}$  describes a closed contour along any closed path in the  $(x', y')$ -plane.

**Definition 3**

Let  $\mathbf{a} = a_x \hat{\mathbf{x}} + a_y \hat{\mathbf{y}}$  be a complex point, a function  $F((\mathbf{r}' - \mathbf{a})^2, \varphi)$  is said to be *c-regular* relatively to  $\mathbf{r}'$  on  $S'_0$  ('c' for 'centered'), if it exists a 4-dimensional complex vicinity of  $S'_0$ , where  $F((\mathbf{r}' - \mathbf{a})^2, \varphi) - F(0, \varphi)$  is analytic relatively to  $((\mathbf{r}' - \mathbf{a})^2)^{\frac{1}{2}} \cos \varphi = (x' - a_x)$  and  $((\mathbf{r}' - \mathbf{a})^2)^{\frac{1}{2}} \sin \varphi = (y' - a_y)$ , and vanishes when  $(\mathbf{r}' - \mathbf{a})^2 \rightarrow 0$ , while  $\frac{\partial(F((\mathbf{r}' - \mathbf{a})^2, \varphi))}{\partial((\mathbf{r}' - \mathbf{a})^2)}$  is bounded. Noticing that the set of points verifying  $(\mathbf{r}' - \mathbf{a})^2 = 0$  is only a 2-dimensional manifold (see definition 2),  $\frac{\partial(F((\mathbf{r}' - \mathbf{a})^2, \varphi))}{\partial((\mathbf{r}' - \mathbf{a})^2)}$  is also analytic relatively to  $(x' - a_x)$  and  $(y' - a_y)$  on  $S'_0$  [28, sect. 23.2].

We then define the condition of continuation for  $F((\mathbf{r}' - \mathbf{a})^2, \varphi)$  relatively to  $\mathbf{a}$ , from  $\mathbf{a} = \mathbf{a}'$  to  $\mathbf{a}$  on  $S'_0$ , as follows : it exists an open connected finite domain  $\mathcal{H}$  of complex points  $\mathbf{a}$  (4-dimensional set) including  $\mathbf{a}'$  and a point of  $S'_0$ , where  $F$  remains *c-regular* on  $S'_0$ .

We remark that, if  $F(t, s)$  is an entire function of  $t$  independent of  $s$ , then the *c-regularity* on  $S'$  is verified for any  $\mathbf{a}$ . If  $F((\mathbf{r}' - \mathbf{a})^2, \varphi)$  is *c-regular* on  $S'_0$ , then it is also the case for  $(\mathbf{r}' - \mathbf{a})^2 (F((\mathbf{r}' - \mathbf{a})^2, \varphi) - F(0, \varphi))$ , and for any primitive  $\int^{(\mathbf{r}' - \mathbf{a})^2} (F(t, \varphi) - F(0, \varphi)) dt$  defined on  $S'_0$  and at points with  $(\mathbf{r}' - \mathbf{a})^2 = 0$ . (Note : considering *c-regular* functions  $(\mathbf{r}' - \mathbf{a})^2 (x' - a)^n$  and  $(\mathbf{r}' - \mathbf{a})^2 (y' - a)^n$ ,  $n \geq 0$ , we note that their partial derivatives with respects to  $(\mathbf{r}' - \mathbf{a})^2$  are  $(1 + n/2)(x' - a)^n$  and  $(1 + n/2)(y' - a)^n$ , and we will generally show how to reduce the problem to the one where we consider *c-regular* functions.)

**Theorem 1**

Let  $S'$  be a subdomain of a complex manifold  $S'_0$ , both simply connected oriented complex manifolds, with  $\mathcal{L} \equiv \partial S'$ , where  $S'_0$  results from a linear transformation of real manifold as in definition 1, and  $F((\mathbf{r}' - \mathbf{a})^2, \varphi)$  be a complex function satisfying the *c-regularity* in  $S'_0$  given in definition 3, then the equality

$$\int_{S'} \frac{\partial(F((\mathbf{r}' - \mathbf{a})^2, \varphi))}{\partial((\mathbf{r}' - \mathbf{a})^2)} dS' = \frac{1}{2} \int_{\mathcal{L}} (F((\mathbf{r}' - \mathbf{a})^2, \varphi) - F(0, \varphi)) [(\hat{\mathbf{z}} \times \frac{\mathbf{r}' - \mathbf{a}}{(\mathbf{r}' - \mathbf{a})^2}) \cdot d\mathbf{r}'] \quad (10)$$

is valid for  $\mathbf{a} \in S'_0$ , and for any  $\mathbf{a}$  verifying the condition of continuation in  $\mathcal{H}$  to  $S'_0$  given in definition 3. This equality is based on analytical continuation from real to complex coordinates, where it is worth noticing that  $(r' - a)^2 = 0$  does no more imply  $r' = a$ .

Proof. The equality (10) corresponds to a continuation of (7) for points with complex coordinates, in the domain of regularity of  $F$ . We first state (10) for  $\mathbf{a} \in S'_0$ . From the

properties of the exterior product and the definition of  $z'_+$  and  $z'_-$  in (8), we can write,

$$\begin{aligned} dS' &= (dx' \wedge dy') = \frac{i}{2}(dz'_+ \wedge dz'_-) = \frac{-i}{4}(z'_- dz'_+ + z'_+ dz'_-) \wedge \left( \frac{dz'_+}{z'_+} - \frac{dz'_-}{z'_-} \right) \\ &= \frac{1}{2}d(z'_+ z'_-) \wedge \frac{-i}{2}d\ln\left(\frac{z'_+}{z'_-}\right) = \frac{1}{2}d(\mathbf{r}' - \mathbf{a})^2 \wedge d\varphi, \end{aligned} \quad (11)$$

From the definition 1,  $S'_0$  results from a linear transformation of a simply connected real manifold relatively to  $(x', y')$  coordinates, and when  $\mathbf{a} \in S'_0$ , it can be mapped from  $\mathbf{a}$  to  $\infty$  at constant  $\varphi$  with straight semi-lines, except on some cut(s) where  $\varphi$  is constant on both sides. In this case, we can integrate in the integral on  $S'$  relatively to  $(\mathbf{r}' - \mathbf{a})^2$  variable from (11), and derive the contour integral along  $\mathcal{L}$  given in (10). Let us notice, that, when  $\mathbf{a} \in S'$ , we integrate from  $\mathbf{r}' = \mathbf{a}$  to a point of  $\mathcal{L}$  at constant  $\varphi$ , while, when  $\mathbf{a} \in S'_0$  but  $\mathbf{a} \notin S'$ , the limits of integration are two points of  $\mathcal{L}$  at constant  $\varphi$ .

From the condition of continuation given in definition 3, we can extend the validity of this equality to any point  $\mathbf{a} = \mathbf{a}'$  belonging to domain  $\mathcal{H}$  which comprises a subset  $S'_a$  of  $S'_0$ . By definition of the c-regularity, the cuts in  $\varphi$  continue to have no influence; the left-hand side of (10) is analytic relatively to complex coordinates  $a_x$  and  $a_y$ ; the right-hand side, continuous at any point of  $\mathcal{H}$ , is analytic when  $(\mathbf{r}' - \mathbf{a})^2 \neq 0$  on  $\mathcal{L}$ . That restriction disappears from the generalized principle of continuation [28, sect. 23.2], and the analyticity extends at any point  $\mathbf{a}$  of  $\mathcal{H}$  by noticing that  $(\mathbf{r}' - \mathbf{a})^2|_{\mathbf{r}' \in \mathcal{L}} = 0$  describes a piecewise analytic  $p$ -dimensional manifold with  $p \leq 3$ . Therefore, considering that we have the analyticity in  $\mathcal{H}$  and that (10) is stated on  $S'_a$ , we can use the identity theorem [28, sect. 6.1], and conclude the validity of (10) for any point of  $\mathcal{H}$ .

Since the integration in (10) is on  $\varphi$  variable, the part of the contour integral, restricted to the factor  $F(0, \varphi)$ , could express in a simple form, and we now analyse it when  $F(0, \varphi) \equiv 1$ .

### Proposition 1

let  $\mathcal{D}_1$  be a smooth semi-line of the complex plane from the origin to  $\infty$ , and  $\mathcal{L}$  a closed contour with complex coordinates  $x'$  and  $y'$ . We have, when  $(\mathbf{r}' - \mathbf{a})^2 \neq 0$  on  $\mathcal{L}$ ,

$$\int_{\mathcal{L}} \left( \widehat{\mathbf{z}} \times \frac{\mathbf{r}' - \mathbf{a}}{(\mathbf{r}' - \mathbf{a})^2} \right) \cdot d\mathbf{r}' = \frac{-i}{2} \int_{\mathcal{L}} d\ln\left(\frac{z'_+}{z'_-}\right) = \pi(N_+ - N_-) \quad (12)$$

where  $N_+$  (resp.  $N_-$ ) is the number of times that the complex  $\frac{z'_+}{z'_-}$  crosses  $\mathcal{D}_1$  in the positive sense (resp. negative).



For a complex surface  $S'$  with complex contour  $\mathcal{L}$  verifying definition 1, i.e. resulting from a invertible linear transformation of a real simply connected surface, we can write, when  $(\mathbf{r}' - \mathbf{a})^2 \neq 0$  on  $\mathcal{L}$ ,

$$\int_{\mathcal{L}} (\widehat{\mathbf{z}} \times \frac{\mathbf{r}' - \mathbf{a}}{(\mathbf{r}' - \mathbf{a})^2}) \cdot d\mathbf{r}' = \pi(\epsilon^+ 1_{S'_+}(a_x + ia_y) + \epsilon^- 1_{S'_-}(a_x - ia_y)), \quad (13)$$

where, we have,

$$\begin{aligned} \epsilon^\pm &= 1 \text{ (resp. } \epsilon^\pm = -1 \text{) for } \mathcal{L}_\pm \text{ positively (resp. negatively) oriented,} \\ \text{with } \mathcal{L}_\pm &= \{(x'_\pm = \text{Re}(x' \pm iy'), y'_\pm = \text{Im}(x' \pm iy')), M(x', y') \in \mathcal{L}\}, \end{aligned} \quad (14)$$

and the functions  $1_{S'_\pm}$  are the indicator functions of the domains  $S'_\pm$  of the complex plane, given by,

$$S'_\pm = \{z, \frac{1}{2i} \int_{\mathcal{L}_\pm} \frac{d(x'_\pm + iy'_\pm)}{x'_\pm + iy'_\pm - z} = \pi\epsilon^\pm\} = \{x' \pm iy', M(x', y') \in S'\}. \quad (15)$$

Proof. From (9), we have, when  $(\mathbf{r}' - \mathbf{a})^2 \neq 0$  on  $\mathcal{L}$ ,

$$\int_{\mathcal{L}} (\widehat{\mathbf{z}} \times \frac{\mathbf{r}' - \mathbf{a}}{(\mathbf{r}' - \mathbf{a})^2}) \cdot d\mathbf{r}' = \frac{-i}{2} \int_{\mathcal{L}} d\ln\left(\frac{z'_+}{z'_-}\right), \quad (16)$$

where the logarithm function is considered by analytic continuation along the closed path covered by the uniform function  $\frac{z'_+}{z'_-}$  on  $\mathcal{L}$ . We then obtain (12), since  $(N_+ - N_-)$  is the number of turns done by  $\frac{z'_+}{z'_-}$  around the origin, along the closed contour  $\mathcal{L}$ . For (13), we write,

$$\begin{aligned} \int_{\mathcal{L}} (\widehat{\mathbf{z}} \times \frac{\mathbf{r}' - \mathbf{a}}{(\mathbf{r}' - \mathbf{a})^2}) \cdot d\mathbf{r}' &= \frac{-i}{2} \int_{\mathcal{L}} \left( \frac{dz'_+}{z'_+} - \frac{dz'_-}{z'_-} \right) \\ &= \frac{1}{2i} \int_{\mathcal{L}_c} \frac{d(x' + iy')}{x' + iy' - (a_x + ia_y)} + \frac{1}{2i} \int_{\mathcal{L}_{c^*}} \frac{d(x' - iy')}{x' - iy' - (a_x - ia_y)} \\ &= \epsilon^+ \pi 1_{S'_c}(a_x + ia_y) + \epsilon^- \pi 1_{S'_{c^*}}(a_x - ia_y), \end{aligned} \quad (17)$$

where the residue theorem on the complex contours  $\mathcal{L}_c$  of the domain  $S'_c = \{x' + iy', M(x', y') \in S'\}$  and  $\mathcal{L}_{c^*}$  of the domain  $S'_{c^*} = \{x' - iy', M(x', y') \in S'\}$  is used. Considering that  $x' \pm iy' = \text{Re}(x' \pm iy') + i\text{Im}(x' \pm iy')$ , we obtain (14) and (15).

Remark : A linear invertible transform of a (closed) simply connected domain is a (closed) simply connected domain (note : for an injective continuous transform, see the Browner's domain invariance theorem).

Proposition 2

Let  $\mathcal{W}$  be an open simply connected subset of the domain of analyticity of

$$\frac{\partial F((\mathbf{r}' - \mathbf{a})^2, \varphi)}{\partial (\mathbf{r}' - \mathbf{a})^2} = s(\mathbf{r}'), \quad (18)$$

relatively to the complex coordinates of  $\mathbf{r}'$ , that includes  $\mathbf{r}' = \mathbf{a}$  and  $S'_0$ . Considering two points  $\mathbf{r}'_{1,2}$  such that  $(\mathbf{r}'_1 - \mathbf{a}) = t_1(\mathbf{r}'_2 - \mathbf{a})$  with  $(\mathbf{r}'_2 - \mathbf{a})^2 \neq 0$ , we suppose that it exists a complex path  $\mathcal{T}_{\mathcal{L}}$  from  $t = t_1$  to  $t = 1$ , defined so that  $t(\mathbf{r}'_2 - \mathbf{a}) + \mathbf{a}$  remains in  $\mathcal{W}$  along  $\mathcal{T}_{\mathcal{L}}$ . In this case, by analytic continuation in an open simply connected subset, we can integrate  $s(\mathbf{r}')$  relatively to  $(\mathbf{r}' - \mathbf{a})^2$  at constant  $\varphi$  and write,

$$F((\mathbf{r}'_2 - \mathbf{a})^2, \varphi) - F((\mathbf{r}'_1 - \mathbf{a})^2, \varphi) = 2(\mathbf{r}'_2 - \mathbf{a})^2 \int_{\mathcal{T}_{\mathcal{L}}} s(t(\mathbf{r}'_2 - \mathbf{a}) + \mathbf{a}) t dt. \quad (19)$$

which could be used in (10) of theorem 1 when  $\mathbf{r}'_1 = \mathbf{a}$ .

Proof. For that we integrate  $s$  along  $\mathcal{T}_{\mathcal{L}}$ , relatively to  $(\mathbf{r}' - \mathbf{a})^2$  variable at constant  $\varphi$ , by letting  $\mathbf{r}' - \mathbf{a} = t(\mathbf{r}'_2 - \mathbf{a})$ , and  $d(\mathbf{r}' - \mathbf{a})^2 = (\mathbf{r}'_2 - \mathbf{a})^2 t dt$ .

Remark 1 :

Let us notice that a complex anisotropic scaling can lead us to work on a complex manifold. If we dilate a real surface  $S'$  by a factor  $p \neq 0$  along the direction  $\mathbf{c}_p$  with,

$$\mathcal{P} : \mathbf{r}' \rightarrow \mathbf{r}'_p = x'_p \hat{\mathbf{c}}_p + y'_p (\hat{\mathbf{z}} \times \hat{\mathbf{c}}_p), \quad x'_p = p(\mathbf{r}' \cdot \hat{\mathbf{c}}_p), \quad y'_p = \mathbf{r}' \cdot (\hat{\mathbf{z}} \times \hat{\mathbf{c}}_p), \quad (20)$$

where  $\hat{\mathbf{c}}_p = \frac{\mathbf{c}_p}{\sqrt{\mathbf{c}_p \cdot \mathbf{c}_p}}$  (with  $\mathbf{c}_p^2 \neq 0$  and  $p \neq 0$ ), we obtain a complex surface  $S'_p$ , with

$$\begin{aligned} dS'_p &= dx'_p \wedge dy'_p = p dx' \wedge dy' = p dS', \\ \mathbf{v} \cdot \mathcal{P}(\mathbf{r}') &= \mathcal{P}(\mathbf{v}) \cdot \mathbf{r}', \quad (\hat{\mathbf{z}} \times \mathcal{P}(\mathbf{v})) \cdot \mathcal{P}(\mathbf{r}') = (\hat{\mathbf{z}} \times p\mathbf{v}) \cdot \mathbf{r}'. \end{aligned} \quad (21)$$

Remark 2 : The choice to orient or not a domain stands out from the following equalities,

$$\int_{\chi([a,b])} f(x) dx = \int_{[a,b]} f(\chi(t)) |\chi'(t)| dt, \quad \int_{\chi(a)}^{\chi(b)} f(x) dx = \int_a^b f(\chi(t)) \chi'(t) dt, \quad (22)$$

by noticing that the first one, contrarily to the second, cannot be applied to complex  $a$  and  $b$ .

Remark 3 : Let us notice that on each straight part of  $\mathcal{L}$  along  $\hat{\mathbf{v}}$ , with  $d\mathbf{r}' = \hat{\mathbf{v}} ds$  and  $\mathbf{r}' - \mathbf{a} = \mathbf{r}_0 + \hat{\mathbf{v}} s$ , we have  $(\hat{\mathbf{z}} \times (\mathbf{r}' - \mathbf{a})) \cdot d\mathbf{r}' = \mathbf{g}_0 ds$ , with constant  $\mathbf{r}_0$  and  $\mathbf{g}_0 = \hat{\mathbf{z}} \times \mathbf{r}_0$ .

### 3.2) Second way

We consider here, as in previous section, the formalism of exterior differential forms, with  $dS' = (dx' \wedge dy')$  [25]-[26, chap.7].

#### Theorem 2

Let  $S'$  be a simply connected oriented complex manifold, satisfying the definition 1, we consider a function  $F((\mathbf{r}' - \mathbf{a})^2, \varphi)$ , analytic relatively to  $((\mathbf{r}' - \mathbf{a})^2)^{\frac{1}{2}} \cos \varphi = (x' - a_x)$  and  $((\mathbf{r}' - \mathbf{a})^2)^{\frac{1}{2}} \sin \varphi = (y' - a_y)$  on  $S'$ . We can write

$$\int_{S'} \mathbf{b} \cdot \text{grad}_{\mathbf{r}'}(F((\mathbf{r}' - \mathbf{a})^2, \varphi) - A_0) dS' = \int_{\mathcal{L}} (F((\mathbf{r}' - \mathbf{a})^2, \varphi) - A_0) (\widehat{\mathbf{z}} \times \mathbf{b}) \cdot d\mathbf{r}' \quad (23)$$

where  $\mathbf{b}$  and  $A_0$  are arbitrary constants, and the cut in  $\varphi$  can be ignored. From

$$\text{grad}_{\mathbf{r}'}(\cdot) = 2(\mathbf{r}' - \mathbf{a}) \frac{\partial(\cdot)}{\partial((\mathbf{r}' - \mathbf{a})^2)} + \frac{(\widehat{\mathbf{z}} \times (\mathbf{r}' - \mathbf{a}))}{(\mathbf{r}' - \mathbf{a})^2} \frac{\partial(\cdot)}{\partial \varphi}, \quad (24)$$

we then have,

$$\begin{aligned} & \int_{S'} (\mathbf{r}' - \mathbf{a}) \cdot \left( \mathbf{b} \frac{\partial(F((\mathbf{r}' - \mathbf{a})^2, \varphi))}{\partial((\mathbf{r}' - \mathbf{a})^2)} - \frac{1}{2} \frac{(\widehat{\mathbf{z}} \times \mathbf{b})}{(\mathbf{r}' - \mathbf{a})^2} \frac{\partial(F((\mathbf{r}' - \mathbf{a})^2, \varphi))}{\partial \varphi} \right) dS' \\ &= \frac{1}{2} \int_{\mathcal{L}} (F((\mathbf{r}' - \mathbf{a})^2, \varphi) - A_0) (\widehat{\mathbf{z}} \times \mathbf{b}) \cdot d\mathbf{r}'. \end{aligned} \quad (25)$$

Proof. In real coordinates, we could use the Stokes theorem [1, chap.2.9], in combination with  $\text{grad}(F((\mathbf{r}' - \mathbf{a})^2, \varphi)) \times \widehat{\mathbf{d}} = \text{curl}(\widehat{\mathbf{d}} F((\mathbf{r}' - \mathbf{a})^2, \varphi))$ , where  $\widehat{\mathbf{d}}$  is a constant vector. More generally, in the formalism of exterior forms for complex manifolds, we prefer here to use gradient in cartesian coordinates with  $dS' = dx' \wedge dy' = -dy' \wedge dx'$ , following

$$\int_{S'} \mathbf{b} \cdot \text{grad}(F) dS' = \int_{S'} b \cdot \widehat{\mathbf{x}} \frac{\partial F}{\partial x'} dx' \wedge dy' - \int_{S'} b \cdot \widehat{\mathbf{y}} \frac{\partial F}{\partial y'} dy' \wedge dx'. \quad (26)$$

From the analyticity of  $F$  relatively to  $x'$  and  $y'$ , the cut in  $\varphi$  is omitted. We integrate  $\frac{\partial F}{\partial x'} dx'$  and  $\frac{\partial F}{\partial y'} dy'$  parts, while noticing  $(\widehat{\mathbf{z}} \times \mathbf{b}) \cdot d\mathbf{r}' = b \cdot \widehat{\mathbf{x}} dy' - b \cdot \widehat{\mathbf{y}} dx'$ , and obtain (23). Then, from

$$\begin{aligned} dF &= \frac{\partial F}{\partial((\mathbf{r}' - \mathbf{a})^2)} d(\mathbf{r}' - \mathbf{a})^2 + \frac{\partial F}{\partial \varphi} d\varphi = \text{grad}_{\mathbf{r}'}(F) \cdot d(\mathbf{r}' - \mathbf{a}), \\ d(\mathbf{r}' - \mathbf{a}) &= \frac{1}{2} \frac{\mathbf{r}' - \mathbf{a}}{(\mathbf{r}' - \mathbf{a})^2} d(\mathbf{r}' - \mathbf{a})^2 + (\widehat{\mathbf{z}} \times (\mathbf{r}' - \mathbf{a})) d\varphi, \end{aligned} \quad (27)$$

we derive (24), which, combined with (23), gives (25).

Remark : Using (23) with  $F((\mathbf{r}' - \mathbf{a})^2, \varphi) = g(\mathbf{r}')F_1((\mathbf{r}' - \mathbf{a})^2)$ , we have,

$$\int_{S'} \mathbf{b} \cdot \text{grad}_{\mathbf{r}'}(g(\mathbf{r}')F_1((\mathbf{r}' - \mathbf{a})^2) - A_0) dS' = \int_{\mathcal{L}} (g(\mathbf{r}')F_1((\mathbf{r}' - \mathbf{a})^2) - A_0)(\hat{\mathbf{z}} \times \mathbf{b}) \cdot d\mathbf{r}', \quad (28)$$

and it is worth noticing that, in asymptotics, when  $F_1((\mathbf{r}' - \mathbf{a})^2)$  is highly oscillatory and  $g(\mathbf{r}')$  remains smooth as  $(\mathbf{r}' - \mathbf{a})^2$  varies, we have,

$$\begin{aligned} \text{grad}_{\mathbf{r}'}(g(\mathbf{r}')F((\mathbf{r}' - \mathbf{a})^2)) &= \text{grad}_{\mathbf{r}'}(g(\mathbf{r}'))F((\mathbf{r}' - \mathbf{a})^2) + 2(\mathbf{r}' - \mathbf{a})g(\mathbf{r}') \frac{\partial F((\mathbf{r}' - \mathbf{a})^2)}{\partial((\mathbf{r}' - \mathbf{a})^2)} \\ &\sim 2(\mathbf{r}' - \mathbf{a})g(\mathbf{r}') \frac{\partial F((\mathbf{r}' - \mathbf{a})^2)}{\partial((\mathbf{r}' - \mathbf{a})^2)}. \end{aligned} \quad (29)$$

### Proposition 3

Let  $F_0((\mathbf{r}' - \mathbf{a})^2)$  be a scalar function, analytic on  $S'$ , and  $\mathbf{F}_{0v}$  a vectorial function, whose each component is analytic on  $S'$ , we have,

$$\begin{aligned} \int_{S'} \mathbf{b} \cdot (\mathbf{r}' - \mathbf{a}) \frac{\partial(F_0((\mathbf{r}' - \mathbf{a})^2))}{\partial((\mathbf{r}' - \mathbf{a})^2)} dS' &= \frac{1}{2} \int_{\mathcal{L}} F_0((\mathbf{r}' - \mathbf{a})^2)(\hat{\mathbf{z}} \times \mathbf{b}) \cdot d\mathbf{r}' \\ \int_{S'} (\mathbf{r}' - \mathbf{a}) \cdot \frac{\partial(\mathbf{F}_{0v}((\mathbf{r}' - \mathbf{a})^2))}{\partial((\mathbf{r}' - \mathbf{a})^2)} dS' &= \frac{1}{2} \int_{\mathcal{L}} (\hat{\mathbf{z}} \times \mathbf{F}_{0v}((\mathbf{r}' - \mathbf{a})^2)) \cdot d\mathbf{r}' \end{aligned} \quad (30)$$

Proof. Using the theorem 2, with  $F((\mathbf{r}' - \mathbf{a})^2, \varphi) = F_0((\mathbf{r}' - \mathbf{a})^2)$  independent of  $\varphi$ , we obtain the first line, then the second line by considering each component of  $\mathbf{F}_{0v}$  as  $F_0$ . On this occasion, let us remark that,

$$\begin{aligned} \text{if } \frac{\partial(F_0((\mathbf{r}' - \mathbf{a})^2))}{\partial((\mathbf{r}' - \mathbf{a})^2)} &= e^{-v(\mathbf{r}' - \mathbf{a})^2} \text{ then } F_0((\mathbf{r}' - \mathbf{a})^2) = \frac{-e^{-v(\mathbf{r}' - \mathbf{a})^2}}{v} + C, \\ \text{if } \frac{\partial(F_0((\mathbf{r}' - \mathbf{a})^2))}{\partial((\mathbf{r}' - \mathbf{a})^2)} &= \frac{e^{-v\tilde{R}}}{\tilde{R}} \text{ then } F_0((\mathbf{r}' - \mathbf{a})^2) = 2 \int^{\tilde{R}} e^{-vt} dt = \frac{-2e^{-v\tilde{R}}}{v} + C, \end{aligned} \quad (31)$$

when  $\tilde{R} = \sqrt{\mathbf{R} \cdot \mathbf{R}} = \sqrt{(\mathbf{r}' - \mathbf{a})^2 + z^2}$ , and  $C, v$  some arbitrary constants.

### Theorem 3

Let  $B_0((\mathbf{r}' - \mathbf{a})^2, \varphi)$  be a function, analytic relatively to  $((\mathbf{r}' - \mathbf{a})^2)^{\frac{1}{2}} \cos \varphi = (x' - a_x)$  and  $((\mathbf{r}' - \mathbf{a})^2)^{\frac{1}{2}} \sin \varphi = (y' - a_y)$  on  $S'$ , we have, for  $n \geq 0$ ,

$$\begin{aligned}
& \mathbf{d}_{n+1} \cdot \int_{S'} \prod_{j=1}^n (\mathbf{d}_j \cdot (\mathbf{r}' - \mathbf{a})) \operatorname{grad}(B_0((\mathbf{r}' - \mathbf{a})^2, \varphi)) dS' \\
&= \int_{\mathcal{L}} \prod_{j=1}^n (\mathbf{d}_j \cdot (\mathbf{r}' - \mathbf{a})) B_0((\mathbf{r}' - \mathbf{a})^2, \varphi) (\widehat{\mathbf{z}} \times \mathbf{d}_{n+1}) \cdot d\mathbf{r}' \\
&- \mathbf{d}_{n+1} \cdot \sum_{m=1}^n \mathbf{d}_m \int_{S'} \prod_{j \neq m}^n (\mathbf{d}_j \cdot (\mathbf{r}' - \mathbf{a})) B_0((\mathbf{r}' - \mathbf{a})^2, \varphi) dS' \tag{32}
\end{aligned}$$

where the  $\mathbf{d}_j$  are constants, and we can let  $\prod_j^0(\dots) = 1$ ,  $\prod_{j \neq m}^1(\dots) = 1$ ,  $\prod_{j \neq m}^0(\dots) = 0$ .

Proof. Since  $f \operatorname{grad}(h) = \operatorname{grad}(fh) - h \operatorname{grad}(f)$  and  $\operatorname{grad}(\mathbf{d}_m \cdot \mathbf{r}') = \mathbf{d}_m$ , we have

$$\begin{aligned}
& \prod_{j=1}^n (\mathbf{d}_j \cdot (\mathbf{r}' - \mathbf{a})) \operatorname{grad}(B_0((\mathbf{r}' - \mathbf{a})^2, \varphi)) \\
&= \operatorname{grad}\left(\prod_{j=1}^n (\mathbf{d}_j \cdot (\mathbf{r}' - \mathbf{a})) B_0((\mathbf{r}' - \mathbf{a})^2, \varphi)\right) - \sum_{m=1}^n \mathbf{d}_m \prod_{j \neq m}^n (\mathbf{d}_j \cdot (\mathbf{r}' - \mathbf{a})) B_0((\mathbf{r}' - \mathbf{a})^2, \varphi) \tag{33}
\end{aligned}$$

Multiplying this equation by  $\mathbf{d}_{n+1}$  and integrating, then using (23), we obtain the proposition. This result will be useful, in particular when  $B_0((\mathbf{r}' - \mathbf{a})^2, \varphi)$  is independent of the variable  $\varphi$ , as we will see in theorem 4.

### 3.3) Why these distinct ways are complementary: some basic results

The first and second theorems permit to consider respective reductions of surface integrals for functions of types  $F((\mathbf{r}' - \mathbf{a})^2, \varphi)$  and  $(\mathbf{r}' - \mathbf{a}) F((\mathbf{r}' - \mathbf{a})^2, \varphi)$ , which, correctly combined, avoid the use of fractional powers of  $(\mathbf{r}' - \mathbf{a})^2$  and of special functions. Let us illustrate it by some basic results.

#### 3.3.1) Exact reduction with successive primitives for polynomial factors

Let us show how to reduce to a contour integral, for arbitrary integer  $n + 1 \geq 0$ ,

$$I_{n+1} = \int_{S'} \prod_{j=1}^{n+1} (\mathbf{d}_j \cdot (\mathbf{r}' - \mathbf{a})) F_0((\mathbf{r}' - \mathbf{a})^2) dS', \tag{34}$$

with a polynomial  $\prod_{j=1}^{n+1} (\mathbf{d}_j \cdot (\mathbf{r}' - \mathbf{a}))$ , with constant vectors  $\mathbf{d}_j$  and  $\prod_{j=1}^0(\dots) = 1$ . We now state, from theorem 1, a specific proposition for the even case  $n + 1 = 2q$ , then, from the addition of theorem 2, a general proposition for arbitrary integer  $n$ . We consider in what follows, that

we are in an open connected domain of the variable  $\mathbf{a}$  including a point of  $S'$ , where  $F_0((\mathbf{r}' - \mathbf{a})^2)$  remains analytic on  $S'$ , which allows the application of theorems 1 and 2.

**Proposition 4**

*We have, from the theorem 1, for  $n + 1 = 2q \geq 0$ ,*

$$\begin{aligned} I_{2q} &= \int_{S'} \prod_{j=1}^{2q} (\mathbf{d}_j \cdot (\mathbf{r}' - \mathbf{a})) F_0((\mathbf{r}' - \mathbf{a})^2) dS' \\ &= \frac{1}{2} \int_{\mathcal{L}} G(\varphi) (B_{2q}((\mathbf{r}' - \mathbf{a})^2) - B_{2q}(0)) \left[ (\hat{\mathbf{z}} \times \frac{\mathbf{r}' - \mathbf{a}}{(\mathbf{r}' - \mathbf{a})^2}) \cdot d\mathbf{r}' \right] \end{aligned} \quad (35)$$

where the  $\mathbf{d}_j$  are constant, the case  $q = 0$  is considered with  $\prod_{j=1}^0 (\dots) \equiv 1$ , and,

$$B_{2q}((\mathbf{r}' - \mathbf{a})^2) = \int^{(\mathbf{r}' - \mathbf{a})^2} t^q F_0(t) dt, \quad G(\varphi)((\mathbf{r}' - \mathbf{a})^2)^q = \prod_{j=1}^{2q} \mathbf{d}_j \cdot (\mathbf{r}' - \mathbf{a}). \quad (36)$$

Proof. Letting  $\frac{\partial(F((\mathbf{r}' - \mathbf{a})^2), \varphi)}{\partial((\mathbf{r}' - \mathbf{a})^2)} = G(\varphi)(\mathbf{r}' - \mathbf{a})^{2q} F_0((\mathbf{r}' - \mathbf{a})^2)$  in theorem 1, we obtain the equality (35), where we notice that  $B_{2q}$  depends on  $q - 1$  successive primitives of  $F_0$ . It is worth noticing that we have,

$$\begin{aligned} B_{2q}((\mathbf{r}' - \mathbf{a})^2) &= \int_0^{(\mathbf{r}' - \mathbf{a})^2} t^q e^{-vt} dt + C \text{ if } F_0((\mathbf{r}' - \mathbf{a})^2) = e^{-v(\mathbf{r}' - \mathbf{a})^2}, \\ B_{2q}((\mathbf{r}' - \mathbf{a})^2) &= 2 \int_{\sqrt{z^2}}^{\tilde{R}} (t^2 - z^2)^q \frac{e^{-vt}}{t^{m-1}} dt + C \text{ if } F_0((\mathbf{r}' - \mathbf{a})^2) = \frac{e^{-v\tilde{R}}}{\tilde{R}^m}, \end{aligned} \quad (37)$$

when  $\tilde{R} = \sqrt{\mathbf{R} \cdot \mathbf{R}} = \sqrt{(\mathbf{r}' - \mathbf{a})^2 + z^2}$ ,  $\mathbf{a}$ ,  $v$ ,  $C$  and  $z$  some constants. In these cases, the function  $B_{2q}((\mathbf{r}' - \mathbf{a})^2) - B_{2q}(0)$  can be expressed in closed form or asymptotically.

A direct use of the theorem 1 for  $I_{n+1}$  with odd  $n + 1$ , would lead to a contour integral with fractional powers of  $(\mathbf{r}' - \mathbf{a})^2$ . To avoid it, we now state a relation that lowers the degree of the polynomial. The repeated use of it, in combination of results for  $n = 0$  (proposition 3) and  $n + 1 = 0$  (proposition 4), allows the reduction without fractional powers.

**Theorem 4**

*We have, for arbitrary integer  $n \geq 0$ ,*

$$\begin{aligned}
 I_{n+1} &= \int_{S'} \mathbf{d}_{n+1} \cdot (\mathbf{r}' - \mathbf{a}) \prod_{j=1}^n (\mathbf{d}_j \cdot (\mathbf{r}' - \mathbf{a})) F_0((\mathbf{r}' - \mathbf{a})^2) dS' \\
 &= \frac{1}{2} \int_{\mathcal{L}} \prod_{j=1}^n (\mathbf{d}_j \cdot (\mathbf{r}' - \mathbf{a})) B_0((\mathbf{r}' - \mathbf{a})^2) (\hat{\mathbf{z}} \times \mathbf{d}_{n+1}) \cdot d\mathbf{r}' \\
 &\quad - \frac{1}{2} \sum_{m=1}^n (\mathbf{d}_{n+1} \cdot \mathbf{d}_m) \int_{S'_{j \neq m}} \prod_{j=1}^n (\mathbf{d}_j \cdot (\mathbf{r}' - \mathbf{a})) B_0((\mathbf{r}' - \mathbf{a})^2) dS' \tag{38}
 \end{aligned}$$

where the  $\mathbf{d}_j$ , numbered from 1 to  $n$ , are arbitrary constant vectors,  $B_0$  is a primitive of  $F_0$ ,  $\prod_j^0(\dots) = 1$ ,  $\prod_{j \neq m}^1(\dots) = 1$ , and  $\prod_{j \neq m}^0(\dots) = 0$ . In the both cases where  $n + 1$  is odd or even, this equation allows, by repeated applications, to reduce  $I_{n+1}$  to a combination of contour integral, depending on  $M = \lfloor (n + 3)/2 \rfloor$  successive primitives of  $F_0$ , denoted  $F_0^{-(q+1)}(x) = \frac{1}{q!} \int^x (x - t)^q F_0(t) dt$ ,  $q = 0, \dots, M - 1$ .

Proof. Using the theorem 3 with the expression (24) of the gradient, we derive (38) of the proposition. The case  $n = 0$  corresponds to proposition 3. The equation (38), after repeated applications, allows the reduction of the polynomial rank, to the rank 1 for odd orders (resp. 0 for even orders) expressed in proposition 3 (resp. in proposition 4). The result then depend on a set of  $\lfloor (n + 1)/2 \rfloor + 1$  (i.e.  $\lfloor (n + 3)/2 \rfloor$ ) successive primitives  $F_0^{-q}$  of  $F_0$ , which can be expressed by the use of integration by parts following  $F_0^{-q}(x) = \frac{1}{q!} \int^x (x - t)^q F_0(t) dt$ .

A generalization of the proposition is derived in chapter 7, by considering a function of  $\mathcal{R}^2$  instead of  $(\mathbf{r}' - \mathbf{a})^2$ , where  $\mathcal{R}^2$  is an arbitrary polynomial of  $x'$  and  $y'$  of rank 2.

Remark 4 :

The reduction to a contour integral is not unique. When we use the theorem 4, the order of the  $\mathbf{d}_j$  could have an influence on the final reduction.

### 3.3.2) On the use of asymptotics for theorem 1

The reduction to a contour integral generally supposes the knowledge of a primitive. When it cannot be expressed in closed form, its expression can often be derived asymptotically.

Lemma 1

Let  $F_l((\mathbf{r}' - \mathbf{a})^2, \varphi) = \frac{e^{-v\tilde{R}}}{\tilde{R}^n}$  be analytic on  $S'$ , with  $\tilde{R} = \sqrt{\mathbf{R} \cdot \mathbf{R}} = \sqrt{(\mathbf{r}' - \mathbf{a})^2 + z^2}$ , constants  $n$  and  $v$  with  $Re(v) > 0$ ,  $n \geq 1$ , and,

$$\frac{\partial(F((\mathbf{r}' - \mathbf{a})^2, \varphi))}{\partial((\mathbf{r}' - \mathbf{a})^2)} \equiv g((\mathbf{r}' - \mathbf{a})^2, \varphi) \frac{\partial(F_1((\mathbf{r}' - \mathbf{a})^2, \varphi))}{\partial((\mathbf{r}' - \mathbf{a})^2)}, \quad (39)$$

$o(e^{-v\tilde{R}})$  from any point with  $\varphi$  definite on  $S'$  as we let  $(\mathbf{r}' - \mathbf{a})^2 \rightarrow \infty$ . Considering  $g((\mathbf{r}' - \mathbf{a})^2, \varphi)$  as an analytic function, respectively to  $x' - a$  and  $y' - a$  variables, with smooth variation on  $S'$  where  $\frac{\partial g((\mathbf{r}' - \mathbf{a})^2, \varphi)}{\partial((\mathbf{r}' - \mathbf{a})^2)}$  is  $O(1)$  for large  $v$ , we can write, for large  $v\tilde{R}$ ,

$$\begin{aligned} & \int_{\infty}^{(\mathbf{r}' - \mathbf{a})^2} \frac{\partial(F(w, \varphi))}{\partial(w)} dw = \\ & = g((\mathbf{r}' - \mathbf{a})^2, \varphi) F_1((\mathbf{r}' - \mathbf{a})^2, \varphi) - \int_{\infty}^{(\mathbf{r}' - \mathbf{a})^2} \frac{\partial g(w, \varphi)}{\partial(w)} F_1(w, \varphi) dw \\ & \sim g((\mathbf{r}' - \mathbf{a})^2, \varphi) F_1((\mathbf{r}' - \mathbf{a})^2, \varphi) + O(1/v), \end{aligned} \quad (40)$$

Proof. The function  $F_1((\mathbf{r}' - \mathbf{a})^2, \varphi)$  is analytic on  $S'$ . We note that

$$\int_{\infty}^{(\mathbf{r}' - \mathbf{a})^2} F_1((\mathbf{r}' - \mathbf{a})^2) (\tilde{R})' d(\mathbf{r}' - \mathbf{a})^2 = \int_{\infty}^{(\mathbf{r}' - \mathbf{a})^2} \frac{e^{-v\tilde{R}}}{\tilde{R}^n} d\tilde{R} \sim -\frac{1}{v} F_1((\mathbf{r}' - \mathbf{a})^2), \quad (41)$$

and considering that the branch point of  $\tilde{R}$  is not on  $S'$ , we then derive the equations from the use of integration by parts.

Proposition 5

Let  $F_1((\mathbf{r}' - \mathbf{a})^2, \varphi) = \frac{e^{-v\tilde{R}}}{\tilde{R}^n}$  be analytic on  $S'$ , with  $\tilde{R} = \sqrt{\mathbf{R} \cdot \mathbf{R}} = \sqrt{(\mathbf{r}' - \mathbf{a})^2 + z^2}$ . For large  $v\tilde{R}$  on  $S'$ , and constants  $n, v, z_1 \neq 0$ , we can write,

$$\begin{aligned} & \int_{S'} h\left(\frac{z_1}{\tilde{R}}, \varphi\right) \frac{\partial(F_1((\mathbf{r}' - \mathbf{a})^2))}{\partial((\mathbf{r}' - \mathbf{a})^2)} dS' \\ & \sim \frac{1}{2} \int_{\mathcal{L}} \left( h\left(\frac{z_1}{\tilde{R}}, \varphi\right) F_1((\mathbf{r}' - \mathbf{a})^2) \left(1 - \frac{1}{v\tilde{R}} \left(\frac{z_1}{\tilde{R}} \frac{h'}{h}\left(\frac{z_1}{\tilde{R}}, \varphi\right)\right)\right) \right. \\ & \quad \left. - F_1(0) \left( h\left(\frac{z_1}{\tilde{R}}, \varphi\right) - \frac{h'\left(\frac{z_1}{\tilde{R}}, \varphi\right)}{v\tilde{R}} \right) \Big|_{(\mathbf{r}' - \mathbf{a})^2=0} \right) (\hat{\mathbf{z}} \times \frac{\mathbf{r}' - \mathbf{a}}{(\mathbf{r}' - \mathbf{a})^2}) \cdot d\mathbf{r}' \end{aligned} \quad (42)$$

when  $\frac{\partial(F((\mathbf{r}' - \mathbf{a})^2, \varphi))}{\partial((\mathbf{r}' - \mathbf{a})^2)} \equiv h\left(\frac{z_1}{\tilde{R}}, \varphi\right) \frac{\partial(F_1((\mathbf{r}' - \mathbf{a})^2, \varphi))}{\partial((\mathbf{r}' - \mathbf{a})^2)}$  and  $h'(w, \varphi) = \partial_w h(w, \varphi)$  is bounded on  $S'$ .

Using  $\partial_{(\mathbf{r}' - \mathbf{a})^2}(\mathbf{R}^2) = 2\tilde{R} (\tilde{R})'$ , we then have,

$$\begin{aligned} & \int_{S'} h\left(\frac{z_1}{\tilde{R}}, \varphi\right) \frac{2\tilde{R} (\tilde{R})' e^{-v\tilde{R}}}{\tilde{R}^{n+1}} dS' \\ & \sim \frac{-1}{v} \int_{\mathcal{L}} \left( h\left(\frac{z_1}{\tilde{R}}, \varphi\right) \frac{e^{-v\tilde{R}}}{\tilde{R}^n} - \left( h\left(\frac{z_1}{\tilde{R}}, \varphi\right) \frac{e^{-vz}}{z^n} \right) \Big|_{(\mathbf{r}' - \mathbf{a})^2=0} \right) (\hat{\mathbf{z}} \times \frac{\mathbf{r}' - \mathbf{a}}{(\mathbf{r}' - \mathbf{a})^2}) \cdot d\mathbf{r}' \end{aligned} \quad (43)$$



Proof. Let us consider the theorem 1 with

$$\frac{\partial(F((\mathbf{r}' - \mathbf{a})^2, \varphi))}{\partial((\mathbf{r}' - \mathbf{a})^2)} = h\left(\frac{z_1}{\tilde{R}}, \varphi\right) \frac{\partial(F_1((\mathbf{r}' - \mathbf{a})^2), \varphi)}{\partial((\mathbf{r}' - \mathbf{a})^2)}, \quad (44)$$

when  $\frac{z_1}{\tilde{R}}$  and  $h'(\frac{z_1}{\tilde{R}}, \varphi)$  are regular and bounded. In this case, we can generally obtained an asymptotic expression of  $F$  if  $F_1$  has steep variation for large  $\tilde{R}$ . For this, we consider the lemma 1, writing,

$$\begin{aligned} & \int \frac{\partial(F((\mathbf{r}' - \mathbf{a})^2, \varphi))}{\partial((\mathbf{r}' - \mathbf{a})^2)} d(\mathbf{r}' - \mathbf{a})^2 = \\ & = h\left(\frac{z_1}{\tilde{R}}, \varphi\right) F_1((\mathbf{r}' - \mathbf{a})^2, \varphi) + \int \frac{F_1((\mathbf{r}' - \mathbf{a})^2, \varphi) (\tilde{R})'}{\tilde{R}} \left(\frac{z_1 h'(\frac{z_1}{\tilde{R}}, \varphi)}{\tilde{R}}\right) d(\mathbf{r}' - \mathbf{a})^2 \\ & \sim h\left(\frac{z_1}{\tilde{R}}, \varphi\right) F_1((\mathbf{r}' - \mathbf{a})^2, \varphi) + \frac{z_1 h'(\frac{z_1}{\tilde{R}}, \varphi)}{\tilde{R}^2} \int F_1((\mathbf{r}' - \mathbf{a})^2, \varphi) (\tilde{R})' d(\mathbf{r}' - \mathbf{a})^2, \end{aligned} \quad (45)$$

with  $h'(w, \varphi) = \partial_w h(w, \varphi)$ ,  $(\tilde{R})' = \partial_{(\mathbf{r}' - \mathbf{a})^2}(\tilde{R})$ . We then obtain (42) from theorem 1 when the integrals remain convergent, then (43).

### 3.3.3) On the use of asymptotics for theorem 2

#### Proposition 6

Let  $w = (\mathbf{r}' - \mathbf{a})^2$ ,  $\mathbf{R}^2 = w + z^2$ , and  $F(w, \varphi) = F_0(w)h(\frac{w^q}{\mathbf{R}^{2q}}G(\varphi))$  be analytic relatively to  $(w)^{\frac{1}{2}}\cos\varphi = (x' - a_x)$  and  $(w)^{\frac{1}{2}}\sin\varphi = (y' - a_y)$  on  $S'$ . When  $F_0(w) = \frac{e^{-v\tilde{R}}}{\tilde{R}^{n-1}}$ , we have,

$$\begin{aligned} & \int_{S'} \mathbf{b} \cdot (\mathbf{r}' - \mathbf{a}) h\left(\frac{(\mathbf{r}' - \mathbf{a})^{2q}}{\mathbf{R}^{2q}} G(\varphi)\right) \frac{\partial F_0((\mathbf{r}' - \mathbf{a})^2)}{\partial((\mathbf{r}' - \mathbf{a})^2)} dS' \\ & \sim \frac{1}{2} \int_{\mathcal{L}} F((\mathbf{r}' - \mathbf{a})^2, \varphi) (\hat{\mathbf{z}} \times \mathbf{b}) \cdot d\mathbf{r}', \end{aligned} \quad (46)$$

for large  $v\tilde{R}$  on  $S'$ , with  $v, \mathbf{b}, z \neq 0$  some constants and  $n, q \geq 1$ ,  $\tilde{R} = \sqrt{\mathbf{R}^2}$ . Generalizing this result to a vector function  $\mathbf{h}_v$  with  $F_v(w, \varphi) = F_0(w)\mathbf{h}_v(\frac{w^q}{\mathbf{R}^{2q}}G(\varphi))$ , we have,

$$\begin{aligned} & \int_{S'} (\mathbf{r}' - \mathbf{a}) \cdot \mathbf{h}_v\left(\frac{(\mathbf{r}' - \mathbf{a})^{2q}}{\mathbf{R}^{2q}} G(\varphi)\right) \frac{\partial F_0((\mathbf{r}' - \mathbf{a})^2)}{\partial((\mathbf{r}' - \mathbf{a})^2)} dS' \\ & \sim \frac{1}{2} \int_{\mathcal{L}} (\hat{\mathbf{z}} \times F_v((\mathbf{r}' - \mathbf{a})^2, \varphi)) \cdot d\mathbf{r}'. \end{aligned} \quad (47)$$

where  $\frac{\partial F_0((\mathbf{r}' - \mathbf{a})^2)}{\partial((\mathbf{r}' - \mathbf{a})^2)} \sim \left(\frac{-vF_0((\mathbf{r}' - \mathbf{a})^2)}{2\tilde{R}}\right)$ . This remains valid as  $\text{grad}_{\mathbf{r}'}(\mathbf{R}^2) = 2(\mathbf{r}' - \mathbf{a}) + o(1)$ .

Proof. Using  $\frac{\partial}{\partial((\mathbf{r}' - \mathbf{a})^2)}(F_0((\mathbf{r}' - \mathbf{a})^2)) = \left(\frac{-vF_0((\mathbf{r}' - \mathbf{a})^2)}{2R}\right)\left(1 + \frac{n}{vR}\right)$ , and

$$\begin{aligned} \frac{\partial}{\partial((\mathbf{r}' - \mathbf{a})^2)}\left(h\left(\frac{(\mathbf{r}' - \mathbf{a})^{2q}}{R^{2q}}G(\varphi)\right)\right) &= \frac{q}{R^2}\left(\frac{z^2(\mathbf{r}' - \mathbf{a})^{2q-2}}{R^{2q}}G(\varphi)h'\left(\frac{(\mathbf{r}' - \mathbf{a})^{2q}}{R^{2q}}G(\varphi)\right)\right), \\ \frac{1}{(\mathbf{r}' - \mathbf{a})^2}\frac{\partial}{\partial\varphi}\left(h\left(\frac{(\mathbf{r}' - \mathbf{a})^{2q}}{R^{2q}}G(\varphi)\right)\right) &= \frac{1}{R^2}\left(\frac{(\mathbf{r}' - \mathbf{a})^{2q-2}}{R^{2q-2}}G'(\varphi)h'\left(\frac{(\mathbf{r}' - \mathbf{a})^{2q}}{R^{2q}}G(\varphi)\right)\right), \end{aligned} \quad (48)$$

we have

$$h\left(\frac{(\mathbf{r}' - \mathbf{a})^{2q}}{R^{2q}}G(\varphi)\right)\text{grad}_{\mathbf{r}'}(F_0((\mathbf{r}' - \mathbf{a})^2)) \sim \text{grad}_{\mathbf{r}'}(F((\mathbf{r}' - \mathbf{a})^2, \varphi)), \quad (49)$$

We can then derive (46) from theorem 2, and use it for each component of  $\mathbf{h}_v$  to derive (47).

#### ***4) Basic reduction in case (a) for pure dipolar illuminations : monostatic Physical Optics diffracted fields and primitives for an exact reduction***

We study here the reduction of Physical Optics radiation integrals belonging to case (a) (defined in section 2) for pure electric and magnetic dipolar illumination, from results of previous sections. A rewriting of the PO integrals, and the expressions of primitives involved in the reduction, are given for flat surfaces. A generalization for non dipolar illuminations and for the case of  $S$  with low curvature will be respectively given in section 5 and 7.

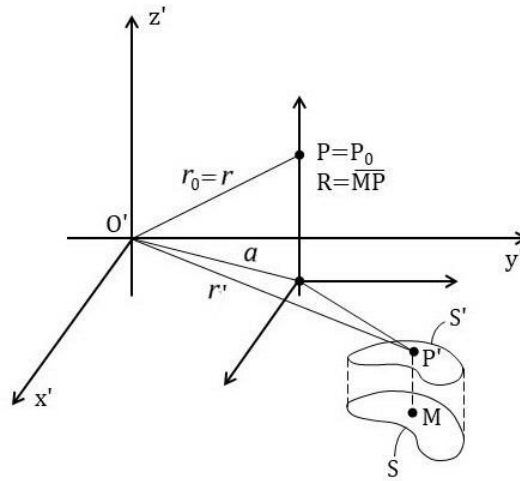


figure 2 : diffraction by  $S$  illuminated by source at  $\mathbf{r}_0 = \mathbf{r}$  (monostatic case)

Our method allows to study the PO diffraction for perfectly and imperfectly conducting surfaces. To simplify the reading, we choose to describe the latter case (surface with impedance boundary conditions) in an appendice.

#### 4.1) Basic expansion of kernels in radiation surface integral expressions

Let us consider the expressions of the electric and magnetic fields,  $\mathbf{E}$  and  $\mathbf{H}$ , radiated by electric and magnetic currents  $\mathbf{J}$  and  $\mathbf{M}$  impressed on a real surface  $S$ , in free space with impedance  $Z_0$  and wavenumber  $k = \frac{2\pi}{\lambda}$ , at the observation point  $(\mathbf{a} + z\hat{\mathbf{z}}) \notin S$  [1], given in (1), where  $\psi(\tilde{R}) = \frac{e^{-ik\tilde{R}}}{\tilde{R}}$ ,  $\tilde{R} = \sqrt{\mathbf{R} \cdot \mathbf{R}}$ ,  $\mathbf{R} = \mathbf{a} + z\hat{\mathbf{z}} - (\mathbf{r}' + z'\hat{\mathbf{z}}) = \tilde{R}\mathbf{u}$ , and  $(\mathbf{r}' + z'\hat{\mathbf{z}})$  is the point of integration on  $S$ . We can write,

$$\begin{aligned} (\text{grad}_r \text{div}_r + k^2)(\psi \mathbf{J}) &= \tilde{R} \left( \frac{\psi'}{\tilde{R}} \right)' (\mathbf{J} \cdot \mathbf{u}) \mathbf{u} + \frac{\psi'}{\tilde{R}} \mathbf{J} + k^2 \psi \mathbf{J}, \\ \text{curl}_r(\psi \mathbf{J}) &= -\mathbf{J} \times \text{grad}_r \psi = \frac{\psi'}{\tilde{R}} \mathbf{R} \times \mathbf{J}, \end{aligned} \quad (50)$$

where  $\psi' = \partial_{\tilde{R}} \psi$  and  $\left( \frac{\psi'}{\tilde{R}} \right)' = \partial_{\tilde{R}} \left( \frac{\psi'}{\tilde{R}} \right)$ ; from  $\Delta \psi = \frac{1}{R^2} (\tilde{R})^3 \frac{\psi'}{\tilde{R}}$ , and  $\tilde{R} \left( \frac{\psi'}{\tilde{R}} \right)' + k^2 \psi = -3 \frac{\psi'}{\tilde{R}}$  with  $(\mathbf{J} \cdot \mathbf{u}) \mathbf{u} = \mathbf{u} \times (\mathbf{u} \times \mathbf{J}) + \mathbf{J}$ , we derive,

$$(\text{grad}_r \text{div}_r + k^2)(\psi \mathbf{J}) = \tilde{R} \left( \frac{\psi'}{\tilde{R}} \right)' \mathbf{u} \times (\mathbf{u} \times \mathbf{J}) - \frac{2\psi'}{\tilde{R}} \mathbf{J}. \quad (51)$$

This gives us, at  $(\mathbf{a} + z\hat{\mathbf{z}}) \notin S$ ,

$$\begin{aligned} \mathbf{E} &= \frac{Z_0}{4\pi i k} \int_S \left( \tilde{R} \left( \frac{\psi'}{\tilde{R}} \right)' \mathbf{u} \times (\mathbf{u} \times \mathbf{J}) - \frac{2\psi'}{\tilde{R}} \mathbf{J} \right) dS - \frac{1}{4\pi} \int_S \frac{\psi'}{\tilde{R}} \mathbf{R} \times \mathbf{M} dS, \\ Z_0 \mathbf{H} &= \frac{Z_0}{4\pi} \int_S \frac{\psi'}{\tilde{R}} \mathbf{R} \times \mathbf{J} dS + \frac{1}{4\pi i k} \int_S \left( \tilde{R} \left( \frac{\psi'}{\tilde{R}} \right)' \mathbf{u} \times (\mathbf{u} \times \mathbf{M}) - \frac{2\psi'}{\tilde{R}} \mathbf{M} \right) dS. \end{aligned} \quad (52)$$

We now study the case where  $S$  is equal to its projection  $S'$  on the plane  $z = 0$ . A generalization for the case of  $S$  with low curvature will be given in section 7.

#### 4.2) Exact reduction to contour integrals for perfectly conducting flat plate

In monostatic case, the primary point source and the observation point have the same position. The Physical Optics radiation integrals in perfectly conducting case ( $\mathbf{M}|_S = 0$ ), for electric and magnetic dipolar illuminations, can be then reworked in such a form that they can be exactly reduced to contour integrals, with all required primitives in closed forms.

##### Lemma 2

*Let us consider a surface integral on  $S'$  which is a combination of integrals of the type,*

$$\int_{S'} B(\mathbf{R}^2) dS', \int_{S'} \frac{B(\mathbf{R}^2)(\mathbf{R}^2 - z^2)}{\mathbf{R}^2} G(\varphi) dS' \text{ and } \int_{S'} (\mathbf{a} - \mathbf{r}') B(\mathbf{R}^2) dS', \quad (53)$$

where  $\mathbf{R}^2 = (\mathbf{r}' - \mathbf{a})^2 + z^2$ ,  $G(\varphi) = \frac{((\mathbf{r}' - \mathbf{a}) \cdot \hat{\mathbf{d}})(\mathbf{r}' - \mathbf{a})}{(\mathbf{r}' - \mathbf{a})^2}$ , with  $z \neq 0$  and  $\hat{\mathbf{d}}$  some constants. If the point  $\mathbf{a}$  belongs to an open connected domain which includes a point of  $S'$ , where  $B(\mathbf{R}^2)$  and  $B(\mathbf{R}^2)/\mathbf{R}^2$  is analytic on  $S'$ , we can, with analytic expressions of primitives of  $B(t)$  and  $B(t)/t$ , reduce the integral to a non-singular contour integral, by using the proposition 4 (derived from theorem 1), and the proposition 3 (derived from theorem 2).

Proof. To reduce the integrals to contour integrals, we let successively  $\frac{\partial(F((\mathbf{r}' - \mathbf{a})^2, \varphi))}{\partial((\mathbf{r}' - \mathbf{a})^2)}$  as  $B(\mathbf{R}^2)$  and  $\frac{B(\mathbf{R}^2)(\mathbf{R}^2 - z^2)}{\mathbf{R}^2} G(\varphi)$  in proposition 4, involving the primitives of  $B(t)$  and  $B(t)/t$ , and  $\frac{\partial(F((\mathbf{r}' - \mathbf{a})^2)}{\partial((\mathbf{r}' - \mathbf{a})^2)}$  as  $B(\mathbf{R}^2)$  in proposition 3, involving the primitive of  $B(t)$ .

#### 4.2.1) Physical Optics diffracted fields and reduction, for an electric dipolar source

##### Theorem 5

The Physical Optics diffracted field, radiated by a perfectly conducting surface  $S'$  of the plane  $z' = 0$ , when illuminated by a electric dipole  $\mathbf{J}_0 = \hat{\mathbf{d}}\delta(\mathbf{r}' - \mathbf{r}_0)$ , can be written, at  $\mathbf{r} = z\hat{\mathbf{z}} + \mathbf{a} \notin S'$ , when  $\mathbf{r}_0 = \mathbf{r}$  (monostatic case), following,

$$\begin{aligned} \mathbf{E} &= \frac{-Z_0 \hat{\mathbf{d}}}{16\pi^2 ik} \int_{S'} z \tilde{\mathbf{R}} \left( \left( \frac{\psi'}{\tilde{\mathbf{R}}} \right)^2 \right)' dS' + \frac{Z_0}{16\pi^2 ik} \int_{S'} z \tilde{\mathbf{R}} \left( \left( \frac{\psi'}{\tilde{\mathbf{R}}} \right)^2 \right)' (\mathbf{u} \cdot \hat{\mathbf{d}}) \mathbf{u} dS' + \\ &+ \frac{Z_0 (\hat{\mathbf{z}} \cdot \hat{\mathbf{d}})}{4\pi^2 ik} \int_{S'} \left( \frac{\psi'}{\tilde{\mathbf{R}}} \right)^2 (\mathbf{a} - \mathbf{r}') dS' + \frac{Z_0 z ((\hat{\mathbf{z}} \cdot \hat{\mathbf{d}}) \hat{\mathbf{z}} - \hat{\mathbf{d}})}{4\pi^2 ik} \int_{S'} \left( \frac{\psi'}{\tilde{\mathbf{R}}} \right)^2 dS', \\ \mathbf{H} &= \frac{-\hat{\mathbf{d}}}{8\pi^2} \times \int_{S'} z \mathbf{R} \left( \frac{\psi'}{\tilde{\mathbf{R}}} \right)^2 dS', \end{aligned} \quad (54)$$

with  $(\cdot)' \equiv \partial_{\tilde{\mathbf{R}}}(\cdot)$ ,  $\psi(\tilde{\mathbf{R}}) = \frac{e^{-ik\tilde{\mathbf{R}}}}{\tilde{\mathbf{R}}}$ ,  $\tilde{\mathbf{R}} = \sqrt{\mathbf{R} \cdot \mathbf{R}}$ ,  $\mathbf{R}^2 = (\mathbf{r}' - \mathbf{a})^2 + z^2$ , and,

$$(\mathbf{u} \cdot \hat{\mathbf{d}}) \mathbf{u} = \frac{(\hat{\mathbf{d}} \cdot \hat{\mathbf{z}}) \hat{\mathbf{z}} z^2}{\mathbf{R}^2} + \frac{(\hat{\mathbf{d}} \cdot \hat{\mathbf{z}} + \hat{\mathbf{z}}(\hat{\mathbf{d}} \cdot (\cdot))) z (\mathbf{a} - \mathbf{r}')}{\mathbf{R}^2} + \frac{\mathbf{R}^2 - z^2}{\mathbf{R}^2} \frac{((\mathbf{r}' - \mathbf{a}) \cdot \hat{\mathbf{d}})(\mathbf{r}' - \mathbf{a})}{(\mathbf{r}' - \mathbf{a})^2} \quad (55)$$

Using the following exact primitives,

$$\begin{aligned} \int \frac{1}{\tilde{\mathbf{R}}} \left( \left( \frac{\psi'}{\tilde{\mathbf{R}}} \right)^2 \right)' d\mathbf{R}^2 &= 2 \frac{(ik + \frac{1}{\tilde{\mathbf{R}}})^2}{\mathbf{R}^4} e^{-i2k\tilde{\mathbf{R}}}, \int \left( \frac{\psi'}{\tilde{\mathbf{R}}} \right)^2 d\mathbf{R}^2 = -\frac{(1 + 2ik\tilde{\mathbf{R}})}{2\mathbf{R}^4} e^{-i2k\tilde{\mathbf{R}}}, \\ \int \tilde{\mathbf{R}} \left( \left( \frac{\psi'}{\tilde{\mathbf{R}}} \right)^2 \right)' d\mathbf{R}^2 &= 2 \left( -k^2 + \frac{3ik}{\tilde{\mathbf{R}}} + \frac{3}{2\mathbf{R}^2} \right) \frac{e^{-i2k\tilde{\mathbf{R}}}}{\mathbf{R}^2}, \end{aligned} \quad (56)$$

the lemma 2 applies, and we can exactly reduce every integral of (54) to a non-singular contour integral whose integrand involves no special function.

Proof. We can write the Physical Optics field with (52), considering that  $\mathbf{J} = 2\hat{\mathbf{z}} \times \mathbf{H}_i$  and  $\mathbf{M} = 0$  on  $S'$ , where  $\mathbf{H}_i$  is radiated by the source  $\mathbf{J}_0 = \hat{\mathbf{d}}\delta(\mathbf{r}' - \mathbf{r}_0)$ , and thus

$$2\pi\mathbf{J} = \hat{\mathbf{z}} \times (\mathbf{R}_0 \times \frac{\psi'_0}{\tilde{R}_0} \hat{\mathbf{d}}) = (z\hat{\mathbf{z}} \cdot \hat{\mathbf{d}}) \frac{\psi'_0}{\tilde{R}_0} \mathbf{R}_0 + z_0 \frac{\psi'_0}{\tilde{R}_0} \hat{\mathbf{d}}, \quad \mathbf{R}_0 = (\mathbf{r}' - \mathbf{r}_0) \quad (57)$$

In the monostatic case, when  $\mathbf{r} = \mathbf{r}_0 = z\hat{\mathbf{z}} + \mathbf{a} \notin S'$  and  $\mathbf{R} = -\mathbf{R}_0 = \tilde{R} \mathbf{u}$ , we can write,

$$\begin{aligned} \mathbf{E} &= \frac{Z_0}{8\pi^2 ik} \int_{S'} z\tilde{R} \left(\frac{\psi'}{\tilde{R}}\right)' \mathbf{u} \times (\mathbf{u} \times \frac{\psi'}{\tilde{R}} \hat{\mathbf{d}}) + 2\left(\frac{\psi'}{\tilde{R}}\right)^2 ((z\hat{\mathbf{z}} \cdot \hat{\mathbf{d}}) \mathbf{R} - z\hat{\mathbf{d}}) dS', \\ &= \frac{-Z_0 \hat{\mathbf{d}}}{16\pi^2 ik} \int_{S'} z\tilde{R} \left(\left(\frac{\psi'}{\tilde{R}}\right)^2\right)' dS' + \frac{Z_0}{16\pi^2 ik} \int_{S'} z\tilde{R} \left(\left(\frac{\psi'}{\tilde{R}}\right)^2\right)' (\mathbf{u} \cdot \hat{\mathbf{d}}) \mathbf{u} dS' + \\ &+ \frac{Z_0 (z\hat{\mathbf{z}} \cdot \hat{\mathbf{d}})}{4\pi^2 ik} \int_{S'} \left(\frac{\psi'}{\tilde{R}}\right)^2 (\mathbf{a} - \mathbf{r}') dS' + \frac{Z_0 ((z\hat{\mathbf{z}} \cdot \hat{\mathbf{d}})z - \hat{\mathbf{d}})}{4\pi^2 ik} \int_{S'} z \left(\frac{\psi'}{\tilde{R}}\right)^2 dS', \end{aligned} \quad (58)$$

and

$$\mathbf{H} = \frac{-1}{8\pi^2} \int_{S'} \left(\frac{\psi'}{\tilde{R}}\right)^2 \mathbf{R} \times (z\hat{\mathbf{z}} \times (\mathbf{R} \times \hat{\mathbf{d}})) dS' = \frac{-\hat{\mathbf{d}}}{8\pi^2} \times \int_{S'} z\mathbf{R} \left(\frac{\psi'}{\tilde{R}}\right)^2 dS', \quad (59)$$

and thus (54). The lemma 2 then applies, and we can reduce the expressions of  $\mathbf{E}$  and  $\mathbf{H}$  to contour integrals, by using the exact primitives given in (56) (derived in appendice A). Let us notice that, for large  $k\tilde{R}$ ,

$$\mathbf{E} \sim \frac{Z_0}{8\pi^2 ik} \int_{S'} \frac{z\tilde{R}}{2} \left(\left(\frac{\psi'}{\tilde{R}}\right)^2\right)' ((\mathbf{u} \cdot \hat{\mathbf{d}}) \mathbf{u} - \hat{\mathbf{d}}) dS', \quad \mathbf{H} \sim \frac{-\hat{\mathbf{d}}}{8\pi^2} \times \int_{S'} z\mathbf{R} \left(\frac{\psi'}{\tilde{R}}\right)^2 dS'. \quad (60)$$

#### 4.2.2) Physical Optics diffracted fields and reduction, for a magnetic dipolar source

##### Theorem 6

*The Physical Optics diffracted field, radiated by a perfectly conducting surface  $S'$  of the plane  $z' = 0$ , when illuminated by a magnetic dipole  $\mathbf{M}_0 = \hat{\mathbf{d}}\delta(\mathbf{r}' - \mathbf{r}_0)$ , can be written, at  $\mathbf{r} = z\hat{\mathbf{z}} + \mathbf{a} \notin S'$ , when  $\mathbf{r}_0 = \mathbf{r}$  (monostatic case), following,*

$$\begin{aligned}
\mathbf{E} &= \frac{1}{8(\pi ik)^2} \int_{S'} z \left( \frac{\psi'}{\tilde{R}} \right)' \left( \frac{\psi'}{\tilde{R}} \right)' (\mathbf{R} \times \hat{\mathbf{d}}) dS' - \\
&\quad - \frac{1}{4(\pi ik)^2} \int_{S'} \hat{\mathbf{z}} \times \left( \tilde{R} \left( \left( \frac{\psi'}{\tilde{R}} \right)^2 \right)' (\mathbf{u}(\mathbf{u} \cdot \hat{\mathbf{d}}) - \hat{\mathbf{d}}) - 2 \left( \frac{\psi'}{\tilde{R}} \right)^2 \hat{\mathbf{d}} \right) dS', \\
\mathbf{H} &= \frac{1}{8\pi^2 ik Z_0} \int_{S'} \frac{z}{2} \tilde{R} \left( \left( \frac{\psi'}{\tilde{R}} \right)^2 \right)' (\hat{\mathbf{d}} - \mathbf{u}(\mathbf{u} \cdot \hat{\mathbf{d}})) - 2 \left( \frac{\psi'}{\tilde{R}} \right)^2 \mathbf{R} \times \hat{\mathbf{d}} dS', \tag{61}
\end{aligned}$$

with  $(\cdot)' \equiv \partial_{\tilde{R}}(\cdot)$ ,  $\psi(\tilde{R}) = \frac{e^{-ik\tilde{R}}}{\tilde{R}}$  and (55). Using the exact primitives given in (56) and,

$$\int \left( \frac{\psi'}{\tilde{R}} \right)' \left( \frac{\psi'}{\tilde{R}} \right)' d\mathbf{R}^2 = ik^3 \frac{\psi^2}{\tilde{R}} + 3 \left( \frac{\psi}{\tilde{R}} \right)^2 \left( \frac{k^2}{2} - \left( ik + \frac{1}{\tilde{R}} \right)^2 \right), \tag{62}$$

the lemma 2 applies, and we can exactly reduce every integral of (61) to non-singular contour integral whose integrand involves no special function.

Proof. We can write the Physical Optics field with (52), considering that the sources  $\mathbf{J} = 2\hat{\mathbf{z}} \times \mathbf{H}_i$  and  $\mathbf{M} = 0$  on  $S'$ , where  $\mathbf{H}_i$  is radiated by  $\mathbf{M}_0 = \hat{\mathbf{d}}\delta(\mathbf{r}' - \mathbf{r}_0)$ , and thus,

$$\mathbf{J} = \frac{1}{2\pi ik Z_0} \hat{\mathbf{z}} \times \left( \tilde{R}_0 \left( \frac{\psi'_0}{\tilde{R}_0} \right)' \mathbf{u}_0 \times (\mathbf{u}_0 \times \hat{\mathbf{d}}) - \frac{2\psi'_0}{\tilde{R}_0} \hat{\mathbf{d}} \right), \quad \mathbf{R}_0 = (\mathbf{r}' - \mathbf{r}_0), \quad \mathbf{u}_0 = \frac{\mathbf{R}_0}{\tilde{R}_0}, \tag{63}$$

In the monostatic case  $\mathbf{R} = -\mathbf{R}_0 = z\hat{\mathbf{z}} + \mathbf{a} - \mathbf{r}'$ , we then obtain,

$$\begin{aligned}
\mathbf{E} &= \frac{1}{8(\pi ik)^2} \int_{S'} z \tilde{R} \left( \frac{\psi'}{\tilde{R}} \right)' (\mathbf{u} \times \left( \frac{\psi'}{\tilde{R}} \right)' \hat{\mathbf{d}}) - \tilde{R} \left( \frac{\psi'}{\tilde{R}} \right)' \hat{\mathbf{z}} \times (\mathbf{u} \times (\mathbf{u} \times \frac{2\psi'}{\tilde{R}} \hat{\mathbf{d}})) dS' - \\
&\quad - \frac{1}{8(\pi ik)^2} \int_{S'} \frac{2\psi'}{\tilde{R}} \hat{\mathbf{z}} \times \left( \tilde{R} \left( \frac{\psi'}{\tilde{R}} \right)' \mathbf{u} \times (\mathbf{u} \times \hat{\mathbf{d}}) - \frac{2\psi'}{\tilde{R}} \hat{\mathbf{d}} \right) dS' \\
&= \frac{1}{8(\pi ik)^2} \int_{S'} z \left( \frac{\psi'}{\tilde{R}} \right)' \left( \frac{\psi'}{\tilde{R}} \right)' (\mathbf{R} \times \hat{\mathbf{d}}) dS' - \\
&\quad - \frac{1}{4(\pi ik)^2} \int_{S'} \hat{\mathbf{z}} \times \left( \tilde{R} \left( \left( \frac{\psi'}{\tilde{R}} \right)^2 \right)' (\mathbf{u}(\mathbf{u} \cdot \hat{\mathbf{d}}) - \hat{\mathbf{d}}) - 2 \left( \frac{\psi'}{\tilde{R}} \right)^2 \hat{\mathbf{d}} \right) dS', \tag{64}
\end{aligned}$$

and

$$\begin{aligned}
\mathbf{H} &= \frac{1}{8\pi^2 ik Z_0} \int_{S'} \frac{\psi'}{\tilde{R}} \mathbf{R} \times \left( \hat{\mathbf{z}} \times \left( \tilde{R} \left( \frac{\psi'}{\tilde{R}} \right)' \mathbf{u} \times (\mathbf{u} \times \hat{\mathbf{d}}) - \frac{2\psi'}{\tilde{R}} \hat{\mathbf{d}} \right) \right) dS' \\
&= \frac{1}{8\pi^2 ik Z_0} \int_{S'} \frac{z}{2} \tilde{R} \left( \left( \frac{\psi'}{\tilde{R}} \right)^2 \right)' (\hat{\mathbf{d}} - \mathbf{u}(\mathbf{u} \cdot \hat{\mathbf{d}})) - 2 \left( \frac{\psi'}{\tilde{R}} \right)^2 \mathbf{R} \times \hat{\mathbf{d}} dS'. \tag{65}
\end{aligned}$$

and thus (61). Considering these expressions with (55), the lemma 2 applies, and we can completely reduce them to contour integrals, by using the primitives given in (56) and (62)

(derived in appendice A). Let us notice that, for large  $k\tilde{R}$ ,

$$\mathbf{E} \sim \frac{-\hat{\mathbf{d}} \times}{8\pi^2(ik)^2} \int_{S'} z \mathbf{R} \left( \left( \frac{\psi'}{\tilde{R}} \right)' \right)^2 dS', \mathbf{H} \sim \frac{1}{8\pi^2 Z_0} \int_{S'} \frac{z(\tilde{R} \left( \left( \frac{\psi'}{\tilde{R}} \right)' \right)^2 (\hat{\mathbf{d}} - \mathbf{u}(\mathbf{u} \cdot \hat{\mathbf{d}}))}{2ik} dS'. \quad (66)$$

### 5) Reduction to contour integrals in case (a) for a perfectly conducting flat plate illuminated by a non pure dipolar illumination

#### 5.1) Asymptotic surface integral expressions for a weighted dipolar illumination

We now consider the incident field, as a combination of radiations for electric and magnetic dipoles  $\mathbf{J}_0 = \hat{\mathbf{d}}_e \delta(\mathbf{r}' - \mathbf{r}_0)$  and  $\mathbf{M}_0 = \hat{\mathbf{d}}_h \delta(\mathbf{r}' - \mathbf{r}_0)$  weighted by  $f_{e,h}(\mathbf{u}_0)$ , following,

$$\begin{aligned} \mathbf{E}_i(\mathbf{r}') &\sim \frac{1}{4\pi ik} (f_e(\mathbf{u}_0) \tilde{R}_0 \left( \frac{\psi'_0}{\tilde{R}_0} \right)' (\mathbf{u}_0 \times (\mathbf{u}_0 \times Z_0 \hat{\mathbf{d}}_e)) - ik f_h(\mathbf{u}_0) \frac{\psi'_0}{\tilde{R}_0} \mathbf{R}_0 \times \hat{\mathbf{d}}_h) \\ &\sim \frac{ik}{4\pi} \frac{e^{-ik\tilde{R}_0}}{\tilde{R}_0} (f_e(\mathbf{u}_0) (\mathbf{u}_0 \times (\mathbf{u}_0 \times Z_0 \hat{\mathbf{d}}_e)) + f_h(\mathbf{u}_0) (\mathbf{u}_0 \times \hat{\mathbf{d}}_h)), \\ \mathbf{H}_i(\mathbf{r}') &\sim \frac{1}{4\pi ik Z_0} (ik f_e(\mathbf{u}_0) \frac{\psi'_0}{\tilde{R}_0} (\mathbf{R}_0 \times Z_0 \hat{\mathbf{d}}_e) + f_h(\mathbf{u}_0) \left( \frac{\psi'_0}{\tilde{R}_0} \right)' \mathbf{R}_0 \times (\mathbf{u}_0 \times \hat{\mathbf{d}}_h)) \\ &\sim \frac{ik}{4\pi Z_0} \frac{e^{-ik\tilde{R}_0}}{\tilde{R}_0} (-f_e(\mathbf{u}_0) (\mathbf{u}_0 \times Z_0 \hat{\mathbf{d}}_e) + f_h(\mathbf{u}_0) (\mathbf{u}_0 \times (\mathbf{u}_0 \times \hat{\mathbf{d}}_h))). \end{aligned} \quad (67)$$

for large  $k\tilde{R}_0$ , where  $\mathbf{R}_0 = \mathbf{r}' - \mathbf{r}_0 = \tilde{R}_0 \mathbf{u}_0$ ,  $\tilde{R}_0 = \sqrt{\mathbf{R}_0 \cdot \mathbf{R}_0}$ , and the functions  $f_{e,h}(\mathbf{u}_0) (\mathbf{u}_0 \times \hat{\mathbf{d}}_h)$  are analytic as  $\mathbf{r}' \in S'$ ,  $\mathbf{r}_0 \notin S'$ , with  $\mathbf{E}_i(\mathbf{r}') \sim -Z_0 \mathbf{u}_0 \times \mathbf{H}_i(\mathbf{r}')$ .

In this case, the near-field monostatic diffracted field expressions at  $\mathbf{r} = \mathbf{r}_0 \notin S'$ , for a perfectly conducting plate  $S'$  illuminated by a source of arbitrary radiation pattern, valid from infinity to a distance qualified to be short relatively to the dimensions of  $S'$ , can be derived from (52) with  $\mathbf{J} = 2\hat{\mathbf{z}} \times \mathbf{H}_i$  and  $\mathbf{M} = 0$  on  $S'$ .

Therefore, we obtain the combination of (60) and (66) weighted by  $f_{e,h}$ , following,

$$\begin{aligned} \mathbf{E}(\mathbf{r}) &\sim \frac{1}{8\pi^2(ik)^2} \left( \int_{S'} z f_e(-\mathbf{u}) \frac{ik}{2} \left( \left( \frac{\psi'}{\tilde{R}} \right)' \right)^2 (\mathbf{R} \times (\mathbf{u} \times Z_0 \hat{\mathbf{d}}_e)) dS' \right. \\ &\quad \left. + \int_{S'} z f_h(-\mathbf{u}) \left( \left( \frac{\psi'}{\tilde{R}} \right)' \right)^2 (\mathbf{R} \times \hat{\mathbf{d}}_h) dS' \right), \\ \mathbf{H}(\mathbf{r}) &\sim \frac{1}{8\pi^2(ik)^2 Z_0} \left( \int_{S'} z f_e(-\mathbf{u}) (ik)^2 \left( \frac{\psi'}{\tilde{R}} \right)' (\mathbf{R} \times Z_0 \hat{\mathbf{d}}_e) dS' \right. \\ &\quad \left. - \int_{S'} z f_h(-\mathbf{u}) \frac{ik}{2} \left( \left( \frac{\psi'}{\tilde{R}} \right)' \right)^2 (\mathbf{R} \times (\mathbf{u} \times \hat{\mathbf{d}}_h)) dS' \right), \end{aligned} \quad (68)$$

for large  $k\tilde{R}$ , with  $\mathbf{r} = \mathbf{r}_0 = \mathbf{a} + z\hat{\mathbf{z}}$ ,  $\tilde{R} = \sqrt{\mathbf{R} \cdot \mathbf{R}}$ ,  $\mathbf{R} = \tilde{R} \mathbf{u} = \mathbf{a} + z\hat{\mathbf{z}} - \mathbf{r}'$ ,  $\mathbf{u} = -\mathbf{u}_0$ . In this frame, let us notice, that, if we want to consider a source with a spherical pattern, instead of a dipole with toric pattern, we can apply the weighting factors  $f_{e,h}(\mathbf{u}_0) = \beta_{e,h} / \sqrt{(\mathbf{u}_0 \times \hat{\mathbf{d}}_{e,h})^2}$ , with  $(\mathbf{u}_0 \times \hat{\mathbf{d}}_{e,h})^2 = 1 - (\mathbf{u}_0 \cdot \hat{\mathbf{d}}_{e,h})^2$ .

## 5.2) Reduction to contour integrals for even and odd weight functions $f_{e,h}$

Thereafter, we reduce  $\mathbf{E}$  in (68) to a contour integral, for  $f_h \equiv 0$  (resp.  $f_e \equiv 0$ ), when  $f_e$  (resp.  $f_h$ ) is even or odd relatively to coordinates of  $\frac{\mathbf{r}' - \mathbf{a}}{R}$ , in the cases where  $\hat{\mathbf{d}}_{e,h} \cdot \hat{\mathbf{z}} = 0$ .

### 5.2.1) Electric case with $f_h \equiv 0$ , $\hat{\mathbf{d}} = \hat{\mathbf{d}}_e$

Let us consider the case with  $f_h \equiv 0$ ,  $\hat{\mathbf{d}} = \hat{\mathbf{d}}_e$ . The plate  $S'$  is then illuminated by,

$$\mathbf{E}_i \sim \frac{ikZ_0}{4\pi} \frac{e^{-ik\tilde{R}_0}}{\tilde{R}_0} f_e(\mathbf{u}_0)(\mathbf{u}_0(\mathbf{u}_0 \cdot \hat{\mathbf{d}}) - \hat{\mathbf{d}}), \quad (69)$$

and the PO diffracted field at  $\mathbf{r} = \mathbf{a} + z\hat{\mathbf{z}} \notin S'$ , when  $\mathbf{r} = \mathbf{r}_0$  (monostatic case), is given by,

$$\mathbf{E} \sim -\frac{Z_0}{8\pi k^2} \frac{i}{\lambda} \int_{S'} z\tilde{R} \left( \left( \frac{\psi'}{\tilde{R}} \right)^2 \right)' f_e(-\mathbf{u}) ((\mathbf{u} \cdot \hat{\mathbf{d}})\mathbf{u} - \hat{\mathbf{d}}) dS'. \quad (70)$$

### Theorem 7

When  $f_e(-\mathbf{u}) = \frac{f(\frac{-z}{\tilde{R}}, \frac{\mathbf{r}' - \mathbf{a}}{\tilde{R}}) + f(\frac{-z}{\tilde{R}}, \frac{\mathbf{a} - \mathbf{r}'}{\tilde{R}})}{2} = \frac{g(\frac{z}{\tilde{R}}, \frac{\mathbf{a} - \mathbf{r}'}{\tilde{R}})}{\sqrt{1 - (\mathbf{u} \cdot \hat{\mathbf{d}})^2}}$  and  $\hat{\mathbf{d}} \cdot \hat{\mathbf{z}} = 0$ , the integral reduces to,

$$\begin{aligned} \mathbf{E} \sim & \frac{-i}{\lambda} \left( \frac{1}{2} \int_{\mathcal{L}} (F_1((\mathbf{r}' - \mathbf{a})^2, \varphi) f_e(-\mathbf{u}) - F_1(0, \varphi) g(1, 0)) [(\hat{\mathbf{z}} \times \frac{\mathbf{r}' - \mathbf{a}}{(\mathbf{r}' - \mathbf{a})^2}) \cdot d\mathbf{r}'] \right) + \\ & + \frac{i}{\lambda} \left( \frac{1}{2} \int_{\mathcal{L}} (F_2((\mathbf{r}' - \mathbf{a})^2) f_e(-\mathbf{u}) - F_2(0) g(1, 0)) [(\hat{\mathbf{z}} \times \hat{\mathbf{d}}) \cdot d\mathbf{r}'] \right), \end{aligned} \quad (71)$$

where  $\mathcal{L} \equiv \partial S'$  is oriented counterclockwise, and the primitives  $F_1$  and  $F_2$  are written,

$$\begin{aligned} F_1 &= \frac{Z_0 z}{8\pi k^2} \int \tilde{R} \left( \left( \frac{\psi'}{\tilde{R}} \right)^2 \right)' \left( \frac{(\mathbf{r}' - \mathbf{a})^2 G(\varphi)}{R^2} - \hat{\mathbf{d}} \right) d\mathbf{R}^2, \quad F_2 = \frac{Z_0 z^2 \hat{\mathbf{z}}}{8\pi k^2} \int \tilde{R} \left( \left( \frac{\psi'}{\tilde{R}} \right)^2 \right)' \frac{1}{R^2} d\mathbf{R}^2 \\ F_1 &\sim -\frac{Z_0}{4\pi} \frac{z e^{-2ik\tilde{R}}}{R^2} \left( \frac{(\mathbf{r}' - \mathbf{a})^2 G(\varphi)}{R^2} - \hat{\mathbf{d}} \right), \quad F_2 \sim -\frac{Z_0}{4\pi} \frac{z^2 \hat{\mathbf{z}} e^{-2ik\tilde{R}}}{R^4}, \end{aligned} \quad (72)$$

for  $z \neq 0$ , with  $G(\varphi) = \frac{((\mathbf{r}' - \mathbf{a}) \cdot \hat{\mathbf{d}})(\mathbf{r}' - \mathbf{a})}{(\mathbf{r}' - \mathbf{a})^2}$ ,  $F_1(0, \varphi) = \frac{Z_0}{4\pi} \frac{\hat{\mathbf{d}} e^{-2ikz}}{z}$  and  $F_2(0) = -\frac{Z_0}{4\pi} \frac{\hat{\mathbf{z}} e^{-2ikz}}{z^2}$ ,  $f_e(-\mathbf{u})$  regular on  $S'$ . The  $VV$  polarization term is given by  $\mathbf{E} \cdot \hat{\mathbf{d}}$  when  $\mathbf{r}_0 \cdot \hat{\mathbf{d}} = 0$ .



Proof. We detail (70), and notice that, when  $\widehat{\mathbf{d}} \cdot \widehat{\mathbf{z}} = 0$ , we have

$$(\mathbf{u} \cdot \widehat{\mathbf{d}})\mathbf{u} - \widehat{\mathbf{d}} = \frac{(\mathbf{r}' - \mathbf{a})^2 G(\varphi)}{R^2} - \widehat{\mathbf{d}} + \frac{(\mathbf{a} - \mathbf{r}') \cdot \widehat{\mathbf{d}}}{R^2} z \widehat{\mathbf{z}}, G(\varphi) = \frac{((\mathbf{r}' - \mathbf{a}) \cdot \widehat{\mathbf{d}})(\mathbf{r}' - \mathbf{a})}{(\mathbf{r}' - \mathbf{a})^2} \quad (73)$$

Considering that  $\frac{g(\frac{z}{\widetilde{R}}, \frac{\mathbf{a}-\mathbf{r}'}{\widetilde{R}})}{\sqrt{1-(\mathbf{u} \cdot \widehat{\mathbf{d}})^2}}$  is a regular even function of  $\frac{\mathbf{a}-\mathbf{r}'}{\widetilde{R}}$  for  $z \neq 0$ , we use the theorem 1 with the proposition 5 for the part of the integrand with  $(\frac{(\mathbf{r}'-\mathbf{a})^2 G(\varphi)}{R^2} - \widehat{\mathbf{d}})$  factor, then the proposition 6 for the part of the integrand with  $(\mathbf{a} - \mathbf{r}') \cdot \widehat{\mathbf{d}}$  factor, and thus derive (71).

For an illumination with an isotropic pattern, we can choose  $g \equiv 1$ , and write,

$$\mathbf{E} \cdot \widehat{\mathbf{d}} \sim - \frac{Z_0}{4\pi} \frac{i}{\lambda} \left( \frac{1}{2} \int_{\mathcal{L}} \frac{(z e^{-2ik\widetilde{R}} \sqrt{1 - (\frac{\mathbf{r}'-\mathbf{a}}{\widetilde{R}} \cdot \widehat{\mathbf{d}})^2} - \frac{e^{-2ikz}}{z})}{(\mathbf{r}' - \mathbf{a})^2} [(\widehat{\mathbf{z}} \times (\mathbf{r}' - \mathbf{a})) \cdot d\mathbf{r}'] \right), \quad (74)$$

which is the  $VV$  polarization term when  $\mathbf{r}_0 \cdot \widehat{\mathbf{d}} = 0$ .

Let us notice that,  $(\widehat{\mathbf{z}} \times (\mathbf{r}' - \mathbf{a})) \cdot \widehat{\mathbf{v}}$  is constant on each straight part of  $\mathcal{L}$  along  $\widehat{\mathbf{v}}$ , and that, for real  $\mathbf{a} \notin S'$ , the parts of the integrals with  $\frac{e^{-2ikz}}{z}$  factor vanish.

Theorem 8

When  $f_e(-\mathbf{u}) = \frac{f(\frac{z}{\widetilde{R}}, \frac{\mathbf{r}'-\mathbf{a}}{\widetilde{R}}) - f(\frac{-z}{\widetilde{R}}, \frac{\mathbf{a}-\mathbf{r}'}{\widetilde{R}})}{2} = \frac{(\mathbf{r}'-\mathbf{a})}{\widetilde{R}} \cdot \frac{h(\frac{z}{\widetilde{R}}, \frac{\mathbf{a}-\mathbf{r}'}{\widetilde{R}})}{\sqrt{1-(\mathbf{u} \cdot \widehat{\mathbf{d}})^2}}$  and  $\widehat{\mathbf{d}} \cdot \widehat{\mathbf{z}} = 0$ , it reduces to,

$$\begin{aligned} \mathbf{E} \sim & - \frac{i}{\lambda} \left( \frac{1}{2} \int_{\mathcal{L}} F_1((\mathbf{r}' - \mathbf{a})^2, \varphi) [(\widehat{\mathbf{z}} \times \mathbf{h}(\frac{z}{\widetilde{R}}, \frac{\mathbf{a} - \mathbf{r}'}{\widetilde{R}})) \cdot d\mathbf{r}'] + \right. \\ & \left. + \frac{1}{2} \int_{\mathcal{L}} F_2((\mathbf{r}' - \mathbf{a})^2)' \left( \frac{\widehat{\mathbf{d}} \cdot (\mathbf{a} - \mathbf{r}') f_e(-\mathbf{u})}{\widetilde{R}} \right) [(\widehat{\mathbf{z}} \times \frac{\mathbf{r}' - \mathbf{a}}{(\mathbf{r}' - \mathbf{a})^2}) \cdot d\mathbf{r}'] \right), \end{aligned} \quad (75)$$

where, for  $z \neq 0$ , the primitives  $F_1$  and  $F_2$  are written following,

$$\begin{aligned} F_1 &= \frac{Z_0 z}{8\pi k^2} \int \left( \left( \frac{\psi'}{\widetilde{R}} \right)^2 \right)' \left( \frac{(\mathbf{r}' - \mathbf{a})^2 G(\varphi)}{R^2} - \widehat{\mathbf{d}} \right) dR^2, F_2 = \frac{Z_0 z^2 \widehat{\mathbf{z}}}{8\pi k^2} \int \left( \left( \frac{\psi'}{\widetilde{R}} \right)^2 \right)' dR^2, \\ F_1 &\sim - \frac{Z_0}{4\pi} \frac{z e^{-2ik\widetilde{R}}}{\widetilde{R} R^2} \left( \frac{(\mathbf{r}' - \mathbf{a})^2 G(\varphi)}{R^2} - \widehat{\mathbf{d}} \right), F_2 \sim - \frac{Z_0}{4\pi} \frac{z^2 \widehat{\mathbf{z}} e^{-2ik\widetilde{R}}}{\widetilde{R} R^2}, \end{aligned} \quad (76)$$

with  $G(\varphi) = \frac{((\mathbf{r}'-\mathbf{a}) \cdot \widehat{\mathbf{d}})(\mathbf{r}'-\mathbf{a})}{(\mathbf{r}'-\mathbf{a})^2}$ ,  $f_e(-\mathbf{u})$  regular on  $S'$ ,  $\mathcal{L} \equiv \partial S'$  oriented counterclockwise. The  $VV$  polarization term is given by  $\mathbf{E} \cdot \widehat{\mathbf{d}}$  when  $\mathbf{r}_0 \cdot \widehat{\mathbf{d}} = 0$ .

Proof. Considering that  $\frac{h(\frac{z}{\widetilde{R}}, \frac{\mathbf{a}-\mathbf{r}'}{\widetilde{R}})}{\sqrt{1-(\mathbf{u} \cdot \widehat{\mathbf{d}})^2}}$  is a regular even function of the coordinates of  $\frac{\mathbf{r}'-\mathbf{a}}{\widetilde{R}}$ ,  $\widehat{\mathbf{d}} \cdot \widehat{\mathbf{z}} = 0$ , we use the proposition 6 and the theorem 1 with proposition 5, and derive (75).

5.2.2) Magnetic case with  $f_e \equiv 0$ ,  $\hat{\mathbf{d}} = \hat{\mathbf{d}}_h$

Let us consider the case with  $f_e \equiv 0$ ,  $\hat{\mathbf{d}} = \hat{\mathbf{d}}_h$ . The plate  $S'$  is then illuminated by,

$$\mathbf{E}_i \sim \frac{ik}{4\pi} (\mathbf{u}_0 \times \hat{\mathbf{d}}) f_h(\mathbf{u}_0) \frac{e^{-ik\tilde{R}_0}}{\tilde{R}_0}, \quad (77)$$

and the PO diffracted field at  $\mathbf{r} = \mathbf{a} + z\hat{\mathbf{z}} \notin S'$ , when  $\mathbf{r} = \mathbf{r}_0$  (manostatic case), satisfies,

$$\mathbf{E} \cdot \mathbf{v}_0 \sim \frac{-z}{8(\pi ik)^2} \int_{S'} \left( \left( \frac{\psi'}{\tilde{R}} \right)' \right)^2 \mathbf{w}_0 \cdot \mathbf{R} f_h(-\mathbf{u}) dS' \quad \text{with } \mathbf{w}_0 = \mathbf{v}_0 \times \hat{\mathbf{d}}. \quad (78)$$

Theorem 9

When  $f_h(-\mathbf{u}) = \frac{f(\frac{-z}{\tilde{R}}, \frac{\mathbf{r}'-\mathbf{a}}{\tilde{R}}) + f(\frac{-z}{\tilde{R}}, \frac{\mathbf{a}-\mathbf{r}'}{\tilde{R}})}{2} = \frac{g(\frac{z}{\tilde{R}}, \frac{\mathbf{a}-\mathbf{r}'}{\tilde{R}})}{\sqrt{1-(\mathbf{u} \cdot \hat{\mathbf{d}})^2}}$  and  $\hat{\mathbf{d}} \cdot \hat{\mathbf{z}} = 0$ , the integral reduces to,

$$\mathbf{E} \cdot \mathbf{v}_0 \sim \frac{ik}{4\pi} \int_{\mathcal{L}} \frac{\frac{ze^{-2ik\tilde{R}} f_h(-\mathbf{u})}{\tilde{R}^2} - \frac{\tilde{R} e^{-2ikz} g(1,0)}{z}}{4\pi(\mathbf{r}' - \mathbf{a})^2} (\hat{\mathbf{z}} \times \frac{-\mathbf{w}_0(\mathbf{r}' - \mathbf{a})^2 + (\mathbf{r}' - \mathbf{a})(zw_{0z})}{\tilde{R}}) \cdot d\mathbf{r}' \quad (79)$$

where  $\mathbf{w}_0 = \mathbf{v}_0 \times \hat{\mathbf{d}}$ . For  $\mathbf{v}_0 = (\hat{\mathbf{d}} \times \frac{\mathbf{r}_0}{r_0})$ , with  $\hat{\mathbf{d}} \cdot \mathbf{r}_0 = \hat{\mathbf{d}} \cdot \hat{\mathbf{z}} = 0$ , we have,

$$\mathbf{E} \cdot \mathbf{v}_0 \sim \frac{ik}{4\pi} \int_{\mathcal{L}} \frac{\frac{ze^{-2ik\tilde{R}} f_h(-\mathbf{u})}{\tilde{R}^2} - \frac{\tilde{R} e^{-2ikz} g(1,0)}{z}}{4\pi(\mathbf{r}' - \mathbf{a})^2} (\hat{\mathbf{z}} \times (\frac{(\mathbf{r}' - \mathbf{a})\tilde{R}}{\tilde{r}_0} - \frac{\mathbf{r}'(\mathbf{r}' - \mathbf{a})^2}{\tilde{R}\tilde{r}_0})) \cdot d\mathbf{r}', \quad (80)$$

denoted the HH polarization term. The contour  $\mathcal{L} \equiv \partial S'$  is oriented counterclockwise.

Proof. Noticing that  $\mathbf{w}_0 \cdot \mathbf{R} = -\mathbf{w}_0 \cdot (\mathbf{r}' - \mathbf{a}) + zw_{0z}$ , and  $f_h(-\mathbf{u}) = \frac{f(\frac{-z}{\tilde{R}}, \frac{\mathbf{r}'-\mathbf{a}}{\tilde{R}}) + f(\frac{-z}{\tilde{R}}, \frac{\mathbf{a}-\mathbf{r}'}{\tilde{R}})}{2}$  is even relatively to the coordinates of  $\frac{\mathbf{r}'-\mathbf{a}}{\tilde{R}}$ , we derive, from the theorem 1 and proposition 6,

$$\begin{aligned} \mathbf{E} \cdot \mathbf{v}_0 &\sim \frac{i}{\lambda} \frac{1}{2} \int_{\mathcal{L}} (F((\mathbf{r}' - \mathbf{a})^2) - F(0)) (\hat{\mathbf{z}} \times (-\mathbf{w}_0 + \frac{\mathbf{r}' - \mathbf{a}}{(\mathbf{r}' - \mathbf{a})^2} (zw_{0z}))) \cdot d\mathbf{r}', \\ F &= \frac{z}{4\pi k^2 ik} \int \left( \left( \frac{\psi'}{\tilde{R}} \right)' \right)^2 f_h(-\mathbf{u}) d\mathbf{R}^2. \end{aligned} \quad (81)$$

From the proposition 5, we derive  $F \sim \frac{ik}{4\pi} \frac{1}{ik} \frac{ze^{-2ik\tilde{R}}}{\tilde{R}^3} f_h(-\mathbf{u})$ , and thus (79). For polarization HH, with  $\mathbf{v}_0 = (\hat{\mathbf{d}} \times \frac{\mathbf{r}_0}{r_0})$  with  $\mathbf{r}_0 \cdot \hat{\mathbf{d}} = 0$  and  $\mathbf{w}_0 = \frac{\mathbf{r}_0}{r_0} = \frac{\mathbf{a} + z\hat{\mathbf{z}}}{r_0}$ , we then derive (83) from,

$$-\mathbf{w}_0(\mathbf{r}' - \mathbf{a})^2 + (\mathbf{r}' - \mathbf{a})(zw_{0z}) = \frac{(-\mathbf{r}' + (\mathbf{r}' - \mathbf{a}))(\mathbf{r}' - \mathbf{a})^2 + (\mathbf{r}' - \mathbf{a})z^2}{\tilde{r}_0} \quad (82)$$

For an illumination with isotropic pattern, i.e  $g \equiv 1$  and  $f_h(-\mathbf{u}) = \frac{1}{\sqrt{1-(\mathbf{u} \cdot \hat{\mathbf{d}})^2}}$ , we have,

$$\mathbf{E} \cdot \mathbf{v}_0 \sim \frac{ik}{4\pi} \int_{\mathcal{L}} \frac{\left( \frac{ze^{-2ik\tilde{R}}}{\mathbf{R}^2 \sqrt{1-(\mathbf{u} \cdot \hat{\mathbf{d}})^2}} - \frac{\tilde{\mathbf{R}} e^{-2ikz}}{z} \right)}{4\pi(\mathbf{r}' - \mathbf{a})^2} (\hat{\mathbf{z}} \times \left( \frac{(\mathbf{r}' - \mathbf{a})\tilde{\mathbf{R}}}{\tilde{r}_0} - \frac{\mathbf{r}'(\mathbf{r}' - \mathbf{a})^2}{\tilde{\mathbf{R}} \tilde{r}_0} \right)). d\mathbf{r}'. \quad (83)$$

Let us notice that,  $(\hat{\mathbf{z}} \times (\mathbf{r}' - \mathbf{a})) \cdot \hat{\mathbf{v}}$  is constant on each straight part of  $\mathcal{L}$  along  $\hat{\mathbf{v}}$ , and that, for real  $\mathbf{a} \notin S'$ , the parts of the integrals with  $\frac{e^{-2ikz}}{z}$  factor vanish.

Theorem 10

When  $f_h(-\mathbf{u}) = \frac{f(\frac{-z}{\tilde{\mathbf{R}}}, \frac{\mathbf{r}' - \mathbf{a}}{\tilde{\mathbf{R}}}) - f(\frac{-z}{\tilde{\mathbf{R}}}, \frac{\mathbf{a} - \mathbf{r}'}{\tilde{\mathbf{R}}})}{2} = \frac{(\mathbf{r}' - \mathbf{a})}{\tilde{\mathbf{R}}} \cdot \frac{h(\frac{z}{\tilde{\mathbf{R}}}, \frac{\mathbf{a} - \mathbf{r}'}{\tilde{\mathbf{R}}})}{\sqrt{1-(\mathbf{u} \cdot \hat{\mathbf{d}})^2}}$  and  $\hat{\mathbf{d}} \cdot \hat{\mathbf{z}} = 0$ , the integral reduces to,

$$\begin{aligned} \mathbf{E} \cdot \mathbf{v}_0 &\sim \frac{ik}{4\pi} \left( \int_{\mathcal{L}} \frac{ze^{-2ik\tilde{R}} \left( \frac{(\mathbf{w}_0 \cdot (\mathbf{a} - \mathbf{r}')) f_h(-\mathbf{u})}{\tilde{\mathbf{R}}} \right)}{4\pi \mathbf{R}^2} (\hat{\mathbf{z}} \times \frac{(\mathbf{r}' - \mathbf{a})}{(\mathbf{r}' - \mathbf{a})^2}) \cdot d\mathbf{r}' \right. \\ &\quad \left. + \int_{\mathcal{L}} \frac{ze^{-2ik\tilde{R}} (zw_{0z})}{4\pi \mathbf{R}^4} (\hat{\mathbf{z}} \times \frac{h(\frac{z}{\tilde{\mathbf{R}}}, \frac{\mathbf{a} - \mathbf{r}'}{\tilde{\mathbf{R}}})}{\sqrt{1-(\mathbf{u} \cdot \hat{\mathbf{d}})^2}}) \cdot d\mathbf{r}' \right). \end{aligned} \quad (84)$$

For  $\mathbf{v}_0 = (\hat{\mathbf{d}} \times \frac{\mathbf{r}_0}{r_0})$  with  $\hat{\mathbf{d}} \cdot \mathbf{r}_0 = 0$ , and  $\mathbf{w}_0 = \frac{\mathbf{r}_0}{r_0} = \frac{\mathbf{a} + z\hat{\mathbf{z}}}{\tilde{r}_0}$ ,  $\mathbf{E} \cdot \mathbf{v}_0$  is the HH polarization term.

Proof.  $\frac{h(\frac{z}{\tilde{\mathbf{R}}}, \frac{\mathbf{a} - \mathbf{r}'}{\tilde{\mathbf{R}}})}{\sqrt{1-(\mathbf{u} \cdot \hat{\mathbf{d}})^2}}$  is a regular even function of the coordinates of  $\frac{\mathbf{r}' - \mathbf{a}}{\tilde{\mathbf{R}}}$  with  $\hat{\mathbf{d}} \cdot \hat{\mathbf{z}} = 0$ . We can then use the proposition 6 and the theorem 1 with proposition 5, and derive (84).

### 6) Reduction in case (b) : complex scaling and exact reduction of radiation integrals with an exponential term whose argument is an arbitrary polynomial of rank 2.

As defined in section 2, we consider the case (b) and study a surface radiation integral whose integrand has an exponential term with an argument that can be approximated by a complex polynomial of rank 2. This form is particularly used when the scatterer  $S'$  is illuminated by a gaussian beam [13]-[15], or when the largest dimension  $D$  of  $S'$  is such that  $D$  and  $\sqrt{kD^3}$  are much smaller than the source and observation distances from  $S'$  [2]-[4].

We consider in this section the exact reduction to a contour integral for the surface integral,

$$\int_{S'} e^{\mathcal{R}^2} dS' \quad \text{with} \quad \mathcal{R}^2 = (\alpha_1 x' + \alpha_2 y') - ik(h_1 x'^2 + h_2 y'^2 + h_3 x' y'), \quad (85)$$

for arbitrary  $\alpha_i$  and  $h_j$ , without any special function, by using complex scaling transformation, which will be considered for more complex integrands in chapter 7.

## 6.1) Transformation of the polynomial argument $\mathcal{R}^2$

### 6.1.1) quadratic reduction

#### Proposition 7

We can write, for any  $h_j$  coefficients with  $h_3^2 \neq 4h_1h_2$ ,

$$h_1x^2 + h_2y^2 + h_3xy = \xi((1 - c_{p,x}^2)x^2 + (1 - c_{p,y}^2)y^2 - 2c_{p,x}c_{p,y}xy) \quad (86)$$

where

$$\begin{aligned} 2c_{p,x}c_{p,y} &= -h_3/\xi, \quad c_{p,x}^2 = 1 - h_1/\xi, \quad c_{p,y}^2 = 1 - h_2/\xi \\ \xi &= \frac{(h_1 + h_2) + \epsilon\sqrt{(h_1 - h_2)^2 + h_3^2}}{2}, \quad \epsilon = +1 \text{ or } -1. \end{aligned} \quad (87)$$

where the choice of  $\epsilon = +1$  or  $-1$  is free, and we can let,

$$\begin{aligned} c_{p,x} &= \epsilon'_x \sqrt{1 - h_1/\xi}, \quad c_{p,y} = \epsilon'_y \sqrt{1 - h_2/\xi}, \\ \epsilon_x'^2 &= \epsilon_y'^2 = 1, \quad \epsilon'_x \epsilon'_y = \frac{-h_3/\xi}{2\sqrt{1 - h_1/\xi}\sqrt{1 - h_2/\xi}}, \end{aligned} \quad (88)$$

or

$$\begin{aligned} c_{p,x} &= \epsilon_x \sqrt{\xi - h_1}/\sqrt{\xi}, \quad c_{p,y} = \epsilon_y \sqrt{\xi - h_2}/\sqrt{\xi}, \\ \epsilon_x^2 &= \epsilon_y^2 = 1, \quad \epsilon_x \epsilon_y = \frac{-h_3}{2\sqrt{\xi - h_1}\sqrt{\xi - h_2}}, \end{aligned} \quad (89)$$

where the case  $h_3 = 0$  is considered in the limit sense. As we can take indifferently  $\mathbf{c}_p$  or  $-\mathbf{c}_p$ , we can choose arbitrarily  $\epsilon_x$  or  $\epsilon_y$  equal to 1. Considering the relation when  $\epsilon$  change of sign, and denoting  $(\mathbf{c}_p)_{\pm 1}$  for  $\mathbf{c}_p$  when  $\epsilon = \pm 1$ , it is worth noticing that,

$$\frac{(\mathbf{c}_{px})_\epsilon}{(\mathbf{c}_{py})_\epsilon} = -\frac{(\mathbf{c}_{py})_{-\epsilon}}{(\mathbf{c}_{px})_{-\epsilon}}, \quad (\mathbf{c}_p)_\epsilon^2 - 1 = \frac{1}{(\mathbf{c}_p)_{-\epsilon}^2 - 1}, \quad (90)$$

and thus,

$$\begin{aligned} (\mathbf{c}_p)_\epsilon &= ((\mathbf{c}_{px})_\epsilon \hat{\mathbf{x}} + (\mathbf{c}_{py})_\epsilon \hat{\mathbf{y}})_\epsilon = \frac{(\mathbf{c}_{py})_\epsilon}{(\mathbf{c}_{px})_{-\epsilon}} (- (\mathbf{c}_{py})_{-\epsilon} \hat{\mathbf{x}} + (\mathbf{c}_{px})_{-\epsilon} \hat{\mathbf{y}}) \\ &= \frac{(\mathbf{c}_{py})_\epsilon}{(\mathbf{c}_{px})_{-\epsilon}} (\hat{\mathbf{z}} \times (\mathbf{c}_p)_{-\epsilon}), \end{aligned} \quad (91)$$

For the particular case where  $h_3 = 0$ , we can consider, after choosing  $\epsilon = +1$  or  $-1$  in definition of  $\xi$ ,

$$\begin{aligned} \xi = h_1, c_{p,x} = 0, c_{p,y} = \sqrt{h_1 - h_2}/\sqrt{h_1}, \widehat{\mathbf{c}}_p = \widehat{\mathbf{y}}, p^2 = 1 - (\mathbf{c}_p)^2 = h_2/h_1, \\ \xi = h_2, c_{p,x} = \sqrt{h_2 - h_1}/\sqrt{h_2}, c_{p,y} = 0, \widehat{\mathbf{c}}_p = \widehat{\mathbf{x}}, p^2 = 1 - (\mathbf{c}_p)^2 = h_1/h_2, \end{aligned} \quad (92)$$

We also note that, when  $\xi \rightarrow 0$  for a particular choice of  $\epsilon$ , we have  $h_3^2 \rightarrow 4h_1h_2$  while  $\xi c_{p,x}^2 \rightarrow -h_1$  and  $\xi c_{p,y}^2 \rightarrow -h_2$ , and thus we have,

$$(h_1x'^2 + h_2y'^2 + h_3x'y') \rightarrow (\sqrt{h_1}x' + \epsilon'\sqrt{h_2}y')^2, (\epsilon')^2 = 1. \quad (93)$$

Proof. Letting,

$$(h_1x'^2 + h_2y'^2 + h_3x'y') = \xi(\mathbf{r}'^2 - (\mathbf{c}_p \cdot \mathbf{r}')^2) \quad (94)$$

we obtain (116) with (87), then  $2c_{p,x}c_{p,y} = -h_3/\xi$ ,  $c_{p,x}^2 = 1 - h_1/\xi$ ,  $c_{p,y}^2 = 1 - h_2/\xi$ . Thus,  $4(1 - h_1/\xi)(1 - h_2/\xi) = (h_3/\xi)^2$  (with  $\xi = 0$  when  $h_3^2 = 4h_1h_2$ ) and we have,

$$\begin{aligned} \xi &= \frac{(h_1 + h_2) + \epsilon\sqrt{(h_1 - h_2)^2 + 4(h_3/2)^2}}{2}, \\ c_{p,x} &= \epsilon'_x\sqrt{1 - h_1/\xi} = \epsilon_x\sqrt{\xi - h_1}/\sqrt{\xi}, c_{p,y} = \epsilon'_y\sqrt{1 - h_2/\xi} = \epsilon_y\sqrt{\xi - h_2}/\sqrt{\xi} \\ \epsilon'_x\epsilon'_y &= \frac{-h_3/\xi}{2\sqrt{1 - h_1/\xi}\sqrt{1 - h_2/\xi}}, \epsilon_x\epsilon_y = \frac{-h_3}{2\sqrt{\xi - h_1}\sqrt{\xi - h_2}}, \epsilon_{x(y)}^2 = \epsilon'_{x(y)}^2 = 1, \end{aligned} \quad (95)$$

where the choice of  $\epsilon = +1$  or  $-1$  is free, and we can take indifferently  $\mathbf{c}_p$  or  $-\mathbf{c}_p$ . We then note that

$$\begin{aligned} \frac{(\mathbf{c}_{px})_\epsilon}{(\mathbf{c}_{py})_\epsilon} &= \frac{(\mathbf{c}_{px})_\epsilon(\mathbf{c}_{py})_\epsilon}{(\mathbf{c}_{py})_\epsilon(\mathbf{c}_{py})_\epsilon} = \frac{-h_3}{2(\xi_\epsilon - h_2)} = \frac{-h_3}{(h_1 - h_2) + \epsilon\sqrt{(h_1 - h_2)^2 + h_3^2}} \\ &= \frac{(h_1 - h_2) - \epsilon\sqrt{(h_1 - h_2)^2 + h_3^2}}{h_3} = -\frac{(\mathbf{c}_{py})_{-\epsilon}}{(\mathbf{c}_{px})_{-\epsilon}}, \\ (\mathbf{c}_p)_\epsilon^2 &= 1 - \frac{-\xi + (h_1 + h_2)}{(h_1 + h_2) + \epsilon\sqrt{(h_1 - h_2)^2 + h_3^2}} = 1 - \frac{(h_1 + h_2) - \epsilon\sqrt{(h_1 - h_2)^2 + h_3^2}}{(h_1 + h_2) + \epsilon\sqrt{(h_1 - h_2)^2 + h_3^2}} \\ &= 1 - \frac{1}{1 - (\mathbf{c}_p)_{-\epsilon}^2} = \frac{(\mathbf{c}_p)_{-\epsilon}^2}{(\mathbf{c}_p)_{-\epsilon}^2 - 1} = 1 + \frac{1}{(\mathbf{c}_p)_{-\epsilon}^2 - 1}. \end{aligned}$$

**Proposition 8**

We can write,

$$\begin{aligned}
 (\alpha_1 x' + \alpha_2 y') - ik(h_1 x'^2 + h_2 y'^2 + h_3 x' y') &= \mathbf{w}' \cdot \mathbf{r}' - ik\xi((\mathbf{r}')^2 - (\mathbf{c}_p \cdot \mathbf{r}')^2) \\
 &= \mathbf{w}'_m \cdot \mathbf{r}'_p - ik\xi(\mathbf{r}'_p)^2 = v(\mathbf{r}'_p - \mathbf{a}_p)^2 + v_1,
 \end{aligned} \tag{96}$$

with  $\mathbf{w}' \cdot \mathbf{r}' = \alpha_1 x' + \alpha_2 y'$ ,  $\mathbf{w}' \cdot \mathbf{r}' = \mathbf{w}'_m \cdot \mathbf{r}'_p$ ,  $\mathbf{c}_p$  defined in proposition 7, and

$$\begin{aligned}
 \mathbf{r}'_p &= (p(\widehat{\mathbf{c}}_p \cdot \mathbf{r}')\widehat{\mathbf{c}}_p + (\mathbf{r}' - \widehat{\mathbf{c}}_p(\widehat{\mathbf{c}}_p \cdot \mathbf{r}'))), \quad \mathbf{c}_p = \sqrt{\mathbf{c}_p^2} \widehat{\mathbf{c}}_p, \\
 \mathbf{a}_p &= \frac{\mathbf{w}'_m}{2ik\xi}, \quad v = -ik\xi, \quad v_1 = \frac{\mathbf{w}'_m{}^2}{4ik\xi}, \quad \mathbf{w}'_m = \frac{(\widehat{\mathbf{c}}_p \mathbf{w}')}{p} \widehat{\mathbf{c}}_p + (\mathbf{w}' - \widehat{\mathbf{c}}_p(\widehat{\mathbf{c}}_p \mathbf{w}')), \\
 \xi &= \frac{(h_1 + h_2) + \epsilon \sqrt{(h_1 - h_2)^2 + h_3^2}}{2} = \frac{2(h_1 h_2 - (h_3/2)^2)}{(h_1 + h_2) - \epsilon \sqrt{(h_1 - h_2)^2 + h_3^2}}, \\
 p &= \sqrt{1 - \mathbf{c}_p^2} = \sqrt{-1 + \frac{h_1 + h_2}{\xi}} = \sqrt{\frac{(h_1 + h_2) - \epsilon \sqrt{(h_1 - h_2)^2 + h_3^2}}{(h_1 + h_2) + \epsilon \sqrt{(h_1 - h_2)^2 + h_3^2}}},
 \end{aligned} \tag{97}$$

with  $\epsilon = +1$  or  $-1$ . The arguments of square roots are taken apart of its cut. This can be considered in general, in the sense of the limit, if  $\xi \rightarrow 0$  or  $\mathbf{c}_p^2 \rightarrow 1$  (as  $h_3^2 \rightarrow 4h_1 h_2$ ), and even for  $\mathbf{c}_p^2 = 0$  (note :  $\mathbf{c}_p^2 = 0$  does not imply  $\mathbf{c}_p = 0$  in complex coordinates), taking  $\mathbf{c}_p^2 \rightarrow 0$ , i.e.  $(h_1 - h_2)^2 + h_3^2 \rightarrow 0$  and  $\xi \rightarrow \frac{(h_1 + h_2)}{2}$ , so that we have,

$$\mathbf{r}'_p \rightarrow \mathbf{r}' - \frac{(\mathbf{c}_p \cdot \mathbf{r}')\mathbf{c}_p}{2} \quad \text{and} \quad \mathbf{w}'_m \rightarrow \mathbf{w}' + \frac{(\mathbf{c}_p \cdot \mathbf{w}')\mathbf{c}_p}{2}. \tag{98}$$

Proof. We choose  $\mathbf{w}'_m$  and  $\mathbf{r}'_p$  verifying,

$$\mathbf{w}'_m \cdot \mathbf{r}'_p = \mathbf{w}' \cdot \mathbf{r}' = \alpha_1 x' + \alpha_2 y', \quad (\mathbf{r}'_p)^2 = ((\mathbf{r}')^2 - (\mathbf{c}_p \cdot \mathbf{r}')^2), \tag{99}$$

as in (97), with  $\widehat{\mathbf{c}}_p = \frac{\mathbf{c}_p}{\sqrt{\mathbf{c}_p \cdot \mathbf{c}_p}}$  and  $\mathbf{c}_p$  defined in proposition 7. We then have,

$$\begin{aligned}
 (\alpha_1 x' + \alpha_2 y') - ik(h_1 x'^2 + h_2 y'^2 + h_3 x' y') &= \\
 &= \mathbf{w}'_m \cdot \mathbf{r}'_p - ik\xi(\mathbf{r}'_p)^2 = -ik\xi\left((\mathbf{r}'_p - \frac{\mathbf{w}'_m}{2ik\xi})^2 - \frac{\mathbf{w}'_m{}^2}{(2ik\xi)^2}\right) = v(\mathbf{r}'_p - \mathbf{a}_p)^2 + v_1,
 \end{aligned} \tag{100}$$

with  $\mathbf{a}_p = \frac{\mathbf{w}'_m}{2ik\xi}$ ,  $v = -ik\xi$ ,  $v_1 = \frac{\mathbf{w}'_m{}^2}{4ik\xi}$ , and thus (97). In (97), we can use that  $p \sim 1 - \mathbf{c}_p^2/2$  when  $\mathbf{c}_p^2 \rightarrow 0$  and derive (98). Notice that  $(p\xi)^2 = h_1 h_2 - (\frac{h_3}{2})^2$ .

## 6.2) Reduction of the surface integral to a contour integral

Following the expression of  $\mathbf{r}'_p$ , we change of variables and derive a reduction of integrals.

### 6.2.1) Complex scaling and definition of $S'_p$

#### Definition 4

Using  $\mathbf{r}'_p$  of (97), we consider  $\mathcal{P}$ , the anisotropic dilation (or scaling) by a factor  $p = \sqrt{1 - \mathbf{c}_p^2}$  along the vector  $\widehat{\mathbf{c}}_p = \mathbf{c}_p / \sqrt{\mathbf{c}_p^2}$  (with  $\mathbf{c}_p^2 \neq 0$ ), given by,

$$\begin{aligned} \mathcal{P}: \mathbf{r}' &= x' \widehat{\mathbf{x}} + y' (\widehat{\mathbf{z}} \times \widehat{\mathbf{x}}) \rightarrow \mathbf{r}'_p = x'_p \widehat{\mathbf{c}}_p + y'_p (\widehat{\mathbf{z}} \times \widehat{\mathbf{c}}_p), \\ \mathcal{P}(\mathbf{r}') &= \mathbf{r}'_p \text{ with } x'_p = p(\mathbf{r}' \cdot \widehat{\mathbf{c}}_p), y'_p = \mathbf{r}' \cdot (\widehat{\mathbf{z}} \times \widehat{\mathbf{c}}_p), \\ \mathcal{P}(\mathbf{c}_p) &= p\mathbf{c}_p, \mathcal{P}(\widehat{\mathbf{z}} \times \mathbf{c}_p) = (\widehat{\mathbf{z}} \times \mathbf{c}_p), \end{aligned} \quad (101)$$

where  $h_3^2 \neq 4h_1h_2$ , i.e.  $p \neq 0$  and  $\mathcal{P}(\mathbf{r}')$  is invertible. Let us notice that we have,

$$\begin{aligned} \text{grad}_{\mathbf{r}'}(g(\mathbf{r}')) &= \mathcal{P}(\text{grad}_{\mathbf{r}'_p}(g(\mathbf{r}'))), \\ \mathcal{P}^2(\mathbf{r}') &= \mathbf{r}' - (\mathbf{c}_p \cdot \mathbf{r}') \mathbf{c}_p, \end{aligned} \quad (102)$$

and that,

$$\begin{aligned} \text{if } p \rightarrow 1, \text{ i.e. } (\mathbf{c}_p)^2 \rightarrow 0 \text{ or } \mathbf{c}_p \rightarrow A_0(\widehat{\mathbf{x}} \pm i\widehat{\mathbf{y}}), \\ \text{then } \mathbf{r}'_p \rightarrow \mathbf{r}' - \frac{(\mathbf{c}_p \cdot \mathbf{r}') \mathbf{c}_p}{2} \text{ and } \mathbf{w}'_m \rightarrow \mathbf{w}' + \frac{(\mathbf{c}_p \cdot \mathbf{w}') \mathbf{c}_p}{2}, \end{aligned} \quad (103)$$

with  $A_0$  a constant (note :  $\widehat{\mathbf{x}} \pm i\widehat{\mathbf{y}} = \frac{\mathbf{e} \pm i\widehat{\mathbf{z}} \times \mathbf{c}}{c_x \mp ic_y}$  for any complex vector  $\mathbf{c}$  with  $c_x \mp ic_y \neq 0$ ).

As in definition 1, we then define a complex manifold  $S'_p$  with contour  $\mathcal{L}_p$  as the transformation by  $\mathcal{P}$  of the real manifold  $S'$  with contour  $\mathcal{L}$ . Letting  $dS' = dx' \wedge dy'$  and  $dS'_p = dx'_p \wedge dy'_p$ , we have  $p dS' = dS'_p$ .

Remark : we note that,

$$\mathcal{P}^{2n+1}(\mathbf{r}') = \mathcal{P}(\mathbf{r}') - (p \sum_{k=0}^{n-1} p^{2k}) (\mathbf{c}_p \cdot \mathbf{r}') \mathbf{c}_p, \quad \mathcal{P}^{2n+2}(\mathbf{r}') = \mathbf{r}' - \left( \sum_{k=0}^n p^{2k} \right) (\mathbf{c}_p \cdot \mathbf{r}') \mathbf{c}_p, \quad (104)$$

with  $\sum_{k=0}^n p^{2k} = \frac{1-p^{2(n+1)}}{1-p^2}$ ,  $n \geq 1$ .

### 6.2.2) First reduction along $\mathcal{L}_p$

#### Proposition 9

Considering  $\mathbf{r}'_p$ ,  $\mathbf{a}_p = \frac{\mathbf{w}'_m}{2ik\xi}$ ,  $v$  and  $p$  in (97), and an arbitrary constant  $A_0$ , we can write

$$\begin{aligned} \int_{S'} e^{(\alpha_1 x' + \alpha_2 y') - ik(h_1 x'^2 + h_2 y'^2 + h_3 x' y')} \cdot dS' &= \frac{1}{2p} \int_{\mathcal{L}_p} \left( \frac{\widehat{\mathbf{z}} \times (\mathbf{r}'_p - \mathbf{a}_p) (e^{\mathbf{w}'_m \cdot \mathbf{r}'_p - ik\xi(\mathbf{r}'_p)^2} - A_0)}{\mathbf{w}'_m \cdot \mathbf{r}'_p - ik\xi(\mathbf{r}'_p)^2 - \frac{\mathbf{w}'_m{}^2}{4ik\xi}} \right) \cdot d\mathbf{r}'_p \\ &+ \frac{(A_0 - e^{\frac{\mathbf{w}'_m{}^2}{4ik\xi}})}{-2ikp\xi} \int_{\mathcal{L}_p} \left( \widehat{\mathbf{z}} \times \frac{\mathbf{r}'_p - \mathbf{a}_p}{(\mathbf{r}'_p - \mathbf{a}_p)^2} \right) \cdot d\mathbf{r}'_p, \end{aligned} \quad (105)$$

where  $\mathbf{w}'_m \cdot \mathbf{r}'_p - ik\xi(\mathbf{r}'_p)^2 - \frac{\mathbf{w}'_m{}^2}{4ik\xi} = v(\mathbf{r}'_p - \mathbf{a}_p)^2$ ,  $v = -ik\xi$ . Let  $\mathcal{D}_1$  be a smooth semi-line of the complex plane from the origin to  $\infty$ , we notice that, when  $(\mathbf{r}'_p - \mathbf{a}_p)^2 \neq 0$  on  $\mathcal{L}$ ,

$$\int_{\mathcal{L}_p} \left( \widehat{\mathbf{z}} \times \frac{\mathbf{r}'_p - \mathbf{a}_p}{(\mathbf{r}'_p - \mathbf{a}_p)^2} \right) \cdot d\mathbf{r}'_p = \frac{-i}{2} \int_{\mathcal{L}_p} d \ln(\chi(\mathbf{r}'_p)) = \pi(N_+ - N_-), \quad (106)$$

where  $N_+$  (resp.  $N_-$ ) is the number of times  $\chi(\mathbf{r}'_p)$  crosses  $\mathcal{D}_1$  in positive sense (resp. negative) with  $\chi(\mathbf{r}'_p) = \frac{z'_{\mathbf{a}+}}{z'_{\mathbf{a}-}}$ ,  $z'_{\mathbf{a}+} = x'_a + iy'_a$ ,  $z'_{\mathbf{a}-} = x'_a - iy'_a$ ,  $\mathbf{r}'_p - \mathbf{a}_p = x'_a \widehat{\mathbf{c}}_p + y'_a (\widehat{\mathbf{z}} \times \widehat{\mathbf{c}}_p)$ .

Proof. From the proposition 8 and  $p dS' = dS'_p$ , we have,

$$\int_{S'} e^{(\alpha_1 x' + \alpha_2 y') - ik(h_1 x'^2 + h_2 y'^2 + h_3 x' y')} dS' = \frac{1}{p} \int_{S'_p} e^{v(\mathbf{r}'_p - \mathbf{a}_p)^2 + v_1} dS'_p. \quad (107)$$

We can then apply the theorem 1 to the surface integral, following,

$$\begin{aligned} \int_{S'_p} e^{v(\mathbf{r}'_p - \mathbf{a}_p)^2 + v_1} dS_p &= \frac{1}{2} \left( \int_{\mathcal{L}_p} \left( \frac{\widehat{\mathbf{z}} \times (\mathbf{r}'_p - \mathbf{a}_p) (e^{-ik\xi(\mathbf{r}'_p - \mathbf{a}_p)^2 + v_1} - A_0)}{v(\mathbf{r}'_p - \mathbf{a}_p)^2} \right) \cdot d\mathbf{r}'_p + \right. \\ &\left. + (A_0 - e^{v_1}) \int_{\mathcal{L}_p} \left( \widehat{\mathbf{z}} \times \frac{\mathbf{r}'_p - \mathbf{a}_p}{v(\mathbf{r}'_p - \mathbf{a}_p)^2} \right) \cdot d\mathbf{r}'_p \right), \end{aligned} \quad (108)$$

and derive (105), which is non-singular for  $A_0 = e^{v_1}$ . We obtain (106) from proposition 2.

Remark 5 : Let the components of  $\mathbf{a}_p$  with large values such that  $\text{Im}(\frac{z'_{\mathbf{a}+}}{z'_{\mathbf{a}-}})$  little varies, and,

$$\max \left| \text{Im} \left( \frac{z'_{\mathbf{a}+}}{z'_{\mathbf{a}-}} - \frac{z'_{\mathbf{a}+}}{z'_{\mathbf{a}-}} \Big|_{\mathbf{r}'_p=0} \right) \right| = \max \left| \text{Im} \left( \frac{1}{2} \ln \left( \frac{1 - \frac{(\mathbf{r}'_p + i(\mathbf{r}'_p \times \widehat{\mathbf{z}})) \cdot \widehat{\mathbf{c}}_p}{(\mathbf{a}_p + i(\mathbf{a}_p \times \widehat{\mathbf{z}})) \cdot \widehat{\mathbf{c}}_p}}{1 - \frac{(\mathbf{r}'_p - i(\mathbf{r}'_p \times \widehat{\mathbf{z}})) \cdot \widehat{\mathbf{c}}_p}{(\mathbf{a}_p - i(\mathbf{a}_p \times \widehat{\mathbf{z}})) \cdot \widehat{\mathbf{c}}_p}} \right) \right) \Big|_{\mathbf{r}'_p \in \mathcal{L}_p} \right| < \pi/2, \quad (109)$$

then  $\frac{z'_{\mathbf{a}+}}{z'_{\mathbf{a}-}}$  does not run round the origin, the winding number is null, and  $N_+ = N_-$  in (106).

6.2.3) Some properties of  $\mathcal{P}(\mathbf{r}') = \mathbf{r}'_p$  to analyse and transform (105)

By definition, we have  $p \mathcal{P}^{-1}(\mathbf{u}) = (\widehat{\mathbf{c}}_p \cdot \mathbf{u}) \widehat{\mathbf{c}}_p + p(\mathbf{u} - \widehat{\mathbf{c}}_p(\widehat{\mathbf{c}}_p \cdot \mathbf{u}))$ , and



$$\mathcal{P}(\mathbf{u}) \cdot \mathbf{v} = \mathbf{u} \cdot \mathcal{P}(\mathbf{v}), \mathcal{P}(\widehat{\mathbf{z}} \times \mathbf{u}) = p \widehat{\mathbf{z}} \times \mathcal{P}^{-1}(\mathbf{u}), \mathcal{P}(\widehat{\mathbf{z}} \times \mathcal{P}(\mathbf{v})) = p(\widehat{\mathbf{z}} \times \mathbf{v}), \quad (110)$$

for any vectors  $\mathbf{u}$  and  $\mathbf{v}$  of the plane. Thus, we have,

$$\mathbf{u} \cdot d\mathbf{r}'_p = \mathcal{P}(\mathbf{u}) \cdot d\mathbf{r}', (\widehat{\mathbf{z}} \times \mathbf{r}'_p) \cdot d\mathbf{r}'_p = p(\widehat{\mathbf{z}} \times \mathbf{r}') \cdot d\mathbf{r}'. \quad (111)$$

Considering (110), we can add, from  $\mathbf{w}'_m \cdot \mathbf{r}'_p = \mathbf{w}' \cdot \mathbf{r}'$  and  $\mathbf{r}'_p = \mathcal{P}(\mathbf{r}')$ , that,

$$\mathcal{P}(\mathbf{w}'_m) = \mathbf{w}', \mathbf{w}'_m = \mathcal{P}^{-1}(\mathbf{w}'), \quad (112)$$

and let from now on,

$$\begin{aligned} \mathcal{M}(\mathbf{w}') &= p^2 \mathcal{P}^{-1}(\mathbf{w}'_m) \text{ verifying } p\mathcal{P}(\widehat{\mathbf{z}} \times \mathbf{w}'_m) = \widehat{\mathbf{z}} \times \mathcal{M}(\mathbf{w}'), \\ \mathcal{N}(\mathbf{w}') &= p \mathbf{w}'_m \text{ verifying } v_1 = \frac{\mathbf{w}'_m \cdot \mathbf{w}'_m}{4ik\xi} = \frac{(\mathcal{N}(\mathbf{w}'))^2}{4ik\xi p^2}, \end{aligned} \quad (113)$$

with  $\mathbf{a}_p = \frac{\mathbf{w}'_m}{2ik\xi}$ .

**Proposition 10**

Let  $(\mathbf{c}_p)_\pm = (\widehat{\mathbf{c}}_p)_\pm \sqrt{(\mathbf{c}_p^2)_\pm}$ ,  $\xi_\pm$ ,  $(\mathcal{P})_\pm$  and  $(\mathbf{r}'_p)_\pm$ , be the respective terms  $(\mathbf{c}_p)$ ,  $\xi$ ,  $\mathcal{P}$  and  $(\mathbf{r}'_p)$  for  $\epsilon = \pm 1$  (factor used in (95) for  $\xi$ ), with  $p_\pm^2 = 1 - (\mathbf{c}_p^2)_\pm = -1 + \frac{h_1+h_2}{\xi_\pm}$ , we notice that,

$$\begin{aligned} p_\pm^2 &= \frac{\xi_{\mp 1}}{\xi_{\pm 1}} = \frac{1}{p_\mp^2}, (\xi_{\pm 1} p_\pm)^2 = (\xi_{\mp 1} p_\mp)^2 = \xi_+ \xi_- = h_1 h_2 - (h_3/2)^2, \\ (\widehat{\mathbf{c}}_p)_\pm &= \pm \epsilon_c (\widehat{\mathbf{z}} \times (\widehat{\mathbf{c}}_p)_\mp), (\mathbf{r}'_p)_\pm = p_\pm (\mathbf{r}'_p)_\mp, (\mathcal{P})_\pm = p_\pm (\mathcal{P})_\mp, \end{aligned} \quad (114)$$

where  $(\epsilon_c)^2 = 1$  and  $\xi_\pm$  is finite. Letting  $\epsilon \equiv \pm$  and  $-\epsilon \equiv \mp$ , we remark that  $\xi_{-\epsilon} = 0$  implies  $h_3^2 = 4h_1 h_2$ ,  $\xi_\epsilon = h_1 + h_2$ , and  $(\mathbf{c}_p^2)_\epsilon = 1$  if  $h_1 + h_2 \neq 0$ , and that  $(\mathbf{c}_p^2)_\epsilon = 1$  implies  $\xi_{-\epsilon} = 0$ ,  $\xi_\epsilon = h_1 + h_2$ , while  $(\mathbf{c}_p^2)_\epsilon = 0$  implies  $(\mathbf{c}_p^2)_{-\epsilon} = 0$ ,  $2\xi_\pm = h_1 + h_2$ ,  $p_\pm = 1$ .

**Proof.** For that, we consider (87), and notice that

$$\begin{aligned} 1 - (\mathbf{c}_p^2)_\pm &= -1 + \frac{h_1 + h_2}{\xi_\pm} = \frac{\xi_\mp}{\xi_\pm} = \frac{1}{1 - (\mathbf{c}_p^2)_\mp}, \\ \frac{(\widehat{\mathbf{c}}_{px})_\epsilon}{(\widehat{\mathbf{c}}_{py})_\epsilon} &= \frac{(\mathbf{c}_{px})_\epsilon (\mathbf{c}_{py})_\epsilon}{(\mathbf{c}_{py})_\epsilon (\mathbf{c}_{py})_\epsilon} = \frac{-h_3}{2(\xi_\epsilon - h_2)} = \frac{-h_3}{(h_1 - h_2) + \epsilon \sqrt{(h_1 - h_2)^2 + h_3^2}} \\ &= \frac{(h_1 - h_2) - \epsilon \sqrt{(h_1 - h_2)^2 + h_3^2}}{h_3} = -\frac{(\mathbf{c}_{py})_{-\epsilon}}{(\mathbf{c}_{px})_{-\epsilon}} = -\frac{(\widehat{\mathbf{c}}_{py})_{-\epsilon}}{(\widehat{\mathbf{c}}_{px})_{-\epsilon}}. \end{aligned} \quad (115)$$

Proposition 11

We can write,

$$\frac{(\widehat{\mathbf{z}} \times \frac{(\mathbf{r}'_p - \mathbf{a}_p)}{(\mathbf{r}'_p - \mathbf{a}_p)^2}) \cdot d\mathbf{r}'_p}{-2ikp\xi} = \frac{\frac{1}{2}(\widehat{\mathbf{z}} \times (\mathbf{r}' - \frac{\mathcal{M}(\mathbf{w}')}{2ik\xi p^2})) \cdot d\mathbf{r}'}{\mathbf{w}' \cdot \mathbf{r}' - ik(h_1x'^2 + h_2y'^2 + h_3x'y') - \frac{(\mathcal{N}(\mathbf{w}'))^2}{4ik\xi p^2}} \quad (116)$$

where,

$$\begin{aligned} \frac{\mathcal{M}(\mathbf{w}')}{2ik\xi p^2} &= \mathcal{P}^{-1}(\mathbf{a}_p) = \frac{1}{2ik(\xi p)^2} \left( (h_2w'_x - \frac{h_3w'_y}{2})\widehat{\mathbf{x}} + (h_1w'_y - \frac{h_3w'_x}{2})\widehat{\mathbf{y}} \right), \\ \frac{(\mathcal{N}(\mathbf{w}'))^2}{4ik\xi p^2} &= \frac{\mathbf{w}'_m{}^2}{4ik\xi} = \frac{1}{4ik(\xi p)^2} (h_2w'_x{}^2 + h_1w'_y{}^2 - h_3w'_xw'_y). \end{aligned} \quad (117)$$

with  $(\xi p)^2 = \xi_+\xi_- = h_1h_2 - (h_3/2)^2$ . Let us notice that when  $h_3 = 0$  and  $h_1 = h_2 \rightarrow 0$ ,

$$\frac{(\widehat{\mathbf{z}} \times \frac{(\mathbf{r}'_p - \mathbf{a}_p)}{(\mathbf{r}'_p - \mathbf{a}_p)^2}) \cdot d\mathbf{r}'_p}{-2ikp\xi} = \frac{(\widehat{\mathbf{z}} \times \mathbf{w}')}{(\mathbf{w}')^2} \cdot d\mathbf{r}' \quad (118)$$

Proof. Considering (96) and (111)-(113), we obtain (116), then we derive (117) from,

$$\begin{aligned} \mathcal{M}(\mathbf{w}') &= (\widehat{\mathbf{c}}_p \cdot \mathbf{w}')\widehat{\mathbf{c}}_p + (1 - \mathbf{c}_p^2)(\mathbf{w}' - \widehat{\mathbf{c}}_p(\widehat{\mathbf{c}}_p \cdot \mathbf{w}')) = (1 - \mathbf{c}_p^2)\mathbf{w}' + (\mathbf{c}_p \cdot \mathbf{w}')\mathbf{c}_p \\ &= (-1 + \frac{h_1 + h_2}{\xi})\mathbf{w}' + (1 - \frac{h_1}{\xi})w'_x\widehat{\mathbf{x}} + (1 - \frac{h_2}{\xi})w'_y\widehat{\mathbf{y}} - \frac{h_3}{2\xi}(w'_y\widehat{\mathbf{x}} + w'_x\widehat{\mathbf{y}}) \\ &= \frac{1}{\xi}(h_2w'_x\widehat{\mathbf{x}} + h_1w'_y\widehat{\mathbf{y}} - \frac{h_3}{2}(w'_y\widehat{\mathbf{x}} + w'_x\widehat{\mathbf{y}})) = \frac{1}{\xi}((h_2w'_x - \frac{h_3w'_y}{2})\widehat{\mathbf{x}} + (h_1w'_y - \frac{h_3w'_x}{2})\widehat{\mathbf{y}}), \\ (\mathcal{N}(\mathbf{w}'))^2 &= (1 - \mathbf{c}_p^2)\mathbf{w}'^2 + (\mathbf{c}_p \cdot \mathbf{w}')^2 = (-1 + \frac{h_1 + h_2}{\xi})(w'_x{}^2 + w'_y{}^2) + \\ &(1 - \frac{h_1}{\xi})w'_x{}^2 + (1 - \frac{h_2}{\xi})w'_y{}^2 - \frac{h_3}{\xi}w'_xw'_y = \frac{1}{\xi}(h_2w'_x{}^2 + h_1w'_y{}^2 - h_3w'_xw'_y). \end{aligned} \quad (119)$$

Proposition 12

Let  $z'_{\mathbf{a}\pm} = (\mathbf{r}'_p - \mathbf{a}_p) \cdot (\widehat{\mathbf{x}} \pm i\widehat{\mathbf{y}}) = x'_a \pm iy'_a$ , where  $\mathbf{r}'_p - \mathbf{a}_p = x'_a\widehat{\mathbf{c}}_p + y'_a(\widehat{\mathbf{z}} \times \widehat{\mathbf{c}}_p)$  with  $\mathbf{r}'_p = \mathcal{P}(\mathbf{r}')$ , then  $z'_{\mathbf{a}\pm} = 0$  on lines  $\mathbf{r}' = \mathcal{P}^{-1}(\mathbf{a}_p) + t\mathbf{u}_c$ ,  $\mathbf{u}_c = \widehat{\mathbf{c}}_p \pm ip(\widehat{\mathbf{z}} \times \widehat{\mathbf{c}}_p)$ . For real  $\mathbf{r}'$ , each equation  $z'_{\mathbf{a}\pm} = 0$  has no or an infinity of solutions if  $\frac{\mp ip\xi + \frac{h_3}{2}}{h_2} (= \frac{h_1}{\pm ip\xi + \frac{h_3}{2}})$  is real, and a single one  $\mathbf{r}_{\pm}$  else. In the latter case, we notice that  $\mathbf{r}_+ = \mathbf{r}_- = \mathcal{P}^{-1}(\mathbf{a}_p)$  if  $\mathcal{P}^{-1}(\mathbf{a}_p)$  is a point with real coordinates, for example when  $\mathbf{a}_p$  and  $\mathcal{P}(\mathbf{r}')$  are real.

Proof. From  $z'_{\mathbf{a}\pm} = (\mathbf{r}'_p - \mathbf{a}_p) \cdot (\widehat{\mathbf{x}} \pm i\widehat{\mathbf{y}})$ ,  $(\widehat{\mathbf{x}} \pm i\widehat{\mathbf{y}})^2 = 0$  and  $\widehat{\mathbf{x}} \pm i\widehat{\mathbf{y}} = \frac{\mathbf{c} \pm i\widehat{\mathbf{z}} \times \mathbf{c}}{c_x \mp ic_y}$  as  $\mathbf{c}^2 \neq 0$ , we have  $z'_{\mathbf{a}\pm} = 0$  as  $\mathbf{r}' = \mathcal{P}^{-1}(\mathbf{a}_p) + t\mathbf{u}_c$ ,  $\mathbf{u}_c = p\mathcal{P}^{-1}(\widehat{\mathbf{c}}_p \pm i(\widehat{\mathbf{z}} \times \widehat{\mathbf{c}}_p))$ , and  $z'_{\mathbf{a}\pm} = \frac{(\mathbf{r}' - \mathcal{P}^{-1}(\mathbf{a}_p)) \cdot \mathbf{v}_c}{c_x \mp ic_y}$  with  $\mathbf{v}_c = \mathcal{P}(\mathbf{c}_p \pm i\widehat{\mathbf{z}} \times \mathbf{c}_p) = p\mathbf{c}_p \pm i\widehat{\mathbf{z}} \times \mathbf{c}_p$ ; we can consider solutions of  $\mathbf{v}_c \cdot \mathbf{r}' = 0$  with

$$\frac{\mathbf{v}_{cx}}{\mathbf{v}_{cy}} = \frac{pc_x \mp ic_y}{pc_y \pm ic_x} = \frac{\mp ip(c_x^2 + c_y^2) + c_x c_y(p^2 - 1)}{p^2 c_y^2 + c_x^2} = \frac{\mp ip - c_x c_y}{1 - c_y^2} = \frac{\mp ip\xi + \frac{h_3}{2}}{h_2} \quad (120)$$

and deduce last statement. Let us remark that  $\mathbf{u}_c \cdot \mathbf{v}_c = 0$  and  $(p\xi)^2 = h_1 h_2 - (\frac{h_3}{2})^2$ .

#### 6.2.4) Derivation of the contour integral along $\mathcal{L}$

##### Theorem 11

We can reduce the integral of an exponential with quadratic argument on  $S'$ , to an integral along the contour  $\mathcal{L}$  of  $S'$ , following,

$$\int_{S'} e^{\mathbf{w}' \cdot \mathbf{r}' - ik(h_1 x'^2 + h_2 y'^2 + h_3 x' y')} dS' = \frac{1}{2} \int_{\mathcal{L}} \left[ \frac{(e^{\mathbf{w}' \cdot \mathbf{r}' - ik(h_1 x'^2 + h_2 y'^2 + h_3 x' y')} - A_0)}{\mathbf{w}' \cdot \mathbf{r}' - ik(h_1 x'^2 + h_2 y'^2 + h_3 x' y') - \frac{\mathbf{w}'_m{}^2}{4ik\xi}} \right] [(\widehat{\mathbf{z}} \times (\mathbf{r}' - \frac{\mathcal{M}(\mathbf{w}')}{2ik\xi p^2})) \cdot d\mathbf{r}'] + \frac{(A_0 - e^{\frac{\mathbf{w}'_m{}^2}{4ik\xi}})}{-2ikp\xi} \int_{\mathcal{L}_p} (\widehat{\mathbf{z}} \times \frac{\mathbf{r}'_p - \mathbf{a}_p}{(\mathbf{r}'_p - \mathbf{a}_p)^2}) \cdot d\mathbf{r}'_p, \quad (121)$$

with  $\frac{\mathcal{M}(\mathbf{w}')}{2ik\xi p^2}$  and  $\frac{\mathbf{w}'_m{}^2}{4ik\xi}$  detailed in proposition 11, arbitrary  $A_0$ , and  $\widehat{\mathbf{y}} = \widehat{\mathbf{z}} \times \widehat{\mathbf{x}}$ . The integral on  $\mathcal{L}$  is non-singular when  $A_0 = e^{\frac{\mathbf{w}'_m{}^2}{4ik\xi}}$ . For  $A_0 \neq e^{\frac{\mathbf{w}'_m{}^2}{4ik\xi}}$ , we notice that, when  $(\mathbf{r}'_p - \mathbf{a}_p)^2 \neq 0$  on  $\mathcal{L}$ ,

$$\int_{\mathcal{L}_p} (\widehat{\mathbf{z}} \times \frac{\mathbf{r}'_p - \mathbf{a}_p}{(\mathbf{r}'_p - \mathbf{a}_p)^2}) \cdot d\mathbf{r}'_p = \frac{-i}{2} \int_{\mathcal{L}} d\ln(\chi(\mathcal{P}(\mathbf{r}'))) = \pi(N_+ - N_-) \quad (122)$$

where  $N_+$  (resp.  $N_-$ ) is the number of times the complex  $\chi(\mathcal{P}(\mathbf{r}'))$  crosses any smooth semi-line  $\mathcal{D}_1$  from the origin to  $\infty$ , in positive sense (resp. negative), with  $\chi(\mathcal{P}(\mathbf{r}')) = \frac{x'_a + iy'_a}{x'_a - iy'_a}$ ,  $\mathcal{P}(\mathbf{r}') - \mathbf{a}_p = x'_a \widehat{\mathbf{c}}_p + y'_a (\widehat{\mathbf{z}} \times \widehat{\mathbf{c}}_p)$ ,  $\mathcal{P}(\mathbf{r}') = \mathbf{r}'_p$ . Let us notice that, when  $\mathbf{a}_p$  is large and satisfies (109), in particular if  $h_1 h_2 \rightarrow (h_3/2)^2$ , we have  $\int_{\mathcal{L}} d\ln(\chi(\mathcal{P}(\mathbf{r}'))) = 0$ .

Proof. For that, we rewrite the proposition 9 with the use of propositions 8 and 11.

Remark 6 :

From (8), we have,

$$\mathbf{r}' = \mathcal{P}^{-1}(\mathbf{a}_p) + h'_p \mathbf{u}_{\varphi_p}, \quad \mathbf{u}_{\varphi_p} = \mathcal{P}^{-1}(\cos\varphi_p \hat{\mathbf{x}} + \sin\varphi_p \hat{\mathbf{y}}), \quad h'_p{}^2 = (\mathbf{r}'_p - \mathbf{a}_p)^2, \quad (123)$$

except on lines  $\mathbf{r}' = \mathcal{P}^{-1}(\mathbf{a}_p) + t\mathbf{u}_c$ ,  $\mathbf{u}_c = \hat{\mathbf{c}}_p \pm ip(\hat{\mathbf{z}} \times \hat{\mathbf{c}}_p)$  with  $z'_{\pm} = 0$  ( $\mathbf{u}_{\varphi_p}$  undefined).

### 7) More general uses of complex scaling

We now consider that the integrand of the surface integral is a function  $g(\mathbf{r}')f(\mathcal{R}^2)$  which verify the condition of continuation given in definition 3, where  $\mathcal{R}^2$  is a polynomial of coordinates. For the reduction of integrals, we define  $S'_p$  of contour  $\mathcal{L}_p$  as a transformation of  $S'$  by a complex scaling  $\mathcal{P}$ , with  $dS' = dx' \wedge dy'$ ,  $dS'_p = dx'_p \wedge dy'_p$ , and  $dS' = dS'_p/p$ .

#### 7.1) Exact reduction of $\int_{S'} g(\mathbf{r}') f(\mathcal{R}^2) dS'$

Let us consider the polynomial  $\mathcal{R}^2$  of order 2 with arbitrary coefficients,

$$\mathcal{R}^2 = d_0 + (\alpha_1 x' + \alpha_2 y') - ik(h_1 x'^2 + h_2 y'^2 + h_3 x' y') = d_0 + v(\mathbf{r}'_p - \mathbf{a}_p)^2 + v_1, \quad (124)$$

following the propositions 8 and 10 of previous chapter, with  $\mathbf{r}'_p = \mathcal{P}(\mathbf{r}')$ .

##### 7.1.1) Reduction when $g \equiv 1$

#### Theorem 12

Let  $F_v$  be a primitive of  $f$ , we can write,

$$\begin{aligned} \int_{S'} f(\mathcal{R}^2) dS' &= \frac{1}{2} \int_{\mathcal{L}} \left[ \frac{F_v(\mathcal{R}^2) - A_0}{\mathcal{R}^2 - \left(\frac{w'_m{}^2}{4ik\xi} + d_0\right)} \right] [(\hat{\mathbf{z}} \times (\mathbf{r}' - \mathcal{P}^{-1}(\mathbf{a}_p))) \cdot d\mathbf{r}'] \\ &+ \frac{A_0 - F_v\left(\frac{w'_m{}^2}{4ik\xi} + d_0\right)}{-2ikp\xi} \int_{\mathcal{L}_p} \left( \hat{\mathbf{z}} \times \frac{\mathbf{r}'_p - \mathbf{a}_p}{(\mathbf{r}'_p - \mathbf{a}_p)^2} \right) \cdot d\mathbf{r}'_p, \end{aligned} \quad (125)$$

where  $A_0$  is an arbitrary constant,  $p = \sqrt{1 - \mathbf{c}_p^2}$ , and,

$$\begin{aligned} \mathcal{P}^{-1}(\mathbf{a}_p) &= \frac{1}{2ik(h_1 h_2 - (h_3/2)^2)} \left( (h_2 w'_x - \frac{h_3 w'_y}{2}) \hat{\mathbf{x}} + (h_1 w'_y - \frac{h_3 w'_x}{2}) \hat{\mathbf{y}} \right), \\ \frac{w'_m{}^2}{4ik\xi} &= \frac{1}{4ik(h_1 h_2 - (h_3/2)^2)} (h_2 w'_x{}^2 + h_1 w'_y{}^2 - h_3 w'_x w'_y), \end{aligned} \quad (126)$$

while  $\int_{\mathcal{L}_p} (\hat{\mathbf{z}} \times \frac{\mathbf{r}'_p - \mathbf{a}_p}{(\mathbf{r}'_p - \mathbf{a}_p)^2}) \cdot d\mathbf{r}'_p$  can be determined with (122), and  $(\xi p)^2 = h_1 h_2 - (h_3/2)^2$ .

Proof. In proposition 9 and theorem 11, we change the exponential function for  $f$ , and let,

$$\frac{F_v(\mathcal{R}^2)}{v} = \int f(d_0 + v(\mathbf{r}'_p - \mathbf{a}_p)^2 + v_1)d(\mathbf{r}'_p - \mathbf{a}_p)^2 = \frac{1}{v} \int f(\mathcal{R}^2)d\mathcal{R}^2. \quad (127)$$

We can then write, for  $S'_p$  and  $\mathcal{L}_p$ , the dilated by  $\mathcal{P}$  of  $S'$  and  $\mathcal{L}$ ,

$$\begin{aligned} \int_{S'} f(\mathcal{R}^2)dS' &= \frac{1}{\sqrt{1 - \mathbf{c}_p^2}} \int_{S'_p} f(d_0 + v(\mathbf{r}'_p - \mathbf{a}_p)^2 + v_1)dS'_p \\ &= \int_{\mathcal{L}_p} \frac{F(\mathcal{R}^2) - A_0}{\mathcal{R}^2 - (v_1 + d_0)} \frac{(\hat{\mathbf{z}} \times (\mathbf{r}'_p - \mathbf{a}_p)) \cdot d\mathbf{r}'_p}{2\sqrt{1 - \mathbf{c}_p^2}} - \frac{i(A_0 - F(v_1 + d_0))}{4v\sqrt{1 - \mathbf{c}_p^2}} \int_{\mathcal{L}_p} d\ln\left(\frac{z'_{\mathbf{a}+}}{z'_{\mathbf{a}-}}\right). \end{aligned} \quad (128)$$

Considering the proposition 11, we then derive (125).

### 7.1.2) Exact reduction when $g(\mathbf{r}')$ is a polynomial of coordinates

Considering that  $\mathbf{r}' = \mathcal{P}^{-1}(\mathbf{r}'_p)$ , and

$$\begin{aligned} \hat{\mathbf{x}} \cdot (\mathbf{r}' - \mathcal{P}^{-1}(\mathbf{a}_p)) &= \hat{\mathbf{x}} \cdot \mathcal{P}^{-1}(\mathbf{r}'_p - \mathbf{a}_p) = \mathcal{P}^{-1}(\hat{\mathbf{x}}) \cdot (\mathbf{r}'_p - \mathbf{a}_p) \\ \hat{\mathbf{y}} \cdot (\mathbf{r}' - \mathcal{P}^{-1}(\mathbf{a}_p)) &= \hat{\mathbf{y}} \cdot \mathcal{P}^{-1}(\mathbf{r}'_p - \mathbf{a}_p) = \mathcal{P}^{-1}(\hat{\mathbf{y}}) \cdot (\mathbf{r}'_p - \mathbf{a}_p) \end{aligned}$$

we notice, from Taylor expansion, that we have, for any polynomial  $g(\mathbf{r}')$  of order  $p$ ,

$$g(\mathbf{r}') = \sum_{j=0}^p \frac{1}{j!} \sum_{k=0}^j \binom{j}{k} \frac{\partial^j g|_{\mathbf{r}'=\mathcal{P}^{-1}(\mathbf{a}_p)}}{\partial x'^k \partial y'^{j-k}} [(\mathcal{P}^{-1}(\hat{\mathbf{x}}) \cdot (\mathbf{r}'_p - \mathbf{a}_p))^k (\mathcal{P}^{-1}(\hat{\mathbf{y}}) \cdot (\mathbf{r}'_p - \mathbf{a}_p))^{j-k}] \quad (129)$$

### Theorem 13

Considering a polynomial  $g(\mathbf{r}')$  of order  $p$ , we can develop the exact reduction of  $\int_{S'} g(\mathbf{r}') f(\mathcal{R}^2) dS'$ . For that, we need the reduction of surface integral

$$I_0 = \int_{S'} f(\mathcal{R}^2) dS'$$

given by theorem 12 in (125), and of the elementary integrals,

$$I_{n+1} = \int_{S'} \mathbf{d}_{n+1} \cdot (\mathbf{r}'_p - \mathbf{a}_p) \prod_{j=1}^n (\mathbf{d}_j \cdot (\mathbf{r}'_p - \mathbf{a}_p)) f(\mathcal{R}^2) dS'$$

for  $1 \leq n + 1 \leq p$ , with arbitrary constants  $\mathbf{d}_j$ . Using the general relation,

$$I_{n+1} = \frac{-1}{2ik\xi} \left( \frac{1}{p} \int_{\mathcal{L}} \prod_{j=1}^n (\mathbf{d}_j \cdot (\mathbf{r}'_p - \mathbf{a}_p)) F_v(\mathcal{R}^2) \mathcal{P}(\hat{\mathbf{z}} \times \mathbf{d}_{n+1}) \cdot d\mathbf{r}' \right. \\ \left. - \sum_{m=1}^n (\mathbf{d}_{n+1} \cdot \mathbf{d}_m) \int_{S'_{j \neq m}} \prod_{j=1}^n (\mathbf{d}_j \cdot (\mathbf{r}'_p - \mathbf{a}_p)) F_v(\mathcal{R}^2) dS' \right), \quad (130)$$

where  $F_v$  is a primitive of  $f$ , and  $\prod_{j=1}^0(\dots) = 1$ ,  $\prod_{j \neq i}^1(\dots) = 1$ , and  $\prod_{j \neq i}^0(\dots) = 0$ , and  $(\xi p)^2 = h_1 h_2 - (h_3/2)^2$ , we note that we decrease the rank of polynomial in remaining surface integral, and we can, by a repeated application of (130), reduce  $I_{n+1}$  to a combination of contour integrals depending on  $M = \lfloor (n+3)/2 \rfloor$  successive primitives of  $f$ , denoted  $\frac{1}{q!} \int^x (x-t)^q f(t) dt$ ,  $q = 0, \dots, M-1$ .

Proof. Taking up the theorem 4,  $(\mathbf{r}'_p - \mathbf{a}_p)$  and  $dS'_p$  instead of  $(\mathbf{r}' - \mathbf{a})$  and  $dS'$ , with  $v = -ik\xi$  and  $dS'_p \equiv p dS'$ , we derive (130). By repeated applications, we can decrease the rank, and using theorem 12, reduce  $I_{n+1}$  to a combination of contour integrals for any  $n \geq 0$ , using repeated integrals of  $f$  (note : let us remark the results given in appendix A for repeated integrals and their asymptotics).

Remark 7 : Let us notice that, with  $v = -ik\xi$  and  $(\xi p)^2 = \xi_+ \xi_- = h_1 h_2 - (h_3/2)^2$ ,

$$\int_{S'} \mathbf{d} \cdot (\mathbf{r}' - \mathcal{P}^{-1}(\mathbf{a}_p)) f(\mathcal{R}^2) dS' = \frac{1}{p} \int_{S'_p} \mathcal{P}^{-1}(\mathbf{d}) \cdot (\mathbf{r}'_p - \mathbf{a}_p) f(\mathcal{R}^2) dS'_p \\ = \frac{1}{2p} \int_{\mathcal{L}} \frac{[F_v(\mathcal{R}^2) - A_0]}{v} [\mathcal{P}(\hat{\mathbf{z}} \times \mathcal{P}^{-1}(\mathbf{d})) \cdot d\mathbf{r}']. \quad (131)$$

where  $\frac{\mathcal{P}(\hat{\mathbf{z}} \times \mathcal{P}^{-1}(\mathbf{d}))}{\xi p} = \frac{(\hat{\mathbf{z}} \times \mathcal{P}^{-2}(\mathbf{d}))}{\xi}$  with  $\frac{\mathcal{P}^{-2}(\mathbf{d})}{\xi} = \frac{\mathbf{d} \cdot \hat{\mathbf{c}}_p}{p^2 \xi} \hat{\mathbf{c}}_p + \frac{\mathbf{d} \cdot (\hat{\mathbf{z}} \times \hat{\mathbf{c}}_p)}{\xi} \hat{\mathbf{z}} \times \hat{\mathbf{c}}_p$ . For  $h_3 = 0$ , we have, using (92),  $\frac{\mathcal{P}^{-2}(\mathbf{d})}{\xi} = \frac{\mathbf{d} \cdot \hat{\mathbf{x}}}{h_1} \hat{\mathbf{x}} + \frac{\mathbf{d} \cdot \hat{\mathbf{y}}}{h_2} \hat{\mathbf{y}}$ .

7.2) Asymptotic reduction of  $\int_{S'} g(\mathbf{r}') f(\mathcal{R}^2) dS'$  when  $f(\mathcal{R}^2) = e^{\mathcal{R}^2}$  or  $f(\mathcal{R}^2) = \frac{e^{\sqrt{\mathcal{R}^2}}}{\sqrt{\mathcal{R}^2}}$ , while  $g(\mathbf{r}')$  is a smooth analytical function on  $S'$ .

7.2.1) the case with  $f(\mathcal{R}^2) = e^{\mathcal{R}^2}$ , when  $\mathcal{R}^2$  is a polynomial of order 2

Let us note  $\mathcal{R}^2$  a polynomial of order 2 as in (124), with  $h'_p{}^2 = (\mathbf{r}'_p - \mathbf{a}_p)^2$ ,  $\mathbf{r}'_p = \mathcal{P}(\mathbf{r}')$ ,

$$\mathcal{R}^2 = d_0 + (\alpha_1 x' + \alpha_2 y') - ik(h_1 x'^2 + h_2 y'^2 + h_3 x' y') = d_0 + v h'_p{}^2 + v_1, \quad (132)$$

and consider asymptotic reductions when  $f(\mathcal{R}^2) = e^{\mathcal{R}^2}$  has large variations on  $S'$  at constant  $\varphi_p$ .

**Proposition 13**

*We consider the proposition 2 with*

$$\frac{\partial F((\mathbf{r}' - \mathbf{a})^2, \varphi)}{\partial (\mathbf{r}' - \mathbf{a})^2} = s(\mathbf{r}') = g(\mathbf{r}')e^{\mathcal{R}^2}, \quad (133)$$

with  $\mathbf{a} = \mathcal{P}^{-1}(\mathbf{a}_p)$  and  $F_{vs} \equiv vF$ , assuming that, at any point  $\mathbf{r}' \in S'$ , it exists a path  $\mathcal{T}_L$  in the  $t$ -complex plane from  $t = 0$  to  $t = 1$ , defined so that  $t(\mathbf{r}' - \mathbf{a}') + \mathbf{a}'$  with  $\mathbf{a}' = \mathcal{P}^{-1}(\mathbf{a}_p)$  remains in an open simply connected subset  $\mathcal{W}_s$  of the domain of analyticity of  $s$ . A primitive  $F_{vs} \equiv vF$  of  $s$  relatively to  $vh_p'^2$ , satisfying

$$\frac{\partial F_{vs}((\mathbf{r}' - \mathbf{a})^2, \varphi)}{v\partial (\mathbf{r}' - \mathbf{a})^2} = s(\mathbf{r}') = g(\mathbf{r}')e^{\mathcal{R}^2}, \quad (134)$$

verifies,

$$F_{vs}(\mathbf{r}') - F_{vs}(\mathbf{a}') = 2vh_p'^2 e^{d_0+v_1} \int_{\mathcal{T}_L} g(t(\mathbf{r}' - \mathbf{a}') + \mathbf{a}') e^{t^2vh_p'^2} t dt. \quad (135)$$

**Proof.** From its definition and proposition 2, the function  $F_{vs}$  verifies

$$\begin{aligned} F_{vs}(\mathbf{r}' = \mathcal{P}^{-1}(\mathbf{a}_p) + h'_p \mathbf{u}_{\varphi_p}) - F_{vs}(\mathbf{r}' = \mathcal{P}^{-1}(\mathbf{a}_p)) &= \int_0^{vh_p'^2} s(\mathcal{P}^{-1}(\mathbf{a}_p) + h' \mathbf{u}_{\varphi_p}) v dh'^2 \\ &= 2v(\mathbf{r}'_p - \mathbf{a}_p)^2 \int_{\mathcal{T}_L} g(t\mathcal{P}^{-1}(\mathbf{r}'_p - \mathbf{a}_p) + \mathcal{P}^{-1}(\mathbf{a}_p)) f(d_0 + v_1 + t^2vh_p'^2) t dt, \end{aligned} \quad (136)$$

For that, we have let  $h' = th'_p$  as a new variable of integration, with  $\mathbf{r}'_p - \mathbf{a}_p = h'_p \mathcal{P}(\mathbf{u}_{\varphi_p})$ ,  $h_p'^2 = (\mathbf{r}'_p - \mathbf{a}_p)^2$ ,  $\mathbf{u}_{\varphi_p} = \mathcal{P}^{-1}(\cos\varphi_p \hat{\mathbf{x}} + \sin\varphi_p \hat{\mathbf{y}})$ , which finally implies (135).

**Theorem 14**

*When the proposition 13 applies, the surface integral  $\int_{S'} g(\mathbf{r}') e^{\mathcal{R}^2} dS'$  can be reduced asymptotically for large  $v$ , by changing in theorem 12,  $f(\mathcal{R}^2)$  for  $g(\mathbf{r}') e^{\mathcal{R}^2}$ ,  $F_v(\mathcal{R}^2)$  for  $F_a(\mathbf{r}')$  and  $F_v(\frac{w_m^2}{4ik\xi} + d_0)$  for 0, where*

$$F_a(\mathbf{r}') = \mathcal{K}_s(\mathbf{r}') + U_s(|vh_p'^2| - b_s)(\mathcal{K}_l(\mathbf{r}')), \quad (137)$$

with  $vh_p'^2 = v(\mathbf{r}'_p - \mathbf{a}_p)^2$ , and, letting  $u(t) = g(t\mathcal{P}^{-1}(\mathbf{r}'_p - \mathbf{a}_p) + \mathcal{P}^{-1}(\mathbf{a}_p))$ ,

$$\begin{aligned} \mathcal{K}_l(\mathbf{r}') &= (u(1) - \frac{\partial_t u(1)}{2vh_p'^2})e^{(vh_p'^2+d_0+v_1)} - (u(t_s) - \frac{\partial_t u(t_s)}{2t_svh_p'^2})e^{(t_s^2vh_p'^2+d_0+v_1)} \\ \mathcal{K}_s(\mathbf{r}') &= 2vh_p'^2 e^{d_0+v_1} \int_0^{t_s} u(t) e^{t^2vh_p'^2} t dt, \end{aligned} \quad (138)$$

with  $t_s = 1 + U_s(|vh_p'^2| - b_s)(\sqrt{\frac{b_s}{|vh_p'^2|}} - 1)$ . The number  $b_s$  is a constant,  $U_s(x) = \begin{cases} 1 & \text{if } x \geq 0 \\ 0 & \text{if } x < 0 \end{cases}$ , while  $u(t)$  remains analytic on the integration path. In (137),  $\mathcal{K}_l$  is the asymptotic expression of the part of integral with  $t^2vh_p'^2 \geq b_s$ , when  $|vh_p'^2| \geq b_s$ , for some large  $b_s$ .

Proof. For large  $t^2vh_p'^2 > b_s$ , we can use an integration by parts in (135), and remark that,

$$\begin{aligned} & 2vh_p'^2 e^{d_0+v_1} \int_{t_s}^1 g(t(\mathbf{r}' - \mathcal{P}^{-1}(\mathbf{a}_p)) + \mathcal{P}^{-1}(\mathbf{a}_p)) (e^{t^2vh_p'^2} t) dt \\ &= e^{d_0+v_1} (u(1)e^{vh_p'^2} - u(t_s)e^{t_s^2vh_p'^2} - \int_{t_s}^1 \partial_t g(t(\mathbf{r}' - \mathcal{P}^{-1}(\mathbf{a}_p)) + \mathcal{P}^{-1}(\mathbf{a}_p))) e^{t^2vh_p'^2} dt \\ &\sim e^{d_0+v_1} ((u(1) - \frac{\partial_t u(1)}{2vh_p'^2})e^{vh_p'^2} - (u(t_s) - \frac{\partial_t u(t_s)}{2vh_p'^2 t_s})e^{t_s^2vh_p'^2}), \end{aligned} \quad (139)$$

Dividing the path of integration in (135) into two parts, we then derive (137).

Remark 8 : From proposition 12, we have, when  $\frac{\mp i\sqrt{h_1 h_2 - (\frac{h_3}{2})^2 + \frac{h_3}{2}}}{h_2}$  are not real, two roots  $\mathbf{r}'_{\pm}$  with  $h_p'^2 = 0$  on the real plane, and if  $g(r_{\pm}) = g(\mathcal{P}^{-1}(a_p))$ ,  $\frac{(r'_p - a_p)(g(r') - g(\mathcal{P}^{-1}(a_p)))}{(r'_p - a_p)^2} < \infty$  from,

$$(r'_p - a_p) \frac{g(r') - g(r_{\pm})}{z'_{a+} z'_{a-}} \Big|_{r \rightarrow r_{\pm}} = \frac{(r'_p - a_p)}{z'_{a\mp}} \left[ \frac{(r - r_{\pm})}{z'_{a\pm}} \cdot \text{grad}_{r'}(g(\mathbf{r}')) \right] \Big|_{r \rightarrow r_{\pm}} < \infty. \quad (140)$$

7.2.2) The case with  $f(\mathcal{R}^2) = \frac{e^{\sqrt{\mathcal{R}^2}}}{(\sqrt{\mathcal{R}^2})^n}$  to generalize the results of chapter 5

If we want to generalize the results of chapter 5 and take account of a moderate curvature of the surface of integration, we have to consider  $(z - z')$  instead of  $z$  in (68), with first order approximation of  $z' = (C_1 x'^2 + C_2 y'^2)$  relatively to  $x'$  and  $y'$  coordinates. The case  $f(\mathcal{R}^2) = \frac{e^{\sqrt{\mathcal{R}^2}}}{(\sqrt{\mathcal{R}^2})^n}$  has already been studied in the propositions 5 and 6, and we now add the special case where,



$$\begin{aligned}
\sqrt{\mathcal{R}^2} &= -ik\sqrt{(x' - a_x)^2 + (y' - a_y)^2 + ((C_1x'^2 + C_2y'^2) - z)^2} \\
&= -ik\sqrt{h'_1(x' - a'_x)^2 + h'_2(y' - a'_y)^2 + z''^2 + \epsilon(x', y')} \\
&= -ik\sqrt{(\mathbf{r}'_p - \mathbf{a}_p)^2 + d_0 + v_1 + \epsilon(x', y')}, \\
\sqrt{\mathcal{R}_0^2} &= -ik\sqrt{h'_1(x' - a'_x)^2 + h'_2(y' - a'_y)^2 + z''^2} \\
&= -ik\sqrt{(\mathbf{r}'_p - \mathbf{a}_p)^2 + d_0 + v_1},
\end{aligned} \tag{141}$$

for  $k$  and  $z'' \gg 1$ , with,

$$\begin{aligned}
h'_j &= 1 - 2zC_j, \quad a'_x = \frac{a_x}{h'_1}, \quad a'_y = \frac{a_y}{h'_2}, \\
z''^2 &= z^2 - \left(\frac{1 - h'_1}{h'_1}a_x^2 + \frac{1 - h'_2}{h'_2}a_y^2\right), \\
\epsilon(x', y') &= (C_1x'^2 + C_2y'^2)^2,
\end{aligned}$$

and  $f(\mathcal{R}^2)$  that remains bounded as  $|k| \rightarrow \infty$ , while  $(\partial_{(\mathbf{r}'_p - \mathbf{a}_p)^2} \frac{\mathcal{R}^2}{-k^2}) - 1$  can be neglected on  $S'$ .

**Theorem 15**

Letting  $f(\mathcal{R}^2) = \frac{e^{2\sqrt{\mathcal{R}^2}}}{\sqrt{\mathcal{R}^2}}$  and  $F_v = e^{2\sqrt{\mathcal{R}^2}}$ , we can use theorem 1, and write from lemma 1 for large  $k$ ,

$$\begin{aligned}
&\int_{S'} g(\mathbf{r}') f(\mathcal{R}_0^2) dS' \\
&\sim \int_{\mathcal{L}} \frac{\frac{1}{2}(g(\mathbf{r}') F_v(\mathcal{R}_0^2(\mathbf{r}')) - g(\mathbf{a}') F_v(\mathcal{R}_0^2(\mathbf{a}')))(\hat{\mathbf{z}} \times (\mathbf{r}' - \mathbf{a}')) \cdot d\mathbf{r}'}{-k^2(h'_1(x' - a'_x)^2 + h'_2(y' - a'_y)^2)},
\end{aligned} \tag{142}$$

Subsequently, we can use  $\frac{g(\mathbf{r}') f(\mathcal{R}^2)}{f(\mathcal{R}_0^2)} = \frac{g(\mathbf{r}') \sqrt{\mathcal{R}_0^2}}{\sqrt{\mathcal{R}^2}} e^{2(\sqrt{\mathcal{R}^2} - \sqrt{\mathcal{R}_0^2})}$  in place of  $g(\mathbf{r}')$  for small  $\epsilon(x', y')$ , and we have,

$$\begin{aligned}
&\int_{S'} g(\mathbf{r}') f(\mathcal{R}^2) dS', \\
&\sim \int_{\mathcal{L}} \frac{\frac{1}{2}(g_1(\mathbf{r}') F_v(\mathcal{R}^2(\mathbf{r}')) - g_1(\mathbf{a}') F_v(\mathcal{R}^2(\mathbf{a}')))(\hat{\mathbf{z}} \times (\mathbf{r}' - \mathbf{a}')) \cdot d\mathbf{r}'}{-k^2(h'_1(x' - a'_x)^2 + h'_2(y' - a'_y)^2)}
\end{aligned} \tag{143}$$

with  $g_1(\mathbf{r}') = \frac{g(\mathbf{r}') \sqrt{\mathcal{R}_0^2}}{\sqrt{\mathcal{R}^2}}$ , when  $(\partial_{(\mathbf{r}'_p - \mathbf{a}_p)^2} \frac{\mathcal{R}^2}{-k^2}) - 1$  can be neglected on  $S'$  and if the  $h'_j$  are both real and of the same sign. When the  $h'_j$  are of different signs, the problem is more complex and we need to consider  $h'_j - i\delta$  instead of  $h'_j$  in the denominator of the integrand, with

$\delta \text{sign}(\arg(ik)) \rightarrow 0^+$ . In order to consider  $\frac{e^{\sqrt{\mathcal{R}^2}}}{(\sqrt{\mathcal{R}^2})^n}$  in place of  $\frac{e^{\sqrt{\mathcal{R}^2}}}{\sqrt{\mathcal{R}^2}}$ , we can let  $\frac{g(\mathbf{r}')}{(\sqrt{\mathcal{R}^2})^{n-1}}$  in place of  $g(\mathbf{r}')$ .

Proof. These expressions are directly derived from the lemma 1. The  $\delta$  term permits to avoid any line of singularities. Concerning the sign of  $\delta$ , we choose it so that  $e^{\sqrt{\mathcal{R}^2}} < \infty$  when  $\mathbf{r}' \rightarrow \infty$  and  $|\arg(ik)| = \pi/2$ , and thus  $0 < \delta \text{sign}(\arg(ik)) \ll 1$ .

This theorem will be used in next section 7.3 and validated in numerical results, letting  $g_1(\mathbf{r}') \sim g(\mathbf{r}')$ . On this occasion, let us notice from (24) that  $\partial_{(\mathbf{r}'_p - \mathbf{a}_p)^2} \equiv \frac{(\mathbf{r}'_p - \mathbf{a}_p)}{2(\mathbf{r}'_p - \mathbf{a}_p)^2} \cdot \text{grad}_{\mathbf{r}'_p}$  and  $\text{grad}_{\mathbf{r}'_p} \equiv \mathcal{P}^{-1}(\text{grad}_{\mathbf{r}'})$ , so that we have,  $\partial_{(\mathbf{r}'_p - \mathbf{a}_p)^2} \equiv \frac{(\mathbf{r}' - \mathcal{P}^{-1}(\mathbf{a}_p))}{2(\mathbf{r}'_p - \mathbf{a}_p)^2} \cdot \text{grad}_{\mathbf{r}'}$ .

Remark : Let us add some remark on the influence of the small real  $\delta$  used in the theorem 15. Considering proposition 11 and 1, we can let,

$$\begin{aligned} (\mathbf{r}'_p - \mathbf{a}'_p)^2 &= (h'_1 - i\delta)(x' - a'_x)^2 + (h'_2 - i\delta)(y' - a'_y)^2 = \\ &= \prod_{\pm} (\sqrt{(h'_1 - i\delta)(x' - a'_x)^2} \pm i\sqrt{(h'_2 - i\delta)(y' - a'_y)^2}), \end{aligned} \quad (144)$$

with  $\delta$  chosen such that  $(h'_1 - i\delta) \neq -(h'_2 - i\delta)$ , and note that,

$$\begin{aligned} \int_{\mathcal{L}} \frac{\frac{1}{2}(\widehat{\mathbf{z}} \times (\mathbf{r}' - \mathbf{a}')) \cdot d\mathbf{r}'}{(h'_1 - i\delta)(x' - a'_x)^2 + (h'_2 - i\delta)(y' - a'_y)^2} &= \frac{1}{2p\xi} \int_{\mathcal{L}_p} (\widehat{\mathbf{z}} \times \frac{\mathbf{r}'_p - \mathbf{a}'_p}{(\mathbf{r}'_p - \mathbf{a}'_p)^2}) \cdot d\mathbf{r}'_p \\ &= \frac{\pi(\epsilon^+ 1_{S'_+}(a_x + ia_y) + \epsilon^- 1_{S'_-}(a_x - ia_y))}{2p\xi}, \end{aligned} \quad (145)$$

where  $\xi = (h'_1 - i\delta)$  and  $p = \sqrt{\frac{(h'_2 - i\delta)}{\xi}} \sim \sqrt{\frac{h'_2}{h'_1}(1 - i\delta(\frac{1}{h'_2} - \frac{1}{h'_1}))}$  (proposition 8). In general, it is worth noticing that we have not to take  $p\xi$  as  $\sqrt{(p\xi)^2}$ , but each term  $p$  and  $\xi$  in a distinct manner. For example, we note that  $(1 - i\delta)\sqrt{\frac{(-1-i\delta)}{(1-i\delta)}}$  is defined, while  $\sqrt{(1 - i\delta)(-1 - i\delta)}$  is not, since  $(1 - i\delta)(-1 - i\delta) = -1 - \delta^2$  is on the cut of the square root function.

### 8) Application to case (c) : reduction of Physical Optics radiation integrals for curved plates

Let us consider the expressions of the electric  $\mathbf{E}$ , radiated by electric currents  $\mathbf{J}$  excited on a real perfectly conducting surface  $S$  by weighted dipolar illuminations defined with  $f_{e,h}$  in (68) of chapter 5, in free space with impedance  $Z_0$  and wavenumber  $k = \frac{2\pi}{\lambda}$ . We take this time a curved plate  $S$  instead of a flat one, defined by

$$\mathbf{r}' + \zeta_s(\mathbf{r}')\widehat{\mathbf{z}}, \zeta_s(\mathbf{r}') = (C_1x'^2 + C_2y'^2), \zeta_s(\mathbf{0}) = 0, \quad (146)$$

with  $C_j < 0$  (resp.  $C_j > 0$ ) for convex (resp. concave) surface inflection. We note that, for purely convex (resp. purely concave) surface with  $C_1C_2 > 0$ ,  $S$  is below (resp. above) the plane  $z = 0$ , while, for saddle surfaces with  $C_1C_2 < 0$ ,  $S$  is partly below and partly above the plane.

In what follows, the surface  $S'$  is the projection of  $S$  on the plane  $z = 0$ ,  $\mathbf{r}'$  is the current point of integration on  $S'$ ,  $\mathcal{L}$  is the contour of  $S'$ ,  $(\mathbf{a} + z\widehat{\mathbf{z}})$  is the observation point, and,

$$\mathbf{R}^2(r') = (x' - a_x)^2 + (y' - a_y)^2 + ((C_1x'^2 + C_2y'^2) - z)^2,$$

is the square of the distance between the current point of  $S$  and the observation point, with the equalities in (141) that applies.

8.1) Reduction to a contour integral for radiation of curved plates when  $f_h \equiv 0$ .

8.1.1) A contour integral at finite but large distance

We consider the Physical Optics (PO) diffracted field at  $\mathbf{r} = \mathbf{a} + z\widehat{\mathbf{z}} \notin S$ , when  $\mathbf{r} = \mathbf{r}_0$  (monostatic case), given by,

$$\mathbf{E} \sim -\frac{Z_0}{8(\pi k)^2} \frac{ik}{2} \int_{S'} [(\widehat{\mathbf{n}} \cdot \mathbf{R})J^s](\mathbf{r}') \widetilde{R} \left( \left( \frac{\psi'}{\widetilde{R}} \right)^2 \right)' f_e(-\mathbf{u}) ((\mathbf{u} \cdot \widehat{\mathbf{d}})\mathbf{u} - \widehat{\mathbf{d}}) dS'. \quad (147)$$

where  $\widehat{\mathbf{n}}J^s = \partial_x \mathbf{M}|_S \wedge \partial_y \mathbf{M}|_S$ ,  $S'$  is the projection of the curved surface  $S$  of points  $\mathbf{M}$  on the plane  $z = 0$ ,  $\psi(\widetilde{R}) = \frac{e^{-ik\widetilde{R}}}{\widetilde{R}}$ ,  $(.)' \equiv \partial_{\widetilde{R}}(.)$ . We use that,

$$\begin{aligned} \mathbf{R}^2(r') &= (1 - 2zC_1)(x' - a'_x)^2 + (1 - 2zC_2)(y' - a'_y)^2 + z''^2 + \epsilon(\mathbf{r}') \\ a'_x &= \frac{a_x}{1 - 2zC_1}, a'_y = \frac{a_y}{1 - 2zC_2}, z''^2 = z^2 - \left( \frac{2zC_1}{1 - 2zC_1} a_x^2 + \frac{2zC_2}{1 - 2zC_2} a_y^2 \right), \\ [(\widehat{\mathbf{n}} \cdot \mathbf{R})J^s](\mathbf{r}') &= -2(C_1x'(a_x - x') + C_2y'(a_y - y')) + z - \zeta_s(\mathbf{r}') \end{aligned} \quad (148)$$

where  $\epsilon(\mathbf{r}') = (\zeta_s(\mathbf{r}'))^2$ ,  $\zeta_s(\mathbf{r}') = (C_1x'^2 + C_2y'^2)^2$ , and let  $\mathbf{R}^2 = (\mathbf{r}'_p - \mathbf{a}_p)^2 + z''^2 + \epsilon(\mathbf{r}')$  with  $\mathbf{a}_p = \mathcal{P}(\mathbf{a}')$ . In what follows, we consider that the term  $\epsilon(x, y)$  has moderate influence on terms in front of exponential terms, and  $\partial_{(\mathbf{r}'_p - \mathbf{a}_p)^2}(\mathbf{R}^2) - 1$  can be neglected on  $S'$ . For that, we take  $z$  sufficiently large so that  $|(C_1x^2 + C_2y^2)|_S \ll z$ .

Theorem 16

Considering the theorem 15, we can write the surface integral (147), when  $k$  is large and  $|C_1x^2 + C_2y^2|_S \ll z$ , as the contour integral,

$$\mathbf{E} \sim -\frac{Z_0 ik}{(4\pi)^2} \int_{\mathcal{L}} \frac{(\frac{e^{-2ik\tilde{R}(\mathbf{r}')}}{R^2(\mathbf{r}')} \mathcal{W}_e(\mathbf{r}') - \frac{e^{-2ik\tilde{R}(\mathbf{a}')}}{R^2(\mathbf{a}')} \mathcal{W}_e(\mathbf{a}'))[(\hat{\mathbf{z}} \times (\mathbf{r}' - \mathbf{a}')) \cdot d\mathbf{r}']}{(1 - 2zC_1 - i\delta)(x' - a'_x)^2 + (1 - 2zC_2 - i\delta)(y' - a'_y)^2}, \quad (149)$$

with  $\tilde{R}(\mathbf{r}') = \sqrt{(\mathbf{r}' - \mathbf{a}')^2 + (\zeta_s(\mathbf{r}') - z)^2}$ ,  $\mathcal{W}_e(\mathbf{r}') = f_e(-\mathbf{u})(\hat{\mathbf{d}} - (\mathbf{u} \cdot \hat{\mathbf{d}})\hat{\mathbf{u}})[(\hat{\mathbf{n}} \cdot \mathbf{R})J^s](\mathbf{r}')$ , and  $\mathcal{L}$  the contour of  $S'$  on the plane  $z = 0$ . The term  $\delta$  is a small positive real constant tending to 0, in particular necessary for the cases where  $(1 - 2zC_1)(1 - 2zC_2) < 0$ .

When  $f_e(-u) = \frac{1}{\sqrt{1-(\mathbf{u} \cdot \hat{\mathbf{d}})^2}}$  and  $\hat{\mathbf{d}} \cdot \hat{\mathbf{z}} = 0$ , we then have the  $VV$  polarisation term  $\mathbf{E} \cdot \hat{\mathbf{d}}$  (already considered for a flat plate in (74)) following,

$$\mathbf{E} \cdot \hat{\mathbf{d}} = -\frac{Z_0 ik}{(4\pi)^2} \int_{\mathcal{L}} \frac{(\frac{[(\hat{\mathbf{n}} \cdot \mathbf{R})J^s](\mathbf{r}')e^{-2ik\tilde{R}(\mathbf{r}')}\sqrt{1-(\frac{\mathbf{r}'-\mathbf{a}'}{\tilde{R}(\mathbf{r}')} \cdot \hat{\mathbf{d}})^2}}{R^2(\mathbf{r}')} - \frac{[(\hat{\mathbf{n}} \cdot \mathbf{R})J^s](\mathbf{a}')e^{-2ik\tilde{R}(\mathbf{a}')}\sqrt{1-(\frac{\mathbf{a}'-\mathbf{a}'}{\tilde{R}(\mathbf{a}')} \cdot \hat{\mathbf{d}})^2}}{R^2(\mathbf{a}')})}{(1 - 2zC_1 - i\delta)(x' - a'_x)^2 + (1 - 2zC_2 - i\delta)(y' - a'_y)^2}[(\hat{\mathbf{z}} \times (\mathbf{r}' - \mathbf{a}')) \cdot d\mathbf{r}']}. \quad (150)$$

Remark 9 : Since  $\frac{\lambda}{2R_c} \ll 1$  with  $R_c = \min(\frac{1}{2|C_j|})$  in PO,  $z_\lambda = \max(\frac{(x^2+y^2)|_S}{\lambda}) < z$  in far field, and  $|C_1x^2 + C_2y^2|_S < z_f \frac{\lambda}{2R_c}$ , we notice that  $|C_1x^2 + C_2y^2|_S \ll z$  for  $z \gg z_\lambda \frac{\lambda}{2R_c}$ .

### 8.1.2) Asymptotic cases : at very large distance or at focal points

For  $z \rightarrow \infty$ , we can take in surface and contour integral,

$$\sqrt{\mathbf{R}^2} \sim \tilde{r}_0 - \frac{\mathbf{r}_0}{\tilde{r}_0} \cdot (\mathbf{r}' + \hat{\mathbf{z}}\zeta_s(\mathbf{r}')), \quad z'' \sim \tilde{r}_0 - \frac{\mathbf{r}_0}{\tilde{r}_0} \cdot (\mathbf{a}' + \hat{\mathbf{z}}\zeta_s(\mathbf{a}')), \quad (151)$$

with  $[(\hat{\mathbf{n}} \cdot \mathbf{R})J^s](\mathbf{r}') \rightarrow -2(C_1x'a_x + C_2y'a_y) + z$ , and thus, the expression (149) becomes,

$$\mathbf{E} \sim -\frac{Z_0 ik}{(4\pi)^2} \frac{e^{-2ik\tilde{r}_0}}{r_0^2} f_e(-\frac{\mathbf{r}_0}{\tilde{r}_0})(\hat{\mathbf{d}} - (\frac{\mathbf{r}_0}{\tilde{r}_0} \cdot \hat{\mathbf{d}})\frac{\mathbf{r}_0}{\tilde{r}_0}) \int_{\mathcal{L}} \frac{\mathcal{H}(\mathbf{r}')e^{2ik(\frac{\mathbf{r}_0}{\tilde{r}_0} \cdot (\mathbf{r}' + \hat{\mathbf{z}}\zeta_s(\mathbf{r}')))} - \mathcal{H}(\mathbf{a}')e^{2ik(\frac{\mathbf{r}_0}{\tilde{r}_0} \cdot (\mathbf{a}' + \hat{\mathbf{z}}\zeta_s(\mathbf{a}')))}}{(\frac{2zC_1-1}{2z} + i\delta)(x' - a'_x)^2 + (\frac{2zC_2-1}{2z} + i\delta)(y' - a'_y)^2}[(\hat{\mathbf{z}} \times (\mathbf{r}' - \mathbf{a}')) \cdot d\mathbf{r}'], \quad (152)$$

where  $\mathcal{H}(\mathbf{r}') = (C_1x'(\frac{a_x}{z}) + C_2y'(\frac{a_y}{z}) - \frac{1}{2})$ .

At focal points, as  $\mathcal{K}_m = \min(1 - 2zC_j)|_{j=1,2} \rightarrow 0$ , we have  $a' \rightarrow \infty$  and  $z''^2 \rightarrow \infty$ , but  $a'/z''^2$  and  $\mathcal{K}_m a'$  tend to finite constant vectors, and we can write for (150),

$$\mathbf{E} \sim \frac{Z_0 ik}{(4\pi)^2} \left( \int_{\mathcal{L}} \frac{\left( \frac{e^{-2ik\tilde{R}(\mathbf{r}')}}{R^2(\mathbf{r}')} \mathcal{W}_e(\mathbf{r}') - \frac{e^{-2ik\tilde{R}(\mathbf{a}')}}{R^2(\mathbf{a}')} \mathcal{W}_e(\mathbf{a}') \right) [(\hat{\mathbf{z}} \times \mathcal{K}_m \mathbf{a}')] \cdot d\mathbf{r}'}{\mathcal{K}_m(\mathbf{R}^2 - z^2 - \epsilon(\mathbf{r}')) + \left( \frac{2zC_1\mathcal{K}_m}{(1-2zC_1)} a_x^2 + \frac{2zC_2\mathcal{K}_m}{(1-2zC_2)} a_y^2 \right)} \right), \quad (153)$$

8.2) Reduction to a contour integral for diffraction by curved plates when  $f_e \equiv 0$ .

8.2.1) A contour integral at finite but large distance

The PO diffracted field at  $\mathbf{r} = \mathbf{a} + z\hat{\mathbf{z}} \notin S'$ , when  $\mathbf{r} = \mathbf{r}_0$  (monostatic case), satisfies,

$$\mathbf{E} \cdot \mathbf{v}_0 \sim \frac{-1}{8(\pi ik)^2} \int_{S'} [(\hat{\mathbf{n}} \cdot \mathbf{R}) J^s](\mathbf{r}') \left( \left( \frac{\psi'}{\tilde{R}} \right)' \right)^2 \mathbf{w}_0 \cdot \mathbf{R} f_h(-\mathbf{u}) dS', \quad (154)$$

where  $\mathbf{v}_0 = \hat{\mathbf{d}} \times \mathbf{w}_0$ .

Considering that  $\mathbf{w}_0 \cdot \mathbf{R} = \mathbf{w}_0 \cdot (\mathbf{R}_1 + \mathbf{R}_2)$  with  $\mathbf{R}_2(\mathbf{r}') = (\mathbf{a} - \mathbf{a}' + (z - \zeta_s(\mathbf{r}'))\hat{\mathbf{z}})$  and  $\mathbf{R}_1(\mathbf{r}') = (\mathbf{a}' - \mathbf{r}')$ , we could use (131) for the part with  $\mathbf{R}_1$  term, as already used in theorem 8, but we here prefer to directly use the theorem 15.

Theorem 17

Considering the theorem 15, we can write the surface integral (154), when  $k$  is large and  $|(C_1 x^2 + C_2 y^2)|_S \ll z$ , as the contour integral,

$$\mathbf{E} \cdot \mathbf{v}_0 \sim \frac{ik}{(4\pi)^2} \int_{\mathcal{L}} \frac{\left( \frac{\mathcal{W}_h(\mathbf{r}') e^{-2ik\tilde{R}(\mathbf{r}')}}{(\tilde{R}(\mathbf{r}'))^3} - \frac{\mathcal{W}_h(\mathbf{a}') e^{-2ik\tilde{R}(\mathbf{a}')}}{(\tilde{R}(\mathbf{a}'))^3} \right) (\hat{\mathbf{z}} \times (\mathbf{r}' - \mathbf{a}')) \cdot d\mathbf{r}'}{(1 - 2zC_1 - i\delta)(x' - a'_x)^2 + (1 - 2zC_2 - i\delta)(y' - a'_y)^2}, \quad (155)$$

where (148) is used,  $\mathcal{W}_h(\mathbf{r}') = f_h(-\mathbf{u}) [(\hat{\mathbf{n}} \cdot \mathbf{R}) J^s](\mathbf{r}') [\mathbf{w}_0 \cdot \mathbf{R}](\mathbf{r}')$ . The HH polarization is then obtained when we let  $\mathbf{w}_0 = \frac{\mathbf{r}_0}{r_0} = \frac{\mathbf{a} + z\hat{\mathbf{z}}}{r_0}$ . We can consider (as in (83) for a flat plate) an isotropic illumination with  $f_h(\mathbf{u}_0) = 1/\sqrt{1 - \left( \frac{(\mathbf{r}' - \mathbf{a}') \cdot \hat{\mathbf{d}}}{\tilde{R}} \right)^2}$ ,  $\hat{\mathbf{d}} \cdot \mathbf{r}_0 = \hat{\mathbf{d}} \cdot \hat{\mathbf{z}} = 0$ .

8.2.2) Asymptotic cases : at very large distance or at focal points

For  $z \rightarrow \infty$ , we use (151) in previous expression, and we can write,

$$\begin{aligned} \mathbf{E} \cdot \mathbf{v}_0 &\sim \frac{ik}{(4\pi)^2} \frac{e^{-2ik\tilde{R}}}{\tilde{r}_0^2} f_h\left(-\frac{\mathbf{r}_0}{\tilde{r}_0}\right) (\mathbf{w}_0 \cdot \mathbf{r}_0) \\ &\int_{\mathcal{L}} \frac{\mathcal{H}(\mathbf{r}') e^{2ik\left(\frac{r_0}{\tilde{r}_0} \cdot (\mathbf{r}' + \hat{\mathbf{z}}\zeta_s(\mathbf{r}'))\right)} - \mathcal{H}(\mathbf{a}') e^{2ik\left(\frac{r_0}{\tilde{r}_0} \cdot (\mathbf{a}' + \hat{\mathbf{z}}\zeta_s(\mathbf{a}'))\right)}}{\left(\frac{2zC_1-1}{2z} + i\delta\right)(x' - a'_x)^2 + \left(\frac{2zC_2-1}{2z} + i\delta\right)(x' - a'_y)^2} (\hat{\mathbf{z}} \times (\mathbf{r}' - \mathbf{a}')) \cdot d\mathbf{r}', \quad (156) \end{aligned}$$

where  $\mathcal{H}(\mathbf{r}') = (C_1 x'(\frac{a_x}{z}) + C_2 y'(\frac{a_y}{z}) - \frac{1}{2})$ .

At focal points, as  $\mathcal{K}_m = \min(|1 - 2zC_j|_{j=1,2}) \rightarrow 0$ , we have  $a' \rightarrow \infty$  and  $z''^2 \rightarrow \infty$ , but  $\mathbf{a}'/z''^2$  tends to a finite constant vector, and we can then write

$$\mathbf{E} \cdot \mathbf{v}_0 \sim \frac{-ik}{(4\pi)^2} \int_{\mathcal{L}} \frac{(\frac{\mathcal{W}_h(\mathbf{r}')e^{-2ik\tilde{R}}}{\tilde{R}\mathbf{R}^2} - \frac{\mathcal{W}_h(\mathbf{a}')e^{-2ik\tilde{R}(\mathbf{a}')}}{(\tilde{R}(\mathbf{a}'))^3})(\hat{\mathbf{z}} \times \mathcal{K}_m \mathbf{a}') \cdot d\mathbf{r}'}{\mathcal{K}_m(\mathbf{R}^2 - z^2 - \epsilon(\mathbf{r}')) + (\frac{2zC_1\mathcal{K}_m}{1-2zC_1}a_x^2 + \frac{2zC_2\mathcal{K}_m}{1-2zC_2}a_y^2)}. \quad (157)$$

Remark : for  $0 < |(cx)^2 + y^2| < \Delta \ll 1$ , we have  $\int_{x'-\Delta/2}^{x'+\Delta/2} \frac{v(x)dx}{(cx)^2+y^2} \simeq \frac{iv(x')}{2cy} [\ln(\frac{iy+cx}{iy-cx})]_{x'-\Delta/2}^{x'+\Delta/2}$ .

### 9) Miscellaneous results

Proposition 14

Considering

$$\begin{aligned} \mathcal{R}^2 &= (\alpha_1 x' + \alpha_2 y') - ik(h_1 x'^2 + h_2 y'^2 + h_3 x' y') \\ &= \mathbf{w}' \cdot \mathbf{r}' - ik(h_1 x'^2 + h_2 y'^2 + h_3 x' y') \end{aligned}$$

we have, for arbitrary constants  $\mathbf{d}$  and  $A_0$ ,

$$\int_{S'} (\mathbf{d} \cdot \mathbf{r}') f(\mathcal{R}^2) dS' = A_1 \cdot (\int_{\mathcal{L}} [F_v(\mathcal{R}^2) - A_0] d\mathbf{r}' - (\hat{\mathbf{z}} \times \mathbf{w}') \int_{S'} f(\mathcal{R}^2) dS'), \quad (158)$$

where  $F$  is a primitive of  $f$  and  $A_1 = \frac{\mathcal{P}(\hat{\mathbf{z}} \times \mathcal{P}^{-1}(\mathbf{d}))}{-2ikp\xi}$ . In the particular case where  $h_1 h_2 - h_3^2/4 \rightarrow 0$ , and

$$(h_1 x'^2 + h_2 y'^2 + h_3 x' y') \rightarrow (\sqrt{h_1} x' + \epsilon' \sqrt{h_2} y')^2, \quad (\epsilon')^2 = 1, \quad (159)$$

when we have,

$$\begin{aligned} p^2 \xi &= \frac{(h_1 + h_2) - \epsilon \sqrt{(h_1 - h_2)^2 + 4(h_3/2)^2}}{2} \rightarrow 0 \\ \text{or } \xi &= \frac{(h_1 + h_2) - \epsilon \sqrt{(h_1 - h_2)^2 + 4(h_3/2)^2}}{2} \rightarrow 0, \end{aligned} \quad (160)$$

the equality (158) has for consequence that,

$$\begin{aligned} \lim_{\xi \rightarrow 0} \int_{S'} f(\mathcal{R}^2) dS' &= \lim_{\xi \rightarrow 0} \frac{\hat{\mathbf{c}}_p}{\hat{\mathbf{c}}_p \cdot (\hat{\mathbf{z}} \times \mathbf{w}')} \cdot \int_{\mathcal{L}} [F(\mathcal{R}^2) - A_0] d\mathbf{r}', \\ \lim_{p^2 \xi \rightarrow 0} \int_{S'} f(\mathcal{R}^2) dS' &= \lim_{p^2 \xi \rightarrow 0} \frac{(\hat{\mathbf{z}} \times \hat{\mathbf{c}}_p)}{(\hat{\mathbf{z}} \times \hat{\mathbf{c}}_p) \cdot (\hat{\mathbf{z}} \times \mathbf{w}')} \cdot \int_{\mathcal{L}} [F(\mathcal{R}^2) - A_0] d\mathbf{r}', \end{aligned} \quad (161)$$

i.e. we have, letting  $\mathbf{v} = ((\xi - h_1)\hat{x} + \frac{-h_3}{2}\hat{y})$  for  $\xi \neq h_1$  or  $(\frac{-h_3}{2}\hat{x} + (\xi - h_2)\hat{y})$  for  $\xi \neq h_2$ ,

$$\begin{aligned} \lim_{\xi \rightarrow 0} \int_{S'} f(\mathcal{R}^2) dS' &= \lim_{\xi \rightarrow 0} \frac{\mathbf{v}}{\mathbf{v} \cdot (\hat{\mathbf{z}} \times \mathbf{w}')} \cdot \int_{\mathcal{L}} [F(\mathcal{R}^2) - A_0] d\mathbf{r}', \\ \lim_{p^2\xi \rightarrow 0} \int_{S'} f(\mathcal{R}^2) dS' &= \lim_{p^2\xi \rightarrow 0} \frac{(\hat{\mathbf{z}} \times \mathbf{v})}{(\hat{\mathbf{z}} \times \mathbf{v}) \cdot (\hat{\mathbf{z}} \times \mathbf{w}')} \cdot \int_{\mathcal{L}} [F(\mathcal{R}^2) - A_0] d\mathbf{r}', \end{aligned} \quad (162)$$

while, for  $\xi = h_1 = h_2$ , which implies  $h_3 = 0$  and  $p^2\xi = \xi$ , we have,

$$(\hat{\mathbf{z}} \times \mathbf{w}') \lim_{\xi \rightarrow 0} \int_{S'} f(\mathcal{R}^2) dS' = \lim_{\xi \rightarrow 0} \int_{\mathcal{L}} [F(\mathcal{R}^2) - A_0] d\mathbf{r}'. \quad (163)$$

Proof. Using that  $\mathbf{d} \cdot \mathbf{r}' = \mathcal{P}^{-1}(\mathbf{d}) \cdot (\mathbf{r}'_p - \mathbf{a}_p) + \mathcal{P}^{-1}(\mathbf{d}) \cdot \mathbf{a}_p$ , then the proposition 3, we derive,

$$\begin{aligned} \int_{S'} \mathbf{d} \cdot \mathbf{r}' f(\mathcal{R}^2) dS' - \mathcal{P}^{-1}(\mathbf{d}) \cdot \mathbf{a}_p \int_{S'} f(\mathcal{R}^2) dS' &= \frac{1}{p} \int_{S'_p} \mathcal{P}^{-1}(\mathbf{d}) \cdot (\mathbf{r}'_p - \mathbf{a}_p) f(\mathcal{R}^2) dS'_p \\ &= \frac{1}{2p} \int_{\mathcal{L}} \frac{[F_v(\mathcal{R}^2) - A_0]}{v} [\mathcal{P}(\hat{\mathbf{z}} \times \mathcal{P}^{-1}(\mathbf{d})) \cdot d\mathbf{r}'], \end{aligned} \quad (164)$$

and thus (158), if we notice that,

$$\begin{aligned} \mathcal{P}^{-1}(\mathbf{d}) \cdot \mathbf{a}_p &= \hat{\mathbf{z}} \times \mathcal{P}^{-1}(\mathbf{d}) \cdot \hat{\mathbf{z}} \times \mathbf{a}_p = \frac{\hat{\mathbf{z}} \times \mathcal{P}^{-1}(\mathbf{d}) \cdot \hat{\mathbf{z}} \times \mathcal{P}^{-1}(\mathbf{w}')}{2ik\xi} \\ &= \frac{\hat{\mathbf{z}} \times \mathcal{P}^{-1}(\mathbf{d}) \cdot \mathcal{P}(\hat{\mathbf{z}} \times \mathbf{w}')}{2ikp\xi} = \frac{\mathcal{P}(\hat{\mathbf{z}} \times \mathcal{P}^{-1}(\mathbf{d})) \cdot (\hat{\mathbf{z}} \times \mathbf{w}')}{2ikp\xi}, \end{aligned} \quad (165)$$

where we have used (110)-(111) with  $\mathbf{a}_p = \frac{\mathcal{P}^{-1}(\mathbf{w}')}{2ik\xi}$ . Considering that,

$$\begin{aligned} \frac{\mathcal{P}(\hat{\mathbf{z}} \times \mathcal{P}^{-1}(\mathbf{d}))}{p\xi} &= \frac{(\hat{\mathbf{z}} \times \mathcal{P}^{-2}(\mathbf{d}))}{\xi}, \\ \frac{\mathcal{P}^{-2}(\mathbf{d})}{\xi} &= \frac{\mathbf{d} \cdot \hat{\mathbf{c}}_p}{p^2\xi} \hat{\mathbf{c}}_p + \frac{\mathbf{d} \cdot (\hat{\mathbf{z}} \times \hat{\mathbf{c}}_p)}{\xi} \hat{\mathbf{z}} \times \hat{\mathbf{c}}_p, \end{aligned} \quad (166)$$

we note that, in right side of (158),  $\frac{\mathcal{P}(\hat{\mathbf{z}} \times \mathcal{P}^{-1}(\mathbf{d}))}{2ikp\xi}$  tends to  $\infty$  for  $\xi \rightarrow 0$  or  $p^2\xi \rightarrow 0$ , while left side of (158) is finite for arbitrary bounded  $\mathbf{d}$ . We can then write,

$$\begin{aligned} \lim_{\xi \rightarrow 0} \hat{\mathbf{c}}_p \cdot [(\hat{\mathbf{z}} \times \mathbf{w}') \int_{S'} f(\mathcal{R}^2) dS' - \int_{\mathcal{L}} [F(\mathcal{R}^2) - A_0] d\mathbf{r}'] &= 0, \\ \lim_{p^2\xi \rightarrow 0} (\hat{\mathbf{z}} \times \hat{\mathbf{c}}_p) \cdot [(\hat{\mathbf{z}} \times \mathbf{w}') \int_{S'} f(\mathcal{R}^2) dS' - \int_{\mathcal{L}} [F(\mathcal{R}^2) - A_0] d\mathbf{r}'] &= 0, \end{aligned} \quad (167)$$

and thus (161). We then obtain (162), if we use (87)-(89) and,

$$\begin{aligned}
c_{p,x} &= \epsilon_x \sqrt{\xi - h_1} / \sqrt{\xi}, \quad c_{p,y} = \epsilon_y \sqrt{\xi - h_2} / \sqrt{\xi}, \\
\epsilon_x^2 &= \epsilon_y^2 = 1, \quad \epsilon_y \epsilon_x = \frac{\epsilon_y}{\epsilon_x} = \frac{\epsilon_x}{\epsilon_y} = \frac{-h_3}{2\sqrt{\xi - h_1}\sqrt{\xi - h_2}}, \\
\xi &= \frac{(h_1 + h_2) + \epsilon \sqrt{(h_1 - h_2)^2 + h_3^2}}{2}, \\
(\mathbf{c}_p)_\epsilon^2 &= 1 - \frac{(h_1 + h_2) - \epsilon \sqrt{(h_1 - h_2)^2 + h_3^2}}{(h_1 + h_2) + \epsilon \sqrt{(h_1 - h_2)^2 + h_3^2}}, \\
p^2 &= 1 - (\mathbf{c}_p)^2, \quad p^2 \xi = \frac{(h_1 + h_2) - \epsilon \sqrt{(h_1 - h_2)^2 + 4(h_3/2)^2}}{2}.
\end{aligned} \tag{168}$$

Remark : let us note, for the expression of  $\frac{\mathcal{P}^{-2}(\mathbf{d})}{\xi}$ , that,

$$\begin{aligned}
(\mathbf{d} \cdot \widehat{\mathbf{c}}_p) \widehat{\mathbf{c}}_p &= \frac{\mathbf{d} \cdot (\sqrt{\xi - h_1} \widehat{x} + \frac{\epsilon_y}{\epsilon_x} \sqrt{\xi - h_2} \widehat{y})}{\epsilon \sqrt{(h_1 - h_2)^2 + h_3^2}} (\sqrt{\xi - h_1} \widehat{x} + \frac{\epsilon_y}{\epsilon_x} \sqrt{\xi - h_2} \widehat{y}) \\
&= \frac{\mathbf{d} \cdot ((\xi - h_1) \widehat{x} + \frac{-h_3}{2} \widehat{y})}{(\xi - h_1) \epsilon \sqrt{(h_1 - h_2)^2 + h_3^2}} ((\xi - h_1) \widehat{x} + \frac{-h_3}{2} \widehat{y}) \\
&= \frac{\mathbf{d} \cdot (\frac{-h_3}{2} \widehat{x} + (\xi - h_2) \widehat{y})}{(\xi - h_2) \epsilon \sqrt{(h_1 - h_2)^2 + h_3^2}} (\frac{-h_3}{2} \widehat{x} + (\xi - h_2) \widehat{y}),
\end{aligned} \tag{169}$$

and,

$$\begin{aligned}
(\mathbf{d} \cdot (\widehat{\mathbf{z}} \times \widehat{\mathbf{c}}_p)) (\widehat{\mathbf{z}} \times \widehat{\mathbf{c}}_p) &= \frac{\mathbf{d} \cdot (\sqrt{\xi - h_1} \widehat{y} - \frac{\epsilon_y}{\epsilon_x} \sqrt{\xi - h_2} \widehat{x})}{\epsilon \sqrt{(h_1 - h_2)^2 + h_3^2}} (\sqrt{\xi - h_1} \widehat{y} - \frac{\epsilon_y}{\epsilon_x} \sqrt{\xi - h_2} \widehat{x}) \\
&= \frac{\mathbf{d} \cdot ((\xi - h_1) \widehat{y} - \frac{-h_3}{2} \widehat{x})}{(\xi - h_1) \epsilon \sqrt{(h_1 - h_2)^2 + h_3^2}} ((\xi - h_1) \widehat{y} - \frac{-h_3}{2} \widehat{x}) \\
&= \frac{\mathbf{d} \cdot (\frac{-h_3}{2} \widehat{y} - (\xi - h_2) \widehat{x})}{(\xi - h_2) \epsilon \sqrt{(h_1 - h_2)^2 + h_3^2}} (\frac{-h_3}{2} \widehat{y} - (\xi - h_2) \widehat{x}).
\end{aligned} \tag{170}$$

Remark : Using (114) of proposition 10, we have, as  $\epsilon = \pm 1$ ,

$$\begin{aligned}
(\widehat{\mathbf{c}}_p)_\pm &= \pm \epsilon_c (\widehat{\mathbf{z}} \times (\widehat{\mathbf{c}}_p)_\mp), \\
(p^2 \xi)_\pm &= (\xi)_\mp, \\
\left( \frac{\mathbf{d} \cdot (\widehat{\mathbf{c}}_p)}{p^2 \xi} (\widehat{\mathbf{c}}_p) \right)_\pm &= \left( \frac{\mathbf{d} \cdot (\widehat{\mathbf{z}} \times \widehat{\mathbf{c}}_p)}{\xi} \widehat{\mathbf{z}} \times \widehat{\mathbf{c}}_p \right)_\mp,
\end{aligned} \tag{171}$$



with  $(\epsilon_c)^2 = 1$ , and we then note that,

$$\frac{\mathcal{P}^{-2}(\mathbf{d})}{\xi} = \frac{\mathbf{d} \cdot \widehat{\mathbf{c}}_p}{p^2 \xi} \widehat{\mathbf{c}}_p + \frac{\mathbf{d} \cdot (\widehat{\mathbf{z}} \times \widehat{\mathbf{c}}_p)}{\xi} \widehat{\mathbf{z}} \times \widehat{\mathbf{c}}_p \text{ is independent of } \epsilon, \quad (172)$$

For example, we have,

$$\begin{aligned} \frac{\widehat{x} \mathcal{P}^{-2}(\widehat{x})}{\xi} &= \frac{(\widehat{x} \cdot \widehat{\mathbf{c}}_p)^2}{p^2 \xi} + \frac{(\widehat{x} \cdot (\widehat{\mathbf{z}} \times \widehat{\mathbf{c}}_p))^2}{\xi} = \frac{1}{\xi (\mathbf{c}_p)_\epsilon^2} \left( \frac{\xi - h_1}{p^2 \xi} + \frac{\xi - h_2}{\xi} \right) \\ &= \frac{1}{\xi (\mathbf{c}_p)_\epsilon^2} \left( \frac{(h_2 - h_1) + \epsilon \sqrt{(h_1 - h_2)^2 + h_3^2}}{(h_1 + h_2) - \epsilon \sqrt{(h_1 - h_2)^2 + h_3^2}} + \frac{(h_1 - h_2) + \epsilon \sqrt{(h_1 - h_2)^2 + h_3^2}}{(h_1 + h_2) + \epsilon \sqrt{(h_1 - h_2)^2 + h_3^2}} \right) \\ &= \frac{1}{\xi (\mathbf{c}_p)_\epsilon^2} \left( \frac{4h_2 \epsilon \sqrt{(h_1 - h_2)^2 + h_3^2}}{((h_1 + h_2) - \epsilon \sqrt{(h_1 - h_2)^2 + h_3^2})((h_1 + h_2) + \epsilon \sqrt{(h_1 - h_2)^2 + h_3^2})} \right) \\ &= \frac{h_2}{(h_1 h_2 - h_3^2/4)}. \end{aligned} \quad (173)$$

**Proposition 15**

Let  $F_v$  be a primitive of  $f$ , we have, for the limit case  $\xi \rightarrow 0$  where  $\mathcal{R}^2 \rightarrow \mathbf{w}' \cdot \mathbf{r}'$ ,

$$\int_{S'} f(\mathbf{w}' \cdot \mathbf{r}') dS' = \frac{(\widehat{\mathbf{z}} \times \mathbf{w}')}{(\mathbf{w}')^2} \cdot \int_{\mathcal{L}} (F_v(\mathbf{w}' \cdot \mathbf{r}') - F_v(0)) d\mathbf{r}'. \quad (174)$$

Proof. We can obtain this equality from (163), or by using  $\text{grad}(F_v(\mathbf{w}' \cdot \mathbf{r}')) = \mathbf{w}' F'_v(\mathbf{w}' \cdot \mathbf{r}')$ .

Remark : Letting  $\mathbf{w}' = a \mathbf{w}'_1$  with  $a \rightarrow 0$ , we note from (174) that,

$$\int_{S'} dS' = \frac{(\widehat{\mathbf{z}} \times \mathbf{w}'_1)}{(\mathbf{w}'_1)^2} \cdot \int_{\mathcal{L}} \mathbf{w}'_1 \cdot \mathbf{r}' d\mathbf{r}'. \quad (175)$$

## 10) Numerical results

10.1) Reduction of a surface integral whose integrand has an exponential term with a quadratic argument

10.1.1) For  $e^{\mathbf{w}' \cdot \mathbf{r}' - ik\xi(\mathbf{r}'^2 - (\mathbf{c}_p \cdot \mathbf{r}')^2)}$  integrand in surface integral

We choose to validate our reduction given in theorem 11, by verifying,

$$\frac{k}{\sqrt{\pi}} \int_{S'} e^{\mathbf{w}' \cdot \mathbf{r}' - ik\xi(|\mathbf{r}'|^2 - (\mathbf{c}_p \cdot \mathbf{r}')^2)} dS' = \frac{k}{2\sqrt{\pi}} \int_{\mathcal{L}} \left[ \frac{(e^{\mathbf{w}' \cdot \mathbf{r}' - ik\xi(|\mathbf{r}'|^2 - (\mathbf{c}_p \cdot \mathbf{r}')^2)} - e^{\frac{\mathbf{w}'_m^2}{4ik\xi}})}{(\mathbf{w}' \cdot \mathbf{r}' - ik\xi(|\mathbf{r}'|^2 - (\mathbf{c}_p \cdot \mathbf{r}')^2)) - \frac{\mathbf{w}'_m^2}{4ik\xi}} \right] [(\hat{\mathbf{z}} \times (\mathbf{r}' - \frac{\mathcal{M}(\mathbf{w}')}{2ik\xi(1 - \mathbf{c}_p^2)})) \cdot d\mathbf{r}'], \quad (176)$$

when  $\mathbf{w}'$  and  $\mathbf{c}_p$  vary, in the figures 3 and 4. We take  $S'$ , a square of 1m side length, whose center is at  $\mathbf{r}_c = (0.5 \hat{\mathbf{x}} + 0.5 \hat{\mathbf{y}})$ ,  $k = 2\pi/\lambda$ ,  $\lambda = 2$  cm,  $\xi = e^{i\Phi}/|\mathbf{r}|$ ,  $|\mathbf{r}| = 1$  m. The term  $\mathbf{c}_p = \sin\theta \cos\varphi \hat{\mathbf{x}} + \sin\theta \sin\varphi \hat{\mathbf{y}}$  is considered for  $\varphi = \pi/4$  as  $\theta$  varies, and  $\mathbf{w}' = 2ik\mathbf{c}_p$ . The modulus of each side in (176) are given in dB, for  $\xi = e^{i\Phi}/|\mathbf{r}|$  with  $\Phi = 0$ ,  $\Phi = -\frac{\pi}{18}$  and  $\Phi = -\frac{\pi}{2}$ . The respective complexity of integrands in surface and line integrals are equal, and the save in computation time grows linearly with the ratio of surface to its contour. We notice the perfect agreement of the respective results (less than 0.01dB).

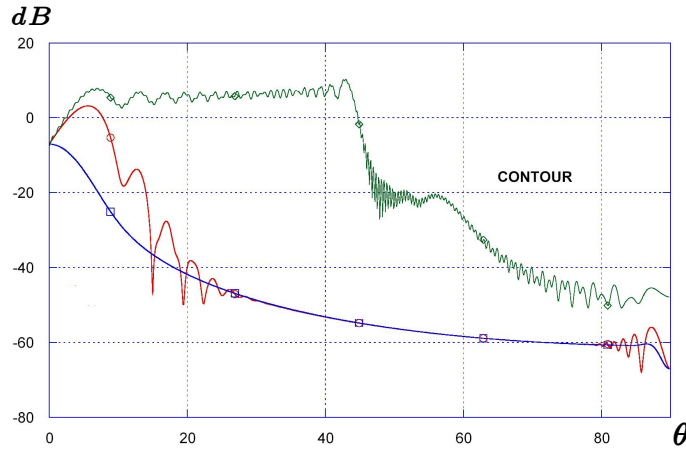


figure 3 : contour integral term in (176), for  $\xi = e^{i\Phi}/|\mathbf{r}|$  with  $\Phi = 0$  (green),  $\Phi = -\frac{\pi}{18}$  (red) and  $\Phi = -\frac{\pi}{2}$  (blue curve)

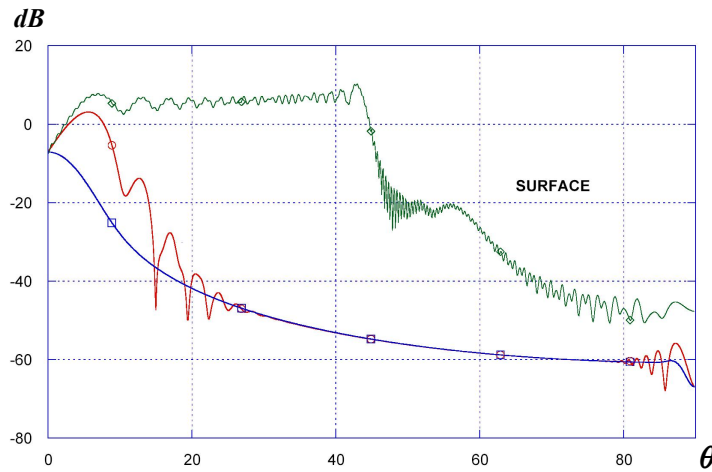


figure 4 : surface integral term in (176), for  $\xi = e^{i\Phi}/|\mathbf{r}|$  with  $\Phi = 0$  (green),  $\Phi = -\frac{\pi}{18}$  (red) and  $\Phi = -\frac{\pi}{2}$  (blue)

10.1.2) For  $\sqrt{a' + b'x' + cx'^2} e^{w'.r' - ik\xi(r'^2 - (c_p.r')^2)}$  integrand in surface integral

We choose to validate our reduction given in theorem 14 with  $b_s = 4$ , for the example with  $\sqrt{a' + b'x' + cx'^2} e^{w'.r' - ik\xi(|r'|^2 - (c_p.r')^2)}$  integrand. The exponential argument is similar to the ones considered in previous section with  $\Phi = -\frac{\pi}{4}$ . The constants  $a'$ ,  $b'$  and  $c'$  considered in figure 5 are respectively,  $a' = 1$  with  $b' = c' = 0$ , and  $a' = 1$  with  $b' = c' = 100$ . The agreement of results given by contour and surface integrals are excellent.

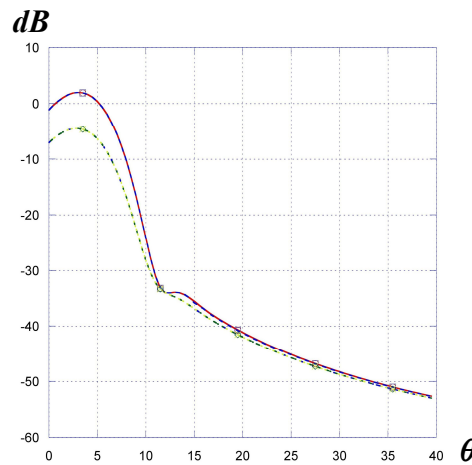


figure 5 : amplitude in dB of the 'surface' integrals with  $\sqrt{a' + b'x' + cx'^2} e^{w'.r' - ik\xi(|r'|^2 - (c_p.r')^2)}$  integrand, and its 'contour' reduction given by theorem 14 : for  $\xi = e^{i\Phi}/|\mathbf{r}|$  with  $\Phi = -\pi/4$ ,  $a' = 1$  with  $b' = c' = 0$  in yellow (contour) and green (surface) curves, and  $a' = 1$  with  $b' = c' = 100$  in blue (contour) and red (surface) curves.

10.2) Near-field radiation surface integral in Physical Optics for a large plate illuminated by a point source with spherical pattern.

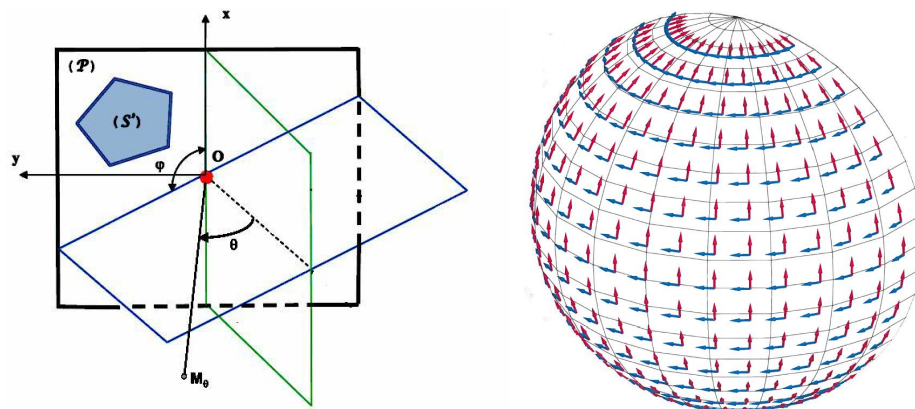


figure 6 : Geometry for the near-field RCS for a source at  $M_\theta$  illuminating  $S'$  with a spherical pattern

### 10.2.1) Large flat plates

We consider the monostatic Physical Optics radiation term  $U = \mathbf{E} \cdot \hat{\mathbf{d}}$  in  $VV$  polarisation (resp.  $U = \mathbf{E} \cdot \mathbf{v}_0$  in  $HH$ ) at constant  $\varphi$  when  $\theta$  varies (fig. 6), for a point source with isotropic pattern, and its reduction (74) (resp. ((83))) from theorem 7 (resp. 9), for a plate  $S'$  of large dimension. We then compare the backscattering pattern  $\frac{\sqrt{4\pi} \tilde{r} U}{|E_i|_{O'}}$  (RCS) given by contour and surface integral expressions, in figures 7 and 8, when we consider a square plate with 1 m side length, shifted from the origin  $O'$  and centered at  $(0, 0.5 \text{ m})$ , observed in the plane  $\varphi = 20 \text{ deg}$ , at the distance  $\tilde{r} = \tilde{r}_0 = 1 \text{ m}$ , for  $\lambda = 2 \text{ cm}$ .

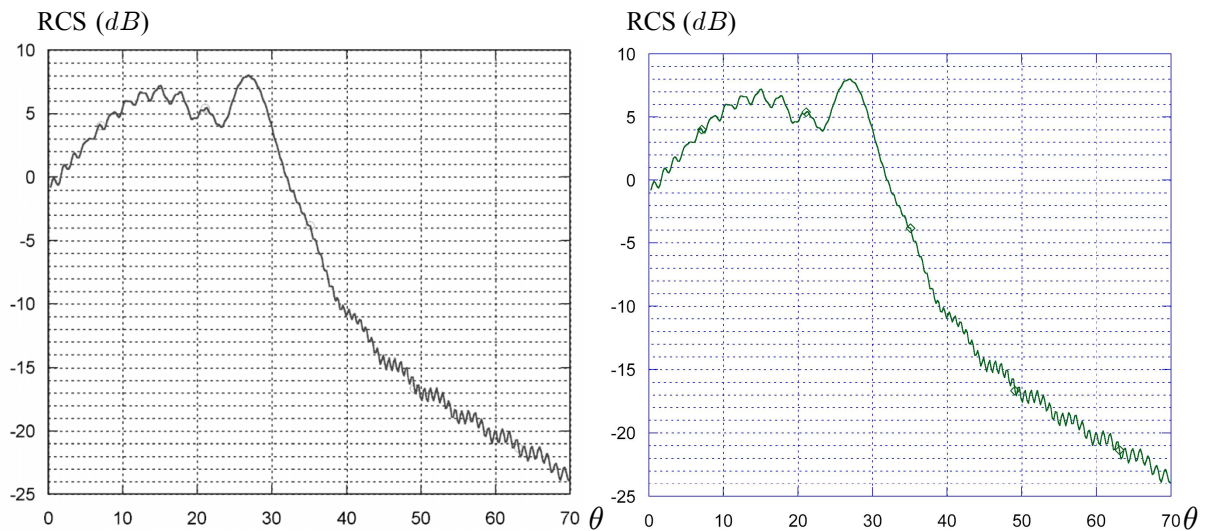


figure 7 : RCS for polarisation  $VV$ , contour (left-hand) and surface (right-hand) integrals

In figure 9, we consider a larger shift of the plate center, disposed at  $(0, 3 \text{ m})$ , and we observe the field at  $\tilde{r} = \tilde{r}_0 = 3.5 \text{ m}$  in the plane  $\varphi = 80 \text{ deg}$ . We distinguish more clearly differences between the polarisations  $VV$  and  $HH$  (in figure 8,  $VV$  (black/green curves) is clearly larger than  $HH$  (brown/red curves)), and notice that the agreement between our contour integral and the initial surface integral continues to be excellent.

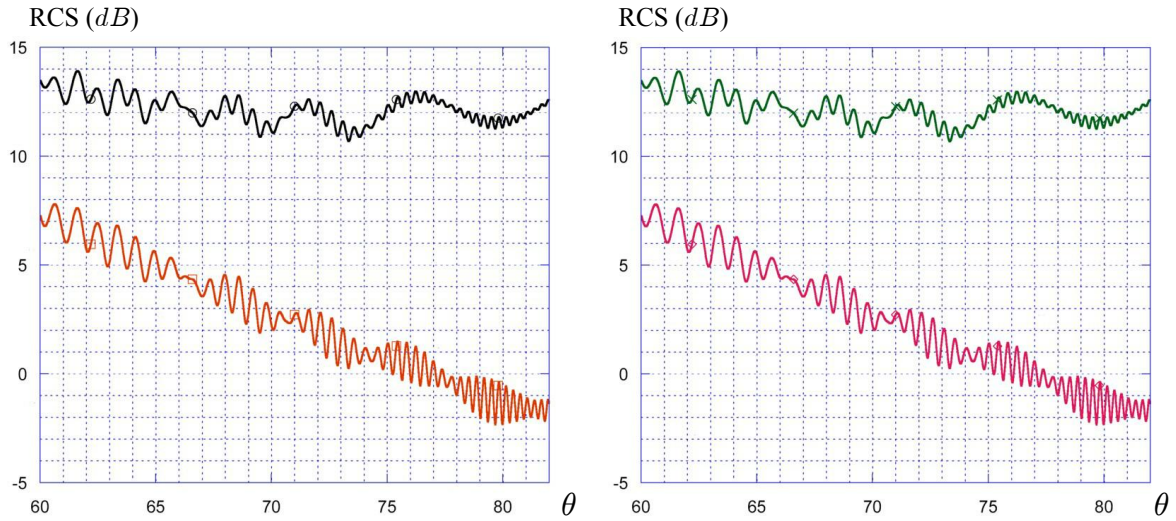


figure 9 : RCS for polarisation VV and HH, contour (left-hand) and surface (right-hand) integrals

### 10.2.2) Large curved plates : convex, concave and saddle surfaces

We consider the monostatic Physical Optics radiation term  $U = \mathbf{E} \cdot \hat{\mathbf{d}}$  in VV polarisation (resp.  $U = \mathbf{E} \cdot \mathbf{v}_0$  in HH) at constant  $\varphi$  when  $\theta$  varies (fig. 6), for a point source with isotropic pattern, and its reduction (150) (resp. (155)) from theorem 16 (resp. 18), for a curved plate  $S$  with side length  $D = 1$  m, centered at the origin, observed in planes  $\varphi = 70^\circ$  and  $20^\circ$ , at different distances, for  $k = 100\pi$  ( $\lambda = 2\text{cm}$ ), as  $|C_{1,2}|D^2 \ll 2z$ . We then compare the backscattering pattern  $\frac{\sqrt{4\pi} \hat{r} U}{|E_i| \sigma'}$  given by contour and surface integral expressions. In figure 10, at large distance  $z = 100\text{m}$  when  $\theta = 0$ , we note the influence of curvature on the width of the plateau, which is similar for convex and concave case, and vanishes when the plate is flat.

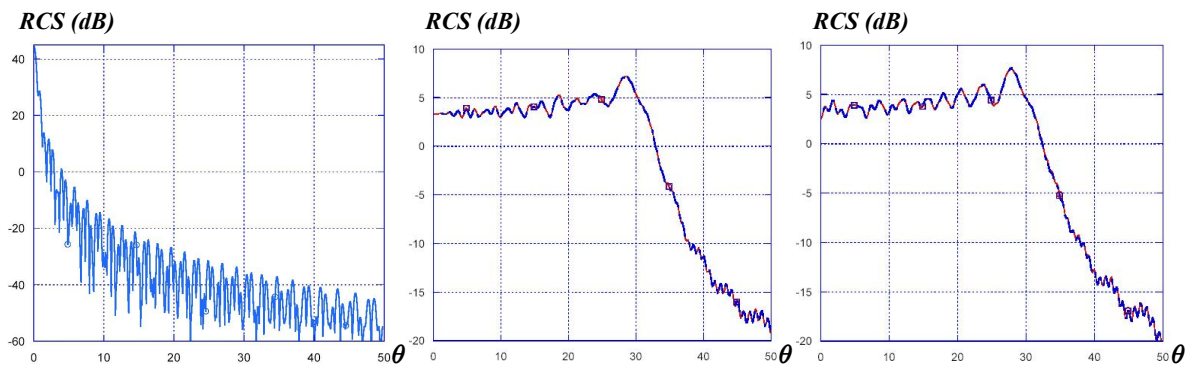


figure 10 : RCS VV at 100m , for flat plate  $C_{1,2} = 0$  (left), for convex plate with  $C_{1,2} = -0.6$  (middle), or concave with  $C_{1,2} = +0.6$  (right), with surface integral (blue) and contour integral (red) (perfect agreement) at  $\varphi = 70^\circ$

In figure 11 and 12, we let vary the curvature in the near field, at a distance of 6m when  $\theta = 0$ , and give the results from surface and contour integrals, in convex and concave cases.

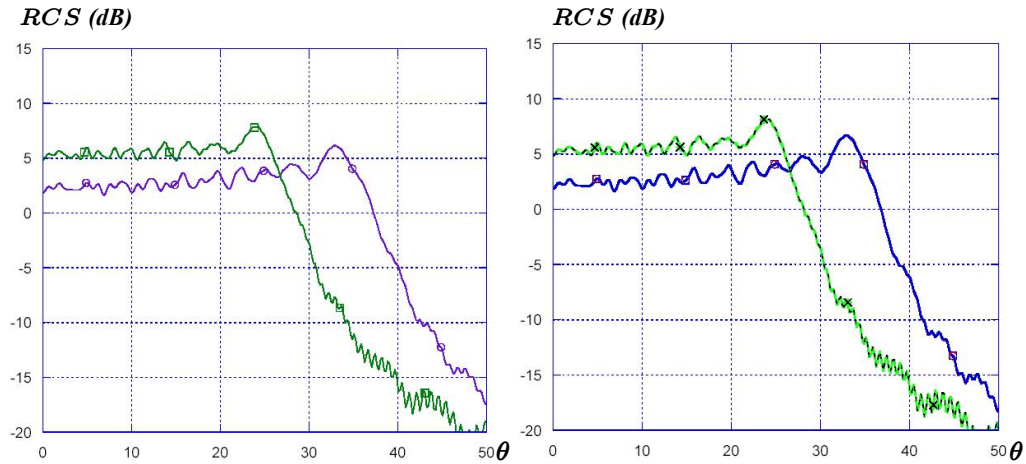


figure 11: RCS VV and HH at 6m for surface integral (on left) et contour integral (on right),  
for  $C_1 = C_2 = -0.4$  (green) et  $C_1 = C_2 = -0.6$  (blue) (purely convex), at  $\varphi = 70^\circ$

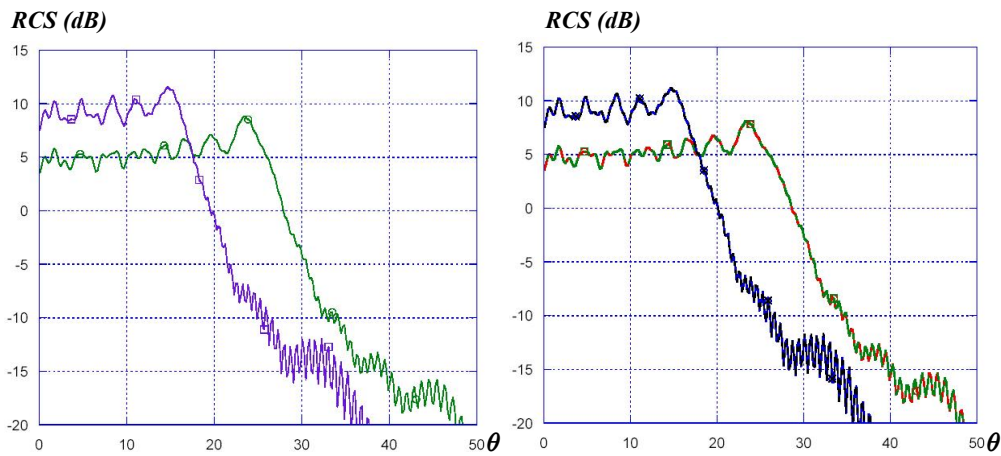


figure 12: RCS VV and HH at 6m for surface integral (on left) et contour integral (on right),  
for  $C_1 = C_2 = 0.4$  (blue) and  $C_1 = C_2 = 0.6$  (green) (purely concave), at  $\varphi = 70^\circ$

At 3m, the figures 13, 14 and 15 show the variation of patterns as we let vary the curvature. In figure 15, we detail the concave case, and show a minimum of the width of the plateau near the focal point when  $1 - 2zC = 0$ , or varies monotonically else. In figure 16, we let vary the distance and observe that the plateau, very different in near field, becomes similar at large distance. In figure 17 and 18, we give results for a saddle surface. The asymmetry of the plate is illustrated by the change of pattern in the cases  $\varphi = 70^\circ$  and  $\varphi = 20^\circ$  in figure 17, and we observe the changes as we let vary the distance in figure 18. In all these cases, we notice the very good agreement of the surface and our contour integrals.

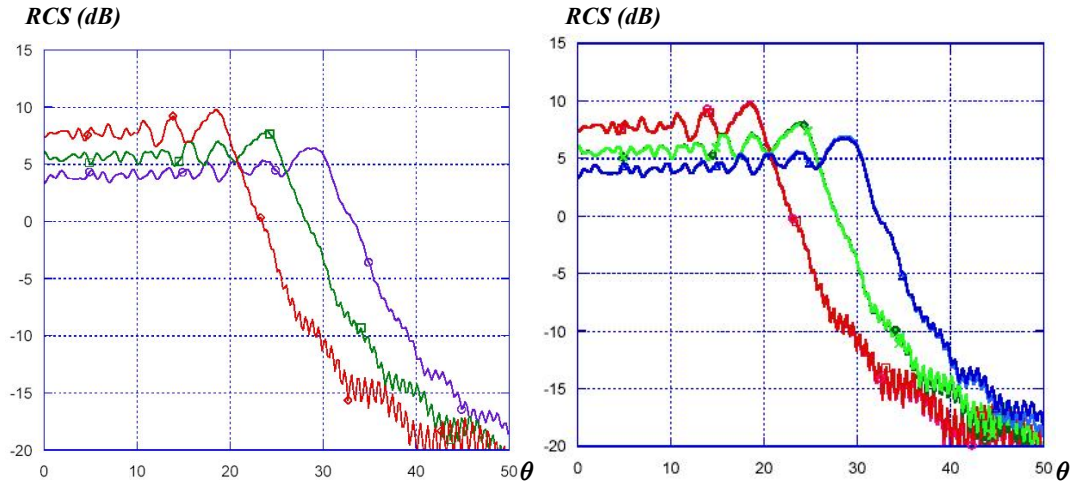


figure 13 : Comparison of RCS VV and HH at 3m for surface integral (on left) et contour integral (on right),  
for  $C_1 = C_2 = -0.2$ (red),  $= -0.3$ (green),  $= -0.4$ (blue) at  $\varphi = 70^\circ$

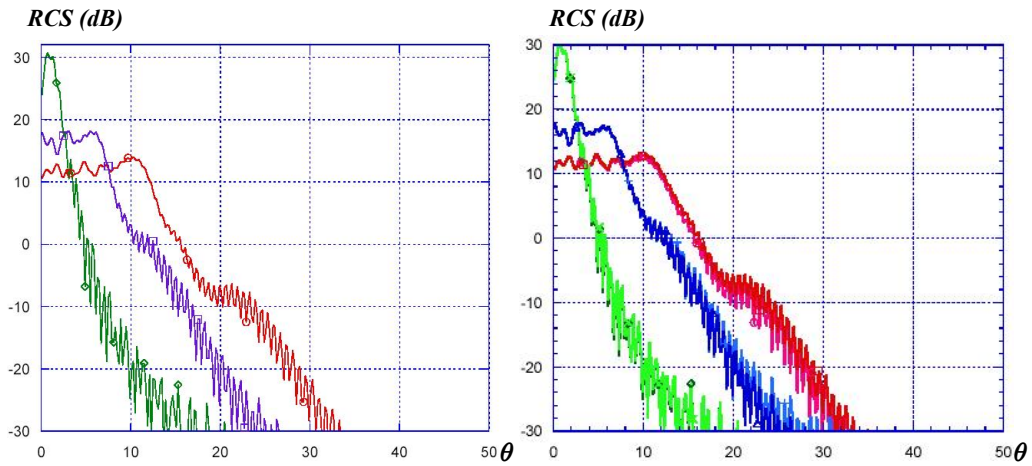


figure 14 : Comparison of RCS VV and HH at 3m for surface integral (on left) et contour integral (on right),  
for  $C_1 = C_2 = 0.2$ (green),  $= 0.3$ (blue),  $= 0.4$ (red) at  $\varphi = 70^\circ$

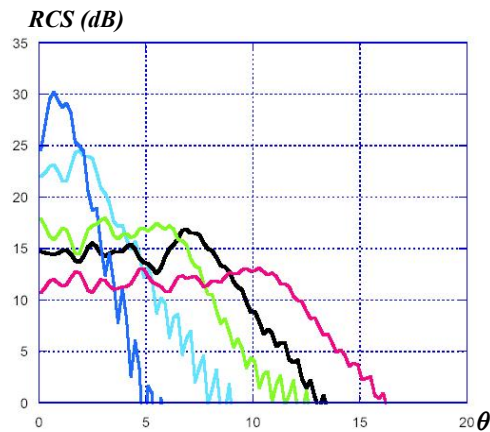


figure 15 : Variation in purely concave cases with  $C_j = C$ , in vicinity of  $1 - 2zC = 0$  (focal case) at 3m,  
 $C = 0.4$  (red),  $C = 0.3$  (vert),  $C = 0.2$  (blue),  $C = 0.1$  (clear blue),  $C = 0$ . (black)

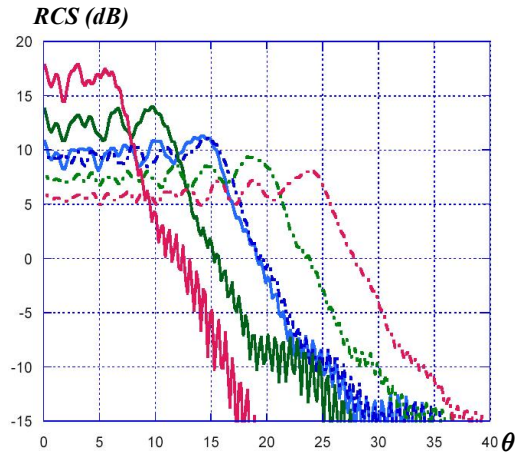


figure 16 : purely convex cases with negative  $C_j$  (dashed lines) ou concave cases with positive  $C_j$  (continuous lines), for  $|C_1| = |C_2| = 0.3$ , at 3m (red), 6m (green), et 100m (blue)

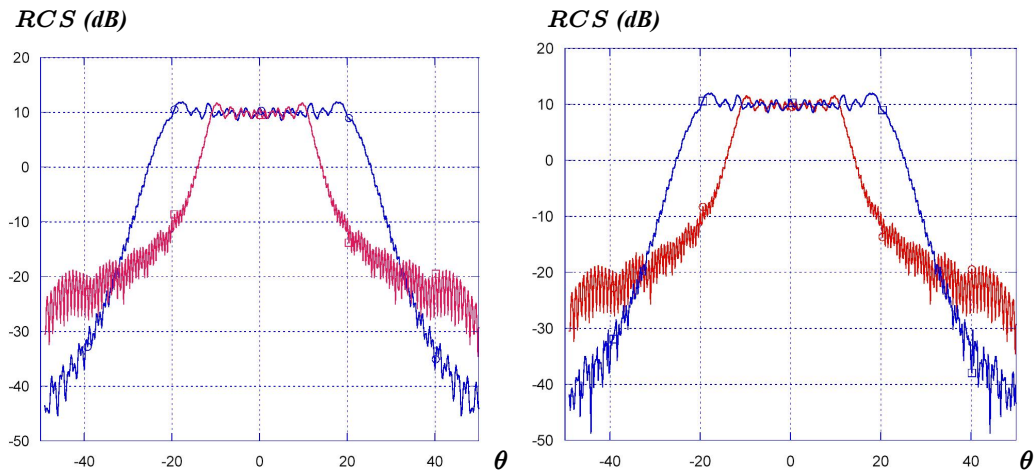


figure 17: Comparison of the RCS VV for surface integral (on left) et contour integral (on right), at 6m, for saddle surface with  $C_1 = 0.3$  et  $C_2 = -0.3$  at  $\varphi = 70^\circ$  (in blue) and at  $\varphi = 20^\circ$  (in red)

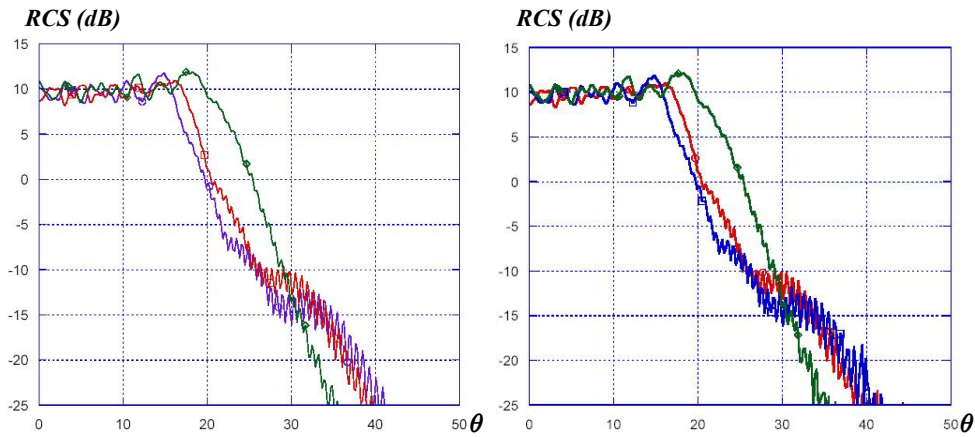


figure 18: Comparison of the RCS VV for surface integral (on left) et contour integral (on right), at 6m, 20m and 100m, for  $C_1 = 0.3$  et  $C_2 = -0.3$  (saddle surface, convex in the plane of incidence at  $\varphi = 70^\circ$ )



## ***11) Conclusion***

How to reduce fast oscillating surface integrals with complex phases, often encountered in wave physics, to a line integral was studied in this paper. So, we have derived a general method of reduction to a non-singular contour integral, for a large class of radiation surface integrals, when the currents are analytically defined on the surface of integration, as in Physical Optics, that can cover a large domain of applications.

We have presented simple near-field expressions in monostatic case, with neither special function nor singularity, that are valid from small to large distance, which applies for arbitrary dimensions of objects, illumination patterns, and impedance boundary conditions (perfectly or imperfectly reflective surfaces), for large planar or moderately curved plates. Secondly, specific original expressions are presented in bistatic cases, in particular for the reduction of surface integrals whose integrands have a phase that is an arbitrary complex polynomial of the coordinates of rank 2, corresponding to quadratic approximations which are commonly used for Green's function or in gaussian beam propagation problems. Because the respective complexity of integrands in surface and line integrals are equal, the save in computation time grows linearly with the ratio of surface to its contour. As made evident by the numerical examples in sect. 8, the presented formulae are highly accurate, and they ought to find wide applications in accelerating the computation.

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## Appendice A : Primitives in theorems 5 and 6, and properties

### A.1) Primitives for the theorem 5

Let us show that,

$$\begin{aligned} \int \frac{1}{\tilde{R}} \left( \left( \frac{\psi'}{\tilde{R}} \right)^2 \right)' d\mathbf{R}^2 &= 2 \left( \frac{\psi'}{\tilde{R}} \right)^2 = 2 \frac{(ik + \frac{1}{\tilde{R}})^2}{\mathbf{R}^4} e^{-i2k\tilde{R}}, \\ \int \left( \frac{\psi'}{\tilde{R}} \right)^2 d\mathbf{R}^2 &= -2 \frac{(1 + 2ik\tilde{R})}{4\mathbf{R}^4} e^{-i2k\tilde{R}} \\ \int \tilde{R} \left( \left( \frac{\psi'}{\tilde{R}} \right)^2 \right)' d\mathbf{R}^2 &= 2 \left( -k^2 + \frac{3ik}{\tilde{R}} + \frac{3}{2\mathbf{R}^2} \right) \frac{e^{-i2k\tilde{R}}}{\mathbf{R}^2} \end{aligned} \quad (177)$$

with  $\tilde{R} = \sqrt{\mathbf{R} \cdot \mathbf{R}}$ ,  $(\cdot)' \equiv \partial_{\tilde{R}}(\cdot)$ ,  $\psi(\tilde{R}) = \frac{e^{-ik\tilde{R}}}{\tilde{R}}$ ,  $\mathbf{R}^2 = (\mathbf{r}' - \mathbf{a})^2 + z^2 \neq 0$ . The first line is trivial since  $d\mathbf{R}^2 = 2\tilde{R} d\tilde{R}$ , while the second and third line need some manipulations.

We remark first that  $\psi$  verifies  $\Delta\psi + k^2\psi = 0$  as  $\mathbf{R}^2 \neq 0$ , and thus,

$$\frac{1}{\mathbf{R}^2} (\tilde{R}^3 \left( \frac{\psi'}{\tilde{R}} \right))' = \tilde{R} \left( \frac{\psi'}{\tilde{R}} \right)' + 3 \frac{\psi'}{\tilde{R}} = -k^2\psi, \quad (178)$$

which multiplied by  $\frac{\psi'}{\tilde{R}}$ , leads us to

$$\tilde{R} \left( \left( \frac{\psi'}{\tilde{R}} \right)^2 \right)' = -k^2 \frac{((\psi)^2)'}{\tilde{R}} - 6 \left( \frac{\psi'}{\tilde{R}} \right)^2. \quad (179)$$

Considering that we have, by integration by parts,

$$\int \tilde{R} \left( \left( \frac{\psi'}{\tilde{R}} \right)^2 \right)' \tilde{R} d\tilde{R} = (\psi')^2 - 2 \int \left( \frac{\psi'}{\tilde{R}} \right)^2 \tilde{R} d\tilde{R} \quad (180)$$

we can use (179), and write,

$$\int -k^2 ((\psi)^2)' d\tilde{R} - \int 6 \left( \frac{\psi'}{\tilde{R}} \right)^2 \tilde{R} d\tilde{R} = (\psi')^2 - 2 \int \left( \frac{\psi'}{\tilde{R}} \right)^2 \tilde{R} d\tilde{R} \quad (181)$$

so that we obtain the second line of (177) following,

$$\int \left( \frac{\psi'}{\tilde{R}} \right)^2 \tilde{R} d\tilde{R} = \frac{1}{4} ((ik\psi)^2 - (\psi')^2) = - \left( 2ik + \frac{1}{\tilde{R}} \right) \frac{\psi^2}{4\tilde{R}} \left( = \frac{(\tilde{R} \psi^2)'}{4\tilde{R}^2} \right) \quad (182)$$

We can then rewrite (180), and derive the third line of (177),

$$\int \tilde{R} \left( \left( \frac{\psi'}{\tilde{R}} \right)^2 \right)' \tilde{R} d\tilde{R} = \frac{3(\psi')^2 - (ik\psi)^2}{2} = \left( -k^2 + \frac{3ik}{\tilde{R}} + \frac{3}{2\tilde{R}^2} \right) \psi^2 \quad (183)$$

## A.2) Primitives for the theorem 6

For the theorem 6, we have to complete (177). For this, we write,

$$\begin{aligned} \int \left( \frac{\psi'}{\tilde{R}} \right)' \left( \frac{\psi'}{\tilde{R}} \right)' d\mathbf{R}^2 &= -2 \int \left( \frac{\psi'}{\tilde{R}} \right)' \left( k^2 \psi + 3 \frac{\psi'}{\tilde{R}} \right) d\tilde{R} = \\ &= \left( -2k^2 \frac{\psi' \psi}{\tilde{R}} + k^2 \int \left( \frac{\psi'}{\tilde{R}} \right)^2 d\mathbf{R}^2 \right) - 3 \left( \frac{\psi'}{\tilde{R}} \right)^2 \\ &= 2k^2 \left( ik + \frac{1}{\tilde{R}} \right) \frac{\psi^2}{\tilde{R}} - k^2 \left( ik + \frac{1}{2\tilde{R}} \right) \frac{\psi^2}{\tilde{R}} - 3 \left( \frac{\psi'}{\tilde{R}} \right)^2 \\ &= k^2 \left( ik + \frac{3}{2\tilde{R}} \right) \frac{\psi^2}{\tilde{R}} - 3 \left( ik + \frac{1}{\tilde{R}} \right)^2 \left( \frac{\psi}{\tilde{R}} \right)^2 \end{aligned} \quad (184)$$

where we have successively used (178), an integration by parts, and (182).

## A3) Some exact and asymptotic properties of primitives $\int_{\infty}^{R^2} \frac{e^{-v\tilde{R}}}{\tilde{R}^n} dR^2$

$$\int_{\infty}^{R^2} \frac{e^{-v\tilde{R}}}{\tilde{R}^n} dR^2 = 2v^{n-2} V_{n-1}(v\tilde{R}) \sim -\frac{2}{v} \frac{e^{-v\tilde{R}}}{\tilde{R}^{n-1}} \left( 1 - \frac{n}{v\tilde{R}} + \dots \right), \text{ if } v\tilde{R} \rightarrow \infty \quad (185)$$

$$V_n(u) = \int_{\infty}^u \frac{e^{-t}}{t^n} dt = -\frac{E_n(u)}{u^{n-1}}, \quad V_n(u) \sim -\frac{e^{-u}}{u^n} \left( 1 - \frac{n}{u} + \frac{n(n+1)}{u^2} - \dots \right) \text{ if } u \rightarrow \infty$$

with  $\tilde{R} = \sqrt{\mathbf{R} \cdot \mathbf{R}}$ , while, for  $p$  successive primitives, we have,

$$\begin{aligned}
& (p-1)! \int_{\infty}^{R'^2} d\mathbf{R}^2 \dots \int_{\infty}^{R'^2} d\mathbf{R}^2 \frac{e^{-v\tilde{R}}}{\tilde{R}^n} = \int_{\infty}^{R'^2} (\mathbf{R}'^2 - \mathbf{R}^2)^{p-1} \frac{e^{-v\tilde{R}}}{\tilde{R}^n} dR^2 \\
& = \sum_{0 \leq m \leq p-1} (-1)^{p-1-m} C_{p-1}^m (\tilde{R}')^{2m} \int_{\infty}^{R'^2} \frac{e^{-v\tilde{R}}}{\tilde{R}^{n-2(p-1-m)}} dR^2 \\
& = 2 \sum_{0 \leq m \leq p-1} (-1)^{p-1-m} C_{p-1}^m (\tilde{R}')^{2m} (v^{n-2(p-1-m)-2} V_{n-2(p-1-m)-1}(v\tilde{R}')) \\
& = -2 \sum_{0 \leq m \leq p-1} (-1)^{p-1-m} C_{p-1}^m (\tilde{R}')^{2m} \left( \frac{E_{n-2(p-1-m)-1}(v\tilde{R}')}{(\tilde{R}')^{n-2(p-1-m)-2}} \right) \\
& = \frac{-2}{(\tilde{R}')^{n-2p}} \sum_{0 \leq m \leq p-1} (-1)^{p-1-m} C_{p-1}^m E_{n-2(p-1-m)-1}(v\tilde{R}') \tag{186}
\end{aligned}$$

where we remark that the first asymptotic orders of the sum, when  $v\tilde{R} \gg 1$ , vanish. Thus, applying this expression, we have for  $p = 2$ ,

$$\begin{aligned}
& \int_{\infty}^{|\mathbf{R}'|^2} d\mathbf{R}^2 \int_{\infty}^{|\mathbf{R}'|^2} d\mathbf{R}^2 \frac{e^{-v\tilde{R}}}{\tilde{R}^n} \sim \\
& \sim \frac{-2}{v} \frac{e^{-v|\mathbf{R}'|}}{\tilde{R}'^{n-(2p-1)}} \sum_{0 \leq m \leq 1} (-1)^{p-1-m} C_{p-1}^m \left( 1 - \frac{n-2(p-m)+1}{v\tilde{R}'} \right) \Big|_{p=2} \\
& \sim \frac{4}{v^2} \frac{e^{-v|\mathbf{R}'|}}{(\tilde{R}')^{n-(2p-2)}} \sum_{0 \leq m \leq 1} (-1)^{p-1-m} m C_{p-1}^m \Big|_{p=2} = \frac{4}{v^2} \frac{e^{-v|\mathbf{R}'|}}{(\tilde{R}')^{n-2}} \tag{187}
\end{aligned}$$

Besides we note that, from,

$$\int_{\infty}^{R'^2} \frac{-ve^{-v\tilde{R}} \left( 1 + \frac{n}{v\tilde{R}} \right)}{\tilde{R}^{n+1}} d\mathbf{R}^2 = \frac{2e^{-v\tilde{R}'}}{\tilde{R}'^n}, \tag{188}$$

we have,

$$(v\tilde{R}')^n e^{v\tilde{R}'} (V_n(v\tilde{R}') + nV_{n+1}(v\tilde{R}')) = -1. \tag{189}$$

#### A4) An asymptotic development of $\tilde{R}$

Let us consider the asymptotics of  $\tilde{R} = \sqrt{\mathbf{R}^2}$  when  $\mathbf{R} = \mathbf{R}_1 - \mathbf{R}_2$  for  $|\mathbf{R}_1| \gg |\mathbf{R}_2|$ . For that, we write,

$$\begin{aligned} \mathbf{R}^2 &= (\mathbf{R}_1 - \mathbf{R}_2)^2 = (\sqrt{\mathbf{R}_1^2} \mathbf{u}_1 - \mathbf{R}_2)^2 \\ &= (\sqrt{\mathbf{R}_1^2} - \mathbf{R}_2 \cdot \mathbf{u}_1)^2 + \mathbf{R}_2^2 - (\mathbf{R}_2 \cdot \mathbf{u}_1)^2, \end{aligned} \quad (190)$$

which leads us to,

$$\begin{aligned} \sqrt{\mathbf{R}^2} &= (\sqrt{\mathbf{R}_1^2} - \mathbf{R}_2 \cdot \mathbf{u}_1) + \frac{\mathbf{R}_2^2 - (\mathbf{R}_2 \cdot \mathbf{u}_1)^2}{\sqrt{\mathbf{R}^2} + (\sqrt{\mathbf{R}_1^2} - \mathbf{R}_2 \cdot \mathbf{u}_1)}, \\ \frac{\sqrt{\mathbf{R}^2}}{\sqrt{\mathbf{R}_1^2}} &= 1 - \frac{\mathbf{R}_2 \cdot \mathbf{u}_1}{\sqrt{\mathbf{R}_1^2}} + \frac{\mathbf{R}_2^2 - (\mathbf{R}_2 \cdot \mathbf{u}_1)^2}{2\sqrt{\mathbf{R}_1^2}(1 + \frac{1}{2}(\frac{\sqrt{\mathbf{R}^2}}{\sqrt{\mathbf{R}_1^2}} - 1 - \frac{\mathbf{R}_2 \cdot \mathbf{u}_1}{\sqrt{\mathbf{R}_1^2}}))}. \end{aligned} \quad (191)$$

Letting  $\delta = \frac{1}{2}(\frac{\sqrt{\mathbf{R}^2}}{\sqrt{\mathbf{R}_1^2}} - 1 - \frac{\mathbf{R}_2 \cdot \mathbf{u}_1}{\sqrt{\mathbf{R}_1^2}})$ , we can then write,

$$\frac{\sqrt{\mathbf{R}^2}}{\sqrt{\mathbf{R}_1^2}} = 1 - \frac{\mathbf{R}_2 \cdot \mathbf{u}_1}{\sqrt{\mathbf{R}_1^2}} + \frac{\mathbf{R}_2^2 - (\mathbf{R}_2 \cdot \mathbf{u}_1)^2}{2\sqrt{\mathbf{R}_1^2}} \left( \sum_{n=0}^N (-\delta)^n + \frac{(-\delta)^{N+1}}{1+\delta} \right). \quad (192)$$

which gives us a complete asymptotics of  $\sqrt{\mathbf{R}^2}$  for  $|\mathbf{R}_1| \gg |\mathbf{R}_2|$  at any order  $N$  with explicit remaining term.

Appendice B : Reduction of Physical Optics radiation integrals for imperfectly conducting surface

In the hypothesis of a point source illuminating at large distance an impedance flat surface of normal  $\hat{\mathbf{z}}$  [29]-[30], we can consider the incident field as a locally plane wave, combination of polarisations  $TE$  ( $\mathbf{E} //$  surface) and  $TM$  ( $\mathbf{H} //$  surface), respectively denoted  $(\mathbf{E}_e^i, \mathbf{H}_e^i)$  and  $(\mathbf{E}_h^i, \mathbf{H}_h^i)$ . The currents  $\mathbf{J}$  and  $\mathbf{M}$ , respectively denoted  $\mathbf{J}_{e(h)}|_S = \hat{\mathbf{z}} \times \mathbf{H}_{e(h)}$  and  $\mathbf{M}_{e(h)}|_S = -\hat{\mathbf{z}} \times \mathbf{E}_{e(h)}$ , then satisfy,

$$\mathbf{J}_{e(h)} = 2b'_{e(h)} \left( \frac{z}{\tilde{R}} \right) (\hat{\mathbf{z}} \times \mathbf{H}_{e(h)}^i), \quad \mathbf{M}_{e(h)} = -a'_{e(h)} \left( \frac{z}{\tilde{R}} \right) (\hat{\mathbf{z}} \times \mathbf{J}_{e(h)}) \quad (193)$$

where  $a'_{e,h}$  and  $b'_{e,h}$  are regular functions. In the case of constant impedance,  $b_{e(h)}$  and  $a_{e(h)}$  are constants, which are  $b'_{e(h)} \equiv 1$  and  $a'_{e(h)} \equiv 0$  in perfectly conducting case. Thereafter, we suppress the indices  $e$  and  $h$  for the use in the expression (52).

B.1) Physical Optics diffracted fields and reduction, for an electric dipolar source

Theorem 18

The Physical Optics field, diffracted by the surface  $S'$  with an impedance condition, illuminated by the electric dipole  $\mathbf{J}_0 = \hat{\mathbf{d}}\delta(\mathbf{r}' - \mathbf{r}_0)$  is given by (52) with (193). It can be written, in monostatic case, when  $\hat{\mathbf{d}} \cdot \hat{\mathbf{z}} = 0$ , following

$$\begin{aligned} \mathbf{E} \sim & \frac{Z_0}{8\pi^2 ik} \int_{S'} \frac{zb'(\frac{z}{\tilde{R}})}{2} \tilde{R} \left( \left( \frac{\psi'}{\tilde{R}} \right)^2 \right)' ((\mathbf{u} \cdot \hat{\mathbf{d}}) \mathbf{u} - \hat{\mathbf{d}}) dS' + \\ & + \frac{1}{8\pi^2} \int_{S'} c\left(\frac{z}{\tilde{R}}\right) (\psi')^2 \left( \frac{\mathbf{R}^2 - z^2}{\mathbf{R}^2} \left( \frac{\mathbf{r}' - \mathbf{a}}{|\mathbf{r}' - \mathbf{a}|} \times \hat{\mathbf{z}} \right) \left( (\mathbf{r}' - \mathbf{a}) \times \hat{\mathbf{z}} \right) \cdot \hat{\mathbf{d}} \right) + \\ & + ((\mathbf{u} \cdot \hat{\mathbf{d}}) \mathbf{u} - \hat{\mathbf{d}}) dS' \end{aligned} \quad (194)$$

for large  $k\tilde{R}$ , with  $c \equiv a'b'$ ,  $(\mathbf{u} \cdot \hat{\mathbf{d}}) \mathbf{u} = \frac{z\hat{\mathbf{z}}(\hat{\mathbf{d}} \cdot (\mathbf{a} - \mathbf{r}'))}{\mathbf{R}^2} + \frac{(\mathbf{R}^2 - z^2)}{\mathbf{R}^2} \frac{(\hat{\mathbf{d}} \cdot (\mathbf{a} - \mathbf{r}'))(\mathbf{a} - \mathbf{r}')}{|\mathbf{a} - \mathbf{r}'|^2}$ . Considering the propositions 5 and 6, we can use,

$$\int (\psi')^2 d\mathbf{R}^2 \sim -\frac{ike^{-i2k\tilde{R}}}{\tilde{R}}, \quad \int z\tilde{R} \left( \left( \frac{\psi'}{\tilde{R}} \right)^2 \right)' d\mathbf{R}^2 \sim -\frac{2k^2 ze^{-i2k\tilde{R}}}{\mathbf{R}^2}, \quad (195)$$

and reduce asymptotically each integral to a non-singular contour integral.

Proof. We consider, using (193), that  $S'$  is illuminated by the radiation of the electrical dipole  $\mathbf{J}_0 = \hat{\mathbf{d}}\delta(\mathbf{r}' - \mathbf{r}_0)$  with  $\hat{\mathbf{d}} \cdot \hat{\mathbf{z}} = 0$  ( $\hat{\mathbf{d}} // \text{plane}$ ), and thus

$$\begin{aligned} 2\pi \mathbf{J}|_S &= b' \left( \frac{z}{\tilde{R}} \right) \hat{\mathbf{z}} \times (\mathbf{R}_0 \times \frac{\psi'_0}{\tilde{R}_0} \hat{\mathbf{d}}), \quad \mathbf{R}_0 = (\mathbf{r}' - \mathbf{r}_0) \\ 2\pi \mathbf{M}|_S &= -c \left( \frac{z}{\tilde{R}} \right) \hat{\mathbf{z}} \times (\hat{\mathbf{z}} \times (\mathbf{R}_0 \times \frac{\psi'_0}{\tilde{R}_0} \hat{\mathbf{d}})), \\ &= -c \left( \frac{z}{\tilde{R}} \right) \left( (\hat{\mathbf{z}} \cdot (\mathbf{R}_0 \times \frac{\psi'_0}{\tilde{R}_0} \hat{\mathbf{d}})) \hat{\mathbf{z}} - (\mathbf{R}_0 \times \frac{\psi'_0}{\tilde{R}_0} \hat{\mathbf{d}}) \right) \end{aligned} \quad (196)$$

where we suppose  $a'(v)$ ,  $b'(v)$ , and  $c(v)$  regular as  $v \in [0, 1]$ . In the case where  $\mathbf{R} = -\mathbf{R}_0 = z\hat{\mathbf{z}} + \mathbf{a} - \mathbf{r}'$  (monostatic case), we then obtain,

$$\begin{aligned} \mathbf{E} \sim & \frac{Z_0}{8\pi^2 ik} \int_{S'} \frac{zb'(\frac{z}{\tilde{R}})}{2} \tilde{R} \left( \left( \frac{\psi'}{\tilde{R}} \right)^2 \right)' \mathbf{u} \times (\mathbf{u} \times \hat{\mathbf{d}}) dS' + \\ & + \frac{1}{8\pi^2} \int_{S'} c\left(\frac{z}{\tilde{R}}\right) \left( \frac{\psi'}{\tilde{R}} \right)^2 \left( -\mathbf{R} \times \hat{\mathbf{z}} (\hat{\mathbf{z}} \cdot (\mathbf{R} \times \hat{\mathbf{d}})) + \mathbf{R} \times (\mathbf{R} \times \hat{\mathbf{d}}) \right) dS' \end{aligned} \quad (197)$$

and thus (194). We can then, following the theorems 1 and 2, and the propositions 5 and 6, reduce the field  $\mathbf{E}$  to a contour integral. For this, we can use (177), and also,

$$\begin{aligned} \int (\psi')^2 d\mathbf{R}^2 &= \int \mathbf{R}^2 \left(\frac{\psi'}{\tilde{R}}\right)^2 d\mathbf{R}^2 = \frac{(\tilde{R} \psi^2)'}{2} - \int \frac{(\tilde{R} \psi^2)'}{2\mathbf{R}^2} d\mathbf{R}^2 \\ &= -\frac{(3 + 2ik\tilde{R})}{2\mathbf{R}^2} e^{-i2k\tilde{R}} - \int \frac{e^{-i2k\tilde{R}}}{2\mathbf{R}^4} d\mathbf{R}^2 = -\frac{ik}{\tilde{R}} e^{-i2k\tilde{R}} \left(1 + O\left(\frac{1}{k\tilde{R}}\right)\right), \end{aligned} \quad (198)$$

## B.2) Physical Optics diffracted fields and reduction, for a magnetic dipolar source

### Theorem 19

The Physical Optics field, diffracted by the surface  $S'$  with an impedance condition, illuminated by the magnetic dipole  $\mathbf{M}_0 = \hat{\mathbf{d}}\delta(\mathbf{r}' - \mathbf{r}_0)$ , is given by (52) with (193). It can be written, in monostatic case, when  $\hat{\mathbf{d}} \cdot \hat{\mathbf{z}} = 0$ , following

$$\begin{aligned} \mathbf{E} &\sim \frac{-z}{8(\pi ik)^2} \hat{\mathbf{d}} \times \int_{S'} b' \left(\frac{z}{\tilde{R}}\right) \mathbf{R} \left(\left(\frac{\psi'}{\tilde{R}}\right)'\right)^2 dS' \\ &\quad - \frac{1}{8\pi^2 ik Z_0} \hat{\mathbf{z}} \times \int_{S'} (\mathbf{R} \times \hat{\mathbf{z}}) \frac{c(\frac{z}{\tilde{R}})}{2} \tilde{R} \left(\left(\frac{\psi'}{\tilde{R}}\right)'\right)^2 \left(\frac{z}{\mathbf{R}^2} (\mathbf{R} \cdot \hat{\mathbf{d}})\right) dS' \\ &\quad - \frac{1}{8\pi^2 ik Z_0} \hat{\mathbf{d}} \times \int_{S'} \mathbf{R} \frac{c(\frac{z}{\tilde{R}})}{2} \tilde{R} \left(\left(\frac{\psi'}{\tilde{R}}\right)'\right)^2 dS' \end{aligned} \quad (199)$$

for large  $k\tilde{R}$ , with  $c \equiv a'b'$ ,  $(\mathbf{R} \times \hat{\mathbf{z}})(\mathbf{R} \cdot \hat{\mathbf{d}}) = (\mathbf{R}^2 - z^2) \frac{(\hat{\mathbf{d}} \cdot (\mathbf{a} - \mathbf{r}'))(\mathbf{a} - \mathbf{r}') \times \hat{\mathbf{z}}}{(\mathbf{a} - \mathbf{r}')^2}$ . Considering the propositions 5 and 6, we can use,

$$\begin{aligned} \int \left(\frac{\psi'}{\tilde{R}}\right)' \left(\frac{\psi'}{\tilde{R}}\right)' d\mathbf{R}^2 &\sim k^2 ik \frac{e^{-i2k\tilde{R}}}{\tilde{R}^3}, \quad \int \tilde{R} \left(\left(\frac{\psi'}{\tilde{R}}\right)'\right)^2 d\mathbf{R}^2 \sim -\frac{2k^2 e^{-i2k\tilde{R}}}{\mathbf{R}^2} \\ \int \frac{1}{\tilde{R}} \left(\left(\frac{\psi'}{\tilde{R}}\right)'\right)^2 d\mathbf{R}^2 &\sim -\frac{2k^2 e^{-i2k\tilde{R}}}{\mathbf{R}^4} \end{aligned} \quad (200)$$

and reduce asymptotically each surface integral to a non-singular contour integral.

Proof. We consider, using (193), that  $S'$  is illuminated by the radiation of the magnetic dipole  $\mathbf{M}_0 = \hat{\mathbf{d}}\delta(\mathbf{r}' - \mathbf{r}_0)$  with  $\hat{\mathbf{d}} \cdot \hat{\mathbf{z}} = 0$  ( $\hat{\mathbf{d}} \parallel \text{plane}$ ), and thus



$$\begin{aligned} \mathbf{J}|_S &= \frac{b'(\frac{z}{\tilde{R}})}{2\pi ik Z_0} \hat{\mathbf{z}} \times (\tilde{R}_0 (\frac{\psi'}{\tilde{R}_0})' \mathbf{u}_0 \times (\mathbf{u}_0 \times \hat{\mathbf{d}}) - \frac{2\psi'}{\tilde{R}} \hat{\mathbf{d}}), \quad \mathbf{R}_0 = (\mathbf{r}' - \mathbf{r}_0), \quad \mathbf{u}_0 = \frac{\mathbf{R}_0}{\tilde{R}_0}, \\ \mathbf{M}|_S &= \frac{-c(\frac{z}{\tilde{R}})}{2\pi ik Z_0} \hat{\mathbf{z}} \times (\hat{\mathbf{z}} \times (\tilde{R}_0 (\frac{\psi'}{\tilde{R}_0})' \mathbf{u}_0 \times (\mathbf{u}_0 \times \hat{\mathbf{d}}) - \frac{2\psi'}{\tilde{R}} \hat{\mathbf{d}})), \end{aligned} \quad (201)$$

where we suppose  $a'(v)$ ,  $b'(v)$ , and  $c(v)$  regular as  $v \in [0, 1]$ . In the case where  $\mathbf{R} = -\mathbf{R}_0 = z\hat{\mathbf{z}} + \mathbf{a} - \mathbf{r}'$  (monostatic case), we then obtain,

$$\begin{aligned} \mathbf{E} &= \frac{1}{8(\pi ik)^2} \int_{S'} z b'(\frac{z}{\tilde{R}}) (\frac{\psi'}{\tilde{R}})' (\frac{\psi'}{\tilde{R}})' (\mathbf{R} \times \hat{\mathbf{d}}) dS' \\ &\quad - \frac{1}{4(\pi ik)^2} \int_{S'} b'(\frac{z}{\tilde{R}}) \hat{\mathbf{z}} \times (\tilde{R} ((\frac{\psi'}{\tilde{R}})^2)' (\mathbf{u}(\mathbf{u} \cdot \hat{\mathbf{d}}) - \hat{\mathbf{d}}) - 2(\frac{\psi'}{\tilde{R}})^2 \hat{\mathbf{d}}) dS' \\ &\quad + \frac{1}{8\pi^2 ik Z_0} \int_{S'} c(\frac{z}{\tilde{R}}) \frac{\psi'}{\tilde{R}} \mathbf{R} \times (\hat{\mathbf{z}} \times (\hat{\mathbf{z}} \times (\tilde{R} (\frac{\psi'}{\tilde{R}})' \mathbf{u} \times (\mathbf{u} \times \hat{\mathbf{d}}) - \frac{2\psi'}{\tilde{R}} \hat{\mathbf{d}}))) dS' \\ &\sim \frac{-z}{8(\pi ik)^2} \hat{\mathbf{d}} \times \int_{S'} b'(\frac{z}{\tilde{R}}) ((\frac{\psi'}{\tilde{R}})')^2 R dS' \\ &\quad + \frac{1}{8\pi^2 ik Z_0} \int c(\frac{z}{\tilde{R}}) \tilde{R} \frac{\psi'}{\tilde{R}} (\mathbf{R} \times \hat{\mathbf{z}}) (\hat{\mathbf{z}} \cdot (\mathbf{u} \times (\mathbf{u} \times \hat{\mathbf{d}}))) (\frac{\psi'}{\tilde{R}})' dS' \\ &\quad - \frac{1}{8\pi^2 ik Z_0} \int c(\frac{z}{\tilde{R}}) \tilde{R} \frac{\psi'}{\tilde{R}} \mathbf{R} \times ((\frac{\psi'}{\tilde{R}})' \mathbf{u} \times (\mathbf{u} \times \hat{\mathbf{d}})) dS' \end{aligned} \quad (202)$$

and thus (199). We can then, following the theorems 1 and 2, and the propositions 5 and 6, reduce the field  $\mathbf{E}$  to a contour integral. For this, we can use (177), (184), and

$$(\mathbf{R} \times \hat{\mathbf{z}}) \tilde{R} \frac{(\mathbf{R} \cdot \hat{\mathbf{d}})}{\mathbf{R}^2} = \frac{\mathbf{R}^2 - z^2}{\tilde{R}} \frac{(\hat{\mathbf{d}} \cdot (\mathbf{a} - \mathbf{r}')) ((\mathbf{a} - \mathbf{r}') \times \hat{\mathbf{z}})}{(\mathbf{a} - \mathbf{r}')^2}. \quad (203)$$

### Appendice C : Concerning theorem 14 for complex cases

Usual methods (as Filon's) can be used for  $\mathcal{K}_s$  with some precaution concerning analyticity of  $u$ . For example, we can let  $\int_0^{t_s} g(s_1 t + s_0) e^{t^2 v h_p'^2} t dt \simeq \sum_{0 \leq t_{s,n} \leq t_s} e^{t_{s,n}^2 v h_p'^2} \int_{t_{s,n-1}}^{t_{s,n}} g(s_1 t + s_0) t dt$ .

Generally, we can assume that  $g(s_1 t + s_0)$  is such that, when the path from 0 to  $t_s$  is real, it does not cross any branch cut of  $g$ . When cuts are present on  $S'$ , we can split  $S'$ , which is equivalent to arbitrarily suppress the part of integration through the cut.

In many cases, a closed form primitive of  $tg(s_1 t + s_0)$  can be used. When  $g(\mathbf{r}') = \sqrt{a' + b x' + c' x'^2}$ , i.e.  $g(s_1 t + s_0) = \sqrt{a + b t + c t^2}$ , the primitive  $H(t) = \int^t t \sqrt{R} dt$  with  $R = a + b t + c t^2 = \frac{(2ct+b)^2 + \Delta'}{4c} = c \prod_{\pm} (t - t_{\pm})$ ,  $t_{\pm} = \frac{b(-1 \pm \sqrt{1 - \frac{4ac}{b^2}})}{2c}$ ,

$\Delta' = 4ac - b^2$ , is given by (see [31] 2.262.2),

$$\begin{aligned}
 H|_{c \neq 0} &= \frac{1}{c^2 \sqrt{c}} \left( \sqrt{c} \sqrt{R} \left( \frac{cR}{3} - \frac{(2ct+b)b}{8} \right) - \frac{b\Delta'}{16} \ln \left( \frac{2\sqrt{c}\sqrt{R} + 2ct + b}{b\sqrt{1 - \frac{4ac}{b^2}}} \right) \right), \\
 H|_{c=0, b \neq 0} &= \frac{2t\sqrt{R^3}}{3b} - \frac{4\sqrt{R^5}}{15b^2}, \quad H|_{c=b=0} = \frac{t^2\sqrt{a}}{2}.
 \end{aligned} \tag{204}$$

Considering a Taylor expansion for  $H|_{|c| \ll |a|, |c| \ll |b|}$  when  $0 \leq t \leq 1$ , we have,

$$\begin{aligned}
 H|_{|c| \ll |a|, |c| \ll |b| \neq 0} &\sim H|_{c=0, b \neq 0} + c \frac{\sqrt{R_{c=0}} \left( \left( \frac{R_{c=0}}{7} - \frac{3a}{5} \right) R_{c=0} + a^2 \right) R_{c=0} - a^3}{b^4}, \\
 H|_{|c| \ll |b| \ll |a|} &\sim H|_{c=0, |b| \ll |a|} + c \int \frac{t^3}{2\sqrt{a+bt}} dt - \frac{c^2}{2} \int \frac{t^5}{4(\sqrt{a+bt})^3} dt \\
 &\sim H|_{c=b=0} + b \frac{t^3}{2\sqrt{a}} \left( \frac{1}{3} - b \frac{t}{16a} \right) + c \frac{t^4}{4\sqrt{a}} \left( \frac{1}{2} - \frac{t}{a} \left( \frac{b}{5} + \frac{ct}{12} \right) \right)
 \end{aligned} \tag{205}$$

In practice, we can consider for (205) that  $u \ll v$  if  $u \leq v/20$ .

Remark : Let  $P_{2n}(z) = z^{2n} + \sum_{k=0}^{2n-1} a_k z^k = z^{2n} (1 + z^{-1} \sum_{k=0}^{2n-1} a_k z^{k+1-2n})$ . For  $g \equiv (P_{2n})^{1/2}$ , it exists a polynomial  $g_0(z) = z^n (1 + \sum_{k=1}^n b_k z^{-k})$  such that  $g(z) = g_0(z) + O(z^{-1})$ .