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## **Novel reduction of complex surface radiation integrals to contour integrals**

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**Summary :** We develop here an efficient method for the reduction of surface radiation integrals to contour integrals, when we suppose known, as in Physical Optics (PO), the analytic expression of surface currents whose electromagnetic radiation is calculated. Although many authors have studied this problem, we present here an original solution, general and practical, leading to simple non-singular contour expressions (without any special function), valid from infinity to a short distance of the source (near-field case), for large plane or moderately curved surfaces :

a) in monostatic case, for the radiation of PO currents carried by a perfectly or imperfectly reflective surface, excited by a point source whose radiation can be of arbitrary pattern, without any approximation in the phase of the integrand,

b) in bistatic case, when the integrand has a complex phase which is a quadratic function of the coordinates, as in the case of gaussian beam propagation or in second order phase approximation for the free space Green function at large distance.

Comparisons with classic surface integration results show excellent agreements for both cases a) and b).

### **1) Introduction**

We develop here an efficient method for the reduction of surface radiation integrals to contour integrals, when we suppose known, as in Physical Optics (PO) [1]-[5], the analytic expression of surface currents whose electromagnetic radiation is calculated. In what follows, we look for solutions in near-field cases, which signifies here that we search for expressions which give a correct result from infinity to a short distance of the sources, regardless of dimensions of the integration surface. Although many authors have studied this type of problems [6]-[21], we present here an original solution, general but practical, leading to simple non-singular expressions [22], in the case of plane or moderately curved surfaces :

a) in monostatic near-field case, for the radiation of PO currents carried by a perfectly or imperfectly reflective surface, excited by a point source whose radiation can be of arbitrary pattern, without any approximation in the phase of the integrand,

b) in bistatic near-field case, when the integrand has a complex phase which is a quadratic function of the coordinates, as for gaussian beam propagation or in second order phase approximation of the radiation Green function at large distance.

The known methods do not possess the appropriate generality to solve our problems, either because they depends too closely on equations of Helmholtz or Maxwell [6]-[11], or they assume a too narrow domain of geometries [12]-[14] or of illuminations [15]-[19], or they

don't allow to consider complex phases [16] or complex surfaces due to a change of variables [6]-[21], or they don't propose general expressions formally non-singular [8]-[11],[18],[21]. The paper is organized as follows. After presenting the problem in chapter 2, we develop our method for an analytic reduction of surface integrals in chapter 3, then detail its application in case a) (chapters 4 and 5) and in case b) (chapter 6). Developed first for plane surfaces of integration, the method is extended to surfaces with low curvature in chapter 7. Comparisons with classic surface integration results show excellent agreements for both cases a) and b) (chapter 8). Some developments are given in appendices, in particular the reduction for imperfectly reflective surfaces.

## 2) Formulation

Let  $S$  be a regular open surface of low curvature, defined in a cartesian coordinate system of origin  $O'$  as a smooth perturbation of a surface  $S'$  of the plane  $z = 0$  with unit normal  $\hat{\mathbf{z}}$ . We study the reduction to a contour integral along  $\partial S'$ , of the electric  $\mathbf{E}$  and magnetic  $\mathbf{H}$  fields radiated by the surface currents  $\mathbf{J}$  and  $\mathbf{M}$  on  $S$  [1, sect. 1.26],

$$\begin{aligned}\mathbf{E} &= \frac{Z_0}{4\pi ik} \int_S (\tilde{R} (\frac{\psi'}{\tilde{R}}))' \mathbf{u} \times (\mathbf{u} \times \mathbf{J}) - \frac{2\psi'}{\tilde{R}} \mathbf{J} dS' - \frac{1}{4\pi} \int_S \frac{\psi'}{\tilde{R}} \mathbf{R} \times \mathbf{M} dS', \\ Z_0 \mathbf{H} &= \frac{Z_0}{4\pi} \int_S \frac{\psi'}{\tilde{R}} \mathbf{R} \times \mathbf{J} dS' + \frac{1}{4\pi ik} \int_S (\tilde{R} (\frac{\psi'}{\tilde{R}}))' \mathbf{u} \times (\mathbf{u} \times \mathbf{M}) - \frac{2\psi'}{\tilde{R}} \mathbf{M} dS',\end{aligned}\quad (1)$$

at the point  $\mathbf{r} = (\mathbf{a} + z\hat{\mathbf{z}}) \notin S$ , in free space of wavenumber  $k$  and impedance  $Z_0$  [1]. The current point of integration on  $S$  is  $(\mathbf{r}' + z'\hat{\mathbf{z}})$ , while  $\psi(\tilde{R}) = \frac{e^{-ik\tilde{R}}}{\tilde{R}}$ ,  $\mathbf{R} = \mathbf{r} - (\mathbf{r}' + z'\hat{\mathbf{z}})$ ,  $\tilde{R} = \sqrt{\mathbf{R} \cdot \mathbf{R}}$ ,  $\mathbf{u} = \frac{\mathbf{R}}{\tilde{R}}$ . We then assume that  $\mathbf{J}$  and  $\mathbf{M}$  are analytically defined. For example, we can consider that the currents, when excited by a point source at  $\mathbf{r}_0$ , are of the form,

$$\sum_n f_n(\mathbf{u}_0) \frac{e^{-ik\tilde{R}_0}}{(k\tilde{R}_0)^n}, \quad (2)$$

where  $\mathbf{R}_0 = (\mathbf{r}' + z'\hat{\mathbf{z}}) - \mathbf{r}_0$ ,  $\tilde{R}_0 = \sqrt{\mathbf{R}_0 \cdot \mathbf{R}_0}$ ,  $\mathbf{u}_0 = \frac{\mathbf{R}_0}{\tilde{R}_0}$ , and the  $f_n$  are regular functions.

In what follows, we define a method that allows to reduce the surface integrals in (1) to simple and efficient non-singular contour integrals, for the following cases :

- the monostatic case where  $\mathbf{r} = \mathbf{r}_0$ , and thus  $\mathbf{R} = -\mathbf{R}_0$  in (1), for a plane or moderately curved surface without any approximation in the phase of the integrand. In this situation, exact or approximated expressions, without any special function, no discontinuity of asymptotics, and no restriction on the dimensions of  $S'$ , are developed.

- the bistatic case ( $\mathbf{r}$  and  $\mathbf{r}_0$  can be distinct), for a surface with low curvature, and an arbitrary illumination (for example, gaussian), when integrals in (1) are approximated in the form,

$$\frac{e^{-ik(\tilde{r}+\tilde{r}_0)}}{\tilde{r}\tilde{r}_0} \int_{S'} e^{(\alpha_1 x' + \alpha_2 y') + (\beta_1 x'^2 + \beta_2 y'^2 + \beta_3 x' y')} t(\mathbf{r}') dS', \quad (3)$$

with some smooth function  $t(\mathbf{r}')$ ,  $\tilde{r} = \sqrt{\mathbf{r} \cdot \mathbf{r}}$ ,  $\tilde{r}_0 = \sqrt{\mathbf{r}_0 \cdot \mathbf{r}_0}$ , and some complex constants  $\alpha_i, \beta_j$ . In this situation, contour integrals without any special function, which are exact, for example when  $t(\mathbf{r}') = \sum_{n,m} t_{n,m} x'^n y'^m$ , or approximated, are derived.

It is worth noticing that we don't suppose that  $\mathbf{J}$  and  $\mathbf{M}$  derive from fields satisfying Helmholtz equation as in [6]-[11], and the currents can be in polynomial form or coming from an illumination with arbitrary smooth pattern, for perfectly or imperfectly reflective surfaces.

We begin to develop, in next chapter,

- two fundamental ways of investigation for the reduction of a surface integral,
  - some basic examples illustrating their complementarity,
- which subsequently leads us to efficient solutions for our problems.

### ***3) A combination of two fundamental ways for an efficient reduction to a contour integral***

The reduction that we propose is carried out by a correct combination of two fundamental ways of investigation and associated basic results, that are perfectly adapted to solve in a simple manner the class of problems that we study.

#### **Definition 1**

*Let the plane  $z = 0$  be defined with unit normal  $\hat{\mathbf{z}}$ , and a unit standard basis composed of  $\hat{\mathbf{x}}$  and  $\hat{\mathbf{y}} = \hat{\mathbf{z}} \times \hat{\mathbf{x}}$ . Each point is denoted  $\mathbf{r}' = x'\hat{\mathbf{x}} + y'\hat{\mathbf{y}}$  in  $(x', y')$  coordinates. In the applications, analytical changes of variables can transform the initial domain of integration with real coordinates to a domain with complex coordinates. The dot and cross products of vectors with complex coordinates are here considered as in real space (without complex conjugation), with  $\mathbf{u} \cdot \mathbf{v} = u_x v_x + u_y v_y$  when  $\mathbf{u} = u_x \hat{\mathbf{x}} + u_y \hat{\mathbf{y}}$  and  $\mathbf{v} = v_x \hat{\mathbf{x}} + v_y \hat{\mathbf{y}}$ .*

*In this situation, we assume that  $S'$  is a (2-dimensional) complex manifold of the plane  $z = 0$ , simply connected and oriented, with piecewise analytical contour  $\mathcal{L} \equiv \partial S'$ , which results from the transformation of a simply connected (2-dimensional) real oriented manifold of the plane  $z = 0$ , by an invertible function  $P$ , that we suppose thereafter linear to simplify.*

*Letting  $P(\mathbf{u}) = P_x(\mathbf{u})\hat{\mathbf{x}} + P_y(\mathbf{u})\hat{\mathbf{y}}$ , we notice that  $P(\mathbf{u}) \cdot P(\mathbf{u}) = \prod_{\pm} (P_x(\mathbf{u}) \pm i P_y(\mathbf{u}))$ .*

### 3.1) First way

#### 3.1.1) Some elements of analysis in real space

Considering the domain of integration in real cylindrical coordinates  $(\rho, \varphi)$ , and a function  $F(\rho^2, \varphi)$  regular at constant  $\varphi$  from the origin to any point of real  $S'$ , we can write,

$$\int_{S'} \frac{\partial F(\rho^2, \varphi)}{\partial(\rho^2)} dS' = \int_{\mathcal{L}} \int_{0+}^{\rho(\varphi)} \frac{\partial F(\rho^2, \varphi)}{\partial(\rho^2)} \rho d\rho d\varphi = \frac{1}{2} \int_{\mathcal{L}} (F(\rho^2, \varphi) - F(0, \varphi)) d\varphi. \quad (4)$$

where  $\mathcal{L}$  is positively oriented. In this expression,  $d\varphi = (\hat{z} \times \frac{\bar{\rho}}{\rho^2}) \cdot d\bar{\rho}$  with  $\bar{\rho}$  the current point of integration, and the cut at  $\varphi = 0$  gives no contribution. We can recover this equality with the Stokes theorem in its common form [1, equ. (34) chap.2.9], by using,

$$\begin{aligned} \text{rot}(\hat{z} \times \frac{\bar{\rho}}{\rho^2} (F(\rho^2, \varphi) - F(0, \varphi))) &= \hat{z} \text{div}(\frac{\bar{\rho}}{\rho^2} (F(\rho^2, \varphi) - F(0, \varphi))) \\ &= \hat{z} \frac{2\partial((\bar{\rho} \cdot \frac{\bar{\rho}}{\rho^2})(F(\rho^2, \varphi) - F(0, \varphi)))}{\partial(\rho^2)} = \hat{z} \frac{2\partial(F(\rho^2, \varphi))}{\partial(\rho^2)}, \end{aligned} \quad (5)$$

when  $F(\rho^2, \varphi) - F(0, \varphi)$  is analytic on  $S'$ , so that we have,

$$\int_{S'} \frac{\partial(F(\rho^2, \varphi))}{\partial(\rho^2)} dS' = \frac{1}{2} \int_{\mathcal{L}} (F(\rho^2, \varphi) - F(0, \varphi)) (\hat{z} \times \frac{\bar{\rho}}{\rho^2}) \cdot d\bar{\rho}. \quad (6)$$

Shifting the origin to real point  $\mathbf{a} = a_x \hat{\mathbf{x}} + a_y \hat{\mathbf{y}}$  (fig. 1) with,

$$\bar{\rho} = \mathbf{r}' - \mathbf{a}, \quad \frac{\mathbf{r}' - \mathbf{a}}{\sqrt{(\mathbf{r}' - \mathbf{a})^2}} = \hat{\mathbf{x}} \cos \varphi + \hat{\mathbf{y}} \sin \varphi, \quad (7)$$

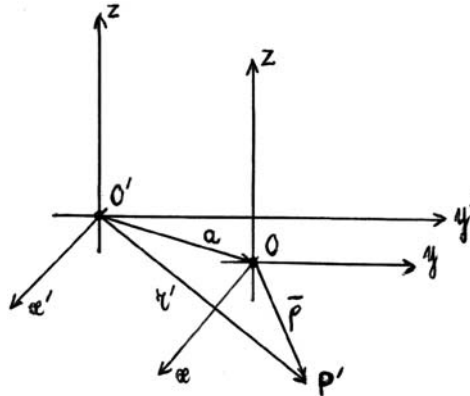


figure 1 : geometry and shift of the origin

we can write, when  $F((\mathbf{r}' - \mathbf{a})^2, \varphi) - F(0, \varphi)$  remains analytic on  $S'$ ,

$$\int_{S'} \frac{\partial(F((\mathbf{r}' - \mathbf{a})^2, \varphi))}{\partial((\mathbf{r}' - \mathbf{a})^2)} dS' = \frac{1}{2} \int_{\mathcal{L}} (F((\mathbf{r}' - \mathbf{a})^2, \varphi) - F(0, \varphi)) (\widehat{\mathbf{z}} \times \frac{\mathbf{r}' - \mathbf{a}}{(\mathbf{r}' - \mathbf{a})^2}) \cdot d\mathbf{r}', \quad (8)$$

with  $\int_{\mathcal{L}} d\varphi = 2\pi$  if  $\mathbf{a} \in S'$  and  $\int_{\mathcal{L}} d\varphi = 0$  if  $\mathbf{a} \notin S'$ , for real  $S'$  and  $\mathbf{a}$ .

### 3.1.2) Reduction in complex coordinates

The Stokes theorem, usually used in a domain with real coordinates, can be extended to the case with complex coordinates [23, p.213] by the formalism of exterior differential forms on complex manifolds, letting  $dS' = (dx' \wedge dy')$  [24],[25, chap.7],[26, chap.3].

#### Definition 2

Considering  $\mathbf{r}' = x' \widehat{\mathbf{x}} + y' \widehat{\mathbf{y}}$  with  $\widehat{\mathbf{y}} = \widehat{\mathbf{z}} \times \widehat{\mathbf{x}}$ , in the complex 4-dimensional space of complex points, and a fixed complex point  $\mathbf{a} = a_x \widehat{\mathbf{x}} + a_y \widehat{\mathbf{y}}$ , we can write,

$$\begin{aligned} z'_\pm &= x' \pm iy' - (a_x \pm ia_y) = (\mathbf{r}' - \mathbf{a}) \cdot (\widehat{\mathbf{x}} \pm i\widehat{\mathbf{y}}), \\ e^{\pm i\varphi} &= \frac{z'_\pm}{(z'_+ z'_-)^{1/2}}, \quad e^{\pm 2i\varphi} = \frac{z'_\pm}{z'_\mp}, \quad \varphi = \mp i \ln\left(\frac{z_\pm}{(z'_+ z'_-)^{1/2}}\right), \\ \mathbf{r}' - \mathbf{a} &= \frac{(z'_+ + z'_-)}{2} \widehat{\mathbf{x}} + \frac{(z'_+ - z'_-)}{2i} \widehat{\mathbf{y}}, \quad (\mathbf{r}' - \mathbf{a})^2 = z'_+ z'_-, \\ \mathbf{r}' - \mathbf{a} &= z'_\pm e^{\mp i\varphi} (\cos(\varphi) \widehat{\mathbf{x}} + \sin(\varphi) \widehat{\mathbf{y}}), \quad (\mathbf{r}' - \mathbf{a})^2 = (z'_\pm e^{\mp i\varphi})^2. \end{aligned} \quad (9)$$

where the functions  $(\cdot)^{1/2}$  and  $\ln(\cdot)$  are considered by analytical continuation. We notice that, in complex coordinates, the set of points verifying  $(\mathbf{r}' - \mathbf{a})^2 = 0$  are the 2-dimensional straight lines  $\mathbf{r}' = \mathbf{a} + t(\widehat{\mathbf{x}} \pm i\widehat{\mathbf{y}})$  where  $z_\pm = 0$ , with complex  $t$ , while we remark that  $\widehat{\mathbf{x}} \pm i\widehat{\mathbf{y}} = \frac{e \pm i\widehat{\mathbf{z}} \times \mathbf{c}}{c_x \mp ic_y}$  for any complex vector  $\mathbf{c}$  with  $\mathbf{c}^2 \neq 0$ . Apart from branch cut(s), we have,

$$d\varphi = \frac{-i}{2} d \ln\left(\frac{z'_+}{z'_-}\right) = (\widehat{\mathbf{z}} \times \frac{\mathbf{r}' - \mathbf{a}}{(\mathbf{r}' - \mathbf{a})^2}) \cdot d\mathbf{r}'. \quad (10)$$

It is worth noticing that  $d\varphi$  is independent of the choice of  $\widehat{\mathbf{x}}$  and  $\widehat{\mathbf{y}} = \widehat{\mathbf{z}} \times \widehat{\mathbf{x}}$ , and it vanishes along any line  $\mathbf{r}' = \mathbf{a} + t\mathbf{v}$ , with variable  $t$  and constant  $\mathbf{v}$ . On this occasion, we remark that  $\frac{z'_+}{z'_-}$  describes a closed contour along any closed path of  $(x', y')$ -plane.

#### Definition 3

Let  $\mathbf{a} = a_x \widehat{\mathbf{x}} + a_y \widehat{\mathbf{y}}$  be a complex point, a function  $F((\mathbf{r}' - \mathbf{a})^2, \varphi)$  is denoted  $c$ -regular on  $S'$  (' $c$ ' for 'centered'), if it exists a 4-dimensional vicinity of  $S'$ , where  $F((\mathbf{r}' - \mathbf{a})^2, \varphi) - F(0, \varphi)$  is analytical relatively to  $((\mathbf{r}' - \mathbf{a})^2)^{\frac{1}{2}} \cos\varphi = (x' - a_x)$  and

$((\mathbf{r}' - \mathbf{a})^2)^{\frac{1}{2}} \sin \varphi = (y' - a_y)$ , and vanishes when  $(\mathbf{r}' - \mathbf{a})^2 \rightarrow 0$ , while  $\frac{\partial(F((\mathbf{r}' - \mathbf{a})^2, \varphi))}{\partial((\mathbf{r}' - \mathbf{a})^2)}$  is bounded. Noticing that the set of points verifying  $(\mathbf{r}' - \mathbf{a})^2 = 0$  is only 2-dimensional (see definition 2),  $\frac{\partial(F((\mathbf{r}' - \mathbf{a})^2, \varphi))}{\partial((\mathbf{r}' - \mathbf{a})^2)}$  is also analytical relatively to  $(x' - a_x)$  and  $(y' - a_y)$  on  $S'$  [27, sect. 23.2]. We remark that  $F((\mathbf{r}' - \mathbf{a})^2, \varphi) = ((\mathbf{r}' - \mathbf{a})^2)^{n+3/2} e^{\pm i\varphi}$  is  $c$ -regular for any complex  $\mathbf{a}$  and integer  $n \geq 0$ .

We then define the condition of continuation for  $F$ , from  $\mathbf{a} = \mathbf{a}'$  to  $S'$ , as follows : it exists an open connected finite domain  $\mathcal{H}$  of complex points  $\mathbf{a}$  (4-dimensional set with  $a_x$  and  $a_y < \infty$ ) which includes  $\mathbf{a}'$  and a point of  $S'$ , where  $F((\mathbf{r}' - \mathbf{a})^2, \varphi)$  remains  $c$ -regular on  $S'$ . We remark that, if  $F(t, s)$  is an entire function of  $t$  independent of  $s$ , then the condition is verified for any  $\mathbf{a}$ . If the condition is satisfied for  $F((\mathbf{r}' - \mathbf{a})^2, \varphi)$  on  $S'$ , then it is also the case for  $(\mathbf{r}' - \mathbf{a})^2 (F((\mathbf{r}' - \mathbf{a})^2, \varphi) - F(0, \varphi))$  on  $S'$ , and any primitive  $\int^{(\mathbf{r}' - \mathbf{a})^2} (F(t, \varphi) - F(0, \varphi)) dt$  defined on  $S'$  and at points with  $(\mathbf{r}' - \mathbf{a})^2 = 0$ .

### Theorem 1

Let  $S'$  be an oriented complex manifold, simply connected, with contour  $\mathcal{L}$ , resulting from a linear transformation of a real manifold as in definition 1, and  $F((\mathbf{r}' - \mathbf{a})^2, \varphi)$  be a complex function satisfying the  $c$ -regularity given in definition 3, then the equality

$$\int_{S'} \frac{\partial(F((\mathbf{r}' - \mathbf{a})^2, \varphi))}{\partial((\mathbf{r}' - \mathbf{a})^2)} dS' = \frac{1}{2} \int_{\mathcal{L}} (F((\mathbf{r}' - \mathbf{a})^2, \varphi) - F(0, \varphi)) \left[ (\hat{\mathbf{z}} \times \frac{\mathbf{r}' - \mathbf{a}}{(\mathbf{r}' - \mathbf{a})^2}) \cdot d\mathbf{r}' \right], \quad (11)$$

is valid for  $\mathbf{a} \in S'$ , and for any  $\mathbf{a}$  verifying the condition of continuation in  $\mathcal{H}$  to  $S'$  given in definition 3. The cut in  $\varphi$  can be ignored in (11), and  $(\mathbf{r}' - \mathbf{a})^2 = 0$  does not imply  $\mathbf{r}' = \mathbf{a}$ .

Proof. The equality (11) corresponds to a continuation of (8) for points with complex coordinates, in the domain of regularity of  $F$ . We first state (11) for  $\mathbf{a} \in S'$ . From the properties of the exterior product and the definition of  $z'_+$  and  $z'_-$  in (9), we can write,

$$\begin{aligned} dS' &= (dx' \wedge dy') = \frac{i}{2} (dz'_+ \wedge dz'_-) = \frac{-i}{4} (z'_- dz'_+ + z'_+ dz'_-) \wedge \left( \frac{dz'_+}{z'_+} - \frac{dz'_-}{z'_-} \right) \\ &= \frac{1}{2} d(z'_+ z'_-) \wedge \frac{-i}{2} d \ln \left( \frac{z'_+}{z'_-} \right) = \frac{1}{2} d(\mathbf{r}' - \mathbf{a})^2 \wedge d\varphi, \end{aligned} \quad (12)$$

From the definition 1,  $S'$  results from a linear transformation of a simply connected real manifold relatively to  $(x', y')$  coordinates, and when  $\mathbf{a} \in S'$ , it can be mapped from  $\mathbf{a}$  to  $\infty$  at constant  $\varphi$  with straight semi-lines, except on some cut(s) where  $\varphi$  is constant on both

sides and  $F$  remains analytic. In this case, we can integrate the integral on  $S'$  relatively to  $(\mathbf{r}' - \mathbf{a})^2$  variable from (12), and derive the contour integral along  $\mathcal{L}$  given in (11).

From the condition of continuation given in definition 3, we can extend the validity of this equality to any point  $\mathbf{a} = \mathbf{a}'$  belonging to domain  $\mathcal{H}$  which comprises a subset  $S'_a$  of  $S'$ . By definition of the c-regularity, the cuts in  $\varphi$  continue to have no influence; the left-hand side of (11) is analytical relatively to complex coordinates  $a_x$  and  $a_y$ ; the right-hand side, continuous at any point of  $\mathcal{H}$ , is analytical when  $(\mathbf{r}' - \mathbf{a})^2 \neq 0$  on  $\mathcal{L}$ . That restriction disappears from the generalized principle of continuation [27, sect. 23.2], and the analyticity extends at any point  $\mathbf{a}$  of  $\mathcal{H}$  by noticing that  $(\mathbf{r}' - \mathbf{a})^2|_{\mathbf{r}' \in \mathcal{L}} = 0$  describes a piecewise analytical  $p$ -dimensional hypersurface ( $p \leq 3$ ). Therefore, considering that we have the analyticity in  $\mathcal{H}$  and that (11) is stated on  $S'_a$ , we can use the identity theorem for functions of several variables [27, sect. 6.1], and conclude the validity of (11) for any point of  $\mathcal{H}$ , in particular at  $\mathbf{a} = \mathbf{a}'$ .

Since the integration in (11) is on  $\varphi$  variable, the integral, restricted to the factor  $F(0, \varphi)$ , could express in a simple form, and we now analyse the case  $F(0, \varphi) \equiv 1$ .

#### Proposition 1

let  $\mathcal{D}_1$  be a smooth semi-line of the complex plane from the origin to  $\infty$ , and  $\mathcal{L}$  a closed contour of the plane  $z = 0$ . We have, when  $(\mathbf{r}' - \mathbf{a})^2 \neq 0$  on  $\mathcal{L}$ ,

$$\int_{\mathcal{L}} (\widehat{\mathbf{z}} \times \frac{\mathbf{r}' - \mathbf{a}}{(\mathbf{r}' - \mathbf{a})^2}) \cdot d\mathbf{r}' = \frac{-i}{2} \int_{\mathcal{L}} d \ln \left( \frac{z'_+}{z'_-} \right) = \pi(N_+ - N_-) \quad (13)$$

where  $N_+$  (resp.  $N_-$ ) is the number of times that  $\frac{z'_+}{z'_-}$  crosses  $\mathcal{D}_1$  in the positive sense (resp. negative). For surface  $S'$  as described in definition 1, i.e. resulting from a linear transformation of a real simply connected surface, we can write, when  $(\mathbf{r}' - \mathbf{a})^2 \neq 0$  on  $\mathcal{L}$ ,

$$\int_{\mathcal{L}} (\widehat{\mathbf{z}} \times \frac{\mathbf{r}' - \mathbf{a}}{(\mathbf{r}' - \mathbf{a})^2}) \cdot d\mathbf{r}' = \pi(\epsilon^+ 1_{S'_c}(a_x + ia_y) + \epsilon^- 1_{S'_{c*}}(a_x - ia_y)), \quad (14)$$

where  $\epsilon^\pm = +1$  or  $-1$ ,  $1_{S'_{c(*)}}$  are the indicator functions of the domain  $S'_{c(*)}$  of the complex plane, with  $S'_c = \{x' + iy', M(x', y') \in S'\}$  and  $S'_{c*} = \{x' - iy', M(x', y') \in S'\}$ . For real simply connected surface  $S'$  positively oriented, we have  $\epsilon^\pm = 1$ .

Proof. From (10), we have, when  $(\mathbf{r}' - \mathbf{a})^2 \neq 0$  on  $\mathcal{L}$ ,



$$\int_{\mathcal{L}} (\widehat{\mathbf{z}} \times \frac{\mathbf{r}' - \mathbf{a}}{(\mathbf{r}' - \mathbf{a})^2}) \cdot d\mathbf{r}' = \frac{-i}{2} \int_{\mathcal{L}} d\ln\left(\frac{z'_+}{z'_-}\right) \quad (15)$$

where the logarithm function is considered by analytical continuation along the closed path covered by the uniform function  $\frac{z'_+}{z'_-}$  on  $\mathcal{L}$ . We then obtain (13), since  $(N_+ - N_-)$  is the number of turns done by  $\frac{z'_+}{z'_-}$  around the origin, along the closed contour  $\mathcal{L}$ . For (14), we write,

$$\begin{aligned} \int_{\mathcal{L}} (\widehat{\mathbf{z}} \times \frac{\mathbf{r}' - \mathbf{a}}{(\mathbf{r}' - \mathbf{a})^2}) d\mathbf{r}' &= \frac{-i}{2} \int_{\mathcal{L}} \left( \frac{dz'_+}{z'_+} - \frac{dz'_-}{z'_-} \right) \\ &= \frac{-i}{2} \int_{\mathcal{L}_c} \frac{d(x' + iy')}{x' + iy' - (a_x + ia_y)} + \frac{-i}{2} \int_{\mathcal{L}_{c^*}} \frac{d(x' - iy')}{x' - iy' - (a_x - ia_y)} \\ &= \epsilon^+ \pi 1_{S'_c}(a_x + ia_y) + \epsilon^- \pi 1_{S'_{c^*}}(a_x - ia_y), \end{aligned} \quad (16)$$

where the residue theorem on the complex contours  $\mathcal{L}_c$  of  $S'_c = \{x' + iy', M(x', y') \in S'\}$  and  $\mathcal{L}_{c^*}$  of  $S'_{c^*} = \{x' - iy', M(x', y') \in S'\}$  is used. For real  $S'$  with contour  $\mathcal{L}$  positively oriented, we have  $\epsilon^\pm = 1$ .

Remark 1 :

As an example, let us notice that a complex anisotropic scaling can lead us to work on a complex manifold. If we dilate a real surface  $S'$  by a factor  $p \neq 0$  along the direction  $\mathbf{c}_p$  with,

$$\mathcal{P} : \mathbf{r}' \rightarrow \mathbf{r}'_p = x'_p \widehat{\mathbf{c}}_p + y'_p (\widehat{\mathbf{z}} \times \widehat{\mathbf{c}}_p), \quad x_p = p(\mathbf{r}' \cdot \widehat{\mathbf{c}}_p), \quad y_p = \mathbf{r}' \cdot (\widehat{\mathbf{z}} \times \widehat{\mathbf{c}}_p), \quad \widehat{\mathbf{c}}_p = \frac{\mathbf{c}_p}{\sqrt{\mathbf{c}_p \cdot \mathbf{c}_p}}, \quad (17)$$

we obtain a complex surface  $S'_p$ . In this case, we have,

$$\begin{aligned} dS'_p &= dx'_p \wedge dy'_p = p dx' \wedge dy' = p dS', \\ \mathbf{v} \cdot d\mathcal{P}(\mathbf{r}') &= \mathcal{P}(\mathbf{v}) \cdot d\mathbf{r}', \quad (\widehat{\mathbf{z}} \times \mathcal{P}(\mathbf{v})) \cdot d\mathcal{P}(\mathbf{r}') = (\widehat{\mathbf{z}} \times p\mathbf{v}) \cdot d\mathbf{r}', \\ (\mathcal{P}_x(\mathbf{r}' - \mathbf{a}) \pm i\mathcal{P}_y(\mathbf{r}' - \mathbf{a})) &= (\mathbf{r}' - \mathcal{P}^{-1}(\mathbf{a})) \cdot \mathcal{P}(\widehat{\mathbf{x}} \pm i\widehat{\mathbf{y}}). \end{aligned} \quad (18)$$

Remark 2 : The choice to orient or not a domain stands out from the following equalities,

$$\int_{\varphi([a,b])} f(x) dx = \int_{[a,b]} f(\varphi(t)) |\varphi'(t)| dt, \quad \int_{\varphi(a)}^{\varphi(b)} f(x) dx = \int_a^b f(\varphi(t)) \varphi'(t) dt, \quad (19)$$

by noticing that the first one, contrarily to the second, cannot be applied to complex  $a$  and  $b$ .

Remark 3 : Let us notice that on each straight part of  $\mathcal{L}$  along  $\widehat{\mathbf{v}}$ , with  $d\mathbf{r}' = \widehat{\mathbf{v}} ds$  and  $\mathbf{r}' - \mathbf{a} = \mathbf{r}_0 + \widehat{\mathbf{v}} s$ , we have  $(\widehat{\mathbf{z}} \times (\mathbf{r}' - \mathbf{a})) \cdot d\mathbf{r}' = \mathbf{g}_0 ds$ , with constant  $\mathbf{r}_0$  and  $\mathbf{g}_0 = \widehat{\mathbf{z}} \times \mathbf{r}_0$ .

### 3.2) Second way

We consider here, as in previous section, the formalism of exterior differential forms, with  $dS' = (dx' \wedge dy')$  [24]-[25, chap.7].

#### Theorem 2

Let  $S'$  be a simply connected oriented complex manifold, satisfying the definition 1, we consider a function  $F((\mathbf{r}' - \mathbf{a})^2, \varphi)$ , analytical relatively to  $((\mathbf{r}' - \mathbf{a})^2)^{\frac{1}{2}} \cos \varphi = (x' - a_x)$  and  $((\mathbf{r}' - \mathbf{a})^2)^{\frac{1}{2}} \sin \varphi = (y' - a_y)$  on  $S'$ . We can write

$$\int_{S'} b \cdot \text{grad}_{\mathbf{r}'}(F((\mathbf{r}' - \mathbf{a})^2, \varphi) - A_0) dS' = \int_{\mathcal{L}} (F((\mathbf{r}' - \mathbf{a})^2, \varphi) - A_0) (\widehat{\mathbf{z}} \times b) \cdot d\mathbf{r}' \quad (20)$$

where  $b$  and  $A_0$  are constants, and the cut in  $\varphi$  can be ignored. From

$$\text{grad}_{\mathbf{r}'}(\cdot) = 2(\mathbf{r}' - \mathbf{a}) \frac{\partial(\cdot)}{\partial((\mathbf{r}' - \mathbf{a})^2)} + \frac{(\widehat{\mathbf{z}} \times (\mathbf{r}' - \mathbf{a}))}{(\mathbf{r}' - \mathbf{a})^2} \frac{\partial(\cdot)}{\partial \varphi}, \quad (21)$$

we then have,

$$\begin{aligned} & \int_{S'} (\mathbf{r}' - \mathbf{a}) \cdot \left( b \frac{\partial(F((\mathbf{r}' - \mathbf{a})^2, \varphi))}{\partial((\mathbf{r}' - \mathbf{a})^2)} - \frac{1}{2} \frac{(\widehat{\mathbf{z}} \times b)}{(\mathbf{r}' - \mathbf{a})^2} \frac{\partial(F((\mathbf{r}' - \mathbf{a})^2, \varphi))}{\partial \varphi} \right) dS' \\ &= \frac{1}{2} \int_{\mathcal{L}} (F((\mathbf{r}' - \mathbf{a})^2, \varphi) - A_0) (\widehat{\mathbf{z}} \times b) \cdot d\mathbf{r}'. \end{aligned} \quad (22)$$

Proof. In real coordinates, we could use the Stokes theorem [1, chap.2.9], in combination with  $\text{grad}(F((\mathbf{r}' - \mathbf{a})^2, \varphi)) \times \widehat{\mathbf{d}} = \text{rot}(\widehat{\mathbf{d}}F((\mathbf{r}' - \mathbf{a})^2, \varphi))$ , where  $\widehat{\mathbf{d}}$  is a constant vector. More generally, in the formalism of exterior forms for complex manifolds, we prefer here to use gradient in cartesian coordinates with  $dS' = dx' \wedge dy' = -dy' \wedge dx'$ , following

$$\int_{S'} b \cdot \text{grad}(F) dS' = \int_{S'} b \cdot \widehat{\mathbf{x}} \frac{\partial F}{\partial x'} dx' \wedge dy' - \int_{S'} b \cdot \widehat{\mathbf{y}} \frac{\partial F}{\partial y'} dy' \wedge dx'. \quad (23)$$

From the analyticity of  $F$  relatively to  $x'$  and  $y'$ , the cut in  $\varphi$  is omitted. We integrate  $\frac{\partial F}{\partial x'} dx'$  and  $\frac{\partial F}{\partial y'} dy'$  parts, while noticing  $(\widehat{\mathbf{z}} \times b) \cdot d\mathbf{r}' = b \cdot \widehat{\mathbf{x}} dy' - b \cdot \widehat{\mathbf{y}} dx'$ , and obtain (20). Then, from

$$\begin{aligned} dF &= \frac{\partial F}{\partial(\mathbf{r}' - \mathbf{a})^2} d(\mathbf{r}' - \mathbf{a})^2 + \frac{\partial F}{\partial \varphi} d\varphi = \text{grad}_{\mathbf{r}'}(F) \cdot d(\mathbf{r}' - \mathbf{a}), \\ d(\mathbf{r}' - \mathbf{a}) &= \frac{1}{2} \frac{\mathbf{r}' - \mathbf{a}}{(\mathbf{r}' - \mathbf{a})^2} d(\mathbf{r}' - \mathbf{a})^2 + (\widehat{\mathbf{z}} \times (\mathbf{r}' - \mathbf{a})) d\varphi, \end{aligned} \quad (24)$$

we derive (21), which, combined with (20), gives (22).

**Proposition 2**

Let  $F_0((\mathbf{r}' - \mathbf{a})^2)$  be a scalar function, analytical on  $S'$ , and  $\mathbf{F}_{0v}$  a vectorial function, whose each component is analytical on  $S'$ , we have,

$$\begin{aligned} \int_{S'} b \cdot (\mathbf{r}' - \mathbf{a}) \frac{\partial(F_0((\mathbf{r}' - \mathbf{a})^2))}{\partial((\mathbf{r}' - \mathbf{a})^2)} dS' &= \frac{1}{2} \int_{\mathcal{L}} F_0((\mathbf{r}' - \mathbf{a})^2) (\widehat{\mathbf{z}} \times b) \cdot d\mathbf{r}' \\ \int_{S'} (\mathbf{r}' - \mathbf{a}) \cdot \frac{\partial(\mathbf{F}_{0v}((\mathbf{r}' - \mathbf{a})^2))}{\partial((\mathbf{r}' - \mathbf{a})^2)} dS' &= \frac{1}{2} \int_{\mathcal{L}} (\widehat{\mathbf{z}} \times \mathbf{F}_{0v}((\mathbf{r}' - \mathbf{a})^2)) \cdot d\mathbf{r}' \end{aligned} \quad (25)$$

Proof. Using the theorem 2, with  $F((\mathbf{r}' - \mathbf{a})^2, \varphi) = F_0((\mathbf{r}' - \mathbf{a})^2)$  independent of  $\varphi$ , we obtain the first line, then the second line by considering each component of  $\mathbf{F}_{0v}$  as  $F_0$ . On this occasion, let us remark that,

$$\begin{aligned} \text{if } \frac{\partial(F_0((\mathbf{r}' - \mathbf{a})^2))}{\partial((\mathbf{r}' - \mathbf{a})^2)} &= e^{-v(\mathbf{r}' - \mathbf{a})^2} \text{ then } F_0((\mathbf{r}' - \mathbf{a})^2) = \frac{-e^{-v(\mathbf{r}' - \mathbf{a})^2}}{v} + C, \\ \text{if } \frac{\partial(F_0((\mathbf{r}' - \mathbf{a})^2))}{\partial((\mathbf{r}' - \mathbf{a})^2)} &= \frac{e^{-v\widetilde{R}}}{\widetilde{R}} \text{ then } F_0((\mathbf{r}' - \mathbf{a})^2) = 2 \int^{\widetilde{R}} e^{-vt} dt = \frac{-2e^{-v\widetilde{R}}}{v} + C, \end{aligned} \quad (26)$$

with  $\widetilde{R} = \sqrt{\mathbf{R} \cdot \mathbf{R}} = \sqrt{(\mathbf{r}' - \mathbf{a})^2 + z^2}$ , and  $C, v$  some constants.

**Proposition 3**

Let  $B_0((\mathbf{r}' - \mathbf{a})^2, \varphi)$  be a function, analytical relatively to  $((\mathbf{r}' - \mathbf{a})^2)^{\frac{1}{2}} \cos \varphi = (x' - a_x)$  and  $((\mathbf{r}' - \mathbf{a})^2)^{\frac{1}{2}} \sin \varphi = (y' - a_y)$  on  $S'$ , we have, for  $n \geq 0$ ,

$$\begin{aligned} \mathbf{d}_{n+1} \cdot \int_{S'} \prod_{j=1}^n (\mathbf{d}_j \cdot (\mathbf{r}' - \mathbf{a})) \text{grad}(B_0((\mathbf{r}' - \mathbf{a})^2, \varphi)) dS' \\ = \int_{\mathcal{L}} \prod_{j=1}^n (\mathbf{d}_j \cdot (\mathbf{r}' - \mathbf{a})) B_0((\mathbf{r}' - \mathbf{a})^2, \varphi) (\widehat{\mathbf{z}} \times \mathbf{d}_{n+1}) \cdot d\mathbf{r}' \\ - \mathbf{d}_{n+1} \cdot \sum_{m=1}^n \mathbf{d}_m \int_{S'} \prod_{j \neq m}^n (\mathbf{d}_j \cdot (\mathbf{r}' - \mathbf{a})) B_0((\mathbf{r}' - \mathbf{a})^2, \varphi) dS' \end{aligned} \quad (27)$$

where the  $\mathbf{d}_j$  are constant, and we can let  $\prod_j^0 (\dots) = 1, \prod_{j \neq m}^1 (\dots) = 1, \prod_{j \neq m}^0 (\dots) = 0$ .

Proof. Since  $f \text{grad}(h) = \text{grad}(fh) - h \text{grad}(f)$  and  $\text{grad}(\mathbf{d}_m \cdot \mathbf{r}') = \mathbf{d}_m$ , we have

$$\begin{aligned} & \prod_{j=1}^n (\mathbf{d}_j \cdot (\mathbf{r}' - \mathbf{a})) \operatorname{grad}(B_0((\mathbf{r}' - \mathbf{a})^2, \varphi)) \\ &= \operatorname{grad} \left( \prod_{j=1}^n (\mathbf{d}_j \cdot (\mathbf{r}' - \mathbf{a})) B_0((\mathbf{r}' - \mathbf{a})^2, \varphi) \right) - \sum_{m=1}^n \mathbf{d}_m \prod_{j \neq m}^n (\mathbf{d}_j \cdot (\mathbf{r}' - \mathbf{a})) B_0((\mathbf{r}' - \mathbf{a})^2, \varphi) \end{aligned} \quad (28)$$

Multiplying this equation by  $\mathbf{d}_{n+1}$  and integrating, then using (20), we obtain the proposition. This result will be useful, in particular when  $B_0((\mathbf{r}' - \mathbf{a})^2, \varphi)$  is independent of the variable  $\varphi$ , as we will see in proposition 5.

### 3.3) Why these distinct ways are complementary: some basic results

The first and second theorems permit to consider respective reductions of surface integrals for functions of types  $F((\mathbf{r}' - \mathbf{a})^2, \varphi)$  and  $(\mathbf{r}' - \mathbf{a})F((\mathbf{r}' - \mathbf{a})^2, \varphi)$ , which, correctly combined, avoid the use of fractional powers of  $(\mathbf{r}' - \mathbf{a})^2$  and of special functions. Let us illustrate it by some basic results.

#### 3.3.1) First basic example

Let us show how to reduce to a contour integral, for arbitrary integer  $n + 1 \geq 0$ ,

$$I_{n+1} = \int_{S'} \prod_{j=1}^{n+1} (\mathbf{d}_j \cdot (\mathbf{r}' - \mathbf{a})) F_0((\mathbf{r}' - \mathbf{a})^2) dS', \quad (29)$$

where the  $\mathbf{d}_j$  are constant vectors. We now state, from theorem 1, a specific proposition for the even case  $n + 1 = 2q$ , then, from the addition of theorem 2, a general proposition for arbitrary integer  $n$ . We consider in what follows, that we are in an open connected domain for the variable  $\mathbf{a}$  including a point of  $S'$ , where  $F_0((\mathbf{r}' - \mathbf{a})^2)$  remains analytical on  $S'$ , which allows the application of theorems 1 and 2.

#### Proposition 4

*We have, from the theorem 1, for  $q \geq 0$ ,*

$$\begin{aligned} I_{2q} &= \int_{S'} \prod_{j=1}^{2q} (\mathbf{d}_j \cdot (\mathbf{r}' - \mathbf{a})) F_0((\mathbf{r}' - \mathbf{a})^2) dS' \\ &= \frac{1}{2} \int_{\mathcal{L}} G(\varphi) (B_{2q}((\mathbf{r}' - \mathbf{a})^2) - B_{2q}(0)) \left[ (\hat{\mathbf{z}} \times \frac{\mathbf{r}' - \mathbf{a}}{(\mathbf{r}' - \mathbf{a})^2}) \cdot d\mathbf{r}' \right] \end{aligned} \quad (30)$$

where the  $\mathbf{d}_j$  are constant, the case  $q = 0$  is considered with  $\prod_{j=1}^0(\dots) \equiv 1$ , and,

$$B_{2q}((\mathbf{r}' - \mathbf{a})^2) = \int_0^{(\mathbf{r}' - \mathbf{a})^2} t^q F_0(t) dt + C, \quad G(\varphi) = \prod_{j=1}^{2q} \left( \frac{\mathbf{d}_j \cdot (\mathbf{r}' - \mathbf{a})}{((\mathbf{r}' - \mathbf{a})^2)^{1/2}} \right). \quad (31)$$

Proof. Letting  $\frac{\partial(F((\mathbf{r}' - \mathbf{a})^2), \varphi)}{\partial((\mathbf{r}' - \mathbf{a})^2)} = G(\varphi)(\mathbf{r}' - \mathbf{a})^{2q} F_0((\mathbf{r}' - \mathbf{a})^2)$  in the theorem 1, we obtain the equality (30), where we notice that  $B_{2q}$  depends on  $q - 1$  successive primitives of  $F_0$ . It is worth noticing that we have,

$$\begin{aligned} B_{2q}((\mathbf{r}' - \mathbf{a})^2) &= \int_0^{(\mathbf{r}' - \mathbf{a})^2} t^q e^{-vt} dt + C \text{ if } F_0((\mathbf{r}' - \mathbf{a})^2) = e^{-v(\mathbf{r}' - \mathbf{a})^2}, \\ B_{2q}((\mathbf{r}' - \mathbf{a})^2) &= 2 \int_{\sqrt{z^2}}^{\tilde{R}} (t^2 - z^2)^q \frac{e^{-vt}}{t^{m-1}} dt + C \text{ if } F_0((\mathbf{r}' - \mathbf{a})^2) = \frac{e^{-v\tilde{R}}}{\tilde{R}^m}, \end{aligned} \quad (32)$$

with  $\tilde{R} = \sqrt{\mathbf{R} \cdot \mathbf{R}} = \sqrt{(\mathbf{r}' - \mathbf{a})^2 + z^2}$ ,  $\mathbf{a}$ ,  $v$ ,  $C$  and  $z$  some constants. In these cases, the function  $B_{2q}$  can be reduced, exactly or asymptotically for large  $v\tilde{R}$ .

A similar method, if used for odd  $n + 1$ , would lead to a contour integral with fractional powers of  $(\mathbf{r}' - \mathbf{a})^2$ . To avoid it, we state a relation that lowers the degree of the polynomial in the surface integral. The repeated use of this relation, in combination of known results for  $n = 0$  (from proposition 2) and  $n + 1 = 0$  (from proposition 4), allows the reduction of  $I_{n+1}$  to a known contour integral for arbitrary  $n + 1 \geq 0$ , without any use of fractional powers.

#### Proposition 5

We have, for  $n \geq 0$ ,

$$\begin{aligned} I_{n+1} &= \int_{S'} \mathbf{d}_{n+1} \cdot (\mathbf{r}' - \mathbf{a}) \prod_{j=1}^n (\mathbf{d}_j \cdot (\mathbf{r}' - \mathbf{a})) F_0((\mathbf{r}' - \mathbf{a})^2) dS' \\ &= \frac{1}{2} \int_{\mathcal{L}} \prod_{j=1}^n (\mathbf{d}_j \cdot (\mathbf{r}' - \mathbf{a})) B_0((\mathbf{r}' - \mathbf{a})^2) (\hat{\mathbf{z}} \times \mathbf{d}_{n+1}) \cdot d\mathbf{r}' \\ &\quad - \frac{1}{2} \sum_{m=1}^n (\mathbf{d}_{n+1} \cdot \mathbf{d}_m) \int_{S'_{j \neq m}} \prod_{j=1}^n (\mathbf{d}_j \cdot (\mathbf{r}' - \mathbf{a})) B_0((\mathbf{r}' - \mathbf{a})^2) dS' \end{aligned} \quad (33)$$

where the  $\mathbf{d}_j$ , numbered from 1 to  $n$ , are arbitrary constant vectors,  $B_0$  is a primitive of  $F_0$ ,  $\prod_j^0(\dots) = 1$ ,  $\prod_{j \neq m}^1(\dots) = 1$ , and  $\prod_{j \neq m}^0(\dots) = 0$ . If  $n + 1$  is odd, or  $n + 1$  is even, this equation allows,

by repeated applications, to reduce  $I_{n+1}$  to a combination of contour integral, depending on  $\lfloor (n + 3)/2 \rfloor$  successive primitives of  $F_0$ .

Proof. Using the proposition 3 with the expression (21) of the gradient, we derive (0) of the proposition. The case  $n = 0$  corresponds to proposition 2. The equation (0), after repeated applications, allows the reduction of the polynomial rank, to the rank 1 for odd orders (resp. 0 for even orders) expressed in proposition 2 (resp. in proposition 4). This result depends on a set of  $\lfloor (n + 1)/2 \rfloor + 1$  (i.e.  $\lfloor (n + 3)/2 \rfloor$ ) successive primitives  $F_0^{-q}$  of  $F_0$ , which can be expressed  $F_0^{-q}(x) = \frac{1}{q!} \int^x (x - t)^q F_0(t) dt$ .

A generalization of the proposition is derived in chapter 7, by considering a function of  $\mathcal{R}^2$  instead of  $(\mathbf{r}' - \mathbf{a})^2$ , where  $\mathcal{R}^2$  is an arbitrary polynomial of  $x'$  and  $y'$  of rank 2.

Remark 4 :

The reduction to a contour integral is not unique, and, when we use the proposition 5, the order of the  $\mathbf{d}_j$  could have an influence on the final reduction.

### 3.3.2) Second basic example

The reduction to a contour integral generally supposes the knowledge of a primitive. When it cannot be expressed in closed form, its expression can often be derived asymptotically.

Proposition 6

Let  $F_1((\mathbf{r}' - \mathbf{a})^2) = \frac{e^{-v\tilde{\mathbf{R}}}}{\tilde{\mathbf{R}}^n}$  be an analytic function on  $S'$ , with  $n, v$ , and  $z \neq 0$  some constants,  $\tilde{\mathbf{R}} = \sqrt{\mathbf{R} \cdot \mathbf{R}} = \sqrt{(\mathbf{r}' - \mathbf{a})^2 + z^2}$ . For large  $v\tilde{\mathbf{R}}$  on  $S'$ , we can write,

$$\begin{aligned} & \int_{S'} h\left(\frac{z}{\tilde{\mathbf{R}}}, \varphi\right) \frac{\partial(F_1((\mathbf{r}' - \mathbf{a})^2))}{\partial((\mathbf{r}' - \mathbf{a})^2)} dS' \\ & \sim \frac{1}{2} \int_{\mathcal{L}} \left( h\left(\frac{z}{\tilde{\mathbf{R}}}, \varphi\right) F_1((\mathbf{r}' - \mathbf{a})^2) \left( 1 - \frac{1}{v\tilde{\mathbf{R}}} \left( \frac{z}{\tilde{\mathbf{R}}} \frac{h'}{h} \left( \frac{z}{\tilde{\mathbf{R}}}, \varphi \right) \right) \right) \right. \\ & \left. - F_1(0) \left( h(1, \varphi) - \frac{h'(1, \varphi)}{vz} \right) \right) (\hat{\mathbf{z}} \times \frac{\mathbf{r}' - \mathbf{a}}{(\mathbf{r}' - \mathbf{a})^2}) \cdot d\mathbf{r}' \end{aligned} \quad (34)$$

where  $h'(w, \varphi) = \partial_w h(w, \varphi)$  is bounded on  $S'$ , and  $\frac{\partial(F((\mathbf{r}' - \mathbf{a})^2, \varphi))}{\partial((\mathbf{r}' - \mathbf{a})^2)} \equiv h\left(\frac{z}{\tilde{\mathbf{R}}}, \varphi\right) \frac{\partial(F_1((\mathbf{r}' - \mathbf{a})^2))}{\partial((\mathbf{r}' - \mathbf{a})^2)}$  allows the application of theorem 1 on  $S'$ . Therefore, we have,

$$\begin{aligned} & \int_{S'} h\left(\frac{z}{\tilde{R}}, \varphi\right) \frac{e^{-v\tilde{R}}}{\tilde{R}^{n+1}} dS' \\ & \sim \frac{-1}{v} \int_{\mathcal{L}} \left( h\left(\frac{z}{\tilde{R}}, \varphi\right) \frac{e^{-v\tilde{R}}}{\tilde{R}^n} - h(1, \varphi) \frac{e^{-vz}}{z^n} \right) (\hat{\mathbf{z}} \times \frac{\mathbf{r}' - \mathbf{a}}{(\mathbf{r}' - \mathbf{a})^2}) \cdot d\mathbf{r}' \end{aligned} \quad (35)$$

where  $z/\tilde{R} = \cos\theta$ ,  $\theta$  is the angle of  $\mathbf{R} = z\hat{\mathbf{z}} + \mathbf{a} - \mathbf{r}'$  with  $\hat{\mathbf{z}}$ .

Proof. Let us consider the theorem 1, when

$$\frac{\partial(F((\mathbf{r}' - \mathbf{a})^2, \varphi))}{\partial((\mathbf{r}' - \mathbf{a})^2)} = h\left(\frac{z}{\tilde{R}}, \varphi\right) \frac{\partial(F_1((\mathbf{r}' - \mathbf{a})^2))}{\partial((\mathbf{r}' - \mathbf{a})^2)}, \quad (36)$$

with  $\frac{z}{\tilde{R}}$  and  $h'(\frac{z}{\tilde{R}}, \varphi)$  regular and bounded. In this case, we can generally obtained an asymptotic expression of  $F$  if  $F_1$  has steep variation for large  $\tilde{R}$ . For this, we notice that,

$$\begin{aligned} & \int \frac{\partial(F((\mathbf{r}' - \mathbf{a})^2, \varphi))}{\partial((\mathbf{r}' - \mathbf{a})^2)} d(\mathbf{r}' - \mathbf{a})^2 = \\ & = h\left(\frac{z}{\tilde{R}}, \varphi\right) F_1((\mathbf{r}' - \mathbf{a})^2) + \int \frac{F_1((\mathbf{r}' - \mathbf{a})^2)}{2\tilde{R}^2} \left( \frac{zh'(\frac{z}{\tilde{R}}, \varphi)}{\tilde{R}} \right) d(\mathbf{r}' - \mathbf{a})^2 \\ & = h\left(\frac{z}{\tilde{R}}, \varphi\right) F_1((\mathbf{r}' - \mathbf{a})^2) + \left( \frac{zh'(\frac{z}{\tilde{R}}, \varphi)}{2\tilde{R}^3} \right) \int F_1((\mathbf{r}' - \mathbf{a})^2) d(\mathbf{r}' - \mathbf{a})^2 + \dots \end{aligned} \quad (37)$$

with  $h'(w, \varphi) = \partial_w h(w, \varphi)$ . Taking  $F_1((\mathbf{r}' - \mathbf{a})^2) = \frac{e^{-v\tilde{R}}}{\tilde{R}^n}$ , which satisfies

$$\int F_1((\mathbf{r}' - \mathbf{a})^2) d(\mathbf{r}' - \mathbf{a})^2 = \int \frac{2e^{-v\tilde{R}}}{\tilde{R}^{n-1}} d\tilde{R} \sim -\frac{2\tilde{R}}{v} F_1((\mathbf{r}' - \mathbf{a})^2), \quad (38)$$

we derive (34) with the asymptotic expression of primitive,

$$\int \frac{\partial(F((\mathbf{r}' - \mathbf{a})^2))}{\partial((\mathbf{r}' - \mathbf{a})^2)} d(\mathbf{r}' - \mathbf{a})^2 \sim h\left(\frac{z}{\tilde{R}}, \varphi\right) F_1((\mathbf{r}' - \mathbf{a})^2) \left( 1 - \frac{1}{v\tilde{R}} \left( \frac{z}{\tilde{R}} \frac{h'}{h} \left( \frac{z}{\tilde{R}}, \varphi \right) \right) \right) \quad (39)$$

in theorem 1, then obtain equation (35) from  $\frac{\partial(F_1((\mathbf{r}' - \mathbf{a})^2))}{\partial(\mathbf{R}^2)} = -\frac{v\tilde{R}}{2} \frac{F_1((\mathbf{r}' - \mathbf{a})^2)}{\mathbf{R}^2} \left( 1 + \frac{n}{v\tilde{R}} \right)$ .

### 3.3.3) Third basic example

Proposition 7

Let  $F(w, \varphi) = F_0(w)h\left(\frac{w^q}{\mathbf{R}^{2q}}G(\varphi)\right)$  be analytical relatively to  $((\mathbf{r}' - \mathbf{a})^2)^{\frac{1}{2}}\cos\varphi = (x' - a_x)$  and  $((\mathbf{r}' - \mathbf{a})^2)^{\frac{1}{2}}\sin\varphi = (y' - a_y)$  on  $S'$ , with  $\tilde{R} = \sqrt{\mathbf{R}^2} = \sqrt{(\mathbf{r}' - \mathbf{a})^2 + z^2}$ ,  $F_0(w) = \frac{e^{-v\tilde{R}}}{\tilde{R}^{n-1}}$ , and  $v, \mathbf{b}, z \neq 0$  some constants, we have,

$$\begin{aligned} & \int_{S'} \mathbf{b} \cdot (\mathbf{r}' - \mathbf{a}) h \left( \frac{(\mathbf{r}' - \mathbf{a})^{2q}}{\mathbf{R}^{2q}} G(\varphi) \right) \frac{\partial F_0((\mathbf{r}' - \mathbf{a})^2)}{\partial((\mathbf{r}' - \mathbf{a})^2)} dS' \\ & \sim \frac{1}{2} \int_{\mathcal{L}} F((\mathbf{r}' - \mathbf{a})^2, \varphi) (\hat{\mathbf{z}} \times \mathbf{b}) \cdot d\mathbf{r}', \end{aligned} \quad (40)$$

for large  $v\tilde{R}$  on  $S'$ , with  $n$  and  $q \geq 1$ . Generalizing this result to a vector function  $h_v$  with  $F_v(w, \varphi) = F_0(w)h_v(\frac{w^q}{\mathbf{R}^{2q}}G(\varphi))$ , we have,

$$\begin{aligned} & \int_{S'} (\mathbf{r}' - \mathbf{a}) \cdot h_v \left( \frac{(\mathbf{r}' - \mathbf{a})^{2q}}{\mathbf{R}^{2q}} G(\varphi) \right) \frac{\partial F_0((\mathbf{r}' - \mathbf{a})^2)}{\partial((\mathbf{r}' - \mathbf{a})^2)} dS' \\ & \sim \frac{1}{2} \int_{\mathcal{L}} (\hat{\mathbf{z}} \times F_v((\mathbf{r}' - \mathbf{a})^2, \varphi)) \cdot d\mathbf{r}', \end{aligned} \quad (41)$$

Proof. Following the hypotheses, we have

$$\begin{aligned} & h \left( \frac{(\mathbf{r}' - \mathbf{a})^{2q}}{\mathbf{R}^{2q}} G(\varphi) \right) \text{grad}_{\mathbf{r}'} (F_0((\mathbf{r}' - \mathbf{a})^2)) = \\ & = \text{grad}_{\mathbf{r}'} (F((\mathbf{r}' - \mathbf{a})^2, \varphi)) + O\left(\frac{F_0((\mathbf{r}' - \mathbf{a})^2)}{\tilde{R}}\right), \end{aligned} \quad (42)$$

where we have used that,

$$\begin{aligned} & \frac{\partial}{\partial((\mathbf{r}' - \mathbf{a})^2)} (F_0((\mathbf{r}' - \mathbf{a})^2)) = \left( \frac{-vF_0((\mathbf{r}' - \mathbf{a})^2)}{2\tilde{R}} \right) \left( 1 + \frac{n}{v\tilde{R}} \right), \\ & \frac{\partial}{\partial((\mathbf{r}' - \mathbf{a})^2)} \left( h \left( \frac{(\mathbf{r}' - \mathbf{a})^{2q}}{\mathbf{R}^{2q}} G(\varphi) \right) \right) = \frac{1}{\mathbf{R}^2} \left( \frac{z^2(\mathbf{r}' - \mathbf{a})^{2q-2}}{\mathbf{R}^{2q}} G(\varphi) h' \left( \frac{(\mathbf{r}' - \mathbf{a})^{2q}}{\mathbf{R}^{2q}} G(\varphi) \right) \right), \\ & \frac{1}{(\mathbf{r}' - \mathbf{a})^2} \frac{\partial}{\partial\varphi} \left( h \left( \frac{(\mathbf{r}' - \mathbf{a})^{2q}}{\mathbf{R}^{2q}} G(\varphi) \right) \right) = \frac{1}{\mathbf{R}^2} \left( \frac{(\mathbf{r}' - \mathbf{a})^{2q-2}}{\mathbf{R}^{2q-2}} G'(\varphi) h' \left( \frac{(\mathbf{r}' - \mathbf{a})^{2q}}{\mathbf{R}^{2q}} G(\varphi) \right) \right), \end{aligned} \quad (43)$$

We can then derive (40) from theorem 2, and use it for each component of  $h_v$  to derive (41).

#### **4) Physical Optics diffracted fields in electromagnetism for a dipolar illumination in monostatic case, and primitives for an exact reduction**

We consider here the reduction to non-singular contour integrals of the Physical Optics diffracted fields for a dipolar illumination in monostatic, from our theoretical results based on theorem 1 and 2. For this, some particular rewriting of the PO radiation integral are developed, and analytical expressions of primitives involved in the reduction are given. Non dipolar illumination will be considered in chapter 5.



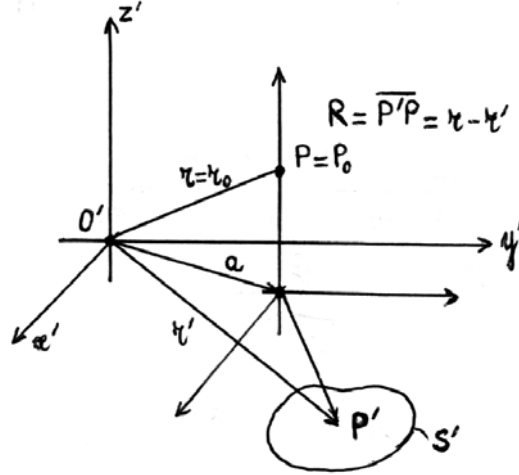


figure 2 : diffraction by  $S'$  illuminated by source at  $\mathbf{r}_0 = \mathbf{r}$  (monostatic case)

Our method allows to study the PO diffraction for perfectly and imperfectly conducting surfaces. To simplify the reading, we choose to describe the latter case (surface with impedance boundary conditions) in appendice.

#### 4.1) A particular writing of surface integrals for the application of our method

##### 4.1.1) Basic expressions

Let us consider the expressions of the electric and magnetic fields,  $\mathbf{E}$  and  $\mathbf{H}$ , radiated by electric and magnetic currents  $\mathbf{J}$  and  $\mathbf{M}$  impressed on a real surface  $S$ , in free space with impedance  $Z_0$  and wavenumber  $k = \frac{2\pi}{\lambda}$ , at the observation point  $\mathbf{r} = \mathbf{a} + z\hat{\mathbf{z}} \notin S$  [1], following,

$$\begin{aligned} \mathbf{E} &= \frac{Z_0}{4\pi ik} \int_S (\text{grad}_r \text{div}_r + k^2)(\psi \mathbf{J}) dS' - \frac{1}{4\pi} \int_S \text{rot}_r(\psi \mathbf{M}) dS', \\ Z_0 \mathbf{H} &= \frac{Z_0}{4\pi} \int_S \text{rot}_r(\psi \mathbf{J}) dS' + \frac{1}{4\pi ik} \int_S (\text{grad}_r \text{div}_r + k^2)(\psi \mathbf{M}) dS', \\ \psi(\tilde{\mathbf{R}}) &= \frac{e^{-ik\tilde{\mathbf{R}}}}{\tilde{\mathbf{R}}}, \mathbf{R} = \mathbf{a} + z\hat{\mathbf{z}} - (\mathbf{r}' + z'\hat{\mathbf{z}}) = \tilde{\mathbf{R}} \mathbf{u}, \tilde{\mathbf{R}} = \sqrt{\mathbf{R} \cdot \mathbf{R}}, \end{aligned} \quad (44)$$

where  $(\mathbf{r}' + z'\hat{\mathbf{z}})$  is the point of integration on  $S$ , and  $(\Delta + k^2)\psi|_{\tilde{\mathbf{R}} \neq 0} = 0$ . We can write,

$$\begin{aligned} (\text{grad}_r \text{div}_r + k^2)(\psi \mathbf{J}) &= \tilde{\mathbf{R}} \left( \frac{\psi'}{\tilde{\mathbf{R}}} \right)' (\mathbf{J} \cdot \mathbf{u}) \mathbf{u} + \frac{\psi'}{\tilde{\mathbf{R}}} \mathbf{J} + k^2 \psi \mathbf{J}, \\ \text{rot}_r(\psi \mathbf{J}) &= -\mathbf{J} \times \text{grad}_r \psi = \frac{\psi'}{\tilde{\mathbf{R}}} \mathbf{R} \times \mathbf{J}, \end{aligned} \quad (45)$$

where  $\psi' = \partial_{\tilde{R}}\psi$  and  $(\frac{\psi'}{\tilde{R}})' = \partial_{\tilde{R}}(\frac{\psi'}{\tilde{R}})$ ; from  $\Delta\psi = \frac{1}{\tilde{R}^2}((\tilde{R})^3\frac{\psi'}{\tilde{R}})$ , and  $\tilde{R}(\frac{\psi'}{\tilde{R}})' + k^2\psi = -3\frac{\psi'}{\tilde{R}}$  with  $(\mathbf{J} \cdot \mathbf{u})\mathbf{u} = \mathbf{u} \times (\mathbf{u} \times \mathbf{J}) + \mathbf{J}$ , we derive,

$$(\text{grad}_{\mathbf{r}} \text{div}_{\mathbf{r}} + k^2)(\psi \mathbf{J}) = \tilde{R} \left( \frac{\psi'}{\tilde{R}} \right)' \mathbf{u} \times (\mathbf{u} \times \mathbf{J}) - \frac{2\psi'}{\tilde{R}} \mathbf{J}. \quad (46)$$

This gives us, for  $\mathbf{r} = \mathbf{a} + z \hat{\mathbf{z}} \notin S$ ,

$$\begin{aligned} \mathbf{E} &= \frac{Z_0}{4\pi i k} \int_S \left( \tilde{R} \left( \frac{\psi'}{\tilde{R}} \right)' \mathbf{u} \times (\mathbf{u} \times \mathbf{J}) - \frac{2\psi'}{\tilde{R}} \mathbf{J} \right) dS' - \frac{1}{4\pi} \int_S \frac{\psi'}{\tilde{R}} \mathbf{R} \times \mathbf{M} dS', \\ Z_0 \mathbf{H} &= \frac{Z_0}{4\pi} \int_S \frac{\psi'}{\tilde{R}} \mathbf{R} \times \mathbf{J} dS' + \frac{1}{4\pi i k} \int_S \left( \tilde{R} \left( \frac{\psi'}{\tilde{R}} \right)' \mathbf{u} \times (\mathbf{u} \times \mathbf{M}) - \frac{2\psi'}{\tilde{R}} \mathbf{M} \right) dS'. \end{aligned} \quad (47)$$

We will study, in this chapter, the case where  $S$  is equal to its projection  $S'$  on the plane  $z = 0$ , and in chapter 7, a generalization for the case of  $S$  with low curvature, where  $(z' - z)^2$  can be approximated by a quadratic polynomial of the coordinates of  $\mathbf{r}'$ .

#### 4.1.2) Physical Optics expressions in monostatic perfectly conducting case ( $\mathbf{M}|_S = 0$ )

In monostatic case, the position of the point source that illuminates the scatterer is the same than the one of the observation point. We show here that the Physical Optics radiation integrals can be then reworked in such a form that they can be exactly reduced to contour integrals, with all required primitives in closed forms and no need of special functions.

##### Proposition 8

*Let us consider a surface integral on  $S'$ , combination of integrals of the type,*

$$\int_{S'} B(\mathbf{R}^2) dS', \int_{S'} \frac{B(\mathbf{R}^2)(\mathbf{R}^2 - z^2)}{\mathbf{R}^2} G(\varphi) dS' \text{ and } \int_{S'} (\mathbf{a} - \mathbf{r}') B(\mathbf{R}^2) dS', \quad (48)$$

*where  $\mathbf{R}^2 = (\mathbf{r}' - \mathbf{a})^2 + z^2$ ,  $G(\varphi) = \frac{((\mathbf{r}' - \mathbf{a}) \cdot \hat{\mathbf{d}})(\mathbf{r}' - \mathbf{a})}{(\mathbf{r}' - \mathbf{a})^2}$ , with  $z$  and  $\hat{\mathbf{d}}$  some constants. Assuming that  $\mathbf{a}$  belongs to an open connected domain of points which includes a point of  $S'$ , where  $B(\mathbf{R}^2)$  is analytical on  $S'$ , we can, with analytical expressions of primitives of the  $B(t)$  and  $B(t)/t$ , reduce the integral to a non-singular contour integral, by using the proposition 4 (derived from theorem 1), and the proposition 2 (derived from theorem 2).*

*Proof.* To reduce the integrals to contour integrals, we let successively  $\frac{\partial(F((\mathbf{r}' - \mathbf{a})^2, \varphi))}{\partial((\mathbf{r}' - \mathbf{a})^2)}$  as  $B(\mathbf{R}^2)$  and  $\frac{B(\mathbf{R}^2)(\mathbf{R}^2 - z^2)}{\mathbf{R}^2} G(\varphi)$  in proposition 4, involving the primitives of  $B(t)$  and  $B(t)/t$ , and  $\frac{\partial(F((\mathbf{r}' - \mathbf{a})^2))}{\partial((\mathbf{r}' - \mathbf{a})^2)}$  as  $B(\mathbf{R}^2)$  in proposition 2, involving the primitive of  $B(t)$ .

4.1.2.1) Physical Optics diffracted fields and reduction, for an electric dipolar source

Theorem 3

The Physical Optics diffracted field, radiated by a perfectly conducting surface  $S'$  of the plane  $z' = 0$ , when illuminated by a electric dipole  $\mathbf{J}_0 = \widehat{\mathbf{d}}\delta(\mathbf{r}' - \mathbf{r}_0)$ , can be written, at  $\mathbf{r} = z\widehat{\mathbf{z}} + \mathbf{a} \notin S'$ , when  $\mathbf{r}_0 = \mathbf{r}$  (monostatic case), following,

$$\begin{aligned} \mathbf{E} &= \frac{-Z_0\widehat{\mathbf{d}}}{16\pi^2ik} \int_{S'} z\widetilde{\mathbf{R}} \left(\left(\frac{\psi'}{\widetilde{\mathbf{R}}}\right)'\right)^2 dS' + \frac{Z_0}{16\pi^2ik} \int_{S'} z\widetilde{\mathbf{R}} \left(\left(\frac{\psi'}{\widetilde{\mathbf{R}}}\right)'\right)^2 (\mathbf{u}\cdot\widehat{\mathbf{d}})\mathbf{u} dS' + \\ &+ \frac{Z_0(\widehat{\mathbf{z}}\cdot\widehat{\mathbf{d}})}{4\pi^2ik} \int_{S'} \left(\frac{\psi'}{\widetilde{\mathbf{R}}}\right)^2 (\mathbf{a} - \mathbf{r}') dS' + \frac{Z_0z((\widehat{\mathbf{z}}\cdot\widehat{\mathbf{d}})\widehat{\mathbf{z}} - \widehat{\mathbf{d}})}{4\pi^2ik} \int_{S'} \left(\frac{\psi'}{\widetilde{\mathbf{R}}}\right)^2 dS', \\ \mathbf{H} &= \frac{-\widehat{\mathbf{d}}}{8\pi^2} \times \int_{S'} z\mathbf{R} \left(\frac{\psi'}{\widetilde{\mathbf{R}}}\right)^2 dS', \end{aligned} \quad (49)$$

with,

$$(\mathbf{u}\cdot\widehat{\mathbf{d}})\mathbf{u} = (\widehat{\mathbf{d}}\cdot\widehat{\mathbf{z}})\widehat{\mathbf{z}} \frac{z^2}{\mathbf{R}^2} + ((\widehat{\mathbf{d}}\cdot\widehat{\mathbf{z}}) + (\widehat{\mathbf{d}}\cdot(\cdot))\widehat{\mathbf{z}}) \frac{z(\mathbf{a} - \mathbf{r}')}{\mathbf{R}^2} + \frac{\mathbf{R}^2 - z^2}{\mathbf{R}^2} \frac{((\mathbf{r}' - \mathbf{a})\cdot\widehat{\mathbf{d}})(\mathbf{r}' - \mathbf{a})}{(\mathbf{r}' - \mathbf{a})^2} \quad (50)$$

and  $\mathbf{R}^2 - z^2 = (\mathbf{r}' - \mathbf{a})^2$ . The proposition 8 applies, and we can exactly reduce every integral to non-singular contour integral whose integrand involves no special function.

Proof. We can write the Physical Optics field with (47), considering that  $\mathbf{J} = 2\widehat{\mathbf{z}} \times \mathbf{H}_i$  and  $\mathbf{M} = 0$  on  $S'$ , where  $\mathbf{H}_i$  is radiated by the source  $\mathbf{J}_0 = \widehat{\mathbf{d}}\delta(\mathbf{r}' - \mathbf{r}_0)$ , and thus

$$2\pi\mathbf{J} = \widehat{\mathbf{z}} \times (\mathbf{R}_0 \times \frac{\psi'_0}{\widetilde{\mathbf{R}}_0} \widehat{\mathbf{d}}) = (\widehat{\mathbf{z}}\cdot\widehat{\mathbf{d}}) \frac{\psi'_0}{\widetilde{\mathbf{R}}_0} \mathbf{R}_0 + z_0 \frac{\psi'_0}{\widetilde{\mathbf{R}}_0} \widehat{\mathbf{d}}, \quad \mathbf{R}_0 = (\mathbf{r}' - \mathbf{r}_0) \quad (51)$$

In the monostatic case, when  $\mathbf{r} = \mathbf{r}_0 = z\widehat{\mathbf{z}} + \mathbf{a} \notin S'$  and  $\mathbf{R} = -\mathbf{R}_0 = \widetilde{\mathbf{R}} \mathbf{u}$ , we can write,

$$\begin{aligned} \mathbf{E} &= \frac{Z_0}{8\pi^2ik} \int_{S'} z\widetilde{\mathbf{R}} \left(\frac{\psi'}{\widetilde{\mathbf{R}}}\right)' \mathbf{u} \times (\mathbf{u} \times \frac{\psi'}{\widetilde{\mathbf{R}}} \widehat{\mathbf{d}}) + 2\left(\frac{\psi'}{\widetilde{\mathbf{R}}}\right)^2 ((\widehat{\mathbf{z}}\cdot\widehat{\mathbf{d}})\mathbf{R} - z\widehat{\mathbf{d}}) dS', \\ &= \frac{-Z_0\widehat{\mathbf{d}}}{16\pi^2ik} \int_{S'} z\widetilde{\mathbf{R}} \left(\left(\frac{\psi'}{\widetilde{\mathbf{R}}}\right)'\right)^2 dS' + \frac{Z_0}{16\pi^2ik} \int_{S'} z\widetilde{\mathbf{R}} \left(\left(\frac{\psi'}{\widetilde{\mathbf{R}}}\right)'\right)^2 (\mathbf{u}\cdot\widehat{\mathbf{d}})\mathbf{u} dS' + \\ &+ \frac{Z_0(\widehat{\mathbf{z}}\cdot\widehat{\mathbf{d}})}{4\pi^2ik} \int_{S'} \left(\frac{\psi'}{\widetilde{\mathbf{R}}}\right)^2 (\mathbf{a} - \mathbf{r}') dS' + \frac{Z_0((\widehat{\mathbf{z}}\cdot\widehat{\mathbf{d}})\widehat{\mathbf{z}} - \widehat{\mathbf{d}})}{4\pi^2ik} \int_{S'} z\left(\frac{\psi'}{\widetilde{\mathbf{R}}}\right)^2 dS', \end{aligned} \quad (52)$$

and

$$\mathbf{H} = \frac{-1}{8\pi^2} \int_{S'} \left(\frac{\psi'}{\tilde{R}}\right)^2 \mathbf{R} \times (\hat{\mathbf{z}} \times (\mathbf{R} \times \hat{\mathbf{d}})) dS' = \frac{-\hat{\mathbf{d}}}{8\pi^2} \times \int_{S'} z \mathbf{R} \left(\frac{\psi'}{\tilde{R}}\right)^2 dS', \quad (53)$$

and thus (49) with (50). The proposition 8 then applies, and we can reduce the expressions of  $\mathbf{E}$  and  $\mathbf{H}$  to contour integrals, by using the following exact primitives,

$$\begin{aligned} \int \frac{1}{\tilde{R}} \left(\left(\frac{\psi'}{\tilde{R}}\right)^2\right)' d\mathbf{R}^2 &= 2 \frac{(ik + \frac{1}{\tilde{R}})^2}{\mathbf{R}^4} e^{-i2k\tilde{R}}, \quad \int \left(\frac{\psi'}{\tilde{R}}\right)^2 d\mathbf{R}^2 = -\frac{(1 + 2ik\tilde{R})}{2\mathbf{R}^4} e^{-i2k\tilde{R}}, \\ \int \tilde{R} \left(\left(\frac{\psi'}{\tilde{R}}\right)^2\right)' d\mathbf{R}^2 &= 2 \left(-k^2 + \frac{3ik}{\tilde{R}} + \frac{3}{2\mathbf{R}^2}\right) \frac{e^{-i2k\tilde{R}}}{\mathbf{R}^2}, \end{aligned} \quad (54)$$

derived in appendice A, which involve no special function. Let us notice that, for large  $k\tilde{R}$ ,

$$\mathbf{E} \sim \frac{Z_0}{8\pi^2 ik} \int_{S'} \frac{z\tilde{R}}{2} \left(\left(\frac{\psi'}{\tilde{R}}\right)^2\right)' ((\mathbf{u} \cdot \hat{\mathbf{d}}) \mathbf{u} - \hat{\mathbf{d}}) dS', \quad \mathbf{H} \sim \frac{-\hat{\mathbf{d}}}{8\pi^2} \times \int_{S'} z \mathbf{R} \left(\frac{\psi'}{\tilde{R}}\right)^2 dS'. \quad (55)$$

#### 4.1.2.2) Physical Optics diffracted fields and reduction, for a magnetic dipolar source

##### Theorem 4

*The Physical Optics diffracted field, radiated by a perfectly conducting surface  $S'$  of the plane  $z' = 0$ , when illuminated by a magnetic dipole  $\mathbf{M}_0 = \hat{\mathbf{d}}\delta(\mathbf{r}' - \mathbf{r}_0)$ , can be written, at  $\mathbf{r} = z\hat{\mathbf{z}} + \mathbf{a} \notin S'$ , when  $\mathbf{r}_0 = \mathbf{r}$  (monostatic case), following,*

$$\begin{aligned} \mathbf{E} &= \frac{1}{8(\pi ik)^2} \int_{S'} z \left(\frac{\psi'}{\tilde{R}}\right)' \left(\frac{\psi'}{\tilde{R}}\right)' (\mathbf{R} \times \hat{\mathbf{d}}) dS' - \\ &\quad - \frac{1}{4(\pi ik)^2} \int_{S'} \hat{\mathbf{z}} \times (\tilde{R} \left(\left(\frac{\psi'}{\tilde{R}}\right)^2\right)' (\mathbf{u}(\mathbf{u} \cdot \hat{\mathbf{d}}) - \hat{\mathbf{d}}) - 2\left(\frac{\psi'}{\tilde{R}}\right)^2 \hat{\mathbf{d}}) dS', \\ \mathbf{H} &= \frac{1}{8\pi^2 ik Z_0} \int_{S'} \frac{z\tilde{R}}{2} \left(\left(\frac{\psi'}{\tilde{R}}\right)^2\right)' (\hat{\mathbf{d}} - \mathbf{u}(\mathbf{u} \cdot \hat{\mathbf{d}})) - 2\left(\frac{\psi'}{\tilde{R}}\right)^2 \mathbf{R} \times \hat{\mathbf{d}} dS', \end{aligned} \quad (56)$$

with (50). The proposition 8 applies, and we can exactly reduce every integral to non-singular contour integral whose integrand involves no special function.

Proof. We can write the Physical Optics field with (47), considering that the sources  $\mathbf{J} = 2\hat{\mathbf{z}} \times \mathbf{H}_i$  and  $\mathbf{M} = 0$  on  $S'$ , where  $\mathbf{H}_i$  is radiated by  $\mathbf{M}_0 = \hat{\mathbf{d}}\delta(\mathbf{r}' - \mathbf{r}_0)$ , and thus,

$$\mathbf{J} = \frac{1}{2\pi ik Z_0} \hat{\mathbf{z}} \times \left(\tilde{R}_0 \left(\frac{\psi'_0}{\tilde{R}_0}\right)' \mathbf{u}_0 \times (\mathbf{u}_0 \times \hat{\mathbf{d}}) - \frac{2\psi'_0}{\tilde{R}_0} \hat{\mathbf{d}}\right), \quad \mathbf{R}_0 = (\mathbf{r}' - \mathbf{r}_0), \quad \mathbf{u}_0 = \frac{\mathbf{R}_0}{\tilde{R}_0}, \quad (57)$$

In the monostatic case  $\mathbf{R} = -\mathbf{R}_0 = z\hat{\mathbf{z}} + \mathbf{a} - \mathbf{r}'$ , we then obtain,

$$\begin{aligned} \mathbf{E} &= \frac{1}{8(\pi ik)^2} \int_{S'} z \tilde{R} \left( \frac{\psi'}{\tilde{R}} \right)' (\mathbf{u} \times \left( \frac{\psi'}{\tilde{R}} \right)' \hat{\mathbf{d}}) - \tilde{R} \left( \frac{\psi'}{\tilde{R}} \right)' \hat{\mathbf{z}} \times (\mathbf{u} \times (\mathbf{u} \times \frac{2\psi'}{\tilde{R}} \hat{\mathbf{d}})) dS' - \\ &- \frac{1}{8(\pi ik)^2} \int_{S'} \frac{2\psi'}{\tilde{R}} \hat{\mathbf{z}} \times \left( \tilde{R} \left( \frac{\psi'}{\tilde{R}} \right)' \mathbf{u} \times (\mathbf{u} \times \hat{\mathbf{d}}) - \frac{2\psi'}{\tilde{R}} \hat{\mathbf{d}} \right) dS' \\ &= \frac{1}{8(\pi ik)^2} \int_{S'} z \left( \frac{\psi'}{\tilde{R}} \right)' \left( \frac{\psi'}{\tilde{R}} \right)' (\mathbf{R} \times \hat{\mathbf{d}}) dS' - \\ &- \frac{1}{4(\pi ik)^2} \int_{S'} \hat{\mathbf{z}} \times \left( \tilde{R} \left( \left( \frac{\psi'}{\tilde{R}} \right)^2 \right)' (\mathbf{u}(\mathbf{u} \cdot \hat{\mathbf{d}}) - \hat{\mathbf{d}}) - 2 \left( \frac{\psi'}{\tilde{R}} \right)^2 \hat{\mathbf{d}} \right) dS', \end{aligned} \quad (58)$$

and

$$\begin{aligned} \mathbf{H} &= \frac{1}{8\pi^2 ik Z_0} \int_{S'} \frac{\psi'}{\tilde{R}} \mathbf{R} \times (\hat{\mathbf{z}} \times \left( \tilde{R} \left( \frac{\psi'}{\tilde{R}} \right)' \mathbf{u} \times (\mathbf{u} \times \hat{\mathbf{d}}) - \frac{2\psi'}{\tilde{R}} \hat{\mathbf{d}} \right)) dS' \\ &= \frac{1}{8\pi^2 ik Z_0} \int_{S'} z \frac{\tilde{R}}{2} \left( \left( \frac{\psi'}{\tilde{R}} \right)^2 \right)' (\hat{\mathbf{d}} - \mathbf{u}(\mathbf{u} \cdot \hat{\mathbf{d}})) - 2 \left( \frac{\psi'}{\tilde{R}} \right)^2 \mathbf{R} \times \hat{\mathbf{d}} dS'. \end{aligned} \quad (59)$$

and thus, (56) with (50). Considering these expressions, the proposition 8 applies, and we can completely reduce them to contour integrals, by using the primitives given in (54) and,

$$\int \left( \frac{\psi'}{\tilde{R}} \right)' \left( \frac{\psi'}{\tilde{R}} \right)' d\mathbf{R}^2 = ik^3 \frac{\psi^2}{\tilde{R}} + 3 \left( \frac{\psi}{\tilde{R}} \right)^2 \left( \frac{k^2}{2} - \left( ik + \frac{1}{\tilde{R}} \right)^2 \right), \quad (60)$$

derived in appendice A, that involve no special function. Let us notice that, for large  $k\tilde{R}$ ,

$$\mathbf{E} \sim \frac{-\hat{\mathbf{d}}}{8\pi^2 (ik)^2} \times \int_{S'} z \mathbf{R} \left( \left( \frac{\psi'}{\tilde{R}} \right)' \right)^2 dS', \quad \mathbf{H} \sim \frac{1}{8\pi^2 Z_0} \int_{S'} \frac{z \left( \tilde{R} \left( \left( \frac{\psi'}{\tilde{R}} \right)^2 \right)' (\hat{\mathbf{d}} - \mathbf{u}(\mathbf{u} \cdot \hat{\mathbf{d}})) \right)}{2ik} dS' \quad (61)$$

### **5) Contour integrals for Physical Optics diffracted fields, in near-field monostatic case, for a perfectly conducting plate illuminated by a source of arbitrary radiation pattern**

In previous chapter, we consider Physical Optics in monostatic case for electric and magnetic dipolar sources, and we now apply our method for a more complex illumination.

#### **5.1) Initial expressions for a weighted dipolar illumination**

We now consider the incident field, as a combination of radiations for electric and magnetic dipoles  $\mathbf{J}_0 = \hat{\mathbf{d}}_e \delta(\mathbf{r}' - \mathbf{r}_0)$  and  $\mathbf{M}_0 = \hat{\mathbf{d}}_h \delta(\mathbf{r}' - \mathbf{r}_0)$  weighted by functions  $f_{e,h}(\mathbf{u}_0)$  where

$\mathbf{u}_0$  denotes the direction of radiation at large distance of the sources, following,

$$\begin{aligned}
\mathbf{E}_i(\mathbf{r}') &\sim \frac{1}{4\pi ik} (f_e(\mathbf{u}_0) \widetilde{R}_0 \left(\frac{\psi'_0}{\widetilde{R}_0}\right)' (\mathbf{u}_0 \times (\mathbf{u}_0 \times Z_0 \widehat{\mathbf{d}}_e)) - ik f_h(\mathbf{u}_0) \frac{\psi'_0}{\widetilde{R}_0} \mathbf{R}_0 \times \widehat{\mathbf{d}}_h) \\
&\sim \frac{ik}{4\pi} \frac{e^{-ik\widetilde{R}_0}}{\widetilde{R}_0} (f_e(\mathbf{u}_0) (\mathbf{u}_0 \times (\mathbf{u}_0 \times Z_0 \widehat{\mathbf{d}}_e)) + f_h(\mathbf{u}_0) (\mathbf{u}_0 \times \widehat{\mathbf{d}}_h)), \\
\mathbf{H}_i(\mathbf{r}') &\sim \frac{1}{4\pi ik Z_0} (ik f_e(\mathbf{u}_0) \frac{\psi'_0}{\widetilde{R}_0} (\mathbf{R}_0 \times Z_0 \widehat{\mathbf{d}}_e) + f_h(\mathbf{u}_0) \left(\frac{\psi'_0}{\widetilde{R}_0}\right)' \mathbf{R}_0 \times (\mathbf{u}_0 \times \widehat{\mathbf{d}}_h)) \\
&\sim \frac{ik}{4\pi Z_0} \frac{e^{-ik\widetilde{R}_0}}{\widetilde{R}_0} (-f_e(\mathbf{u}_0) (\mathbf{u}_0 \times Z_0 \widehat{\mathbf{d}}_e) + f_h(\mathbf{u}_0) (\mathbf{u}_0 \times (\mathbf{u}_0 \times \widehat{\mathbf{d}}_h))). \tag{62}
\end{aligned}$$

for large  $k\widetilde{R}_0$ , where  $\widetilde{R}_0 = \sqrt{\mathbf{R}_0 \cdot \mathbf{R}_0}$ ,  $\mathbf{R}_0 = \mathbf{r}' - \mathbf{r}_0 = \widetilde{R}_0 \mathbf{u}_0$ . In this case, the near-field monostatic diffracted field expressions at  $\mathbf{r} = \mathbf{r}_0 \notin S'$ , for a perfectly conducting plate  $S'$  illuminated by a source of arbitrary radiation pattern, valid from infinity to a short distance independent of the dimensions of  $S'$ , can be derived from (47) with  $\mathbf{J} = 2\widehat{\mathbf{z}} \times \mathbf{H}_i$  and  $\mathbf{M} = 0$  on  $S'$ , as the combination of (55) and (61) weighted by  $f_{e,h}$ , following,

$$\begin{aligned}
\mathbf{E} &\sim \frac{1}{8\pi^2 (ik)^2} \left( \int_{S'} z f_e(-\mathbf{u}) \frac{ik}{2} \left(\left(\frac{\psi'}{\widetilde{R}}\right)'\right)^2 (\mathbf{R} \times (\mathbf{u} \times Z_0 \widehat{\mathbf{d}}_e)) dS' \right. \\
&\quad \left. + \int_{S'} z f_h(-\mathbf{u}) \left(\left(\frac{\psi'}{\widetilde{R}}\right)'\right)^2 (\mathbf{R} \times \widehat{\mathbf{d}}_h) dS' \right), \\
\mathbf{H} &\sim \frac{1}{8\pi^2 (ik)^2 Z_0} \left( \int_{S'} z f_e(-\mathbf{u}) (ik)^2 \left(\frac{\psi'}{\widetilde{R}}\right)^2 (\mathbf{R} \times Z_0 \widehat{\mathbf{d}}_e) dS' \right. \\
&\quad \left. - \int_{S'} z f_h(-\mathbf{u}) \frac{ik}{2} \left(\left(\frac{\psi'}{\widetilde{R}}\right)'\right)^2 (\mathbf{R} \times (\mathbf{u} \times \widehat{\mathbf{d}}_h)) dS' \right), \tag{63}
\end{aligned}$$

for large  $k\widetilde{R}$ , with  $\mathbf{r} = \mathbf{r}_0 = \mathbf{a} + z\widehat{\mathbf{z}}$ ,  $\widetilde{R} = \sqrt{\mathbf{R} \cdot \mathbf{R}}$ ,  $\mathbf{R} = \widetilde{R} \mathbf{u} = \mathbf{a} + z\widehat{\mathbf{z}} - \mathbf{r}'$ ,  $\mathbf{u} = -\mathbf{u}_0$ . In this frame, let us notice, that, if we want to consider a source with a spherical pattern, instead of a dipole with toric pattern, we can apply the weighting factors  $f_{e,h}(\mathbf{u}_0) = \beta_{e,h} / \sqrt{(\mathbf{u}_0 \times \widehat{\mathbf{d}})^2}$ , with  $(\mathbf{u}_0 \times \widehat{\mathbf{d}})^2 = 1 - (\mathbf{u}_0 \cdot \widehat{\mathbf{d}})^2$ .

## 5.2) Reduction to contour integrals for a weighted dipolar illumination

Thereafter, we reduce  $\mathbf{E}$  in (63) to a contour integral, for  $f_h \equiv 0$  (resp.  $f_e \equiv 0$ ), when  $f_e$  (resp.  $f_h$ ) is even or odd relatively to coordinates of  $\frac{\mathbf{r}' - \mathbf{a}}{R}$ , in the cases where  $\widehat{\mathbf{d}}_{e,h} \cdot \widehat{\mathbf{z}} = 0$ .

### 5.2.1) Electric case with $f_h \equiv 0$ , $\widehat{\mathbf{d}} = \widehat{\mathbf{d}}_e$

Let us consider the case with  $f_h \equiv 0$ ,  $\widehat{\mathbf{d}} = \widehat{\mathbf{d}}_e$ . The plate  $S'$  is then illuminated by,

$$\mathbf{E}_i \sim \frac{ikZ_0}{4\pi} \frac{e^{-ik\tilde{R}_0}}{\tilde{R}_0} f_e(\mathbf{u}_0)(\mathbf{u}_0(\mathbf{u}_0 \cdot \hat{\mathbf{d}}) - \hat{\mathbf{d}}), \quad (64)$$

and the PO diffracted field at  $\mathbf{r} = \mathbf{a} + z\hat{\mathbf{z}} \notin S'$ , when  $\mathbf{r} = \mathbf{r}_0$  (monostatic case), is given by,

$$\mathbf{E} \sim -\frac{Z_0}{8\pi k^2} \frac{i}{\lambda} \int_{S'} z\tilde{R} \left( \left( \frac{\psi'}{\tilde{R}} \right)^2 \right)' f_e(-\mathbf{u}) ((\mathbf{u} \cdot \hat{\mathbf{d}})\mathbf{u} - \hat{\mathbf{d}}) dS'. \quad (65)$$

Theorem 5

The integral (65) reduces, when  $\hat{\mathbf{d}} \cdot \hat{\mathbf{z}} = 0$  and  $f_e(-\mathbf{u}) = \frac{f(\frac{-z}{\tilde{R}}, \frac{\mathbf{r}' - \mathbf{a}}{\tilde{R}}) + f(\frac{-z}{\tilde{R}}, \frac{\mathbf{a} - \mathbf{r}'}{\tilde{R}})}{2} = \frac{g(\frac{z}{\tilde{R}}, \frac{\mathbf{a} - \mathbf{r}'}{\tilde{R}})}{\sqrt{1 - (\mathbf{u} \cdot \hat{\mathbf{d}})^2}}$ , to

$$\begin{aligned} \mathbf{E} \sim & \frac{-i}{\lambda} \left( \frac{1}{2} \int_{\mathcal{L}} (F_1((\mathbf{r}' - \mathbf{a})^2, \varphi) f_e(-\mathbf{u}) - F_1(0, \varphi) g(1, 0)) [(\hat{\mathbf{z}} \times \frac{\mathbf{r}' - \mathbf{a}}{(\mathbf{r}' - \mathbf{a})^2}) \cdot d\mathbf{r}'] \right) + \\ & + \frac{i}{\lambda} \left( \frac{1}{2} \int_{\mathcal{L}} (F_2((\mathbf{r}' - \mathbf{a})^2) f_e(-\mathbf{u}) - F_2(0) g(1, 0)) [(\hat{\mathbf{z}} \times \hat{\mathbf{d}}) \cdot d\mathbf{r}'] \right), \end{aligned} \quad (66)$$

where  $\mathcal{L} \equiv \partial S'$  is oriented counterclockwise, and the primitives  $F_1$  and  $F_2$  are written,

$$\begin{aligned} F_1 &= \frac{Z_0 z}{8\pi k^2} \int \tilde{R} \left( \left( \frac{\psi'}{\tilde{R}} \right)^2 \right)' \left( \frac{(\mathbf{r}' - \mathbf{a})^2 G(\varphi)}{\mathbf{R}^2} - \hat{\mathbf{d}} \right) d\mathbf{R}^2, \quad F_2 = \frac{Z_0 z^2 \hat{\mathbf{z}}}{8\pi k^2} \int \tilde{R} \left( \left( \frac{\psi'}{\tilde{R}} \right)^2 \right)' \frac{1}{\mathbf{R}^2} d\mathbf{R}^2 \\ F_1 &\sim -\frac{Z_0}{4\pi} \frac{ze^{-2ik\tilde{R}}}{\mathbf{R}^2} \left( \frac{(\mathbf{r}' - \mathbf{a})^2 G(\varphi)}{\mathbf{R}^2} - \hat{\mathbf{d}} \right), \quad F_2 \sim -\frac{Z_0}{4\pi} \frac{z^2 \hat{\mathbf{z}} e^{-2ik\tilde{R}}}{\mathbf{R}^4}, \end{aligned} \quad (67)$$

for  $z \neq 0$ , with  $G(\varphi) = \frac{((\mathbf{r}' - \mathbf{a}) \cdot \hat{\mathbf{d}})(\mathbf{r}' - \mathbf{a})}{(\mathbf{r}' - \mathbf{a})^2}$ ,  $F_1(0, \varphi) = \frac{Z_0}{4\pi} \frac{\hat{\mathbf{d}} e^{-2ikz}}{z}$  and  $F_2(0) = -\frac{Z_0}{4\pi} \frac{\hat{\mathbf{z}} e^{-2ikz}}{z^2}$ ,  $f_e(-\mathbf{u})$  regular on  $S'$ . This gives us the VV polarization term  $\mathbf{E} \cdot \hat{\mathbf{d}}$ , when  $\mathbf{r}_0 \cdot \hat{\mathbf{d}} = 0$ .

Proof. We detail (65), and notice that, when  $\hat{\mathbf{d}} \cdot \hat{\mathbf{z}} = 0$ , we have

$$(\mathbf{u} \cdot \hat{\mathbf{d}})\mathbf{u} - \hat{\mathbf{d}} = \left( \frac{(\mathbf{r}' - \mathbf{a})^2 G(\varphi)}{\mathbf{R}^2} - \hat{\mathbf{d}} \right) + \left( \frac{(\mathbf{a} - \mathbf{r}') \cdot \hat{\mathbf{d}}}{\mathbf{R}^2} \right) z\hat{\mathbf{z}}, \quad G(\varphi) = \frac{((\mathbf{r}' - \mathbf{a}) \cdot \hat{\mathbf{d}})(\mathbf{r}' - \mathbf{a})}{(\mathbf{r}' - \mathbf{a})^2} \quad (68)$$

Considering that  $\frac{g(\frac{z}{\tilde{R}}, \frac{\mathbf{a} - \mathbf{r}'}{\tilde{R}})}{\sqrt{1 - (\mathbf{u} \cdot \hat{\mathbf{d}})^2}}$  is a regular even function of the coordinates of  $\frac{\mathbf{a} - \mathbf{r}'}{\tilde{R}}$ , we can use the theorem 1 with the proposition 6 for the part of the integrand with  $\left( \frac{(\mathbf{r}' - \mathbf{a})^2 G(\varphi)}{\mathbf{R}^2} - \hat{\mathbf{d}} \right)$  factor, then the proposition 7 for the part of the integrand with  $(\mathbf{a} - \mathbf{r}') \cdot \hat{\mathbf{d}}$  factor, and we derive (66).

For an illumination with an isotropic pattern, we can choose  $g \equiv 1$ , and write,

$$\mathbf{E} \cdot \hat{\mathbf{d}} \sim -\frac{Z_0}{4\pi} \frac{i}{\lambda} \left( \frac{1}{2} \int_{\mathcal{L}} \frac{\left( \frac{ze^{-2ik\tilde{R}}}{\mathbf{R}^2} \sqrt{1 - \left( \frac{(\mathbf{r}' - \mathbf{a}) \cdot \hat{\mathbf{d}}}{\tilde{R}} \right)^2} - \frac{e^{-2ikz}}{z} \right)}{(\mathbf{r}' - \mathbf{a})^2} [(\hat{\mathbf{z}} \times (\mathbf{r}' - \mathbf{a})) \cdot d\mathbf{r}'] \right), \quad (69)$$

which is to the  $VV$  polarization term in this case when  $\mathbf{r}_0 \cdot \hat{\mathbf{d}} = 0$ .

Let us notice that,  $(\hat{\mathbf{z}} \times (\mathbf{r}' - \mathbf{a})) \cdot \hat{\mathbf{v}}$  is constant on each straight part of  $\mathcal{L}$  along  $\hat{\mathbf{v}}$ , and that, for real  $\mathbf{a} \notin S'$ , the parts of the integrals (74)-(75) with  $\frac{e^{-2ikz}}{z}$  factor vanish.

Theorem 6

The integral (65), when  $\hat{\mathbf{d}} \cdot \hat{\mathbf{z}} = 0$  and  $f_e(-\mathbf{u}) = \frac{f(\frac{z}{\tilde{R}}, \frac{r'-a}{\tilde{R}}) - f(\frac{-z}{\tilde{R}}, \frac{a-r'}{\tilde{R}})}{2} = \frac{(r'-a)}{\tilde{R}} \cdot \frac{h(\frac{z}{\tilde{R}}, \frac{a-r'}{\tilde{R}})}{\sqrt{1-(\mathbf{u} \cdot \hat{\mathbf{d}})^2}}$ , reduces to,

$$\begin{aligned} \mathbf{E} \sim & -\frac{i}{\lambda} \left( \frac{1}{2} \int_{\mathcal{L}} F_1((\mathbf{r}' - \mathbf{a})^2, \varphi) [(\hat{\mathbf{z}} \times \mathbf{h}(\frac{z}{\tilde{R}}, \frac{\mathbf{a} - \mathbf{r}'}{\tilde{R}})) \cdot d\mathbf{r}'] + \right. \\ & \left. + \frac{1}{2} \int_{\mathcal{L}} F_2((\mathbf{r}' - \mathbf{a})^2)' \left( \frac{\hat{\mathbf{d}} \cdot (\mathbf{a} - \mathbf{r}') f_e(-\mathbf{u})}{\tilde{R}} \right) [(\hat{\mathbf{z}} \times \frac{\mathbf{r}' - \mathbf{a}}{(\mathbf{r}' - \mathbf{a})^2}) \cdot d\mathbf{r}'] \right), \end{aligned} \quad (70)$$

where, for  $z \neq 0$ , the primitives  $F_1$  and  $F_2$  are written following,

$$\begin{aligned} F_1 &= \frac{Z_0 z}{8\pi k^2} \int \left( \left( \frac{\psi'}{\tilde{R}} \right)^2 \right)' \left( \frac{(\mathbf{r}' - \mathbf{a})^2 G(\varphi)}{R^2} - \hat{\mathbf{d}} \right) dR^2, \quad F_2 = \frac{Z_0 z^2 \hat{\mathbf{z}}}{8\pi k^2} \int \left( \left( \frac{\psi'}{\tilde{R}} \right)^2 \right)' dR^2, \\ F_1 &\sim -\frac{Z_0 z e^{-2ik\tilde{R}}}{4\pi \tilde{R} R^2} \left( \frac{(\mathbf{r}' - \mathbf{a})^2 G(\varphi)}{R^2} - \hat{\mathbf{d}} \right), \quad F_2 \sim -\frac{Z_0 z^2 \hat{\mathbf{z}} e^{-2ik\tilde{R}}}{4\pi \tilde{R} R^2}, \end{aligned} \quad (71)$$

with  $G(\varphi) = \frac{((\mathbf{r}' - \mathbf{a}) \cdot \hat{\mathbf{d}})(\mathbf{r}' - \mathbf{a})}{(\mathbf{r}' - \mathbf{a})^2}$ ,  $f_e(-\mathbf{u})$  regular on  $S'$ ,  $\mathcal{L} \equiv \partial S'$  oriented counterclockwise.

This gives us the  $VV$  polarization term,  $\mathbf{E} \cdot \hat{\mathbf{d}}$ , when  $\mathbf{r}_0 \cdot \hat{\mathbf{d}} = 0$ .

Proof. Considering that  $\frac{h(\frac{z}{\tilde{R}}, \frac{a-r'}{\tilde{R}})}{\sqrt{1-(\mathbf{u} \cdot \hat{\mathbf{d}})^2}}$  is a regular even function of the coordinates of  $\frac{\mathbf{r}' - \mathbf{a}}{\tilde{R}}$ ,  $\hat{\mathbf{d}} \cdot \hat{\mathbf{z}} = 0$ , we use the proposition 7 and the theorem 1 with proposition 6, to derive (70).

5.2.2) Magnetic case with  $f_e \equiv 0$ ,  $\hat{\mathbf{d}} = \hat{\mathbf{d}}_h$

Let us consider the case with  $f_e \equiv 0$ ,  $\hat{\mathbf{d}} = \hat{\mathbf{d}}_h$ . The plate  $S'$  is then illuminated by,

$$\mathbf{E}_i \sim \frac{ik}{4\pi} (\mathbf{u}_0 \times \hat{\mathbf{d}}) f_h(\mathbf{u}_0) \frac{e^{-ik\tilde{R}_0}}{\tilde{R}_0}, \quad (72)$$

and the PO diffracted field at  $\mathbf{r} = \mathbf{a} + z\hat{\mathbf{z}} \notin S'$ , when  $\mathbf{r} = \mathbf{r}_0$  (manostatic case), satisfies,

$$\mathbf{E} \cdot \mathbf{v}_0 \sim \frac{-z}{8(\pi ik)^2} \int_{S'} \left( \left( \frac{\psi'}{\tilde{R}} \right)' \right)^2 \mathbf{w}_0 \cdot \mathbf{R} f_h(-\mathbf{u}) dS' \quad \text{with } \mathbf{w}_0 = \mathbf{v}_0 \times \hat{\mathbf{d}}. \quad (73)$$



Theorem 7

The integral (73), when  $\widehat{\mathbf{d}} \cdot \widehat{\mathbf{z}} = 0$  and  $f_h(-\mathbf{u}) = \frac{f(\frac{-z}{\widetilde{R}}, \frac{\mathbf{r}'-\mathbf{a}}{\widetilde{R}}) + f(\frac{-z}{\widetilde{R}}, \frac{\mathbf{a}-\mathbf{r}'}{\widetilde{R}})}{2} = \frac{g(\frac{z}{\widetilde{R}}, \frac{\mathbf{a}-\mathbf{r}'}{\widetilde{R}})}{\sqrt{1-(\mathbf{u} \cdot \widehat{\mathbf{d}})^2}}$ , reduces to,

$$\mathbf{E} \cdot \mathbf{v}_0 \sim \frac{ik}{4\pi} \int_{\mathcal{L}} \frac{\left( \frac{ze^{-2ik\widetilde{R}} f_h(-\mathbf{u})}{\widetilde{R}^2} - \frac{\widetilde{R} e^{-2ikz} g(1,0)}{z} \right)}{4\pi(\mathbf{r}' - \mathbf{a})^2} (\widehat{\mathbf{z}} \times \frac{-\mathbf{w}_0(\mathbf{r}' - \mathbf{a})^2 + (\mathbf{r}' - \mathbf{a})(zw_{0z})}{\widetilde{R}}). d\mathbf{r}' \quad (74)$$

where  $\mathbf{w}_0 = \mathbf{v}_0 \times \widehat{\mathbf{d}}$ . For  $\mathbf{v}_0 = (\widehat{\mathbf{d}} \times \frac{\mathbf{r}_0}{r_0})$ , with  $\widehat{\mathbf{d}} \cdot \mathbf{r}_0 = \widehat{\mathbf{d}} \cdot \widehat{\mathbf{z}} = 0$ , we have,

$$\mathbf{E} \cdot \mathbf{v}_0 \sim \frac{ik}{4\pi} \int_{\mathcal{L}} \frac{\left( \frac{ze^{-2ik\widetilde{R}} f_h(-\mathbf{u})}{\widetilde{R}^2} - \frac{\widetilde{R} e^{-2ikz} g(1,0)}{z} \right)}{4\pi(\mathbf{r}' - \mathbf{a})^2} (\widehat{\mathbf{z}} \times \left( \frac{(\mathbf{r}' - \mathbf{a})\widetilde{R}}{\widetilde{r}_0} - \frac{\mathbf{r}'(\mathbf{r}' - \mathbf{a})^2}{\widetilde{R} \widetilde{r}_0} \right)). d\mathbf{r}', \quad (75)$$

denoted the HH polarization term. The contour  $\mathcal{L} \equiv \partial S'$  is oriented counterclockwise.

Proof. We can write (73) with  $w_0 \cdot \mathbf{R} = -\mathbf{w}_0 \cdot (\mathbf{r}' - \mathbf{a}) + zw_{0z}$ . Noticing that the function  $f_h(-\mathbf{u}) = \frac{f(\frac{-z}{\widetilde{R}}, \frac{\mathbf{r}'-\mathbf{a}}{\widetilde{R}}) + f(\frac{-z}{\widetilde{R}}, \frac{\mathbf{a}-\mathbf{r}'}{\widetilde{R}})}{2}$  is even relatively to the coordinates of  $\frac{\mathbf{r}'-\mathbf{a}}{\widetilde{R}}$ , we derive, from the theorem 1 and the proposition 7,

$$\begin{aligned} \mathbf{E} \cdot \mathbf{v}_0 &\sim \frac{i}{\lambda} \left( \frac{1}{2} \int_{\mathcal{L}} (F((\mathbf{r}' - \mathbf{a})^2) - F(0)) (\widehat{\mathbf{z}} \times (-\mathbf{w}_0 + \frac{\mathbf{r}' - \mathbf{a}}{(\mathbf{r}' - \mathbf{a})^2} (zw_{0z}))) \right). d\mathbf{r}', \\ F &= \frac{z}{4\pi k^2 ik} \int \left( \left( \frac{\psi'}{\widetilde{R}} \right)' \right)^2 f_h(-\mathbf{u}) d\mathbf{R}^2. \end{aligned} \quad (76)$$

From the proposition 6, we derive  $F \sim \frac{ik}{4\pi} \frac{1}{ik} \frac{ze^{-2ik\widetilde{R}}}{\widetilde{R}^3} f_h(-\mathbf{u})$ , and thus (74). For the polarization HH,  $\mathbf{v}_0 = (\widehat{\mathbf{d}} \times \frac{\mathbf{r}_0}{r_0})$  with  $\mathbf{r}_0 \cdot \widehat{\mathbf{d}} = 0$ , and  $\mathbf{w}_0 = \frac{\mathbf{r}_0}{r_0} = \frac{\mathbf{a} + z\widehat{\mathbf{z}}}{r_0}$ ; We then have,

$$\begin{aligned} \widehat{\mathbf{z}} \times \frac{-\mathbf{w}_0(\mathbf{r}' - \mathbf{a})^2 + (\mathbf{r}' - \mathbf{a})(zw_{0z})}{\widetilde{R}} &= \widehat{\mathbf{z}} \times \left( \frac{(-\mathbf{r}' + (\mathbf{r}' - \mathbf{a}))(\mathbf{r}' - \mathbf{a})^2 + (\mathbf{r}' - \mathbf{a})z^2}{\widetilde{R} \widetilde{r}_0} \right) \\ &= \widehat{\mathbf{z}} \times \left( -\frac{\mathbf{r}'(\mathbf{r}' - \mathbf{a})^2}{\widetilde{R} \widetilde{r}_0} + \frac{\widetilde{R}}{\widetilde{r}_0} (\mathbf{r}' - \mathbf{a}) \right), \end{aligned} \quad (77)$$

and we obtain (75). For an illumination with an isotropic pattern, we choose  $g \equiv 1$  and  $f_h(-\mathbf{u}) = \frac{1}{\sqrt{1-(\mathbf{u} \cdot \widehat{\mathbf{d}})^2}}$ , and we have,

$$\mathbf{E} \cdot \mathbf{v}_0 \sim \frac{ik}{4\pi} \int_{\mathcal{L}} \frac{\left( \frac{ze^{-2ik\widetilde{R}}}{\widetilde{R}^2 \sqrt{1-(\mathbf{u} \cdot \widehat{\mathbf{d}})^2}} - \frac{\widetilde{R} e^{-2ikz}}{z} \right)}{4\pi(\mathbf{r}' - \mathbf{a})^2} (\widehat{\mathbf{z}} \times \left( \frac{(\mathbf{r}' - \mathbf{a})\widetilde{R}}{\widetilde{r}_0} - \frac{\mathbf{r}'(\mathbf{r}' - \mathbf{a})^2}{\widetilde{R} \widetilde{r}_0} \right)). d\mathbf{r}'. \quad (78)$$

Let us notice that,  $(\widehat{\mathbf{z}} \times (\mathbf{r}' - \mathbf{a})) \cdot \widehat{\mathbf{v}}$  is constant on each straight part of  $\mathcal{L}$  along  $\widehat{\mathbf{v}}$ , and that, for real  $\mathbf{a} \notin S'$ , the parts of the integrals (74)-(75) with  $\frac{e^{-2ikz}}{z}$  factor vanish.

Theorem 8

The integral (73), when  $\widehat{\mathbf{d}} \cdot \widehat{\mathbf{z}} = 0$  and  $f_h(-\mathbf{u}) = \frac{f(\frac{-z}{R}, \frac{r'-a}{R}) - f(\frac{-z}{R}, \frac{a-r'}{R})}{2} = \frac{(r'-a)}{R} \cdot \frac{h(\frac{z}{R}, \frac{a-r'}{R})}{\sqrt{1-(\mathbf{u} \cdot \widehat{\mathbf{d}})^2}}$ , reduces to,

$$\begin{aligned} \mathbf{E} \cdot \mathbf{v}_0 &\sim \frac{ik}{4\pi} \left( \int_{\mathcal{L}} \frac{ze^{-2ik\tilde{R}} \left( \frac{(\mathbf{w}_0 \cdot (\mathbf{a}-\mathbf{r}')) f_h(-\mathbf{u})}{\tilde{R}} \right)}{4\pi \mathbf{R}^2} (\widehat{\mathbf{z}} \times \frac{(\mathbf{r}' - \mathbf{a})}{(\mathbf{r}' - \mathbf{a})^2}) \cdot d\mathbf{r}' \right. \\ &\quad \left. + \int_{\mathcal{L}} \frac{ze^{-2ik\tilde{R}} (z w_{0z})}{4\pi \mathbf{R}^4} (\widehat{\mathbf{z}} \times \frac{h(\frac{z}{R}, \frac{a-r'}{R})}{\sqrt{1-(\mathbf{u} \cdot \widehat{\mathbf{d}})^2}}) \cdot d\mathbf{r}' \right). \end{aligned} \quad (79)$$

For  $\mathbf{v}_0 = (\widehat{\mathbf{d}} \times \frac{\mathbf{r}_0}{r_0})$  with  $\widehat{\mathbf{d}} \cdot \mathbf{r}_0 = 0$ , and  $\mathbf{w}_0 = \frac{\mathbf{r}_0}{r_0} = \frac{\mathbf{a} + z\widehat{\mathbf{z}}}{r_0}$ ,  $\mathbf{E} \cdot \mathbf{v}_0$  is the HH polarization term.

Proof.  $\frac{h(\frac{z}{R}, \frac{a-r'}{R})}{\sqrt{1-(\mathbf{u} \cdot \widehat{\mathbf{d}})^2}}$  is a regular even function of the coordinates of  $\frac{r'-a}{R}$  with  $\widehat{\mathbf{d}} \cdot \widehat{\mathbf{z}} = 0$ . We can then use the proposition 7 and the theorem 1 with proposition 6, and derive (79).

### 6) Reduction of a surface radiation integral with an exponential term whose argument is an arbitrary complex polynomial of rank 2.

We study a surface radiation integral whose integrand has an exponential term with an argument that can be approximated by a complex polynomial of rank 2 (see [2]-[4],[14]), as for a plate of largest dimension  $D$ , illuminated by :

- a point source, with  $D$  and  $\sqrt{kD^3} \ll$  source and observation distances from the plate,
- a gaussian shaped field.

We consider first the reduction to a contour integral for the surface integral,

$$\int_{S'} e^{\mathcal{R}^2} dS' \quad \text{with} \quad \mathcal{R}^2 = (\alpha_1 x' + \alpha_2 y') - ik(h_1 x'^2 + h_2 y'^2 + h_3 x' y'), \quad (80)$$

for arbitrary  $\alpha_i$  and  $h_j$ . Our method allows the exact reduction without any use of special functions. The case of a more complex integrand will be considered in chapter 7.

#### 6.1) Transformation of the polynomial argument $\mathcal{R}^2$

##### 6.1.1) quadratic form factorization

Proposition 9

We can write, for any  $h_j$  coefficients with  $h_3^2 \neq 4h_1 h_2$ ,

$$h_1x^2 + h_2y^2 + h_3xy = \xi((1 - c_{p,x}^2)x^2 + (1 - c_{p,y}^2)y^2 - 2c_{p,x}c_{p,y}xy) \quad (81)$$

where

$$\begin{aligned} 2c_{p,x}c_{p,y} &= -h_3/\xi, \quad c_{p,x}^2 = 1 - h_1/\xi, \quad c_{p,y}^2 = 1 - h_2/\xi \\ \xi &= \frac{(h_1 + h_2) + \epsilon\sqrt{(h_1 - h_2)^2 + h_3^2}}{2}, \quad \epsilon = +1 \text{ or } -1. \end{aligned} \quad (82)$$

Proof. Letting,

$$(h_1x'^2 + h_2y'^2 + h_3x'y') = \xi(\mathbf{r}'^2 - (\mathbf{c}_p \cdot \mathbf{r}')^2) \quad (83)$$

we obtain (81) with (82), then  $2c_{p,x}c_{p,y} = -h_3/\xi$ ,  $c_{p,x}^2 = 1 - h_1/\xi$ ,  $c_{p,y}^2 = 1 - h_2/\xi$ , and  $4(1 - h_1/\xi)(1 - h_2/\xi) = (h_3/\xi)^2$ , with  $\xi = 0$  when  $h_3^2 = 4h_1h_2$ . Thus, we have,

$$\begin{aligned} \xi &= \frac{(h_1 + h_2) + \epsilon\sqrt{(h_1 - h_2)^2 + 4(h_3/2)^2}}{2}, \\ c_{p,x} &= \epsilon_x\sqrt{1 - h_1/\xi}, \quad c_{p,y} = \epsilon_y\sqrt{1 - h_2/\xi}, \end{aligned} \quad (84)$$

where the choice of  $\epsilon = +1$  or  $-1$  is free, and,  $\epsilon_x^2 = \epsilon_y^2 = 1$  with  $\epsilon_x\epsilon_y = \frac{-h_3/\xi}{2\sqrt{1-h_1/\xi}\sqrt{1-h_2/\xi}}$ .

Proposition 10

We can write,

$$\begin{aligned} (\alpha_1x' + \alpha_2y') - ik(h_1x'^2 + h_2y'^2 + h_3x'y') &= \mathbf{w}' \cdot \mathbf{r}' - ik\xi((\mathbf{r}')^2 - \mathbf{c}_p^2(\widehat{\mathbf{c}}_p \mathbf{r}')^2) \\ &= \mathbf{w}'_m \cdot \mathbf{r}'_p - ik\xi(\mathbf{r}'_p)^2 = v(\mathbf{r}'_p - \mathbf{a}_p)^2 + v_1, \end{aligned} \quad (85)$$

with  $\mathbf{c}_p$  defined in proposition 9, and

$$\begin{aligned} \mathbf{r}'_p &= (p(\widehat{\mathbf{c}}_p \cdot \mathbf{r}')\widehat{\mathbf{c}}_p + (\mathbf{r}' - \widehat{\mathbf{c}}_p(\widehat{\mathbf{c}}_p \cdot \mathbf{r}'))), \quad \mathbf{c}_p = \sqrt{\mathbf{c}_p^2} \widehat{\mathbf{c}}_p, \quad p = \sqrt{1 - \mathbf{c}_p^2}, \\ \mathbf{a}_p &= \frac{\mathbf{w}'_m}{2ik\xi}, \quad v = -ik\xi, \quad v_1 = \frac{\mathbf{w}'_m{}^2}{4ik\xi}, \quad \mathbf{w}'_m = \frac{(\widehat{\mathbf{c}}_p \mathbf{w}')}{p} \widehat{\mathbf{c}}_p + (\mathbf{w}' - \widehat{\mathbf{c}}_p(\widehat{\mathbf{c}}_p \mathbf{w}')), \end{aligned} \quad (86)$$

considered at the limit when  $\xi = 0$ ,  $\mathbf{c}_p^2 = 1$  ( $h_3^2 = 4h_1h_2$ ), or  $\mathbf{c}_p^2 \rightarrow 0$  ( $(h_1 - h_2)^2 + h_3^2 = 0$ ). For  $\mathbf{c}_p^2 \rightarrow 0$ , we notice that  $\mathbf{r}'_p \rightarrow \mathbf{r}' - \frac{(\mathbf{c}_p \cdot \mathbf{r}')\mathbf{c}_p}{2}$  and  $\mathbf{w}'_m \rightarrow \mathbf{w}' + \frac{(\mathbf{c}_p \cdot \mathbf{w}')\mathbf{c}_p}{2}$ .

Proof. We choose  $\mathbf{w}'_m$  and  $\mathbf{r}'_p$  verifying,

$$\mathbf{w}'_m \cdot \mathbf{r}'_p = \mathbf{w}' \cdot \mathbf{r}' = \alpha_1x' + \alpha_2y', \quad (\mathbf{r}'_p)^2 = ((\mathbf{r}')^2 - (\mathbf{c}_p \cdot \mathbf{r}')^2), \quad (87)$$

as in (86), with  $\widehat{\mathbf{c}}_p = \frac{\mathbf{c}_p}{\sqrt{\mathbf{c}_p \cdot \mathbf{c}_p}}$  and  $\mathbf{c}_p$  defined in proposition 9. We then have,

$$\begin{aligned}
& (\alpha_1 x' + \alpha_2 y') - ik(h_1 x'^2 + h_2 y'^2 + h_3 x' y') = \\
& = \mathbf{w}'_m \cdot \mathbf{r}'_p - ik\xi(\mathbf{r}'_p)^2 = -ik\xi\left(\left(\mathbf{r}'_p - \frac{\mathbf{w}'_m}{2ik\xi}\right)^2 - \frac{\mathbf{w}'_m{}^2}{(2ik\xi)^2}\right) = v(\mathbf{r}'_p - \mathbf{a}_p)^2 + v_1, \quad (88)
\end{aligned}$$

with  $\mathbf{a}_p = \frac{\mathbf{w}'_m}{2ik\xi}$ ,  $v = -ik\xi$ ,  $v_1 = \frac{\mathbf{w}'_m{}^2}{4ik\xi}$ . Notice that  $\xi = 0$  or  $\mathbf{c}_p^2 = 1$  implies  $h_3^2 = 4h_1h_2$ .

## 6.2) Reduction of the surface integral to a contour integral

Following the expression of  $\mathbf{r}'_p$ , we change of variables and derive a reduction of integrals.

### 6.2.1) definition of $S'_p$

Using  $\mathbf{r}'_p$  of (86), we consider  $\mathcal{P}$ , the anisotropic dilation (or scaling) by a factor  $p = \sqrt{1 - \mathbf{c}_p^2}$  along the vector  $\widehat{\mathbf{c}}_p = \mathbf{c}_p / \sqrt{\mathbf{c}_p^2}$  with  $\mathbf{c}_p^2 \neq 0$ , given by,

$$\mathcal{P} : \mathbf{r}' = x' \widehat{\mathbf{x}} + y' (\widehat{\mathbf{z}} \times \widehat{\mathbf{x}}) \rightarrow \mathbf{r}'_p = x'_p \widehat{\mathbf{c}}_p + y'_p (\widehat{\mathbf{z}} \times \widehat{\mathbf{c}}_p), \quad x'_p = p(\mathbf{r}' \cdot \widehat{\mathbf{c}}_p), \quad y'_p = \mathbf{r}' \cdot (\widehat{\mathbf{z}} \times \widehat{\mathbf{c}}_p) \quad (89)$$

where  $h_3^2 \neq 4h_1h_2$ , i.e.  $\mathbf{c}_p^2 \neq 1$  and  $\mathcal{P}(\mathbf{r}')$  is invertible. A complex manifold  $S'_p$  with contour  $\mathcal{L}_p$  is defined like in definition 1, as the transformation by  $\mathcal{P}$  of the real oriented manifold  $S'$  with contour  $\mathcal{L}$ . Letting  $dS' = dx' \wedge dy'$  and  $dS'_p = dx'_p \wedge dy'_p$ , we have  $p dS' = dS'_p$ .

### 6.2.2) First reduction along $\mathcal{L}_p$

#### Proposition 11

Considering  $\mathbf{r}'_p$ ,  $v$ ,  $\mathbf{a}_p = \frac{\mathbf{w}'_m}{2ik\xi}$ , and  $p$  from (86), and an arbitrary constant  $A_0$ , we can write

$$\begin{aligned}
\int_{S'} e^{(\alpha_1 x' + \alpha_2 y') - ik(h_1 x'^2 + h_2 y'^2 + h_3 x' y')} \cdot dS' &= \frac{1}{2p} \int_{\mathcal{L}_p} \left( \frac{\widehat{\mathbf{z}} \times (\mathbf{r}'_p - \mathbf{a}_p) (e^{\mathbf{w}'_m \cdot \mathbf{r}'_p - ik\xi(\mathbf{r}'_p)^2} - A_0)}{\mathbf{w}'_m \cdot \mathbf{r}'_p - ik\xi(\mathbf{r}'_p)^2 - \frac{\mathbf{w}'_m{}^2}{4ik\xi}} \right) \cdot d\mathbf{r}'_p \\
&+ \frac{(A_0 - e^{\frac{\mathbf{w}'_m{}^2}{4ik\xi}})}{-2ikp\xi} \int_{\mathcal{L}_p} \left( \widehat{\mathbf{z}} \times \frac{\mathbf{r}'_p - \mathbf{a}_p}{(\mathbf{r}'_p - \mathbf{a}_p)^2} \right) \cdot d\mathbf{r}'_p, \quad (90)
\end{aligned}$$

where  $\mathbf{w}'_m \cdot \mathbf{r}'_p - ik\xi(\mathbf{r}'_p)^2 - \frac{\mathbf{w}'_m{}^2}{4ik\xi} = v(\mathbf{r}'_p - \mathbf{a}_p)^2$ . Let  $\mathcal{D}_1$  a smooth semi-line of the complex plane from the origin to  $\infty$ , we notice that, for  $(\mathbf{r}'_p - \mathbf{a}_p)^2 \neq 0$  on  $\mathcal{L}$ ,

$$\int_{\mathcal{L}_p} \left( \widehat{\mathbf{z}} \times \frac{\mathbf{r}'_p - \mathbf{a}_p}{(\mathbf{r}'_p - \mathbf{a}_p)^2} \right) \cdot d\mathbf{r}'_p = \frac{-i}{2} \int_{\mathcal{L}_p} d \ln(g(\mathbf{r}'_p)) = \pi(N_+ - N_-), \quad (91)$$

where  $N_+$  (resp.  $N_-$ ) describes the number of times  $g(\mathbf{r}'_p)$  crosses  $\mathcal{D}_1$  in positive sense (resp. negative), with  $g(\mathbf{r}'_p) = \frac{z'_{\mathbf{a}+}}{z'_{\mathbf{a}-}}$ ,  $z'_{\mathbf{a}+} = x'_a + iy'_a$ ,  $z'_{\mathbf{a}-} = x'_a - iy'_a$ ,  $\mathbf{r}'_p - \mathbf{a}_p = x'_a \hat{\mathbf{c}}_p + y'_a (\hat{\mathbf{z}} \times \hat{\mathbf{c}}_p)$ .

Proof. From the proposition 10 and  $p dS' = dS'_p$ , we have,

$$\int_{S'} e^{v(\alpha_1 x' + \alpha_2 y') - ik(h_1 x'^2 + h_2 y'^2 + h_3 x' y')} dS' = \frac{1}{p} \int_{S'_p} e^{v(\mathbf{r}'_p - \mathbf{a}_p)^2 + v_1} dS'_p. \quad (92)$$

We can then apply the theorem 1 to the surface integral, following,

$$\begin{aligned} \int_{S'_p} e^{v(\mathbf{r}'_p - \mathbf{a}_p)^2 + v_1} dS'_p &= \frac{1}{2} \left( \int_{\mathcal{L}_p} \left( \frac{\hat{\mathbf{z}} \times (\mathbf{r}'_p - \mathbf{a}_p) (e^{-ik\xi(\mathbf{r}'_p - \mathbf{a}_p)^2 + v_1} - A_0)}{v(\mathbf{r}'_p - \mathbf{a}_p)^2} \right) \cdot d\mathbf{r}'_p + \right. \\ &\left. + (A_0 - e^{v_1}) \int_{\mathcal{L}_p} \left( \hat{\mathbf{z}} \times \frac{\mathbf{r}'_p - \mathbf{a}_p}{v(\mathbf{r}'_p - \mathbf{a}_p)^2} \right) \cdot d\mathbf{r}'_p \right), \end{aligned} \quad (93)$$

and derive (90). We obtain (91) from proposition 1.

Remark 5 : Let the components of  $\mathbf{a}_p$  with large values such that  $\text{Im}(\frac{z'_{\mathbf{a}+}}{z'_{\mathbf{a}-}})$  little varies, and,

$$\max |\text{Im}(\frac{z'_{\mathbf{a}+}}{z'_{\mathbf{a}-}} - \frac{z'_{\mathbf{a}+}}{z'_{\mathbf{a}-}}|_{\mathbf{r}'_p=0})| = \max |\text{Im}(\frac{1}{2} \ln(\frac{1 - \frac{(\mathbf{r}'_p + i(\mathbf{r}'_p \times \hat{\mathbf{z}})) \cdot \hat{\mathbf{c}}_p}{(\mathbf{a}_p + i(\mathbf{a}_p \times \hat{\mathbf{z}})) \cdot \hat{\mathbf{c}}_p}}{1 - \frac{(\mathbf{r}'_p - i(\mathbf{r}'_p \times \hat{\mathbf{z}})) \cdot \hat{\mathbf{c}}_p}{(\mathbf{a}_p - i(\mathbf{a}_p \times \hat{\mathbf{z}})) \cdot \hat{\mathbf{c}}_p}}))|_{\mathbf{r}'_p \in \mathcal{L}_p} < \pi/2, \quad (94)$$

then  $\frac{z'_{\mathbf{a}+}}{z'_{\mathbf{a}-}}$  does not run round the origin, the winding number is null, and  $N_+ = N_-$  in (91).

6.2.3) Some properties of  $\mathcal{P}(\mathbf{r}') = \mathbf{r}'_p$  to analyse and transform (90)

Considering the definition (89), we have  $\mathcal{P}(\mathbf{u}) \cdot \mathbf{v} = \mathbf{u} \cdot \mathcal{P}(\mathbf{v})$ , and  $\mathcal{P}(\hat{\mathbf{z}} \times \mathcal{P}(\mathbf{u})) = p(\hat{\mathbf{z}} \times \mathbf{u})$  from  $\mathcal{P}(\hat{\mathbf{z}} \times \mathbf{u}) = \hat{\mathbf{z}} \times ((\hat{\mathbf{c}}_p \cdot \mathbf{u}) \hat{\mathbf{c}}_p + p(\mathbf{u} - \hat{\mathbf{c}}_p(\hat{\mathbf{c}}_p \cdot \mathbf{u})))$ , which leads us to,

$$\mathbf{u} \cdot d\mathbf{r}'_p = \mathcal{P}(\mathbf{u}) \cdot d\mathbf{r}', \quad (\hat{\mathbf{z}} \times \mathbf{r}'_p) \cdot d\mathbf{r}'_p = p(\hat{\mathbf{z}} \times \mathbf{r}') \cdot d\mathbf{r}'. \quad (95)$$

We notice that  $p \hat{\mathbf{z}} \times \mathbf{w}'_m = \mathcal{P}(\hat{\mathbf{z}} \times \mathbf{w}')$ ,  $\mathbf{w}'_m = \mathcal{P}^{-1}(\mathbf{w}')$ , and, from now on, we let,

$$\mathcal{M}(\mathbf{w}') = \mathcal{P}(\hat{\mathbf{z}} \times \mathcal{N}(\mathbf{w}')) \times \hat{\mathbf{z}}, \quad \mathcal{N}(\mathbf{w}') = p \mathbf{w}'_m = ((\hat{\mathbf{c}}_p \cdot \mathbf{w}') \hat{\mathbf{c}}_p + p(\mathbf{w}' - \hat{\mathbf{c}}_p(\hat{\mathbf{c}}_p \cdot \mathbf{w}'))). \quad (96)$$

Proposition 12

Let  $(\mathbf{c}_p)_{\pm 1} = (\hat{\mathbf{c}}_p)_{\pm 1} \sqrt{(\mathbf{c}_p^2)_{\pm 1}}$ ,  $\xi_{\pm 1}$ ,  $(\mathcal{P})_{\pm 1}$  and  $(\mathbf{r}'_p)_{\pm 1}$ , be the respective terms  $(\mathbf{c}_p)$ ,  $\xi$ ,  $\mathcal{P}$  and  $(\mathbf{r}'_p)$  for  $\epsilon = \pm 1$  (factor used in (84) for  $\xi$ ), with  $p_{\pm}^2 = 1 - (\mathbf{c}_p^2)_{\pm 1}$ , we notice that,

$$p_{\pm}^2 = \frac{\xi_{\mp 1}}{\xi_{\pm 1}} = \frac{1}{p_{\mp}^2}, (\xi_{\pm 1} p_{\pm})^2 = (\xi_{\mp 1} p_{\mp})^2 = \xi_+ \xi_- = h_1 h_2 - (h_3/2)^2, \\ (\widehat{\mathbf{c}}_p)_{\pm 1} = \pm \epsilon_c (\widehat{\mathbf{z}} \times (\widehat{\mathbf{c}}_p)_{\mp 1}), (\mathbf{r}'_p)_{\pm 1} = p_{\pm} (\mathbf{r}'_p)_{\mp 1}, (\mathcal{P})_{\pm 1} = p_{\pm} (\mathcal{P})_{\mp 1}, \quad (97)$$

where  $(\epsilon_c)^2 = 1$  and  $\xi_{\pm 1}$  is finite. So,  $\xi_{-\epsilon} = 0$  implies  $h_3^2 = 4h_1 h_2$ ,  $\xi_{\epsilon} = h_1 + h_2$ ,  $(\mathbf{c}_p^2)_{\epsilon} = 1$  if  $h_1 + h_2 \neq 0$ ,  $(\mathbf{c}_p^2)_{\epsilon} = 1$  implies  $\xi_{-\epsilon} = 0$ , and  $(\mathbf{c}_p^2)_{\epsilon} = 0$  implies  $\xi_{\pm 1} = h_1 + h_2$ ,  $p_{\pm} = 1$ .

Proof. Considering (82), we have

$$1 - (\mathbf{c}_p^2)_{\pm 1} = -1 + \frac{h_1 + h_2}{\xi_{\pm 1}} = \frac{\xi_{\mp 1}}{\xi_{\pm 1}} = \frac{1}{1 - (\mathbf{c}_p^2)_{\mp 1}}, \\ \frac{(\widehat{\mathbf{c}}_{px})_{\epsilon}}{(\widehat{\mathbf{c}}_{py})_{\epsilon}} = \frac{(c_{px})_{\epsilon} (c_{py})_{\epsilon}}{(c_{py})_{\epsilon} (c_{py})_{\epsilon}} = \frac{-h_3}{2(\xi_{\epsilon} - h_2)} = \frac{-h_3}{(h_1 - h_2) + \epsilon \sqrt{(h_1 - h_2)^2 + h_3^2}} \\ = \frac{(h_1 - h_2) - \epsilon \sqrt{(h_1 - h_2)^2 + h_3^2}}{h_3} = -\frac{(c_{py})_{-\epsilon}}{(c_{px})_{-\epsilon}} = -\frac{(\widehat{\mathbf{c}}_{py})_{-\epsilon}}{(\widehat{\mathbf{c}}_{px})_{-\epsilon}}. \quad (98)$$

Proposition 13

We can write,

$$\frac{(\widehat{\mathbf{z}} \times \frac{(\mathbf{r}'_p - \mathbf{a}_p)}{(\mathbf{r}'_p - \mathbf{a}_p)^2}) \cdot d\mathbf{r}'_p}{-2ikp\xi} = \frac{\frac{1}{2}(\widehat{\mathbf{z}} \times (\mathbf{r}' - \frac{\mathcal{M}(\mathbf{w}')}{2ik\xi(1 - \mathbf{c}_p^2)})) \cdot d\mathbf{r}'}{\mathbf{w}' \cdot \mathbf{r}' - ik(h_1 x'^2 + h_2 y'^2 + h_3 x' y') - \frac{(\mathcal{N}(\mathbf{w}'))^2}{4ik\xi p^2}} \quad (99)$$

where,

$$\frac{\mathcal{M}(\mathbf{w}')}{2ik\xi p^2} = \frac{1}{2ik\xi_+ \xi_-} ((h_2 w'_x - \frac{h_3 w'_y}{2}) \widehat{\mathbf{x}} + (h_1 w'_y - \frac{h_3 w'_x}{2}) \widehat{\mathbf{y}}), \\ \frac{(\mathcal{N}(\mathbf{w}'))^2}{4ik\xi p^2} = \frac{\mathbf{w}'_m{}^2}{4ik\xi} = \frac{1}{4ik\xi_+ \xi_-} (h_2 w_x'^2 + h_1 w_y'^2 - h_3 w'_x w'_y). \quad (100)$$

Proof. Considering (85) and (95)-(96), we obtain (99), then (100) from  $(\xi p)^2 = \xi_+ \xi_-$  and,

$$\mathcal{M}(\mathbf{w}') = (\widehat{\mathbf{c}}_p \cdot \mathbf{w}') \widehat{\mathbf{c}}_p + (1 - \mathbf{c}_p^2)(\mathbf{w}' - \widehat{\mathbf{c}}_p (\widehat{\mathbf{c}}_p \cdot \mathbf{w}')) = (1 - \mathbf{c}_p^2) \mathbf{w}' + (\mathbf{c}_p \cdot \mathbf{w}') \mathbf{c}_p \\ = (-1 + \frac{h_1 + h_2}{\xi}) \mathbf{w}' + (1 - \frac{h_1}{\xi}) w'_x \widehat{\mathbf{x}} + (1 - \frac{h_2}{\xi}) w'_y \widehat{\mathbf{y}} - \frac{h_3}{2\xi} (w'_y \widehat{\mathbf{x}} + w'_x \widehat{\mathbf{y}}) \\ = \frac{1}{\xi} (h_2 w'_x \widehat{\mathbf{x}} + h_1 w'_y \widehat{\mathbf{y}} - \frac{h_3}{2} (w'_y \widehat{\mathbf{x}} + w'_x \widehat{\mathbf{y}})) = \frac{1}{\xi} ((h_2 w'_x - \frac{h_3 w'_y}{2}) \widehat{\mathbf{x}} + (h_1 w'_y - \frac{h_3 w'_x}{2}) \widehat{\mathbf{y}}), \\ (\mathcal{N}(\mathbf{w}'))^2 = (1 - \mathbf{c}_p^2) \mathbf{w}'^2 + (\mathbf{c}_p \cdot \mathbf{w}')^2 = (-1 + \frac{h_1 + h_2}{\xi}) (w_x'^2 + w_y'^2) + \\ (1 - \frac{h_1}{\xi}) w_x'^2 + (1 - \frac{h_2}{\xi}) w_y'^2 - \frac{h_3}{\xi} w'_x w'_y = \frac{1}{\xi} (h_2 w_x'^2 + h_1 w_y'^2 - h_3 w'_x w'_y). \quad (101)$$

Proposition 14

Let be  $z'_{\mathbf{a}\pm} = (\mathbf{r}'_p - \mathbf{a}_p) \cdot (\hat{\mathbf{x}} \pm i\hat{\mathbf{y}}) = x'_a \pm iy'_a$ , where  $\mathbf{r}'_p - \mathbf{a}_p = x'_a \hat{\mathbf{c}}_p + y'_a (\hat{\mathbf{z}} \times \hat{\mathbf{c}}_p)$  with  $\mathbf{r}'_p = \mathcal{P}(\mathbf{r}')$ , then  $z'_{\mathbf{a}\pm} = 0$  on lines  $\mathbf{r}' = \mathcal{P}^{-1}(\mathbf{a}_p) + t\mathbf{u}_c$ ,  $\mathbf{u}_c = \hat{\mathbf{c}}_p \pm ip(\hat{\mathbf{z}} \times \hat{\mathbf{c}}_p)$ . For real  $\mathbf{r}'$ , each equation  $z'_{\mathbf{a}\pm} = 0$  has no or an infinity of solutions if  $\frac{\mp ip\xi + \frac{h_3}{2}}{h_2} (= \frac{h_1}{\pm ip\xi + \frac{h_3}{2}})$  is real, and a single one  $\mathbf{r}_{\pm}$  else, with  $\mathbf{r}_+ = \mathbf{r}_- = \mathcal{P}^{-1}(\mathbf{a}_p)$  if  $\mathbf{a}_p$  and  $\mathcal{P}(\mathbf{r}')$  are real.

Proof. From  $z'_{\mathbf{a}\pm} = (\mathbf{r}'_p - \mathbf{a}_p) \cdot (\hat{\mathbf{x}} \pm i\hat{\mathbf{y}})$ ,  $(\hat{\mathbf{x}} \pm i\hat{\mathbf{y}})^2 = 0$  and  $\hat{\mathbf{x}} \pm i\hat{\mathbf{y}} = \frac{\mathbf{c} \pm i\hat{\mathbf{z}} \times \mathbf{c}}{c_x \mp ic_y}$  as  $\mathbf{c}^2 \neq 0$ , we have  $z'_{\mathbf{a}\pm} = 0$  as  $\mathbf{r}' = \mathcal{P}^{-1}(\mathbf{a}_p) + t\mathbf{u}_c$ ,  $\mathbf{u}_c = p\mathcal{P}^{-1}(\hat{\mathbf{c}}_p \pm i(\hat{\mathbf{z}} \times \hat{\mathbf{c}}_p))$ , and  $z'_{\mathbf{a}\pm} = \frac{(\mathbf{r}' - \mathcal{P}^{-1}(\mathbf{a}_p)) \cdot \mathbf{v}_c}{c_x \mp ic_y}$  with  $\mathbf{v}_c = \mathcal{P}(\mathbf{c}_p \pm i\hat{\mathbf{z}} \times \mathbf{c}_p) = p\mathbf{c}_p \pm i\hat{\mathbf{z}} \times \mathbf{c}_p$ ; we can consider solutions of  $\mathbf{v}_c \cdot \mathbf{r}' = 0$  with

$$\frac{\mathbf{v}_{cx}}{\mathbf{v}_{cy}} = \frac{pc_x \mp ic_y}{pc_y \pm ic_x} = \frac{\mp ip(c_x^2 + c_y^2) + c_x c_y (p^2 - 1)}{p^2 c_y^2 + c_x^2} = \frac{\mp ip - c_x c_y}{1 - c_y^2} = \frac{\mp ip\xi + \frac{h_3}{2}}{h_2} \quad (102)$$

and deduce last statement. Let us remark that  $\mathbf{u}_c \cdot \mathbf{v}_c = 0$  and  $(p\xi)^2 = h_1 h_2 - (\frac{h_3}{2})^2$ .

#### 6.2.4) Derivation of the contour integral along $\mathcal{L}$

Theorem 9

We can reduce the integral of an exponential with quadratic argument on  $S'$ , to an integral along the contour  $\mathcal{L}$  of  $S'$ , following,

$$\int_{S'} e^{\mathbf{w}' \cdot \mathbf{r}' - ik(h_1 x'^2 + h_2 y'^2 + h_3 x' y')} dS' = \frac{1}{2} \int_{\mathcal{L}} \left[ \frac{(e^{\mathbf{w}' \cdot \mathbf{r}' - ik(h_1 x'^2 + h_2 y'^2 + h_3 x' y')} - A_0)}{\mathbf{w}' \cdot \mathbf{r}' - ik(h_1 x'^2 + h_2 y'^2 + h_3 x' y') - \frac{\mathbf{w}'_m{}^2}{4ik\xi}} \right] [(\hat{\mathbf{z}} \times (\mathbf{r}' - \frac{\mathcal{M}(\mathbf{w}')}{2ik\xi p^2})) \cdot d\mathbf{r}'] + \frac{(A_0 - e^{\frac{\mathbf{w}'_m{}^2}{4ik\xi}})}{-2ikp\xi} \int_{\mathcal{L}_p} (\hat{\mathbf{z}} \times \frac{\mathbf{r}'_p - \mathbf{a}_p}{(\mathbf{r}'_p - \mathbf{a}_p)^2}) \cdot d\mathbf{r}'_p, \quad (103)$$

with arbitrary  $A_0$ , and  $\frac{\mathcal{M}(\mathbf{w}')}{2ik\xi p^2}$ ,  $\frac{\mathbf{w}'_m{}^2}{4ik\xi}$  detailed in proposition 13. Let  $\mathcal{D}_1$  be a smooth semi-line of the complex plane from the origin to  $\infty$ , we notice that, for  $(\mathcal{P}(\mathbf{r}') - \mathbf{a}_p)^2 \neq 0$  on  $\mathcal{L}$ ,

$$\int_{\mathcal{L}_p} (\hat{\mathbf{z}} \times \frac{\mathbf{r}'_p - \mathbf{a}_p}{(\mathbf{r}'_p - \mathbf{a}_p)^2}) \cdot d\mathbf{r}'_p = \frac{-i}{2} \int_{\mathcal{L}} d\ln(g(\mathcal{P}(\mathbf{r}'))) = \pi(N_+ - N_-) \quad (104)$$

where  $N_+$  (resp.  $N_-$ ) describes the number of times  $g(\mathcal{P}(\mathbf{r}'))$  crosses  $\mathcal{D}_1$  in positive sense (resp. negative), with  $g(\mathcal{P}(\mathbf{r}')) = \frac{x'_a + iy'_a}{x'_a - iy'_a}$ ,  $\mathcal{P}(\mathbf{r}') - \mathbf{a}_p = x'_a \hat{\mathbf{c}}_p + y'_a (\hat{\mathbf{z}} \times \hat{\mathbf{c}}_p)$ ,  $\mathcal{P}(\mathbf{r}') = \mathbf{r}'_p$ . When  $\mathbf{a}_p$  is large and satisfies (94), in particular if  $h_1 h_2 \rightarrow (h_3/2)^2$ ,  $\int_{\mathcal{L}} d\ln(g(\mathcal{P}(\mathbf{r}'))) = 0$ .

Proof. For that, we rewrite the proposition 11 with the use of propositions 10 and 13.

### 7) More general cases of applications

If we want to generalize the results of chapter 5 and take account of a moderate curvature of the surface of integration, we have to consider  $(z - z')$  instead of  $z$  in (63), with first order approximation of  $z'$  relatively to  $x'$  and  $y'$  coordinates. Thus, the term  $\mathbf{R}^2$  is changed for a polynomial  $\mathcal{R}^2$  of order 2 with arbitrary coefficients,

$$\mathcal{R}^2 = d_0 + (\alpha_1 x' + \alpha_2 y') - ik(h_1 x'^2 + h_2 y'^2 + h_3 x' y') = d_0 + v(\mathbf{r}'_p - \mathbf{a}_p)^2 + v_1, \quad (105)$$

following the proposition 10, with  $\mathbf{r}'_p = \mathcal{P}(\mathbf{r}')$  defined in (89). We thereafter consider the case where the integrand of the surface integral is a function  $g(\mathbf{r}')f(\mathcal{R}^2)$ , which, expressed relatively to  $\mathbf{r}'_p - \mathbf{a}_p$  variable, verifies the condition of continuation given in definition 3 (even for  $k \rightarrow \infty$ ). For the reduction to a contour integral, we then follow the process developed in chapter 6. As previously, we define  $S'_p$  of contour  $\mathcal{L}_p$  as the transformation of the manifold  $S'$  by  $\mathcal{P}$ , with  $dS' = dx' \wedge dy'$ ,  $dS'_p = dx'_p \wedge dy'_p$ , and  $dS' = dS'_p/p$ .

#### 7.1) Reduction of $\int_{S'} g(\mathbf{r}') f(\mathcal{R}^2) dS'$ when $g \equiv 1$

##### Theorem 10

Considering  $F$  as a primitive of  $f$ , we can write,

$$\begin{aligned} \int_{S'} f(\mathcal{R}^2) dS' &= \frac{1}{2} \int_{\mathcal{L}} \left[ \frac{F(\mathcal{R}^2) - A_0}{\mathcal{R}^2 - (\frac{w'_m{}^2}{4ik\xi} + d_0)} \right] \left[ (\hat{\mathbf{z}} \times (\mathbf{r}' - \frac{\mathcal{M}(\mathbf{w}')}{2ik\xi(1 - \mathbf{c}_p^2)}) \right] \cdot d\mathbf{r}' \\ &+ \frac{A_0 - F(\frac{w'_m{}^2}{4ik\xi} + d_0)}{-2ikp\xi} \int_{\mathcal{L}_p} (\hat{\mathbf{z}} \times \frac{\mathbf{r}'_p - \mathbf{a}_p}{(\mathbf{r}'_p - \mathbf{a}_p)^2}) \cdot d\mathbf{r}'_p, \end{aligned} \quad (106)$$

where  $A_0$  is a constant,  $p = \sqrt{1 - \mathbf{c}_p^2}$ , and,

$$\begin{aligned} \frac{\mathcal{M}(\mathbf{w}')}{2ik\xi p^2} &= \frac{1}{2ik(h_1 h_2 - (h_3/2)^2)} \left( (h_2 w'_x - \frac{h_3 w'_y}{2}) \hat{\mathbf{x}} + (h_1 w'_y - \frac{h_3 w'_x}{2}) \hat{\mathbf{y}} \right), \\ \frac{w'_m{}^2}{4ik\xi} &= \frac{1}{4ik(h_1 h_2 - (h_3/2)^2)} (h_2 w'_x{}^2 + h_1 w'_y{}^2 - h_3 w'_x w'_y). \end{aligned} \quad (107)$$

and  $\int_{\mathcal{L}_p} (\hat{\mathbf{z}} \times \frac{\mathbf{r}'_p - \mathbf{a}_p}{(\mathbf{r}'_p - \mathbf{a}_p)^3}) \cdot d\mathbf{r}'_p$  can be determined with (104).

Proof. In proposition 11 and theorem 9, we change the exponential function for  $f$ , and let,



$$\frac{F(\mathcal{R}^2)}{v} = \int f(d_0 + v(\mathbf{r}'_p - \mathbf{a}_p)^2 + v_1)d(\mathbf{r}'_p - \mathbf{a}_p)^2 = \frac{1}{v} \int f(\mathcal{R}^2)d\mathcal{R}^2. \quad (108)$$

We can then write, for  $S'_p$  and  $\mathcal{L}_p$ , the dilated by  $\mathcal{P}$  of  $S'$  and  $\mathcal{L}$ ,

$$\begin{aligned} \int_{S'} f(\mathcal{R}^2)dS' &= \frac{1}{\sqrt{1 - \mathbf{c}_p^2}} \int_{S'_p} f(d_0 + v(\mathbf{r}'_p - \mathbf{a}_p)^2 + v_1)dS'_p \\ &= \int_{\mathcal{L}_p} \frac{F(\mathcal{R}^2) - A_0}{\mathcal{R}^2 - (v_1 + d_0)} \frac{[(\hat{\mathbf{z}} \times (\mathbf{r}'_p - \mathbf{a}_p)) \cdot d\mathbf{r}'_p]}{2\sqrt{1 - \mathbf{c}_p^2}} - \frac{i(A_0 - F(v_1 + d_0))}{4v\sqrt{1 - \mathbf{c}_p^2}} \int_{\mathcal{L}_p} d\ln\left(\frac{z'_{\mathbf{a}_+}}{z'_{\mathbf{a}_-}}\right). \end{aligned} \quad (109)$$

Considering the proposition 13, we then derive (106) with (107).

## 7.2) Reduction of $\int_{S'} g(\mathbf{r}') f(\mathcal{R}^2) dS'$ when $g(\mathbf{r}')$ is a polynomial of coordinates

From Taylor expansion and  $\mathbf{r}' = \mathcal{P}^{-1}(\mathbf{r}'_p)$ , we have, for any polynomial  $g \equiv g_n$  of order  $n$ ,

$$g_n(\mathbf{r}') = \sum_{j=0}^n \frac{1}{j!} \sum_{k=0}^j \binom{j}{k} \frac{\partial^j g|_{\mathbf{r}'=\mathcal{P}^{-1}(\mathbf{a}_p)}}{\partial x'^k \partial y'^{j-k}} [(\mathcal{P}^{-1}(\hat{\mathbf{x}}) \cdot (\mathbf{r}'_p - \mathbf{a}_p))^k (\mathcal{P}^{-1}(\hat{\mathbf{y}}) \cdot (\mathbf{r}'_p - \mathbf{a}_p))^{j-k}] \quad (110)$$

### Theorem 11

Considering  $I_{n+1} = \int_{S'} \mathbf{d}_{n+1} \cdot (\mathbf{r}'_p - \mathbf{a}_p) \prod_{j=1}^n (\mathbf{d}_j \cdot (\mathbf{r}'_p - \mathbf{a}_p)) f(\mathcal{R}^2) dS'$  with constant  $\mathbf{d}_j$ , we have,

$$\begin{aligned} I_{n+1} &= \frac{-1}{2ik\xi} \left( \frac{1}{p} \int_{\mathcal{L}} \prod_{j=1}^n (\mathbf{d}_j \cdot (\mathbf{r}'_p - \mathbf{a}_p)) F(\mathcal{R}^2) \mathcal{P}(\hat{\mathbf{z}} \times \mathbf{d}_{n+1}) \cdot d\mathbf{r}' \right. \\ &\quad \left. - \sum_{m=1}^n (\mathbf{d}_{n+1} \cdot \mathbf{d}_m) \int_{S'_{j \neq m}} \prod_{j=1}^n (\mathbf{d}_j \cdot (\mathbf{r}'_p - \mathbf{a}_p)) F(\mathcal{R}^2) dS' \right), \end{aligned} \quad (111)$$

for  $n \geq 0$ , where  $F$  is a primitive of  $f$ , and  $\prod_{j=1}^0 (\dots) = 1$ ,  $\prod_{j \neq i}^1 (\dots) = 1$ , and  $\prod_{j \neq i}^0 (\dots) = 0$ . This allows, by repeated applications, to reduce  $I_{n+1}$  to a combination of contour integrals, depending on  $\lfloor (n+3)/2 \rfloor$  successive primitives of  $f$ .

Proof. Taking up the proposition 5, we consider  $(\mathbf{r}'_p - \mathbf{a}_p)$  and  $dS'_p$  instead of  $(\mathbf{r}' - \mathbf{a})$  and  $dS'$ , and derive (111). Noticing that, by repeated applications, we decrease the polynomial rank, we can reduce  $I_{n+1}$  to a combination of contour integrals for arbitrary  $n \geq 0$ .

Remark 6 :

Let us notice that  $\text{grad}_{\mathbf{r}'}(g_n(x', y')) = \mathcal{P}(\text{grad}_{\mathbf{r}'_p}(g_n(\mathbf{r}')))$ , and that, from (9),

$$\mathbf{r}' = \mathcal{P}^{-1}(\mathbf{a}_p) + h'_p \mathbf{u}_{\varphi_p}, \quad \mathbf{u}_{\varphi_p} = \mathcal{P}^{-1}(\cos\varphi_p \hat{\mathbf{x}} + \sin\varphi_p \hat{\mathbf{y}}), \quad h'_p = z'_{\mathbf{a}_{\pm}} e^{\mp i\varphi_p}, \quad (112)$$

except on lines  $\mathbf{r}' = \mathcal{P}^{-1}(\mathbf{a}_p) + t\mathbf{u}_c$ ,  $\mathbf{u}_c = \hat{\mathbf{c}}_p \pm ip(\hat{\mathbf{z}} \times \hat{\mathbf{c}}_p)$  with  $z'_{\mathbf{a}_{\pm}} = 0$  ( $\mathbf{u}_{\varphi_p}$  undefined).

### 7.3) Asymptotic reduction of $\int_{S'} g(\mathbf{r}') f(\mathcal{R}^2) dS'$

We now consider asymptotic reductions when  $\mathcal{R}^2$  has large variations respectively to  $g(\mathbf{r}')$ . The case with  $f(\mathcal{R}^2) = \frac{e^{\sqrt{\mathcal{R}^2}}}{(\sqrt{\mathcal{R}^2})^n}$  has already been studied in the propositions 6 and 7, and we now consider the case with  $f(\mathcal{R}^2) = e^{\mathcal{R}^2}$ , for a large class of smooth functions  $g(\mathbf{r}')$ .

#### Theorem 12

Let us write  $g(\mathbf{r}') = g_0(\mathbf{r}') + g_1(\mathbf{r}')$ , where  $g_0$  is a polynomial of coordinates. We assume two cases (A) and (B) :  $(\mathbf{r}'_p - \mathbf{a}_p)^2 = 0$  at two points  $\mathbf{r}'_{\pm}$  in case (A) (see proposition 14), and  $g_1$  verifies,

$$(A) \quad g_1(\mathbf{r}'_{\pm}) - g_1(\mathcal{P}^{-1}(\mathbf{a}_p)) = 0 \text{ such that } \frac{(\mathbf{r}'_p - \mathbf{a}_p)(g_1(\mathbf{r}') - g_1(\mathcal{P}^{-1}(\mathbf{a}_p)))}{(\mathbf{r}'_p - \mathbf{a}_p)^2} < \infty,$$

$$(B) \quad g_1(\mathbf{r}') = o(1) \text{ as } \mathbf{r}' \gg 1. \quad (113)$$

Considering  $\int_{S'} g(\mathbf{r}') e^{\mathcal{R}^2} dS'$  with  $\mathcal{R}^2 = v h_p'^2 + v_1 + d_0$ ,  $h_p'^2 = (\mathbf{r}'_p - \mathbf{a}_p)^2$ , we can split it in two terms. The first one  $\int_{S'} g_0(\mathbf{r}') e^{\mathcal{R}^2} dS'$  can be calculated with the theorems 10 and 11. The second one  $\int_{S'} g_1(\mathbf{r}') e^{\mathcal{R}^2} dS'$  can be reduced asymptotically for large  $v$ , by changing in theorem 10,  $f(\mathcal{R}^2)$  for  $f_1(\mathbf{r}') = g_1(\mathbf{r}') e^{\mathcal{R}^2}$ , and  $F(\mathbf{r}')$  for  $F_1(\mathbf{r}')$ , an asymptotic expression of primitive of  $f_1$  relatively to  $\mathcal{R}^2$  at constant  $\varphi_p$ , which verifies  $F_1(\mathbf{r}') \sim g_1(\mathbf{r}') (e^{\mathcal{R}^2} - e^{l_0})$  for large  $v h_p'^2$  and  $\frac{(\mathbf{r}'_p - \mathbf{a}_p)(F_1(\mathbf{r}') - F_1(\mathcal{P}^{-1}(\mathbf{a}_p)))}{(\mathbf{r}'_p - \mathbf{a}_p)^2} < \infty$  on  $S'$ .

We can then consider, for large  $v$ , that, in case (A),

$$F_1(\mathbf{r}') = g_1(\mathbf{r}') (e^{\mathcal{R}^2} - e^{l_0}), \quad (114)$$

while, in case (B),

$$F_1(\mathbf{r}') = V(|v h_p'^2|) (g_1(\mathbf{r}') (e^{\mathcal{R}^2} - e^{l_0})) + (1 - V(|v h_p'^2|)) (e^{\mathcal{R}^2} \int_0^{v h_p'^2} g_1(\mathbf{r}') d(v h'^2) + C),$$

$$C = \int_{l_0}^{\mathcal{R}^2|_{h_p'=0}} g_1(\mathbf{r}') e^{\mathcal{R}^2} d\mathcal{R}^2 \sim g_1(\mathbf{r}')|_{h_p'^2=-1/v\pi} (e^{v_1+d_0} - e^{l_0}) \text{ if } h'_p g_1(\mathbf{r}')|_{h'_p \rightarrow \infty} \neq 0, \quad (115)$$

with  $V(t) \sim \frac{(t/\pi)^2}{1+(t/\pi)^2}$  as  $t = o(1)$  or  $t \gg 1$ ,  $\mathbf{r}'$  given in (112), and a large constant  $l_0 = vl_1 + v_1 + d_0$  that could tend to infinity. The path of integration from  $l_0$  to  $\mathcal{R}^2$  is assumed such that  $g_1(\mathbf{r}')e^{\mathcal{R}^2}$  remains analytic and  $\text{Re}(l_0) \ll \text{Re}(\mathcal{R}^2)$  on the path.

Proof. The first integral term  $\int_{S'} g_0(\mathbf{r}') f(\mathcal{R}^2) dS'$  in case (A) and (B) can be reduced from theorems 11 and 10. For the second integral with  $g_1$ , we use theorem 10, with a primitive  $F_1$  at constant  $\varphi_p$  of  $g_1(\mathbf{r}') f(\mathcal{R}^2)$  in place of a primitive of  $f(\mathcal{R}^2)$ . In case (A), we notice that

$$\frac{\partial g_1(\mathbf{r}')}{\partial h_p'^2} \Big|_{r'=r_\pm} = \frac{g_1(\mathbf{r}') - g_1(r_\pm)}{z'_{\mathbf{a}+} z'_{\mathbf{a}-}} \Big|_{r \rightarrow r_\pm} = \frac{[\frac{(r-r_\pm)}{z'_{\mathbf{a}\pm}} \cdot \text{grad}_{r'}(g_1(\mathbf{r}'))]}{z'_{\mathbf{a}\mp}} \Big|_{r \rightarrow r_\pm} \text{ and } \frac{\partial g_1(\mathbf{r}')}{\partial h_p'^2} \Big|_{r' \neq r_\pm} < \infty,$$

on  $S'$ , and thus  $\frac{\partial e^{vh_p'^2} g(\mathbf{r}')}{\partial (h_p'^2)} \sim v e^{vh_p'^2} g(\mathbf{r}')$  on  $S'$  as  $v$  is large, so that  $F_1(\mathbf{r}') \sim g_1(\mathbf{r}')(e^{\mathcal{R}^2} - e^{l_0})$ . In case (B), the condition  $g_1(\mathbf{r}') = o(1)$  implies that  $g_1(\mathbf{r}')$  vanishes when  $h_p' \rightarrow \infty$ . Using (112), we have  $\frac{\partial e^{vh_p'^2} g_1(\mathbf{r}')}{\partial (vh_p'^2)} = e^{vh_p'^2} g_1(\mathbf{r}') + \frac{\mathbf{u}_{\varphi_p} \cdot \text{grad}_{r'}(g_1)}{2vh_p'} \sim e^{vh_p'^2} g(\mathbf{r}')$  for large  $vh_p'^2$ , and thus,

$$\int_{vl_1}^{vh_p'^2} g_1(\mathbf{r}') e^{\mathcal{R}^2} d(vh'^2) \sim g_1(\mathbf{r}') \int_{l_0}^{\mathcal{R}^2|_{h'=h_p'}} e^{\mathcal{R}^2} d(\mathcal{R}^2), \quad (116)$$

with  $\mathcal{R}^2 = vh'^2 + v_1 + d_0$  on the integration path, while, for small  $vh_p'^2$  but  $v$  is large,

$$\int_{vl_1}^{vh_p'^2} g_1(\mathbf{r}') e^{\mathcal{R}^2} d(vh'^2) \sim e^{\mathcal{R}^2} \int_0^{vh_p'^2} g_1(\mathbf{r}') d(vh'^2) + C, \quad (117)$$

where  $l_0 = vl_1 + v_1 + d_0$ ,  $l_1 = l_2^2$ , and  $C = \int_{l_0}^{\mathcal{R}^2|_{h_p'=0}} g_1(\mathbf{r}') e^{\mathcal{R}^2} d\mathcal{R}^2$ ; the path of integration from  $l_0$  to  $\mathcal{R}^2$  is assumed with a constant  $l_1$  such that  $g(\mathbf{r}')e^{\mathcal{R}^2}$  remains analytic and  $\text{Re}(l_0) < \text{Re}(\mathcal{R}^2)$  on the path. We can then combine these results and derive (115) for  $F_1$ . To simplify  $C$  for large  $v$  as  $h_p' g_1(\mathbf{r}')|_{h_p' \rightarrow \infty} \neq 0$ , we consider  $\int_{-\infty}^{vh_p'^2} \frac{e^{vh'^2}}{\sqrt{1+ch'^2}} d(vh'^2)$  given by,

$$\begin{aligned} \sqrt{\frac{-v}{c}} e^{-\frac{v}{c}} \sqrt{\pi} \text{erfc}\left(\sqrt{\frac{-v}{c}}(1+ch_p'^2)\right) &\sim \frac{e^{vh_p'^2}}{\sqrt{1+ch_p'^2}} \text{ when } vh'^2 \text{ is large,} \\ &\sim \frac{vh_p'^2}{\sqrt{1+ch_p'^2}} + 1 \text{ when } vh_p'^2 \text{ is small but } \frac{v}{c} \text{ is large,} \\ &\sim e^{vh_p'^2} \frac{2v}{c} \sqrt{1+ch_p'^2} + \sqrt{\frac{-v\pi}{c}} \text{ when } vh_p'^2 \text{ is small but } \frac{v}{c} \text{ is small.} \end{aligned}$$

It agrees with (116)-(117) if  $C \sim g_1(\mathbf{r}')|_{h_p'^2=-1/v\pi}$  from which we derive last line of (115).

#### 7.4) Miscellaneous results

##### Proposition 15

We have, for arbitrary constants  $\mathbf{d}$  and  $A_0$ ,

$$\int_{S'} (\mathbf{d} \cdot \mathbf{r}') f(\mathcal{R}^2) dS' = A_1 \cdot \left( \int_{\mathcal{L}} [F(\mathcal{R}^2) - A_0] d\mathbf{r}' - (\widehat{\mathbf{z}} \times \mathbf{w}') \int_{S'} f(\mathcal{R}^2) dS' \right), \quad (118)$$

where  $F$  is a primitive of  $f$  and  $A_1 = \frac{\mathcal{P}(\widehat{\mathbf{z}} \times \mathcal{P}^{-1}(\mathbf{d}))}{-2ikp\xi}$ . This implies, in particular,

$$(\widehat{\mathbf{z}} \times \mathbf{w}') \int_{S'} f(\mathcal{R}^2) dS' |_{\xi \rightarrow 0} = \int_{\mathcal{L}} [F(\mathcal{R}^2) - A_0] d\mathbf{r}' |_{\xi \rightarrow 0}. \quad (119)$$

Proof. Using that  $\mathbf{d} \cdot \mathbf{r}' = \mathcal{P}^{-1}(\mathbf{d}) \cdot (\mathbf{r}'_p - \mathbf{a}_p) + \mathcal{P}^{-1}(\mathbf{d}) \cdot \mathbf{a}_p$ , then the proposition 2, we derive,

$$\begin{aligned} \int_{S'} \mathbf{d} \cdot \mathbf{r}' f(\mathcal{R}^2) dS' - \mathcal{P}^{-1}(\mathbf{d}) \cdot \mathbf{a}_p \int_{S'} f(\mathcal{R}^2) dS' &= \frac{1}{p} \int_{S'_p} \mathcal{P}^{-1}(\mathbf{d}) \cdot (\mathbf{r}'_p - \mathbf{a}_p) f(\mathcal{R}^2) dS'_p \\ &= \frac{1}{2p} \int_{\mathcal{L}} \frac{[F(\mathcal{R}^2) - A_0]}{v} [\mathcal{P}(\widehat{\mathbf{z}} \times \mathcal{P}^{-1}(\mathbf{d})) \cdot d\mathbf{r}'], \end{aligned} \quad (120)$$

and thus (118) if we notice that  $\mathcal{P}^{-1}(\mathbf{d}) \cdot \mathbf{a}_p = \frac{\mathcal{P}(\widehat{\mathbf{z}} \times \mathcal{P}^{-1}(\mathbf{d})) \cdot (\widehat{\mathbf{z}} \times \mathbf{w}')}{2ikp\xi}$ . For  $\xi \rightarrow 0$ ,  $\mathbf{d} = \widehat{\mathbf{z}} \times \mathbf{c}_p$  implies  $\frac{\mathcal{P}(\widehat{\mathbf{z}} \times \mathcal{P}^{-1}(\mathbf{d}))}{2ikp\xi} \rightarrow \infty$ ; right side of (118) being bounded, we then derive (119).

##### Proposition 16

Considering  $F$  as the primitive of  $f$ , we have, for the limit case  $\xi \rightarrow 0$  where  $\mathcal{R}^2 \rightarrow \mathbf{w}' \cdot \mathbf{r}'$ ,

$$\int_{S'} f(\mathbf{w}' \cdot \mathbf{r}') dS' = \frac{(\widehat{\mathbf{z}} \times \mathbf{w}')}{(\mathbf{w}')^2} \int_{\mathcal{L}} (F(\mathbf{w}' \cdot \mathbf{r}') - F(0)) d\mathbf{r}', \quad (121)$$

Proof. We can obtain this equality from (119), or by using  $\text{grad}(F(\mathbf{w}' \cdot \mathbf{r}')) = \mathbf{w}' F'(\mathbf{w}' \cdot \mathbf{r}')$ .

## 8) Numerical results

### 8.1) Numerical results concerning the reduction of a surface integral whose integrand has an exponential term with a quadratic argument

We choose to validate our reduction to a contour integral given in theorem 9, by verifying,

$$\frac{k}{\sqrt{\pi}} \int_{S'} e^{\mathbf{w}' \cdot \mathbf{r}' - ik\xi(|\mathbf{r}'|^2 - (\mathbf{c}_p \cdot \mathbf{r}')^2)} dS' = \frac{k}{2\sqrt{\pi}} \int_{\mathcal{L}} \left[ \frac{(e^{\mathbf{w}' \cdot \mathbf{r}' - ik\xi(|\mathbf{r}'|^2 - (\mathbf{c}_p \cdot \mathbf{r}')^2)} - A_1)}{(\mathbf{w}' \cdot \mathbf{r}' - ik\xi(|\mathbf{r}'|^2 - (\mathbf{c}_p \cdot \mathbf{r}')^2)) - \frac{\mathbf{w}'_m \cdot \mathbf{r}'}{4ik\xi}} \right] [(\hat{\mathbf{z}} \times (\mathbf{r}' - \frac{\mathcal{M}(\mathbf{w}')}{2ik\xi(1 - \mathbf{c}_p^2)})) \cdot d\mathbf{r}'], \quad (122)$$

where  $\xi$ ,  $\mathbf{c}_p$  and  $\mathbf{w}'$  are arbitrary complex constants,  $A_1 = e^{\frac{\mathbf{w}'_m \cdot \mathbf{r}'}{4ik\xi}}$ . For this, we compare the amplitudes of both integrals when  $\mathbf{w}'$  and  $\mathbf{c}_p$  vary. For the applications, we notice that for large values of  $\mathbf{a}_p$  satisfying (94), we can let  $A_1 = 0$ .

We take  $S'$ , a square of 1m side length, whose center is at  $\mathbf{r}_c = 0.5\hat{\mathbf{x}} + 0.5\hat{\mathbf{y}}$ ,  $k = 2\pi/\lambda$ ,  $\lambda = 2\text{cm}$ ,  $\xi = e^{i\Phi}/|\mathbf{r}|$ ,  $|\mathbf{r}| = 1\text{m}$ . The term  $\mathbf{c}_p = \sin\theta\cos\varphi\hat{\mathbf{x}} + \sin\theta\sin\varphi\hat{\mathbf{y}}$  is considered for  $\varphi = \pi/4$  as  $\theta$  varies, in the case where  $\mathbf{w}' = 2ik\mathbf{c}_p$ , in figures 3 and 4. The modulus of each side in (124) are given in  $dB$  (decibels), for  $\xi = e^{i\Phi}/|\mathbf{r}|$  with  $\Phi = 0$  (green),  $\Phi = -\frac{\pi}{18}$  (red) and  $\Phi = -\frac{\pi}{2}$  (blue), where we notice the perfect agreement of the results.

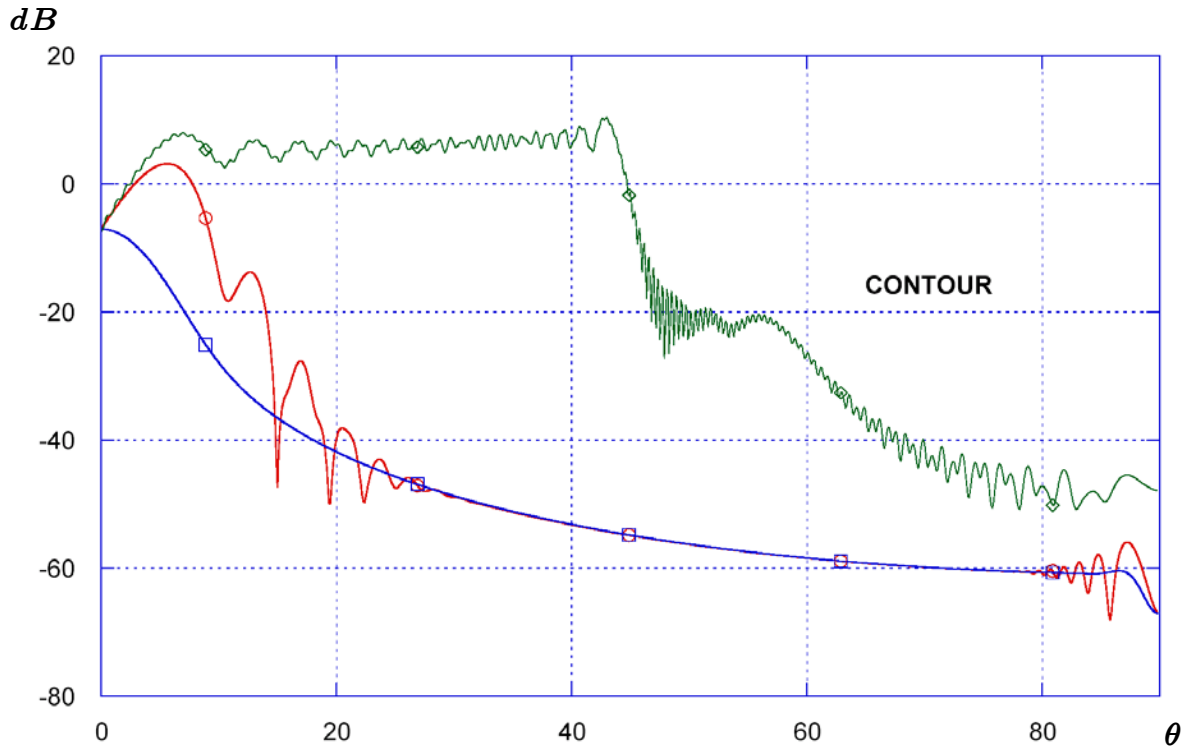


figure 3 : amplitude in dB of the contour integral term in (123)

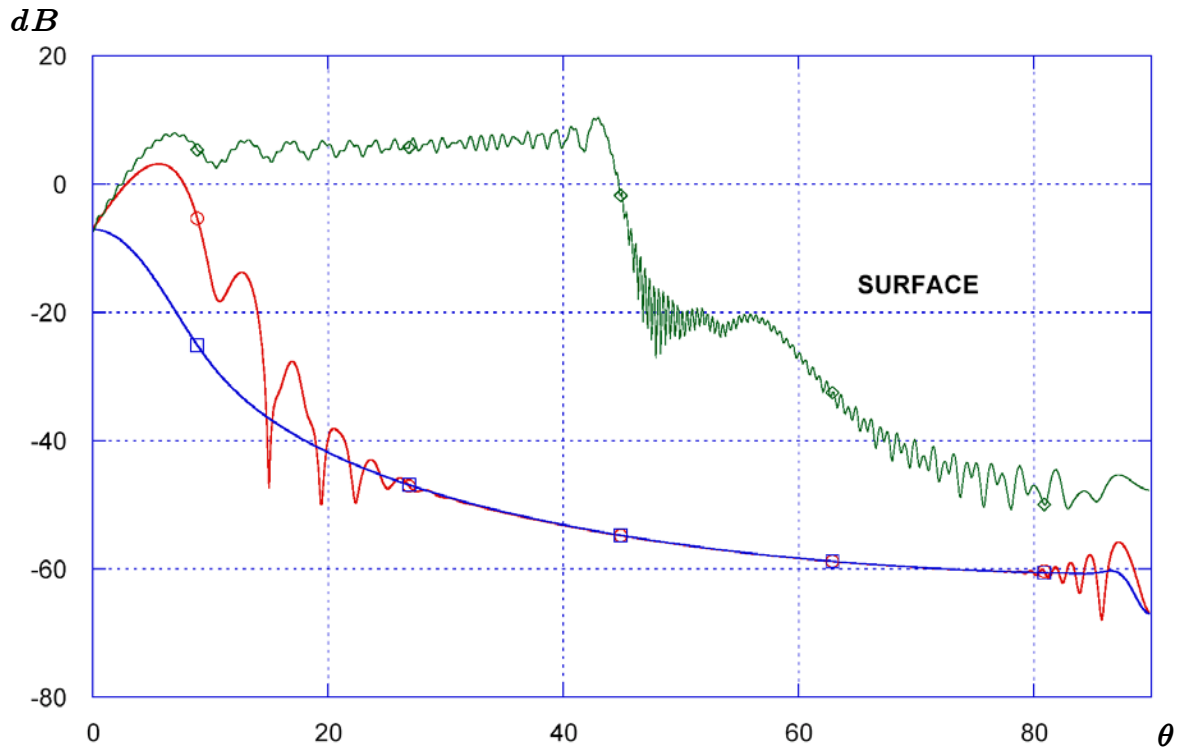


figure 4 : amplitude in dB of the surface integral term in (124)

**8.2) Numerical results concerning the reduction of near-field radiation surface integral in Physical Optics, for a large plate illuminated by a point source with spherical pattern.**

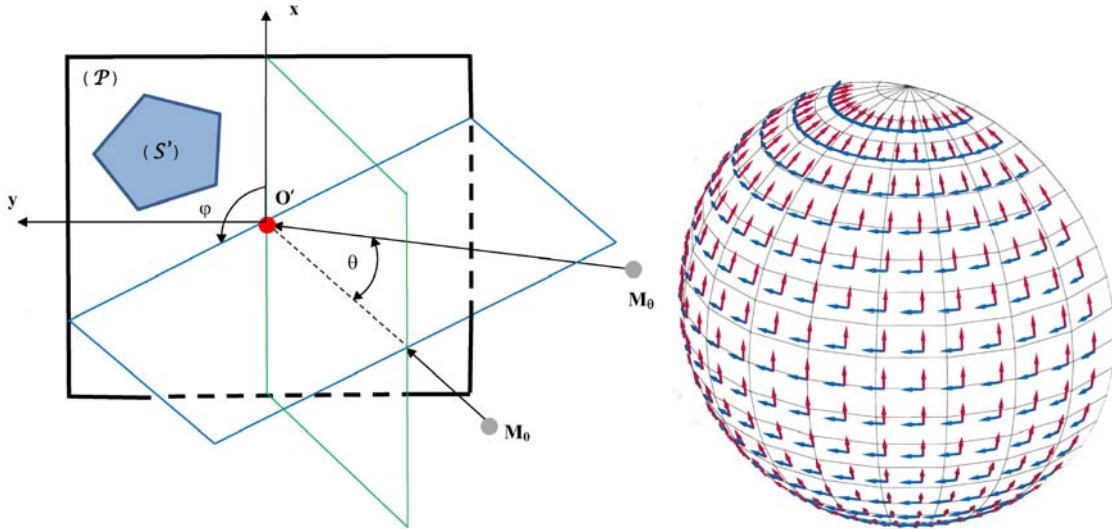


figure 5 : Geometry for the near-field backscattering of  $S'$  in Physical Optics (left-hand), for a source at  $M_0$  illuminating  $S'$  with a spherical pattern (right-hand)

Following the theorem 5 and 7 with  $g = 1$ , we consider the monostatic Physical Optics radiation term  $U = \mathbf{E} \cdot \hat{\mathbf{d}}$  in polarisation  $VV$  (resp.  $U = \mathbf{E} \cdot \mathbf{v}_0$  in  $HH$ ) at constant  $\varphi$  when  $\theta$  varies (fig. 5), for a point source with isotropic pattern, and its reduction (69) (resp. (78)), for a plate  $S'$  of large dimension. We then compare contour and surface integral expressions, in figures 6 and 7, for the backscattering pattern  $\frac{\sqrt{4\pi}U}{|E_{z_i}|_{O'}}$  in  $dB$ , when we consider a square plate with 1m side length, shifted from the origin  $O'$  and centered at  $(0, 0.5m)$ , observed in the plane  $\varphi = 20deg$ , at the distance  $\tilde{r} = \tilde{r}_0 = 1$  m, for  $\lambda = 2cm$ .

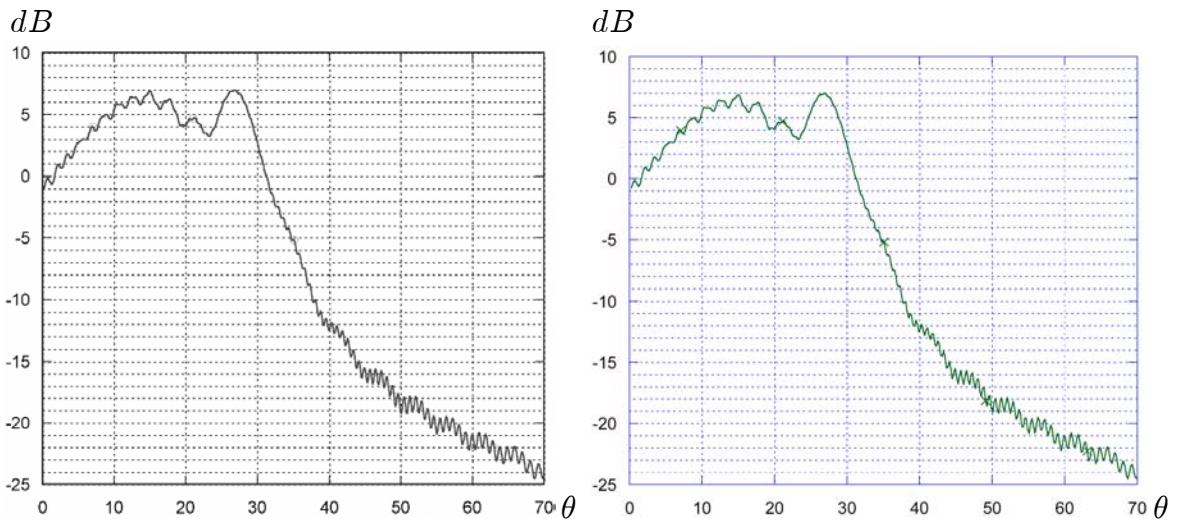


figure 6 : polarisation  $HH$ , contour (left-hand) and surface (right-hand) integrals

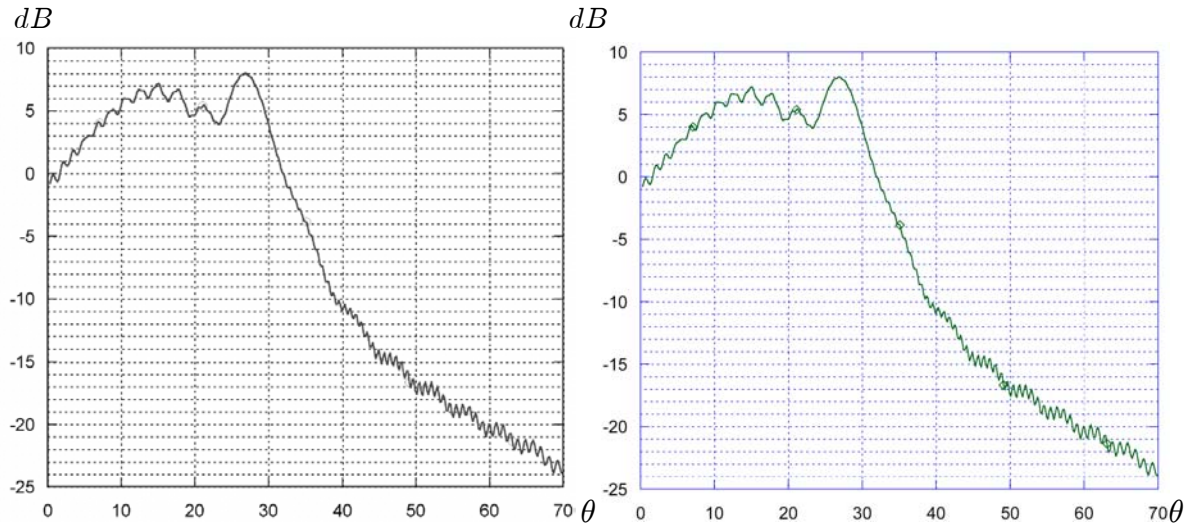


figure 7 : polarisation  $VV$ , contour (left-hand) and surface (right-hand) integrals

In figure 8, we consider a larger shift of the plate center, disposed at  $(0, 3m)$ , and observes at  $\tilde{r} = \tilde{r}_0 = 3.5m$  in the plane  $\varphi = 80deg$ . We distinguish more clearly differences between the polarisations  $HH$  and  $VV$  (in figure 8,  $VV$  is clearly larger than  $HH$ ), while the agreement between our contour integral and the surface integral continues to be excellent.

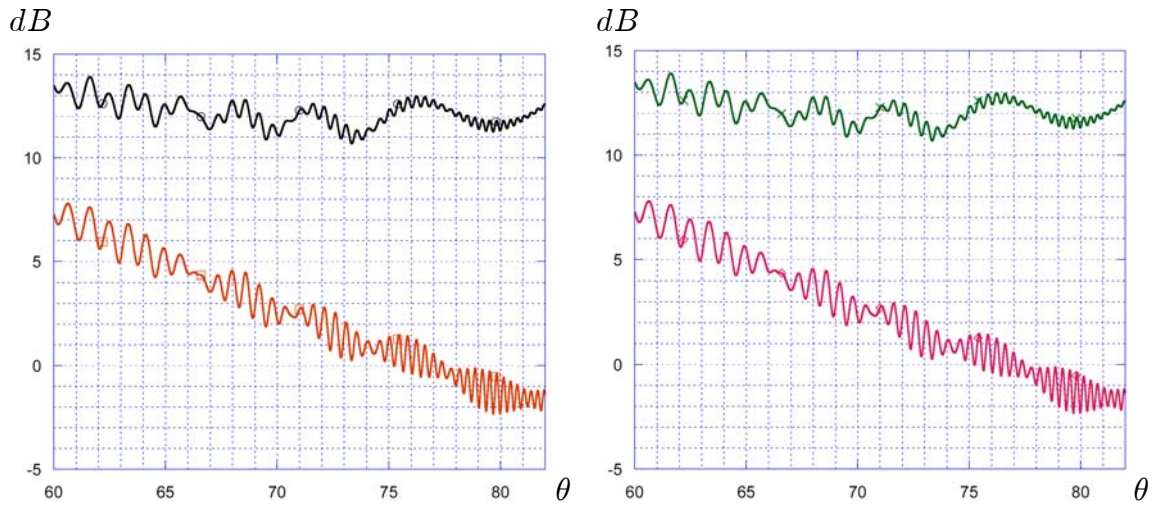


figure 8 : polarisation  $VV$  and  $HH$ , contour (left-hand) and surface (right-hand) integrals



## **9) Conclusion**

We have derived a general method of reduction to a non-singular contour integral, for a large class of radiation surface integrals, when the currents are analytically defined on the surface of integration, as in Physical Optics.

Thus, near-field expressions, valid from small to large distance, are developed for arbitrary

- dimensions of objects,

- illumination patterns,

- impedance boundary conditions (perfectly or imperfectly reflective surfaces).

for a large plane plates in monostatic case, then extended for a moderate curvature of plates.

For the applications to bistatic cases, specific original expressions are also presented for the reduction of surface integrals whose integrands have a complex phase that is an arbitrary complex polynomial of the coordinates of rank 2, which permits calculus for quadratic approximations of phase in Green's functions or arbitrary gaussian beam illuminations.

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Appendice A : Primitives for the theorems 3 and 4, and properties

A.1) Primitives for the theorem 3

Let us show that,

$$\begin{aligned} \int \frac{1}{\tilde{R}} \left( \left( \frac{\psi'}{\tilde{R}} \right)^2 \right)' dR^2 &= 2 \left( \frac{\psi'}{\tilde{R}} \right)^2 = 2 \frac{(ik + \frac{1}{\tilde{R}})^2}{\tilde{R}^4} e^{-i2k\tilde{R}}, \\ \int \left( \frac{\psi'}{\tilde{R}} \right)^2 dR^2 &= -2 \frac{(1 + 2ik\tilde{R})}{4\tilde{R}^4} e^{-i2k\tilde{R}} \\ \int \tilde{R} \left( \left( \frac{\psi'}{\tilde{R}} \right)^2 \right)' dR^2 &= 2 \left( -k^2 + \frac{3ik}{\tilde{R}} + \frac{3}{2R^2} \right) \frac{e^{-i2k\tilde{R}}}{R^2} \end{aligned} \quad (125)$$

with  $\tilde{R} = \sqrt{R.R}$ . The first line is trivial, while the second and third line need some manipulation on  $\psi(\tilde{R}) = \frac{e^{-ik\tilde{R}}}{\tilde{R}}$ . We remark first that  $\psi$  verifies  $\Delta\psi + k^2\psi = 0$ , and thus,

$$\frac{1}{R^2} (\tilde{R}^3 \left( \frac{\psi'}{\tilde{R}} \right))' = \tilde{R} \left( \frac{\psi'}{\tilde{R}} \right)' + 3 \frac{\psi'}{\tilde{R}} = -k^2\psi, \quad (126)$$

which multiplied by  $\frac{\psi'}{\tilde{R}}$ , leads us to

$$\tilde{R} \left( \left( \frac{\psi'}{\tilde{R}} \right)^2 \right)' = -k^2 \frac{((\psi)^2)'}{\tilde{R}} - 6 \left( \frac{\psi'}{\tilde{R}} \right)^2. \quad (127)$$

Considering that, by integration by parts,

$$\int \tilde{R} \left( \left( \frac{\psi'}{\tilde{R}} \right)^2 \right)' \tilde{R} d\tilde{R} = (\psi')^2 - 2 \int \left( \frac{\psi'}{\tilde{R}} \right)^2 \tilde{R} d\tilde{R} \quad (128)$$

we can use (127), and write,

$$\int -k^2 ((\psi)^2)' d\tilde{R} - \int 6 \left( \frac{\psi'}{\tilde{R}} \right)^2 \tilde{R} d\tilde{R} = (\psi')^2 - 2 \int \left( \frac{\psi'}{\tilde{R}} \right)^2 \tilde{R} d\tilde{R} \quad (129)$$

so that we obtain the second line of (125) following,

$$\int \left( \frac{\psi'}{\tilde{R}} \right)^2 \tilde{R} d\tilde{R} = \frac{1}{4} ((ik\psi)^2 - (\psi')^2) = - \left( 2ik + \frac{1}{\tilde{R}} \right) \frac{\psi^2}{4\tilde{R}} \left( = \frac{(\tilde{R} \psi^2)'}{4R^2} \right) \quad (130)$$

We can then rewrite (128), and derive the third line of (125),

$$\int \tilde{R} \left( \left( \frac{\psi'}{\tilde{R}} \right)^2 \right)' \tilde{R} d\tilde{R} = \frac{3(\psi')^2 - (ik\psi)^2}{2} = \left( -k^2 + \frac{3ik}{\tilde{R}} + \frac{3}{2R^2} \right) \psi^2 \quad (131)$$

A.2) Primitives for the theorem 4

For the theorem 4, we have to complete (125). For this, we write,

$$\begin{aligned}
 \int \left(\frac{\psi'}{\widetilde{R}}\right)' \left(\frac{\psi'}{\widetilde{R}}\right)' dR^2 &= -2 \int \left(\frac{\psi'}{\widetilde{R}}\right)' \left(k^2 \psi + 3 \frac{\psi'}{\widetilde{R}}\right) d\widetilde{R} = \\
 &= \left(-2k^2 \frac{\psi' \psi}{\widetilde{R}} + k^2 \int \left(\frac{\psi'}{\widetilde{R}}\right)^2 dR^2\right) - 3 \left(\frac{\psi'}{\widetilde{R}}\right)^2 \\
 &= 2k^2 \left(ik + \frac{1}{\widetilde{R}}\right) \frac{\psi^2}{\widetilde{R}} - k^2 \left(ik + \frac{1}{2\widetilde{R}}\right) \frac{\psi^2}{\widetilde{R}} - 3 \left(\frac{\psi'}{\widetilde{R}}\right)^2 \\
 &= k^2 \left(ik + \frac{3}{2\widetilde{R}}\right) \frac{\psi^2}{\widetilde{R}} - 3 \left(ik + \frac{1}{\widetilde{R}}\right)^2 \left(\frac{\psi}{\widetilde{R}}\right)^2 \\
 &= k^2 ik \frac{\psi^2}{\widetilde{R}} + 3 \left(\frac{\psi}{\widetilde{R}}\right)^2 \left(\frac{k^2}{2} - \left(ik + \frac{1}{\widetilde{R}}\right)^2\right)
 \end{aligned} \tag{132}$$

where we have used (126), an integration by parts, then (130).

A3) Some exact and asymptotic properties of primitives  $\int_{\infty}^{R^2} \frac{e^{-v\widetilde{R}}}{\widetilde{R}^n} dR^2$

$$\begin{aligned}
 \int_{\infty}^{R^2} \frac{e^{-v\widetilde{R}}}{\widetilde{R}^n} dR^2 &= 2v^{n-2} V_{n-1}(v\widetilde{R}) \sim -\frac{2}{v} \frac{e^{-v\widetilde{R}}}{\widetilde{R}^{n-1}} \left(1 - \frac{n}{v\widetilde{R}} + \dots\right), \text{ if } v\widetilde{R} \rightarrow \infty \tag{133} \\
 V_n(u) &= \int_{\infty}^u \frac{e^{-t}}{t^n} dt = -\frac{E_n(u)}{u^{n-1}}, \quad V_n(u) \sim -\frac{e^{-u}}{u^n} \left(1 - \frac{n}{u} + \frac{n(n+1)}{u^2} - \dots\right) \text{ if } u \rightarrow \infty
 \end{aligned}$$

with  $\widetilde{R} = \sqrt{\mathbf{R} \cdot \mathbf{R}}$ , while, for primitives of order  $p \geq 2$ , we have,

$$\begin{aligned}
 (p-1)! \int_{\infty}^{R^2} d\mathbf{R}^2 \dots \int_{\infty}^{R^2} d\mathbf{R}^2 \frac{e^{-v\widetilde{R}}}{\widetilde{R}^n} &= \int_{\infty}^{R^2} (\mathbf{R}'^2 - \mathbf{R}^2)^{p-1} \frac{e^{-v\widetilde{R}}}{\widetilde{R}^n} dR^2 \\
 &= \sum_{0 \leq m \leq p-1} (-1)^{p-1-m} C_{p-1}^m (\widetilde{R}')^{2m} \int_{\infty}^{R^2} \frac{e^{-v\widetilde{R}}}{\widetilde{R}^{n-2(p-1-m)}} dR^2 \\
 &= 2 \sum_{0 \leq m \leq p-1} (-1)^{p-1-m} C_{p-1}^m (\widetilde{R}')^{2m} (v^{n-2(p-1-m)-2} V_{n-2(p-1-m)-1}(v\widetilde{R}')) \\
 &= -2 \sum_{0 \leq m \leq p-1} (-1)^{p-1-m} C_{p-1}^m (\widetilde{R}')^{2m} \left(\frac{E_{n-2(p-1-m)-1}(v\widetilde{R}')}{(\widetilde{R}')^{n-2(p-1-m)-2}}\right) \\
 &= \frac{-2}{(\widetilde{R}')^{n-2p}} \sum_{0 \leq m \leq p-1} (-1)^{p-1-m} C_{p-1}^m E_{n-2(p-1-m)-1}(v\widetilde{R}')
 \end{aligned} \tag{134}$$

where we remark that the first asymptotic orders of the sum, when  $v\tilde{R} \gg 1$ , vanish. So, applying this expression, we have for  $p = 2$ ,

$$\begin{aligned}
 & \int_{\infty}^{|R'|^2} d\mathbf{R}^2 \int_{\infty}^{|R'|^2} d\mathbf{R}^2 \frac{e^{-v\tilde{R}}}{\tilde{R}^n} \sim \\
 & \sim \frac{-2}{v} \frac{e^{-v|R'|}}{\tilde{R}'^{n-(2p-1)}} \sum_{0 \leq m \leq 1} (-1)^{p-1-m} C_{p-1}^m \left(1 - \frac{n-2(p-m)+1}{v\tilde{R}'}\right) \Big|_{p=2} \\
 & \sim \frac{4}{v^2} \frac{e^{-v|R'|}}{(\tilde{R}')^{n-(2p-2)}} \sum_{0 \leq m \leq 1} (-1)^{p-1-m} m C_{p-1}^m \Big|_{p=2} = \frac{4}{v^2} \frac{e^{-v|R'|}}{(\tilde{R}')^{n-2}} \quad (135)
 \end{aligned}$$

We also notice that,

$$\begin{aligned}
 & \int_{\infty}^{R'^2} \frac{-ve^{-v\tilde{R}}}{\tilde{R}^{n+1}} \left(1 + \frac{n}{v\tilde{R}}\right) d\mathbf{R}^2 = \frac{2e^{-v\tilde{R}'}}{\tilde{R}'^n} \\
 & = -2v^n (V_n(v\tilde{R}') + nV_{n+1}(v\tilde{R}')). \quad (136)
 \end{aligned}$$

### Appendice B : Monostatic PO for imperfectly conducting surface whose impedance (or admittance) can vary on the surface

In the hypothesis of a point source illuminating at large distance an impedance plane surface of normal  $\hat{\mathbf{z}}$  [28]-[29], we can consider the incident field as a locally plane wave, combination of polarisations  $TE$  ( $\mathbf{E} //$  surface) and  $TM$  ( $\mathbf{H} //$  surface), respectively denoted  $(\mathbf{E}_e^i, \mathbf{H}_e^i)$  and  $(\mathbf{E}_h^i, \mathbf{H}_h^i)$ . The currents  $\mathbf{J}$  and  $\mathbf{M}$ , respectively denoted  $\mathbf{J}_{e(h)}|_S = \hat{\mathbf{z}} \times \mathbf{H}_{e(h)}$  and  $\mathbf{M}_{e(h)}|_S = -\hat{\mathbf{z}} \times \mathbf{E}_{e(h)}$ , then satisfy,

$$\mathbf{J}_{e(h)} = 2b'_{e(h)} \left(\frac{z}{\tilde{R}}\right) (\hat{\mathbf{z}} \times \mathbf{H}_{e(h)}^i), \quad \mathbf{M}_{e(h)} = -a'_{e(h)} \left(\frac{z}{\tilde{R}}\right) (\hat{\mathbf{z}} \times \mathbf{J}_{e(h)}) \quad (137)$$

where  $a'_{e,h}$  and  $b'_{e,h}$  are regular functions. In the case of constant impedance,  $b_{e(h)}$  and  $a_{e(h)}$  are constants, and  $b'_{e(h)} \equiv 1$  and  $a'_{e(h)} \equiv 0$  in perfectly conducting case. Thereafter, we suppress the indices  $e$  and  $h$  for the use in the expressions (47) with (137).

#### B.1) Physical Optics diffracted fields and reduction, for an electric dipolar source

Theorem 13 :

*The Physical Optics field, diffracted by the surface  $S'$  with an impedance condition, illuminated by the electric dipole  $\mathbf{J}_0 = \hat{\mathbf{d}}\delta(\mathbf{r}' - \mathbf{r}_0)$  is given by (47) with (137). It can be*

written, in monostatic case, when  $\widehat{\mathbf{d}} \cdot \widehat{\mathbf{z}} = 0$ , following

$$\begin{aligned} \mathbf{E} &\sim \frac{Z_0}{8\pi^2 ik} \int_{S'} \frac{zb'(\frac{z}{\widetilde{R}})}{2} \widetilde{R} \left( \left( \frac{\psi'}{\widetilde{R}} \right)^2 \right)' ((\mathbf{u} \cdot \widehat{\mathbf{d}}) \mathbf{u} - \widehat{\mathbf{d}}) dS' + \\ &+ \frac{1}{8\pi^2} \int_{S'} c\left(\frac{z}{\widetilde{R}}\right) (\psi')^2 \left( \frac{\mathbf{R}^2 - z^2}{\mathbf{R}^2} \left( \frac{\mathbf{r}' - \mathbf{a}}{(\mathbf{r}' - \mathbf{a})^2} \times \widehat{\mathbf{z}} \right) \left( ((\mathbf{r}' - \mathbf{a}) \times \widehat{\mathbf{z}}) \cdot \widehat{\mathbf{d}} \right) + \right. \\ &\left. + ((\mathbf{u} \cdot \widehat{\mathbf{d}}) \mathbf{u} - \widehat{\mathbf{d}}) \right) dS' \end{aligned} \quad (138)$$

for large  $k\widetilde{R}$ , with  $c \equiv a'b'$ ,  $(\mathbf{u} \cdot \widehat{\mathbf{d}}) \mathbf{u} = \frac{z\widehat{\mathbf{z}}(\widehat{\mathbf{d}} \cdot (\mathbf{a} - \mathbf{r}'))}{\mathbf{R}^2} + \frac{(\mathbf{R}^2 - z^2)(\widehat{\mathbf{d}} \cdot (\mathbf{a} - \mathbf{r}'))(\mathbf{a} - \mathbf{r}')}{\mathbf{R}^2(\mathbf{a} - \mathbf{r}')^2}$ , and,

$$\begin{aligned} \int c\left(\frac{z}{\widetilde{R}}\right) (\psi')^2 d\mathbf{R}^2 &= - \frac{ikc\left(\frac{z}{\widetilde{R}}\right) e^{-i2k\widetilde{R}}}{\widetilde{R}} \left(1 + O\left(\frac{1}{k\widetilde{R}}\right)\right), \\ \int zb\left(\frac{z}{\widetilde{R}}\right) \widetilde{R} \left( \left( \frac{\psi'}{\widetilde{R}} \right)^2 \right)' d\mathbf{R}^2 &= - \frac{2k^2 zb\left(\frac{z}{\widetilde{R}}\right) e^{-i2k\widetilde{R}}}{\mathbf{R}^2} \left(1 + O\left(\frac{1}{k\widetilde{R}}\right)\right), \end{aligned} \quad (139)$$

We can then apply the propositions 6 and 7 to reduce each integral to non singular contour integral without any special function.

Proof. We consider, using (137), that  $S'$  is illuminated by the radiation of the electrical dipole  $\mathbf{J}_0 = \widehat{\mathbf{d}} \delta(\mathbf{r}' - \mathbf{r}_0)$  with  $\widehat{\mathbf{d}} \cdot \widehat{\mathbf{z}} = 0$  ( $\widehat{\mathbf{d}} // \text{plane}$ ), and thus

$$\begin{aligned} 2\pi \mathbf{J}|_S &= b' \left( \frac{z}{\widetilde{R}} \right) \widehat{\mathbf{z}} \times (\mathbf{R}_0 \times \frac{\psi'_0}{\widetilde{R}_0} \widehat{\mathbf{d}}), \quad \mathbf{R}_0 = (\mathbf{r}' - \mathbf{r}_0) \\ 2\pi \mathbf{M}|_S &= -c \left( \frac{z}{\widetilde{R}} \right) \widehat{\mathbf{z}} \times (\widehat{\mathbf{z}} \times (\mathbf{R}_0 \times \frac{\psi'_0}{\widetilde{R}_0} \widehat{\mathbf{d}})), \\ &= -c \left( \frac{z}{\widetilde{R}} \right) \left( (\widehat{\mathbf{z}} \cdot (\mathbf{R}_0 \times \frac{\psi'_0}{\widetilde{R}_0} \widehat{\mathbf{d}})) \widehat{\mathbf{z}} - (\mathbf{R}_0 \times \frac{\psi'_0}{\widetilde{R}_0} \widehat{\mathbf{d}}) \right) \end{aligned} \quad (140)$$

where we suppose  $a'(v)$ ,  $b'(v)$ , and  $c(v)$  regular for  $v \in [0, 1]$ . In the case where  $\mathbf{R} = -\mathbf{R}_0 = z\widehat{\mathbf{z}} + \mathbf{a} - \mathbf{r}'$  (monostatic case), we then obtain,

$$\begin{aligned} \mathbf{E} &\sim \frac{Z_0}{8\pi^2 ik} \int_{S'} \frac{zb'(\frac{z}{\widetilde{R}})}{2} \widetilde{R} \left( \left( \frac{\psi'}{\widetilde{R}} \right)^2 \right)' \mathbf{u} \times (\mathbf{u} \times \widehat{\mathbf{d}}) dS' + \\ &+ \frac{1}{8\pi^2} \int_{S'} c\left(\frac{z}{\widetilde{R}}\right) \left( \frac{\psi'}{\widetilde{R}} \right)^2 \left( -\mathbf{R} \times \widehat{\mathbf{z}} (\widehat{\mathbf{z}} \cdot (\mathbf{R} \times \widehat{\mathbf{d}})) + \mathbf{R} \times (\mathbf{R} \times \widehat{\mathbf{d}}) \right) dS' \end{aligned} \quad (141)$$

and thus (138). We can then, following the theorems 1 and 2, and the propositions 6 and 7, reduce the field  $\mathbf{E}$  to a contour integral. For this, we can use (125), and also,

$$\begin{aligned} \int (\psi')^2 d\mathbf{R}^2 &= \int \mathbf{R}^2 \left(\frac{\psi'}{\tilde{R}}\right)^2 d\mathbf{R}^2 = \frac{(\tilde{R} \psi^2)'}{2} - \int \frac{(\tilde{R} \psi^2)'}{2\tilde{R}^2} d\mathbf{R}^2 \\ &= -\frac{(3 + 2ik\tilde{R})}{2\tilde{R}^2} e^{-i2k\tilde{R}} - \int \frac{e^{-i2k\tilde{R}}}{2\tilde{R}^4} d\mathbf{R}^2 = -\frac{ik}{2\tilde{R}} e^{-i2k\tilde{R}} \left(1 + O\left(\frac{1}{k\tilde{R}}\right)\right), \end{aligned} \quad (142)$$

since  $\int \left(\frac{\psi'}{\tilde{R}}\right)^2 d\mathbf{R}^2 = \frac{(\tilde{R} \psi^2)'}{2\tilde{R}^2}$  and  $\int \frac{(\tilde{R} \psi^2)'}{2\tilde{R}^2} d\mathbf{R}^2 = \psi^2 + \int \frac{\psi^2}{2\tilde{R}^2} d\mathbf{R}^2$ .

## B.2) Physical Optics diffracted fields and reduction, for a magnetic dipolar source

Theorem 14 :

*The Physical Optics field, diffracted by the surface  $S'$  with an impedance condition, illuminated by the magnetic dipole  $\mathbf{M}_0 = \hat{\mathbf{d}}\delta(\mathbf{r}' - \mathbf{r}_0)$ , is given by (47) with (137). It can be written, in monostatic case, when  $\hat{\mathbf{d}} \cdot \hat{\mathbf{z}} = 0$ , following*

$$\begin{aligned} \mathbf{E} &\sim \frac{-z}{8(\pi ik)^2} \hat{\mathbf{d}} \times \int b' \left(\frac{z}{\tilde{R}}\right) \mathbf{R} \left(\left(\frac{\psi'}{\tilde{R}}\right)'\right)^2 dS' \\ &\quad - \frac{1}{8\pi^2 ik Z_0} \hat{\mathbf{z}} \times \int (\mathbf{R} \times \hat{\mathbf{z}}) \frac{c(\frac{z}{\tilde{R}})}{2} \tilde{R} \left(\left(\frac{\psi'}{\tilde{R}}\right)'\right)^2 \left(\frac{z}{\tilde{R}^2} (\mathbf{R} \cdot \hat{\mathbf{d}})\right) dS' \\ &\quad - \frac{1}{8\pi^2 ik Z_0} \hat{\mathbf{d}} \times \int \mathbf{R} \frac{c(\frac{z}{\tilde{R}})}{2} \tilde{R} \left(\left(\frac{\psi'}{\tilde{R}}\right)'\right)^2 dS' \end{aligned} \quad (143)$$

for large  $k\tilde{R}$ , with  $c \equiv a'b'$ ,  $(\mathbf{R} \times \hat{\mathbf{z}})(\mathbf{R} \cdot \hat{\mathbf{d}}) = (\mathbf{R}^2 - z^2) \frac{(\hat{\mathbf{d}} \cdot (\mathbf{a} - \mathbf{r}'))(\mathbf{a} - \mathbf{r}') \times \hat{\mathbf{z}}}{(\mathbf{a} - \mathbf{r}')^2}$  and

$$\begin{aligned} \int b' \left(\frac{z}{\tilde{R}}\right) \left(\frac{\psi'}{\tilde{R}}\right)' \left(\frac{\psi'}{\tilde{R}}\right)' d\mathbf{R}^2 &= b' \left(\frac{z}{\tilde{R}}\right) k^2 ik \frac{e^{-i2k\tilde{R}}}{\tilde{R}^3} \left(1 + O\left(\frac{1}{k\tilde{R}}\right)\right) \\ \int c \left(\frac{z}{\tilde{R}}\right) \tilde{R} \left(\left(\frac{\psi'}{\tilde{R}}\right)'\right)^2 d\mathbf{R}^2 &= -\frac{2k^2 c \left(\frac{z}{\tilde{R}}\right) e^{-i2k\tilde{R}}}{\tilde{R}^2} \left(1 + O\left(\frac{1}{k\tilde{R}}\right)\right) \\ \int c \left(\frac{z}{\tilde{R}}\right) \frac{1}{\tilde{R}} \left(\left(\frac{\psi'}{\tilde{R}}\right)'\right)^2 d\mathbf{R}^2 &= -\frac{2k^2 c \left(\frac{z}{\tilde{R}}\right) e^{-i2k\tilde{R}}}{\tilde{R}^4} \left(1 + O\left(\frac{1}{k\tilde{R}}\right)\right) \end{aligned} \quad (144)$$

*We can then apply the propositions 6 and 7 to reduce each integral to non singular contour integral without any special function.*

Proof. We consider, using (137), that  $S'$  is illuminated by the radiation of the magnetic dipole  $\mathbf{M}_0 = \hat{\mathbf{d}}\delta(\mathbf{r}' - \mathbf{r}_0)$  with  $\hat{\mathbf{d}} \cdot \hat{\mathbf{z}} = 0$  ( $\hat{\mathbf{d}} // \text{plane}$ ), and thus



$$\begin{aligned} \mathbf{J}|_S &= \frac{b'(\frac{z}{\tilde{R}})}{2\pi ik Z_0} \hat{\mathbf{z}} \times (\tilde{R}_0 (\frac{\psi'}{\tilde{R}_0})' \mathbf{u}_0 \times (\mathbf{u}_0 \times \hat{\mathbf{d}}) - \frac{2\psi'}{\tilde{R}} \hat{\mathbf{d}}), \quad \mathbf{R}_0 = (\mathbf{r}' - \mathbf{r}_0), \quad \mathbf{u}_0 = \frac{\mathbf{R}_0}{\tilde{R}_0}, \\ \mathbf{M}|_S &= \frac{-c(\frac{z}{\tilde{R}})}{2\pi ik Z_0} \hat{\mathbf{z}} \times (\hat{\mathbf{z}} \times (\tilde{R}_0 (\frac{\psi'}{\tilde{R}_0})' \mathbf{u}_0 \times (\mathbf{u}_0 \times \hat{\mathbf{d}}) - \frac{2\psi'}{\tilde{R}} \hat{\mathbf{d}})), \end{aligned} \quad (145)$$

where we suppose  $a'(v)$ ,  $b'(v)$ , and  $c(v)$  regular for  $v \in [0, 1]$ . In the case where  $\mathbf{R} = -\mathbf{R}_0 = z\hat{\mathbf{z}} + \mathbf{a} - \mathbf{r}'$  (monostatic case), we then obtain,

$$\begin{aligned} \mathbf{E} &= \frac{1}{8(\pi ik)^2} \int_{S'} z b'(\frac{z}{\tilde{R}}) (\frac{\psi'}{\tilde{R}})' (\frac{\psi'}{\tilde{R}})' (\mathbf{R} \times \hat{\mathbf{d}}) dS' \\ &\quad - \frac{1}{4(\pi ik)^2} \int_{S'} b'(\frac{z}{\tilde{R}}) \hat{\mathbf{z}} \times (\tilde{R} ((\frac{\psi'}{\tilde{R}})^2)' (\mathbf{u}(\mathbf{u} \cdot \hat{\mathbf{d}}) - \hat{\mathbf{d}}) - 2(\frac{\psi'}{\tilde{R}})^2 \hat{\mathbf{d}}) dS' \\ &\quad + \frac{1}{8\pi^2 ik Z_0} \int_{S'} c(\frac{z}{\tilde{R}}) \frac{\psi'}{\tilde{R}} \mathbf{R} \times (\hat{\mathbf{z}} \times (\hat{\mathbf{z}} \times (\tilde{R} (\frac{\psi'}{\tilde{R}})' \mathbf{u} \times (\mathbf{u} \times \hat{\mathbf{d}}) - \frac{2\psi'}{\tilde{R}} \hat{\mathbf{d}}))) dS' \\ &\sim \frac{-z}{8(\pi ik)^2} \hat{\mathbf{d}} \times \int_{S'} b'(\frac{z}{\tilde{R}}) ((\frac{\psi'}{\tilde{R}})')^2 \mathbf{R} dS' \\ &\quad + \frac{1}{8\pi^2 ik Z_0} \int c(\frac{z}{\tilde{R}}) \tilde{R} \frac{\psi'}{\tilde{R}} (\mathbf{R} \times \hat{\mathbf{z}}) (\hat{\mathbf{z}} \cdot (\mathbf{u} \times (\mathbf{u} \times \hat{\mathbf{d}}))) (\frac{\psi'}{\tilde{R}})' dS' \\ &\quad - \frac{1}{8\pi^2 ik Z_0} \int c(\frac{z}{\tilde{R}}) \tilde{R} \frac{\psi'}{\tilde{R}} \mathbf{R} \times ((\frac{\psi'}{\tilde{R}})' \mathbf{u} \times (\mathbf{u} \times \hat{\mathbf{d}})) dS' \end{aligned} \quad (146)$$

and thus (143). We can then, following the theorems 1 and 2, and the propositions 6 and 7, reduce the field  $\mathbf{E}$  to a contour integral. For this, we can use (125), (132), and

$$(\mathbf{R} \times \hat{\mathbf{z}}) \tilde{R} \frac{(\mathbf{R} \cdot \hat{\mathbf{d}})}{\mathbf{R}^2} = \frac{\mathbf{R}^2 - z^2}{\tilde{R}} \frac{(\hat{\mathbf{d}} \cdot (\mathbf{a} - \mathbf{r}')) ((\mathbf{a} - \mathbf{r}') \times \hat{\mathbf{z}})}{(\mathbf{a} - \mathbf{r}')^2}. \quad (147)$$