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Electromagnetic particles.

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Résumé

The covariant derivative of the 4-components electromagnetic potential in a flat Minkowski spacetime is split into its antisymmetric and symmetric parts. While the former is well known to describe the electromagnetic field, we show that the latter describes the associated particles. When symmetry principles are applied to the invariants in operations of the Poincaré group, one finds equations which describe the structure of the particles. Both parts of the tensor unify the concept of matter-wave duality. Charge and mass are shown to be associated to the potential.

1 Introduction.

When Born and Infeld published their work[1] on the foundations of non linear electromagnetism in the year 1934, their aim was to describe the electron and more generally the physical world from a purely electromagnetic theory. At that time, quantum mechanics was already developed but these authors wrote in the introduction of their article that it was in opposition to the ideas sustaining their work. The study which is developed here was motivated by the fact that both theories are not opposed, in fact they are complementary, simply because quantum mechanics is linear in the Hilbert space. Nonlinear electromagnetism has never been abandoned since[2] and led to interesting applications in cosmology [3]. The following work presents a broader view of the subject : instead of starting from the electromagnetic field, the corresponding potential in a flat Minkowski space is considered. The fundamental object is the covariant derivative of this potential. The antisymmetric part is the well known electromagnetic tensor. The symmetric part can be diagonalized and when constraints resulting from symmetry operations are applied to its invariants, one finds an equation for the potentials. Its solutions reveal a concentration of energy around the origin which corresponds to the structure of particles. An essential result is the explanation of the wave-matter duality which naturally arises from the intermingled parts of the tensor. We describe these calculations below and give examples of solutions and applications.

2 Frame of the theory.

The theory is based on few ingredients :

- 1- A point M is defined in the flat Minkowski spacetime¹ by its 4 coordinates x^i where $x^0 = ct$ and $x^{1,2,3} = x, y, z$ in the cartesian frame with an origin O . The associated vector $\overrightarrow{OM} = x^i \vec{e}_i$ is defined with respect to the orthonormalized basis \vec{e}_i . We choose the metric $\eta = (+, -, -, -)$, the common dimension is length.
- 2- To each point is associated an electromagnetic potential A^i where $A^0 = \phi/c$, is the usual scalar potential and $A^{1,2,3} = A^x, A^y, A^z$ are the components of the vector potential.
- 3- The fundamental tensor is taken to be the covariant derivative² (the gradient tensor) of \tilde{A}_i :

$$a_{ij} = \frac{\partial \tilde{A}_i}{\partial x^j} - \Gamma_{ij}^m \tilde{A}_m \quad . \quad (1)$$

It is simpler to use the cartesian basis first where Christoffel's symbols Γ_{ij}^m cancel. When this tensor is split into its symmetric and antisymmetric parts, one is led to study the role of both. The latter part is the usual

1. Einstein summation convention is used. Upper and lower indices respectively refer to contravariant and covariant components.

2. a_{ij} is the covariant form

field tensor on which classical electromagnetism is based. We name the former the particle tensor and the following development is essentially devoted to the study of some of its properties.

4- We divide the potential into two parts : the first describes an isolated particle while the second describes the space in which it is embedded. This second part originates from the fundamental noise : this field has random properties, it is also isotropic and it can be represented by a diagonal tensor the modulus of which can be used as a reference for the amplitude of A^i . The following study will essentially be concerned by the isolated particle.

5- The next ingredient is the Poincaré group of coordinates transformations. We will associate the transformations of the Poincaré group to the idealized isolated single entity (field/particle), or cluster of entities, which can be characterized by similar properties, such as a given velocity, a symmetry of rotation or translation. The field tensor and the particle tensor do not mix in a coordinate change and their invariants are the fundamental quantities which will provide the description of the system.

We describe below how a fundamental equation can be deduced from these ingredients. The solutions are the components of the 4-potential. These components are mixed together in a non-linear way to form invariants of the system. One of this invariant is related to the density of energy. We give illustrations for a series of spherically symmetric solutions. An application of the theory to make the link with Maxwell's equations is finally given with use of the principle of least action. Here, the elementary electric charge is shown to be a characteristic of only a single type of solution.

3 The fundamental tensor.

3.1 Notation.

The electromagnetic potential (4-vector) A^i is defined by its contravariant components in the direct space. We will write the covector \tilde{A} in the reciprocal space. Its components are : $\tilde{A}_{0,1,2,3} = \phi/c, -\tilde{A}_x, -\tilde{A}_y, -\tilde{A}_z$. The dimension of \tilde{A} and A is that of an ordinary electric potential ($MLT^{-1}Q^{-1}$). The scalar product is a Lorentz invariant :

$$|A|^2 = A^i \tilde{A}_i = \eta^{km} \tilde{A}_m \tilde{A}_k = \eta_{km} A^m A^k = (\phi/c)^2 - \tilde{A}_x^2 - \tilde{A}_y^2 - \tilde{A}_z^2 = (\phi/c)^2 - A^{x2} - A^{y2} - A^{z2} \quad . \quad (2)$$

The components of the potential vector $[A^i]$ (tensor type (0,1)) and the covector $[\tilde{A}_i]$ (type (1,0)) have different signs.

Our fundamental object is the covariant derivative written in a cartesian frame :

$$[a_{ij}] = D[\tilde{A}] = \begin{bmatrix} \phi_{,t} & \phi_{,x} & \phi_{,y} & \phi_{,z} \\ -\tilde{A}_{x,t} & -\tilde{A}_{x,x} & -\tilde{A}_{x,y} & -\tilde{A}_{x,z} \\ -\tilde{A}_{y,t} & -\tilde{A}_{y,x} & -\tilde{A}_{y,y} & -\tilde{A}_{y,z} \\ -\tilde{A}_{z,t} & -\tilde{A}_{z,x} & -\tilde{A}_{z,y} & -\tilde{A}_{z,z} \end{bmatrix} \quad . \quad (3)$$

This is the standard matrix representation of $D[\tilde{A}]$ where the first index in $a_{ij} = \partial \tilde{A}_i / \partial x^j$ is the line index and the second the column index.

We have used the compressed notation for the partial derivatives in the cartesian frame :

$$\begin{aligned} \phi_{,t} &\equiv \frac{\partial(\phi/c)}{c\partial t} \quad , \quad \phi_{,x} \equiv \frac{\partial\phi/c}{\partial x} \quad , \quad \phi_{,y} \equiv \frac{\partial\phi/c}{\partial y} \quad , \quad \phi_{,z} \equiv \frac{\partial\phi/c}{\partial z} \\ \tilde{A}_{x,t} &\equiv \frac{\partial\tilde{A}_x}{c\partial t} \quad , \quad \tilde{A}_{x,x} \equiv \frac{\partial\tilde{A}_x}{\partial x} \quad , \quad \tilde{A}_{x,y} \equiv \frac{\partial\tilde{A}_x}{\partial y} \quad , \quad \tilde{A}_{x,z} \equiv \frac{\partial\tilde{A}_x}{\partial z} \end{aligned} \quad (4)$$

and the same for the derivatives of \tilde{A}_y and \tilde{A}_z .

3.2 The field and the particle tensors.

$[a_{ij}]$ can be split into its symmetric and antisymmetric parts³ :

$$[a_{ij}] = [s_{(ij)}] + [f_{[ij]}] \quad (5)$$

3. This splitting makes sense for covariant or contravariant tensors because their symmetry remains invariant in a coordinate change. This property does not apply to mixed tensors.

with :

$$s_{ij} = \frac{1}{2} (a_{ij} + a_{ji}) \quad \text{and} : \quad f_{(ij)} = \frac{1}{2} (a_{ij} - a_{ji}) \quad (6)$$

The antisymmetric part is explicitly written :

$$[f_{ij}] = 0.5 \begin{bmatrix} 0 & \phi_{,x} + \tilde{A}x_{,t} & \phi_{,y} + \tilde{A}y_{,t} & \phi_{,z} + \tilde{A}z_{,t} \\ -\tilde{A}x_{,t} - \phi_{,x} & 0 & -\tilde{A}x_{,y} + \tilde{A}y_{,x} & -\tilde{A}x_{,z} + \tilde{A}z_{,x} \\ -\tilde{A}y_{,t} - \phi_{,y} & -\tilde{A}y_{,x} + \tilde{A}x_{,y} & 0 & -\tilde{A}y_{,z} + \tilde{A}z_{,y} \\ -\tilde{A}z_{,t} - \phi_{,z} & -\tilde{A}z_{,x} + \tilde{A}x_{,z} & -\tilde{A}z_{,y} + \tilde{A}y_{,z} & 0 \end{bmatrix} . \quad (7)$$

apart the factor 0.5, $[f_{ij}]$ is the transpose of the standard electromagnetic field tensor :

$$[F_{ij}] = \begin{bmatrix} 0 & E_x/c & E_y/c & E_z/c \\ -E_x/c & 0 & -B_z & B_y \\ -E_y/c & B_z & 0 & -B_x \\ -E_z/c & -B_y & B_x & 0 \end{bmatrix} . \quad (8)$$

The electric (E_x, E_y, E_z) and magnetic (B_x, B_y, B_z) fields are defined from the derivatives of the 4-potential covector components :

$$E_x/c = -\frac{\partial \tilde{A}_x}{c \partial t} - \frac{\partial \phi/c}{\partial x} , \quad E_y/c = -\frac{\partial \tilde{A}_y}{c \partial t} - \frac{\partial \phi/c}{\partial y} , \quad E_z/c = -\frac{\partial \tilde{A}_z}{c \partial t} - \frac{\partial \phi/c}{\partial z} . \quad (9)$$

and :

$$B_x = \frac{\partial \tilde{A}_z}{\partial y} - \frac{\partial \tilde{A}_y}{\partial z} , \quad B_y = \frac{\partial \tilde{A}_x}{\partial z} - \frac{\partial \tilde{A}_z}{\partial x} , \quad B_z = \frac{\partial \tilde{A}_y}{\partial x} - \frac{\partial \tilde{A}_x}{\partial y} . \quad (10)$$

The identity : $\partial f_{lm}/\partial x^k + \partial f_{mk}/\partial x^l + \partial f_{kl}/\partial x^m = 0$, when developed, leads to the first set of Maxwell equations.

The symmetric part of $[a_{ij}]$ is :

$$[s_{ij}] = 0.5 \begin{bmatrix} 2\phi_{,t} & \phi_{,x} - \tilde{A}x_{,t} & \phi_{,y} - \tilde{A}y_{,t} & \phi_{,z} - \tilde{A}z_{,t} \\ -\tilde{A}x_{,t} + \phi_{,x} & -2\tilde{A}x_{,x} & -\tilde{A}x_{,y} - \tilde{A}y_{,x} & -\tilde{A}x_{,z} - \tilde{A}z_{,x} \\ -\tilde{A}y_{,t} + \phi_{,y} & -\tilde{A}y_{,x} - \tilde{A}x_{,y} & -2\tilde{A}y_{,y} & -\tilde{A}y_{,z} - \tilde{A}z_{,y} \\ -\tilde{A}z_{,t} + \phi_{,z} & -\tilde{A}z_{,x} - \tilde{A}x_{,z} & -\tilde{A}z_{,y} - \tilde{A}y_{,z} & 2\tilde{A}z_{,z} \end{bmatrix} . \quad (11)$$

We will name $[s_{ij}]$ the "particle tensor" to balance $[f_{ij}]$ which is the field tensor.

As a conclusion of this introductory section, we note that in traditional[6] or modern textbooks[7] the study of electromagnetism generally begins by a definition of fields. The electromagnetic potential is defined later. However, in [8], the potentials are first defined and then the field is introduced from the expression of the force this potential exerts on a charged particle. Both ways are equivalent as long as we are interested in F_{ij} (or f_{ij}) only. Starting the theory from $[a_{ij}]$ is clearly more general and agrees with the Aharonov-Bohm effect which shows that potentials are more fundamental than fields. The tensor $[s_{ij}]$ is absent from standard electromagnetism. It is replaced by phenomenological quantities like the electric charge Q . An objective of our work was to discover if there is a relation between them and the components of $[s_{ij}]$.

3.3 The Lagrangian of Electromagnetism.

Standard electromagnetism is based on the use of the field tensor f_{ij} and the principle of least action applied to the Lagrangian $1/4\mu_0 f^{ij} f_{ij}$. It is thus necessary to consider a similar Lagrangian which takes the particle part into account :

$$\mathcal{L} = \frac{1}{4\mu_0} a^{ij} a_{ij} , \quad (12)$$

One sees that the contracted product $a^{ij} a_{ij}$ can be split into two parts, one of them being the standard tensor of energy of the electromagnetic field :

$$a^{ij} a_{ij} = (s^{ij} + f^{ij}) (s_{ij} + f_{ij}) = s^{ij} s_{ij} + f^{ij} f_{ij} , \quad (13)$$

If we use the general expression (3) for $[a_{ij}]$, the Lagrangian writes⁴ :

$$\begin{aligned} \dot{\mathcal{L}} = \frac{1}{4\mu_0} & (\phi_{,t}^2 + Ax_{,x}^2 + Ay_{,y}^2 + Az_{,z}^2 - Ax_{,t}^2 - Ay_{,t}^2 - Az_{,t}^2 \\ & + Ax_{,y}^2 + Ay_{,x}^2 + Ax_{,z}^2 + Az_{,x}^2 + Ay_{,z}^2 + Az_{,y}^2 - \phi_{,x}^2 - \phi_{,y}^2 - \phi_{,z}^2) \quad , \quad (14) \end{aligned}$$

One can thus divide \mathcal{L} into two parts :
-the part which characterizes the "particle" :

$$\begin{aligned} \mathcal{L}_p = \frac{1}{4\mu_0} & (\phi_{,t}^2 + Ax_{,x}^2 + Ay_{,y}^2 + Az_{,z}^2 + 0.5(Ax_{,y} + Ay_{,x}) + 0.5(Ax_{,z} + Az_{,x})^2 \\ & + 0.5(Ay_{,z} + Az_{,y})^2 - 0.5(Ax_{,t} - \phi_{,x})^2 - 0.5(Ay_{,t} - \phi_{,y})^2 - 0.5(Az_{,t} - \phi_{,z})^2) \quad , \quad (15) \end{aligned}$$

- and the part which characterizes the "field" :

$$\begin{aligned} \mathcal{L}_f = \frac{1}{2\mu_0} & ((Ax_{,y} - Ay_{,x})^2 + (Ax_{,z} - Az_{,x})^2 + (Ay_{,z} - Az_{,y})^2 \\ & - (Ax_{,t} + \phi_{,x})^2 - (Ay_{,t} + \phi_{,y})^2 - (Az_{,t} + \phi_{,z})^2) = \frac{1}{2\mu_0} (B^2 - (E/c)^2) \quad , \quad (16) \end{aligned}$$

The associated tensor of moments is :

$$[M^{ij}] = \frac{\partial \mathcal{L}}{\partial a_{ij}} = 2 \begin{bmatrix} \phi_{,t} & -\phi_{,x} & -\phi_{,y} & -\phi_{,z} \\ -\tilde{A}x_{,t} & \tilde{A}x_{,x} & \tilde{A}x_{,y} & \tilde{A}x_{,z} \\ -\tilde{A}y_{,t} & \tilde{A}y_{,x} & \tilde{A}y_{,y} & \tilde{A}y_{,z} \\ -\tilde{A}z_{,t} & \tilde{A}z_{,x} & \tilde{A}z_{,y} & \tilde{A}z_{,z} \end{bmatrix} \quad . \quad (17)$$

Application of the operator ∇ to $[M^{ij}]^T$ gives Lagrange's equations :

$$\frac{\partial \phi_{,t}}{c\partial t} - \frac{\partial \tilde{A}x_{,t}}{\partial x} - \frac{\partial \tilde{A}y_{,t}}{\partial y} - \frac{\partial \tilde{A}z_{,t}}{\partial z} = 0 \quad . \quad (18)$$

Permuting the order of integration gives :

$$\frac{\partial}{c\partial t} (\phi_{,t} - \partial \tilde{A}x_{,x} - \partial \tilde{A}y_{,y} - \partial \tilde{A}z_{,z}) = 0 \quad . \quad (19)$$

One finds an equation which expresses that the Lorentz gauge is a constant. We will come back this equation later.

The second equation is :

$$-\frac{\partial \tilde{A}x_{,t}}{c\partial t} + \frac{\partial \tilde{A}x_{,x}}{\partial x} - \frac{\partial \tilde{A}y_{,y}}{\partial y} - \frac{\partial \tilde{A}z_{,z}}{\partial z} = 0 \quad . \quad (20)$$

This is the d'Alembert equation for the component $\tilde{A}x$ (see eq.(29)). The last 2 equations give the same result for the other components.

One sees that this Lagrangian does not give Maxwell's equations with sources as one could have expected. We will use another Lagrangian for this purpose. However, this examination of this Lagrangian will be useful in the following.

4 Diagonalization of the symmetric part.

A symmetric tensor can be diagonalized which means that the 10 components of s_{ij} are reduced to 4 in the eigenbasis : these are the components of a 4-pseudo-vector which will characterize the particle. The diagonalization process generally brings an element $\tilde{A}_{i,j}$ into $\tilde{A}_{i,j} - \Gamma_{ij}^m \tilde{A}_m$ in the new system of coordinates where only diagonal elements with $i = j$ survive. It can be done in few essential steps⁵ :

4. Here the distinction between A and \tilde{A} does not matter.

5. Operations of the Poincaré group are : continuous translation in space or time (changing the origin of space or time does not change the result of an experiment (this is the invariance our everyday life is based on!), continuous rotations of coordinates in space, Lorentz transformations connecting two uniformly moving frames of coordinates (boosts), parity and time-reversal transformations.

1- A Lorentz boost brings the reference frames of the observer and the particle together.

2- A translation of axis brings the origins at the same point.

3- A transformation of axis in the geometrical space brings the eigensystem. If the particle is described by an ellipsoid, a rotation brings it on its principal axis. The following study will be restricted to this case.

We will now work on the hypothesis that a cartesian frame exists in which the tensor is diagonal. This hypothesis will allow the description of the particles and the diagonalization process can be reversed to obtain a rotated, translated or boosted tensor which will represent various physical situations.

The diagonalized $[s_{ij}]$ writes :

$$[s_{ij}] = \begin{bmatrix} \dot{\Phi}_{,t} & 0 & 0 & 0 \\ 0 & -\tilde{\mathbf{A}}_{\mathbf{x},\mathbf{x}} & 0 & 0 \\ 0 & 0 & -\tilde{\mathbf{A}}_{\mathbf{y},\mathbf{y}} & 0 \\ 0 & 0 & 0 & -\tilde{\mathbf{A}}_{\mathbf{z},\mathbf{z}} \end{bmatrix} . \quad (21)$$

the different quantities are written in boldface characters to stress the fact that they belong to the eigensystem of coordinates. Here, space and time become independent variables.

Non diagonal elements of (11) nullify in this system which leads to the relations :

$$\begin{aligned} -\tilde{\mathbf{A}}_{\mathbf{x},t} + \dot{\Phi}_{,x} = 0 & \quad , \quad -\tilde{\mathbf{A}}_{\mathbf{y},t} + \dot{\Phi}_{,y} = 0 & \quad , \quad -\tilde{\mathbf{A}}_{\mathbf{z},t} + \dot{\Phi}_{,z} = 0 \\ \tilde{\mathbf{A}}_{\mathbf{y},x} + \tilde{\mathbf{A}}_{\mathbf{x},y} = 0 & \quad , \quad \tilde{\mathbf{A}}_{\mathbf{z},x} + \tilde{\mathbf{A}}_{\mathbf{x},z} = 0 & \quad , \quad \tilde{\mathbf{A}}_{\mathbf{z},y} + \tilde{\mathbf{A}}_{\mathbf{y},z} = 0 \end{aligned} . \quad (22)$$

The first set of equations can be written :

$$\frac{\partial \vec{\mathbf{A}}}{\partial t} = \overrightarrow{\text{grad}}\Phi \quad (23)$$

An equivalent equation can be written with the components of the vector :

$$\frac{\partial \vec{\mathbf{A}}}{\partial t} = -\overrightarrow{\text{grad}}\Phi \quad (24)$$

The electric field is defined from $\vec{\mathbf{A}}$, it becomes in the eigensystem :

$$\vec{E} = -\frac{\partial \vec{\mathbf{A}}}{\partial t} - \overrightarrow{\text{grad}}\Phi = -2\frac{\partial \vec{\mathbf{A}}}{\partial t} = -2\overrightarrow{\text{grad}}\Phi \quad (25)$$

The magnetic field nullifies in this system.

5 The fundamental equation of electromagnetic particles.

We are interested now in the invariants associated to a coordinate transformation. The mixed tensors associated to a_{ij} , s_{ij} and f_{ij} are obtained with the use of the rising operator $\eta^{km} : a^i_j = \eta^{im} a_{mj}$. These tensors are defined in the direct space (they need only contravariant components of the potential and the coordinates) and they are characterized by a conservation of their trace and their determinant in a coordinate change : these are two of the four invariants which are the coefficients of the associated characteristic polynomial.

One has :

$$[s^i_j] = \begin{bmatrix} \Phi_{,t} & 0 & 0 & 0 \\ 0 & \mathbf{A}_{\mathbf{x},\mathbf{x}} & 0 & 0 \\ 0 & 0 & \mathbf{A}_{\mathbf{y},\mathbf{y}} & 0 \\ 0 & 0 & 0 & \mathbf{A}_{\mathbf{z},\mathbf{z}} \end{bmatrix} . \quad (26)$$

Its invariants I_1, I_2, I_3, I_4 are :

$$\begin{aligned} I_1 &= \Phi_{,t} + \dot{\mathbf{A}}_{\mathbf{x},\mathbf{x}} + \dot{\mathbf{A}}_{\mathbf{y},\mathbf{y}} + \dot{\mathbf{A}}_{\mathbf{z},\mathbf{z}} \\ I_2 &= \Phi_{,t}(\dot{\mathbf{A}}_{\mathbf{x},\mathbf{x}} + \dot{\mathbf{A}}_{\mathbf{y},\mathbf{y}} + \dot{\mathbf{A}}_{\mathbf{z},\mathbf{z}}) + \dot{\mathbf{A}}_{\mathbf{x},\mathbf{x}} \dot{\mathbf{A}}_{\mathbf{y},\mathbf{y}} + \dot{\mathbf{A}}_{\mathbf{x},\mathbf{x}} \dot{\mathbf{A}}_{\mathbf{z},\mathbf{z}} + \dot{\mathbf{A}}_{\mathbf{y},\mathbf{y}} \dot{\mathbf{A}}_{\mathbf{z},\mathbf{z}} \\ I_3 &= \dot{\mathbf{A}}_{\mathbf{y},\mathbf{y}} \dot{\mathbf{A}}_{\mathbf{z},\mathbf{z}}(\Phi_{,t} + (\dot{\mathbf{A}}_{\mathbf{x},\mathbf{x}}) + \Phi_{,t}(\dot{\mathbf{A}}_{\mathbf{x},\mathbf{x}}(\dot{\mathbf{A}}_{\mathbf{y},\mathbf{y}} + \dot{\mathbf{A}}_{\mathbf{z},\mathbf{z}})) \\ I_4 &= \Phi_{,t} \dot{\mathbf{A}}_{\mathbf{x},\mathbf{x}} \dot{\mathbf{A}}_{\mathbf{y},\mathbf{y}} \dot{\mathbf{A}}_{\mathbf{z},\mathbf{z}} \end{aligned} \quad (27)$$

I_1 is the trace \mathcal{T} , I_4 is the determinant of $[s^i_j]$.

Let us focus our attention on the trace $\mathcal{T} = I_1$. This is the 4-divergence of A^i , it is a scalar density⁶. Expression $\mathcal{T} = \text{div}[A] = \text{Invariant}$ has been obtained in the cartesian frame, but this is a tensor equation which equally applies to any system of coordinates where Christoffel coefficients generally occur.

When the expression I_1 is fully developed, one sees that it is the *Lorentz gauge* when $\mathcal{T} = 0$. In a cartesian frame of reference :

$$\mathcal{T} = \frac{\partial A^i}{\partial x^i} = \frac{\partial \phi}{c^2 \partial t} + \frac{\partial A^x}{\partial x} + \frac{\partial A^y}{\partial y} + \frac{\partial A^z}{\partial z} = \frac{\partial \phi}{c^2 \partial t} + \text{div} \vec{A} \quad , \quad (28)$$

Anticipating on the following results, we will see that one has effectively $I = 0$.

The trace of $[a^i_j]$ being the same as that of $[s^i_j]$, one concludes that **the Lorentz gauge expresses the invariance of the trace of the covariant derivative of the 4-potential in a Poincaré transformation.**

Another fundamental equation can be derived from \mathcal{T} if one applies the d'Alembert operator $\square = \eta^{mn} \partial_n \partial_m$ to $[A^i]$:

$$\square[A^i] = \frac{\partial^2 \phi / c}{c^2 \partial t^2} - \Delta \vec{A} = 0 \quad , \quad (29)$$

Δ is the Laplacian in the geometrical space.

If we now use the invariance of \mathcal{T} with respect to the origin of coordinates (space and time) together with eqs.(23) in the eigenspace, we obtain the fundamental equation which will give the structure of the electromagnetic particles :

$$\frac{\partial \mathcal{T}}{c \partial \mathbf{t}} = \frac{\partial}{c \partial \mathbf{t}} \left(\frac{\partial \Phi / c}{c \partial \mathbf{t}} + \text{div} \vec{A} \right) = 0 \quad , \quad (30)$$

The order of derivation can be commuted which gives :

$$\frac{\partial^2 (\Phi / c)}{c^2 \partial \mathbf{t}^2} + \frac{\partial}{\partial \mathbf{x}} \frac{\partial \mathbf{A} \mathbf{x}}{c \partial \mathbf{t}} + \frac{\partial}{\partial \mathbf{y}} \frac{\partial \mathbf{A} \mathbf{y}}{c \partial \mathbf{t}} + \frac{\partial}{\partial \mathbf{z}} \frac{\partial \mathbf{A} \mathbf{z}}{c \partial \mathbf{t}} = 0 \quad , \quad (31)$$

and using :

$$\frac{\partial \vec{A}}{\partial \mathbf{t}} = - \overrightarrow{\text{grad}} \Phi \quad (32)$$

one gets :

$$\frac{\partial^2 (\Phi / c)}{c^2 \partial \mathbf{t}^2} - \frac{\partial^2 (\Phi / c)}{\partial \mathbf{x}^2} - \frac{\partial^2 (\Phi / c)}{\partial \mathbf{y}^2} - \frac{\partial^2 (\Phi / c)}{\partial \mathbf{z}^2} = 0 \quad , \quad (33)$$

Let us take as an hypothesis that Φ varies sinusoidally in time with an angular frequency [5] ω . Its amplitude Φ^0 obeys an Helmholtz equation :

$$\frac{\omega^2}{c^2} \Phi^0 + \Delta \Phi^0 = 0 \quad , \quad (34)$$

Solutions Φ_i^0 of (34) are well known and are used in the following section. A general solution will thus be a linear combination of them. The sign of Φ_i^0 and its amplitude is undetermined yet. One should keep in mind that these calculations have been done with the contravariant components of the potential vector $(\phi/c, A^x, A^y, A^z)$: An equivalent relation of (23) can be written with the covariant components :

$$\frac{\partial \vec{A}}{\partial \mathbf{t}} = \overrightarrow{\text{grad}} \Phi \quad (35)$$

where \vec{A} is the potential covector.

The goal of these calculations was to explore some properties of the symmetric part of the fundamental tensor. Up to now, we have reached several important consequences :

- A particular basis $\mathbf{t}, \mathbf{x}, \mathbf{y}, \mathbf{z}$ exists. This basis can be named the eigenbasis of the particle in which the particle tensor $[\mathbf{r}^i_j]$ is represented only by 4 quantities along the diagonal which define a 4-pseudo-vector.
- The invariance of the trace of \mathbf{a}^i_j leads (without any calculation) to the Lorentz gauge (divergence of this

6. When \mathcal{T} will be multiplied by a scalar capacity such as a 4-volume, the product will be a true scalar.

4-pseudo-vector).

- Application of symmetries leads to a wave-type equation for the scalar potential. Solutions of this equation will describe the structure of the particles.
- The fundamental tensor in the eigenbasis is the basic tool which will permit to study later the dynamics of different systems in stationary or moving frames.

6 Electromagnetic particles.

6.1 Expressions of the potentials.

Solutions Φ_i^0 of eq.(34) , are proportional to the products of spherical Bessel functions $j_n(x)$ with spherical harmonics $Y_\ell^m(\theta, \phi)$, where $x = \omega r/c$, r being the radial coordinate (note the different typography between this x and the coordinate x). These solutions offer a way to classify electromagnetic particles. We consider solutions which are invariant in time reversal. The characteristic function of a particle is basically described by the formulas :

$$\phi_n(x, \theta, \phi) \propto \pm j_n(x) Y_\ell^m(\theta, \varphi) \cos \omega t \quad \text{for } n \text{ even,} \quad (36)$$

and :

$$\phi_n(x, \theta, \phi) \propto \pm j_n(x) Y_\ell^m(\theta, \varphi) \sin \omega_n t \quad \text{for } n \text{ odd,} \quad (37)$$

The behavior of $j_n(x)$ in the vicinity of $x = 0$ is :

$$j_n(x) \xrightarrow{x \rightarrow 0} \frac{x^n}{1.3.5..(2n+1)} \quad , \quad (38)$$

The solutions remain finite when $x \rightarrow 0$. Only the scalar potential of the fundamental one ($n=0$) is maximum at $x = 0$ where the others vanish (one can say that they are "hollow"). A general solution can be split into even or odd parts. Both types remain invariant in the transformation $\omega_n \rightarrow -\omega_n$.

The asymptotic behavior of even solutions $j_0(x)$, $j_2(x)$, $J_4(x)$... when x is large is $\pm \sin x/x$. For odd solutions it is $\pm \cos x/x$. However, the sign of successive solutions is alternated which results in the cancellation of a sum like $j_{2p} + j_{2(p+1)}$ far from the origin.

Now we will focus our attention on solutions with spherical symmetries ($\ell = 0$) . These solutions will be labelled by n .

Let us introduce the proportionality factors \mathcal{A}_n in order to have explicit expressions for a future use. These factors represent two physical quantities : the absolute value of the potential difference and a fundamental dimension which is $[\mathcal{A}_n] = ML^2T^{-2}Q^{-1}$ in the standard nomenclature.

Potentials for solution J_n are (contravariant components) :

$$\begin{aligned} \frac{\phi_n}{c} = \pm \frac{\mathcal{A}_n}{c} J_n \cos \omega_n t & \quad \text{for } n \text{ even} & \quad \text{and} & \quad \frac{\phi_n}{c} = \pm \frac{\mathcal{A}_n}{c} J_n \sin \omega_n t & \quad \text{for } n \text{ odd.} \\ \mathcal{A}_n = \mp \frac{\mathcal{A}_n}{c} J'_n \sin \omega_n t & & & & \mathcal{A}_n = \pm \frac{\mathcal{A}_n}{c} J'_n \cos \omega_n t \end{aligned} \quad (39)$$

$A_n(x, t)$ is deduced⁷ from $\phi_n(x, t)$ using eq (23). The potential vector is purely radial for this spherically-symmetric solution. The first three solutions are :

Fundamental solution J_0 :

$$\begin{aligned} \phi_0/c &= \pm \frac{\mathcal{A}_0}{c} \frac{\sin x}{x} \cos \omega t \\ A_0(x, t) &= \mp \mathcal{A}_0 \frac{1}{c} \left[\frac{\cos(x)}{x} - \frac{\sin(x)}{x^2} \right] \sin \omega t \quad , \end{aligned}$$

Solution J_1 :

$$\begin{aligned} \phi_1/c &= \pm \frac{\mathcal{A}_1}{c} \left(-\frac{\cos x}{x} + \frac{\sin x}{x^2} \right) \sin \omega t \\ A_1(x, t) &= \pm \mathcal{A}_1 \frac{1}{c} \left(\frac{2 \cos(x)}{x^2} + \sin(x)(1/x - 2/x^3) \right) \cos \omega t \quad , \end{aligned}$$

7. The relation : $\partial/\partial r = k_n \partial/\partial(k_n r) = \omega_n/c \partial/\partial x$ is used.

Solution J_2 :

$$\begin{aligned}\phi_2/c &= \pm \frac{\mathcal{A}_2}{c} \left(\frac{3-x^2}{x^3} \sin x - \frac{3 \cos x}{x^2} \right) \cos \omega t \\ A_2 &= \mp \frac{\mathcal{A}_2}{c} \left[\left(\frac{4}{x^2} - \frac{9}{x^4} \right) \sin x + \left(\frac{9}{x^3} - \frac{1}{x} \right) \cos x \right] \sin \omega t \quad ,\end{aligned}$$

One should note that the spatial part of the solution is a Bessel function which can be expressed as products of the oscillating functions $\sin x$ and $\cos x$ with polynomials $P(1/x)$. This property can be used to show that each solution is the sum of an incoming and an outgoing wave.

6.2 System invariants.

Once the solutions are obtained in the spherical coordinate system, their study is simpler in this system. The standard relations between the coordinates x, y, z of a point M in the cartesian system and its spherical coordinates $v = (r, \theta, \phi)$ in the geometrical space are :

$$x = r \sin \theta \cos \phi \quad , \quad y = r \sin \theta \sin \phi \quad , \quad z = r \cos \theta \quad , \quad (40)$$

The local spherical coordinates system at M is built from the tangent vectors : $\partial \vec{OM} / \partial v$. In this basis, the components of a vector \vec{V} can be written V^1, V^2, V^3 . The spatial metric tensor is :

$$[g_{ij}] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & r^2 & 0 \\ 0 & 0 & r^2 \sin^2 \theta \end{bmatrix} \quad \text{and} \quad [g^{ij}] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1/r^2 & 0 \\ 0 & 0 & 1/(r^2 \sin^2 \theta) \end{bmatrix} \quad (41)$$

However, it is more comfortable to work in the physical basis where the basis vectors are normalized because here, the components have the same dimension and the metric tensor contains only 1 in the diagonal. Relations between the components are :

$$V^r = V^1 \quad , \quad V^\theta = r V^2 \quad , \quad V^\phi = r \sin \theta V^3 \quad . \quad (42)$$

The fundamental tensor is :

$$[\hat{a}_j^i] = \frac{\partial A^i}{\partial u^j} + \Gamma_{jm}^i A^m \quad , \quad (43)$$

where \hat{a}_j^i is the element i, j of the tensor, $A^i = (\phi/c, A^r, A^\theta, A^\phi)$, $u^j = (ct, r, \theta, \phi)$ and Christoffel's coefficient Γ_{jm}^i are expressed in this system. In this section we will be interested only in solutions with spherical symmetry, *i.e.*, solutions characterized by $\ell = 0$ where A^θ and A^ϕ nullify. The fundamental tensor writes under its two forms in the normalized physical basis :

$$[\hat{a}_{ij}] = \begin{bmatrix} \phi_{,t} & \phi_{,r} & 0 & 0 \\ -\tilde{A}_{r,t} & -\tilde{A}_{r,r} & 0 & 0 \\ 0 & 0 & -\frac{\tilde{A}_r}{r} & 0 \\ 0 & 0 & 0 & -\frac{\tilde{A}_r}{r} \end{bmatrix} \quad \text{and} \quad [\hat{a}_j^i] = \begin{bmatrix} \phi_{,t} & \phi_{,r} & 0 & 0 \\ A^r_{,t} & A^r_{,r} & 0 & 0 \\ 0 & 0 & \frac{A^r}{r} & 0 \\ 0 & 0 & 0 & \frac{A^r}{r} \end{bmatrix} \quad (44)$$

In this basis the metric tensor is $\eta_{ij} = (1, -1, -1, -1)$ on the diagonal and all elements \hat{a}_j^i (and \hat{a}_{ij}) have the same dimension.

The electromagnetic particles and the associated fields can be described by the invariants of $[\hat{a}_j^i]$. As symmetric (particle) and antisymmetric (field) parts of $[\hat{a}_{ij}]$ do not mix in a coordinate change, they have also their own invariants.

The splitting of $[\hat{a}_{ij}]$ into $([\hat{s}_{ij}])$ and $([\hat{f}_{ij}])$ gives for the symmetric (particle) part :

$$[\hat{s}_{ij}] = \frac{1}{2} \begin{bmatrix} 2\phi_{,t} & \phi_{,r} - \tilde{A}_{r,t} & 0 & 0 \\ -\tilde{A}_{r,t} + \phi_{,r} & -2\tilde{A}_{r,r} & 0 & 0 \\ 0 & 0 & -2\frac{\tilde{A}_r}{r} & 0 \\ 0 & 0 & 0 & -2\frac{\tilde{A}_r}{r} \end{bmatrix} = \begin{bmatrix} \phi_{,t} & 0 & 0 & 0 \\ 0 & -\tilde{A}_{r,r} & 0 & 0 \\ 0 & 0 & -\frac{\tilde{A}_r}{r} & 0 \\ 0 & 0 & 0 & -\frac{\tilde{A}_r}{r} \end{bmatrix} \quad (45)$$

Eq.(35) ($\phi_{,r} = \tilde{A}_{r,t}$) can be used to verify that $[\hat{s}_{ij}]$ is diagonal in the spherical system of coordinates. The associated field tensor is :

$$[\hat{f}_{ij}] = \frac{1}{2} \begin{bmatrix} 0 & \phi_{,r} + \tilde{A}_{r,t} & 0 & 0 \\ -\tilde{A}_{r,t} - \phi_{,r} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & \phi_{,r} & 0 & 0 \\ -\phi_{,r} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} = \frac{1}{2} \begin{bmatrix} 0 & -E_{r,t}/c & 0 & 0 \\ E_{r,t}/c & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad (46)$$

which shows that the field is purely radial electric : as we already know, the magnetic field vanishes in the eigensystem of the particle.

6.3 Invariants.

We will now express the invariants associated to each solution. Let us write them for the spherically symmetric solutions J_n . The mixed tensor $[\hat{a}^i_j]$ is :

$$[\hat{a}^i_j] = \begin{bmatrix} \phi_{,t} & \phi_{,r} & 0 & 0 \\ A^r_{,t} & A^r_{,r} & 0 & 0 \\ 0 & 0 & \frac{A^r}{r} & 0 \\ 0 & 0 & 0 & \frac{A^r}{r} \end{bmatrix} \quad (47)$$

$[\hat{a}^i_j]$ is characterized by 4 invariants which are :

$$\begin{aligned} I_1 &= \phi_{,t} + A^r_{,r} + 2\frac{A^r}{r} & (48) \\ I_2 &= \frac{A_r^2}{r^2} + \phi_{,t}A^r_{,r} - \phi_{,r}A^r_{,t} + 2\frac{A^r}{r} (\phi_{,t} + A^r_{,r}) \\ I_3 &= 2\frac{A^r}{r} (\phi_{,r}A^r_{,t} - \phi_{,t}A^r_{,r}) - \frac{A_r^2}{r^2} (\phi_{,t} + A^r_{,r}) \\ I_4 &= \frac{A^{r2}}{r^2} (\phi_{,t}A^r_{,r} - \phi_{,r}A^r_{,t}) & (49) \end{aligned}$$

A physical interpretation of these invariants is obtained from their dimension : a term like a_{ij}^2/μ_0 is a density of energy. It follows that I_2 is proportional to a density of energy and I_4 to the square of it.

Splitting $[\hat{a}_{ij}] = [\hat{s}_{ij}] + [\hat{f}_{ij}]$ into its symmetric and antisymmetric parts gives the 4 invariants associated to $[\hat{s}_{ij}]$:

$$\begin{aligned} I_1^s &= \phi_{,t} + A^r_{,r} + 2\frac{A^r}{r} & (50) \\ I_2^s &= \frac{A_r^2}{r^2} + \phi_{,t}A^r_{,r} + 2\frac{A^r}{r} (\phi_{,t} + A^r_{,r}) \\ I_3^s &= 2\frac{A^r}{r} (\phi_{,t}A^r_{,r} + \frac{A_r^2}{r^2} (\phi_{,t} + A^r_{,r})) \\ I_4^s &= \frac{A^{r2}}{r^2} (\phi_{,t}A^r_{,r}) & (51) \end{aligned}$$

There is a single field invariant :

$$I_2^f = (\phi_{,r} - A^r_{,r})^2 = (E/c)^2 \quad (52)$$

We will now compute these expressions for solutions (39). Explicit formulas for even solutions are :

$$\frac{A_n}{r} = -\frac{\mathcal{A}_n \omega_n}{c^2} \frac{J'_n}{x} \sin \omega_n t \quad (53)$$

$$\phi_{,t} = -\frac{\mathcal{A}_n \omega_n}{c^2} J_n \sin \omega_n t \quad (54)$$

$$\phi_{,r} = \frac{\mathcal{A}_n \omega_n}{c^2} J'_n \cos \omega_n t \quad (55)$$

$$A_{r,r} = -\frac{\mathcal{A}_n \omega_n}{c^2} J''_n \sin \omega_n t \quad (56)$$

$$A_{r,t} = -\frac{\mathcal{A}_n \omega_n}{c^2} J'_n \cos \omega_n t \quad (57)$$

J'_n is the symbol for the derivative of J_n with respect to x (one has $x = k_n r = \omega_n/c r$) and J''_n for the second derivative.

One verifies that the trace $I_1 = 0$ because $J''_n + 2J'_n/x + J_n = 0$ is the equation for the spherical Bessel function when $\ell = 0$ (Lorentz gauge).

We will now integrate I_2, I_3 and I_4 over the whole space-time. The spatial volume element is : $dv = r^2 dr \sin \theta d\theta d\varphi$ and r varies from 0 to infinity, θ from 0 to π and φ from 0 to 2π .

ϕ and A_r being oscillating functions of time, one sees that the integrated value of I_3 over time vanishes. The integral of I_2 over space diverges.

Invariant I_4 deserves a special attention because its integral converges. Its dimension showed that it is proportional to the square of a density of energy. One can thus associate a density of energy to it with the use of an *ad hoc* constant including μ_0 as in the case of the preceding Lagrangian $\mathcal{L} = 1/4\mu_0 a^{ij} a_{ij}$ and a reference of density⁸. I_4 can be developed :

$$I_4 = \frac{1}{r^2} \frac{(\mathcal{A}_n \omega_n)^4}{c^8} J_n'^2 \sin^2 \omega_n t (-J_n J_n'' \sin^2 \omega_n t - J_n'^2 \cos^2 \omega_n t) \quad (58)$$

The mean value of I_4 over a period of time is :

$$\begin{aligned} \bar{I}_4 &= -\frac{1}{r^2} \frac{(\mathcal{A}_n \omega_n)^4}{c^8} J_n'^2 \int_0^{2\pi/\omega_n} \sin^2 \omega_n t (-J_n J_n'' \sin^2 \omega_n t - J_n'^2 \cos^2 \omega_n t) \\ &= \frac{1}{r^2} \frac{\pi}{4} \frac{(\mathcal{A}_n \omega_n)^4}{c^8} J_n'^2 (3J_n J_n'' + J_n'^2) \end{aligned} \quad (59)$$

\bar{I}_4 is an invariant of $[\hat{a}^i_j]$ and contains a part \bar{I}_4^s which is an invariant of $[\hat{s}^i_j]$ and another part \bar{I}_4^f related to the field :

$$\bar{I}_4^s = \frac{1}{r^2} \frac{\pi}{4} \frac{(\mathcal{A}_n \omega_n)^4}{c^8} J_n'^2 (3J_n J_n'') \quad (60)$$

$$\bar{I}_4^f = \frac{1}{r^2} \frac{\pi}{4} \frac{(\mathcal{A}_n \omega_n)^4}{c^8} J_n'^4 \quad (61)$$

Integration over space gives :

$$W_n = \int_0^\infty r^2 dr \int_0^\pi \sin \theta d\theta \int_0^{2\pi} d\phi \bar{I}_4 \quad (62)$$

$$= \pi^2 \frac{\mathcal{A}_n^4 \omega_n^3}{c^7} \int_0^\infty dx (3J_n J_n'^2 J_n'' + J_n'^4) = \pi^2 \frac{\mathcal{A}_n^4 \omega_n^3}{c^7} \int_0^\infty dx \frac{\partial}{\partial x} (J_n J_n'^3) = 0 \quad (63)$$

The integral vanishes because J_n and J'_n vanish for $x \rightarrow \infty$ and either J_n or J'_n nullifies at $x = 0$.

W_n contains two parts : the first is the integrated \bar{I}_4^s , the invariant of the symmetric part $[\hat{s}_{ij}]$:

$$W_{p(n)} = 3\pi^2 \frac{\mathcal{A}_n^4 \omega_n^3}{c^7} \int_0^\infty dx J_n J_n'^2 J_n'' \quad (64)$$

This part is thus associated to the energy of the particle.

The other part can be named the "field part" :

$$W_{f(n)} = \pi^2 \frac{\mathcal{A}_n^4 \omega_n^3}{c^7} \int_0^\infty dx J_n'^4 \quad (65)$$

One obtains the same formula for odd solutions.

$W_n = 0$ implies that $W_{p(n)}$ and $W_{f(n)}$ are equal with opposite signs.

It is important to note that if a solution J_n is replaced by a linear combination $F = \sum_n a_n J_n$, the same formula (62) occurs.

The following curves describe the densities \bar{I}_4^s and \bar{I}_4^f vs x for a series of solutions. There is a peculiarity for $n = 1$, the term $J_1 J_1'^2 J_1''/r^2$ in I_4 (eq.(59)) goes to infinity when $r \rightarrow 0$: When $x = kr \rightarrow 0$, one has the limit :

$$\frac{1}{r^2} J_1'^4 \rightarrow \frac{1}{r^2} \left(\frac{1}{81} - \frac{2x^2}{135} + \dots \right) \quad (66)$$

8. such as $\hbar\omega_n/r_0^3, r_0^3$ being a volume of reference.

While the density of energy diverges near $r = 0$, its integral $W_{f(n)}$ remains finite and is concentrated around the origin.

One observes a concentration of the density of energy \bar{I}_4^s of the particle around the origin for $n = 0$ and $n = 1$. Other solutions have a hollow character and one notes also their spreading and their decrease when n increases. However, the horizontal and vertical scales of these curves are not the same, they depend upon \mathcal{A}_n and ω_n and the comparison between them cannot be done at this stage.

Figures 1-12 display these quantities for the solutions $J_0, J_1, J_2, J_3, J_{10}$ and J_{11} . One observes a concentration of $W_{p(n)}$ around the origin for J_0 and J_1 , and an hollow character of the other structures. One notes also their spreading and their decrease when n increases. One notes also some oscillating character with a change of sign which implies that the energy can become imaginary. For J_1 , we have illustrated the evolution of the energy inside a sphere of radius x instead of \bar{I}_4^f .

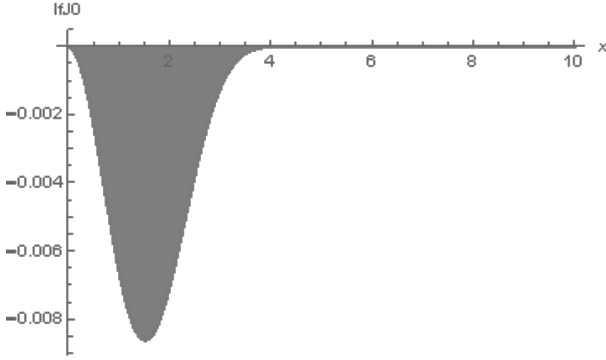


FIGURE 1 – Fundamental solution J_0 : Field invariant \bar{I}_4^f (arbitrary units).

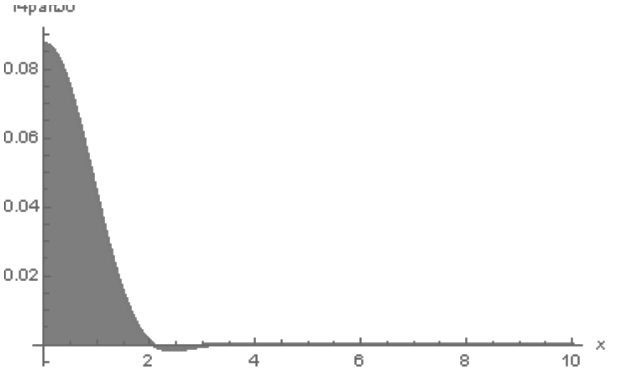


FIGURE 2 – Fundamental solution J_0 : Particle invariant \bar{I}_4^s (arbitrary units).

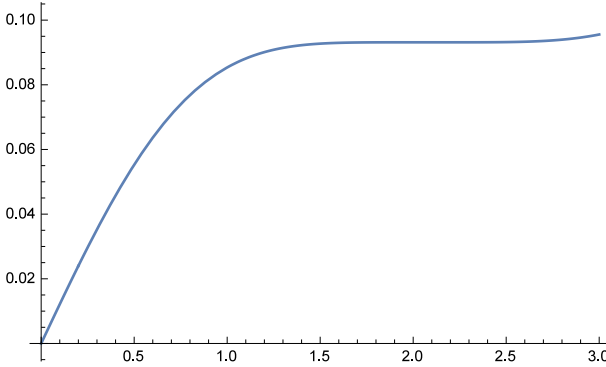


FIGURE 3 – Solution J_1 : Evolution of the electric energy around the origin. (arbitrary units).

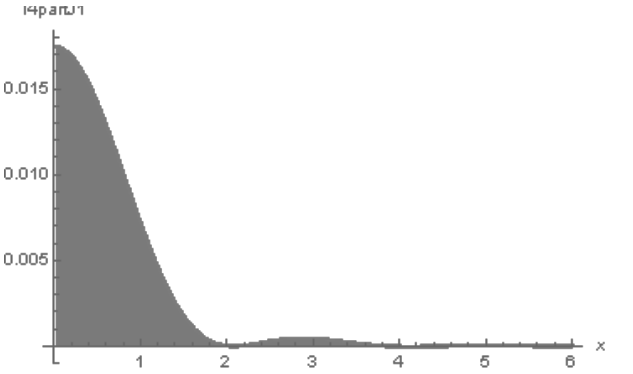


FIGURE 4 – Solution J_1 : Particle invariant \bar{I}_4^s (arbitrary units).

Invariants depend upon x (or r , the distance to the center). However, \bar{I}_4^s , when integrated over the whole space, becomes a number which is a characteristics of the particle. \mathcal{I}_4^s is a quantity proportional to the square of the energy of the particle. In the eigensystem, this energy is purely potential, it follows that its integral is thus related to the particle rest-mass. The following table shows this integral for solutions $J_0, J_1, J_2, J_6, J_{10}$ and J_{11} . Ratios $W_{p(n)}/W_{p(0)}$ are also given, when amplitudes \mathcal{A}_n and frequencies ω_n are equal (which is certainly not the case for real particles). The decrease of this ratio when n increases is very fast, showing that particles become lighter and lighter when n increases.

6.4 Gauge Invariants.

The trace (eq.28) :

$$I = \frac{\partial \phi}{c^2 \partial t} + \text{div} \vec{A} \quad , \quad (67)$$

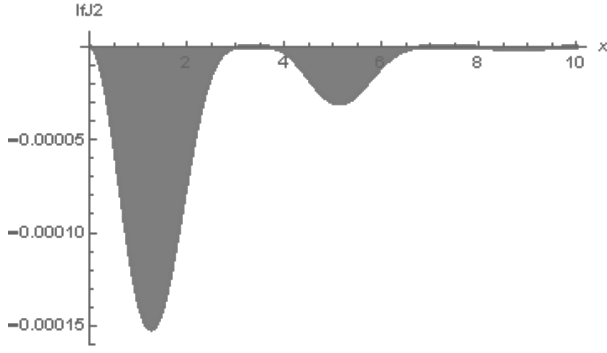


FIGURE 5 – Solution J_2 : Field invariant \bar{I}_4^f (arbitrary units).

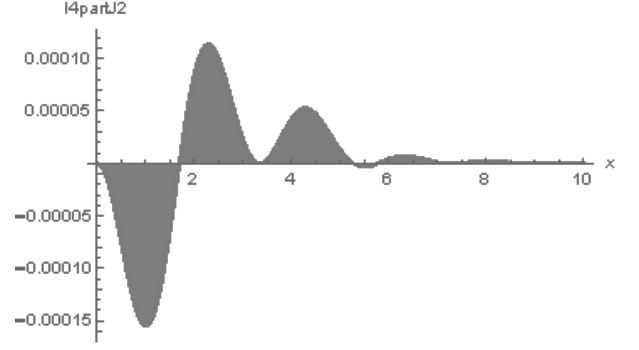


FIGURE 6 – Solution J_2 : Particle invariant \bar{I}_4^s (arbitrary units).

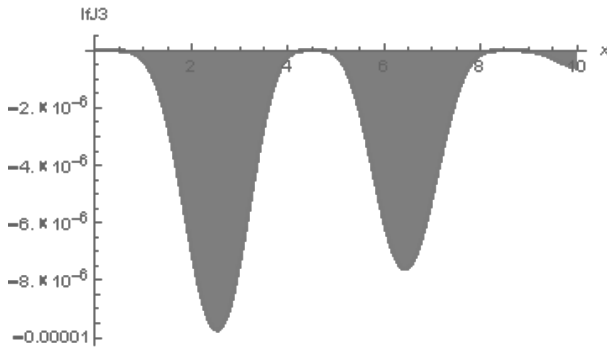


FIGURE 7 – Solution J_3 : Field invariant \bar{I}_4^f (arbitrary units).

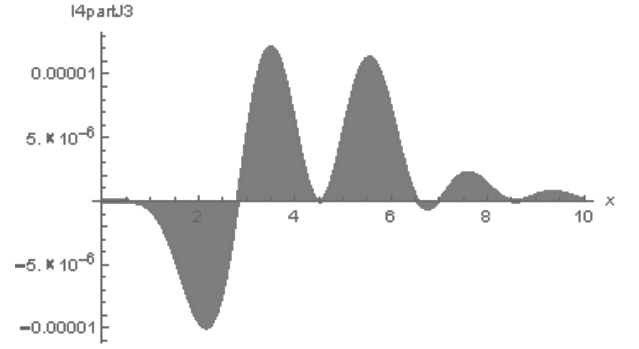


FIGURE 8 – Solution J_3 : Particle invariant \bar{I}_4^s (arbitrary units).

is gauge invariant. I does not change when ϕ/c is replaced by $\phi/c - \partial\mathcal{V}/c\partial t$ and \vec{A} by $\vec{A} + \overrightarrow{\text{grad}}\mathcal{V}$ provided the scalar function \mathcal{V} obeys Lorentz's condition :

$$\frac{\partial^2 \mathcal{V}}{c^2 \partial t^2} - \Delta \mathcal{V} = 0 \quad . \quad (68)$$

This is exactly eq.(33) for the scalar potential ϕ . It follows that if $(\phi_0/c, \vec{A}_0)$ is a solution,

$$\phi_1/c = \phi_0/c - \partial(\phi_0/c)/c\partial t \quad , \quad \vec{A}_1 = \vec{A}_0 + \overrightarrow{\text{grad}}(\phi_0/c) \quad , \quad (69)$$

is also a solution This is an example of the application of the gauge invariance. A second example is given by a plane wave :

$$\mathcal{V} = \mathcal{V}_0 e^{i(\omega t \pm kr)} \quad , \quad (70)$$

which obeys eq.(68) as well.

7 Far field behavior of the solutions.

The far field is defined from the relation $x \gg 1$, or $r \gg 1/k$. In this region, the derivatives of $(\sin x)/x$ and $(\cos x)/x$ are approximated by $\cos x/x$ and $-\sin x/x$ respectively. The scalar potential contains the spherical Bessel functions $J_n(x)$. It follows that even functions behave like $\sin x/x$, with a plus or minus sign following the parity of n : $J_{2n} \rightarrow (-1)^n \sin x/x$. A sum of two successive solutions nullifies in far field if they have the same amplitude.

The longitudinal part of the potential vector contains the functions $\partial(j_n(x))/\partial x$ and thus $A_r^{(n)}$ behaves like $\cos x/x$ with a + or - sign.

The transversal component A_θ does not contain any terms in $1/x = 1/kr$, only higher order terms. It follows that the potential vector becomes longitudinal⁹ far from the particle.

9. One should say, "almost longitudinal", it remains terms in $1/(kr)^2$ and higher powers

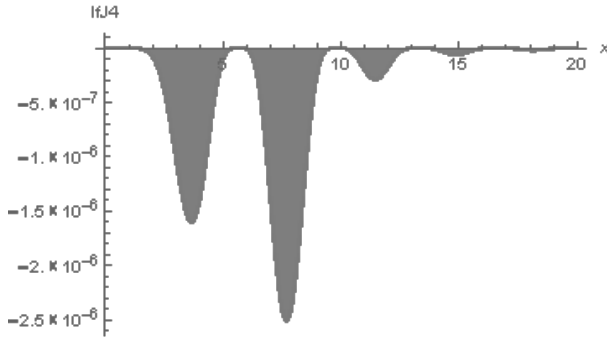


FIGURE 9 – Solution J_4 : Field invariant \bar{I}_4^f (arbitrary units).

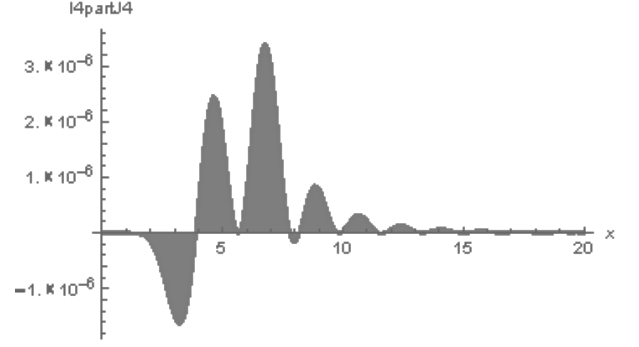


FIGURE 10 – Solution J_4 : Particle invariant \bar{I}_4^s (arbitrary units).

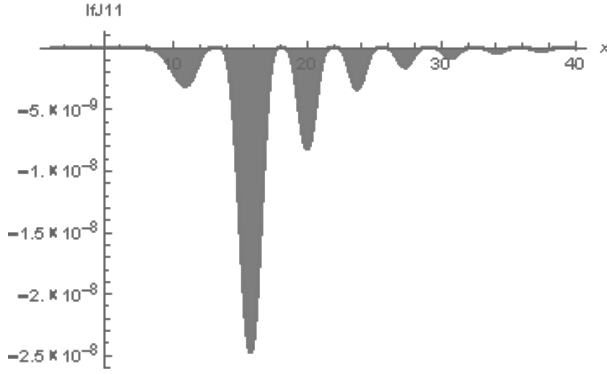


FIGURE 11 – Solution J_{11} : Field invariant \bar{I}_4^f (arbitrary units).

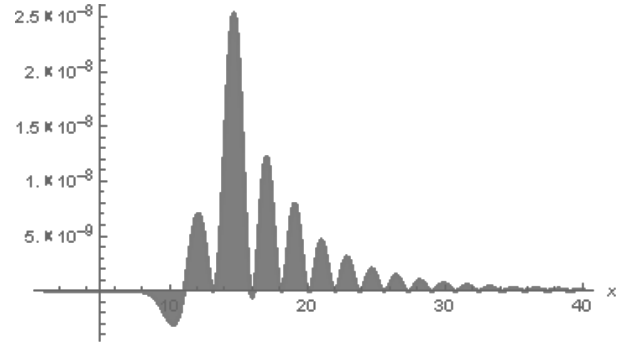


FIGURE 12 – Solution J_{11} : Particle invariant \bar{I}_4^s (arbitrary units).

The far field even solutions are :

$$\begin{aligned} \frac{\Phi(J_{2p}(x))}{c} &\propto (-1)^p \frac{c}{\omega} \frac{\sin x}{x} Y_{2p}^m(\theta, \phi) e^{im\phi} \cos(\omega t) \\ A(J_{2p}(x)) &\propto -(-1)^p \frac{c}{\omega} \frac{\cos x}{x} Y_{2p}^m(\theta, \phi) e^{im\phi} \sin(\omega t) \quad , \end{aligned} \quad (71)$$

The electromagnetic field can thus be expressed in the long range, far from the particle. The electric field is radial, it writes :

$$E_r = -2 \frac{\partial \Phi}{\partial r} \propto -2k \frac{\partial j_n(kr)}{\partial(kr)} Y_n^m(\theta, \phi) \cos(\omega t) \quad , \quad (72)$$

with $\partial j_n(kr)/\partial kr = \pm \cos x/x$ or $\pm \sin x/x$ following the solution J_n . However, as it will be seen in the next section, E_r is **not** the "static" field of standard electrostatics.

The magnetic field is expressed by $\vec{B} = \text{rot} \vec{A}$ and vanishes in the eigensystem of coordinates.

The far-field tensor for a + sign even solution is approximated by :

$$[\hat{a}^i_j] \approx \frac{\mathcal{A}_n \omega_n}{c^2} \begin{bmatrix} -\frac{\sin x}{x} \sin \omega_n t & \frac{\cos x}{x} \cos \omega_n t & 0 & 0 & 0 & 0 \\ -\frac{\cos x}{x} \cos \omega_n t & \frac{\sin x}{x} \sin \omega_n t & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{\cos x}{x} \sin \omega_n t & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{\cos x}{x} \sin \omega_n t & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{\cos x}{x} \sin \omega_n t \end{bmatrix} \quad (73)$$

This tensor is very simple and can be used to study the interaction of two particles separated by a large distance x where little pieces of spherical waves are plane waves. For both even or odd solutions, the field is a combination of a progressive outgoing and incoming spherical wave because :

$$\sin x/x \cos \omega t = \frac{1}{2x} (\sin(\omega t + x) - \sin(\omega t - x)) \quad \text{for even solutions,} \quad (74)$$

TABLE 1 – Integrated Invariant \mathcal{I}_4^s .

Solution	J_0	J_1	J_2	J_6	J_{10}	J_{11}
$W_{p(n)}$	0.5905957	0.15212	0.0231833	0.00157	0.000396	0.0002985
ratio	1	3.88	25.47	577	1491	1978

and :

$$\cos x/x \sin \omega t = \frac{1}{2x} (\sin(\omega t + x) + \sin(\omega t - x)) \quad \text{for odd solutions,} \quad (75)$$

8 Application of the least action principle.

The second set of Maxwell equations relates the field to its sources. The aim of this section is to show that while the field is described by its tensor (antisymmetric part of $[a_{ij}]$), the source terms originate from its coupling with the particle part (the symmetric part of $[a_{ij}]$). The relations between these components are obtained by an application of the principle of least action. These relations are Maxwell equations, they are obtained from Lagrange equations.

8.1 Electrostatics.

Standard theory of electromagnetism begins by a study of the laws of electrostatics where fields and potentials are independent of time. This is in contrast with the above expressions where they depend upon time and space through the sine or cosine rapidly varying functions. We have noted before that the typical frequency of an electromagnetic particle is about 10^{20} Hz, the order of magnitude of a typical wavelength being a picometer. It is clear that an ordinary experiment has no access to these quantities. We thus define the electrostatic potential to be the root mean square of ϕ/c . It can be expressed in terms of solutions (39). For example :

$$\frac{\phi_n}{c} = \frac{\mathcal{A}_n}{c} J_n \cos \omega_n t = \frac{\mathcal{A}_n}{c} (P(1/x) \sin x + Q(1/x) \cos x) \cos \omega_n t \quad , \quad (76)$$

where $P(1/x$ and $Q(1/x$ are polynomials of $1/x$ which reduce to $1/x$ in the far field. It follows that the ordinary electrostatic potential V is :

$$V = RMS(\phi) = \mathcal{A}_n \frac{1}{x} \quad , \quad (77)$$

and the electrostatic field :

$$E_{el.} = -\frac{\partial \phi}{\partial r} = -\frac{\omega}{c} \mathcal{A}_n \frac{1}{x^2} \quad , \quad (78)$$

This formula expresses the connection between the semi-phenomenological electrostatic field and the potentials. The fields $E_{el.}$ and $E_r = -\partial \tilde{A}_r / \partial t - \partial \phi / \partial r$ are both radial.

8.2 Lagrangian.

The principle of least action will be used now in order to recover Maxwell equations. The elementary action is $\mathcal{L} dv$ where \mathcal{L} is the lagrangian density and dv the elementary volume. One has $dv = \sqrt{\eta} d\tau$ where $\sqrt{\eta}$ is the square root of the absolute value of the determinant of the metric tensor. $\sqrt{\eta}$ is a scalar density, $d\tau = dx^0 dx^1 dx^2 dx^3$ is a scalar capacity. The product of both gives a true scalar dv , invariant in a coordinate transformation. As the action is invariant, \mathcal{L} has also to be an invariant. A first try, as we have seen before in section 3.3 is $a^{ij} a_{ij} / (4\mu_0)$. It corresponds to the choice done in standard electromagnetism where it leads to the second set of Maxwell equations if source terms, the phenomenological charges and currents, are included. Here we would like to explore another expression for \mathcal{L} which does not need such terms. The simplest choice for \mathcal{L} is the determinant of the tensor of the particle embedded in its surrounding noise :

$$\mathcal{L} = \frac{1}{\mu_0} \sqrt{|a^i_j + a_0 \eta^i_j|} \quad , \quad (79)$$

where $a^i_j = \eta^{ik} a_{kj}$ is the basic mixed tensor and a_0 a constant having the dimension of a potential over a length. The factor $1/\mu_0$ and the square root are necessary for dimension constraints. The first reason which led us to add this diagonal supplementary term is the expression used by Born and Infeld[10] who

introduced the metric to "represent the space" in their fundamental tensor. Considering only the far field of a single particle at a given point (far from the center of the particle) cannot correspond to a real, physical situation where this field has an amplitude much smaller than the surrounding noise created by the multitude of other particles of the universe. One difference between this noisy floor (or ceiling!) and the particle field is the incoherence of the former and the coherence of the latter. The classical description of noise belongs to stochastic electromagnetism where it is considered to be a superposition of random fields. This superposition is homogeneous, isotropic and stationary (independent of coordinates ct, x, y, z) with a random phase. Far from a particle, the modulus of the scalar and vectorial potentials are equal ($|\mathcal{A}_n/(cx)|$). Adding all these characteristics together, one arrives at the representation of the noise by $a_0\eta_j^i$. Consequences of the interaction of noise and particles will not be analyzed here, let us only say that bare particles and particles dressed by noise may have different properties in the same way dressed or bare atoms have (Lamb shift).

We will thus use the determinant :

$$\mathcal{D} = \begin{vmatrix} a_0 + \phi_{,t} & \phi_{,x} & \phi_{,y} & \phi_{,z} \\ A^x_{,t} & a_0 + A^x_{,x} & A^x_{,y} & A^x_{,z} \\ A^y_{,t} & A^y_{,x} & a_0 + A^y_{,y} & A^y_{,z} \\ A^z_{,t} & A^z_{,x} & A^z_{,y} & a_0 + A^z_{,z} \end{vmatrix} . \quad (80)$$

The expression for the developed determinant contains terms which are products containing 0, 1, 2, 3 or 4 derivatives of the potential of the particle. It follows that it is very different in the vicinity of the origin (the center of the particle) where amplitudes of the potentials are huge with respect to a_0 and far from it where they are small and where terms of order 3 and 4 can be neglected as compared to the others. We will consider first the far field case where equations are linear.

8.3 Linear case.

In this case, products containing 3 or 4 potential derivatives are neglected in the expression of \mathcal{D} which reduces to :

$$\mathcal{D} \approx a_0^4 + a_0^3 (\phi_{,t} + A^x_{,x} + A^y_{,y} + A^z_{,z}) + a_0^2 \{ -A^x_{,y} A^y_{,x} - A^x_{,z} A^z_{,x} - A^y_{,z} A^z_{,y} + A^x_{,x} A^z_{,z} + A^x_{,x} A^y_{,y} + A^y_{,y} A^z_{,z} + A^x_{,x} \phi_{,t} + A^y_{,y} \phi_{,t} + A^z_{,z} \phi_{,t} - A^x_{,t} \phi_x - A^y_{,t} \phi_y - A^z_{,t} \phi_z \} . \quad (81)$$

One can see now that the introduction of the surrounding noise in \mathcal{D} is necessary as it allows to classify the terms following their magnitude, which is impossible otherwise.

Lagrange equations can be written now, taking into account that \mathcal{L} does not depend explicitly of A^j (only its derivatives) :

$$\frac{\partial}{\partial x^j} \left(\frac{\partial \mathcal{L}}{\partial (\partial A^i / \partial x^j)} \right) = 0 , \quad (82)$$

\mathcal{L} does not depend explicitly on x^j which gives :

$$\frac{\partial}{\partial x^j} \left(\frac{\partial \mathcal{D}}{\partial (\partial A^i / \partial x^j)} \right) = 0 , \quad (83)$$

This expression introduces the moments M_i^j :

$$M_i^j = \frac{\partial \mathcal{L}}{\partial A^i_j} , \quad (84)$$

The tensor $[M_i^j]$ is obtained at lowest orders :

$$[M_i^j] = \begin{bmatrix} a_0^3 + a_0^2(A^x_{,x} + A^y_{,y} + A^z_{,z}) & -a_0^2 A^x_{,t} & -a_0^2 A^y_{,t} & -a_0^2 A^z_{,t} \\ -a_0^2 \phi_{,x} & a_0^3 + a_0^2(\phi_{,t} + A^y_{,y} + A^z_{,z}) & -a_0^2 A^y_{,x} & -a_0^2 A^z_{,x} \\ -a_0^2 \phi_{,y} & -a_0^2 A^x_{,y} & a_0^3 + a_0^2(\phi_{,t} + A^x_{,x} + A^z_{,z}) & -a_0^2 A^z_{,y} \\ -a_0^2 \phi_{,z} & -a_0^2 A^x_{,z} & -a_0^2 A^y_{,z} & a_0^3 + a_0^2(\phi_{,t} + A^y_{,y} + A^x_{,x}) \end{bmatrix} . \quad (85)$$

The set of Maxwell equations is obtained from the "product" $\nabla^j M_i^j = \nabla^j M^T_j{}^i$ where :
 $\nabla^j = (\partial/c\partial t, \partial/\partial x, \partial/\partial y, \partial/\partial z)$.

For each value of i , we have to take the divergence of a 4-vector. Taking into account that a_0 is a constant whose derivatives vanish, these equations are :

- For $i = 1$:

$$\frac{\partial}{\partial t} (A_{,x}^x + A_{,y}^y + A_{,z}^z) - \frac{\partial \phi_{,x}}{\partial x} - \frac{\partial \phi_{,y}}{\partial y} - \frac{\partial \phi_{,z}}{\partial z} = 0 \quad (86)$$

which can be written :

$$\frac{\partial}{\partial x} \left(\frac{\partial A^x}{\partial t} - \frac{\partial \phi/c}{\partial x} \right) + \frac{\partial}{\partial y} \left(\frac{\partial A^y}{\partial t} - \frac{\partial \phi/c}{\partial y} \right) + \frac{\partial}{\partial z} \left(\frac{\partial A^z}{\partial t} - \frac{\partial \phi/c}{\partial z} \right) = 0 \quad (87)$$

This equation can be rewritten with the components of the covector in order to introduce the components of the electric field :

$$\begin{aligned} 0 &= \frac{\partial}{\partial x} \left(-\frac{\partial \tilde{A}_x}{\partial t} - \frac{\partial \phi/c}{\partial x} \right) + \frac{\partial}{\partial y} \left(-\frac{\partial \tilde{A}_y}{\partial t} - \frac{\partial \phi/c}{\partial y} \right) + \frac{\partial}{\partial z} \left(-\frac{\partial \tilde{A}_z}{\partial t} - \frac{\partial \phi/c}{\partial z} \right) \\ &= \frac{1}{c} \operatorname{div} \vec{E} \end{aligned}$$

This result is the first Maxwell equation without sources.

- For $i = 2$ one obtains :

$$-\frac{\partial}{\partial t} A_{,t}^x + \frac{\partial}{\partial x} (A_{,y}^y + A_{,z}^z + \phi_{,t}) - \frac{\partial A_{,y}^x}{\partial y} - \frac{\partial A_{,z}^x}{\partial z} = 0 \quad (88)$$

After rearranging terms and introducing covectors, one gets :

$$\begin{aligned} 0 &= \frac{\partial}{\partial t} \left(\frac{\partial \phi/c}{\partial x} - \frac{\partial A^x}{\partial t} \right) + \frac{\partial}{\partial y} \left(\frac{\partial A^y}{\partial x} - \frac{\partial A^x}{\partial y} \right) + \frac{\partial}{\partial z} \left(\frac{\partial A^z}{\partial x} - \frac{\partial A^x}{\partial z} \right) \\ &= \frac{\partial}{\partial t} \left(\frac{\partial \phi/c}{\partial x} + \frac{\partial \tilde{A}_x}{\partial t} \right) - \frac{\partial}{\partial y} [\vec{\operatorname{rot}} \tilde{A}]_z + \frac{\partial}{\partial z} [\vec{\operatorname{rot}} \tilde{A}]_y \\ &= \frac{\partial}{\partial t} (E_x/c) + \frac{\partial B_y}{\partial z} - \frac{\partial B_z}{\partial y} \end{aligned} \quad (89)$$

This result, combined with those obtained with $i = 3$ and 4, leads to the usual Maxwell equation :

$$\operatorname{rot} \vec{B} = \frac{1}{c^2} \frac{\partial E_x}{\partial t} \quad (90)$$

We arrive at the conclusion that the linear form of Maxwell equations results from the truncation of the Lagrangian. Keeping the non-linear terms leads to supplementary terms, i.e., the source terms. This is what we compute in the next section.

8.4 Non-linear terms.

This section is devoted to the calculation of the charge associated to an electromagnetic particle. The idea is that the non-linear part of the Lagrangian, which was neglected in the preceding section, is responsible for a source term in equation (86) or in Gauss's law $\operatorname{div} E = \rho/\epsilon_0$. The charge should thus be computed from an integration over space-time of this equation. As this integration is more easily done in the spherical system of coordinates, it is this system which will be used now.

The non-linear determinant which appears in the Lagrangian is expressed in the spherically symmetric case in the noisy environment :

$$\mathcal{D} = \begin{vmatrix} a_0 + \phi_{,t} & \phi_{,r} & 0 & 0 \\ A_{,t}^r & a_0 + A_{,r}^r & 0 & 0 \\ 0 & 0 & a_0 + \frac{A^r}{r} & 0 \\ 0 & 0 & 0 & a_0 + \frac{A^r}{r} \end{vmatrix} = \left(a_0 + \frac{A^r}{r} \right)^2 [(a_0 + \phi_{,t})(a_0 + A_{,r}^r) - \phi_{,r} A_{,t}^r] \quad (91)$$

One is thus led to split \mathcal{D} into two parts $\mathcal{D} = \mathcal{D}_1 + \mathcal{D}_2$:

$$\mathcal{D}_1 = a_0^4 + a_0^3 \left(2 \frac{A^r}{r} + \phi_{,t} + A^r_{,r} \right) + a_0^2 \left(\frac{A^{r2}}{r^2} + \phi_{,t} A^r_{,r} - \phi_{,r} A^r_{,t} + 2 \frac{A^r}{r} (\phi_{,t} + A^r_{,r}) \right) \quad (92)$$

\mathcal{D}_1 leads to linear Maxwell equations in the far field region as in the preceding section. The second part is the nonlinear part :

$$\mathcal{D}_2 = a_0 \left(2 \frac{A^r}{r} (\phi_{,t} A^r_{,r} - \phi_{,r} A^r_{,t}) + \frac{A^{r2}}{r^2} (\phi_{,t} + A^r_{,r}) \right) + \frac{A^{r2}}{r^2} (\phi_{,t} A^r_{,r} - \phi_{,r} A^r_{,t}) \quad (93)$$

We are interested in rewriting the first Maxwell equation. We thus need the moments :

$$p_1^1 = \frac{\partial \mathcal{D}}{\partial \phi_{,t}} = a_0^2 (A^r_{,r} + 2A^r/r) + \left[a_0 \left(2 \frac{A^r}{r} A^r_{,r} + \frac{A^{r2}}{r^2} \right) + \frac{A^{r2}}{r^2} A^r_{,r} \right] + a_0^3 \quad (94)$$

$$p_1^2 = \frac{\partial \mathcal{D}}{\partial A^r_{,t}} = -a_0^2 \phi_{,r} + \left[-a_0 2 \frac{A^r}{r} \phi_{,r} - \frac{A^{r2}}{r^2} \phi_{,r} \right] \quad (95)$$

The nonlinear parts are put between braces. Now we use the expression of the divergence of a vector \vec{U} in the spherical coordinates of the geometrical space :

$$\text{div} \vec{U} = \frac{1}{r^2} \frac{\partial(r^2 U_r)}{\partial r} + \frac{1}{r} \frac{\partial(U_\theta)}{\partial \theta} + \frac{1}{r \sin \theta} \frac{\partial(U_\phi)}{\partial \phi} + \frac{\cos \theta}{r \sin \theta} U_\theta \quad , \quad (96)$$

which shows that the operators we need in ∇^j are : $\nabla^j = \partial/c\partial t, 1/r^2 \partial(r^2 U_r)/\partial r$. Maxwell's equation is :

$$a_0^2 \frac{\partial}{c\partial t} (A^r_{,r} + 2A^r/r) - \frac{a_0^2}{r^2} \frac{\partial}{\partial r} (r^2 \phi_{,r}) = - \frac{\partial}{c\partial t} \left[a_0 \left(2 \frac{A^r}{r} A^r_{,r} + \frac{A^{r2}}{r^2} \right) + \frac{A^{r2}}{r^2} A^r_{,r} \right] + \frac{1}{r^2} \frac{\partial}{\partial r} (2r a_0 A^r \phi_{,r} + A^{r2} \phi_{,r}) \quad (97)$$

The l.h.s. is clearly $a_0^2 \text{div} \vec{E}/c$ expressed in the spherical system. Now it is time to compare this expression with the standard Maxwell's equation which links $\text{div} \vec{E}$ with the density of charges ρ :

$$\text{div} \vec{E} = - \frac{c}{a_0^2} \frac{\partial}{c\partial t} \left[a_0 \left(2 \frac{A^r}{r} A^r_{,r} + \frac{A^{r2}}{r^2} \right) + \frac{A^{r2}}{r^2} A^r_{,r} \right] + \frac{c}{a_0^2} \frac{1}{r^2} \frac{\partial}{\partial r} (2r a_0 A^r \phi_{,r} + A^{r2} \phi_{,r}) = \frac{\rho}{\epsilon_0} \quad (98)$$

One thus obtain a relation between the phenomenological quantity ρ and the potential. One sees that it can be used to obtain the elementary charge Q .

8.5 The elementary electric charge.

Equation (98), is a function of space with a sinusoidal time dependance. We will write $\rho = |\rho| \sin \omega t$ and compute the elementary charge with an integration over space.

For this purpose, let us use the preceding odd solutions (39) :

$$\begin{aligned} \frac{\phi_n}{c} &= \frac{\mathcal{A}_n}{c} J_n(x) \sin \omega_n t \\ A_n &= \frac{\mathcal{A}_n}{c} J'_n(x) \cos \omega_n t \end{aligned}$$

in (98). Details of calculations follow :

$$\begin{aligned} a_0 \left(2 \frac{A^r}{r} A^r_{,r} + \frac{A^{r2}}{r^2} \right) &= a_0 \cos^2 \omega_n t \frac{(\mathcal{A}_n \omega_n)^2}{c^4} \left(2 \frac{J'_n}{x} J''_n + \frac{J_n'^2}{x^2} \right) \\ \frac{A^{r2}}{r^2} A^r_{,r} &= \frac{(\mathcal{A}_n \omega_n)^3}{c^6} \frac{J_n'^2}{x^2} J''_n \cos^3 \omega_n t \end{aligned} \quad (99)$$

and :

$$\begin{aligned}
-\frac{c}{a_0^2} \frac{\partial}{c \partial t} \left[a_0 \left(2 \frac{A^r}{r} A_{,r} + \frac{A^{r2}}{r^2} \right) + \frac{A^{r2}}{r^2} A_{,r} \right] = \\
-\frac{c}{a_0^2} \left\{ a_0 \frac{2\omega_n}{c} \sin \omega_n t \cos \omega_n t \frac{(\mathcal{A}_n \omega_n)^2}{c^4} \left(2 \frac{J'_n}{x} J''_n + \frac{J'^2_n}{x^2} \right) \right. \\
\left. - \frac{(\mathcal{A}_n \omega_n)^3}{c^6} \frac{J'^2_n}{x^2} J''_n \frac{3\omega_n}{c} \cos^2 \omega_n t \sin \omega_n t \right\} \quad (100)
\end{aligned}$$

The second part of the equation gives :

$$\begin{aligned}
\frac{c}{a_0^2} \frac{1}{r^2} \frac{\partial}{\partial r} \left(2r a_0 A^r \phi_{,r} + A^{r2} \phi_{,r} \right) = \\
\frac{c}{a_0^2} \frac{1}{r^2} \frac{\partial}{\partial r} \left(-2r a_0 \frac{\mathcal{A}_n^2 \omega_n}{c^3} J'^2_n \sin \omega_n t \cos \omega_n t + \frac{\mathcal{A}_n^3 \omega_n}{c^4} J'^3_n \cos^2 \omega_n t \sin \omega_n t \right) \quad (101)
\end{aligned}$$

The mean value over a period of time of the modulus of $\rho/\epsilon_0 = |\rho|/\epsilon_0 \sin \omega_n t$ is :

$$|\rho|/\epsilon_0 = -\frac{c}{a_0^2} \left\{ -\frac{1}{2} \frac{(\mathcal{A}_n \omega_n)^3}{c^6} \frac{J'^2_n}{x^2} J''_n \frac{3\omega_n}{c} \right\} + \frac{c}{a_0^2} \frac{1}{r^2} \frac{\partial}{\partial r} \left(\frac{1}{2} \frac{\mathcal{A}_n^3 \omega_n}{c^4} J'^3_n \right) \quad (102)$$

This charge density can be integrated over space to give the total charge :

$$|Q|/\epsilon_0 = \int_0^\infty r^2 dr \int_0^\pi \sin \theta d\theta \int_0^{2\pi} d\varphi |\bar{\rho}|/\epsilon_0 = 4\pi \int_0^\infty r^2 dr |\rho|/\epsilon_0 \quad . \quad (103)$$

These integrals are easy to compute :

$$\begin{aligned}
|Q|/\epsilon_0 &= \frac{4\pi c}{a_0^2} \left\{ \frac{1}{2} \frac{3(\mathcal{A}_n^3 \omega_n)}{c^4} \int_0^\infty J'^2_n J''_n dx \right\} + \frac{4\pi c}{a_0^2} \left(\frac{1}{2} \frac{\mathcal{A}_n^3 \omega_n}{c^4} |J'^3_n|_\infty \right) \\
&= \frac{4\pi c}{a_0^2} \left\{ \frac{1}{2} \frac{3(\mathcal{A}_n^3 \omega_n)}{c^4} \frac{1}{3} |J'^3_n|_\infty \right\} + \frac{4\pi c}{a_0^2} \left(\frac{1}{2} \frac{\mathcal{A}_n^3 \omega_n}{c^4} |J'^3_n|_\infty \right) \quad (104)
\end{aligned}$$

$$= \frac{4\pi}{a_0^2} \frac{\mathcal{A}_n^3 \omega_n}{c^3} J'^3_n(0) \quad . \quad (105)$$

It remains the value of J'^3_n at the origin only, as J'^3_n nullifies at infinity. **However, the only value of n for which $J'^3_n(x=0) \neq 0$ is $n=1$ and $J'_1(x=0) = 1/3$** (see formula (38)). **One arrives at the conclusion that the charge of an electromagnetic particle is characteristic of the solution $n=1$.** In other words, a particle is charged if its representative potential $\sum J_n$ contains the solution J_1 . The consequence is fundamental : a non-charged particle can exist but is unable to interact with an electromagnetic field. It is very tempting to associate such particles to the mystery of the black matter.

One obtains the simple formula which relates the charge to the potential :

$$\frac{|Q|}{\epsilon_0} = \frac{4\pi}{27a_0^2} \left(\frac{\mathcal{A}_1}{c} \right)^3 \omega_1 \quad , \quad (106)$$

Each term in the r.h.s. of the equation ¹⁰ has a physical meaning :

- a_0^2 is the modulus square of the vacuum noise,
- \mathcal{A}_1 is the amplitude of the vector potential of the solution $n=1$ at the center of the particle and
- ω_1 is the pulsation corresponding to that solution.

It is important to remember that if a solution J_n is replaced by a linear combination $F = \sum_n a_n J_n$, the same formula (104) occurs, which means that if a particle is charged, its potentials necessarily contain J_1 . It means also that a charge is necessarily a multiple of J_1 .

One should also note that our choice of the Lagrangian is not the only one which leads to this result : a function of \mathcal{L} like $a + b\mathcal{L}^c$ does the job.

We have thus shown in this section that the electric charge is related to the electromagnetic potential : the electric charge is no longer a phenomenological quantity, it belongs now to the properties of this potential.

¹⁰. Note the dimension of this equation : $M L^3 T^{-2} Q^{-1}$.

9 Conclusion.

This article has presented an introduction to the study of electromagnetic particles based on an extension of Born and Infeld's ideas. The theory is completely classical : Minkowski's space-time is full of a something called an "electromagnetic potential". Its 16 partial derivatives are the components of its gradient tensor which is the fundamental entity at each point. This tensor describes an "electromagnetic landscape". Its antisymmetric part contains the usual field tensor whose components are the basic fields of standard classical electromagnetism. The symmetric part is the particle tensor. Both parts are intrinsically mingled together but their symmetric or antisymmetric character makes them to remain separated in a coordinate change which explains why this symmetric part has been ignored and replaced by phenomenological quantities. These tensors are characterized by invariants under transformations belonging to the Poincaré group. The symmetric part can be diagonalized which leads to an equation whose solutions are the components of the 4-potential. We have given here the distribution of the squared density of energy around the origin which describe the particle and the field aspect of some solutions. We have tried to connect standard electromagnetism to our calculations : Non-linear Maxwell equations are recovered from an application of the principle of least action to a Lagrangian corresponding to an invariant for a particle embedded in noise. This Lagrangian is not the standard Lagrangian of usual electromagnetism and resembles more to that used in[1]. One of this equation shows that the elementary charge Q is associated to a single solution which means that a charged particle should at least contain this solution. While the starting point of our study was very simple (inclusion of the symmetric part of the tensor), some conclusions are quite surprising : the existence of non-charged particles which are looked for to explain black matter in cosmology is a striking example. It is clear that the properties of several solutions put together have to be studied now, a first goal being the identification with the electron, the muon and the tau families. Our study has voluntary been limited to the static aspect of a particle, a landscape of a desert with a steady hill or mountain which expresses itself by a longitudinal vibration around. Several chapters can be foreseen among which the study of the stability under the influence of the fundamental noise, together with the study of birth and death of particles themselves. Non-standard hypothesis have been made in the course of this work : Among them we have associated a high frequency to the oscillating field-particle : implications of this hypothesis are also, interesting to develop. Applications to the study of two-body or many-body problems are also fundamental for the exploration of atoms and nuclei as well as the description of collisions. Up to now, we have not found any counter-argument against this theory which allows some confidence in its future developments.

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- [5] ω is supposed to be too high to be directly measured : for instance $\omega \geq 10^{20}$ rd/s. See the discussion in [9] . In this view, pure electrostatic, or time-independant fields do not exist : measurements exhibit only mean values taken over a laps of time enormous with respect to $2\pi/\omega$. It follows that de Broglie wavelength $\lambda = h/p = h/mv$ corresponds to a Doppler shift ($\omega/2\pi v/c$). One thus obtain $\omega/2\pi = mc^2/h$. For the electron, it corresponds to a frequency $\omega/2\pi = 1.235 \cdot 10^{20}$ Hz. This hypothesis implies that frequency measurements detect only beatings between the frequency emitted by the moving electron and the floor frequency noise emitted by the other (almost static) electrons. Results obtained in self-interference particle experiments are not opposed to this view : the particle interferes with its image given by the experimental device.
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- [9] V. F. Weisskopf, Recent Developments in the Theory of the Electron, Rev. Mod. Phys., 21, N^o 3, pp. 305-315 (1949)

[10] The Lagrangian which was used by Eddington and then by Born and Infeld is $\mathcal{L} = \sqrt{-|a_{ij}|}$ where the determinant $|a_{ij}|$ is the sum of the metric and the field tensors.