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Y-Calculus: A language for real Matrices derived from the ZX-Calculus

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Abstract. The ZX-Calculus is a powerful diagrammatic language devoted to represent complex quantum evolutions. But the advantages of quantum computing still exist when working with rebits, and evolutions with real coefficients. Some models explicitly use rebits, but the ZX-Calculus can not handle these evolutions as it is.

Hence, we define an alternative language solely dealing with real matrices, with a new set of rules. We show that three of its non-trivial rules are not derivable from the others and we prove that the language is complete for the $\frac{\pi}{2}$-fragment. We define a generalisation of the Hadamard node, and exhibit two interpretations from and to the ZX-Calculus, showing the consistency between the two languages.

1 Introduction

The ZX-Calculus, introduced by Coecke and Duncan [3], allows us to represent and reason with complex quantum evolutions. Its diagrams are universal, meaning that for any quantum transformation, there exists a ZX-diagram that represents it.

Two of its nodes are parametrised by angles. Restricting the language to some particular sets of angles allows us to represent the *real stabiliser quantum mechanics* – angles that are multiples of π, also called π-fragment –, the *stabiliser quantum mechanics* – $\frac{\pi}{2}$-fragment – or the *Clifford+T quantum mechanics* – $\frac{\pi}{4}$-fragment.

One major downside of the diagrammatic approach is that two different diagrams may represent the same matrix. To palliate this problem, the ZX-Calculus comes with a set of transformation rules that preserve the represented matrix: the language is *sound*.

The converse of soundness is *completeness*. The language would be complete if, for any two diagrams that represent the same matrix, they could be transformed into one-another only using the transformation rules allowed by the language. The ZX-Calculus is not complete in general [6], but some of its fragments are. The π-fragment and the $\frac{\pi}{2}$-fragment are both complete [4,1], but the completeness for the $\frac{\pi}{4}$-fragment is still an open question, all the more important that, unlike the other two, this fragment is *approximately universal*, meaning that any quantum evolution can be approximated with arbitrarily good precision with this fragment.

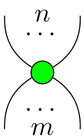
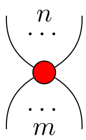
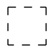
With the ZX-Calculus, some real transformations can only be obtained by composition of complex ones. We define an alternative language, the Y-Calculus, that only deals with real matrices, by losing the angles and a node of the ZX-Calculus, and adding another angle-parametrised node. We give a set of rules to this language, and prove that three of its non-trivial axioms are not derivable from the others. We establish a link between the $\frac{\pi}{2}$-fragment of the Y-Calculus and the π-fragment of the ZX-Calculus, and thanks to the completeness of the latter, we prove the $\frac{\pi}{2}$-fragment of the Y-Calculus is complete.

This link allows us to define a Hadamard node – present in the ZX-Calculus, but not initially in the Y-Calculus– and a rule of the Y-Calculus gives us a hint on a way of generalising this node to any arity. We finally exhibit an interpretation from the Y-Calculus to the ZX-Calculus, which shows the consistency of the two languages, and another interpretation from the ZX-Calculus to the Y-Calculus, which not only demonstrates that any quantum evolution can be efficiently simulated with rebits, but also that we can extract the real and imaginary parts of a ZX-diagram, which also leads to an elegant demonstration of the universality of the Y-Calculus.

2 Y-Calculus

2.1 Diagrams and standard interpretation

A Y-diagram $D : k \to l$ with k inputs and l outputs is generated by:

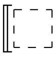
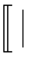
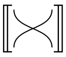
$R_Z^{(n,m)} : n \to m$		$R_X^{(n,m)} : n \to m$	
$R_Y(\alpha) : 1 \to 1$		$e : 0 \to 0$	
$\mathbb{I} : 1 \to 1$		$\sigma : 2 \to 2$	
$\epsilon : 2 \to 0$		$\eta : 0 \to 2$	

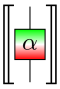
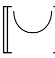
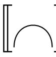
where $n, m \in \mathbb{N}$ and $\alpha \in \mathbb{R}$

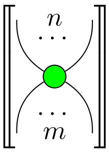
- Spacial Composition: for any $D_1 : a \to b$ and $D_2 : c \to d$, $D_1 \otimes D_2 : a + c \to b + d$ consists in placing D_1 and D_2 side by side, D_2 on the right of D_1.
- Sequential Composition: for any $D_1 : a \to b$ and $D_2 : b \to c$, $D_2 \circ D_1 : a \to c$ consists in placing D_1 on the top of D_2, connecting the outputs of D_1 to the inputs of D_2.

The standard interpretation of the Y-diagrams associates any diagram $D : n \to m$ with a linear map $[\![D]\!] : \mathbb{R}^{2^n} \to \mathbb{R}^{2^m}$ inductively defined as follows:

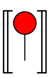
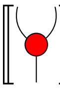
$$[\![D_1 \otimes D_2]\!] := [\![D_1]\!] \otimes [\![D_2]\!] \qquad [\![D_2 \circ D_1]\!] := [\![D_2]\!] \circ [\![D_1]\!]$$

$[\![$  $]\!] := (1)$ $\qquad$ $[\![$  $]\!] := \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ $\qquad$ $[\![$  $]\!] := \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$

$[\![$  $]\!] := \begin{pmatrix} \cos(\alpha/2) & -\sin(\alpha/2) \\ \sin(\alpha/2) & \cos(\alpha/2) \end{pmatrix}$ $\qquad$ $[\![$  $]\!] := \begin{pmatrix} 1 & 0 & 0 & 1 \end{pmatrix}$ $\qquad$ $[\![$  $]\!] := \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix}$

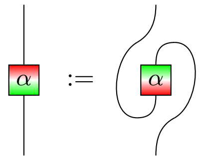
$[\![$  $]\!] := (2)$ $\qquad$ $[\![$  $]\!] := 2^m \left\{ \overbrace{\begin{pmatrix} 1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & \cdots & 0 & 1 \end{pmatrix}}^{2^n} \right.$

If $H = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$, for any $a, b \geq 0$, $\left[\!\!\left[R_X^{(a,b)} \right]\!\!\right] = H^{\otimes b} \circ \left[\!\!\left[R_Z^{(a,b)} \right]\!\!\right] \circ H^{\otimes a}$ (where $M^{\otimes 0} = (1)$ and $M^{\otimes k} = M \otimes M^{\otimes k-1}$ for any $k \in \mathbb{N}^*$). E.g.,

$[\![$  $]\!] := (2)$ $\qquad$ $[\![$  $]\!] = \sqrt{2} \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ $\qquad$ $[\![$  $]\!] = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{pmatrix}$

2.2 Calculus

We define a set of basic transformations of Y-diagrams that preserve the matrices they represent. These axioms are expressed in figure 1, where the upside-down box is defined as:



More generally, we assume that only topology matters, meaning the wires can be bent at will.

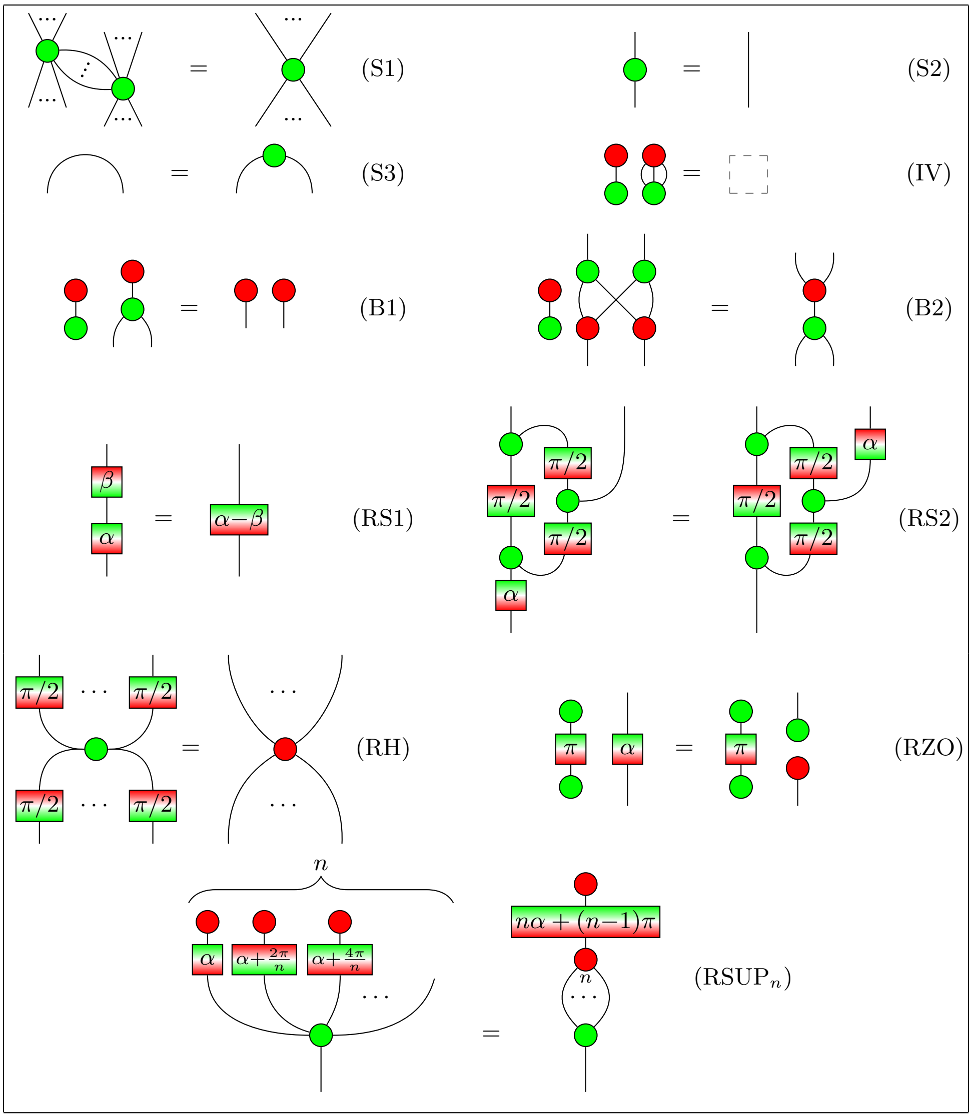
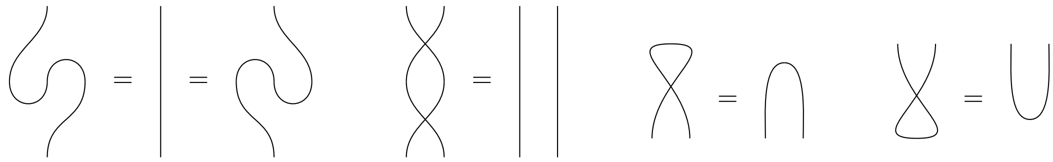


Fig. 1. Rules for the **Y-Calculus** with scalars. All of these rules also hold when flipped upside-down, or with the colours red and green swapped and the real-boxes flipped. The right-hand side of (IV) is an empty diagram. $(\cdots)$ denote zero or more wires, while $(\cdot^{\cdot^{\cdot}})$ denote one or more wires.

Example 1.



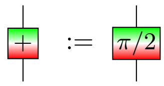
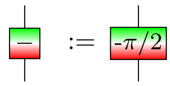
Therefore, two vertices connected by an horizontal wire have meaning.

Theorem 1. *All these equalities are sound, meaning that*

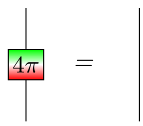
$$(Y \vdash D_1 = D_2) \quad \Rightarrow \quad (\llbracket D_1 \rrbracket = \llbracket D_2 \rrbracket)$$

When we can show that a diagram D_1 is equal to another one, D_2, using a succession of equalities of this set, we write $Y \vdash D_1 = D_2$. Given that the rules are sound, this means that $\llbracket D_1 \rrbracket = \llbracket D_2 \rrbracket$. The rules can obviously be applied to any subdiagram, meaning, for any diagram D:

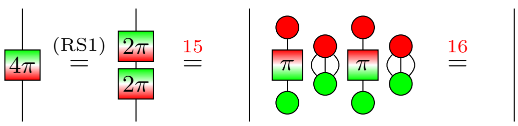
$$(Y \vdash D_1 = D_2) \quad \Rightarrow \quad \begin{cases} (Y \vdash D_1 \circ D = D_2 \circ D) \ \wedge \ (Y \vdash D \circ D_1 = D \circ D_2) \\ (Y \vdash D_1 \otimes D = D_2 \otimes D) \wedge (Y \vdash D \otimes D_1 = D \otimes D_2) \end{cases}$$

Notation: The boxes with $\pm\frac{\pi}{2}$ angles will be written  and  in order to simplify some lemmas and proofs.

Theorem 2. *The real boxes are* 4π*-periodical:*



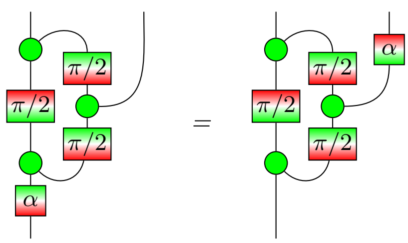
Proof. Using the rule (RS1) and lemmas 15 and 16:



3 Minimality

In this section, we prove the necessity of some rules i.e. we show that some axioms are not deducible from the others. A rule (R) is necessary when $Y \setminus \{(R)\} \nvdash (R)$.

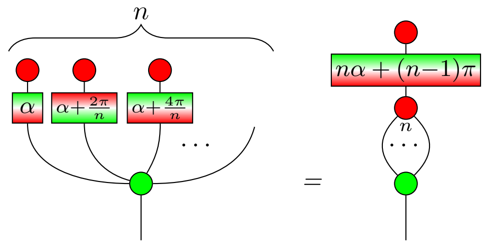
Proposition 1.



(RS3) *cannot be derived from the other rules in any* $\frac{\pi}{2n}$*-fragment* ($n \in \mathbb{N}^*$).

Proof. In appendix at page 19.

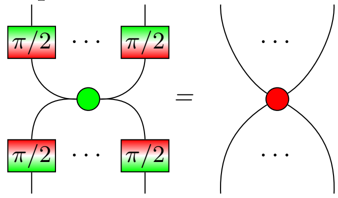
Proposition 2.



(RS$_n$) *is necessary when $n \geq 3$ is prime, and only the rule for prime numbers are present in the set of axioms:* $Y \setminus \{(RS_n)_{n\notin\mathbb{P}}\} \setminus \{(RS_p)\} \nvdash (RS_p)$

Proof. In appendix at page 20.

Proposition 3.



(RH) *cannot be derived from the other rules.*

Proof. In appendix at page 22.

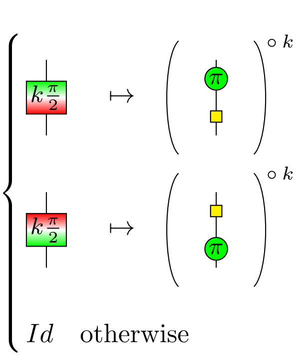
4 Completeness of the $\frac{\pi}{2}$-fragment

Proposition 4. *The $\frac{\pi}{2}$-fragment of the Y-Calculus ($\mathrm{Y}_{\frac{\pi}{2}}$) is complete.*

Proof. The idea of the proof is to show that $\mathrm{Y}_{\frac{\pi}{2}}$ and the real stabiliser ZX-Calculus (ZX$_r$) [4] deal with the same matrices and have the same expressivity. The ZX$_r$ is defined at page 22.

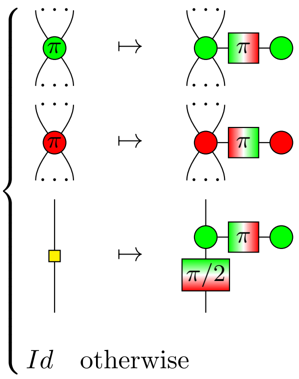
To do so, we define the interpretations:

$[\![.]\!]^{\mathrm{Y}_{\frac{\pi}{2}}\to ZX_r}$:



and

$[\![.]\!]^{ZX_r\to \mathrm{Y}_{\frac{\pi}{2}}}$:



for $k \geq 0$ with $D^{\circ 0} = I$ and $D^{\circ l} = D^{\circ l-1} \circ D$ for $l \geq 2$.
It is important to notice that the rule (RSUP$_n$) is not an axiom of the language $\mathrm{Y}_{\frac{\pi}{2}}$. Indeed, In order to be in the $\frac{\pi}{2}$-fragment, only (RSUP$_2$) and (RSUP$_4$) matter, but (RSUP$_4$) can be obtained from (RSUP$_2$), and (RSUP$_2$) can be derived from the other rules whenever α is a multiple of $\frac{\pi}{2}$. It is no use to prove that (RSUP$_2$) and (RSUP$_4$) are derivable from the new set of rules, because we will prove that the language is complete, hence any semantically correct equation can be derived.

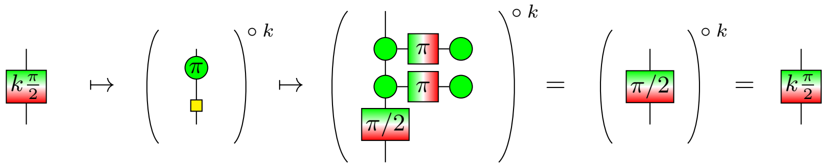
The two interpretations both preserve the equalities of the sets of rules of respectively $\mathrm{Y}_{\frac{\pi}{2}}$ and ZX$_r$ (details page 24). One can easily show that they also preserve the semantics:

$$\left[\!\left[[\![.]\!]^{\mathrm{Y}_{\frac{\pi}{2}}\to ZX_r} \right]\!\right] = [\![.]\!] = \left[\!\left[[\![.]\!]^{ZX_r\to \mathrm{Y}_{\frac{\pi}{2}}} \right]\!\right]$$

Moreover, for any $\mathrm{Y}_{\frac{\pi}{2}}$-diagram D:

$$\mathrm{Y}_{\frac{\pi}{2}} \vdash D = \left[\!\left[[\![D]\!]^{\mathrm{Y}_{\frac{\pi}{2}}\to ZX_r} \right]\!\right]^{ZX_r\to \mathrm{Y}_{\frac{\pi}{2}}}$$

Indeed, using lemma 11 and (RS1):



The reasoning is the same for the upside-down box, and otherwise, the composition of the interpretations is the identity.

Now, let D_1 and D_2 be two $\mathrm{Y}_{\frac{\pi}{2}}$-diagrams such that $[\![D_1]\!] = [\![D_2]\!]$. The two interpretations preserve the semantics, so: $\left[\!\left[[\![D_1]\!]^{\mathrm{Y}_{\frac{\pi}{2}}\to ZX_r}\right]\!\right] = \left[\!\left[[\![D_2]\!]^{\mathrm{Y}_{\frac{\pi}{2}}\to ZX_r}\right]\!\right]$.

Since ZX_r is complete [4], $ZX_r \vdash [\![D_1]\!]^{\mathrm{Y}_{\frac{\pi}{2}}\to ZX_r} = [\![D_2]\!]^{\mathrm{Y}_{\frac{\pi}{2}}\to ZX_r}$.

Moreover, $\mathrm{Y}_{\frac{\pi}{2}}$ proves all the equalities of the ZX_r, so:
$\mathrm{Y}_{\frac{\pi}{2}} \vdash \left[\!\left[[\![D_1]\!]^{\mathrm{Y}_{\frac{\pi}{2}}\to ZX_r}\right]\!\right]^{ZX_r\to\mathrm{Y}_{\frac{\pi}{2}}} = \left[\!\left[[\![D_2]\!]^{\mathrm{Y}_{\frac{\pi}{2}}\to ZX_r}\right]\!\right]^{ZX_r\to\mathrm{Y}_{\frac{\pi}{2}}}$.

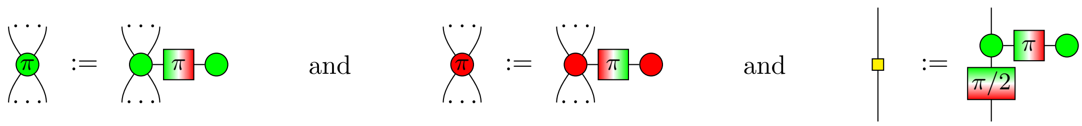
Finally, since $\mathrm{Y}_{\frac{\pi}{2}}$ proves that the composition of the two interpretations is the identity,

$$\mathrm{Y}_{\frac{\pi}{2}} \vdash D_1 = \left[\!\left[[\![D_1]\!]^{\mathrm{Y}_{\frac{\pi}{2}}\to ZX_r}\right]\!\right]^{ZX_r\to\mathrm{Y}_{\frac{\pi}{2}}} = \left[\!\left[[\![D_2]\!]^{\mathrm{Y}_{\frac{\pi}{2}}\to ZX_r}\right]\!\right]^{ZX_r\to\mathrm{Y}_{\frac{\pi}{2}}} = D_2$$

which proves the completeness of $\mathrm{Y}_{\frac{\pi}{2}}$.

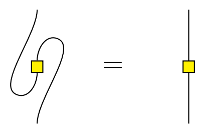
5 Hadamard Generalisation

We have seen in the previous section an interpretation that transforms a π dot and a Hadamard yellow box into real boxes. Since everything works well with it, we would like to introduce the following notations in the Y-Calculus:

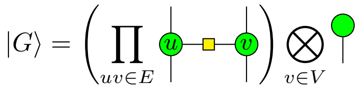


With this notation, the previous section shows that the Y-Calculus proves all the equalities in figure 2.

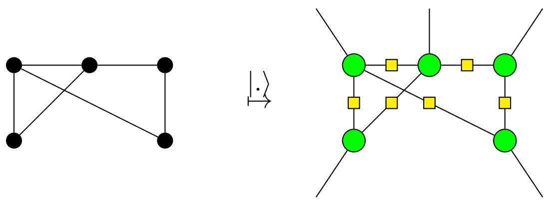
Using lemma 9, one can easily show that:

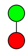


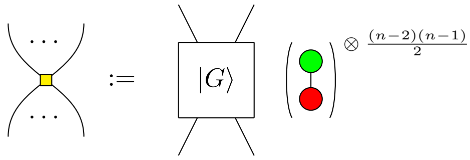
Definition 1. *Let $G = (V, E)$ be an undirected graph. The **graph state** $|G\rangle$ is defined by*



Example 2.

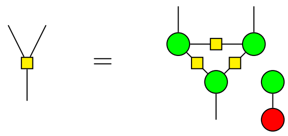


Definition 2. *Let $G = (V, E)$ be an undirected graph. $|G\rangle$ is **complete** iff $E = \{uv \mid u, v \in V, u \neq v\}$. Such a diagram with n inputs/outputs and with $\frac{(n-2)(n-1)}{2}$ times the scalar*  *will be represented by the following node, called Hadamard:*



We may sometimes parametrise the node with its arity.

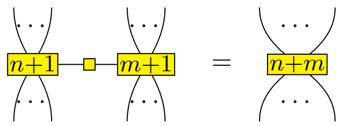
Example 3.



Remark 1. When the arity of the node is 2, we end up with the Hadamard yellow box defined above, so the notation is consistent.

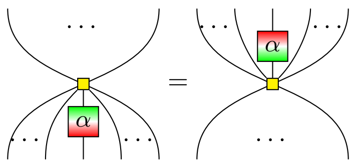
Remark 2. The Hadamard node with any of its wires swapped is equivalent to the node itself, because it represents a complete graph state.

Proposition 5. *Two Hadamard nodes linked by a 2-Hadamard merge into a bigger Hadamard node.*



Proof. The idea is to use the lemma 4 on the wire that links the two "big" yellow boxes, and remark that the result is a bigger complete graph state. Moreover, with the choice of scalars in the definition of the Hadamard box, they add up nicely.

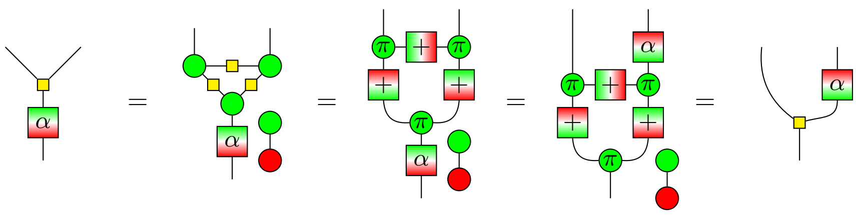
Proposition 6. *A real box can rotate around a Hadamard node, on any of its wires.*



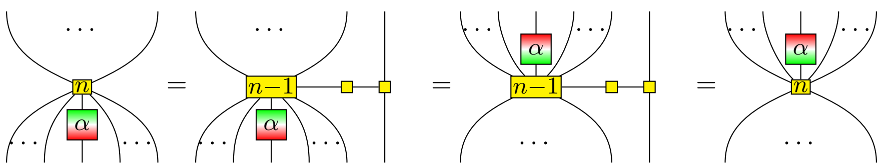
Proof. By induction on the arity of the Hadamard node.

$n = 2$ uses the lemma 9.

$n = 3$, using the decomposition of the Hadamard box, 9 and (RS2):



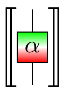
$n \geq 4$: We assume the result is true for $n - 1$:



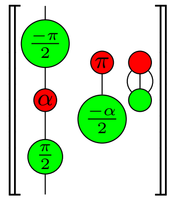
Notice that the choice of the two "excluded" wires is totally arbitrary, so we just have to choose two wires that are not involved with the real box α.

6 From Y-Calculus to ZX-Calculus

We can express any real rotation with a composition of complex rotations allowed by the ZX-calculus – which is reminded in appendix at page 26. More specifically, we can show that:

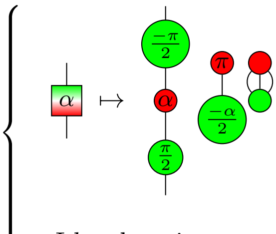


$$= \begin{pmatrix} \cos(\alpha/2) & -\sin(\alpha/2) \\ \sin(\alpha/2) & \cos(\alpha/2) \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & i \end{pmatrix} \begin{pmatrix} \cos(\alpha/2) & -i\sin(\alpha/2) \\ -i\sin(\alpha/2) & \cos(\alpha/2) \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -i \end{pmatrix} =$$



Hence:

$[\![.]\!]^{Y\to ZX}$:



Id otherwise

is an application from the Y-Calculus to the ZX-Calculus that preserves the semantics.

Proposition 7. *The interpretation* $[\![.]\!]^{Y\to ZX}$ *preserves all the rules of the Y-Calculus, so:*

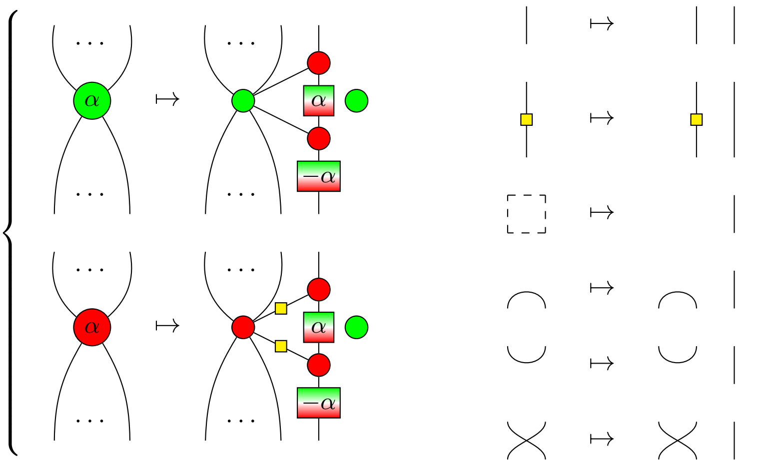
$$\forall D_1, D_2, \quad (Y \vdash D_1 = D_2) \quad \Rightarrow \quad (ZX \vdash [\![D_1]\!]^{Y\to ZX} = [\![D_2]\!]^{Y\to ZX})$$

Proof. In appendix at page 27

7 Simulating the ZX-Calculus with the Y-Calculus

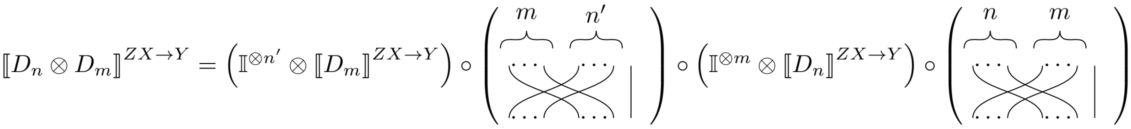
We can transform any complex number in a 2×2 real matrix containing the real and imaginary parts of the initial number. Doing so for all the coefficients of a complex matrix, we end up with a twice as big real matrix, but in the ZX and Y-Calculus, it just amounts to having one additional wire. This is the idea behind the interpretation that allows to simulate the ZX-Calculus with the Y-Calculus:

$[\![.]\!]^{ZX\to Y}$:

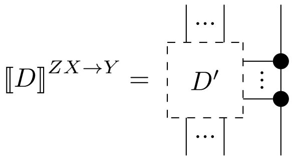


Here, if the diagram on the left represents the matrix $A+iB$, then the one on the right represents $A\otimes I_2 + B \otimes \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$.

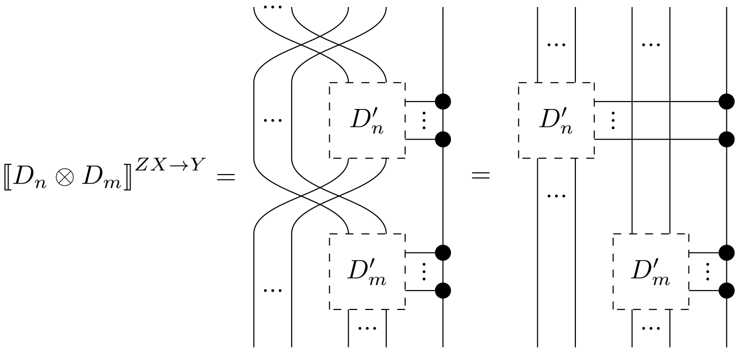
Spacial Composition: The interpretation also changes the way two side by side diagrams are represented: $[\![. \otimes .]\!]^{ZX\to Y} \neq [\![.]\!]^{ZX\to Y} \otimes [\![.]\!]^{ZX\to Y}$. Instead, the two interpreted diagrams share the last wire, called *control wire*. Given D_n a ZX-diagram with n inputs and n' outputs, and D_m a ZX-diagram with m inputs, the interpretation of D_n side-by-side with D_m is:



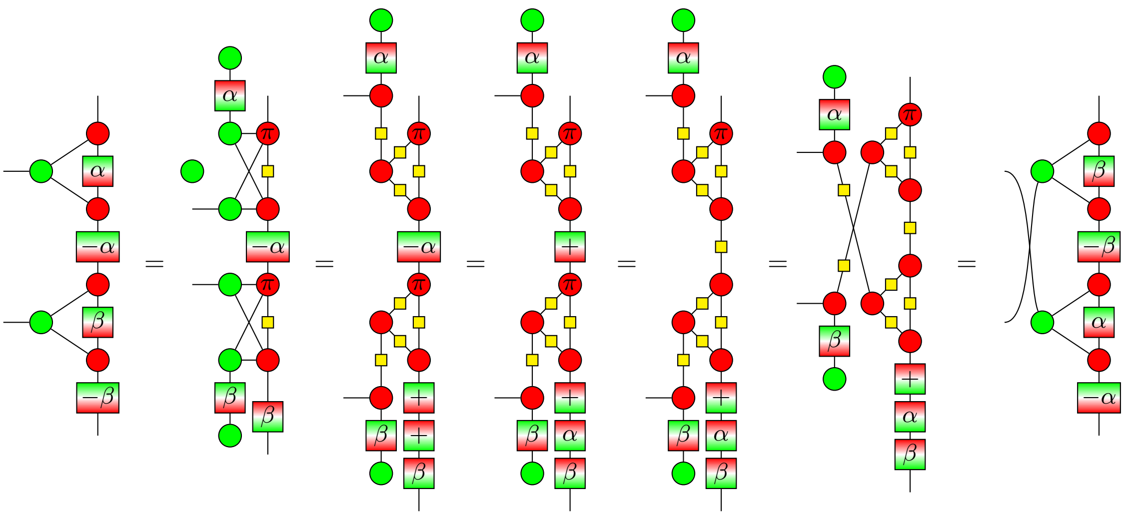
Assuming the interpretation of D is written this way:



We can roughly see the spacial composition as:



All the subdiagrams generated by the interpretation can commute on the control wire. Indeed, using lemma 19, proposition 6, lemma 9 and remark 2:



Now with this result, we can show:

- $[\![(A_1 \otimes B_1) \circ (A_2 \otimes B_2)]\!]^{ZX\to Y} = [\![(A_1 \circ A2) \otimes (B_1 \circ B2)]\!]^{ZX\to Y}$ if the number of outputs of A_2 (resp. B_2) corresponds to the number of inputs of A_1 (resp. B_1)
- $[\![(D_1 \otimes D_2) \otimes D_3]\!]^{ZX\to Y} = [\![D_1 \otimes (D_2 \otimes D_3)]\!]^{ZX\to Y}$
- $[\![e \otimes D]\!]^{ZX\to Y} = [\![D \otimes e]\!]^{ZX\to Y} = [\![D]\!]^{ZX\to Y}$
- $[\![(D_1 \otimes D_2) \circ \sigma]\!]^{ZX\to Y} = [\![\sigma \circ (D_2 \otimes D_1)]\!]^{ZX\to Y}$ for any 1-input/1-output diagrams D_1 and D_2

– Any topological property of the ZX-Calculus is preserved.

Proposition 8. *All the rules of the ZX-Calculus – see figure 3 – are preserved with the interpretation* $[\![.]\!]^{ZX\to Y}$.

Proof. In appendix at page 28.

Proposition 9. *For any diagram D:*

$$\left[\!\!\left[[\![D]\!]^{ZX\to Y}\right]\!\!\right] = \mathrm{Re}([\![D]\!]) \otimes I_2 + \mathrm{Im}([\![D]\!]) \otimes \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

Proof. By induction on the diagram:
• **Base Cases:** Showing the result for a green or red dot with only one wire is just a bit of computation. Then, using (S1), the result can be extended to a green/red dot of any arity. The result is obvious for all other generators.
• **Sequential Composition:** Let two diagrams D_1, D_2, and four real matrices A_1, B_1, A_2, B_2 such that:

$$[\![D_1]\!] = A_1 + iB_1 \qquad \textit{and} \qquad [\![D_2]\!] = A_2 + iB_2$$

We suppose that the result is true for D_1 and D_2:

$$\left[\!\!\left[[\![D_1]\!]^{ZX\to Y}\right]\!\!\right] = A_1 \otimes I_2 + B_1 \otimes \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \qquad \textit{and} \qquad \left[\!\!\left[[\![D_2]\!]^{ZX\to Y}\right]\!\!\right] = A_2 \otimes I_2 + B_2 \otimes \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

On the one hand:

$$\begin{aligned}
\left[\!\!\left[[\![D_2 \circ D_1]\!]^{ZX\to Y}\right]\!\!\right] &= \left[\!\!\left[[\![D_2]\!]^{ZX\to Y}\right]\!\!\right] \circ \left[\!\!\left[[\![D_1]\!]^{ZX\to Y}\right]\!\!\right] \\
&= \left(A_2 \otimes I_2 + B_2 \otimes \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\right) \circ \left(A_1 \otimes I_2 + B_1 \otimes \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\right) \\
&= ((A_2 \circ A_1) - (B_2 \circ B_1)) \otimes I_2 + ((A_2 \circ B_1) + (B_2 \circ A_1)) \otimes \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}
\end{aligned}$$

On the other hand:

$$\begin{aligned}
[\![D_2 \circ D_1]\!] &= (A_2 + iB_2) \circ (A_1 + iB_1) \\
&= (A_2 \circ A_1) - (B_2 \circ B_1) + i(A_2 \circ B_1) + (B_2 \circ A_1)
\end{aligned}$$

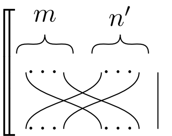
And thus:

$$\left[\!\!\left[[\![D_2 \circ D_1]\!]^{ZX\to Y}\right]\!\!\right] = \mathrm{Re}\left([\![D_2 \circ D_1]\!]\right) \otimes I_2 + \mathrm{Im}\left([\![D_2 \circ D_1]\!]\right) \otimes \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

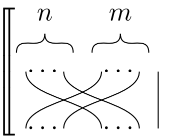
• **Spacial Composition:** With the same diagrams and matrices (we still assume that the result is true for D_1 and D_2).

On the one hand (m being the number of inputs of D_2 and D_1 having n inputs and n' outputs):

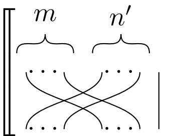
$$\left[\!\left[[\![D_1 \otimes D_2]\!]^{ZX\to Y} \right]\!\right] = \left(I_2^{\otimes n'} \otimes \left[\!\left[[\![D_2]\!]^{ZX\to Y} \right]\!\right] \right) \circ \left[\!\left[\vphantom{X} \right.\right.$$



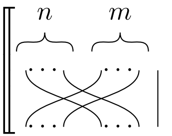
$$\left.\left. \vphantom{X} \right]\!\right] \circ \left(I_2^{\otimes m} \otimes \left[\!\left[[\![D_1]\!]^{ZX\to Y} \right]\!\right] \right) \circ \left[\!\left[\vphantom{X} \right.\right.$$



$$\left.\left. \vphantom{X} \right]\!\right] = \left(I_2^{\otimes n'} \otimes A_2 \otimes I_2 + I_2^{\otimes n'} \otimes B_2 \otimes \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \right) \circ \left[\!\left[\vphantom{X} \right.\right.$$



$$\left.\left. \vphantom{X} \right]\!\right] \circ \left(I_2^{\otimes m} \otimes A_1 \otimes I_2 + I^{\otimes m} \otimes B_1 \otimes \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \right) \circ \left[\!\left[\vphantom{X} \right.\right.$$



$$\left.\left. \vphantom{X} \right]\!\right]$$

$$\begin{aligned} &= \left(I_2^{\otimes n'} \otimes A_2 \otimes I_2 + I_2^{\otimes n'} \otimes B_2 \otimes \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \right) \\ &\quad \circ \left(A_1 \otimes I_2^{\otimes m} \otimes I_2 + B_1 \otimes I_2^{\otimes m} \otimes \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \right) \\ &= ((A_1 \otimes A_2) - (B_1 \otimes B_2)) \otimes I_2 + ((A_1 \otimes B_2) + (B_1 \otimes A_2)) \otimes \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \end{aligned}$$

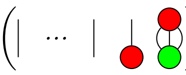
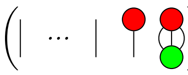
On the other hand:

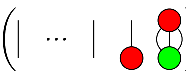
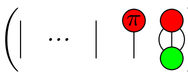
$$[\![D_1 \otimes D_2]\!] = (A_1 \otimes A_2) - (B_1 \otimes B_2) + i((A_1 \otimes B_2) + (B_1 \otimes A_2))$$

Thus:

$$\left[\!\left[[\![D_1 \otimes D_2]\!]^{ZX\to Y} \right]\!\right] = \mathrm{Re}\left([\![D_1 \otimes D_2]\!]\right) \otimes I_2 + \mathrm{Im}\left([\![D_1 \otimes D_2]\!]\right) \otimes \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

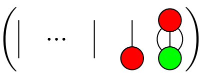
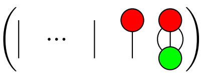
Corollary 1. *Let D be a ZX-diagram, and the interpretation $[\![.]\!]^\natural$ be either $[\![.]\!]^{ZX\to Y}$ or $\left[\!\left[[\![.]\!]^{ZX\to Y} \right]\!\right]^{Y\to ZX}$. Let us define $\mathrm{Re}(D)$ and $\mathrm{Im}(D)$ as follows:*

$$\mathrm{Re}(D) = \Big($$  $$\Big) \circ [\![D]\!]^\natural \circ \Big($$  $$\Big)$$

$$\mathrm{Im}(D) = \Big($$  $$\Big) \circ [\![D]\!]^\natural \circ \Big($$  $$\Big)$$

Then $[\![\mathrm{Re}(D)]\!] = \mathrm{Re}([\![D]\!])$ and $[\![\mathrm{Im}(D)]\!] = \mathrm{Im}([\![D]\!])$

Proof. Let A and B be two real matrices such that $[\![D]\!] = A + iB$.

$$\Big($$  $$\Big) \circ [\![D]\!]^{ZX\to Y} \circ \Big($$  $$\Big)$$

$$\begin{aligned} &= (I \otimes (1\ 0)) \circ \left(A \otimes I_2 + B \otimes \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \right) \circ \left(I \otimes \begin{pmatrix} 1 \\ 0 \end{pmatrix} \right) \\ &= A \otimes \left((1\ 0)\, I_2 \begin{pmatrix} 1 \\ 0 \end{pmatrix} \right) + B \otimes \left((1\ 0) \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \right) = A \end{aligned}$$

The proof is the same for the imaginary part.

Proposition 10. *The Y-Calculus is universal for real quantum transformations:*

$$\forall M \in \mathbb{R}^{2^n} \times \mathbb{R}^{2^m}, \exists D \in Y, [\![D]\!] = M$$

Proof. Let $M \in \mathbb{R}^{2^n} \times \mathbb{R}^{2^m}$. Since the ZX-Calculus is universal, there exists a ZX-diagram D_{ZX} such that $[\![D_{ZX}]\!] = M$.
Let D be the Y-diagram defined as $D = \text{Re}(D_{ZX})$, with Re defined with $[\![.]\!]^{ZX\to Y}$. Then:

$$[\![D]\!] = [\![\text{Re}(D_{ZX})]\!] = \text{Re}([\![D_{ZX}]\!]) = \text{Re}(M) = M$$

Hence, $\forall M \in \mathbb{R}^{2^n} \times \mathbb{R}^{2^m}, \exists D \in Y, [\![D]\!] = M$, which proves the universality.

Proposition 11. *Let S be a set of angles, and ZX_S (resp. Y_S) the fragment of the ZX (resp. Y) that only uses angles in S. If ZX_S is approximately universal, then so is Y_S.*

Proof. Let $M \in \mathbb{R}^{2^n} \times \mathbb{R}^{2^m}$, $\epsilon > 0$ and S such that the ZX_S is approximately universal. Then, there exists a diagram of the ZX_S, D_{ZX}, such that $\|[\![D_{ZX}]\!] - M\| \le \epsilon$. Let D be the Y-diagram of the S-fragment defined as $D = \text{Re}(D_{ZX})$ – we shall notice that the interpretation $[\![.]\!]^{ZX\to Y}$ does not change the fragment that needs be considered. Then:

$$\begin{aligned}\|[\![D]\!] - M\| &= \|[\![\text{Re}(D_{ZX})]\!] - M\| = \|\text{Re}([\![D_{ZX}]\!]) - M\| \\ &= \|\text{Re}([\![D_{ZX}]\!] - M)\| \le \|[\![D_{ZX}]\!] - M\| \le \epsilon\end{aligned}$$

References

1. Miriam Backens (2014): *The ZX-calculus is complete for stabilizer quantum mechanics. New Journal of Physics* 16(9), p. 093021, doi:10.1088/1367-2630/16/9/093021.
2. Myriam Backens & Simon Perdrix & Quanlong Wang (2016): *A simplified Stabilizer zx-calculus.* http://arxiv.org/abs/1602.04744.
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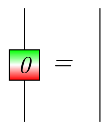
4. Ross Duncan & Simon Perdrix (2013): *Pivoting makes the zx-calculus complete for real stabilizers.* http://arxiv.org/abs/1307.7048.
5. Emmanuel Jeandel & Simon Perdrix & Renaud Vilmart (2016): *Generalised Supplementarity and new rule for Empty Diagrams to Make the ZX-Calculus More Expressive.*
6. Christian Schröder de Witt & Vladimir Zamdzhiev (2014): *The ZX-Calculus is incomplete for quantum mechanics. Electronic Proceedings in Theoretical Computer Science* 270(1), pp. 285-292. https://arxiv.org/abs/1404.3633.

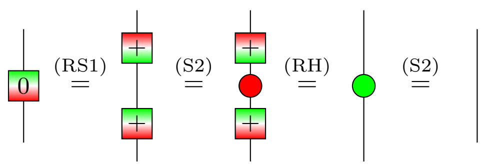
8 Appendix

8.1 Lemmas

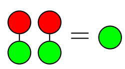
Lemma 1. *A box with angle 0 is a mere wire.*



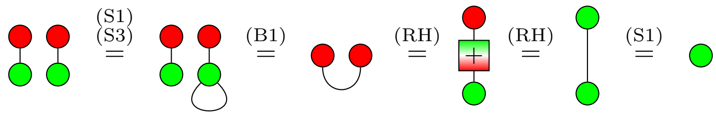
Proof. Using (RS1), (S2) and (RH):



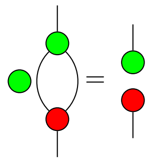
Lemma 2. *A node with no edge equals two "bicolor" scalars.*



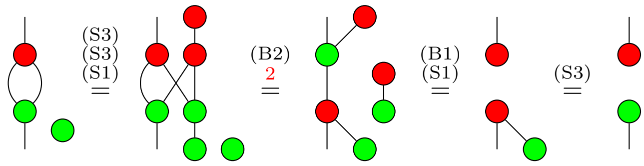
Proof. Using rules (S1), (S3), (B1), (RH):



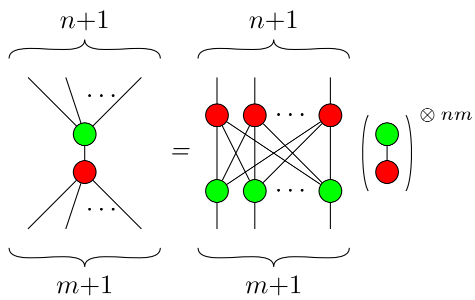
Lemma 3. *We have the Hopf Law:*



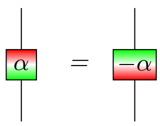
Proof. Using the rules (B1), (B2), (S3), (IV) and lemma 2:



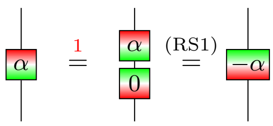
Lemma 4. *The rule* (B2) *has a generalised version, derivable from* (B2) *and* (S1).



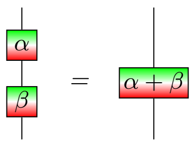
Lemma 5. *The upside-down box α is the upright box with angle $-\alpha$.*



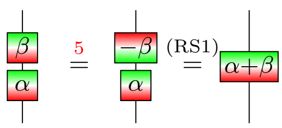
Proof. Using 1 and (RS1):



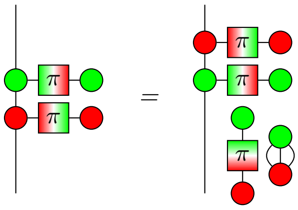
Lemma 6. *Two connected upright boxes merge with the sum of the two angles.*



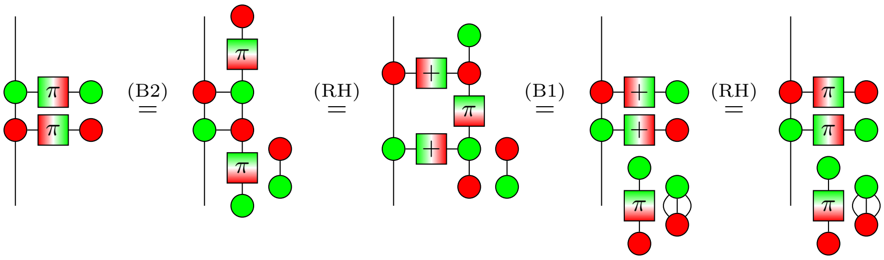
Proof. Using lemma 5 and (RS1):



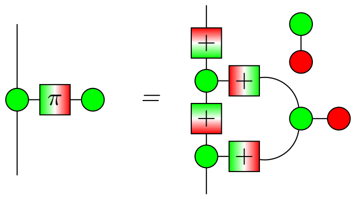
Lemma 7. *The two hanging π branches with inverted colors commute up to a scalar.*



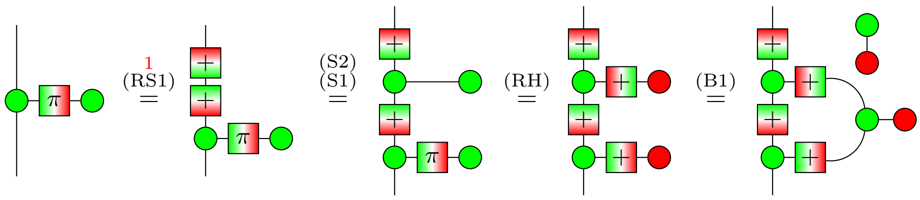
Proof. Using (B2), (RH), (B1):



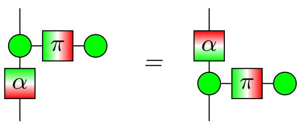
Lemma 8. *The π hanging branch can be decomposed, making a "$\pi/2$ boxes triangle" appear.*



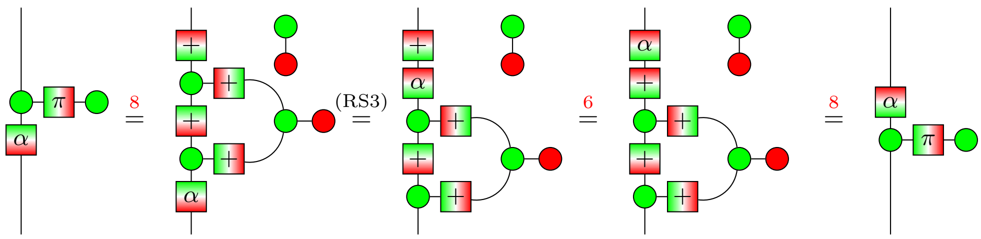
Proof. Using 1, (RS1), (S2), (S1), (RH), (B1):



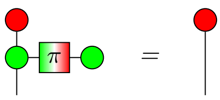
Lemma 9. *A π-branch can "cross" a real box, changing its orientation.*



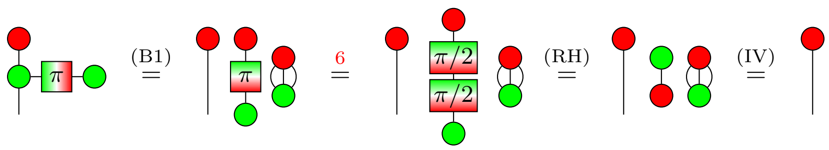
Proof. Using 8, (RS2) and 6:



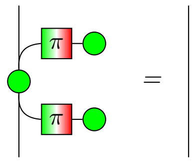
Lemma 10. *A red state followed by a "green" π hanging branch is equal to the mere red state.*



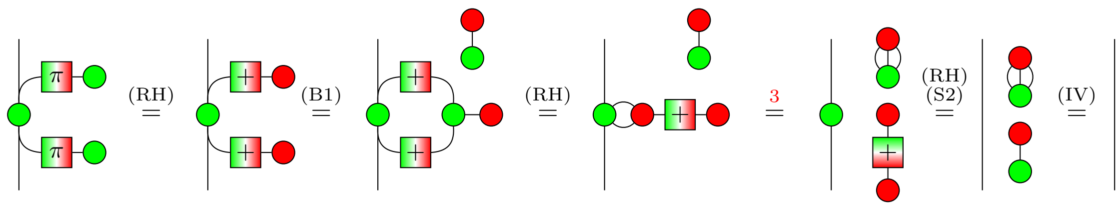
Proof. Using (B1), 6, (RH), and (IV):



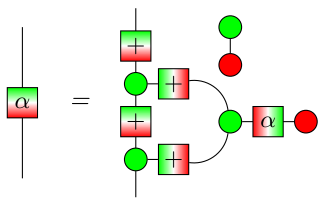
Lemma 11. *Two hanging π branches of the same color give the identity.*



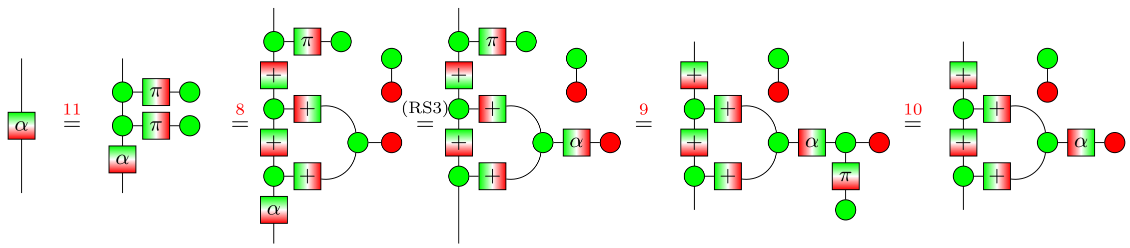
Proof. Using (RH), (B1), the Hopf law 3 and (IV):



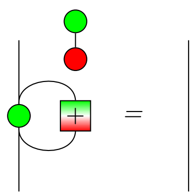
Lemma 12. *Using the π-branch decomposition, we can separate a real box from its main wire.*



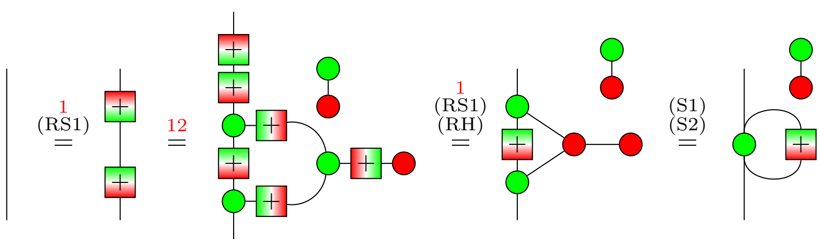
Proof. Using 11, 8, (RS2), 9, 10:



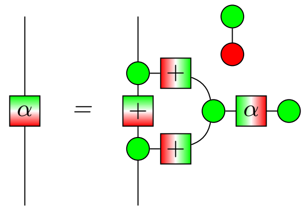
Lemma 13. *A $\frac{\pi}{2}$-loop on a wire is just a wire, up to a scalar.*



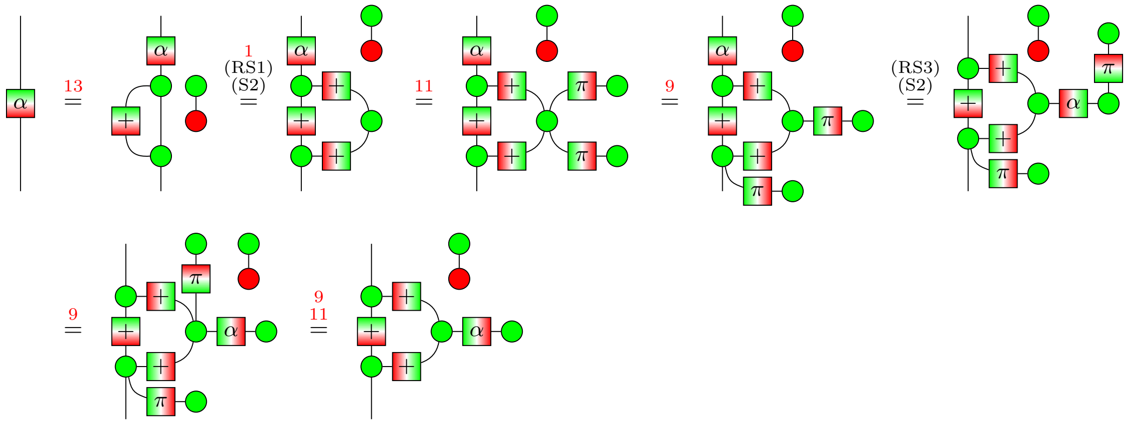
Proof. Using 1, (RS1), 12, (RH), (S1), (S2):



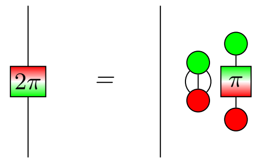
Lemma 14. *We can separate a box from its wire in another way than in lemma 12.*



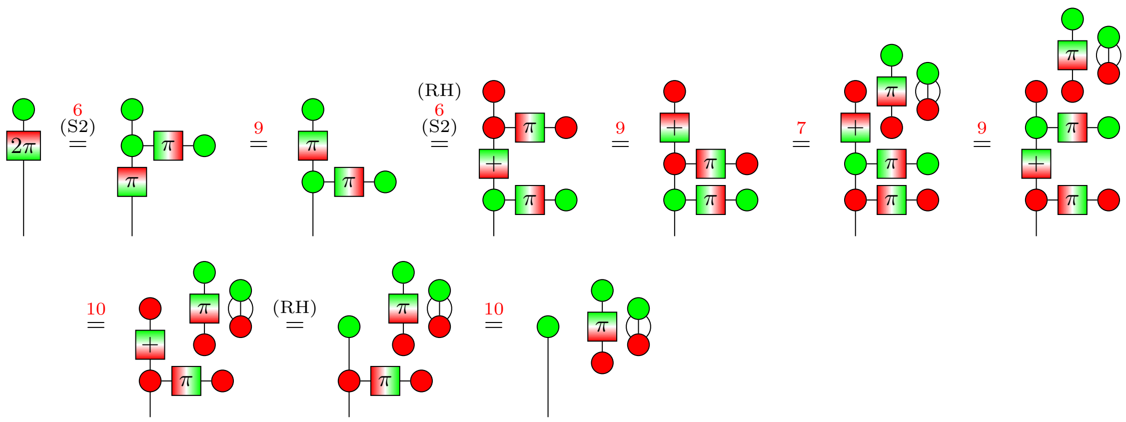
Proof. Using 1, (RS1), 13, 11, 9 and (RS2):



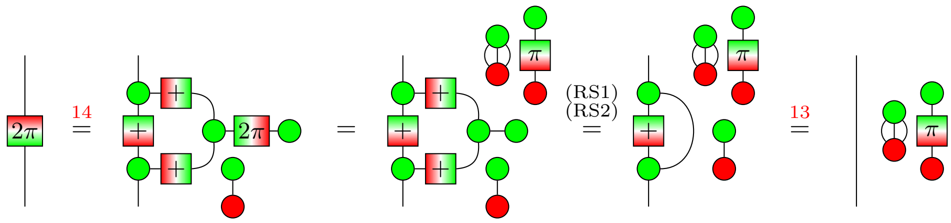
Lemma 15. *The 2π-box is the identity, up to some scalar.*



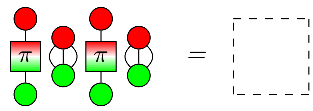
Proof. First, we prove it on the green state, using 6, 9, (RH), 7 and (B1):



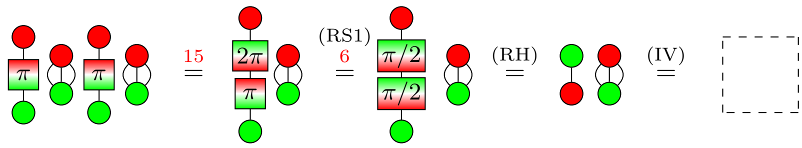
Now, in the general case, using 14, the previous result and 13:



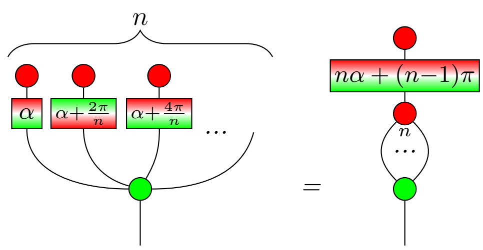
Lemma 16. *Two copies of the previous scalar result in an empty diagram.*



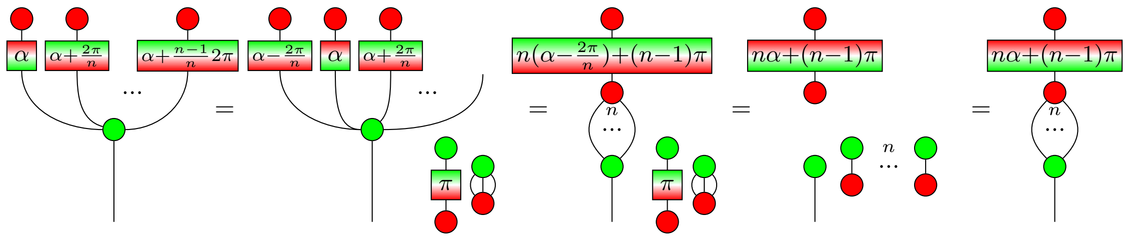
Proof. Using the previous lemma (from right to left), (RS1), 6 (RH) and (IV):



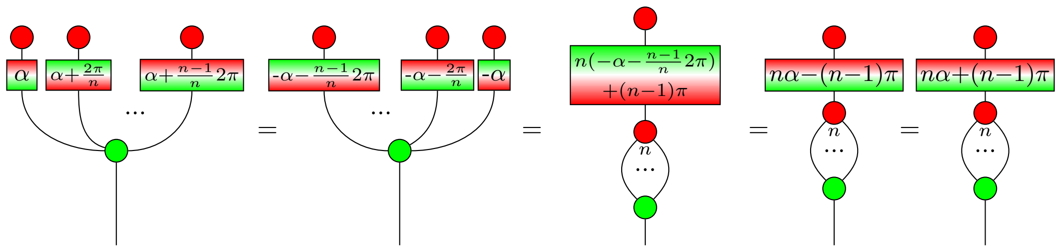
Lemma 17. *The rule* (RSUP$_n$) *is still true when all the boxes are upside-down:*



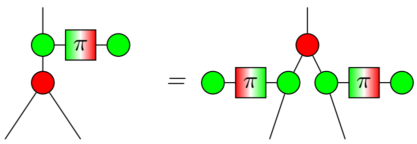
Proof. – If n is even, using lemmas 15 and 3 and the rule (RSUP$_n$):



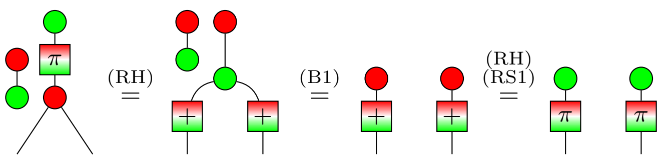
– If n is odd, using 5 and (RSUP$_n$), and remarking that $2(n-1)\pi$ is a multiple of 4π:



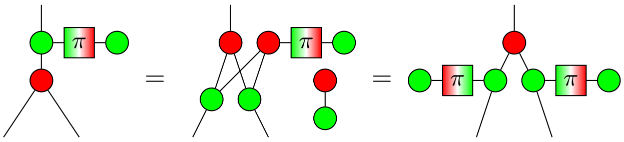
Lemma 18.



Proof. First, using (RH) and (B1):

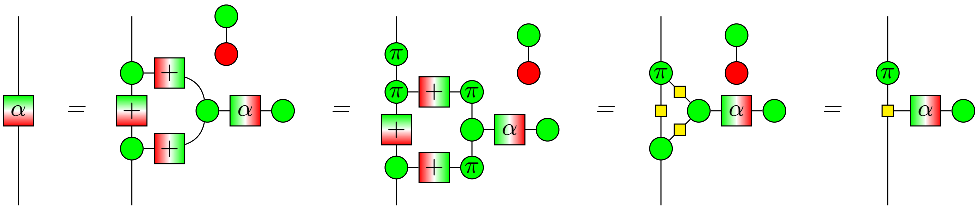


Then, using (B2) and the previous result:

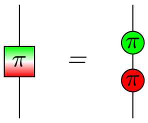


We now assume the existence of the nodes Hadamard and π defined in section 5.

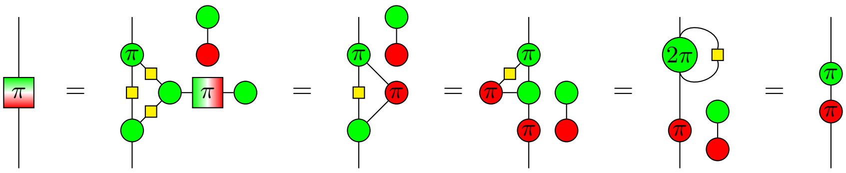
Lemma 19. *The lemma 14 can be rewritten with Hadamard:*



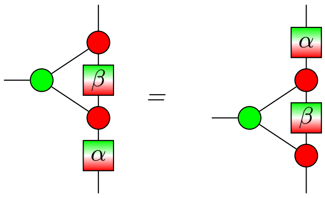
Lemma 20. *A real box π is a green π-dot followed by a red one.*



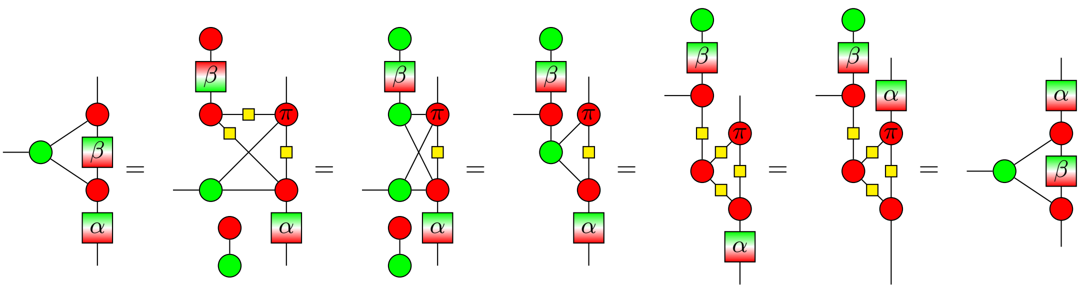
Proof. Using 19, (H), 18 and (HL):



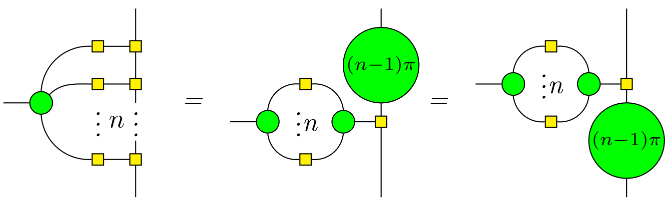
Lemma 21.



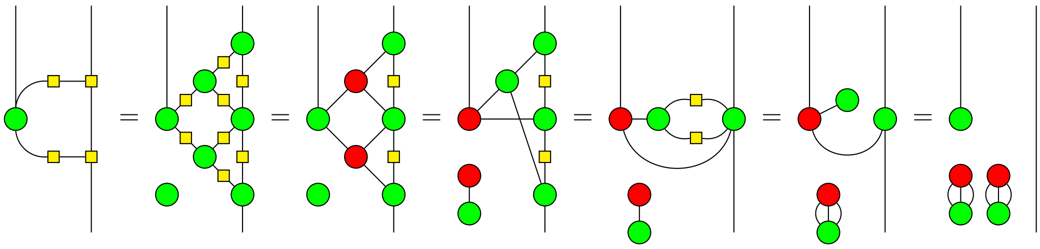
Proof.



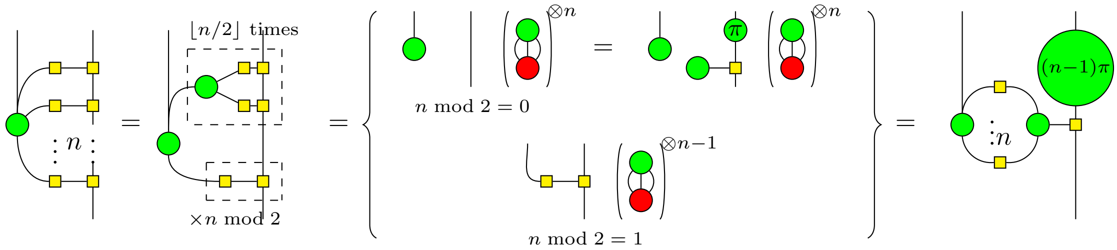
Lemma 22.



Proof. First, when $n = 2$, using (H), (B2), (S1), 3 and (B1):



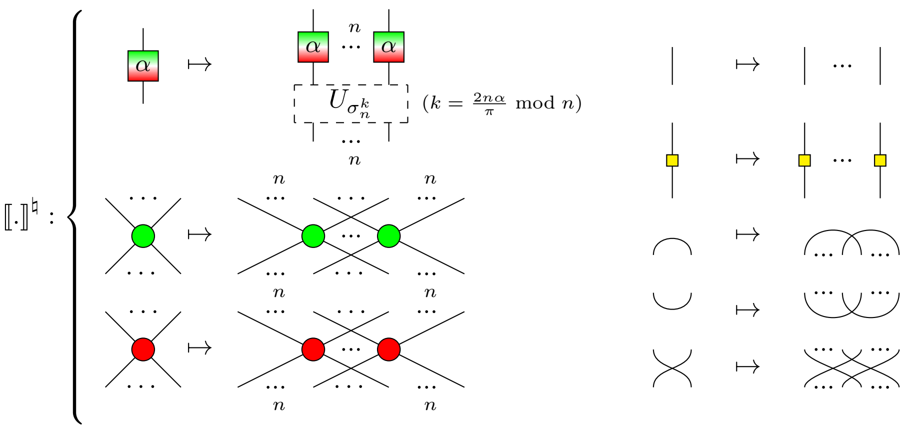
Then, using (S1), the previous result and 3:



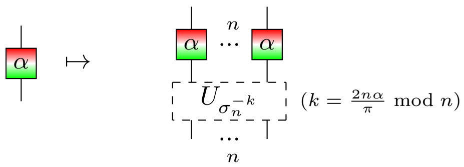
8.2 Minimality

Proof (Proposition 1). Let us consider the circular permutation $\sigma_n : k \mapsto (k+1) \bmod n$, ($k \in [\![0, n-1]\!]$).
First, notice that: $\forall p \in \mathbb{Z}, \quad \sigma_n^p : k \mapsto k + p \bmod n$.

We define a gate that has n inputs and n outputs: $U_{\sigma_n^p}$, which maps the k-th input to the $\sigma_n^p(k)$-th output.
We can notice that $[\![U_{\sigma_n^p}]\!] \circ [\![U_{\sigma_n^q}]\!] = [\![U_{\sigma_n^p \circ \sigma_n^q}]\!] = \left[\!\!\left[U_{\sigma_n^{p+q[n]}}\right]\!\!\right]$.
We can also notice that $[\![R_Y(\alpha)]\!]^{\otimes n} \circ [\![U_{\sigma_n^p}]\!] = [\![U_{\sigma_n^p}]\!] \circ [\![R_Y(\alpha)]\!]^{\otimes n}$
We now consider the following interpretation:



Where $[\![D_1 \otimes D_2]\!]^\natural = [\![D_1]\!]^\natural \otimes [\![D_2]\!]^\natural$ and $[\![D_1 \circ D_2]\!]^\natural = [\![D_1]\!]^\natural \circ [\![D_2]\!]^\natural$ for any two diagrams D_1 and D_2.
One can check that:

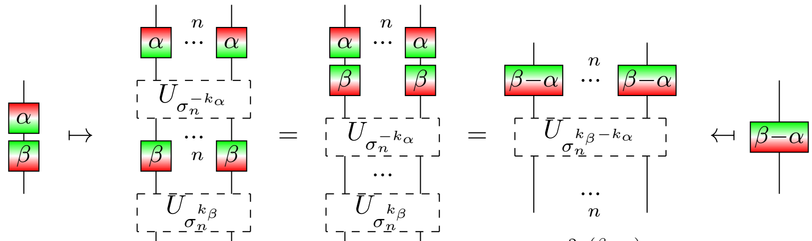


(S1), (S2), (S3), (IV), (B1) and (B2) obviously hold since no real box is used in these axioms.

(RSUP$_n$) holds: the interpretation only swaps identical hanging branches, which changes nothing.

(RH) holds: $\sigma_n^0 = I^{\otimes n}$.

(RS1) holds:

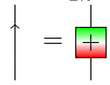


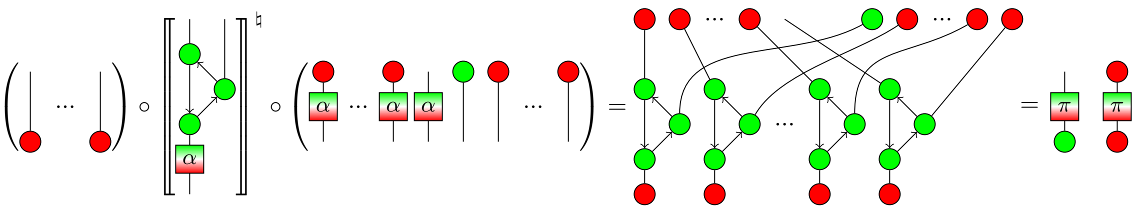
$k_\beta - k_\alpha = \frac{2n(\beta-\alpha)}{\pi} \bmod n$

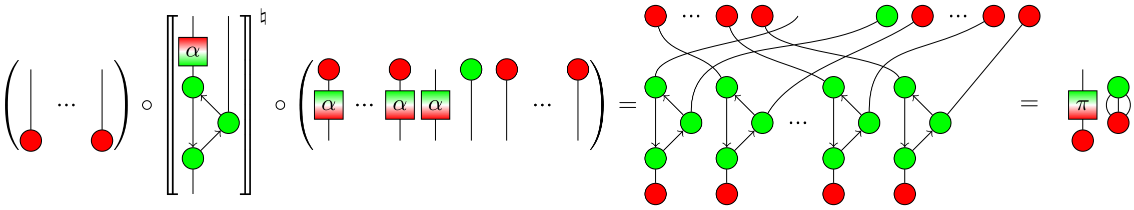
$k_\alpha = \frac{2n\alpha}{\pi} \bmod n$ $\quad$ $k_\alpha = \frac{2n\alpha}{\pi} \bmod n$

$k_\beta = \frac{2n\beta}{\pi} \bmod n$ $\quad$ $k_\beta = \frac{2n\beta}{\pi} \bmod n$

(RS2) **does not hold**: for $\alpha = \frac{\pi}{2n} \bmod \frac{\pi}{2}$, i.e. $k = 1$:

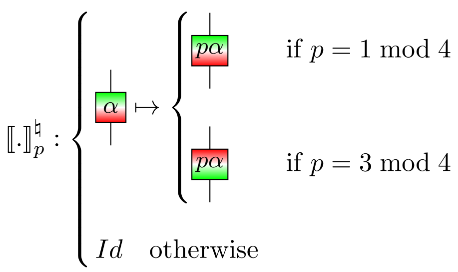
Let us write to simplify: 





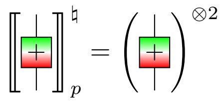
If (RS2) were derivable from the other rules, its interpretation would hold, hence (RS2) is necessary in any $\frac{\pi}{2n}$-fragment.

Proof (Proposition 2). Let $\mathbb{P}$ be the set of prime numbers, and $p \in \mathbb{P}$, $p \geq 3$. Let us consider the following interpretation:

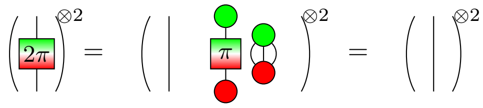


and build the interpretation $([\![.]\!]_p^\natural)^{\otimes 2}$. This interpretation obviously holds for (S1), (S2), (S3), (B1) and (B2) because no real box is involved in these rules. It is also easy to see that it holds for (RS1).

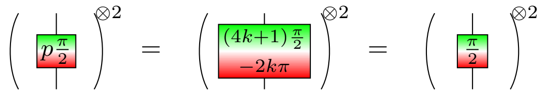
Then we can show that:



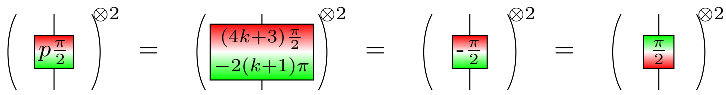
First, thanks to lemmas 15 and 16:



If $p = 4k+1$ then, subtracting k times 2π to the boxes thanks to the previous result:



and if $p = 4k+3$, then:

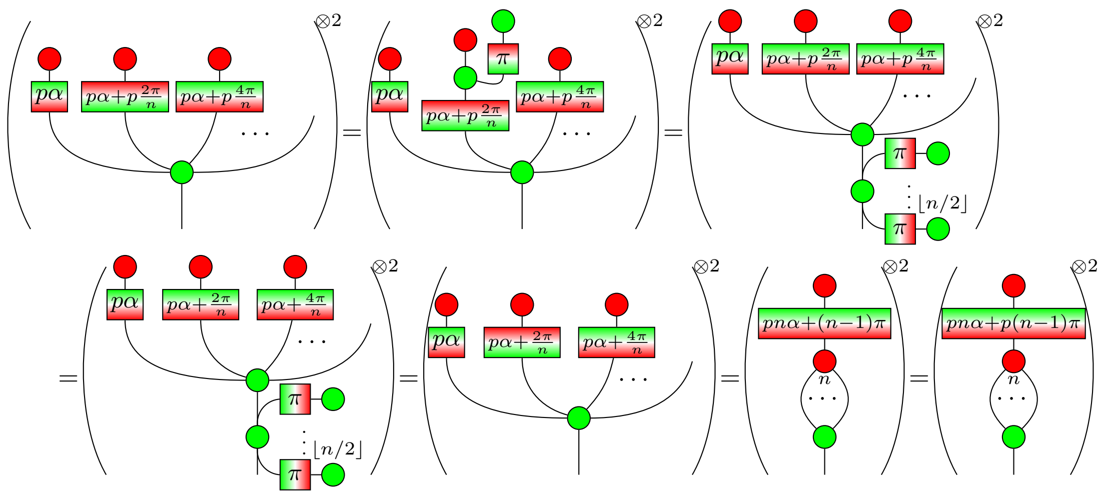


Hence, both (RS2) and (RH) hold for this interpretation.

Now, let $n \in \mathbb{P}$, $n \neq p$. Then $n \wedge p = 1$ and thus:

$$\left\{ p\alpha + p\frac{2k\pi}{n} \,\middle|\, k \in [\![0; n-1]\!] \right\} = \left\{ p\alpha + \frac{2k\pi}{n} \,\middle|\, k \in [\![0; n-1]\!] \right\}$$

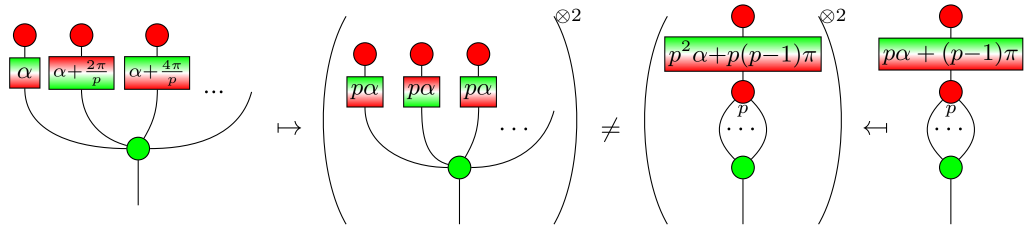
and so, if $p = 1 \bmod 4$, using lemmas 10 and 9, and (RSUP_n):



The reasoning is the same when $p = 3 \bmod 4$, so the rule (RSUP_n) with $n \in \mathbb{P}$, $n \neq p$ holds for this interpretation.

Finally, the rule (RSUP_p) does not hold:

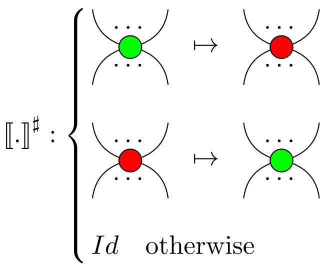
If $p = 1 \bmod 4$, then:



The two interpretations are different for any multiple of $\frac{\pi}{2p}$. Again, the reasoning is the same when $p = 3 \bmod 4$.

Since (RSUP_p) is the only rule that does not hold with this interpretation, it is necessary.

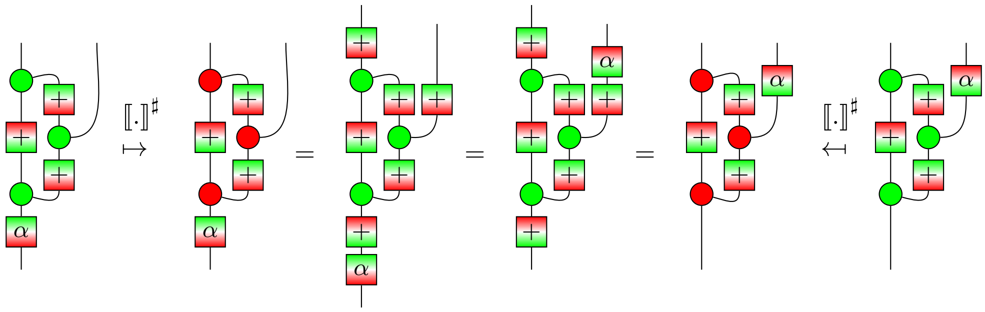
Proof (Proposition 3). Let us consider the interpretation:



and build the interpretation $(\llbracket . \rrbracket^\sharp)^{\otimes 2}$.

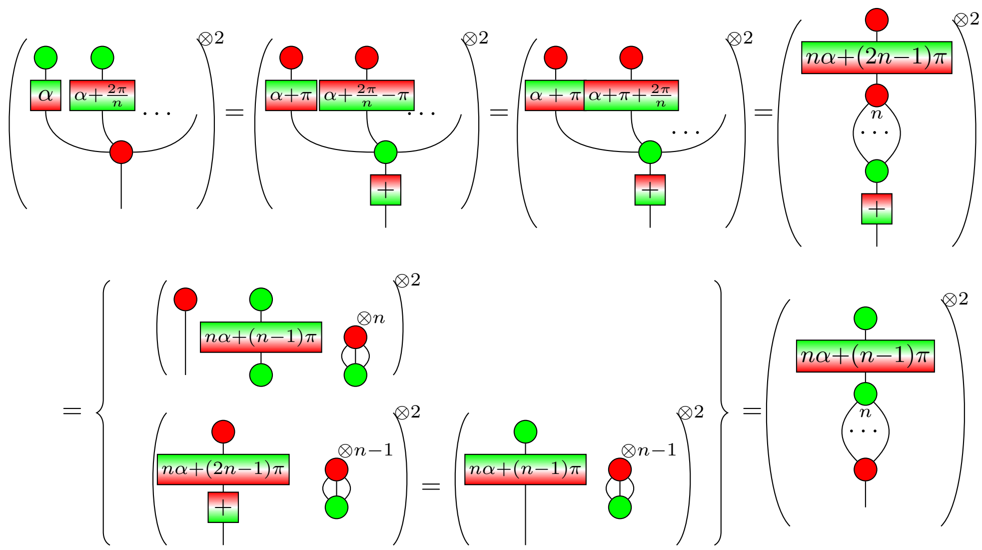
This interpretation obviously holds for (S1), (S2), (S3), (B1) and (B2) because no real box is involved in these rules, and all the rules hold when the colours are swapped and the boxes are flipped. (RS1) also holds, for no green or red dot appears here.

The rule (RS2) holds. Using (RH), (RS1) and (RS2):

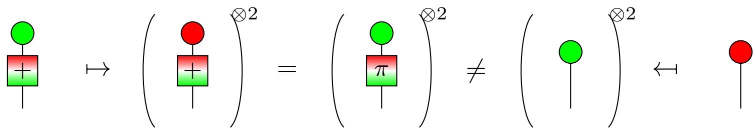


It then obviously holds for $(\llbracket . \rrbracket^\sharp)^{\otimes 2}$.

The rule (RSUP_n) holds.

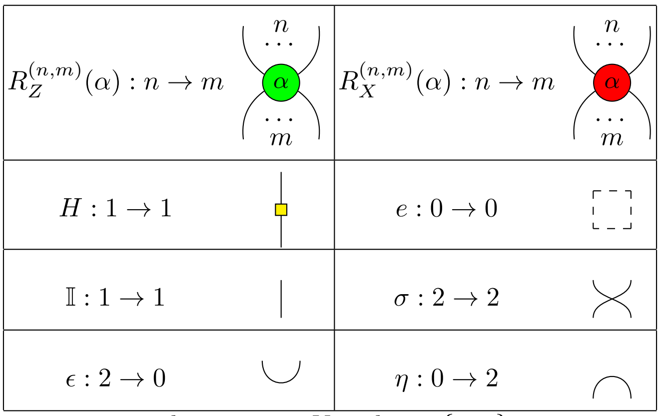


Finally, the rule (RH) does not hold. Indeed for dots of arity 1:



8.3 The completeness of the $\frac{\pi}{2}$-fragment

The real stabiliser ZX-Calculus



where $n, m \in \mathbb{N}$ and $\alpha \in \{0; \pi\}$

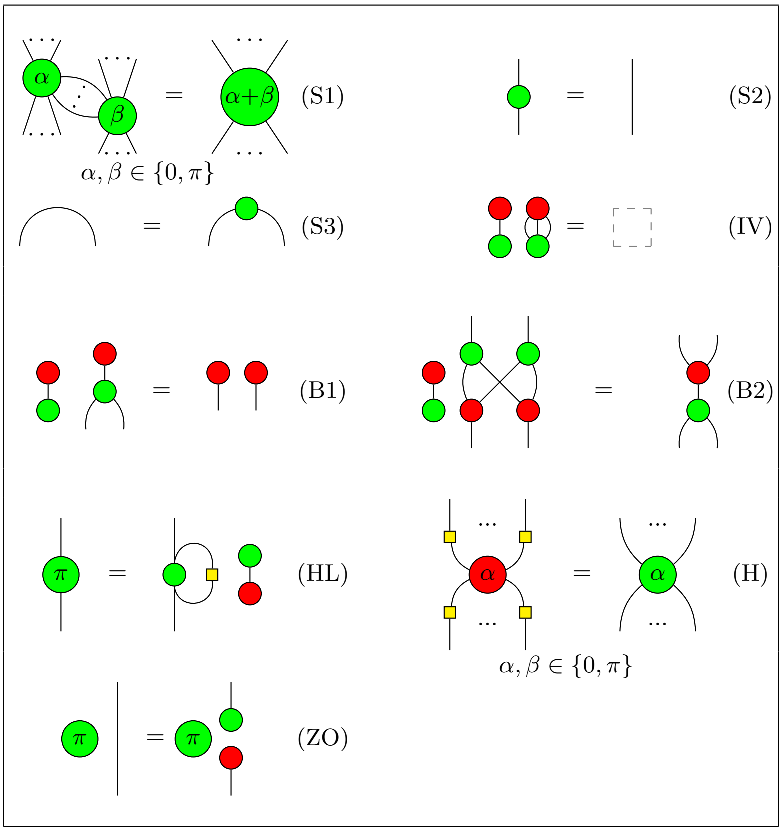
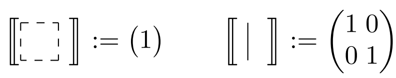
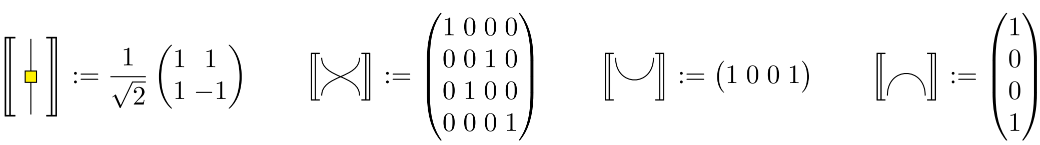


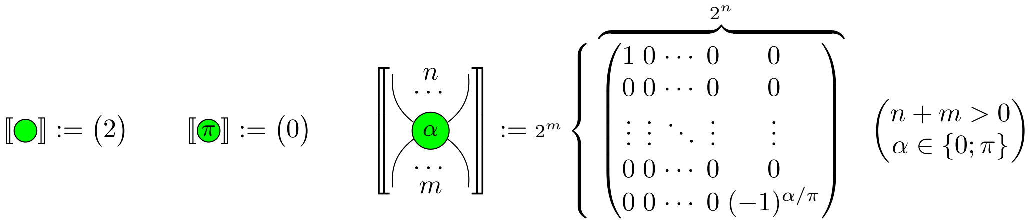
Fig. 2. Rules for the **real stabiliser ZX-calculus** with scalars [4]. All of these rules also hold when flipped upside-down, or with the colours red and green swapped. The right-hand side of (IV) is an empty diagram. $(\cdots)$ denote zero or more wires, while $(\therefore)$ denote one or more wires. In any dot, 2π can be replaced by 0.

The standard interpretation of the real stabiliser ZX-diagrams associates to any diagram $D : n \to m$ a linear map $[\![D]\!] : \mathbb{R}^{2^n} \to \mathbb{R}^{2^m}$ inductively defined as follows:

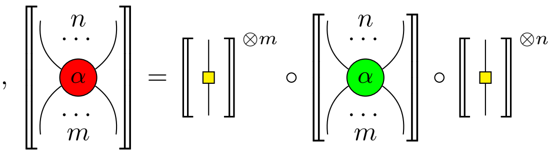
$$[\![D_1 \otimes D_2]\!] := [\![D_1]\!] \otimes [\![D_2]\!] \qquad [\![D_2 \circ D_1]\!] := [\![D_2]\!] \circ [\![D_1]\!]$$







For any $n, m \geq 0$ and $\alpha \in \{0;\pi\}$,

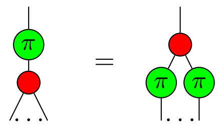


$\big($where $M^{\otimes 0} = \big(1\big)$ and $M^{\otimes k} = M \otimes M^{\otimes k-1}$ for any $k \in \mathbb{N}^*\big)$.

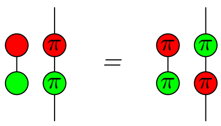
The rules of the real stabiliser ZX-Calculus are shown in figure 2.

From these rules, we can derive [4]:

Lemma 23.



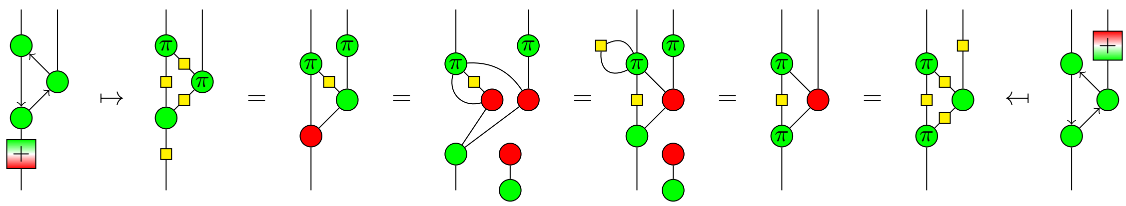
Lemma 24.



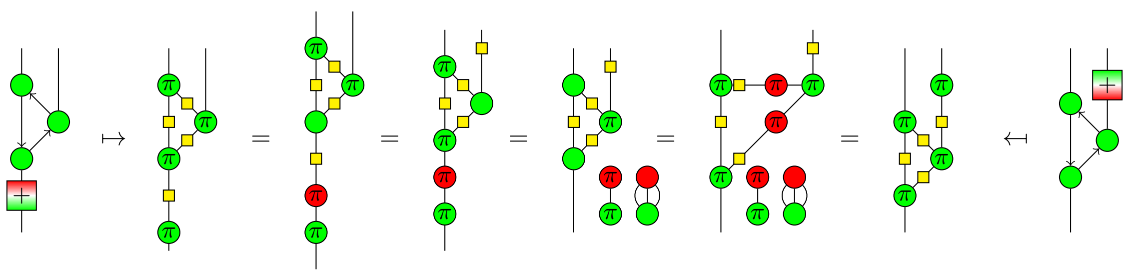
Lemma 25. *The dot π has an absorbing property for any scalar i.e. any diagram with* 0 *input and* 0 *output.*

$[\![.]\!]^{\mathrm{Y}\frac{\pi}{2}\to \boldsymbol{ZX_r}}$ **preserves the rules:** The rules (S1), (S2), (S3), (IV), (B1), (B2) obviously hold since no real box appears in them. (RS1) also holds, quite immediately.

(RS2) holds thanks to the pivoting [4]. Using (H), (B2), (HL), 23:

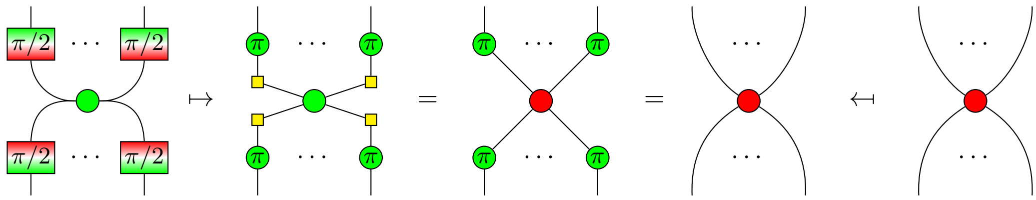


then, using (H), the previous result, lemmas 24 and 23, and (S1):

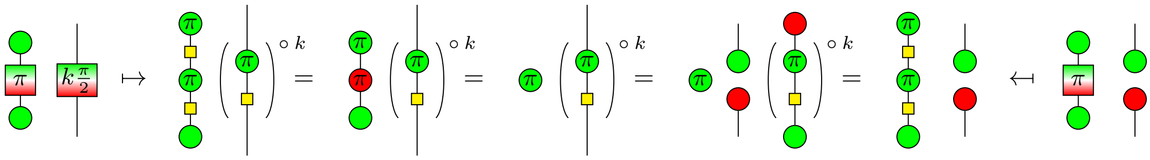


and the result for $k\pi/2$ is obtained by applying k times the results above.

(RH) holds. Using (H), 23 and the 2π-periodicity of green dots:



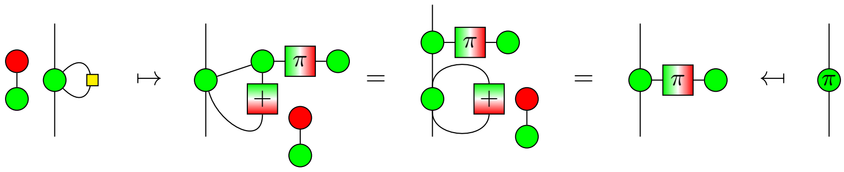
(RZO) holds. Using (H), 23, (ZO) and 25:



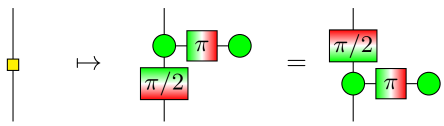
$[\![.]\!]^{\boldsymbol{ZX_r}\to\mathrm{Y}_{\frac{\pi}{2}}}$ **preserves the rules:** First, the rules (S2), (S3), (IV), (B1) and (B2) obviously hold because no yellow box and no angle are involved.

(S1) obviously holds when either α or β is null. When both are π, then the lemma 11 is used to show (S1) holds

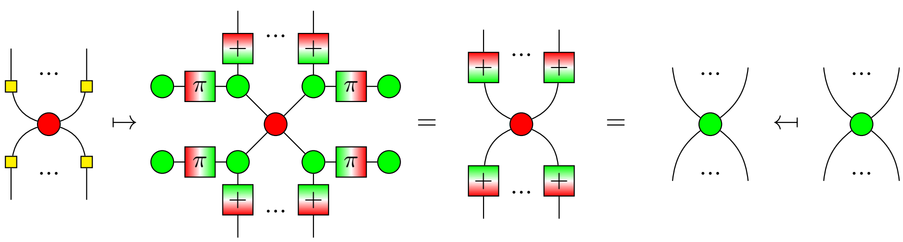
(HL) holds. Indeed, using (RS1) and 13:



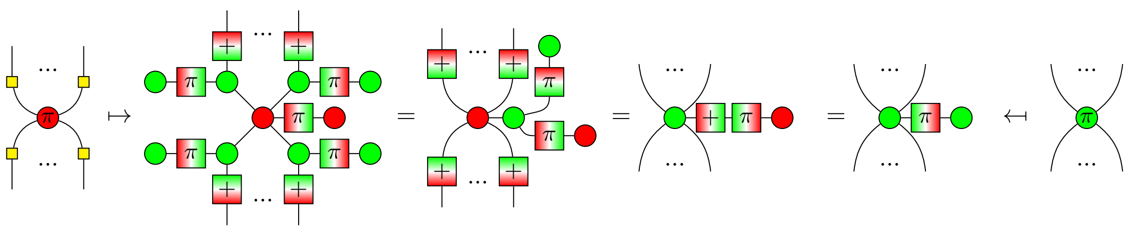
Noticing that:



(H) holds if $\alpha = 0$. Indeed, using 18, 11 and (RH):



(H) holds if $\alpha = \pi$. Indeed, using 18, 9, 10, 11 and (RH):



8.4 The ZX-Calculus

$R_Z^{(n,m)}(\alpha) : n \to m$	$R_X^{(n,m)}(\alpha) : n \to m$
$H : 1 \to 1$	$e : 0 \to 0$
$\mathbb{I} : 1 \to 1$	$\sigma : 2 \to 2$
$\epsilon : 2 \to 0$	$\eta : 0 \to 2$

where $n, m \in \mathbb{N}$ and $\alpha \in \mathbb{R}$

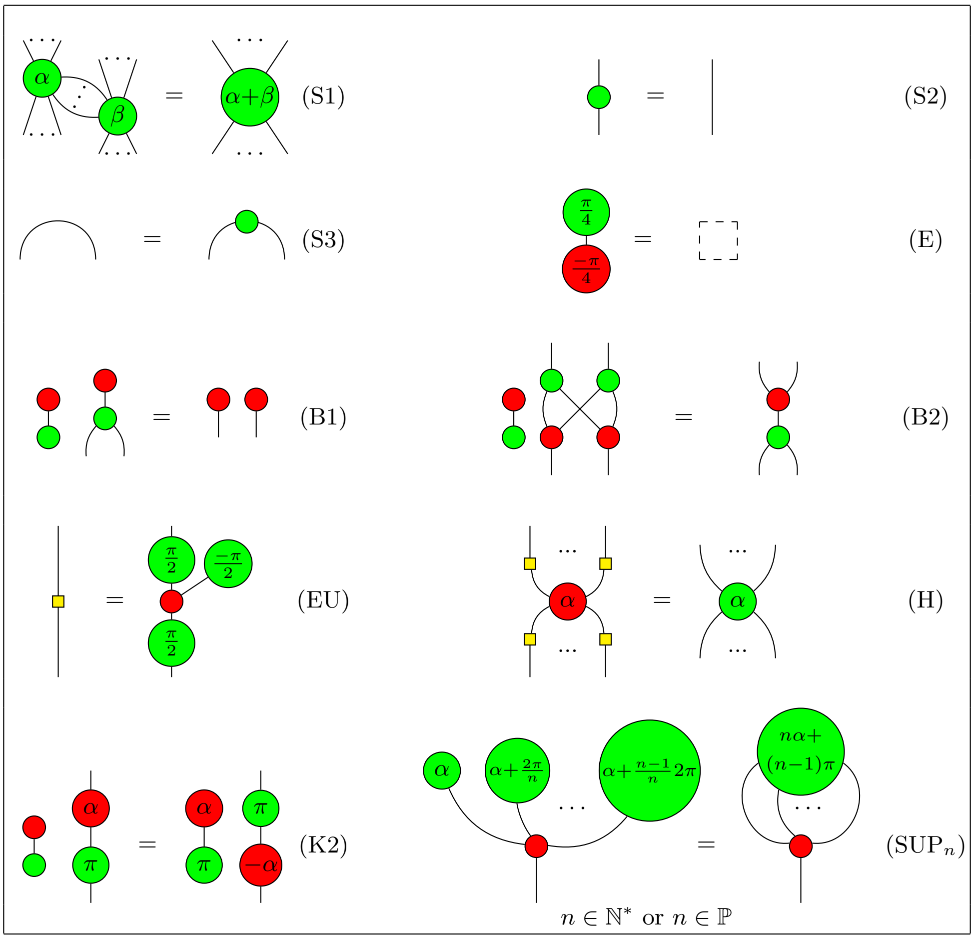
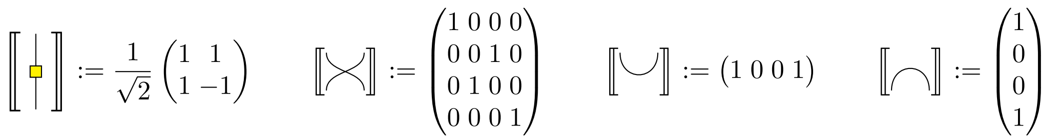
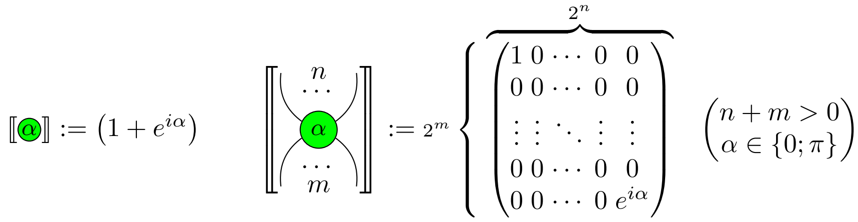


Fig. 3. Set of rules for the ZX-calculus [5] with scalars. All of these rules also hold when flipped upside-down, or with the colours red and green swapped. The right-hand side of (IV) is an empty diagram. $(\cdots)$ denote zero or more wires, while $(\cdot^{\cdot^{\cdot}})$ denote one or more wires.

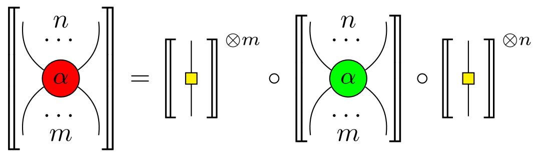
The standard interpretation of the real stabiliser ZX-diagrams associates to any diagram $D : n \to m$ a linear map $[\![D]\!] : \mathbb{R}^{2^n} \to \mathbb{R}^{2^m}$ inductively defined as follows:

$$[\![D_1 \otimes D_2]\!] := [\![D_1]\!] \otimes [\![D_2]\!] \qquad [\![D_2 \circ D_1]\!] := [\![D_2]\!] \circ [\![D_1]\!] \qquad [\![\;\;]\!] := (1) \qquad [\![\,|\,]\!] := \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$





For any $n, m \geq 0$ and $\alpha \in \{0;\pi\}$,

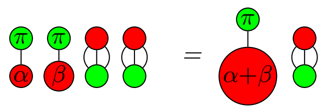


$\big($where $M^{\otimes 0} = \big(1\big)$ and $M^{\otimes k} = M \otimes M^{\otimes k-1}$ for any $k \in \mathbb{N}^*\big)$.

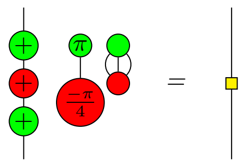
The transformation rules of the ZX-calculus are expressed in the figure 3. It is to be noticed that this set of rules needs that $\pi/4$ is in the fragment we are working with. If not, the rule (E) is unusable and is to be replaced by the rules (ZO) and (IV) present in figure 2.

From these rules can be derived the lemmas:

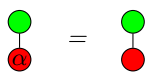
Lemma 26.



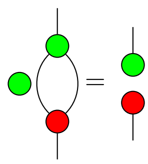
Lemma 27.



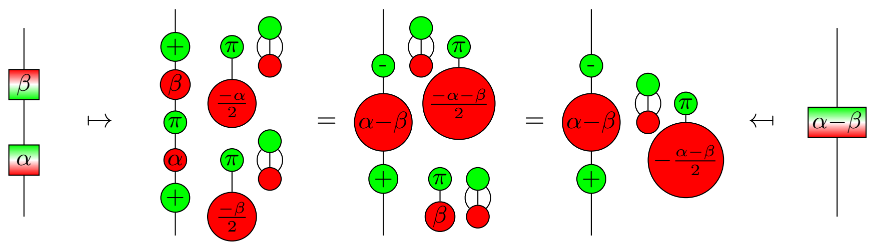
Lemma 28.



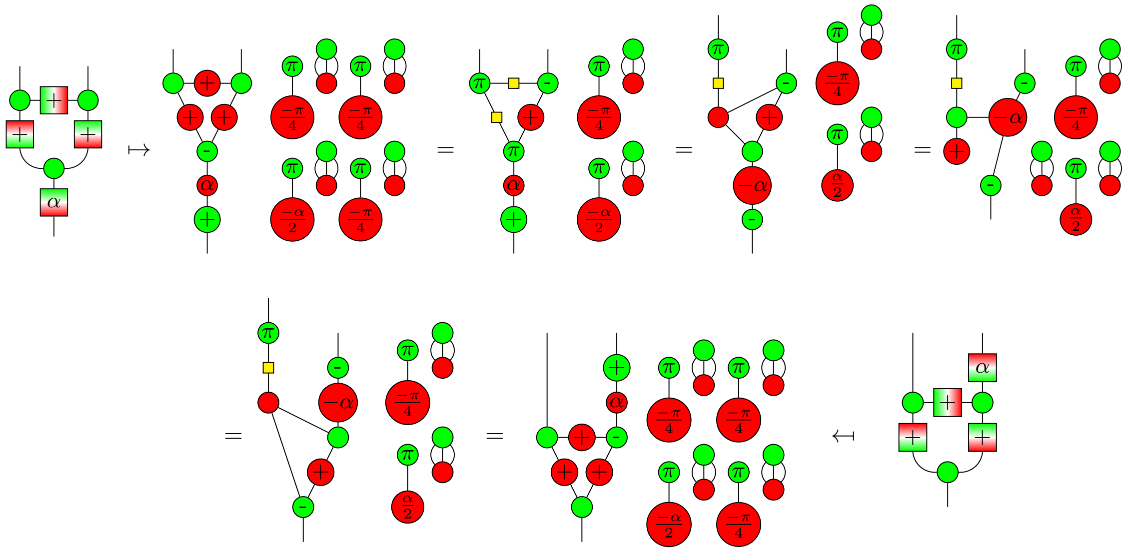
Lemma 29.



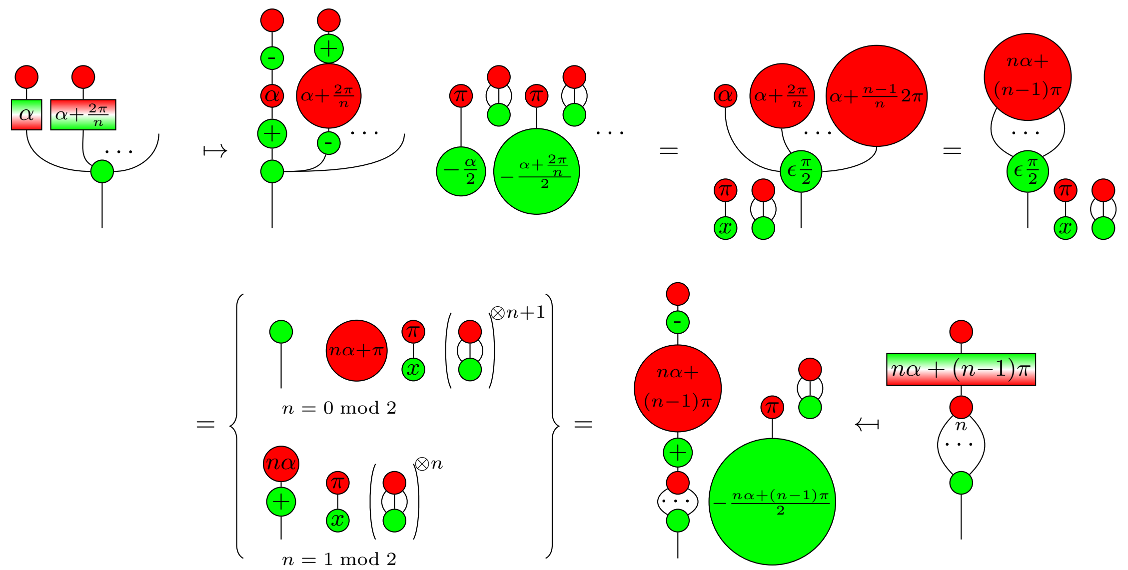
Proof (Proposition 7). (S1), (S2), (S3), (B1) and (B2) obviously hold. (ZO) also holds, the demonstration is the same as for $[\![.]\!]^{\mathrm{Y}_{\frac{\pi}{2}} \to ZX_r}$.
(RS1) holds. Using (K2) and lemma 26:



(RS2) holds. Using lemma 27, (S1), (H), (K2), (B2):



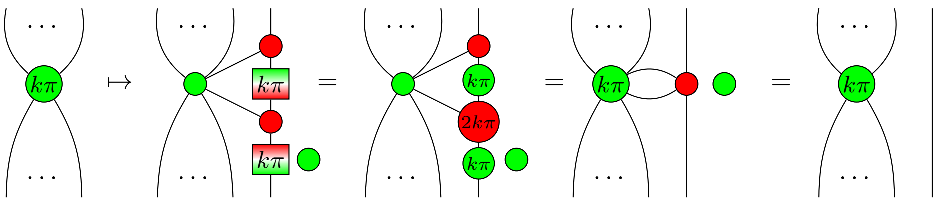
(RSUP_n) holds. Using (B1), 28, 26, (S1), (SUP_n), 29:



with:

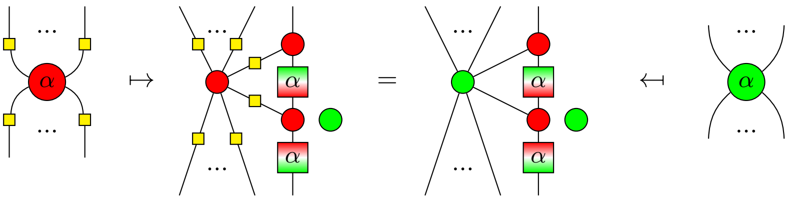
$$\epsilon\frac{\pi}{2} = (n \bmod 2)\frac{\pi}{2} \qquad \text{and} \qquad x = \sum_{k=0}^{n-1} -\frac{\alpha + \frac{2k\pi}{n}}{2} = -\frac{n\alpha + \frac{2\pi(n-1)n}{2n}}{2} = -\frac{n\alpha + (n-1)\pi}{2}$$

Proof (Proposition 8). First notice that:

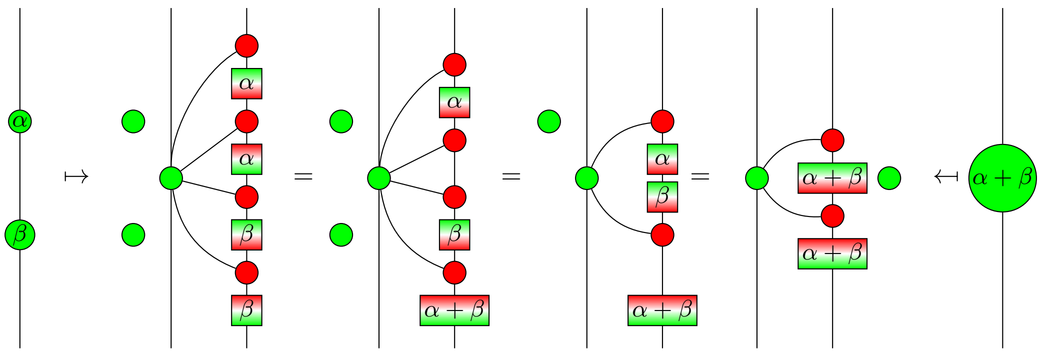


The result is the same with a red dot. Hence, all the rules that only display red and green dots of angles 0 – (S2), (S3), (B1), (B2)– are obviously preserved.

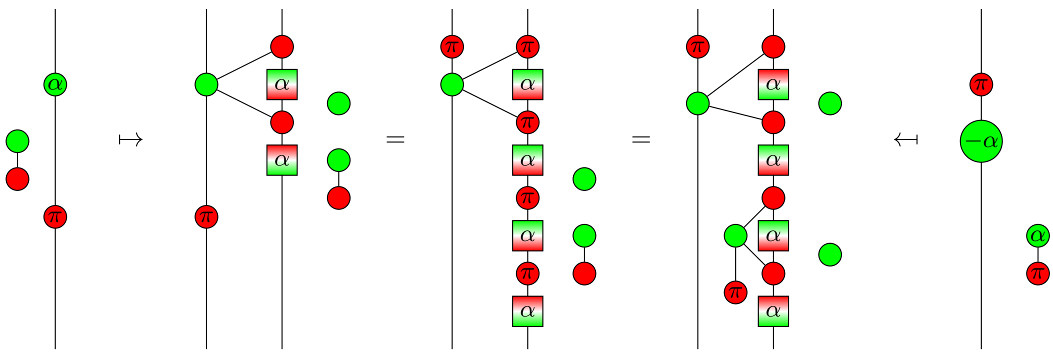
(H) holds:



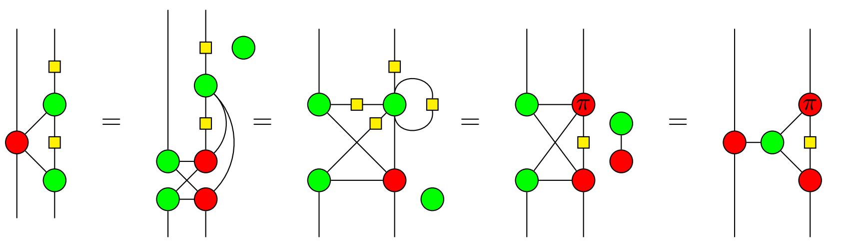
(S1) hold. Using lemmas 21, 6 and 3:



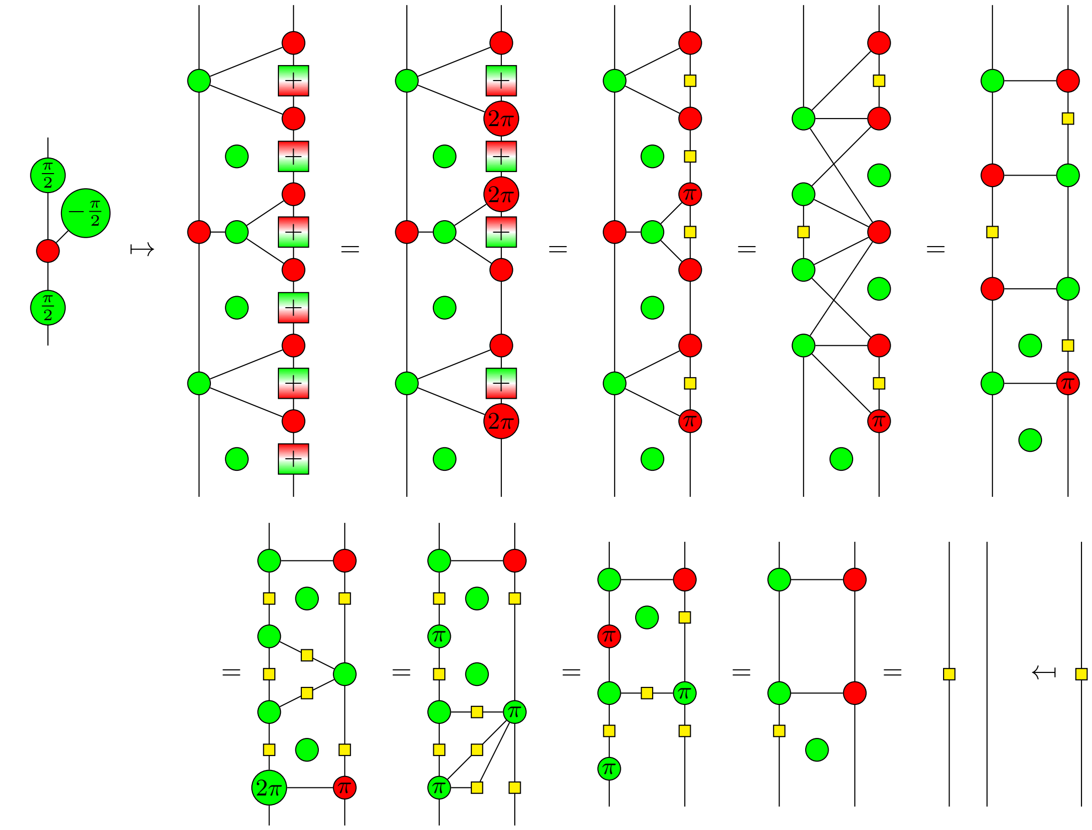
(K2) holds. Using 18, 1, (RS1), 9 and 11:



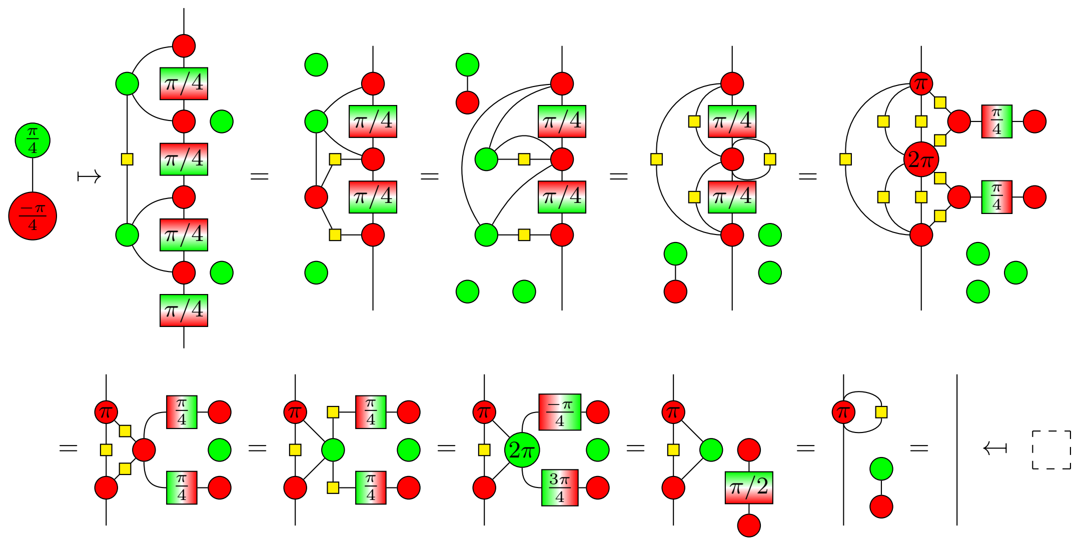
(EU) holds. First notice that:



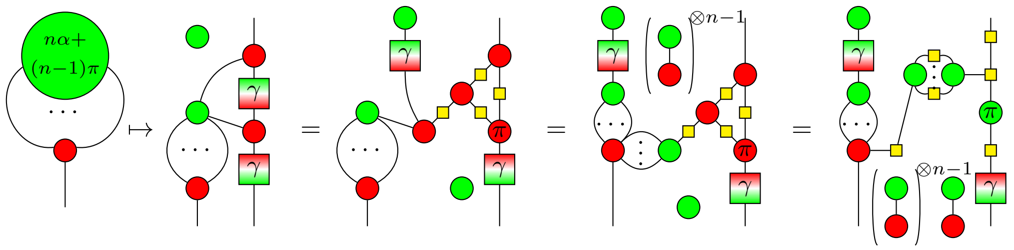
Then:

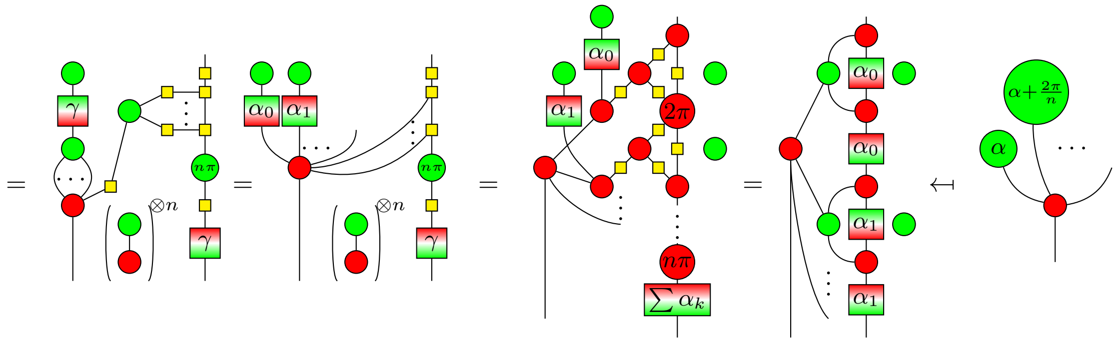


(E) holds. Using 21, (H), (B2), (HL), 3, 11, (S2), the Hadamard decomposition, (RSUP_n), and (RH):



(SUP_n) holds. Using 19, 4, (RH), 22, (RSUP_n), (RS1), 6, 21:





with $\alpha_k = \alpha + \frac{2k\pi}{n}$ and $\gamma = \sum \alpha_k = n\alpha + (n-1)\pi$.